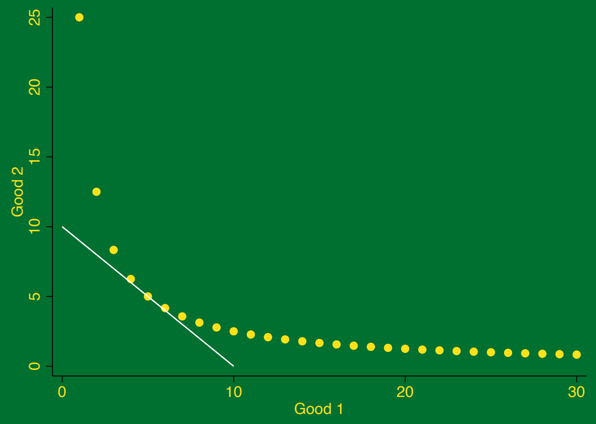
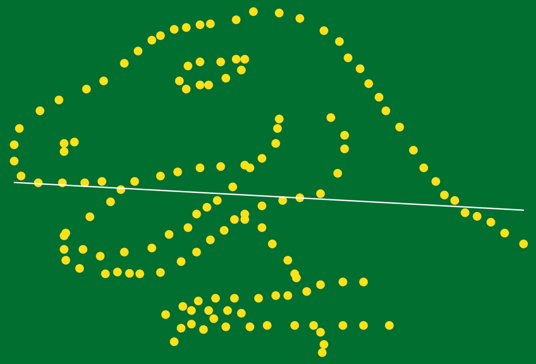
Utility Maximization





Prof Jonathan Davis

UNIVERSITY OF OREGON

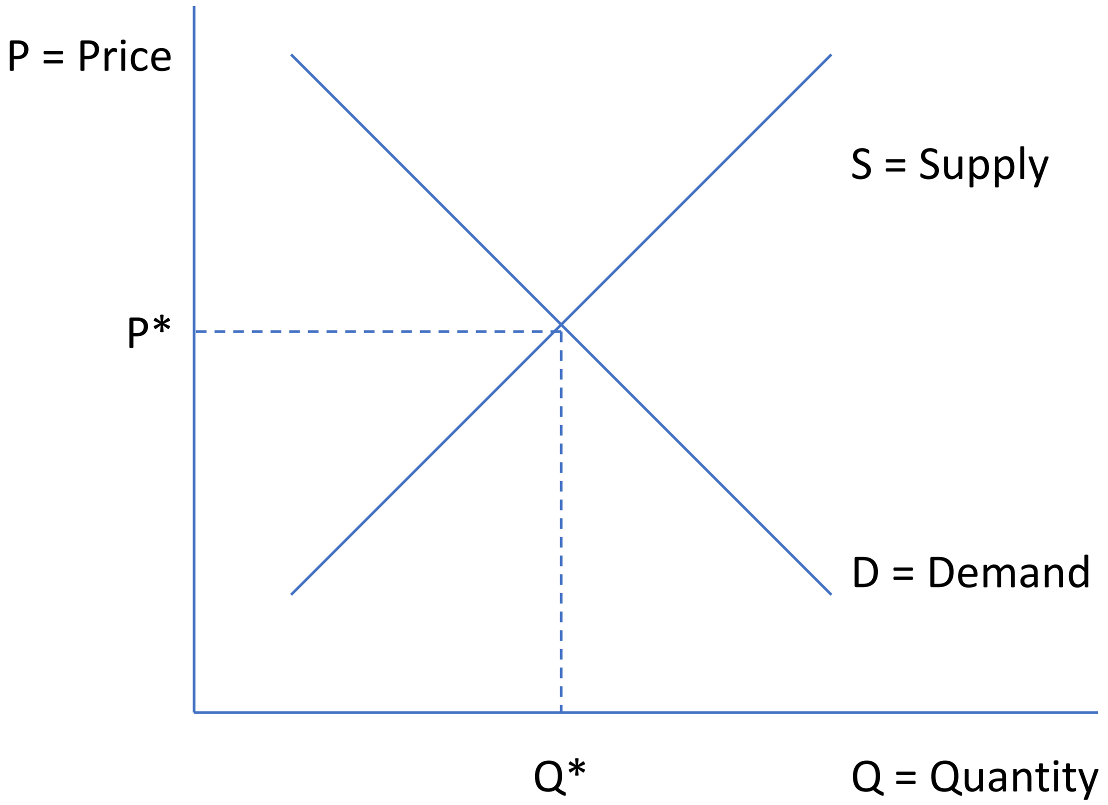
Upcoming Deadlines

Budget Constraints and Preferences HW: **Friday, November 12th, 5pm**

Utility Maximization: **Friday, November 24th** at **5pm**

Demand (Next slide deck): **Friday, December 3rd** at **5pm**

How do people decide what they want?



Our Model of Consumer Theory

We’ve now seen the two ingredients in our model:

1. Budget Constraint

2. Preferences (given by a utility function)

How people make choices?

Intuitively: Of the bundles I can afford, which makes me happiest?

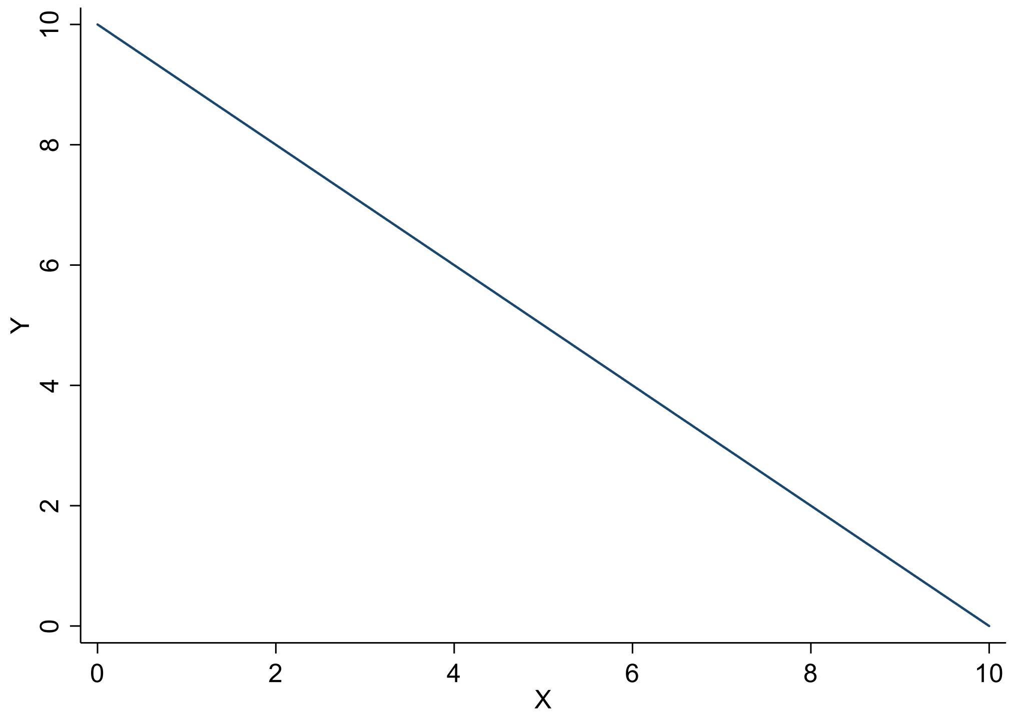
Mathematically:

$$\max_{X,Y} U(X, Y)$$

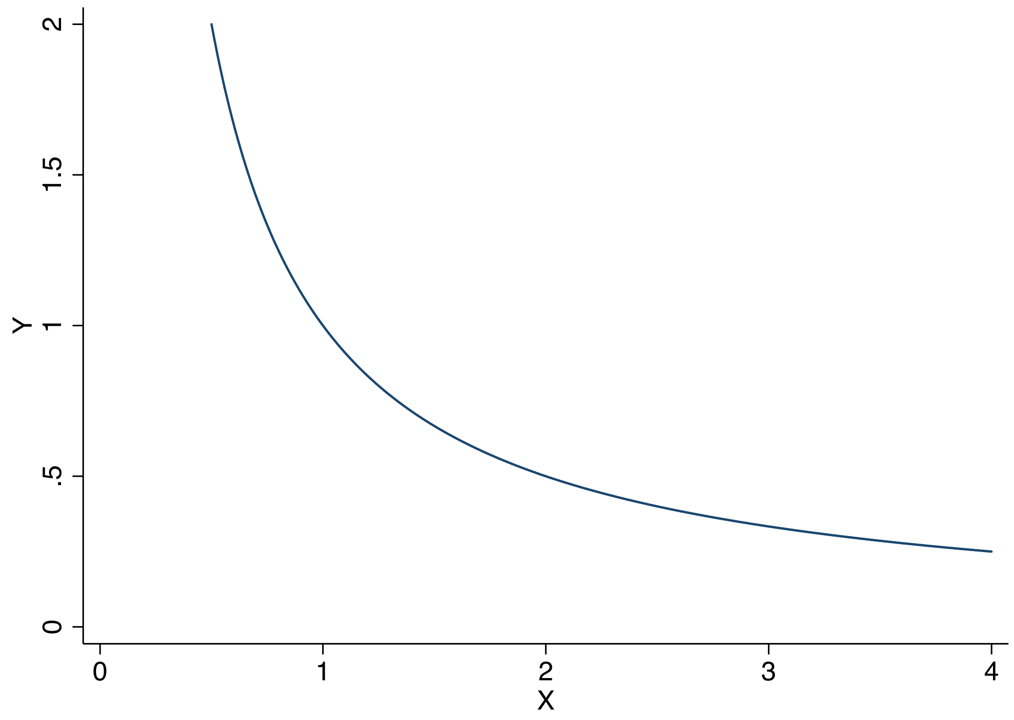
$$such\ that\ p_X X + p_Y Y \leq M$$

Graphing the Two Ingredients

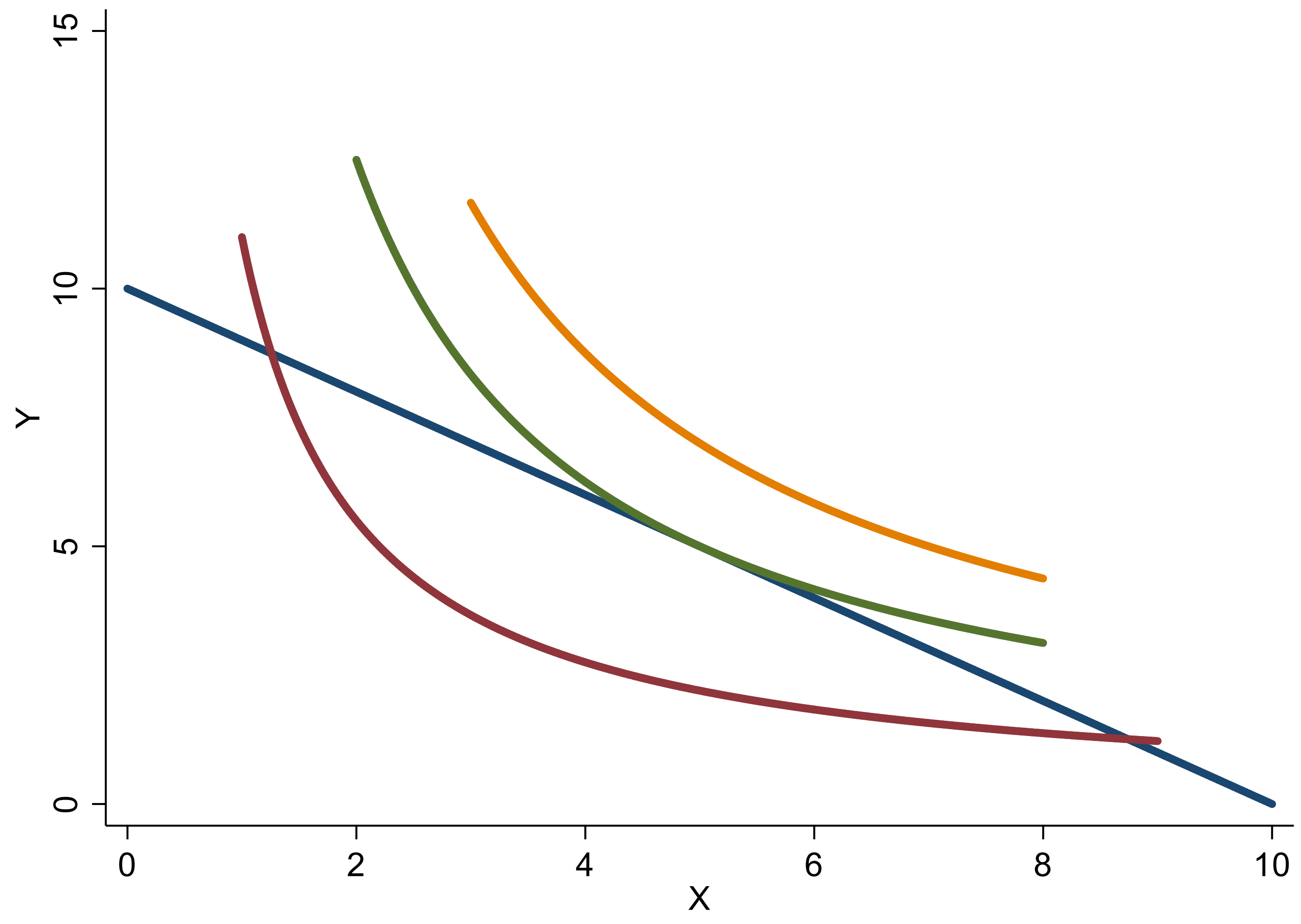
Budget Constraint



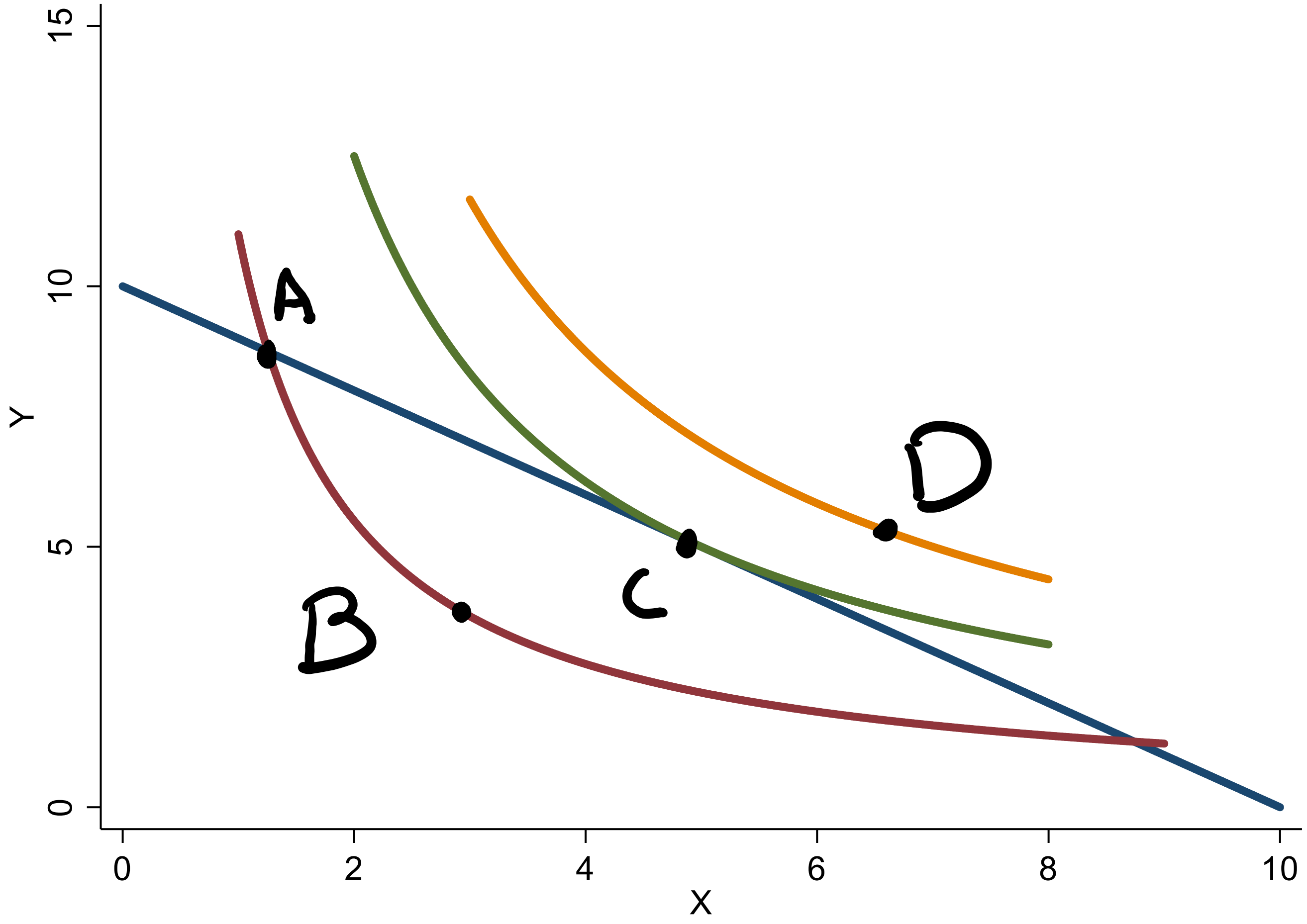
Preferences



Combining the two ingredients



iClicker: Which bundle is optimal for the consumer?



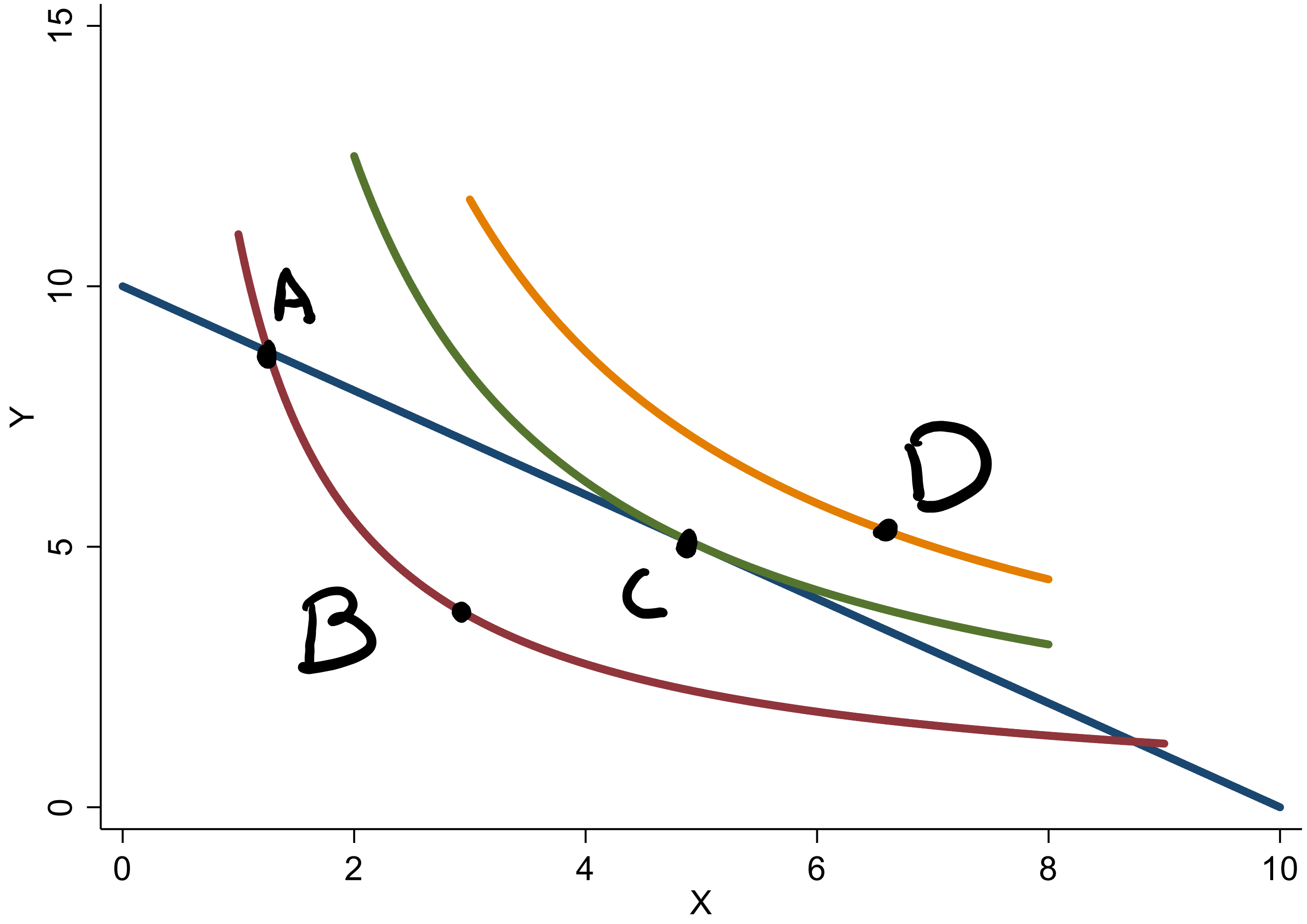
Let’s think through a few bundles

A:

B:

C:

D: Not optimal. **Why?**



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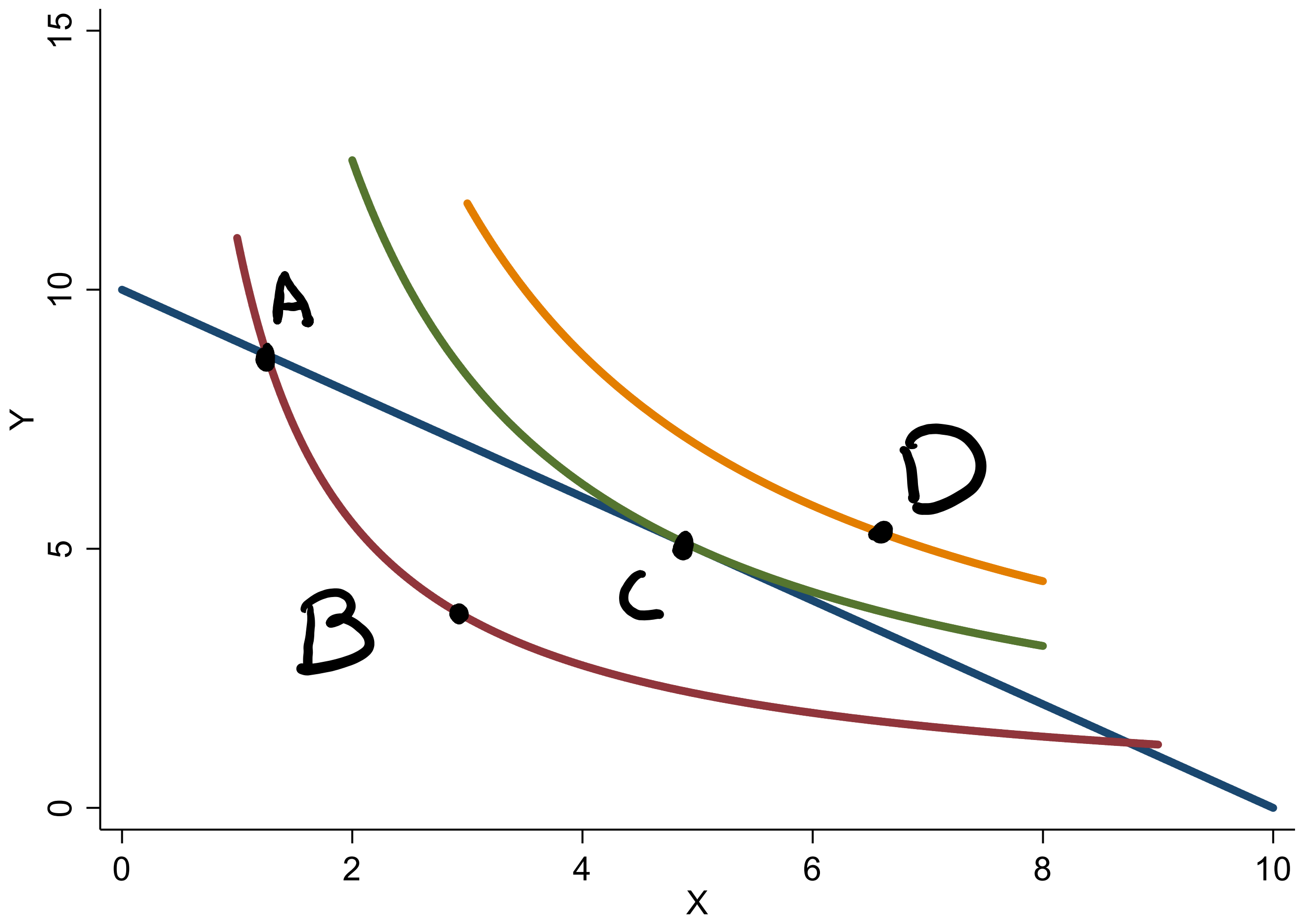


Let’s think through a few bundles

A:

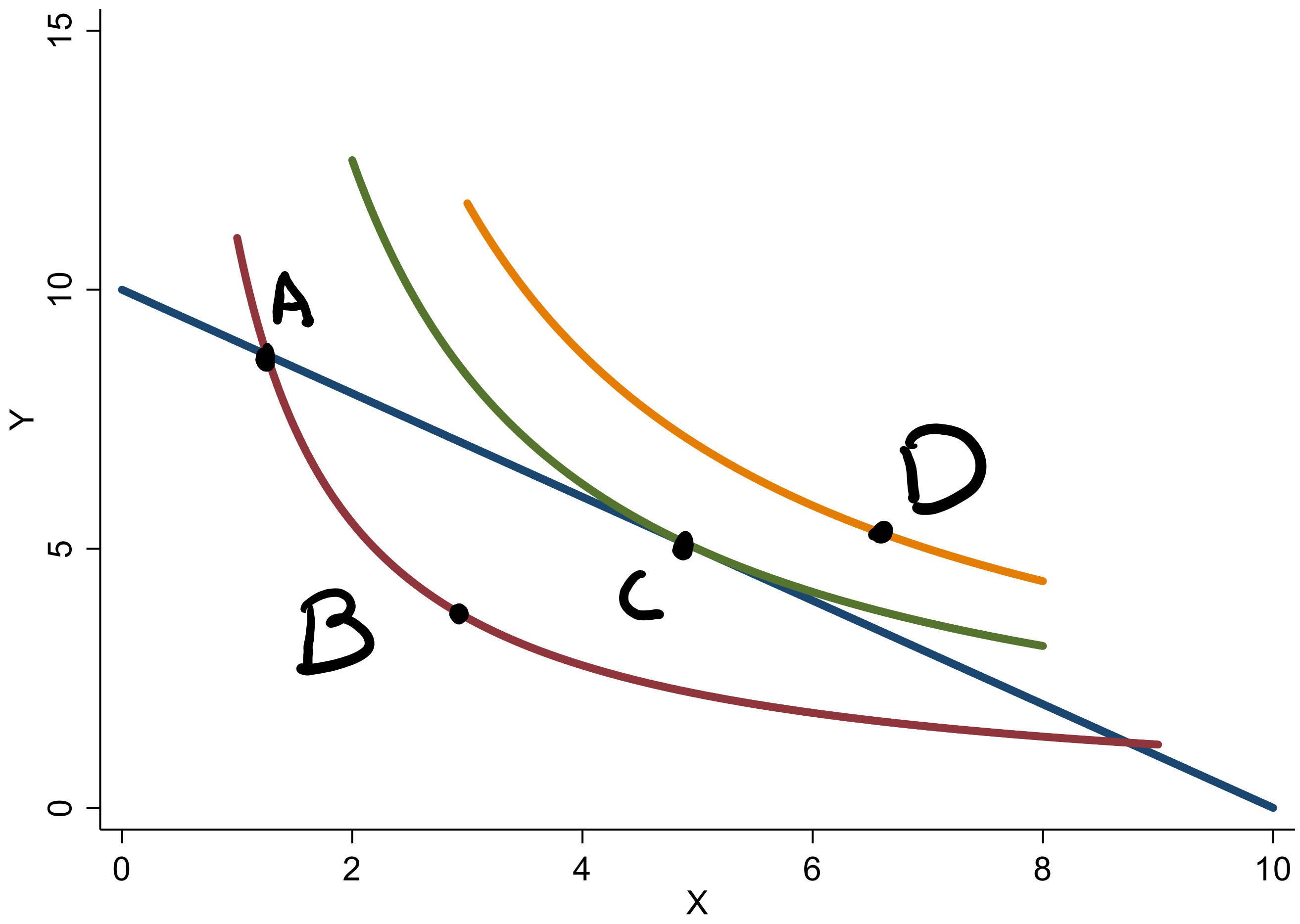
B:

C:

D: Not optimal. **Why?**

D would be great! It’s on the highest indifference curve.

Unfortunately, it is above budget constraint and so is infeasible.



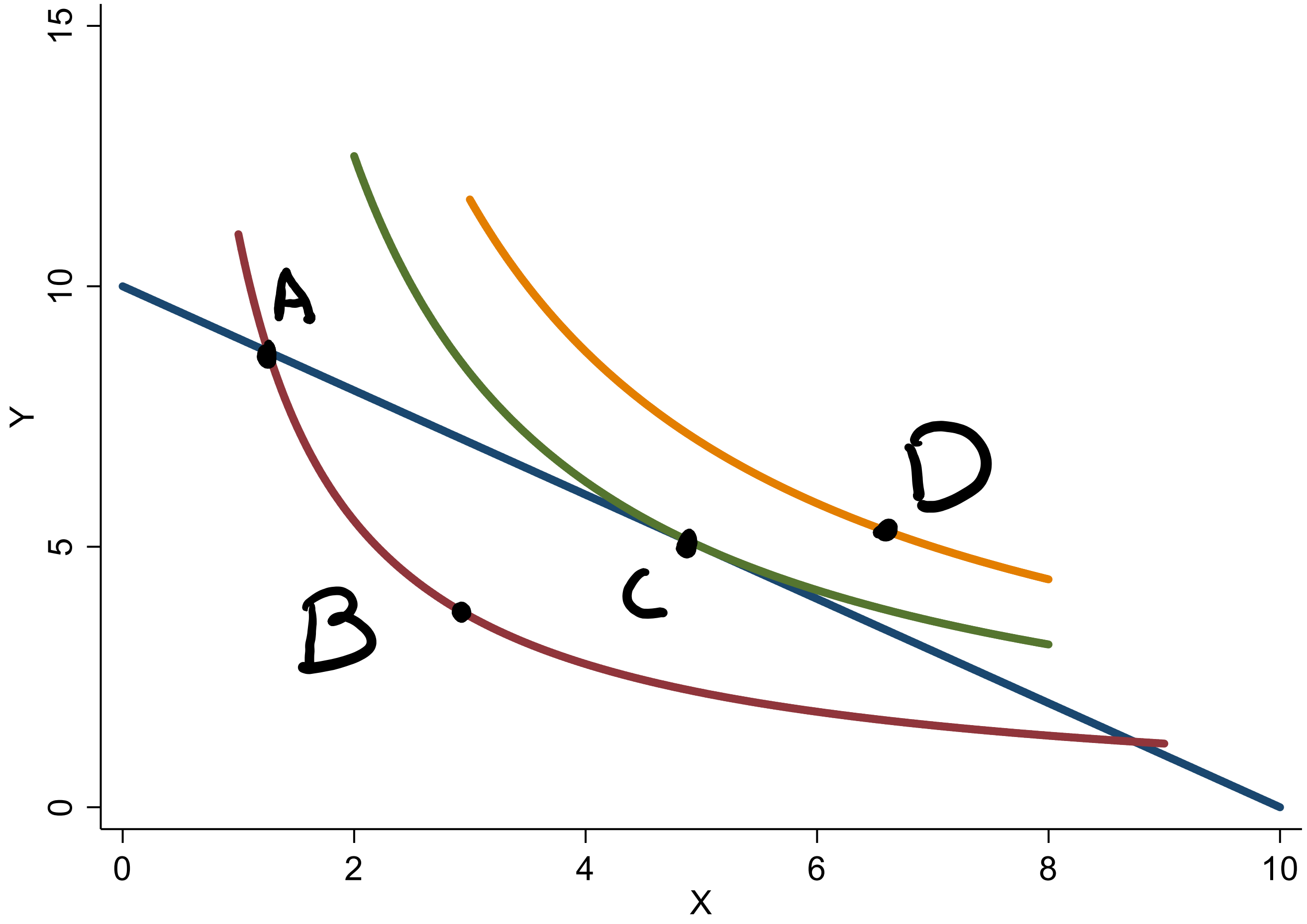
Let’s think through a few bundles

A:

B:

C:

D: Not affordable.



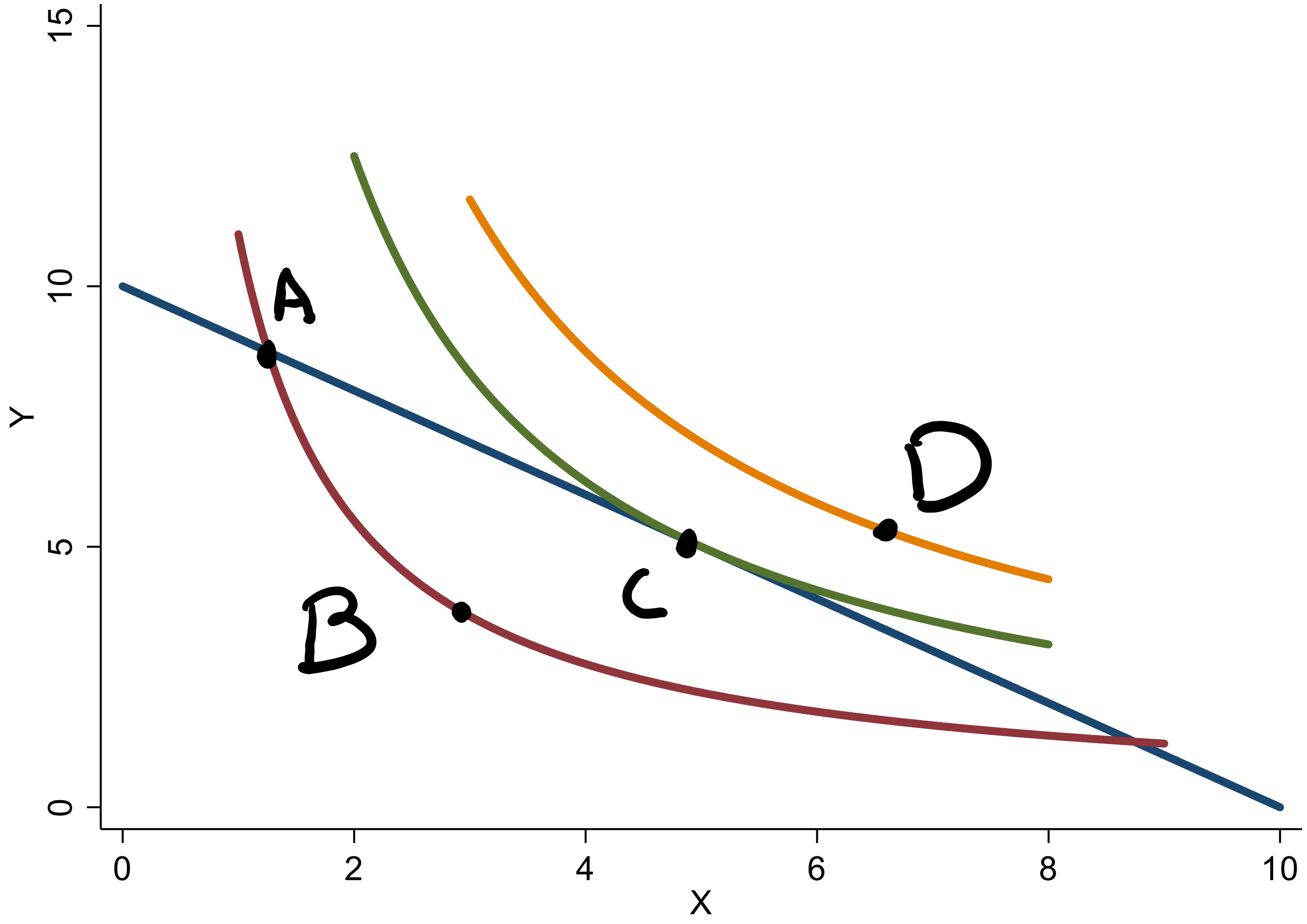
Let's think through a few bundles

A:

B: **Not optimal. Why?**

C:

D: Not affordable.



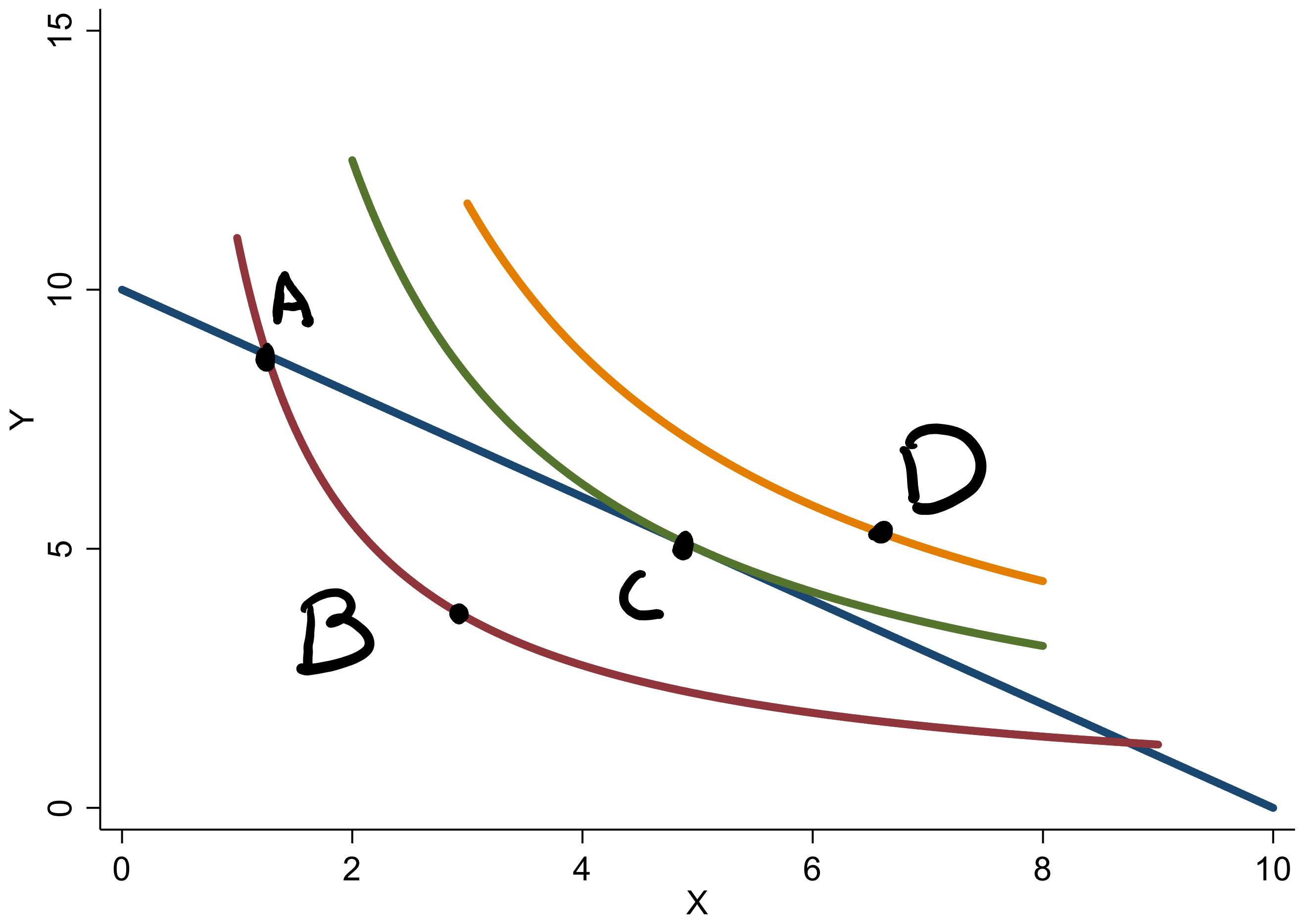
Let’s think through a few bundles

A:

B: Not optimal because it isn’t on the budget constraint so the consumer can buy more X, more Y, or more of both!

C:

D: Not affordable.



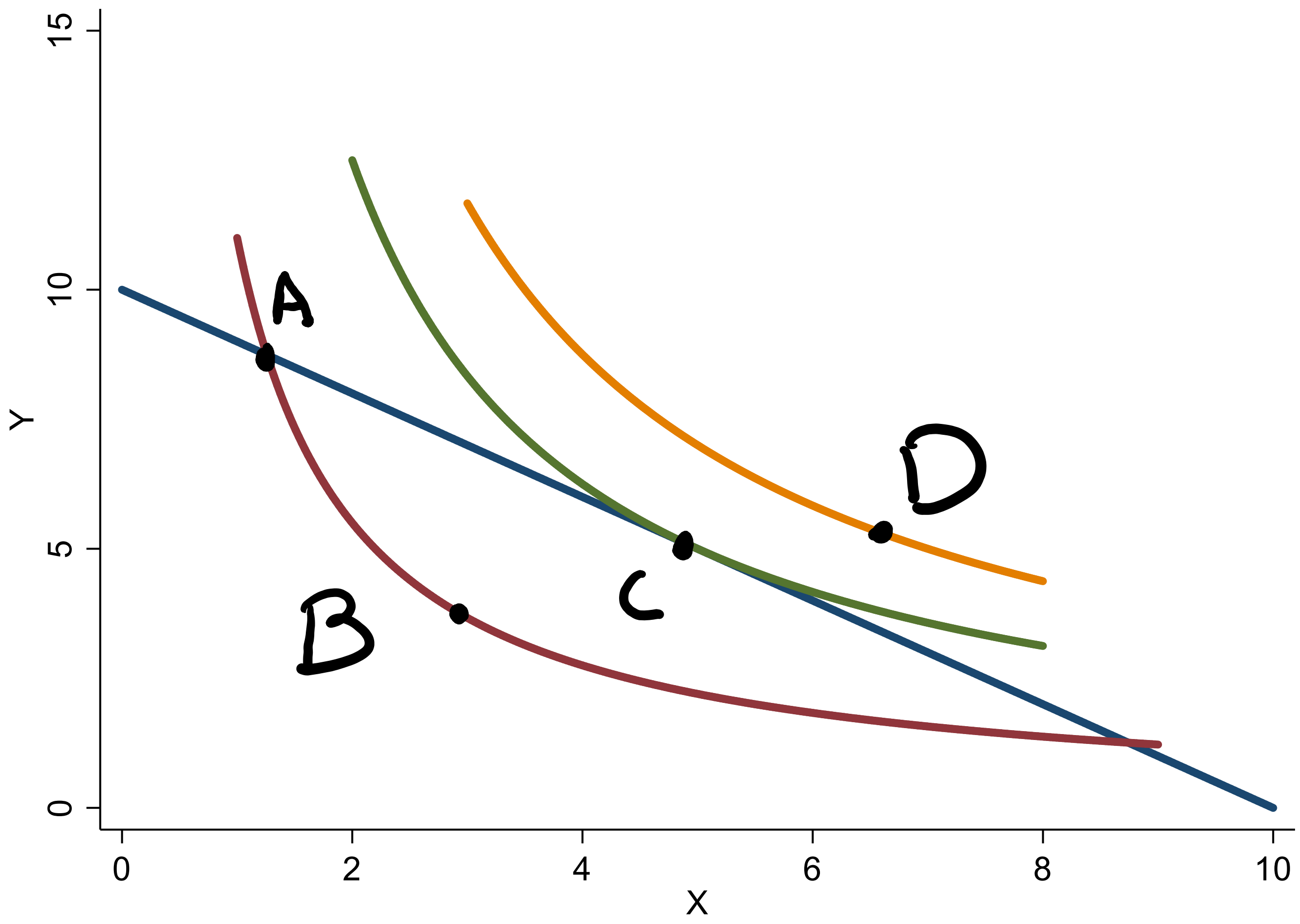
Let’s think through a few bundles

A: **Nope. Why?**

B: Not optimal because it isn’t on the budget constraint so the consumer can buy more X, more Y, or more of both!

C:

D: Not affordable.



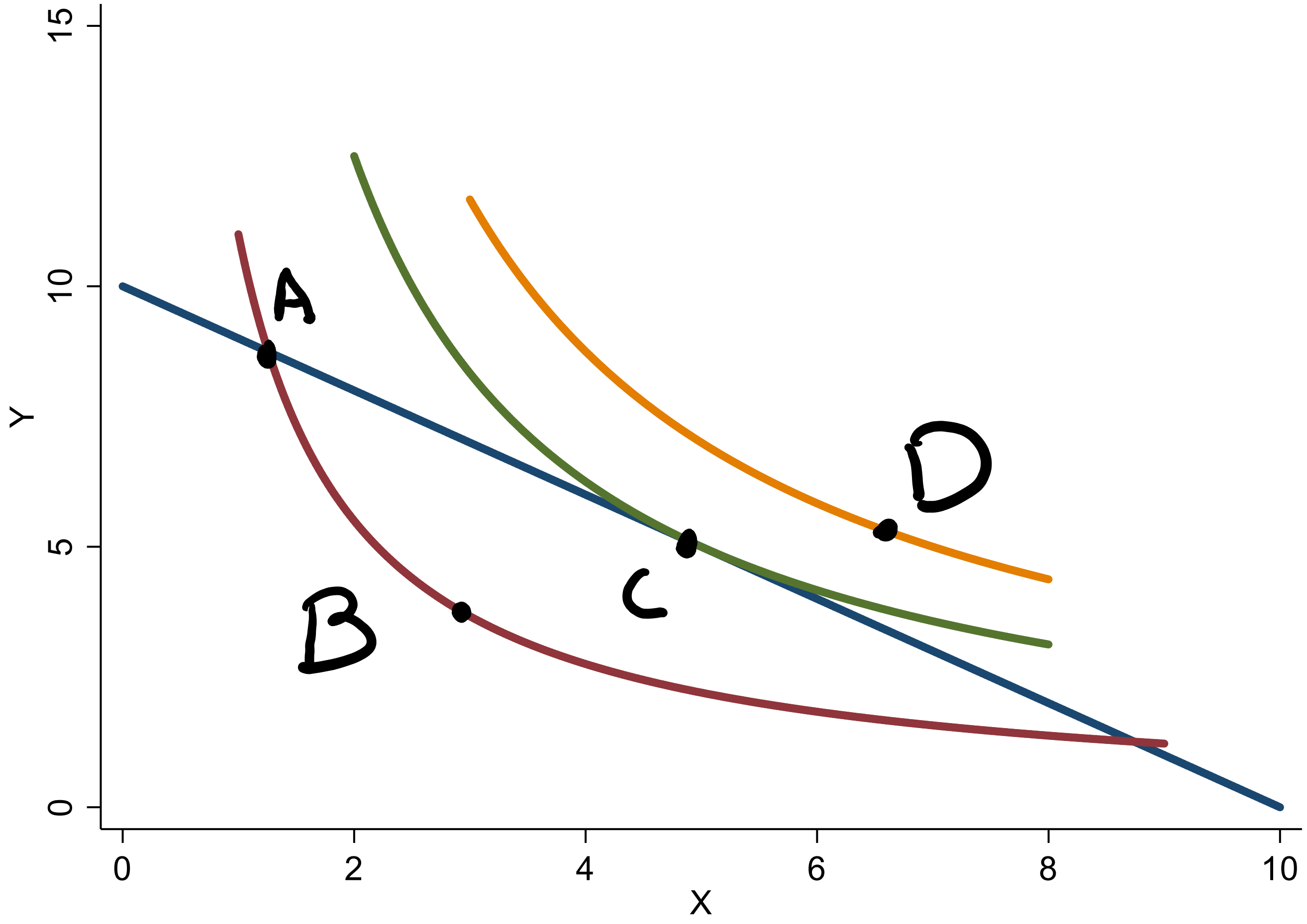
Let's think through a few bundles

A: On the same indifference curve as B and we know the consumer can get more utility than B.

B: Not optimal because it isn't on the budget constraint so the consumer can buy more X, more Y, or more of both!

C:

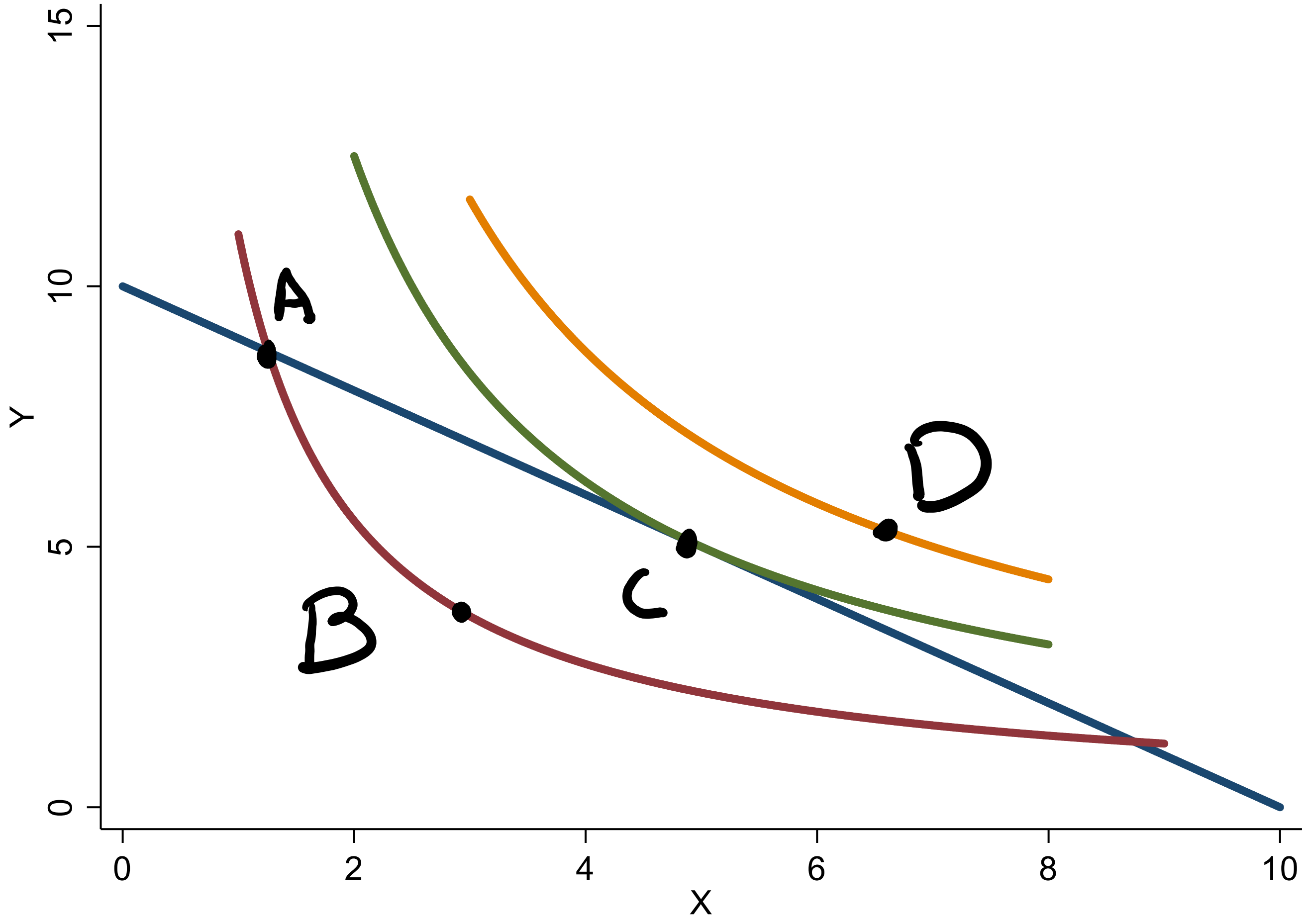
D: Not affordable.



Let’s think through a few bundles

From D: Can’t be on indifference curve that never touches B.C. since unaffordable.

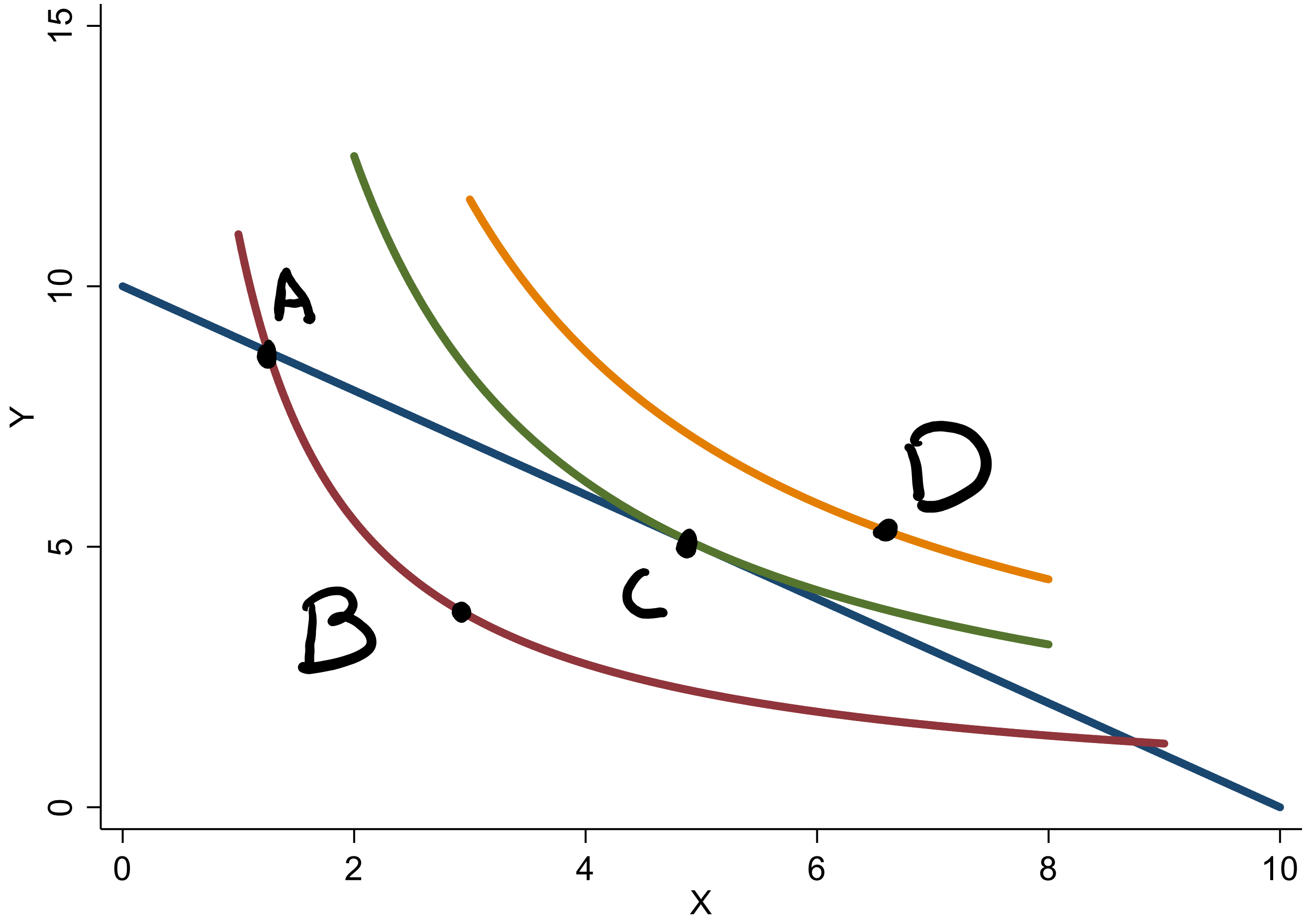
From A: Can’t be on indifference curve that touches B.C. twice because that indifference curve gives same utility as bundle that leaves money on table.



The Optimal Bundle

C has to be the optimal bundle because it is on the highest indifference curve that touches budget constraint.

Touches budget constraint exactly once.

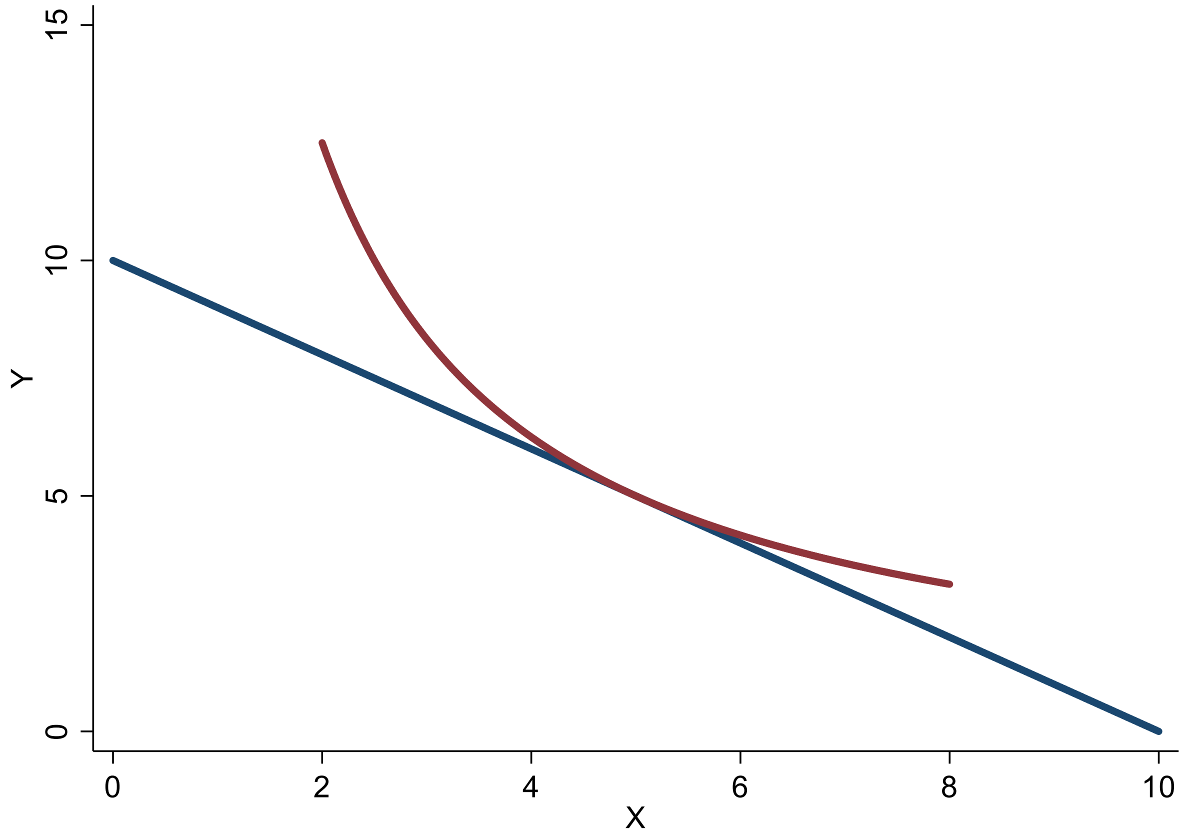


Tangent Lines

When a non-linear curve touches a linear line at exactly one point, we say that the linear line is **tangent** to the non-linear line.

The non-linear line has the same **rate of change** as the linear line at this point.

Or more mathematically:



Derivative of Non-Linear Line = Slope of Linear Line at that point

iClicker: What is the slope of the budget constraint?

A. -1

B. 1

C. $-\frac{p_X}{p_Y}$

D. $-\frac{p_Y}{p_X}$

E. $\mathrm{M}-\frac{p_Y}{p_X}$

iClicker: What is the slope of the budget constraint?

A. -1

B. 1

C. $-\frac{p_X}{p_Y}$

D. $-\frac{p_Y}{p_X}$

E. $\mathrm{M}-\frac{p_Y}{p_X}$

$$M = p_X X + p_Y Y$$

Solve for Y to get equation of budget constraint:

Subtract $p_X X$ from both sides:

$$M - p_X X = p_Y Y$$

Divide both sides by p_Y:

$$\frac{M}{p_Y} - \frac{p_X}{p_Y} X = p_Y Y$$

The Optimality Condition

Marginal Rate of Substitution = |Slope of Budget Constraint|

$$\frac{MU_X}{MU_Y} = \frac{p_X}{p_Y}$$

The Optimality Condition

Marginal Rate of Substitution = |Slope of Budget Constraint|

$$\frac{MU_X}{MU_Y} = \frac{p_X}{p_Y}$$

Note this is exactly equivalent to:

$$\frac{MU_Y}{MU_X} = \frac{p_Y}{p_X}$$

Interpreting the Optimality Condition

$$\frac{MU_X}{MU_Y} = \frac{p_X}{p_Y}$$

Multiplying both sides by MU_Y/p_X gives:

$$\frac{MU_X}{p_X} = \frac{MU_Y}{p_Y}$$

Interpretation: At the optimum bundle, the individual must get the same marginal utility from their last dollar spent on each good.

Old Exam Question

Glen and Ben both drink coffee and coke every day.

Glen likes coffee a lot more than Ben and Ben likes coke more than Glen.

Coffee is $3 a cup and coke is $1 a can.

At the market equilibrium, how many cans of coke is Ben willing to give up for a cup of coffee?

How many cans of coke is Glen willing to give up for a cup of coffee?

Implication of Optimality Condition

$$MRS = \frac{p_X}{p_Y}$$

The marginal rate of substitution is how much of good Y a person is willing to give up to get another unit of X.

But this has to equal the price ratio if they are buying both goods!

Implication of Optimality Condition

How many cans of coke are they willing to give up for coffee?

Answer

Y good is coke because they are giving it up for coffee

X good is therefore coffee

Coffee is $3 a cup and coke is $1 a can

$$\Rightarrow MRS = \frac{3}{1} = 3$$

Ben and Glen are willing to give up 3 cans of coke for a coffee!

Utility Functions

The Standard Case

Cobb-Douglas: $U(X,Y) = X^{\alpha}Y^{\beta}$

Cases that might throw you off

Quasi-Linear: $U(X,Y) = \alpha \log(X) + \beta Y$

Perfect Substitutes: $U(X,Y) = \alpha X + \beta Y$

Perfect Complements: $U(X,Y) = \min(\alpha X, \beta Y)$

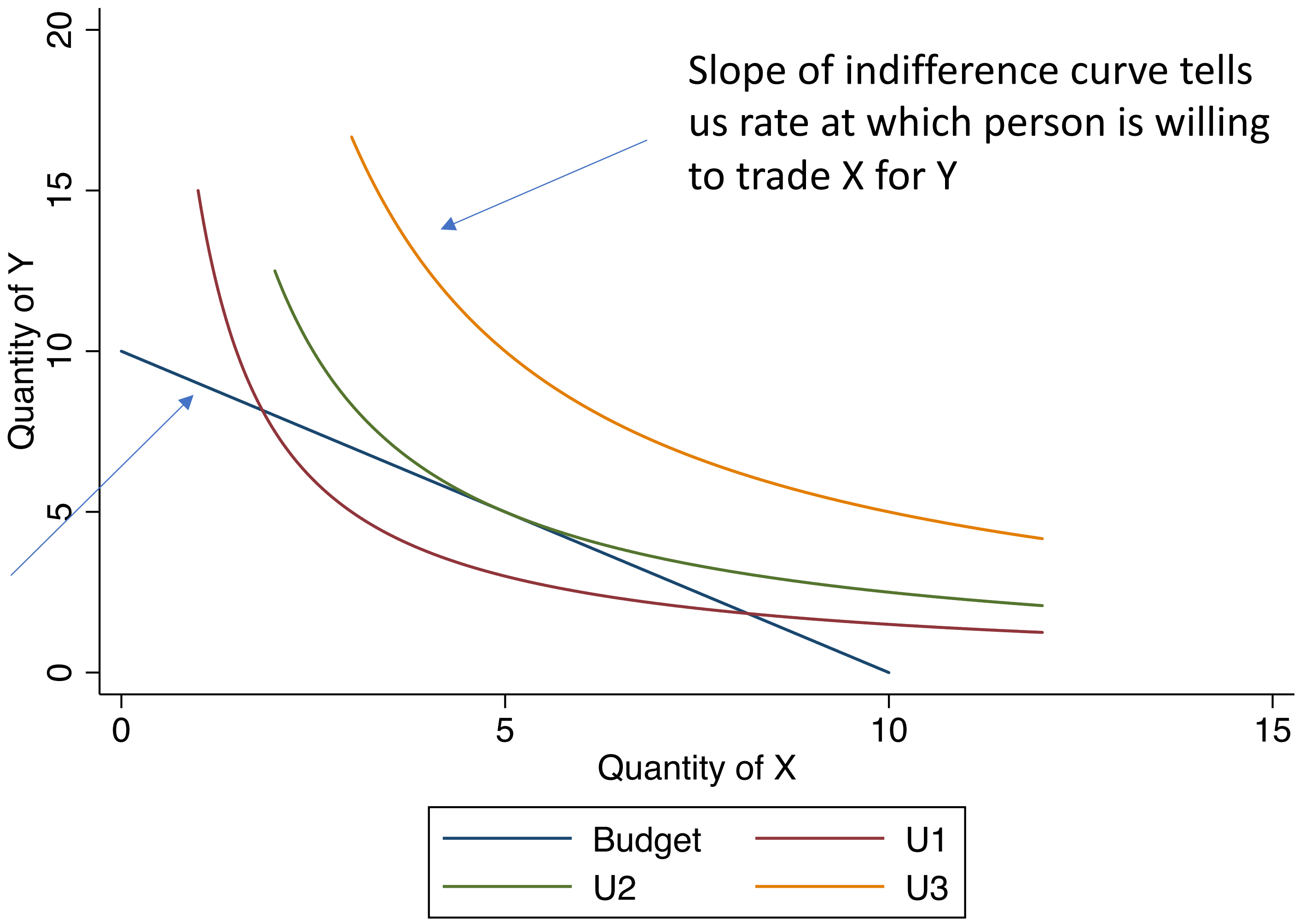
Example: Suppose Ed's utility is $U(X, Y) = XY$.

Ed has \$10 and X costs \$1 and Y costs \$1.

What should Ed buy?

How we will solve it

Slope of budget constraint tells us rate at which market trades X for Y



Two Equations and Two Unknowns

Two unknowns: X^* and Y^*

Two equations:

1. $\frac{MU_X}{MU_Y} = \frac{p_X}{p_Y}$
2. $10 = 1X + 1Y$

Step 1. Calculate marginal utilities for MRS.

Find marginal utility by taking partial derivative:

- -Use power rule from calculus ($\frac{d}{dx}X^n = nX^{n-1}$)
- -Treat other variable as constant

$$U(X,Y) = XY = X^1Y^1$$

$$MU_X = [1X^{1-1}]Y = Y$$

$$MU_Y = X[1Y^{1-1}] = X$$

Step 2: Plug marginal utilities in to MRS.

$$MRS = \frac{MU_X}{MU_Y} = \frac{Y}{X}$$

Step 3: Set MRS equal to slope of budget constraint

$$\frac{MU_X}{MU_Y} = \frac{p_X}{p_Y}$$

Note that this implies: $\frac{MU_X}{p_X} = \frac{MU_Y}{p_Y}$

What does that equation tell us?

Step 3: Set equal to slope of budget constraint

Don't get tricked by fact that $\boldsymbol{MU_X = Y}$ and $\boldsymbol{MU_Y = X}$!

$\boldsymbol{p_X}$ is still in numerator!

$$\frac{MU_X}{MU_Y} = \frac{p_X}{p_Y}$$

Step 3: Set equal to slope of budget constraint

Don't get tricked by fact that $\boldsymbol{MU_X = Y}$ and $\boldsymbol{MU_Y = X}$!

$\boldsymbol{p_X}$ is still in numerator!

$$\frac{MU_X}{MU_Y} = \frac{p_X}{p_Y}$$

$$\frac{Y}{X} = \frac{p_X}{p_Y}$$

Step 3: Set equal to slope of budget constraint

Don’t get tricked by fact that $\boldsymbol{MU_X = Y}$ and $\boldsymbol{MU_Y = X}$!

$\boldsymbol{p_X}$ **is still in numerator!**

$$\frac{MU_X}{MU_Y} = \frac{p_X}{p_Y}$$

$$\frac{Y}{X} = \frac{p_X}{p_Y}$$

$$\frac{Y}{X} = \frac{1}{1}$$

Step 3: Set equal to slope of budget constraint

Don't get tricked by fact that $\boldsymbol{MU_X = Y}$ and $\boldsymbol{MU_Y = X}$!

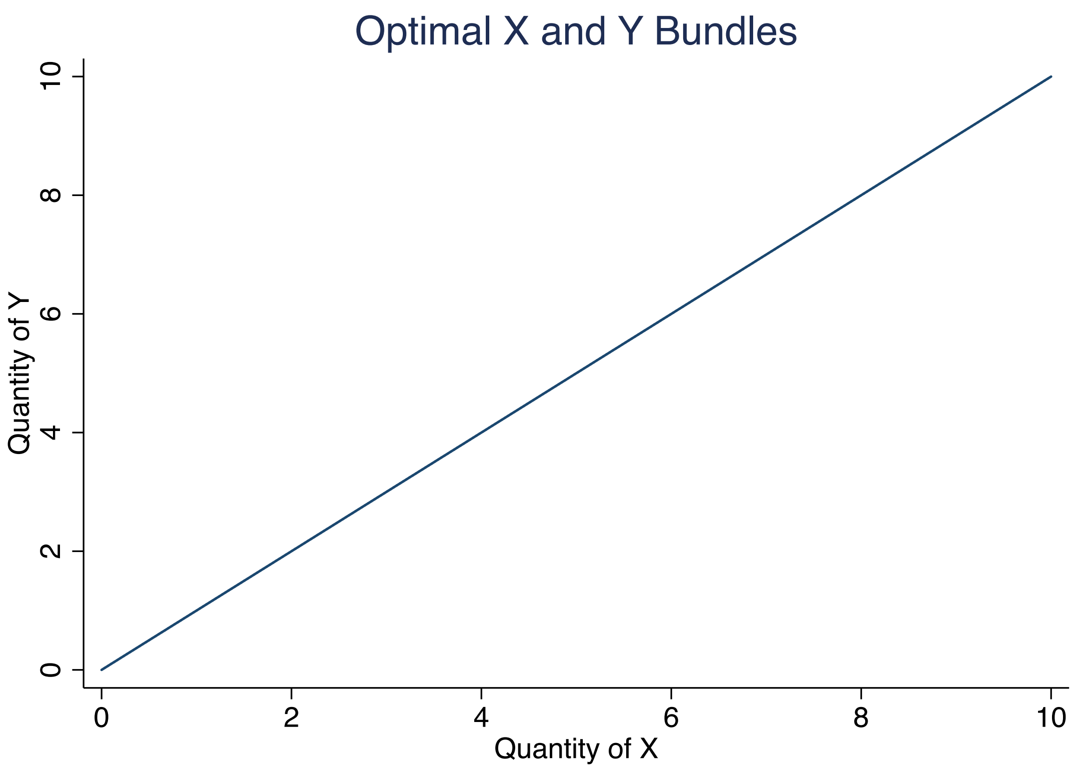
$\boldsymbol{p_X}$ is still in numerator!

$$\frac{MU_X}{MU_Y} = \frac{p_X}{p_Y}$$

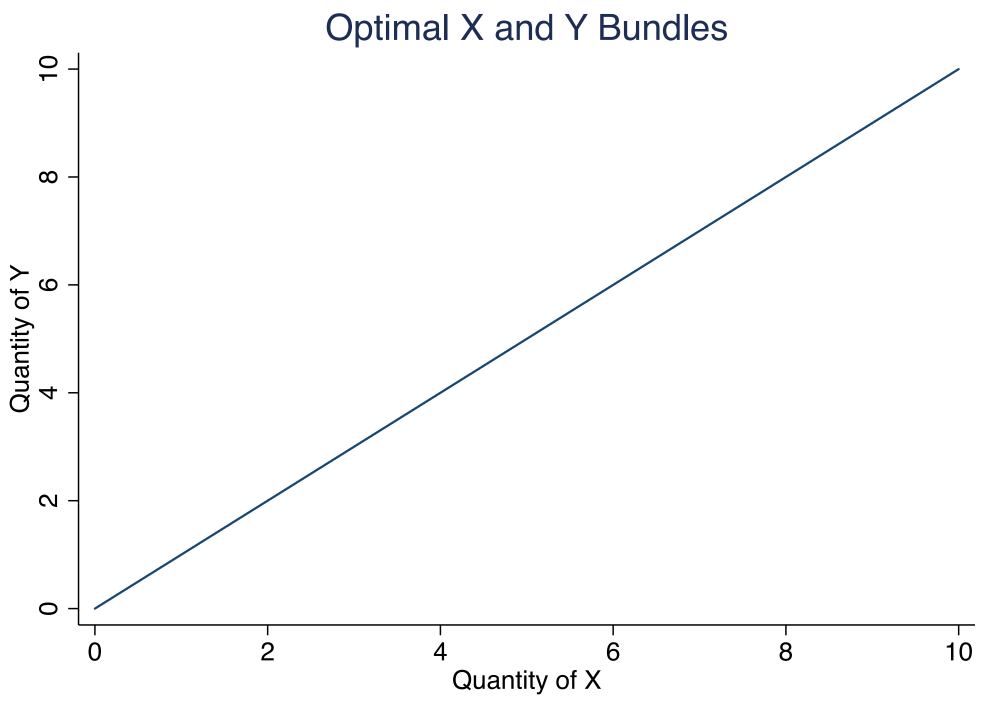
$$\frac{Y^*}{X^*} = \frac{p_X}{p_Y}$$

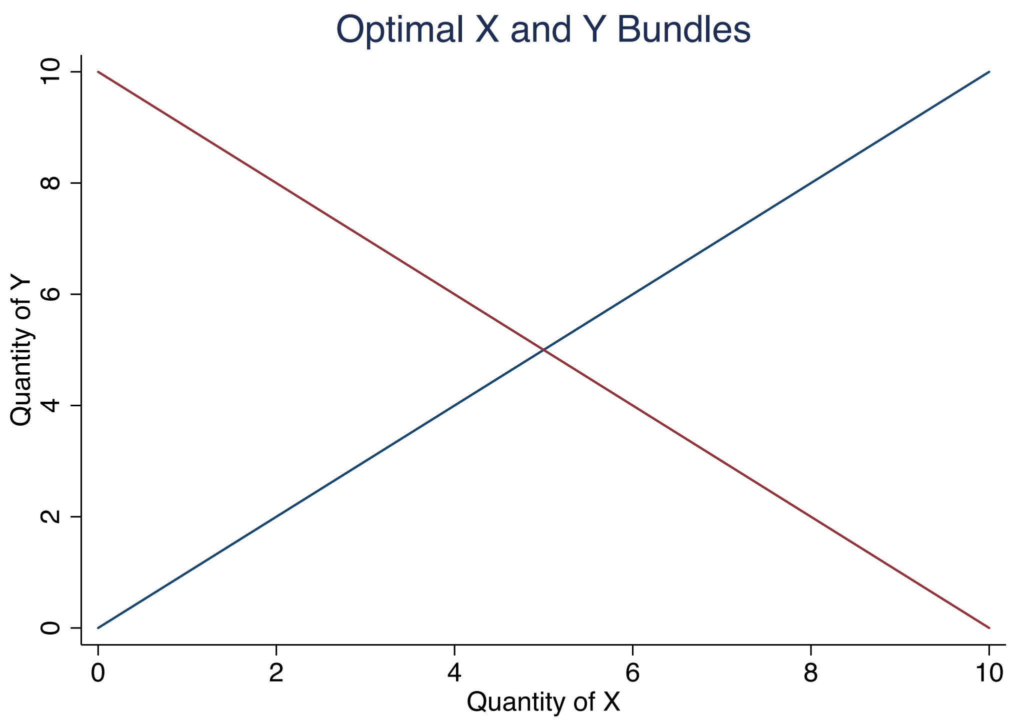
$$\frac{Y^*}{X^*} = \frac{1}{1} \Rightarrow \boldsymbol{Y^* = X^*}$$

Graphically, we've found the optimal X and Y bundles at all levels of income.



Find answer by finding intersection with budget constraint





Step 4: Find that point by plugging equation in to budget constraint

$$10 = X + Y$$

Found $Y^* = X^*$ in step 3:

$$10 = X^* + X^*$$

$$10 = 2X^*$$

$$X^* = 5$$

Step 5: Don’t forget about other good!

We know $Y^* = X^*$ and $X^* = 5$.

So $Y^* = 5$.

Answer: X*=5 and Y*=5

Note for homework: This is the Lagrangian Method

$$\max_{X,Y} U(X,Y)$$
$$s.t\; p_x X + p_Y Y = M$$

Lagrangian method says to write this as:

$$\mathcal{L} = U(X,Y) + \lambda(M - p_X X - p_Y Y)$$

First-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial X} = MU_X - \lambda p_X = 0 \rightarrow MU_X = \lambda p_X$$

$$\frac{\partial \mathcal{L}}{\partial Y} = MU_Y - \lambda p_Y = 0 \rightarrow MU_Y = \lambda p_Y$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M = p_X X + p_Y Y$$

Where $\frac{\partial \mathcal{L}}{\partial X}$ and $\frac{\partial \mathcal{L}}{\partial Y}$ imply the optimality condition: $\frac{MU_X}{p_X} = \frac{MU_Y}{p_Y}$

iClicker: Suppose Adaline's utility is $U(X, Y) = X^{0.5}Y^2$. Adaline has \$20 and X costs \$1 and Y costs \$2. What should Adaline buy?

A. $X^* = 4, Y^* = 8$

B. $X^* = 8, Y^* = 4$

C. $X^* = 6, Y^* = 6$

D. $X^* = \frac{20}{17}, Y^* = \frac{160}{17}$

E. $X^* = \frac{160}{17}, Y^* = \frac{20}{17}$

iClicker: Suppose Adaline's utility is $U(X, Y) = X^{0.5}Y^2$. Adaline has \$20 and X costs \$1 and Y costs \$2. What should Adaline buy?

A. $\boldsymbol{X^* = 4, Y^* = 8}$

B. $X^* = 8, Y^* = 4$

C. $X^* = 6, Y^* = 6$

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E. $X^* = \frac{160}{17}, Y^* = \frac{20}{17}$

Two Equations and Two Unknowns

Two unknowns: X^* and Y^*

Two equations:

1. $\frac{MU_X}{MU_Y} = \frac{p_X}{p_Y}$
2. $20 = 1X + 2Y$

Step 1. Calculate marginal utilities for MRS.

Find marginal utility by taking partial derivative:

-Use power rule from calculus ($\frac{d}{dx}X^n = nX^{n-1}$)

-Treat other variable as constant

$$U(X, Y) = X^{0.5}Y^2$$

$$MU_X = [0.5X^{0.5-1}]Y^2 = 0.5X^{-0.5}Y^2$$

$$MU_Y = X^{0.5}[2Y^{2-1}] = 2X^{0.5}Y$$

Step 2: Plug marginal utilities in to MRS.

$$MRS = \frac{MU_X}{MU_Y} = \frac{0.5X^{-0.5}Y^2}{2X^{0.5}Y}$$

$$MRS = \frac{0.5}{2}\frac{X^{-0.5}}{X^{0.5}}\frac{Y^2}{Y}$$

$$MRS = \frac{1}{4}\frac{Y}{X}$$

Step 3: Set equal to slope of budget constraint

Don't get tricked by fact that $\boldsymbol{MU_X = Y}$ and $\boldsymbol{MU_Y = X}$!

$\boldsymbol{p_X}$ is still in numerator!

$$\frac{MU_X}{MU_Y} = \frac{p_X}{p_Y}$$

Step 3: Set equal to slope of budget constraint

Don't get tricked by fact that $\boldsymbol{MU_X = Y}$ and $\boldsymbol{MU_Y = X}$!

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$$\frac{1}{4}\frac{Y}{X} = \frac{1}{2}$$

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Don't get tricked by fact that $\boldsymbol{MU_X} = \boldsymbol{Y}$ and $\boldsymbol{MU_Y} = \boldsymbol{X}$!

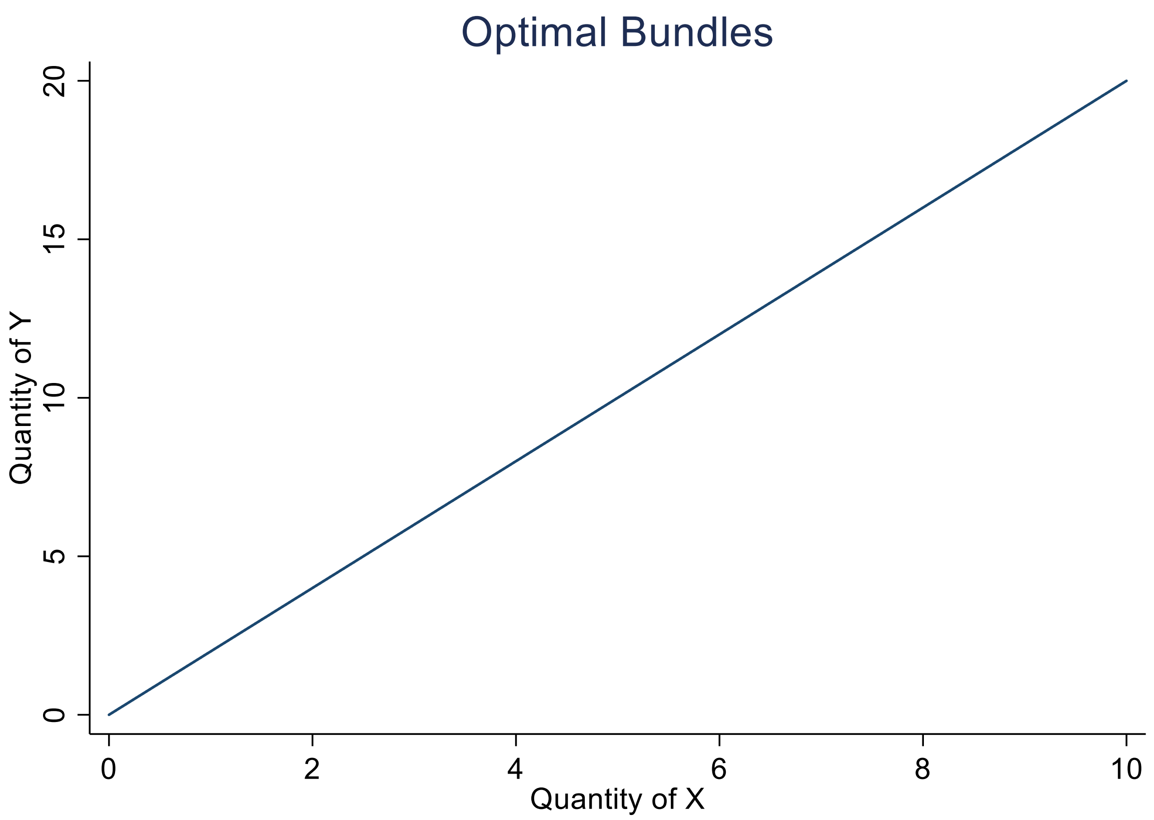
$\boldsymbol{p_X}$ is still in numerator!

$$\frac{MU_X}{MU_Y} = \frac{p_X}{p_Y}$$

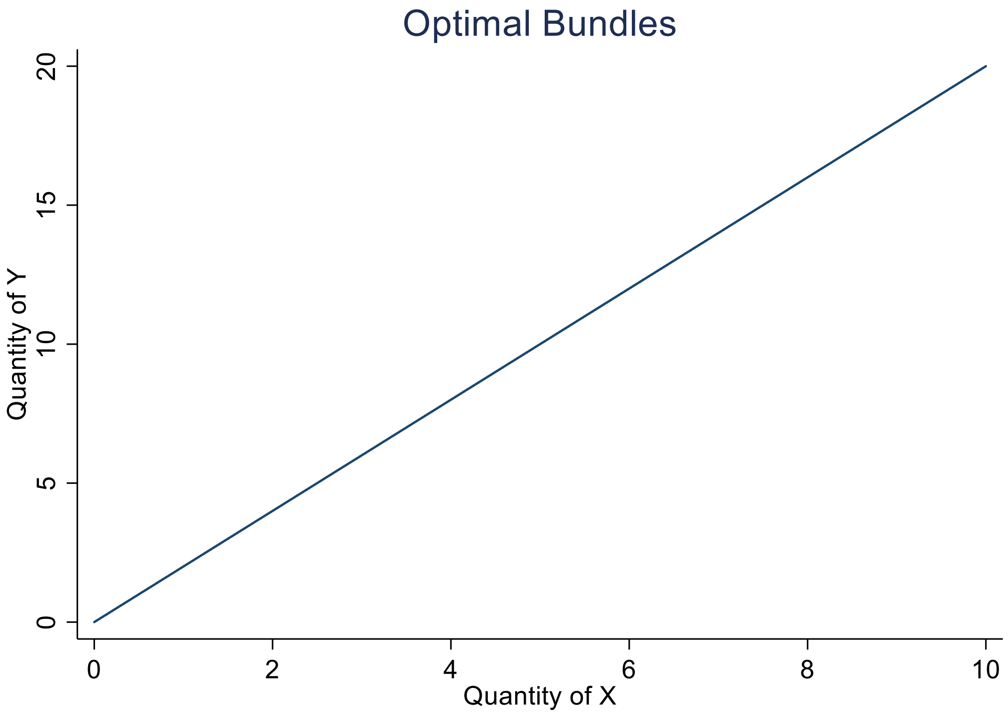
$$\frac{1}{4}\frac{Y^*}{X^*} = \frac{1}{2}$$

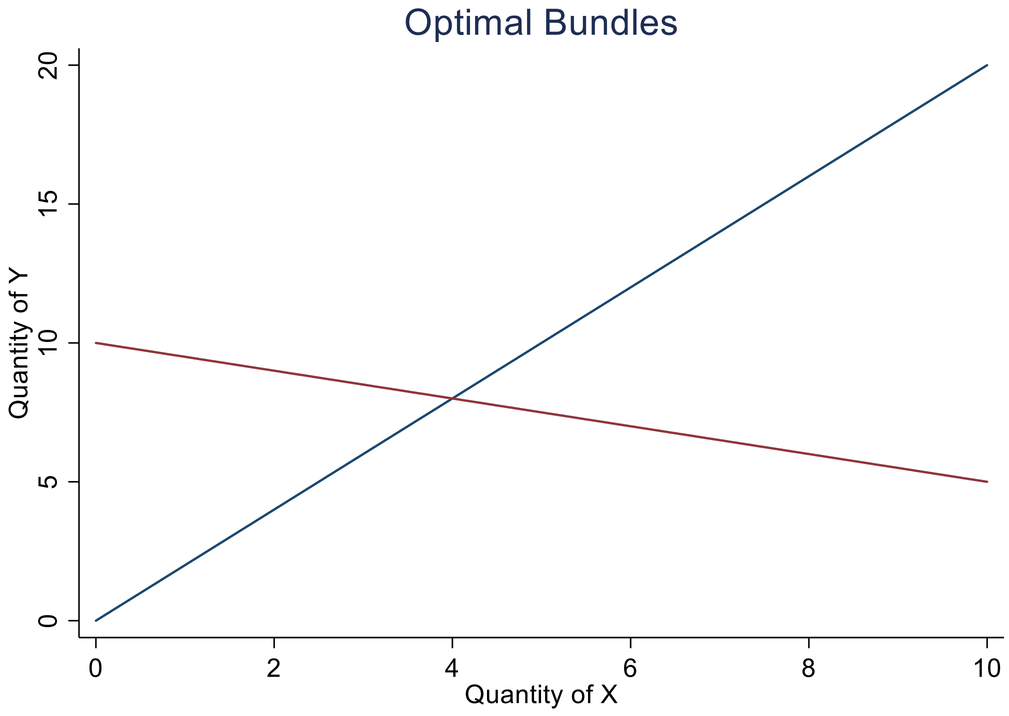
$$\Rightarrow \boldsymbol{Y^* = 2X^*}$$

Graphically, we’ve found the optimal X and Y bundles at all levels of income.



Find answer by finding intersection with budget constraint





Step 4: Find that point by plugging equation in to budget constraint

$$20 = X + 2Y$$

Found $Y^* = 2X^*$ in step 3:

$$20 = X^* + 4X^*$$

$$20 = 5X^*$$

$$X^* = \frac{20}{5} = 4$$

Step 5: Don't forget about other good!

We know $Y^* = 2X^*$ and $X^* = 4$.

So $Y^* = 8$.

Answer: X*=4 and Y*=8

Perfect Substitutes

iClicker: Kim's preferences over 8oz (E) and 16oz (S) lattes are given

$$U(E,S) = E + 2S$$

8oz lattes are \$2 and 16oz lattes are \$3. She has \$5. What should she buy?

A. $E^* = 2.5, S^* = 0$

B. $E^* = 2, S^* = 0$

C. $E^* = 1, S^* = 1$

D. $E^* = 0, S^* = 1$

E. $E^* = 0, S^* = \frac{5}{3}$

iClicker: Kim's preferences over 8oz (E) and 16oz (S) lattes are given

$$U(E, S) = E + 2S$$

8oz lattes are \$2 and 16oz lattes are \$3. She has \$5. What should she buy?

A. $E^* = 2.5, S^* = 0$

B. $E^* = 2, S^* = 0$

C. $E^* = 1, S^* = 1$

D. $E^* = 0, S^* = 1$

E. $\boldsymbol{E^* = 0, S^* = \frac{5}{3}}$

Step 1. Calculate marginal utilities for MRS.

Find marginal utility by taking partial derivative:

$$U(E, S) = E + 2S$$

$$MU_E = 1$$

$$MU_S = 2$$

Step 2: Plug marginal utilities in to MRS.

$$MRS = \frac{MU_E}{MU_S} = \frac{1}{2}$$

Step 3: Set MRS equal to the price ratio.

$$MRS = \frac{1}{2} = \frac{p_E}{p_S} = \frac{2}{3}.$$

Step 3: Set MRS equal to the price ratio.

$$MRS = \frac{1}{2} = \frac{p_E}{p_S} = \frac{2}{3}.$$

Huh, it is never true that $\frac{1}{2} = \frac{2}{3}$.

What do we do now?

Step 4: Calculate marginal utility per dollar!

Marginal utility per dollar from 8oz lattes: $\frac{MU_E}{p_E} = \frac{1}{2}$.

Marginal utility per dollar from 16oz lattes: $\frac{MU_S}{p_S} = \frac{2}{3}$.

Buy only the good with higher marginal utility per dollar!

Step 5: Use budget constraint to figure out how much

$$5 = E + 2S$$

$$5 = (0) + 2S^*$$

$$S^* = \frac{5}{2} = 2.5$$

Perfect Complements

Marty's preferences over left and right shoes are given by $U(L, R) = \min(L, R)$. Shoes cost \$20 each and he has \$100. What should he buy?

Calculate marginal utilities?

Don't need to take derivative of $\min(L, R)$.

We set the marginal rate of substitution = price ratio to find out optimal ratio to consume goods.

With perfect complements we know the ratio!

$$L^* = R^*$$

Calculate marginal utilities?

Don't need to take derivative of $\min(L, R)$.

We set the marginal rate of substitution = price ratio to find out optimal ratio to consume goods.

With perfect complements we know the ratio!

$$L^* = R^*$$

Or in general, for $\min(\alpha X, \beta Y)$ the optimal ratio is $\alpha X^* = \beta Y^*$.

Plug optimal ratio in to budget constraint

$$100 = 20L + 20R$$

Plugging in $L^* = R^*$:

$$100 = 20L^* + 20L^* = 40L^*$$

Solving for L^*:

$$L^* = \frac{100}{40} = 2.5$$

So $R^* = 2.5$ too.

Be careful of word problems!

Martha loves to eat chili, especially an award-winning chili recipe that calls for using 2 tablespoons of chili powder for 1 pound of ground buffalo. What is Martha's utility as a function of tablespoons of chili powder (P) and pounds of ground buffalo (B)?

Be careful of word problems!

Martha loves to eat chili, especially an award-winning chili recipe that calls for using 2 tablespoons of chili powder for 1 pound of ground buffalo. What is Martha's utility as a function of tablespoons of chili powder (*P*) and pounds of ground buffalo (*B*)?

- Perfect Complements because using in exactly fixed proportions
- Needs 1 pound of buffalo (B) for every 2 tablespoons of powder (P)
- This implies $P^* = 2B^*$
- Easy to get this backwards because need 1B and 2P, but this implies $B = 2P$!

iClicker: The three-legged Ork, a space creature from the universe Warhammer, wears 1 right shoe and 2 left shoes. Which set of market bundles provides an Ork with the highest level of utility?

A. 15 right shoes and 14 left shoes
B. 12 right shoes and 12 left shoes
C. 10 right shoes and 16 left shoes
D. 6 right shoes and 12 left shoes

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A. 15 right shoes and 14 left shoes
B. 12 right shoes and 12 left shoes
C. 10 right shoes and 16 left shoes
D. 6 right shoes and 12 left shoes

Need 2 left shoes for every one right shoe, or:

$$2R = L.$$

This is perfect complements utility with

$$U = \min(2R, L).$$

Can check each:

A. $\min(2\times15,14) = 14$
B. $\min(2\times12,12) = 12$
C. $\min(2\times10,16) = 16$
D. $\min(2\times12,12) = 12$

What this shows us

$$U = \min(2R, L)$$

So the optimal ratio is $2R^* = L^*$.

10 right shoes and 16 left shoes

- This bundle is wasteful, so it is a mistake. Wealthier people can make a mistake and still have higher utility.

6 right shoes and 12 left shoes

- This bundle is not wasteful but yields lower utility than (10,16).

What we’ve learned

Solve utility maximization problems by picking quantities where marginal utility per dollar is same for both goods

Cobb-Douglas is standard case.

When marginal utility per dollar is higher for one good, should only buy that good! This is usually case for perfect substitutes utility and is sometimes the case with quasi-linear utility.

With perfect complements, buy as much as can afford in fixed proportions.

My daughter drew this and I’m not a monster so I didn’t delete it!