Unit 14: Series

Contents

1 Power series: an example

e.g. I want to define a function with this equation:

$$g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n3^n}$$

For which $x \in \mathbb{R}$ is $g(x)$ convergent? We can use the Ratio Test.

$$\text{Call } a_n = \frac{x^n}{n3^n}$$
$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{\left|\frac{x^{n+1}}{(n+1)3^{n+1}}\right|}{\left|\frac{x^n}{n3^n}\right|}$$
$$= \frac{|x|}{3}$$

- If $|x| < 3$, then $L = \frac{|x|}{3} < 1$. By Ratio Test, $g(x)$ is absolutely convergent.
- If $|x| > 3$, then $L = \frac{|x|}{3} > 1$. By Ratio test, $g(x)$ is divergent.

We don't know what happens at $x = -3$ or $x = 3$ yet.

$$g(3) = \sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \frac{1}{n} = \infty \text{ (p-series with } p = 1)$$
$$g(-3) = \sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \frac{(-1)^n}{n} \text{ convergent (by AST)}$$

Then, at $x = -3$, $g(x)$ is conditionally convergent, and 3, it is divergent.

To answer the original question, the domain of $g = [-3, 3)$ = the INTERVAL OF CONVERGENCE. 3 = the RADIUS OF CONVERGENCE.

2 Power series: the main theorem

Motivation

- Polynomials are nice
- What about "infinite polynomials"?

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

$$\text{or} \quad f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

- **e.g.** :
  - $g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n 3^n}$ has domain $[-3, 3)$
  - $h(x) = \sum_{n=0}^{\infty} x^n$ has domain $(-1, 1)$

Definition 2.1.
Let $a \in \mathbb{R}$.
A power series centered at a is a function f defined by an equation like

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

where $c_0, c_1, c_2, \dots \in \mathbb{R}$.

$$\text{Domain } f = \{x \in \mathbb{R} \;:\; \text{teh series } f(x) \text{ is convergent}\}$$

Note: $a \in$ Domain f

Ultimate goal: write common functions as power series.

Theorem 2.1.
Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series centered at $a \in \mathbb{R}$.

1. *The domain of f is an interval centered at a:*

$$\begin{array}{ccc} (a-R, a+R) & (a-R, a+R] & \mathbb{R} \\ [a-R, a+R] & [a-R, a+R) & \{a\} \end{array}$$

   - *We call this domain the interval of convergence (IC) of f.*
   - *We call its radius the radius of convergence.* $0 \leq R \leq \infty$

2. - *In the **interior** of the IC, the series is absolutely convergent.*
   - *In the **exterior** of the IC, the series is divergent.*
   - *At the endpoints (if any), anything may happen.*

3. *In the interior of the IC, power series can be "treated like polynomials". They can be added, multiplied, composed...*
   In particular, they can be differentiated or integrated "term by term", and this does not change the radius of convergence.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n x^n &&= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ... \\ f'(x) &= \sum_{n=0}^{\infty} c_n n x^{n-1} &&= c_1 + 2c_2 x + 3c_3 x^2 + ... \\ \int_0^x f(t)dt &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} &&= c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + ... \end{aligned}$$

Goals

1. Write as many functions as possible as power series
   $\longrightarrow$ Taylor series
2. Use that to make limits, integrals, estimations, differential equations, physics,...easier.

3 Taylor polynomials—the definition with the limit

Goal: approximate functions with polynomials.

f: function

$a \in \text{Domain } f$

P: polynomial

I want $P(x) \approx f(x)$ when x is close to a. Example: the tangent line. But what is a “good approximation near a”?
R: ”remainder” or ”error” $R(x) = f(x) - P(x)$. I want R to be small. This means we need $\lim_{x \to a} R(x) = 0$ “fast”. For instance, there are many lines with remainder 0, but the tangent line’s remainder approaches 0 the “fastest”. But how do we measure how fast the limit is?
We notice that the larger exponent polynomials approach 0 faster. We compare the remainder with powers of x.

Definition 3.1.
Let f and g be continuous functions at 0.
Let $n \in \mathbb{N}$.
We say that <u>g is an approximation for f near 0 of order n</u> when

$$\lim_{x \to 0} \frac{f(x) - g(x)}{x^n} = 0$$

This means that $f(x) = g(x) + R(x)$ and as $x \to 0$,

$$R(x) \to 0 \quad \text{faster than} \quad x^n \to 0$$

Definition 3.2.
Let $a \in \mathbb{R}$. Let f and g be continuous functions at a.
Let $n \in \mathbb{N}$.
We say that <u>g is an approximation for f near a of order n</u> when

$$\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n} = 0$$

First definition of Taylor polynomial

Definition 3.3.

Let $a \in \mathbb{R}$.

Let f be a continuous function defined at and near a.

Let $n \in \mathbb{N}$.

The n-th Taylor polynoial for f at a is a polynomial P_n

- *...that is an approximation for f near a of order n:*

$$\lim_{x \to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0$$

- *with degree at most n*

4 Taylor polynomials—the definition with the derivatives

Definition 4.1.
A function f is called …

- C^0 *when f is continuous*
- C^1 *when f' exists and is continuous*
- C^2 *when f' and f'' exist and are continuous*
- …
- C^n *when $f', f'', \dots, f^{(n)}$ exist and are continuous*
- C^∞ *when all derivatives exist (and are continuous)*

For now, assume f and g are C^∞. Can I transform the condition

$$\lim_{x\to a} \frac{f(x)-g(x)}{(x-a)^n} = 0$$

into a condition about their derivatives?

- Call $L = \lim_{x\to a} \frac{f(x)-g(x)}{(x-a)^n}$
  - If $f(a) - g(a) \neq 0$, then "$L = \frac{\text{not } 0}{0} = \pm\infty$".
  - So, assume $f(a) = g(a)$. We get $0/0$. Use L'hôpital's.
- $L \overset{*}{=} \lim_{x\to a} \frac{f'(x)-g'(x)}{n(x-a)^{n-1}}$
  - If $f'(a) - g'(a) \neq 0$, then "$L = \frac{\text{not } 0}{0} = \pm\infty$"
  - So, assume $f'(a) = g'(a)$. We get $0/0$. Use L'hôpital's.
- $L \overset{*}{=} \lim_{x\to a} \frac{f''(x)-g''(x)}{n(n-1)(x-a)^{n-2}} \dots$
- After using L'hôpital's rule n times, we get

$$L \overset{*}{=} \lim_{x\to a} \frac{f^{(n)}(x)-g^{(n)}(x)}{n!} = \frac{f^{(n)}(a)-g^{(n)}(a)}{n!} \qquad \text{since the derivatives are continuous}$$

$$L = 0 \iff \begin{cases} f(a) & = g(a) \\ f'(a) & = g'(a) \\ \dots & \\ f^{(n-1)}(a) & = g^{(n-1)}(a) \\ f^{(n)}(a) & = g^{(n)}(a) \end{cases}$$

I have used that f and g were C^n.

Theorem 4.1.

Let $a \in \mathbb{R}$. Let $n \in \mathbb{N}$.

Let f and g be C^n functions at a.

The following are equivalent:

1. $\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n} = 0$
   ("g is a good approximation for f near a")

2. $f(a) = g(a), f'(a) = g'(a), \ldots, f^n(a) = g^n(a)$
   ("g and f have the same first few derivatives at a")

This proof could be made more formal by using induction.

Definition 4.2.

Let $a \in \mathbb{R}$.

Let $n \in \mathbb{N}$.

Let f be a C^n function at a.

The n-th Taylor polynomial for f at a is

- *a polynomial P_n such that*
- $P_n(a) = f(a), P_n'(a) = f'(a), \ldots P_n^{(n)}(a) = f^{(n)}(a)$
- *with degree at most n.*

This definition is more useful that the original, limit definition for constructing these polynomials. However, we should keep the original definition in mind—it tells us why Taylor polynomials make good approximations.

Notice that this is not a completely new idea: the first Taylor polynomial according to this definition is $y = P_1(x)$, or the tangent line.

5 Taylor polynomials—the formula

Recall:

The n-th Taylor polynomial P_n for the function f at $a \in \mathbb{R}$...

- is an approximation for f near a of order n:

$$\lim_{x \to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0$$

- equivalently, has same value and first n derivatives as f at a:

$$P_n(a) = f(a),\ P_n'(a) = f'(a),\ \ldots,\ P_n^{(n)}(a) = f^{(n)}(a)$$

For simplicity, begin with:

Case $a = 0$

$$P_n(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$$

$$\begin{aligned} P_n(0) &= f(0) \\ P_n'(0) &= f'(0) \\ P_n''(0) &= f''(0) \\ &\ldots \\ P_n^{(n)}(0) &= f^{(n)}(0) \end{aligned}$$

So,

$$\begin{aligned} P_n^{(k)}(0) = k! \cdot c_k &= f^{(k)} \\ c_k &= \frac{f^{(k)}(0)}{k!} \end{aligned}$$

$$\boxed{P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k}$$

Now, how do we

Generalize a**?** Instead of

$$P_n(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$$

write

$$P_n(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3 + \cdots + b_n(x-a)^n$$

We obtain

$$\boxed{P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}$$

Definition 5.1.
Third definition of Taylor polynomial

- *Let* $a \in \mathbb{R}$
- *Let* $n \in \mathbb{N}$
- *Let* f *be a* C^n *function at* a.

The *n-th Taylor polynomial* *for* f *at* a *is*

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Notes:

- degree $P_n \leq n$
- The Taylor polynomials of a function are unique

We can see that the higher degree the Taylor polynomial is, the more accurate the approximation.

Definition 5.2.
Taylor series

- *Let* $a \in \mathbb{R}$
- *Let* f *be a* C^∞ *function at* a

The *Taylor series* *for* f *at* a *is the power series*

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Equivalently, $\forall k \in \mathbb{N},\ S^{(k)}(a) = f^{(k)}(a)$

Ideal case: $f(x) = S(x)$; we call such functions *analytic*.

Note: “Maclaurin series” means “Taylor series at 0”.

6 The four main Maclaurin series

- The n-th Taylor polynomial for f at a:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- The Taylor series for f at a:

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- Maclaurin series is a Taylor series at 0
- f is analytic when $f(x) = S(x)$

e.g. Maclaurin series for $f(x) = e^x$

- For all $k \in \mathbb{N},\ f^{(k)}(x) = e^x,\ f^{(k)}(0) = 1.$

$$\begin{aligned} S(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \end{aligned}$$

e.g. Taylor series for $f(x) = e^x$ at c

To write $f(x) = e^x$ in terms of powers of $(x-c)$ ..., we can use the substitution

$$\boxed{u = x - c}$$

...write f in terms of powers of u instead

$$e^x = e^{c+u} = e^c \cdot e^u = e^c \sum_{n=0}^{\infty} \frac{u^n}{n!}$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n}$$

e.g. Maclaurin series for $g(x) = \sin x$

- $g(x) = \sin x$
- $g'(x) = \cos x$
- $g''(x) = -\sin x$
- $g'''(x) = -\cos x$
- $g^4(x) = \sin x$

- $g(0) = 0$
- $g'(0) = 1$
- $g''(0) = 0$
- $g'''(0) = -1$
- ...

$$
\begin{aligned}
S(x) &= \sum_{n=0}^{\infty} g^{(0)}(0) \frac{x^n}{n!} \\
&= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\
&= \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}
\end{aligned}
$$

sin is analytic, so $\sin x = S(x)$

e.g. Maclaurin series for $h(x) = \cos x$ Left as exercise

The four main Maclaurin series

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots && \text{for all } x \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots && \text{for all } x \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots && \text{for all } x \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots && \text{for } |x| < 1
\end{aligned}
$$

7 Analytic functions and the remainder theorems

An analytic function is equal to its Taylor series...

- ...centered where?
- ...on which domain?

> **Definition 7.1.**
> *Let f be a C^∞ function defined on an open interval I.*
>
> - *Let $a \in I$. Let $S_a(x)$ be the Taylor series of f at a.*
>   *We say that* $\boxed{f \text{ is analytic at } a}$ *when*
>
> $$\exists \text{ an open interval } J_a \text{ centered at } a \text{ such that}$$
> $$\forall x \in J_a, \quad f(x) = S_a(x)$$
>
> - *We say that* $\boxed{f \text{ is analytic}}$ *when*
>
> $$\forall a \in I, \quad f \text{ is analytic at } a$$

We often carelessly summarize this as "the function equals its Taylor series".

Some results about analytic functions (that I won't prove)

1. Polynomials are analytic
2. Sums, products, quotients* and composition of analytic functions are analytic
3. Derivatives and antiderivatives of analytic functions are analytic
4. In the interior of the interval of convergence, a power series "can be manipulated like a polynomial"
5. The Taylor series of a function at a point is unique

> **Goal**: prove that
>
> $$f(x) = e^x \quad \text{and} \quad g(x) = \sin x$$
>
> are analytic.

Why only these two functions? Other elementary functions can be written in terms of these two functions, polynomials, and the operations given.

How will we prove a function is analytic?

Let f be a function defined on an open interval I. Let $a \in I$.

- If f is C^n, we can write the n-th Taylor polynomial P_n at a:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k, \qquad \boxed{f(x) = P_n(x) + R_n(x)}$$

- If f is C^∞, we can write the Taylor series S at a:

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k, \qquad \boxed{\lim_{n\to\infty} P_n(x) = S(x)}$$

- f is analytic when, in addition,

$$f(x) = S(x) \qquad \boxed{\text{This is equivalent to } \lim_{n\to\infty} R_n(x) = 0}$$

  Didn't we know this already? No.

  1. We know, from the definition of Taylor polynomial:

  $$\text{for fixed } n, \quad \lim_{x\to a} \frac{R_n(x)}{(x-a)^n} = 0$$

  2. We want, for the function to be analytic:

  $$\text{for fixed } x, \quad \lim_{n\to\infty} R_n(x) = 0$$

How to estimate the remainder of a Taylor polynomial?

Theorem 7.1.
Typical remainder theorem

- *IF (hypotheses)*
- *THEM (some formula for $R_n(x)$)*

There are at least three versions:

1. Lagrange's form
2. Cauchy's form
3. Integral form

They are all proven with the MEAN VALUE THEOREM or ROLLE'S THEOREM.

Theorem 7.2.

Lagrange's remainder theorem

- *Let I be an open interval. Let $a \in I$.*
- *Let $n \in \mathbb{N}$.*
- *Let f be a C^{n+1} function on I.*
- *Let $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ be its n-th Taylor polynomial.*
- *Let $R_n(x) = f(x) - P_n(x)$ be the remainder.*

THEN

$$\exists \xi \text{ between } a \text{ and } x \text{ such that } R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

How is a remainder theorem useful?

1. To prove a function is analytic:

$$\lim_{n \to \infty} R_n(x) = 0$$

2. To estimate:

   Approximate $f(x)$ by $P_n(x)$ and bound the error.

8 A proof that the exponential function is analytic

Goal: prove that $f(x) = e^x$ is analytic at 0

- $f(x) = e^x,\ a = 0$. Since this function is C^∞

$$e^x = P_n(x) + R_n(x)$$

- $P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ is the n-th Taylor polynomial.
- $R_n(x)$ is the n-th remainder.

We need to show $\boxed{\forall x \in \mathbb{R},\ \lim_{n\to\infty} R_n(x) = 0}$

- Then, as a consequence,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

To prove that the exponential equals its Maclaurin series, all we need to show is that the remainders approach 0 as $n \to \infty$.

Proof.

- Fix $x \in \mathbb{R}$.
- Use Lagrange's remainder theorem for $f(x) = e^x$ and $a = 0$.
  Then, $(\forall n \in \mathbb{N})\quad \exists \xi$ between 0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} = \frac{e^\xi}{(n+1)!} x^{n+1}$$

- Then ...?

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \left[e^\xi \frac{x^{n+1}}{(n+1)!} \right] = \cancel{e^\xi \left[\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \right]}$$

- No. ξ is not a constant. I depends on n and x.
  All we know about ξ is that it is between 0 and x.
- Case 1: $x > 0$. Then $0 < \xi < x$, so

$$0 \le R_n(x) = e^\xi \frac{x^{n+1}}{(n+1)!} \le e^x \frac{x^{n+1}}{(n+1)!}$$

  - From BIG THEOREM $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$
  - From SQUEEZE THEOREM $\lim_{n\to\infty} R_n(x) = 0$

- Case 2: $x < 0$. (Left as exercise)

∎

9 How to write functions as power series quickly

The slow method

1. Start with a C^∞ function f
2. Obtain the Taylor series at a:

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

3. Use the remainder theorems to prove

$$\lim_{n\to\infty} R_n(x) = 0$$

4. Then $f(x) = S(x)$

The better method

The tool
In the interior of the interval of convergence, a power series can be manupulated like a polynomial.

e.g. Write these functions as power series:

1. $f(x) = e^{-x}$ at $a = 0$
2. $f(x) = x^3 \sin x^2$ at $a = 0$
3. $f(x) = \frac{1}{x}$ at $a = 0$
4. $f(x) = \ln(1+x)$ at $a = 0$

Solutions

1. Write $f(x) = e^{-x}$ as a power series at $a = 0$
   This is close to e^x, which we already know how to write as a power series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x$$

   We want e^{-x}, so we just replace x with $-x$ everywhere.

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \quad \text{for all } x$$

2. Write $f(x) = x^3 \sin x^2$ as a power series at $a = 0$

This is close to $\sin x$.

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \text{for all } x \\
\sin x^2 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n^2+1)!} x^{2(2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2} \\
f(x) = x^3 \sin x^2 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+5} \qquad \text{for all } x
\end{aligned}$$

3. Write $f(x) = \frac{1}{x}$ as a power series at 3

   Change of variable $\boxed{u = x - 3}$

$$\begin{aligned}
\frac{1}{x} &= \frac{1}{u+3} \qquad \left(\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1 \right) \\
&= \frac{1}{3} \frac{1}{1 + (u/3)} = \frac{1}{3} \frac{1}{1 - (-u/3)} = \frac{1}{3} \sum_{n=0}^{\infty} (-\frac{u}{3})^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} u^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (x-3)^n
\end{aligned}$$

Valid when $\left|-\frac{u}{3}\right| < 1$. This is the same thing as saying that $|u| < 3$. $|x - 3| < 3$.

10 Lograithm as a power series

How to write log as power series? Let $g(x) = \ln x$. Domain $g = (0, \infty)$

- Option 1: Taylor series centered at $a > 0$ of $g(x) = \ln x$.
- Write $f(x) = \ln(1 + x)$ as power series at 0 .

$$\begin{aligned} f'(x) &= \frac{1}{1+x} \quad \left(\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1 \right) \\ &= \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \quad (\text{for } |x| < 1) \\ f(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C \quad (\text{for } |x| < 1) \end{aligned}$$

Evaluate at $x = 0$:

$$f(0) = 0 + C \quad C = 0$$

$$\begin{aligned} \ln(1+x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \\ &= \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m} \end{aligned}$$

Remember, true only for $|x| < 1$.

Taylor applications: estimations

e.g. Estimate $\sqrt{e}$ with error < 0.001.

- Let $f(x) = e^x$. Want $f(\frac{1}{2})$.
- $\frac{1}{2} = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{2^k} = P_n(\frac{1}{2}) + R_n(\frac{1}{2})$
- $P_n(\frac{1}{2}) = \sum_{k=0}^{n} \frac{1}{k!}\frac{1}{2^k}$
- Want $\left|R_n(\frac{1}{2})\right| < 0.001$

Use Lagrange's remainder theorem!

$$\exists \xi \in (0, \frac{1}{2}) \quad \text{such that} \quad R_n\left(\frac{1}{2}\right) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\left[\frac{1}{2} - 0\right]^{n+1}$$

$$0 < R_n(\frac{1}{2}) < \frac{e^{\frac{1}{2}}}{(n+1)!}\frac{1}{2^{n+1}} < \frac{2}{2^{n+1}(n+1)!} < 0.001$$

Need $2^n(n+1)! > 1000$; $n = 4$ works: $2^5 \cdot 5! = 16 \cdot 120 > 1000$. Estimation for $\sqrt{e}$:

$$P_4(\frac{1}{2}) = \sum_{k=0}^{4} \frac{1}{k!2^k} = \cdots = \frac{211}{128} \approx 1.65843\ldots$$

11 Taylor applications: integrals

- Compute $I = \int_0^3 e^{-x^2} dx$.
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.
- Since there is no domain restriction, we replace x with $-x^2$.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)}{n!} \qquad \text{for all } x$$

12 Taylor applications: limits

$$\begin{aligned}
&\lim_{x\to 0}\frac{\sin x - x}{x^3}\\
&=\lim_{x\to 0}\frac{\left[x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7\right]+\dots-x}{x^3}\\
&=\lim_{x\to 0}\frac{\left[-\frac{1}{3!}x^3+\frac{1}{5!}x^5+\dots\right]}{x^3}=\lim_{x\to 0}\left[-\frac{1}{6}+\frac{1}{120}x^2+\dots\right]\\
&=-\frac{1}{6}
\end{aligned}$$

We can notice here that the only term that really mattered was the first.

e.g. 2:

$$\lim_{x\to 0}\frac{3x^2-e^{x^2}+\cos(2x)}{x\sin x-\ln(1+x^2)}$$

Numerator

- (+) $3x^2$
- (-) $e^{x^2}=1+x^2+\frac{1}{2!}(x^2)^2+\frac{1}{3!}(x^2)^3+\dots$
- (+) $\cos(2x)=1-\frac{1}{2!}(2x)^2+\frac{1}{4!}(2x)^4-\frac{1}{6!}(2x)^6+\dots$

$$3x^2-e^{x^2}+\cos(2x)=[-1+1]+\left[3-1-\frac{4}{2}\right]x^2+\left[-\frac{1}{2}+\frac{2^4}{4!}\right]x^4+\dots=\frac{1}{6}x^4+\text{higher order terms}$$

Denominator

- (+) $x\sin x=x[x-\frac{1}{3!}x^3+\frac{1}{5!}x^5]-\dots$
- (-) $\ln(1+x^2)=x^2-\frac{1}{2}(x^2)^2+\frac{1}{3}(x^2)^3-\dots$

$$x\sin x-\ln(1+x^2)=[1-1]x^2+\left[-1\frac{1}{6}+\frac{1}{2}\right]x^4+\dots=\frac{1}{3}x^4+\text{hiogher order terms}$$

$$\begin{aligned}
&\lim_{x\to 0}\frac{3x^2-e^{x^2}+\cos(2x)}{x\sin x-\ln(1+x^2)}\\
&=\lim_{x\to 0}\frac{\frac{1}{6}x^4+\text{higher order terms}}{\frac{1}{3}x^4+\text{higher order terms}}\\
&=\frac{\frac{1}{6}}{\frac{1}{3}}=\frac{1}{2}
\end{aligned}$$

13 Taylor applications: adding series

e.g. 1: compute $A = \sum_{n=1}^{\infty} \frac{n}{2^n}$

- Want $\sum\limits_{n=1}^{\infty} nx^n$ when $x = \frac{1}{2}$. This is almost, but not quite, the geometric series—$\sum_{n=0}^{\infty} x^n$, except there's an additional n.

$$\begin{aligned}\sum_{n=1}^{\infty} nx^{n-1} &= \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}\\ \sum_{n=1}^{\infty} nx^n &= \frac{x}{(1-x)^2}\end{aligned}$$

- Evaluate at $x = \frac{1}{2} : \sum_{n=1}^{\infty} n\frac{1}{2^n} = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2$

e.g. 2: compute $B = \sum_{n=0}^{\infty} \frac{2^n}{(n+2)n!}$

- Want $\sum\limits_{n=0}^{\infty} \frac{x^n}{(n+2)n!}$ when $x = 2$.

- Know $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

$$\begin{aligned}x^{n+1} &= \frac{x^{n+2}}{(n+2)}\\ \int \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}dx &= \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!} + C\\ \int xe^x x = \int \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}dx &= \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!} + C(x-1)e^x \qquad\qquad = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!} + C\end{aligned}$$

Evaluate at $x = 0 : -1 = 0 + C; C = -1$

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!} &= (x-1)e^x + 1\\ \sum_{n=0}^{\infty} \frac{x^n}{(n+2)n!} &= \frac{(x-1)e^x + 1}{x^2}\end{aligned}$$

Evaluate at $x = 2 : \sum\limits_{n=0}^{\infty} \frac{2^n}{(n+2)n!} = \frac{e^2+1}{4}$

14 Taylor applications: physics

- Use physics principles to derive equations that regulate the behaviour of a system
- The equations are too complicated!
- Approximate euqations with Taylor polynomials
- Profit.

Kinetic energy of a particle

- m_0 : mass "at rest"
- v : velocity
- c : speed of light
- m : mass
- Classical physics:

$$T = \frac{1}{2} m_0 v^2.$$

- Relativity:

$$T = mc^2 - m_0 c^2 = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} - m_0 c^2.$$

Relativity is the better model, but classical physics is a model that works perfectly fine when velocities are small. More precisely, we should be able to obtain the classical equation as an approximation when $\frac{v}{c}$ is small.
Indeed, the classical expression for kinetic energy is the smallest nonzero Taylor polynomial of the relativistic expression for kinetic energy.

$$\begin{aligned}
T &= m_0 c^2 \left[\frac{1}{\sqrt{1 - (\frac{v}{c})^2}} - 1 \right] \quad \text{as a function of } v \\
f(x) &= (1-x)^{-\frac{1}{2}} \quad \text{isolate the essence of the function} \\
&\approx f(0) + f'(x) \cdot x \\
&= 1 + \frac{1}{2} x \\
\frac{1}{\sqrt{1-x^2}} &\approx 1 + \frac{1}{2} x^2 \\
T &= m_0 c^2 \left[\frac{1}{\sqrt{1 - (\frac{v}{c})^2}} - 1 \right] \approx m_0 c^2 \left[\frac{1}{2} \left(\frac{v}{c} \right)^2 \right] \quad \text{substituting} \\
&= \frac{1}{2} m_0 v^2.
\end{aligned}$$

$$\vec{v} = \vec{u} - \vec{f}, ., ., .., ., .$$