

# Unit 13: Series

## Contents

1	Infinite sums: a cautionary tale	1
2	The definition of infinite sum	2
3	Telescopic series	3
4	Examples of divergent series from the definition	4
5	Geometric series	5
6	Linearity of series	6
7	The tail of a series	7
8	A necessary condition for convergence of series	8
9	Positive series	9
10	The integral test	10
11	Integral test examples	11
12	Comparison tests for series	12
13	Alternating series	13
14	Estimating the value of an alternating series	15
15	Absolute convergence vs conditional convergence	16
16	Proof of the absolute convergence test	17
17	Ratio test	18

## 1 Infinite sums: a cautionary tale

We cannot take infinite sums as if they were finite sums.

e.g.:

$$\begin{aligned}S &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \\xS &= x + x^2 + x^3 + x^4 + \dots \\S - xS &= 1 \\S &= \frac{1}{1-x}\end{aligned}$$

When  $x = 2$  :

$$\begin{aligned}S &= \frac{1}{1-2} = -1 \\S &= 1 + 2 + 4 + 8 + \dots\end{aligned}$$

e.g.2:

$$\begin{aligned}T &= \sum_{n=0}^{\infty} (-1)^n \\&= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots\end{aligned}$$

“We can group the terms like this:”

$$\begin{aligned}T &= (1 - 1) + (1 - 1) + (1 - 1) + \dots \\&= 0 + 0 + 0 + \dots = 0\end{aligned}$$

Alternatively,

$$\begin{aligned}T &= 1(-1 + 1) + (-1 + 1) + (-1 + 1) + \dots \\&= 1 + 0 + 0 + 0 + \dots = 1 \\T &= 0 = 1???\end{aligned}$$

### Infinite sums: the right way

- What does adding up infinitely many numbers mean? Define what an infinite sum—a “series”—is.
- When is a series equal to a number? i.e., when is a series convergent?
- Which properties of finite sums carry over to infinite sums?

## 2 The definition of infinite sum

A series is an infinite sum:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

A series is an infinite list:

$$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \dots$$

$$\text{e.g. } S = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1 ?$$

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$\vdots$

$$S_k = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^k} = \sum_{n=1}^k \frac{1}{2^n} = 1 - \frac{1}{2^k}$$

$$S = \lim_{k \rightarrow \infty} S_k = 1$$

To sum the series  $\sum_{n=1}^{\infty} a_n$ ,

- First construct the sequence of *partial sums*  $\{S_k\}_{k=1}^{\infty}$
- Then, compute its limit:

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$\dots$

$$S_k = a_1 + \dots + a_k = \sum_{n=1}^k a_n$$

$$\sum_{n=1}^{\infty} a_n \quad \lim_{k \rightarrow \infty} S_k$$

This series is *convergent* when this limit exists. Otherwise, it is *divergent*.

### 3 Telescopic series

Want to calculate  $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ . Find a formula for the first  $k$  terms, then take the limit as  $k$  goes to infinity.

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{1}{n^2+n} \\ S_1 &= \frac{1}{2} \\ S_2 &= S_1 + \frac{1}{6} = \frac{2}{3} \\ S_3 &= S_2 + \frac{1}{12} = \frac{3}{4} \end{aligned}$$

Conjecture:  $\forall k \geq 1, \quad S_k = \frac{k}{k+1}$

Proof by induction.

**Another way we could've obtained this:**

$$\begin{aligned} S_k &= \sum_{n=1}^k \frac{1}{n^2+n} = \sum_{n=1}^k \frac{1}{n(n+1)} \\ &= \sum_{n=1}^k \left[ \frac{1}{n} - \frac{1}{n+1} \right] \\ &= \left[ 1 - \frac{1}{2} \right] + \left[ \frac{1}{2} - \frac{1}{3} \right] + \left[ \frac{1}{3} - \frac{1}{4} \right] + \cdots + \left[ \frac{1}{k} - \frac{1}{k+1} \right] \\ &= 1 - \frac{1}{k+1} = \frac{k}{k+1} \end{aligned}$$

Then,  $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \lim_{k \rightarrow \infty} \frac{k}{k+1} = 1$

## 4 Examples of divergent series from the definition

$$\sum_{n=1}^{\infty} 1$$

$$\sum_{n=0}^{\infty} (-1)^n$$

e.g.  $S = \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \dots = \infty$

$$S = \lim_{k \rightarrow \infty} S_k$$

$$S_k = \sum_{n=1}^k 1 = 1 + 1 + \dots + 1 = k$$

$$S = \lim_{k \rightarrow \infty} k = \infty$$

Alternatively,  $\sum_{n=1}^{\infty} 1$  is divergent.

e.g.  $S = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \dots$

Then,

$$S = \lim_{k \rightarrow \infty} S_k$$

$$S_k = \sum_{n=0}^{\infty} (-1)^n$$

$$S_0 = 1$$

$$S_1 = 1 - 1 = 0$$

$$S_2 = 1 - 1 + 1 = 1$$

$$S_3 = 1 - 1 + 1 - 1 = 0$$

...

$$S_k = \begin{cases} 0 & \text{if } k \text{ odd} \\ 1 & \text{if } k \text{ even} \end{cases}$$

So  $\lim_{k \rightarrow \infty} S_k$  DNE.

Alternatively,  $\sum_{n=0}^{\infty} (-1)^n$  is divergent.

## 5 Geometric series

A geometric series is one of this form:  $\sum_{n=0}^{\infty} x^n$  where  $x$  is a constant.

Let  $x \in \mathbb{R}$ .  $S = \sum_{n=0}^{\infty} x^n = \lim_{k \rightarrow \infty} S_k$ , where  $S_k = \sum_{n=0}^k x^n$ .

$$\begin{aligned} S_k &= 1 + x + x^2 + \cdots + x^k \\ xS_k &= x + x^2 + x^3 + \cdots + x^{k+1} && \text{multiply by } x \\ S_k - xS_k &= 1 - x^{k+1} \\ S_k &= \frac{1 - x^{k+1}}{1 - x} && \text{if } x \neq 1 \end{aligned}$$

Now take the limit as  $k$  approaches  $\infty$

$$\begin{aligned} \sum_{n=0}^{\infty} x^n &= \lim_{k \rightarrow \infty} S_k \\ &= \lim_{k \rightarrow \infty} \frac{1 - x^{k+1}}{1 - x} \\ &= \frac{1 - \lim_{k \rightarrow \infty} x^{k+1}}{1 - x} && \text{by limit laws} \end{aligned}$$

$$\sum_{n=0}^{\infty} x^n = \begin{cases} \frac{1}{1-x} & \text{if } -1 < x < 1 \\ \text{DIVERGENT} & \text{otherwise} \end{cases}$$

## 6 Linearity of series

This refers to these two properties:

$$\sum_{n=0}^{\infty} (a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$$

$$\sum_{n=0}^{\infty} (ca_n) = c \sum_{n=0}^{\infty} a_n$$

Finite sums can be reordered at-will, but infinite sums cannot.

### Theorem 6.1.

IF  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$  are both convergent,

THEN  $\sum_{n=0}^{\infty} (a_n + b_n)$  is also convergent and

$$\sum_{n=0}^{\infty} (a_n + b_n) = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n$$

### Theorem 6.2.

Let  $c \in \mathbb{R}$ .

IF  $\sum_{n=0}^{\infty} a_n$  is convergent,

THEN  $\sum_{n=0}^{\infty} (ca_n)$  is also convergent, and

$$\sum_{n=0}^{\infty} (ca_n) = c \sum_{n=0}^{\infty} a_n$$

*Proof.* for Theorem 6.1

$$\begin{aligned} \sum_{n=0}^{\infty} a_n &= \lim_{k \rightarrow \infty} S_k, & \text{where } S_k &= \sum_{n=0}^k a_n \\ \sum_{n=0}^{\infty} b_n &= \lim_{k \rightarrow \infty} T_k, & \text{where } T_k &= \sum_{n=0}^k b_n \\ \sum_{n=0}^{\infty} (a_n + b_n) &= \lim_{k \rightarrow \infty} R_k, & \text{where } R_k &= \sum_{n=0}^k (a_n + b_n) \end{aligned}$$

By properties of finite sums,  $R_k = S_k + T_k$

By hypothesis, the first two limits exist. Then, by the limit laws,

$$\lim_{k \rightarrow \infty} R_k = \lim_{k \rightarrow \infty} S_k + \lim_{k \rightarrow \infty} T_k$$

■

This is a template for series—how they’re proven.

1. Write the infinite sum as a limit of partial sums—finite sums
2. Use the properties known to be true for finite sums
3. Pass to the limit

## 7 The tail of a series

We often talk about the tail of a series when studying infinite sums. Let's compare  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} a_n$ .

**Theorem 7.1.**

$\sum_{n=0}^{\infty} a_n$  is convergent  $\iff \sum_{n=1}^{\infty} a_n$  is convergent

Moreover, in that case

$$\left[ \sum_{n=0}^{\infty} a_n \right] = a_0 + \left[ \sum_{n=1}^{\infty} a_n \right]$$

Proof involves writing the series as a limit of finite sums, then we know the property to be true for finite sums.

Because this is true, we can write  $\sum_n a_n$  is convergent / divergent ; in other words, we don't need to specify where it starts.

**Theorem.**

Typical theorem:

IF for all  $n \in \mathbb{N}$ , something about  $a_n$

THEN the series  $\sum_n a_n$  is convergent.

**Theorem.**

Generalized theorem:

IF  $\exists n_0 \in \mathbb{N}$  s.t.  $\forall n \in \mathbb{N}, \quad n \geq n_0 \implies$  something about  $a_n$

THEN the series  $\sum_n a_n$  is convergent.



## 8 A necessary condition for convergence of series

*Sequences vs series:* recall that a series is defined as the limit of a sequence—the sequence of partial sums.

$$\text{So } \sum_{n=0}^{\infty} a_n = \lim_{k \rightarrow \infty} S_k, \quad \text{where } S_k = \sum_{n=0}^k a_n$$

$$\sum_{n=0}^{\infty} a_n \text{ is convergent} \iff \{S_n\}_{n=0}^{\infty} \text{ is convergent}$$

But what is the relation between  $\sum_{n=0}^{\infty} a_n$  and  $\{a_n\}_{n=1}^{\infty}$ ? I expect that if the series of  $a$  is convergent, then the sequence of  $a$ —not  $S$ —is convergent to 0.

### **Theorem 8.1.**

IF the series  $\sum_{n=0}^{\infty} a_n$  is convergent,  
THEN  $\lim_{n \rightarrow \infty} a_n = 0$

How to use in practice?

1. IF  $\lim_{n \rightarrow \infty} a_n = 0$   
THEN the series  $\sum_{n=0}^{\infty} a_n$  may be convergent or divergent.
2. IF  $\lim_{n \rightarrow \infty} a_n \neq 0$   
THEN the series  $\sum_{n=0}^{\infty} a_n$  is divergent—the contrapositive.

*Proof.*

Assume the series  $\sum_{n=0}^{\infty} a_n$  is convergent. wts  $\lim_{n \rightarrow \infty} a_n = 0$ .

This means the following limit exists:

$$S = \lim_{k \rightarrow \infty} S_k, \quad \text{where } S_k = \sum_{n=0}^k a_n$$

Notice that for every  $n \geq 1$ :

$$a_n = S_n - S_{n-1} = S - S = 0$$

Then we can use the limit laws:

$$\lim_{n \rightarrow \infty} a_n = \left[ \lim_{n \rightarrow \infty} S_n \right] - \left[ \lim_{n \rightarrow \infty} S_{n-1} \right]$$

■

**Note:** we can only quickly conclude that the series is divergent from this, never that it is convergent.

## 9 Positive series

We like positive series.

- A series  $\sum_{n=0}^{\infty} a_n$  is *positive* when  $\forall n \in \mathbb{N}, a_n > 0$ .
- A series  $\sum_{n=0}^{\infty} a_n$  is *negative* when  $\forall n \in \mathbb{N}, a_n < 0$ .
- A series  $\sum_{n=0}^{\infty} a_n$  is *non-negative* when  $\forall n \in \mathbb{N}, a_n \geq 0$ .

An *arbitrary* series may be  $\begin{cases} \text{convergent} \\ \text{divergent} \begin{cases} \text{to } \infty \\ \text{to } -\infty \\ \text{"oscillating"} \end{cases} \end{cases}$       A **positive** series may be  $\begin{cases} \text{convergent} \\ \text{divergent to } \infty \end{cases}$

There are only two options.

*Proof.*

In general,  $\sum_{n=0}^{\infty} a_n = \lim_{k \rightarrow \infty} S_k$ , where  $S_k = \sum_{n=0}^k a_n$ .

Assume the series  $\sum_{n=0}^{\infty} a_n$  is positive. Then the sequence  $\{S_n\}_{n=0}^{\infty}$  is increasing:  $S_{n+1} - S_n = a_{n+1}$

Use the Monotone Convergence Theorem:

- An increasing, bounded sequence is convergent
- An increasing, unbounded sequence is divergent to  $\infty$

■

For **POSITIVE SERIES ONLY**, we can use shorthand notation.

- $\sum_n^{\infty} a_n = \infty$  means "divergent"
- $\sum_n^{\infty} a_n < \infty$  means "convergent"

The same applies to non-negative series.

The same applies to eventually non-negative series.

### How does this help?

To prove a positive series is convergent, we only have to prove it does not diverge to  $\infty$ .

Useful theorems which *all rely on the fact that a positive series can only be convergent or infinity*:

1. Integral test
2. Basic comparison test
3. Limit comparison test

## 10 The integral test

Let  $f$  be a continuous function on  $[1, \infty)$ . What is the relation between  $\sum_{n=1}^{\infty} f(n)$  and  $\int_1^{\infty} f(x)dx$ ?

A series and an improper integral are defined in a similar manner.

$$\sum_{n=1}^{\infty} f(n) = \lim_{k \rightarrow \infty} \sum_{n=1}^k f(n) \qquad \int_1^{\infty} f(x)dx = \lim_{b \rightarrow \infty} \int_1^b f(x)dx$$

Is there a relation? There is a tool for integrals that we don't have for series in that we can calculate the value of an improper integral—if we have an expression for its antiderivative.

I will assume:

- $f$  is positive – then  $\sum_{n=1}^{\infty} f(n)$  and  $\int_1^{\infty} f(x)dx$  are convergent or  $\infty$

There is not a possibility of oscillation

- $f$  is decreasing

Plan: find a relation between proper integrals and finite sums, then take the limit.

From a graph, we can tell that  $\int_1^5 f(x)dx$  is bounded below by  $\sum_{n=1}^4 f(n)$ . In order to be able to do this, we need the function to be decreasing, so that the  $f(x)$  at the left endpoint would be the maximum on that interval.

We can also do the same to bound the area below. Then  $\int_1^5 f(x)dx \geq \sum_{n=2}^5 f(n)$ . Then, we arrive at the expression

$$\begin{aligned} \sum_{n=2}^5 f(n) &\leq \int_1^5 f(x)dx \leq \sum_{n=1}^4 f(n) \\ \sum_{n=2}^N f(n) &\leq \int_1^N f(x)dx \leq \sum_{n=1}^{N-1} f(n) \quad \text{since we could've picked any other integer} \\ \sum_{n=2}^{\infty} f(n) &\leq \int_1^{\infty} f(x)dx \leq \sum_{n=1}^{\infty} f(n) \quad \text{by taking the limit as } n \rightarrow \infty \end{aligned}$$

Then, if the right series—which bounds the integral above—is convergent, then the integral must also be convergent. If the integral—which bounds the left series above—is convergent, then that series must also be convergent. Since the function is positive, this has to be convergent or  $\infty$ . Then, we have proven that

### Theorem 10.1.

Let  $a \in \mathbb{R}$

Let  $f$  be a continuous, positive, decreasing function on  $[a, \infty)$

Then

$$\int_a^{\infty} f(x)dx \text{ is convergent} \iff \sum_n f(n) \text{ is convergent}$$

Denoted with  $\int_a^{\infty} f(x)dx \sim \sum_n f(n)$

## 11 Integral test examples

**p-series** For which values of  $p > 0$  is the series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  convergent? Use integral test.

Let  $f(x) = \frac{1}{x^p}$ , then  $f$  is continuous, positive, and decreasing.

By integral test,  $\sum_{n=1}^{\infty} \frac{1}{n^p} \sim \int_1^{\infty} \frac{1}{x^p} dx$ .

We know that  $\int_1^{\infty} \frac{1}{x^p} dx$  is convergent iff  $p > 1$ . Thus,  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is convergent iff  $p > 1$ .

**other** Is  $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$  convergent? It's positive, decreasing, continuous for  $x \geq 2$ .

By integral test,  $\sum_{n=2}^{\infty} \frac{1}{n \ln n} \sim \int_2^{\infty} \frac{1}{x \ln x} dx$ . We can find the antiderivative for this function.

$$\begin{aligned} \int_2^{\infty} \frac{1}{x \ln x} dx &= \lim_{b \rightarrow \infty} \left[ \int_2^b \frac{1}{x \ln x} dx \right] = \lim_{b \rightarrow \infty} \left[ \ln \ln x \right]_2^b \\ &= \lim_{b \rightarrow \infty} [\ln \ln b - \ln \ln 2] = \infty \end{aligned}$$

So by the integral test, the series diverges.

## 12 Comparison tests for series

Comparison tests for series and improper integrals are the same. Results based on:

- A positive series may only be convergent or divergent to  $\infty$
- To prove a positive series is convergent, we only need to prove it is not  $\infty$

### Theorem 12.1.

#### ***BCT for series***

Let  $\sum_n^\infty a_n$  and  $\sum_n^\infty b_n$  be two series.

- Assume, for every  $n \in \mathbb{N}$ ,  $a \leq a_n \leq b_n$
- THEN
  - IF  $\sum_n^\infty a_n = \infty$ , THEN  $\sum_n^\infty b_n = \infty$ .
  - IF  $\sum_n^\infty b_n < \infty$ , THEN  $\sum_n^\infty a_n < \infty$ .

### Theorem 12.2.

#### ***LCT for series***

Let  $\sum_n^\infty a_n$  and  $\sum_n^\infty b_n$  be two positive series.

- IF the limit  $L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$  exists—is a number—and  $L > 0$ .
- THEN

$$\sum_n^\infty a_n \quad \text{and} \quad \sum_n^\infty b_n$$

are both convergent or both divergent.

## 13 Alternating series

**Definition:** A series  $\sum_n a_n$  is alternating when  $\forall n, a_n a_{n+1} < 0$

This means the terms “alternate” between positive and negative.

**Example:**

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots$$

$$S_1 = 1$$

$$S_2 = 1 - \frac{1}{2}$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{3}$$

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$

...

We notice from a graph that  $S_2 < S_4 < S_6 < \dots < S_5 < S_3 < S_1$

$\{S_{2n}\}_n$  is increasing and bounded above (by  $S_1$ ).

$\{S_{2n+1}\}$  is decreasing and bounded below (by  $S_2$ ).

Then by MCT, both are convergent.

Call  $A = \lim_{n \rightarrow \infty} a_n$ ,  $B = \lim_{n \rightarrow \infty} b_{2n+1}$ , then

$$S_{2n+1} = S_{2n} + \frac{1}{2n+1} \quad \text{we can use the limit laws, since all three terms have limits}$$

$$\lim_{n \rightarrow \infty} S_{2n+1} = \lim_{n \rightarrow \infty} S_{2n} + \lim_{n \rightarrow \infty} \frac{1}{2n+1}$$

$$B = A + 0$$

**Theorem 13.1.**

**Lemma:**

Let  $\{c_n\}_n^\infty$  be a sequence.

- IF the sequences of even and odd terms

$$\{c_{2n}\}_n^\infty \quad \text{and} \quad \{c_{2n+1}\}_n^\infty$$

are convergent to the same limit

- THEN the full sequence  $\{c_n\}_n^\infty$  is also convergent to the same limit.

**Theorem 13.2.**

***Alternating series test:***

*Consider a series of the form*

$$\sum_n^{\infty} (-1)^n b_n \quad \text{or} \quad \sum_n^{\infty} (-1)^{n+1} b_n$$

*IF*

1.  $\forall n, \quad b_n > 0$
2. *the sequence  $\{b_n\}_n^{\infty}$  is decreasing*
3.  $\lim_{n \rightarrow \infty} b_n = 0$

*THEN the series is convergent.*

## 14 Estimating the value of an alternating series

Estimate the value of  $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$  with an error smaller than 0.001. First, we need to show that it is convergent—it satisfies the hypotheses of the alternating series theorem. By AST, the series is convergent.

The actual value is  $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = \lim_{k \rightarrow \infty} S_k$ . This suggests that we can use  $S_k$  as an estimate.

Estimate:  $S_k = \sum_{n=1}^k \frac{(-1)^n}{n^4}$  for some large  $k$ ? Which value of  $k$ ?

We need the error to be smaller than a precise number. Error of estimation:  $|S - S_k|$ .

### Theorem 14.1.

*Alternating series theorem, part 2: consider a series of the form*

$$\sum_{n=1}^{\infty} (-1)^n b_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

- IF it satisfies the same three hypotheses as before
- THEN  $|S - S_k| < b_{k+1}$

where  $S_k$  is the  $k$ -th partial sum of the series.

Then, error of estimation:  $|S - S_k| < \frac{1}{(k+1)^4}$ . We need to choose  $k$  so that  $\frac{1}{(k+1)^4} < 0.001$ . Pick  $k$  to be 5.

Estimate:  $-1 + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} \approx -0.94753\dots$



## 15 Absolute convergence vs conditional convergence

Is  $\sum_n \frac{\sin n}{n^2}$  convergent?

What is the relation between the series  $\sum_n a_n$  and  $\sum_n |a_n|$ ?

### Theorem 15.1.

#### Absolute convergence test

Let  $\sum_n a_n$  be a series.

- IF the series  $\sum_n |a_n|$  is convergent
- THEN the series  $\sum_n a_n$  is convergent

We can look at  $\sum_n \frac{|\sin n|}{n^2}$ . Since it is a positive series, we can use comparison tests.

$0 \leq \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$ . We know that  $\sum_n \frac{1}{n^2}$  is convergent, so by BCT,  $\sum_n \frac{|\sin n|}{n^2}$  is also convergent. By ABSOLUTE CONVERGENCE TEST,  $\frac{\sin n}{n^2}$  is also convergent.

e.g. Let  $p_n = \begin{cases} 1/n & \text{if } n \text{ is prime} \\ -1/n & \text{otherwise} \end{cases}$  is  $\sum_n p_n$  convergent?

First, we look at  $\sum_n |p_n|$ .  $\sum_n |p_n| = \sum_n \frac{1}{n} = \infty$ .

The absolute convergence test does not apply, so we do not know if this series is convergent or divergent.

### Definition 15.1.

A convergent series  $\sum_n a_n$  is...

- absolutely convergent when  $\sum_n |a_n|$  is also convergent.
- conditionally convergent when  $\sum_n |a_n| = \infty$ .

For instance,  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  is conditionally convergent. By the alternating series test, it is convergent.  $\sum_{n=1}^{\infty} \frac{1}{n}$ , its absolute value, however, is divergent.

For instance,  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$  is absolutely convergent. By the alternating series test, it is convergent.  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ , its absolute value, is also convergent.

## 16 Proof of the absolute convergence test

**Notation:** “P.T” means “positive terms”, “N.T” means “negative terms”. For instance,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \dots \\ \sum_{n=1}^{\infty} \text{P.T} &= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots \end{aligned}$$

Essentially, we are replacing the negative terms with zeros.

*Proof.*

Assume  $\sum_n |a_n| < \infty$ .

Then,  $\sum_n |\text{P.T}| \leq \sum_n |a_n|$  and  $\sum_n |\text{N.T}| \leq \sum_n |a_n|$ .

By BCT,  $\sum_n |\text{P.T}| < \infty$  and  $\sum_n |\text{N.T}| < \infty$

Therefore,  $\sum_n \text{P.T}$  and  $\sum_n \text{N.T}$  are convergent.

By key observation  $\sum_n a_n = \sum_n \text{P.T} + \sum_n \text{N.T}$ . Thus  $\sum_n a_n$  is convergent. ■

## 17 Ratio test

To determine if a series is convergent or divergent, by computing a limit.

Let  $\sum_n^\infty a_n$  be a series. Assume  $\forall n, a_n \neq 0$ . Assume the limit

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \text{exists or is } \infty$$