Unit 13: Series

Contents

1 Power series: an example

e.g. I want to define a function with this equation:

$$g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n3^n}$$

For which $x \in \mathbb{R}$ is $g(x)$ convergent? We can use the Ratio Test.

$$\begin{aligned}
\text{Call } a_n &= \frac{x^n}{n3^n} \\
L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \frac{\left|\dfrac{x^{n+1}}{(n+1)3^{n+1}}\right|}{\left|\dfrac{x^n}{n3^n}\right|} \\
&= \frac{|x|}{3}
\end{aligned}$$

- If $|x| < 3$, then $L = \frac{|x|}{3} < 1$. By Ratio Test, $g(x)$ is absolutely convergent.
- If $|x| > 3$, then $L = \frac{|x|}{3} > 1$. By Ratio test, $g(x)$ is divergent.

We don't know what happens at $x = -3$ or $x = 3$ yet.

$$\begin{aligned}
g(3) &= \sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \frac{1}{n} = \infty \text{ (p-series with } p = 1) \\
g(-3) &= \sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \frac{(-1)^n}{n} \text{ convergent (by AST)}
\end{aligned}$$

Then, at $x = -3$, $g(x)$ is conditionally convergent, and 3, it is divergent.
To answer the original question, the domain of $g = [-3, 3)$ = the INTERVAL OF CONVERGENCE. 3 = the RADIUS OF CONVERGENCE.

2 Power series: the main theorem

Motivation

- Polynomials are nice
- What about "infinite polynomials"?

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

$$\text{or} \quad f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

- **e.g.** :
  - $g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n 3^n}$ has domain $[-3, 3)$
  - $h(x) = \sum_{n=0}^{\infty} x^n$ has domain $(-1, 1)$

Definition 2.1.
Let $a \in \mathbb{R}$.
A power series centered at a is a function f defined by an equation like

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

where $c_0, c_1, c_2, \dots \in \mathbb{R}$.

$$\text{Domain } f = \{x \in \mathbb{R} \ : \ \text{teh series } f(x) \text{ is convergent}\}$$

Note: $a \in \text{Domain } f$

Ultimate goal: write common functions as power series.

Theorem 2.1.
Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series centered at $a \in \mathbb{R}$.

1. *The domain of f is an interval centered at a:*

$$\begin{array}{ccc} (a-R, a+R) & (a-R, a+R] & \mathbb{R} \\ [a-R, a+R] & [a-R, a+R) & \{a\} \end{array}$$

   - *We call this domain the interval of convergence (IC) of f.*
   - *We call its radius the radius of convergence.* $\quad 0 \leq R \leq \infty$

2. - *In the **interior** of the IC, the series is absolutely convergent.*
   - *In the **exterior** of the IC, the series is divergent.*
   - *At the endpoints (if any), anything may happen.*

3. *In the interior of the IC, power series can be "treated like polynomials". They can be added, multiplied, composed...*
   In particular, they can be differentiated or integrated "term by term", and this does not change the radius of convergence.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n x^n &&= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \\ f'(x) &= \sum_{n=0}^{\infty} c_n n x^{n-1} &&= c_1 + 2c_2 x + 3c_3 x^2 + \dots \\ \int_0^x f(t)dt &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} &&= c_0 x + c_1 \frac{x^2}{2} + c_2 \frac{x^3}{3} + \dots \end{aligned}$$

Goals

1. Write as many functions as possible as power series
   $\longrightarrow$ Taylor series
2. Use that to make limits, integrals, estimations, differential equations, physics,...easier.

3 Taylor polynomials—the definition with the limit

Goal: approximate functions with polynomials.

f: function

$a \in \text{Domain } f$

P: polynomial

I want $P(x) \approx f(x)$ when x is close to a. Example: the tangent line. But what is a "good approximation near a"?
R: "remainder" or "error" $\quad R(x) = f(x) - P(x)$. I want R to be small. This means we need $\lim_{x\to a} R(x) = 0$ "fast". For instance, there are many lines with remainder 0, but the tangent line's remainder approaches 0 the "fastest". But how do we measure how fast the limit is?
We notice that the larger exponent polynomials approach 0 faster. We compare the remainder with powers of x.

Definition 3.1.
Let f and g be continuous functions at 0.
Let $n \in \mathbb{N}$.
We say that <u>g is an approximation for f near 0 of order n</u> when

$$\lim_{x\to 0} \frac{f(x) - g(x)}{x^n} = 0$$

This means that $f(x) = g(x) + R(x)$ and as $x \to 0$,

$$R(x) \to 0 \quad \text{faster than} \quad x^n \to 0$$

Definition 3.2.
Let $a \in \mathbb{R}$. Let f and g be continuous functions at a.
Let $n \in \mathbb{N}$.
We say that <u>g is an approximation for f near a of order n</u> when

$$\lim_{x\to a} \frac{f(x) - g(x)}{(x-a)^n} = 0$$

First definition of Taylor polynomial

Definition 3.3.

Let $a \in \mathbb{R}$.

Let f *be a continuous function defined at and near* a.

Let $n \in \mathbb{N}$.

The *n-th Taylor polynoial for f at a* *is a polynomial* P_n

- *...that is an approximation for* f *near* a *of order* n*:*

$$\lim_{x \to a} \frac{f(x) - P_n(x)}{(x - a)^n} = 0$$

- *with degree at most n*

4 Taylor polynomials—the definition with the derivatives

Definition 4.1.
A function f is called …

- C^0 *when f is continuous*
- C^1 *when f' exists and is continuous*
- C^2 *when f' and f'' exist and are continuous*
- …
- C^n *when $f', f'', \dots, f^{(n)}$ exist and are continuous*
- C^∞ *when all derivatives exist (and are continuous)*

For now, assume f and g are C^∞. Can I transform the condition

$$\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n} = 0$$

into a condition about their derivatives?

- Call $L = \lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n}$
  - If $f(a) - g(a) \neq 0$, then "$L = \frac{\text{not } 0}{0} = \pm\infty$".
  - So, assume $f(a) = g(a)$. We get $0/0$. Use L'hôpital's.
- $L \overset{*}{=} \lim_{x \to a} \frac{f'(x) - g'(x)}{n(x-a)^{n-1}}$
  - If $f'(a) - g'(a) \neq 0$, then "$L = \frac{\text{not } 0}{0} = \pm\infty$"
  - So, assume $f'(a) = g'(a)$. We get $0/0$. Use L'hôpital's.
- $L \overset{*}{=} \lim_{x \to a} \frac{f''(x) - g''(x)}{n(n-1)(x-a)^{n-2}} \dots$
- After using L'hôpital's rule n times, we get

$$L \overset{*}{=} \lim_{x \to a} \frac{f^{(n)}(x) - g^{(n)}(x)}{n!} = \frac{f^{(n)}(a) - g^{(n)}(a)}{n!} \qquad \text{since the derivatives are continuous}$$

$$L = 0 \iff \begin{cases} f(a) & = g(a) \\ f'(a) & = g'(a) \\ \dots & \\ f^{(n-1)}(a) & = g^{(n-1)}(a) \\ f^{(n)}(a) & = g^{(n)}(a) \end{cases}$$

I have used that f and g were C^n.

Theorem 4.1.

Let $a \in \mathbb{R}$. Let $n \in \mathbb{N}$.

Let f and g be C^n functions at a.

The following are equivalent:

1. $\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n} = 0$
   ("g is a good approximation for f near a")

2. $f(a) = g(a), f'(a) = g'(a), \ldots, f^n(a) = g^n(a)$
   ("g and f have the same first few derivatives at a")

This proof could be made more formal by using induction.

Definition 4.2.

Let $a \in \mathbb{R}$.

Let $n \in \mathbb{N}$.

Let f be a C^n function at a.

The <u>n-th Taylor polynomial for f at a</u> is

- *a polynomial P_n such that*
- $P_n(a) = f(a), P_n'(a) = f'(a), \ldots P_n^{(n)}(a) = f^{(n)}(a)$
- *with degree at most n.*

This definition is more useful that the original, limit definition for constructing these polynomials. However, we should keep the original definition in mind—it tells us why Taylor polynomials make good approximations.

Notice that this is not a completely new idea: the first Taylor polynomial according to this definition is $y = P_1(x)$, or the tangent line.

5 Taylor polynomials—the formula

Recall:

The n-th Taylor polynomial P_n for the function f at $a \in \mathbb{R}$...

- is an approximation for f near a of order n:

$$\lim_{x \to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0$$

- equivalently, has same value and first n derivatives as f at a:

$$P_n(a) = f(a),\ P_n'(a) = f'(a),\ \ldots,\ P_n^{(n)}(a) = f^{(n)}(a)$$

For simplicity, begin with:

Case $a = 0$

$$P_n(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$$

$$\begin{aligned} P_n(0) &= f(0) \\ P_n'(0) &= f'(0) \\ P_n''(0) &= f''(0) \\ &\ldots \\ P_n^{(n)}(0) &= f^{(n)}(0) \end{aligned}$$

So,

$$\begin{aligned} P_n^{(k)}(0) = k! \cdot c_k &= f^{(k)} \\ c_k &= \frac{f^{(k)}(0)}{k!} \\ \boxed{P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k} \end{aligned}$$

Now, how do we

Generalize a**?** Instead of

$$P_n(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$$

write

$$P_n(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3 + \cdots + b_n(x-a)^n$$

We obtain

$$\boxed{P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}$$

Definition 5.1.
Third definition of Taylor polynomial

- *Let* $a \in \mathbb{R}$
- *Let* $n \in \mathbb{N}$
- *Let* f *be a* C^n *function at* a.

The n*-th Taylor polynomial for* f *at* a *is*

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Notes:

- degree $P_n \leq n$
- The Taylor polynomials of a function are unique

We can see that the higher degree the Taylor polynomial is, the more accurate the approximation.

Definition 5.2.
Taylor series

- *Let* $a \in \mathbb{R}$
- *Let* f *be a* C^∞ *function at* a

The Taylor series for f *at* a *is the power series*

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Equivalently, $\forall k \in \mathbb{N},\ S^{(k)}(a) = f^{(k)}(a)$

Ideal case: $f(x) = S(x)$; we call such functions *analytic*.

Note: “Maclaurin series” means “Taylor series at 0”.