Infinite sums: a cautionary tale

We cannot take infinite sums as if they were finite sums.

e.g.:

$$
\begin{aligned}
S &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \\
xS &= x + x^2 + x^3 + x^4 + n \dots \\
S - xS &= 1 \\
S &= \frac{1}{1-x}
\end{aligned}
$$

When $x = 2$:

$$
\begin{aligned}
S &= \frac{1}{1-2} = -1 \\
S &= 1 + 2 + 4 + 8 + \dots
\end{aligned}
$$

e.g.2:

$$
\begin{aligned}
T &= \sum_{n=0}^{\infty} (-1)^n \\
&= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots
\end{aligned}
$$