Yang-Mills equation on the conformal compactification of the Anti de Sitter spacetime

Edison Alejandro Montoya Gómez

Abstract

It is presented the construction of the equations of motion for the Yang-Mills (YM) field minimally coupled to the gravitational field, where the space-time metric is the Anti de Sitter (AdS) metric. The idea is to use the conformal invariance of the YM theory, to rewrite the equations of motion on the spherically symmetric AdS space-time in a compact (non-physical) space-time.

1 Introduction

we will show that the Yang-Mills (YM) action (and therefore the equation of motion derived from it) is conformal invariant. This means that our solutions will be the same if the metrics are related by a conformal transformation. Using this fact we will change the description of our theory on a fixed background being the AdS space-time to a theory a compact non-physical space-time which contains the AdS space-time and reproduce its physical solutions with the correct boundary conditions. It is assumed that the space-time structure is given, in our case the background metric will be the AdS metric or at conformal transformation of it.

2 Yang Mills action

The actions for a YM theory minimally coupled to the gravitational field can be written as

$$S = \alpha \int \mathrm{Tr}(F^{\mu\nu} F_{\mu\nu}) \sqrt{-g} d^4x \tag{1}$$

where α is a constant that includes the coupling constant. The symbol Tr is the trace in the Lie algebra of the internal group, that in our case is assumed to be $SU(2)$.
First we will show that the action is conformally invariant. If we make the transformation $g_{\mu\nu} \to \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ the action changes as

$$\begin{aligned} S = \alpha \int \mathrm{Tr}(g^{\mu\beta} g^{\gamma\nu} F_{\beta\gamma} F_{\mu\nu}) \sqrt{-g} d^4x \quad &\Rightarrow \quad S = \alpha \int \mathrm{Tr}(\hat{g}^{\mu\beta} \hat{g}^{\gamma\nu} F_{\beta\gamma} F_{\mu\nu}) \sqrt{-\hat{g}} d^4x \\ &\Rightarrow \quad S = \alpha \int \mathrm{Tr}(\Omega^{-2} g^{\mu\beta} \Omega^{-2} g^{\gamma\nu} F_{\beta\gamma} F_{\mu\nu}) \sqrt{-\Omega^8 g} d^4x \\ &\Rightarrow \quad S = \alpha \int \mathrm{Tr}(g^{\mu\beta} g^{\gamma\nu} F_{\beta\gamma} F_{\mu\nu}) \sqrt{-g} d^4x \end{aligned}$$

Therefore, the action is conformally invariant.

3 AdS metric

Now, we are interested in rewrite the AdS metric, spherically symmetric,

$$ds^2 = -\left(1 - \frac{\Lambda r^2}{3}\right)dt^2 + \frac{1}{\left(1 - \frac{\Lambda r^2}{3}\right)}dr^2 + r^2 d\Omega^2 \tag{2}$$

where Λ can be interpreted as the cosmological constant, with $\Lambda < 0$. It also defines a lenght scale $L := \sqrt{-3/\Lambda}$, then the AdS metric can be rewritten as

$$ds^2 = -\left(1 + \frac{r^2}{L^2}\right)dt^2 + \frac{1}{\left(1 + \frac{r^2}{L^2}\right)}dr^2 + r^2 d\Omega^2 \tag{3}$$

where Ω is the solid angle in the 2-sphere. If we make the change of variables $\frac{r}{L} = \tan\Psi$, then the metric changes as

$$\begin{aligned} ds^2 &= -(1 + \tan^2\Psi)dt^2 + \frac{1}{(1 + \tan^2\Psi)}\sec^4\Psi L^2 d\Psi^2 + \tan^2\Psi L^2 d\Omega^2 & (4)\\ &= \sec^2\Psi\left[-dt^2 + L^2 d\Psi^2 + \sin^2\Psi L^2 d\Omega^2\right] & (5)\end{aligned}$$

If we redefine the time as $t = L\tau$, the metric takes the form

$$ds^2 = L^2\sec^2\Psi\left[-d\tau^2 + d\Psi^2 + \sin^2\Psi d\Omega^2\right] \tag{6}$$

which shows that the AdS metric (6) is conformally related to the metric

$$d\hat{s}^2 = -d\tau^2 + d\Psi^2 + \sin^2\Psi d\Omega^2 \tag{7}$$

where now, the coordinate Ψ take values in the interval $[0, \pi/2)$, the idea is work with this metric and include the "point" $\pi/2$, such that, the new space-time $\hat{M}$ is compact and include the AdS space-time. Note that, the spatial metric with $\Psi \in [0, \pi/2]$ correspond to a hemisphere of the 3-sphere S^3. To be clear, when we add the "point" $\Psi = \pi/2$ we are really adding a 2-sphere S^2 to the boundary at each time slice, this can be seen taken $\Psi = \pi/2$ and by fixing time, the metric reduces to $d\hat{s}^2 = d\Omega$, the S^2 metric. Then the topology of the boundary (null infinite) is $\mathbb{R}_t \times S^2$.
As we shown before, the YM theory is invariant under conformal transformations, then we will use this fact to work with the metric (7) and not with the original AdS metric (3). Note that, if we work on a space-time that is spherically symmetric, we can integrated over the angle Ω and the theory reduces to an effective theory in 2 dimensions, which simplifies the problem.

4 Equations of motion

In order to derive the equations of motion for the YM field on the space-time given by the metric (7) we have different methods, namely,

1. Use the general action (1), make the variation with respect to the gauge field and get the general equations. Later use explicitly the metric and the gauge field in the spherically symmetric form.

2. Fix the metric in the action and make the variation with respect to the gauge field to get the equations of motion. Later use the explicit form of the gauge field in the spherically symmetric form.

3. Fix the metric in the action and write the gauge field in its spherically symmetric form. Later make the variation with respect to the gauge field.

4. Fix the metric, write the gauge field in its spherically symmetric form and integrated over the angle Ω (using that we are in the spherically symmetric case), this reduces the action to an effective action in two dimensions. Later make the variation with respect to the gauge field.

5. Fix the metric in the action, write the gauge field in its spherically symmetric form and fix the gauge. This means that, we are starting from a theory that visit each gauge orbit once. Later make the variation with respect to the gauge field.

6. Fix the metric in the action, write the gauge field in its spherically symmetric form, fix the gauge and integrated over the angle Ω. Later make the variation with respect to the gauge field.

In reference [1], it is write the simplify action (Eq. 3.16) where the metric is choose to be spherically symmetric and the action is reduced to an effective action in two dimension, the gauge field is written in its spherically symmetric form and the equations of motion derived, i.e., it is used the method 4. In order to understand the notation, assumptions and calculations in [1], we will write explicitly the equations of motion for YM with the metric (3). Later we will derived the equations of motion for the YM field on the conformal compactification of the AdS space-time, given by the metric (6), in this case we will use the method 6.

4.1 YM on the AdS space-time

To begin with, we identify the metric $\tilde{g}$,

$$\tilde{g} = -\left(1 + \frac{r^2}{L^2}\right)dt^2 + \frac{1}{\left(1 + \frac{r^2}{L^2}\right)}dr^2 \tag{8}$$

Therefore, $\beta = 0, S = 1, N = (1 + \frac{r^2}{L^2})$. The gauge field is written as

$$A = a\tau_r + [1 - \mathbb{R}e(\omega)]\hat{\star}d\tau_r + \mathbb{I}m(\omega)d\tau_r \tag{9}$$

where $\tau_r = \hat{e}_r \cdot \bar{\Sigma}/(2i)$ and $\hat{\star}$ denotes the Hodge dual with respect to the metric $\hat{g}$.
The physical solutions that we are interested are the ones where $a = 0$ and $\mathbb{I}m(\omega) = 0$, then the gauge field reduces to

$$A = [1 - \mathbb{R}e(\omega)]\hat{\star}d\tau_r \tag{10}$$

and the field strength is,

$$F = -\mathbb{R}e(D\omega) \wedge \hat{\star}d\tau_r + (\omega^2 - 1)\tau_r d\Omega \tag{11}$$

where $D\omega = d\omega$ (exterior derived).
Now we can rewrite the equation (3.21) in [1], that is given by

$$D_a(SD^a\omega) = \frac{S}{r^2}(|\omega|^2 - 1)\omega \tag{12}$$

We saw that $S = 1$ and D_a represents the covariant derived, then Eq. (12) can be rewritten as,

$$\begin{aligned}
\frac{1}{r^2}(\omega^2 - 1)\omega &= \nabla_\alpha \nabla^\alpha \omega \\
&= \nabla_\alpha(\tilde{g}^{\alpha\mu}\partial_\mu \omega) \\
&= \partial_t(\tilde{g}^{tt}\partial_t\omega) + \Gamma^t_{tt}\tilde{g}^{tt}\partial_t\omega + \Gamma^t_{tr}\tilde{g}^{rr}\partial_r\omega + \partial_r(\tilde{g}^{rr}\partial_r\omega) + \Gamma^r_{rt}\tilde{g}^{tt}\partial_t\omega + \Gamma^r_{rr}\tilde{g}^{rr}\partial_r\omega
\end{aligned}$$

using that

$$\tilde{g}^{tt} = -\frac{1}{1+\frac{r^2}{L^2}} \tag{13}$$

$$\tilde{g}^{rr} = 1 + \frac{r^2}{L^2} \tag{14}$$

$$\Gamma^t_{tt} = 0 \tag{15}$$

$$\Gamma^t_{tr} = \frac{\frac{r}{L^2}}{1+\frac{r^2}{L^2}} \tag{16}$$

$$\Gamma^r_{rr} = -\frac{\frac{r}{L^2}}{1+\frac{r^2}{L^2}} \tag{17}$$

$$\Gamma^r_{rt} = 0 \tag{18}$$

we get

$$\frac{1}{r^2}(\omega^2 - 1)\omega = -\frac{1}{1+\frac{r^2}{L^2}}\partial_t^2\omega + \frac{r}{L^2}\partial_r\omega + \partial_r\left[\left(1+\frac{r^2}{L^2}\right)\partial_r\omega\right] - \frac{r}{L^2}\partial_r\omega \tag{19}$$

Finally

$$\frac{1}{r^2}(\omega^2 - 1)\omega = -\frac{1}{1+\frac{r^2}{L^2}}\partial_t^2\omega + \frac{2r}{L^2}\partial_r\omega + \left(1+\frac{r^2}{L^2}\right)\partial_r^2\omega \tag{20}$$

This is the equation of motion for the YM field on the AdS space-time. Assuming spherical symmetry, $a = 0$ and ω real.

4.2 YM on the conformal compactification of the AdS space-time

We want the equation of motion for the YM field on the space-time given by the metric (7) (denoted as $\hat{g}$), which is the conformal compactification of the AdS metric. To do this, we will use three different methods in order to have a clear notion of what the problem is. Also, it will be useful to understand the physics and the mathematical structure of the equations. To begin with, we will give general arguments about the conformal invariance for the YM equations and find the equations in the metric that we want, later we show that these equations agree with the transform equations using the change of variables to find the metric (7), finally we derive the equations of motion from the action using the method 6 presented at the beginning of this section.

Conformal Invariance

As we show before the YM theory minimally coupled to the gravitational field is conformally invariant, therefore its equations of motion are the same for a set of conformal related metrics. We can used this fact to rewrite the equation (12) in terms of the metric $\hat{g}$, to do this we first note that $S = 1$ for $\hat{g}$, second the derivatives are covariant derivatives then there is a dependence of the metric but an independence of the total equation respect to the conformal transformations, finally the term r^2 must be understand as the function that comes with $d\Omega$ (the S^2 metric), i.e., $r = r(\tau, \psi)$ an in the metric $\hat{g}$ takes the form $r = \sin\psi$. With all this, we can say that Eq. (12) transform as

$$\nabla_\alpha \nabla^\alpha \omega(t,r) = \frac{1}{r^2}[\omega^2(t,r) - 1]\omega(t,r) \quad \Rightarrow \quad \hat{\nabla}_\alpha \hat{\nabla}^\alpha \omega(\tau,\psi) = \frac{1}{r^2(\tau,\psi)}[\omega^2(\tau,\psi) - 1]\omega(\tau,\psi)$$

where the hats represents the covariant derive compatible with the metric $\hat{g}$, then using the explicit form of the metric (12), we get

$$\begin{aligned} \hat{\nabla}_\alpha \hat{\nabla}^\alpha \omega &= \frac{1}{r^2}(\omega^2 - 1)\omega && (21) \\ -\partial_\tau^2 \omega + \partial_\psi^2 \omega &= \frac{1}{\sin^2\psi}(\omega^2 - 1)\omega && (22) \end{aligned}$$

This is the equation of motion for the YM field on the fix background given by $\hat{g}$.

Change of varibles

The arguments used in the previous section to get the equation of motion were heuristic, now we want to prove that the result is in agreement with making in Eq. (20) the same change of variables that we did to define the metric $\hat{g}$. The change of variables was

$$t = L\tau \quad \text{and} \quad r = L\tan\psi \tag{23}$$

therefore

$$\begin{aligned} \partial_t^2 &= L^{-2}\partial_\tau^2 && (24) \\ \partial_r &= \frac{L^{-1}}{\sec^2\psi}\partial_\psi && (25) \\ \partial_r^2 &= \frac{L^{-2}}{\sec^4\psi}(\partial_\psi^2 - 2\tan\psi\partial_\psi) && (26) \end{aligned}$$

If we implement the change of variables (23) in Eq. (20) we get

$$\frac{1}{L^2\tan^2\psi}(\omega^2 - 1)\omega = -\frac{1}{\sec^2\psi}\partial_t^2\omega + \frac{2\tan\psi}{L}\partial_r\omega + \sec^2\psi\partial_r^2\omega \tag{27}$$

and using Eqs. (24,25,26) to change the derivatives

$$\begin{aligned} \frac{1}{L^2\tan^2\psi}(\omega^2 - 1)\omega &= -\frac{1}{\sec^2\psi}L^{-2}\partial_\tau^2\omega + \frac{2\tan\psi}{L}\frac{L^{-1}}{\sec^2\psi}\partial_\psi\omega + \sec^2\psi\frac{L^{-2}}{\sec^4\psi}(\partial_\psi^2 - 2\tan\psi\partial_\psi)\omega \\ \frac{1}{\sin^2\psi}(\omega^2 - 1)\omega &= -\partial_\tau^2\omega + 2\tan\psi\partial_\psi\omega + (\partial_\psi^2 - 2\tan\psi\partial_\psi)\omega \\ \frac{1}{\sin^2\psi}(\omega^2 - 1)\omega &= -\partial_\tau^2\omega + \partial_\psi^2\omega && (28) \end{aligned}$$

The last equation is the same that Eq. (22). Then we arrive to the same result by different ways.

Action

Now we want to prove that the equation of motion for the YM field (Eqs. (22) or (28)) are the same that comes from the action. In order to show this, we use the effective action from reference [1] (Eq. (3.16)), this saves us to write the gauge field in its spherically symmetric form and do the integral over the angle Ω. The effective action is

$$S = -\frac{4\pi}{e^2}\int\left(\frac{r^2}{2}\tilde{g}(f,f) + \tilde{g}(D\omega, \bar{D\omega}) + \frac{(|\omega|^2-1)^2}{2r^2}\right)\tilde{\eta} \tag{29}$$

where $f = da, D\omega = d\omega - ia\omega$, if we fix the gauge $f = da = 0, D\omega = d\omega = \bar{D}\omega$ (setting $f = da = 0$ is more than a gauge choice, it eliminates the "electric" part of the field) and if we fix the metric: $r = \sin\psi$, $\tilde{g} = -d\tau^2 + d\psi^2$, $\tilde{\eta} = \sqrt{-\tilde{g}}\, d^2x = d\tau d\psi$. Then the action reduces to

$$S = -\frac{4\pi}{e^2}\int\left(-(\partial_\tau\omega)^2 + (\partial_\psi\omega)^2 + \frac{(\omega^2-1)^2}{2\sin^2\psi}\right)d\tau d\psi \tag{30}$$

Note that we already implement the steps listed in method 6, now remains make the variation of the action with respect to ω

$$\begin{aligned}
\delta S &= -\frac{4\pi}{e^2}\int\left(-2(\partial_\tau\omega)\partial_\tau\delta\omega + 2(\partial_\psi\omega)\partial_\psi\delta\omega + \frac{2(\omega^2-1)\omega\delta\omega}{\sin^2\psi}\right)d\tau d\psi \\
&= -\frac{8\pi}{e^2}\int\left(-\partial_\tau[(\partial_\tau\omega)\delta\omega] + (\partial_\tau^2\omega)\delta\omega + \partial_\psi[(\partial_\psi\omega)\delta\omega] - (\partial_\psi^2\omega)\delta\omega + \frac{(\omega^2-1)\omega\delta\omega}{\sin^2\psi}\right)d\tau d\psi \\
&= -\frac{8\pi}{e^2}\left[-\int d\psi[(\partial_\tau\omega)\delta\omega]\Big|_{\tau_0}^{\tau_1} + \int d\tau[(\partial_\psi\omega)\delta\omega]\Big|_{\psi_0}^{\psi_1}\right. \\
&\qquad \left.\int\left(\partial_\tau^2\omega - \partial_\psi^2\omega + \frac{(\omega^2-1)\omega}{\sin^2\psi}\right)\delta\omega d\tau d\psi\right]
\end{aligned}$$

Assuming that ω is fix at the boundary of the spacetime region we are integrating, then $\delta\omega = 0$ at boundary and therefore the first two integrals are zero. Using that the spacetime region is arbitrary and the variation inside de region is different from zero, we conclude from the third integral that

$$\partial_\tau^2\omega - \partial_\psi^2\omega + \frac{(\omega^2-1)\omega}{\sin^2\psi} = 0 \tag{31}$$

This is the equation of motion for the gauge field ω, which is equal to the previous equations (22,28)

5 Numerical Implementation

In order to implement numerically the YM equation we rewrite it as a set of first order differential equations, by defining the new variables $\pi = \partial_t\omega$ and $\Sigma = \partial_x\omega$. Then Eq. (31)

turns in the set of equations

$$\begin{aligned} \partial_t \omega &= \pi \qquad (32) \\ \partial_t \Sigma &= \partial_x \pi \qquad (33) \\ \partial_t \pi &= \partial_x \Sigma - \frac{(\omega^2 - 1)\omega}{\sin^2 x} \qquad (34) \end{aligned}$$

where we redefine $\tau \to t$ and $\psi \to x$, it make that the equation looks like the wave equation with source. To integrate this set of partial differential equations we use the finite differences method on a 1 dimensional grid and evolve it in time. The spatial derivatives are replace by differences. Also, it has to by impose initial conditions on ω, Σ, π and boundary conditions on ω, Σ, π and $\partial_t\omega$, $\partial_t\pi$, $\partial_t\Sigma$ need to be imposed.
The initial conditions are

$$\begin{aligned} \omega(0,x) &= \omega_0 e^{-(x-x_0)^2/\sigma^2} + 1 \qquad (35) \\ \Sigma(0,x) &= -\frac{2}{\sigma^2}\omega_0(x-x_0)e^{(x-x_0)^2/\sigma^2} \qquad (36) \\ \pi(0,x) &= 0 \qquad (37) \end{aligned}$$

with $\omega_0 = 1, x_0 = \pi/4, \sigma^2 = \pi/100$. The boundary conditions are

$$\begin{aligned} \omega(t,\pi/2) &= 1 \qquad (38) \\ \pi(t,\pi/2) &= 0 \qquad (39) \\ \partial_t\omega(t,\pi/2) &= \pi(t,\pi/2) \qquad (40) \\ \partial_t\pi(t,\pi/2) &= 0 \qquad (41) \\ \partial_t\Sigma(t,\pi/2) &= \partial_x\pi(t,\pi/2) \qquad (42) \end{aligned}$$

The set $(t, x = 0)$ is not a boundary, it is the origin, so we need to impose regularity conditions because the quantity $1/\sin^2 x$ blows up at $x = 0$. The regularity conditions are

$$\begin{aligned} \omega(t,0) &= 1 \qquad (43) \\ \pi(t,0) &= 0 \qquad (44) \\ \partial_t\omega(t,0) &= \pi(t,0) \qquad (45) \\ \partial_t\pi(t,0) &= 0 \qquad (46) \\ \partial_t\Sigma(t,0) &= \partial_x\pi(t,0) \qquad (47) \end{aligned}$$

To solves numerically this system we will use an open source library for problems in 1+1 dimensions that use the finite differences method [2].

References

[1] Olivier Sarbach, "On the Generalization of the Regge-Wheeler Equation for Self-Gravitating Matter Fields", www.ifm.umich.mx/~sarbach/publications/total.pdf

[2] https://github.com/eamonto/yang-mills-on-ads