Unit 13: Series

Contents

1 Infinite sums: a cautionary tale

We cannot take infinite sums as if they were finite sums.
e.g.:

$$
\begin{aligned}
S &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \\
xS &= x + x^2 + x^3 + x^4 + n \dots \\
S - xS &= 1 \\
S &= \frac{1}{1-x}
\end{aligned}
$$

When $x = 2$:

$$
\begin{aligned}
S &= \frac{1}{1-2} = -1 \\
S &= 1 + 2 + 4 + 8 + \dots
\end{aligned}
$$

e.g.2:

$$
\begin{aligned}
T &= \sum_{n=0}^{\infty} (-1)^n \\
&= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots
\end{aligned}
$$

"We can group the terms like this:"

$$
\begin{aligned}
T &= (1-1) + (1-1) + (1-1) + \dots \\
&= 0 + 0 + 0 + \dots = 0
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
T &= 1(-1+1) + (-1+1) + (-1+1) + \dots \\
&= 1 + 0 + 0 + 0 + \dots = 1
\end{aligned}
$$

$$T = 0 = 1???$$

Infinite sums: the right way

- What does adding up infinitely many numbers mean? Define what an infinite sum—a "series"—is.
- When is a series equal to a number? i.e., when is a series convergent?
- Which properties of finite sums carry over to infinite sums?

2 The definition of infinite sum

A series is an infinite sum:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$$

A sequence is an infinite list:

$$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \ldots$$

e.g. $S = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$?

$$\begin{aligned}
S_1 &= \frac{1}{2} \\
S_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\
S_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\
&\vdots \\
S_k &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} = \sum_{n=1}^{k} \frac{1}{2^n} = 1 - \frac{1}{2^k} \\
S &= \lim_{k \to \infty} S_k = 1
\end{aligned}$$

To sum the series $\sum_{n=1}^{\infty} a_n$,

- First construct the sequence of *partial sums* $\{S_k\}_{k=1}^{\infty}$

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\ldots \\
S_k &= a_1 + \cdots + a_k = \sum_{n=1}^{k} a_n
\end{aligned}$$

- Then, compute its limit:

$$\sum_{n=1}^{\infty} a_n \quad \lim_{k \to \infty} S_k$$

This series is *convergent* when this limit exists. Otherwise, it is *divergent.*

3 Telescopic series

Want to calculate $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$. Find a formula for the first k terms, then take the limit as k goes to infinity.

$$
\begin{aligned}
S &= \sum_{n=1}^{\infty} \frac{1}{n^2+n} \\
S_1 &= \frac{1}{2} \\
S_2 &= S_1 + \frac{1}{6} = \frac{2}{3} \\
S_3 &= S_2 + \frac{1}{12} = \frac{3}{4}
\end{aligned}
$$

Conjecture: $\forall k \geq 1, \quad S_k \, \frac{k}{k+1}$
Proof by induction.

Another way we could've obtained this:

$$
\begin{aligned}
S_k = \sum_{n=1}^{k} \frac{1}{n^2+n} &= \sum_{n=1}^{k} \frac{1}{n(n+1)} \\
&= \sum_{n=1}^{k} \left[\frac{1}{n} - \frac{1}{n+1}\right] \\
&= \left[1 - \cancel{\frac{1}{2}}\right] + \left[\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right] + \left[\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right] + \dots + \left[\cancel{\frac{1}{k}} - \frac{1}{k+1}\right] \\
&= 1 - \frac{1}{k+1} = \frac{k}{k+1}
\end{aligned}
$$

Then, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \lim_{k\to\infty} \frac{k}{k+1} = 1$

4 Examples of divergent series from the definition

$\sum_{n=1}^{\infty} 1$	$\sum_{n=0}^{\infty} (-1)^n$

e.g. $S = \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots = \infty$

$$\begin{aligned} S &= \lim_{k \to \infty} S_k \\ S_k &= \sum_{n=1}^{k} 1 = 1 + 1 + \cdots + 1 = k \\ S &= \lim_{k \to \infty} k = \infty \end{aligned}$$

Alternatively, $\sum_{n=1}^{\infty} 1$ is divergent.

e.g. $S = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \ldots$

$$\begin{aligned} S &= \lim_{k \to \infty} S_k \\ S_k &= \sum_{n=0}^{\infty} (-1)^n \\ S_0 &= 1 \\ S_1 &= 1 - 1 = 0 \\ S_2 &= 1 - 1 + 1 = 1 \\ S_3 &= 1 - 1 + 1 - 1 = 0 \\ &\ldots \end{aligned}$$

Then,

$$S_k = \begin{cases} 0 & \text{if } k \text{ odd} \\ 1 & \text{if } k \text{ even} \end{cases}$$

So $\lim_{k \to \infty} S_k$ DNE.

Alternatively, $\sum_{n=0}^{\infty} (-1)^n$ is divergent.

5 Geometric series

A geometric series is one of this form: $\boxed{\sum_{n=0}^{\infty} x^n}$ where x is a constant.

Let $x \in \mathbb{R}$. $S = \sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} S_k$, where $S_k = \sum_{n=0}^{k} x^n$.

$$\begin{aligned}
S_k &= 1 + x + x^2 + \cdots + x^k \\
xS_k &= x + x^2 + x^3 + \cdots + x^{k+1} && \text{multiply by } x \\
S_k - xS_k &= 1 - x^{k+1} \\
S_k &= \frac{1 - x^{k+1}}{1 - x} && \text{if } x \neq 1
\end{aligned}$$

Now take the limit as k approaches ∞

$$\begin{aligned}
\sum_{n=0}^{\infty} x^n &= \lim_{k\to\infty} S_k \\
&= \lim_{k\to\infty} \frac{1 - x^{k+1}}{1 - x} \\
&= \frac{1 - \lim_{k\to\infty} x^{k+1}}{1 - x} && \text{by limit laws}
\end{aligned}$$

$$\boxed{\sum_{n=0}^{\infty} x^n = \begin{cases} \dfrac{1}{1-x} & \text{if } -1 < x < 1 \\ \text{DIVERGENT} & \text{otherwise} \end{cases}}$$

6 Linearity of series

Linearity is true for finite sums, and is also true for infinite sums. This refers to these two properties:

$$\sum_{n=0}^{\infty}(a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n \qquad\qquad \sum_{n=0}^{\infty}(ca_n) = c\sum_{n=0}^{\infty} a_n$$

Finite sums can be reordered at-will, but infinite sums cannot.

Theorem 6.1.

IF $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *are both convergent,*

THEN $\sum_{n=0}^{\infty}(a_n + b_n)$ *is also convergent and*

$$\sum_{n=0}^{\infty}(a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$$

Theorem 6.2.

Let $c \in \mathbb{R}$.

IF $\sum_{n=0}^{\infty} a_n$ *is convergent,*

THEN $\sum_{n=0}^{\infty}(ca_n)$ *is also convergent, and*

$$\sum_{n=0}^{\infty}(ca_n) = c\sum_{n=0}^{\infty} a_n$$

Proof. for *Theorem 6.1*

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n &= \lim_{k\to\infty} S_k, &\quad \text{where } S_k &= \sum_{n=0}^{\infty} a_n \\
\sum_{n=0}^{\infty} b_n &= \lim_{k\to\infty} T_k, &\quad \text{where } T_k &= \sum_{n=0}^{\infty} b_n \\
\sum_{n=0}^{\infty}(a_n + b_n) &= \lim_{k\to\infty} R_k, &\quad \text{where } R_k &= \sum_{n=0}^{\infty}(a_n + b_n) \\
&\text{By properties of finite sums,} & R_k &= S_k + T_k
\end{aligned}$$

By hypothesis, *the first two limits exist.* Then, by the limit laws,

$$\lim_{k\to\infty} R_k \lim_{k\to\infty} S_k + \lim_{k\to\infty} T_k$$

∎

This is a template for series—how they're proven.

1. Write the infinite sum as a limit of partial sums—finite sums
2. Use the properties known to be true for finite sums
3. Pass to the limit

7 The tail of a series

We often talk about the tail of a series when studying infinite sums. Let's compare $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n$.

Theorem 7.1.
$\sum_{n=0}^{\infty} a_n$ is convergent $\iff$ $\sum_{n=1}^{\infty} a_n$ is convergent
Moreover, in that case

$$\left[\sum_{n=0}^{\infty} a_n\right] = a_0 + \left[\sum_{n=1}^{\infty} a_n\right]$$

Proof involves writing the series as a limit of finite sums, then we know the property to be true for finite sums.

Because this is true, we can write $\boxed{\sum_{n}^{\infty} a_n \text{ is convergent / divergent}}$; in other words, we don't need to specify where it starts.

Theorem.
Typical theorem:
IF for all $n \in \mathbb{N}$, $\boxed{\textit{something about } a_n}$
THEN the series $\sum_{n}^{\infty} a_n$ is convergent.

Theorem.
Generalized theorem:
IF $\exists n_0 \in \mathbb{N}$ S.T. $\forall n \in \mathbb{N}, \quad n \geq n_0 \implies \boxed{\textit{something about } a_n}$
THEN the series $\sum_{n}^{\infty} a_n$ is convergent.

8 A necessary condition for convergence of series

Sequences vs series: recall that a series is defined as the limit of a sequence—the sequence of partial sums.

So $\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} S_k, \quad$ where $\sum_{n=0}^{k} a_n$

$$\boxed{\sum_{n=0}^{\infty} a_n \text{ is convergent} \iff \{S_n\}_{n=0}^{\infty} \text{ is convergent}}$$

But what is the relation between $\sum_{n=0}^{\infty} a_n$ and $\{a_n\}_{n=1}^{\infty}$? I expect that if the series of a is convergent, then the sequence of a—not S—is convergent to 0.

> **Theorem 8.1.**
> *IF the series* $\sum_{n=0}^{\infty} a_n$ *is convergent,*
> *THEN* $\lim_{n\to\infty} a_n = 0$

How to use in practice?

1. IF $\lim_{n\to\infty} a_n = 0$
   THEN the series $\sum_{n=0}^{\infty} a_n$ may be convergent or divergent.
2. IF $\lim_{n\to\infty} a_n \neq 0$
   THEN the series $\sum_{n=0}^{\infty} a_n$ is divergent—the contrapositive.

Proof.
Assume the series $\sum_{n=0}^{\infty} a_n$ is convergent. WTS $\lim_{n\to\infty} a_n = 0$.
This means the following limit exists:

$$S = \lim_{k\to\infty} S_k, \qquad \text{where} S_k = \sum_{n=0}^{\infty} a_n$$

Notice that for every $n \geq 1$:

$$a_n = S_n - S_{n-1} = S - S = 0$$

Then we can use the limit laws:

$$\lim_{n\to\infty} a_n = \left[\lim_{n\to\infty} S_n\right] - \left[\lim_{n\to\infty} S_{n-1}\right]$$

■

Note: we can only quickly conclude that the series is divergent from this, never that it is convergent.

9 Positive series

We like positive series.

- A series $\sum_{n=0}^{\infty} a_n$ is *positive* when $\forall n \in \mathbb{N},\ a_n > 0$.
- A series $\sum_{n=0}^{\infty} a_n$ is *negative* when $\forall n \in \mathbb{N},\ a_n < 0$.
- A series $\sum_{n=0}^{\infty} a_n$ is *non-negative* when $\forall n \in \mathbb{N},\ a_n \geq 0$.

$$\text{An } \textit{arbitrary} \text{ series may be } \begin{cases} \text{convergent} \\ \text{divergent} \begin{cases} \text{to } \infty \\ \text{to } -\infty \\ \text{“oscillating”} \end{cases} \end{cases} \qquad \text{A } \textbf{positive} \text{ series may be } \begin{cases} \text{convergent} \\ \text{divergent to } \infty \end{cases}$$

There are only two options.

Proof.

In general, $\displaystyle\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} S_k$, where $\displaystyle S_k = \sum_{n=0}^{k} a_n$.

Assume the series $\displaystyle\sum_{n=0}^{\infty} a_n$ is positive. Then the sequence $\{S_n\}_{n=0}^{\infty}$ is increasing: $S_{n+1} - S_n = a_{n+1}$

Use the Monotone Convergence Theorem:

- An $\boxed{\text{increasing, bounded}}$ sequence is $\boxed{\text{convergent}}$
- An $\boxed{\text{increasing, unbounded}}$ sequence is $\boxed{\text{divergent to } \infty}$

∎

For POSITIVE SERIES ONLY, we can use shorthand notation.

- $\displaystyle\sum_{n}^{\infty} a_n = \infty$ means “divergent”
- $\displaystyle\sum_{n}^{\infty} a_n < \infty$ means “convergent”

The same applies to non-negative series.

The same applies to eventually non-negative series.

How does this help?

To prove a positive series is convergent, we only have to prove it does not diverge to ∞.

Useful theorems which *all rely on the fact that a positive series can only be convergent or infinity*:

1. Integral test
2. Basic comparison test
3. Limit comparison test

10 The integral test

Let f be a continuous function on $[1, \infty)$. What is the relation between $\sum_{n=1}^{\infty} f(n)$ and $\int_{x}^{\infty} f(x)dx$?

A series and an improper integral are defined in a similar manner.

$$\sum_{n=1}^{\infty} f(n) = \lim_{k\to\infty} \sum_{n=1}^{k} f(n) \qquad\qquad \int_1^{\infty} f(x)dx = \lim_{b\to\infty} \int_1^{b} f(x)dx$$

Is there a relation? There is a tool for integrals that we don't have for series in that we can calculate the value of an improper integral—if we have an expression for its antiderivative.

I will assume:

- f is positive – then $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)dx$ are convergent or ∞

  There is not a possibility of oscillation

- f is decreasing

Plan: find a relation between proper integrals and finite sums, then take the limit.

From a graph, we can tell that $\int_1^5 f(x)dx$ is bounded below by $\sum_{n=1}^{4} f(n)$. In order to be able to do this, we need the function to be decreasing, so that the $f(x)$ at the left endpoint would be the maximum on that interval.

We can also do the same to bound the area below. Then $\int_1^5 f(x)dx \geq \sum_{n=2}^{5} f(n)$. Then, we arrive at the expression

$$\begin{aligned}
&\sum_{n=2}^{5} f(n) \leq \int_1^5 f(x)dx \leq \sum_{n=1}^{4} f(n) \\
&\sum_{n=2}^{N} f(n) \leq \int_1^N f(x)dx \leq \sum_{n=1}^{N-1} f(n) \quad \text{since we could've picked any other integer} \\
&\sum_{n=2}^{\infty} f(n) \leq \int_1^{\infty} f(x)dx \leq \sum_{n=1}^{\infty} f(n) \quad \text{by taking the limit as } n \to \infty
\end{aligned}$$

Then, if the right series—which bounds the integral above—is convergent, then the integral must also be convergent. If the integral—which bounds the left series above—is convergent, then that series must also be convergent. Since the function is positive, this has to be convergent or ∞. Then, we have proven that

> **Theorem 10.1.**
> *Let $a \in \mathbb{R}$*
> *Let f be a* continuous, positive, decreasing *function on [a, ∞)*
> *Then*
>
> $$\int_a^{\infty} f(x)dx \quad \textit{is convergent} \quad \iff \quad \sum_{n}^{\infty} f(n) \quad \textit{is convergent}$$
>
> *Denoted with $\int_a^{\infty} f(x)dx \sim \sum_n^{\infty} f(n)$*

11 Integral test examples

p-series For which values of $p > 0$ is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent? Use integral test.

Let $f(x) = \dfrac{1}{x^p}$, then f is continuous, positive, and decreasing.

By integral test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \sim \int_1^{\infty} \frac{1}{x^p} dx$.

We know that $\displaystyle\int_1^{\infty} \frac{1}{x^p} dx$ is convergent iff $p > 1$. Thus, $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent iff } p > 1}$.

other Is $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ convergent? It's positive, decreasing, continuous for $x \geq 2$.

By integral test, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n} \sim \int_2^{\infty} \frac{1}{x \ln x} dx$. We can find the antiderivative for this function.

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x \ln x} dx &= \lim_{b \to \infty} \left[\int_2^{b} \frac{1}{x \ln x} dx \right] = \lim_{b \to \infty} \left[\ln \ln x \Big|_2^b \right] \\
&= \lim_{b \to \infty} [\ln \ln b - \ln \ln 2] = \infty
\end{aligned}$$

So by the integral test, the series diverges.

12 Comparison tests for series

Comparison tests for series and improper integrals are the same. Results based on:

- A positive series may only be convergent or divergent to ∞
- To prove a positive series is convergent, we only need to prove it is not ∞

Theorem 12.1.

BCT for series

Let $\sum_n^\infty a_n$ *and* $\sum_n^\infty b_n$ *be two series.*

- *Assume, for every* $n \in \mathbb{N}$, $a \leq a_n \leq b_n$
- *THEN*
  - *IF* $\displaystyle\sum_n^\infty a_n = \infty$, *THEN* $\displaystyle\sum_n^\infty b_n = \infty$.
  - *IF* $\displaystyle\sum_n^\infty b_n < \infty$, *THEN* $\displaystyle\sum_n^\infty a_n < \infty$.

Theorem 12.2.

LCT for series

Let $\sum_n^\infty a_n$ *and* $\sum_n^\infty b_n$ *be two positive series.*

- *IF the limit* $L = \lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$ *exists—is a number—and* $L > 0$.
- *THEN*

$$\sum_n^\infty a_n \quad \textit{and} \quad \sum_n^\infty b_n$$

  are both convergent or both divergent.

13 Alternating series

Definition: A series $\sum_{n}^{\infty} a_n$ is alternating when $\forall n, a_n a_{n+1} < 0$

This means the terms "alternate" between positive and negative.

Example:

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \ldots \\ S_1 &= 1 \\ S_2 &= 1 - \frac{1}{2} \\ S_3 &= 1 - \frac{1}{2} + \frac{1}{3} \\ S_4 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \\ &\ldots \end{aligned}$$

We notice from a graph that $S_2 < S_4 < S_6 < \cdots < S_5 < S_3 < S_1$

$\{S_{2n}\}_n$ is increasing and bounded above (by S_1).

$\{S_{2n+1}\}$ is decreasing and bounded below (by S_2).

Then by MCT, both are convergent.

Call $A = \lim_{n\to\infty} a_n$, $B = \lim_{n\to\infty} b_{2n+1}$, then

$$\begin{aligned} S_{2n+1} &= S_{2n} + \frac{1}{2n+1} \qquad \text{we can use the limit laws, since all three terms have limits} \\ \lim_{n\to\infty} S_{2n+1} &= \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} \frac{1}{2n+1} \\ B &= A + 0 \end{aligned}$$

Theorem 13.1.

Lemma:

Let $\{c_n\}_n^{\infty}$ be a sequence.

- *IF the sequences of even and odd terms*

$$\{c_{2n}\}_n^{\infty} \quad \text{and} \quad \{c_{2n+1}\}_n^{\infty}$$

  are convergent to the same limit

- *THEN the full sequence $\{c_n\}_n^{\infty}$ is also convergent to the same limit.*

Theorem 13.2.

Alternating series test:

Consider a series of the form

$$\sum_{n}^{\infty}(-1)^n b_n \quad \textit{or} \quad \sum_{n}^{\infty}(-1)^{n+1} b_n$$

IF

1. $\forall n, \quad b_n > 0$
2. *the sequence* $\{b_n\}_n^\infty$ *is decreasing*
3. $\lim_{n\to\infty} b_n = 0$

THEN the series is convergent.

14 Estimating the value of an alternating series

Estimate the value of $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ with an error smaller than 0.001. First, we need to show that it is convergent—it satisfies the hypotheses of the alternating series theorem. By AST, the series is convergent.

The actual value is $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = \lim_{k\to\infty} S_k$. This suggests that we can use S_k as an estimate.

Estimate: $S_k = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ for some large k? Which value of k?

We need the error to be smaller than a precise number. Error of estimation: $|S - S_k|$.

Theorem 14.1.

Alternating series theorem, part 2*: consider a series of the form*

$$\sum_{n}^{\infty} (-1)^n b_n \quad \textit{or} \quad \sum_{n}^{\infty} (-1)^{n+1} b_n$$

- *IF it satisfies the same three hypotheses as before*
- *THEN* $|S - S_k| < b_{k+1}$

where S_k is the k-th partial sum of the series.

Then, error of estimation: $|S - S_k| < \frac{1}{(k+1)^4}$. We need to choose k so that $\frac{1}{(k+1)^4} < 0.001$. Pick k to be 5.

Estimate: $-1 + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} \approx -0.94753....$

15 Absolute convergence vs conditional convergence

Is $\sum_{n}^{\infty} \frac{\sin n}{n^2}$ convergent?

What is the relation between the series $\sum_{n}^{\infty} a_n$ and $\sum_{n}^{\infty} |a_n|$?

Theorem 15.1.

Absolute convergence test

Let $\sum_{n}^{\infty} a_n$ be a series.

- *IF the series $\sum_{n}^{\infty} |a_n|$ is convergent*
- *THEN the series $\sum_{n}^{\infty} a_n$ is convergent*

We can look at $\sum_{n}^{\infty} \frac{|\sin n|}{n^2}$. Since it is a positive series, we can use comparison tests.

$0 \leq \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$. We know that $\sum_{n}^{\infty} \frac{1}{n^2}$ is convergent, so by BCT, $\sum_{n}^{\infty} \frac{|\sin n|}{n^2}$ is also convergent. By ABSOLUTE CONVERGENCE TEST, $\frac{\sin n}{n^2}$ is also convergent.

e.g. Let $p_n = \begin{cases} 1/n & \text{if } n \text{ is prime} \\ -1/n & \text{otherwise} \end{cases}$ is $\sum_{n}^{\infty} p_n$ convergent?

First, we look at $\sum_{n}^{\infty} |p_n|$. $\sum_{n}^{\infty} |p_n| = \sum_{n}^{\infty} \frac{1}{n} = \infty$.

The absolute convergence test does not apply, so we do not know if this series is convergent or divergent.

Definition 15.1.

A convergent series $\sum_{n}^{\infty} a_n$ is...

- *absolutely convergent when $\sum_{n}^{\infty} |a_n|$ is also convergent.*
- *conditionally convergent when $\sum_{n}^{\infty} |a_n| = \infty$.*

For instance, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent. By the alternating series test, it is convergent. $\sum_{n=1}^{\infty} \frac{1}{n}$, its absolute value, however, is divergent.

For instance, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent. By the alternating series test, it is convergent. $\sum_{n=1}^{\infty} \frac{1}{n^2}$, its absolute value, is also convergent.

16 Proof of the absolute convergence test

Notation: "P.T" means "positive terms", "N.T" means "negative terms". For instance,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \dots$$
$$\sum_{n=1}^{\infty} \text{P.T} = 1 \qquad + \frac{1}{3} \qquad + \frac{1}{5} \qquad + \frac{1}{7} + \dots$$

Essentially, we are replacing the negative terms with zeros.

Proof.
Assume $\sum_{n}^{\infty} |a_n| < \infty$.
Then, $\sum_{n}^{\infty} |\text{P.T}| \le \sum_{n}^{\infty} |a_n|$ and $\sum_{n}^{\infty} |\text{N.T}| \le \sum_{n}^{\infty} |a_n|$.
By BCT, $\sum_{n}^{\infty} |\text{P.T}| < \infty$ and $\sum_{n}^{\infty} |\text{N.T}| < \infty$
Therefore, $\sum_{n}^{\infty} \text{P.T}$ and $\sum_{n}^{\infty} \text{N.T}$ are convergent.
By key observation $\sum_{n}^{\infty} a_n = \sum_{n}^{\infty} \text{P.T} + \sum_{n}^{\infty} \text{N.T}$. Thus $\sum_{n}^{\infty} a_n$ is convergent. ■

17 Ratio test

To determine if a series is convergent or divergent, by computing a limit.

Let $\sum_{n}^{\infty} a_n$ be a series. Assume $\forall n, a_n \neq 0$. Assume the limit

$$L = \lim_{n \to \infty} \left| \frac{a_{n_+1}}{a_n} \right| \quad \text{exists or is } \infty$$