Unit 13: Series

Contents

1 Power series: an example

e.g. I want to define a function with this equation:

$$g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n3^n}$$

For which $x \in \mathbb{R}$ is $g(x)$ convergent? We can use the Ratio Test.

$$\begin{aligned}
\text{Call } a_n &= \frac{x^n}{n3^n} \\
L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \frac{\left|\dfrac{x^{n+1}}{(n+1)3^{n+1}}\right|}{\left|\dfrac{x^n}{n3^n}\right|} \\
&= \frac{|x|}{3}
\end{aligned}$$

- If $|x| < 3$, then $L = \frac{|x|}{3} < 1$. By Ratio Test, $g(x)$ is absolutely convergent.
- If $|x| > 3$, then $L = \frac{|x|}{3} > 1$. By Ratio test, $g(x)$ is divergent.

We don't know what happens at $x = -3$ or $x = 3$ yet.

$$\begin{aligned}
g(3) &= \sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \frac{1}{n} = \infty \text{ (p-series with } p = 1) \\
g(-3) &= \sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \frac{(-1)^n}{n} \text{ convergent (by AST)}
\end{aligned}$$

Then, at $x = -3$, $g(x)$ is conditionally convergent, and 3, it is divergent.
To answer the original question, the domain of $g = [-3, 3)$ = the INTERVAL OF CONVERGENCE. 3 = the RADIUS OF CONVERGENCE.

2 Power series: the main theorem

Motivation

- Polynomials are nice
- What about "infinite polynomials"?

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

$$\text{or} \quad f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

- **e.g.** :
  - $g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n 3^n}$ has domain $[-3, 3)$
  - $h(x) = \sum_{n=0}^{\infty} x^n$ has domain $(-1, 1)$

Definition 2.1.
Let $a \in \mathbb{R}$.
A power series centered at a is a function f defined by an equation like

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

where $c_0, c_1, c_2, \dots \in \mathbb{R}$.

$$\text{Domain } f = \{x \in \mathbb{R} \ : \ \text{teh series } f(x) \text{ is convergent}\}$$

Note: $a \in \text{Domain } f$

Ultimate goal: write common functions as power series.

Theorem 2.1.
Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series centered at $a \in \mathbb{R}$.

1. *The domain of f is an interval centered at a:*

$$\begin{array}{ccc} (a-R, a+R) & (a-R, a+R] & \mathbb{R} \\ [a-R, a+R] & [a-R, a+R) & \{a\} \end{array}$$

   - *We call this domain the interval of convergence (IC) of f.*
   - *We call its radius the radius of convergence.* $0 \leq R \leq \infty$

2. - *In the **interior** of the IC, the series is absolutely convergent.*
   - *In the **exterior** of the IC, the series is divergent.*
   - *At the endpoints (if any), anything may happen.*

3. *In the interior of the IC, power series can be "treated like polynomials". They can be added, multiplied, composed...*
   In particular, they can be differentiated or integrated "term by term", and this does not change the radius of convergence.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n x^n &&= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots \\ f'(x) &= \sum_{n=0}^{\infty} c_n n x^{n-1} &&= c_1 + 2c_2 x + 3c_3 x^2 + \ldots \\ \int_0^x f(t)dt &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} &&= c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + \ldots \end{aligned}$$

Goals

1. Write as many functions as possible as power series
   $\longrightarrow$ Taylor series
2. Use that to make limits, integrals, estimations, differential equations, physics,...easier.

3 Taylor polynomials—the definition with the limit

Goal: approximate functions with polynomials.

f: function

$a \in$ domain f

P: polynomial

I want $P(x) \approx f(x)$ when x is close to a. Example: the tangent line. But what is a “good approximation near a”?

R: ”remainder” or ”error” $\quad R(x) = f(x) - P(x)$. I want R to be small.