HW 5

1. To be covariance stationary,

① $E[Z_t]$ doesn't depend on t

and ② $cov(r_t, r_{t-j}) = r_j$ exists and doesn't depend on t

① $E[Z_t] = E[Y_t] + E[X]$

$= \mu + 0$

$= \mu$

doesn't depend on t

② $r_0 = Var(Z_t)$

$= Var(Y_t) + Var(X)$

$= \sigma_Y^2 + 1$

$r_j = cov(Z_t, Z_{t-j})$

$= cov(Y_t + X, Y_{t-j} + X)$

$= \overset{\to 0}{\cancel{cov(Y_t, Y_{t-j})}} + \overset{\to 0}{\cancel{cov(Y_t, X)}} + \overset{\to 0}{\cancel{cov(Y_{t-j}, X)}} + cov(X, X)$

$= E[X^2]$

$= 1$

$\Rightarrow$ So Z_t is covariance stationary

For Z_t to be ergodic

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} r_j = 0$$

$$= \lim_{n\to\infty} \frac{1}{n} \Sigma_{j=1}^{n} 1 = 1$$

So z_t is not ergodic

2. $y_t = (1 + 2.4L + 0.8L^2)\epsilon_t$

$E[\epsilon_t] = 0$

$$E[\epsilon_t \epsilon_\tau] = \begin{cases} 1 & t = \tau \\ 0 & o/w \end{cases}$$

$$E[y_t] = E[\epsilon_t + 2.4\epsilon_{t-1} + 0.8\epsilon_{t-2}]$$
$$= 0$$

$$\gamma_0 = var(y_t) = E[(y_t - E[y_t])^2]$$
$$= E[y_t^2]$$
$$= E[(\epsilon_t + 2.4\epsilon_{t-1} + 0.8\epsilon_{t-2})^2]$$
$$= E[\epsilon_t^2] + 2.4^2 E[\epsilon_{t-2}^2] + 0.8^2 E[\epsilon_{t-2}^2]$$

using $E[\epsilon_t \epsilon_\tau] = 0$ if $t \neq \tau$

$$= 1 + 2.4^2 + 0.8^2$$
$$= 7.4$$

$$\gamma_1 = cov(y_t, y_{t-1})$$
$$= E[(\epsilon_t + 2.4\epsilon_{t-1} + 0.8\epsilon_{t-2})(\epsilon_{t-1} + 2.4\epsilon_{t-2} + 0.8\epsilon_{t-3})]$$
$$= 2.4 E[\epsilon_{t-1}^2] + 0.8 \cdot 2.4 E[\epsilon_{t-2}^2]$$

$= 4.32$

$\gamma_2 = \text{cov}(y_t, y_{t-2})$

$= E[(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2})(\varepsilon_{t-2} + 2.4\varepsilon_{t-3} + 0.8\varepsilon_{t-4})]$

$= 0.8$

all other autocovariance terms are 0.

Since b/ the mean and autocovariances are time independent, y is covariance stationary

3. $(1 - 1.1L + 0.18L^2)\, y_t = \varepsilon_t$

$E[\varepsilon_t] = 0$

$E[\varepsilon_t \varepsilon_\tau] = \begin{cases} 1 & t = \tau \\ 0 & o/w \end{cases}$

Calculate the roots

$(1 - 0.9L)(1 - 0.2L) = 0$

$\Rightarrow L = 1/0.9 = 1.11$

$L = 1/0.2 = 5$

b/ are greater than 1, so the AR(2) process is covariance stationary

$$\gamma_0 = var(y_t) = E[y_t^2]$$
$$= E[y_t(1.1\, y_{t-1} - 0.18 y_{t-2} + \varepsilon_t)]$$
$$= 1.1\gamma_1 - .18\gamma_2 + 1$$

$$\gamma_1 = cov(y_t, y_{t-1})$$
$$= E[y_{t-1}(1.1\, y_{t-1} - 0.18 y_{t-2} + \varepsilon_t)]$$
$$= 1.1\gamma_0 - 0.18\gamma_1$$

$$\gamma_2 = cov(y_t, y_{t-2})$$
$$= 1.1\gamma_1 - 0.18\gamma_0$$

$$\Rightarrow \gamma_0 = 7.8894$$
$$\gamma_1 = 7.3545$$
$$\gamma_2 = 6.6699$$

4, $y_t = c + \phi y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}$

$\varepsilon_t \sim iid(0, \sigma^2)$

$\phi(L) y_t = \theta(L)\varepsilon_t$

$\phi(L) = 1 - \phi L \qquad \theta(L) = 1 + \theta L$

1) To be stationary, ϕ has to be inside the complex unit circle

to be invertible the same has to be true for θ

Intuitively, a stationary process stops depending on time at some point.
An invertible process is a discounted sum of its previous error terms

2) $(1-\phi L)(y_t-\mu) = (1+\theta L)\epsilon_t$

$$y_t-\mu = (1-\phi L)^{-1}(1+\theta L)\epsilon_t$$
$$= \psi(L)\epsilon_t$$

$$(1+\theta L) = (1+\phi L)\psi(L)$$
$$= (1-\phi L)(\psi_0+\psi_1 L+\psi_2 L^2+\dots)$$
$$= \psi_0 + (\psi_1-\phi\psi_0)L + (\psi_2-\phi\psi_1)L^2+\dots$$
$$+ (\psi_j - \phi\psi_{j-1})L^{j-1} + \dots$$

$$\psi_0 = 1$$
$$\psi_j - \phi\psi_{j-1} = \begin{cases}\theta & \text{if } j=1\\ 0 & j>1\end{cases}$$

$$\psi_j = \phi^{j-1}(\phi+\theta)$$
$$= \phi^j + \theta\phi^{j-1}$$

3) $E[y_t] = c + \phi E[y_{t-1}] + E[\varepsilon_t] + \theta E[\varepsilon_{t-1}]$

$$= c + \phi E[y_t]$$

$$\Rightarrow E[y_t] = \frac{c}{1-\phi} = \mu$$

$x_t = y_t - \mu$

$$\Rightarrow x_t = \phi x_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$\gamma_j = E[x_t x_{t-j}]$$

$$= E[x_{t-j}(\phi x_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1})]$$

$$= \phi \gamma_{j-1} + E[\varepsilon_t x_{t-j}] + \theta E[\varepsilon_{t-1} x_{t-j}]$$

$$\Rightarrow E[\varepsilon_t x_{t-j}] = \begin{cases} E[\varepsilon_t^2] = \sigma^2 & \text{for } j=0 \\ 0 & j \geq 1 \end{cases}$$

$$E[\varepsilon_{t-1} x_{t-j}] = \begin{cases} \phi E[\varepsilon_{t-1} x_{t-1}] + \theta E[\varepsilon_{t-1}^2] \\ \quad = (\phi + \theta)\sigma^2 & \text{for } j=0 \\ E[\varepsilon_{t-1}^2] = \sigma^2 & \text{for } j=1 \\ 0 & j \geq 2 \end{cases}$$

$$\Rightarrow \gamma_0 = \phi \gamma_1 + \sigma^2 + \theta(\phi + \theta)\sigma^2$$

$$= \phi \gamma_1 + (1 + \theta^2 + \phi\theta)\sigma^2$$

$$\gamma_1 = \phi \gamma_0 + \theta \sigma^2$$

$$\gamma_j = \phi \gamma_{j-1} \quad \text{for } j \geq 2$$

$$\Rightarrow \gamma_0 = \frac{1+\theta^2+2\phi\theta}{1-\phi^2}\sigma^2$$

$$\gamma_1 = \frac{(\phi+\theta)(1+\phi\theta)}{1-\phi^2}\sigma^2$$

$$\rho_1 = \frac{(\phi+\theta)(1+\phi\theta)}{1+\theta^2+2\phi\theta}$$

$$\rho_j = \frac{\gamma_j}{\gamma_{j-1}} = \phi\rho_{j-1}$$

4) if $\phi = -\theta$,

$\gamma_0 = \sigma^2$

$\gamma_j = 0$

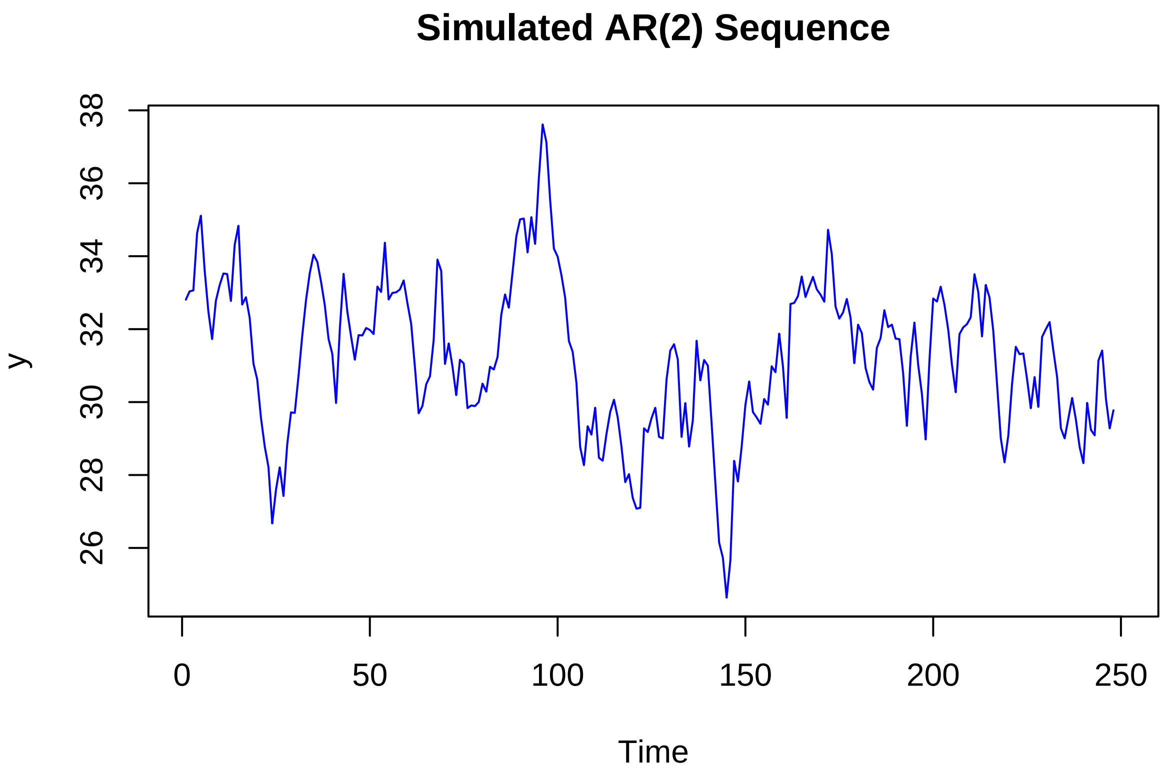
$\rho_j = 0$ for $j \geq 1$

HW 5

2023-05-15

1)

```r
# Set parameters
n = 250

# Set seed
set.seed(123)

# Generate white noise. e ~ N(0,1)
ep = rnorm(n, 0, 1)

# Generate the AR(2) sequence
y = rep(NA, n+2)
y = as.ts(y)
y[1] = 31.25 # Use fixed point as starting seed
y[2] = 31.25

# Simulate sequene
for (sim in 3:n){
  y[sim] = 2.5 + 1.1*y[sim-1] - 0.18*y[sim-2] + ep[sim]
}

# Get rid of starting seed
y = y[-c(1,2)] |> as.ts()

# Plot the sequence
plot(y, col = 'blue', main = 'Simulated AR(2) Sequence')
```



It looks covariance stationary since the sequence stays around fairly steady, and doesn't expload in value.

2)

```r
# Vector of coefficients for AR(2) sequence
poly_y = c(1, -1.1, 0.18)

# Calculate the roots
(roots = polyroot(poly_y))
```

```
## [1] 1.111111+0i 5.000000-0i
```

```r
# Companion F matrix
f_matrix = matrix(c(1.1, -.18, 1, 0), nrow = 2, ncol = 2, byrow = TRUE)

# Calculate eigenvalues
(eigen_values = eigen(f_matrix)$value)
```

```
## [1] 0.9 0.2
```

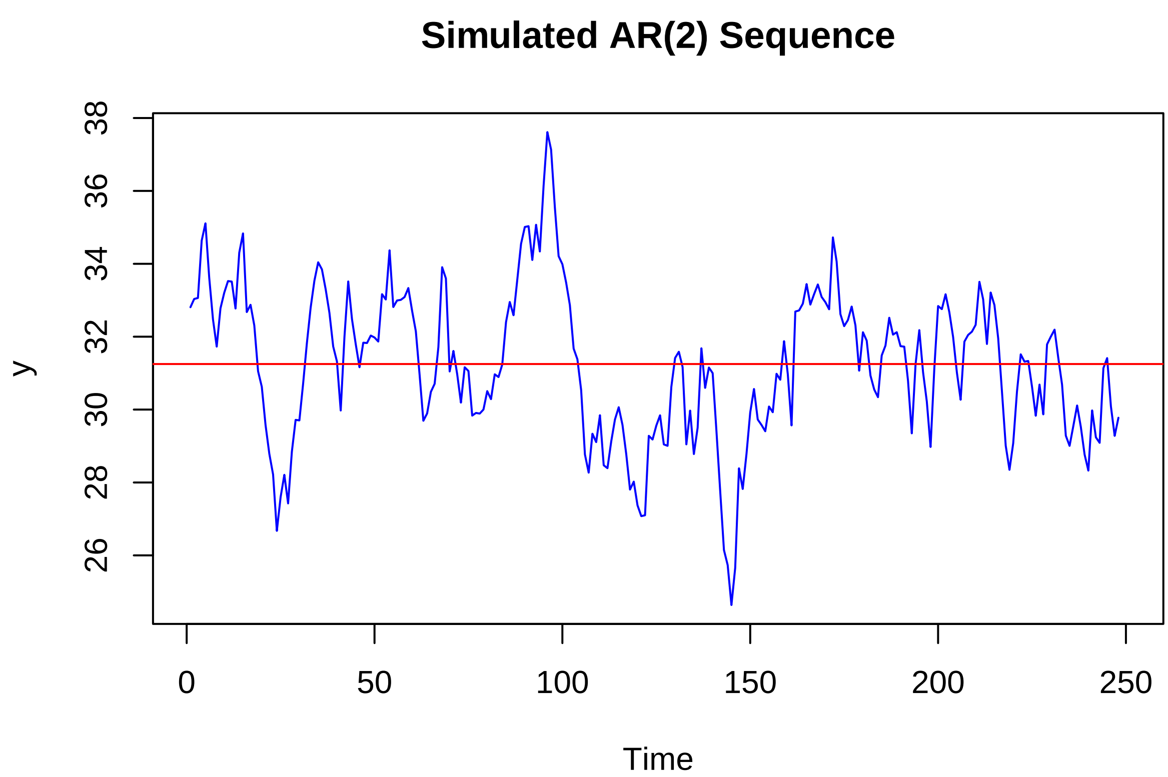
Both roots are outside the complex unit circle, 1.1111111 and 5, so the processs is stationary.

3)

```r
# Calculate the unconditional mean
(mean_y = 2.5/(1 -1.1 +0.18))
```

```
## [1] 31.25
```

```r
# Put mean on graph
plot(y, col = 'blue', main = 'Simulated AR(2) Sequence')
abline(h = mean_y, col = 'red')
```

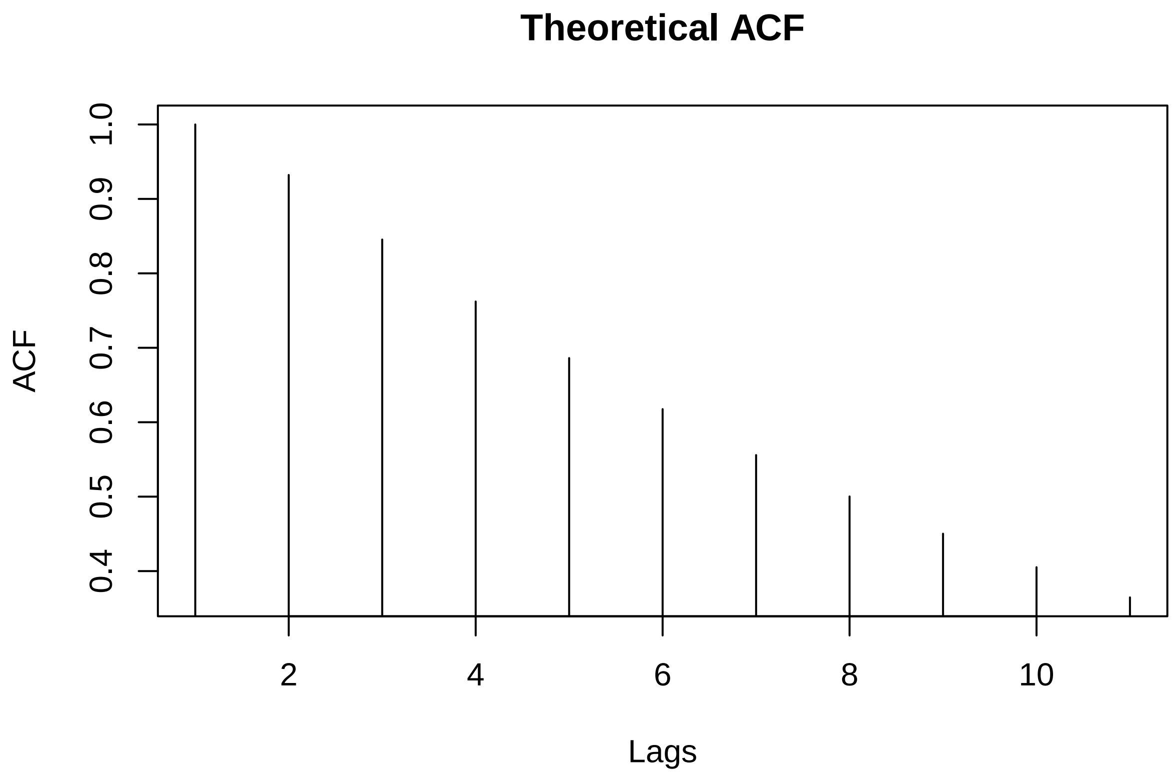


4)

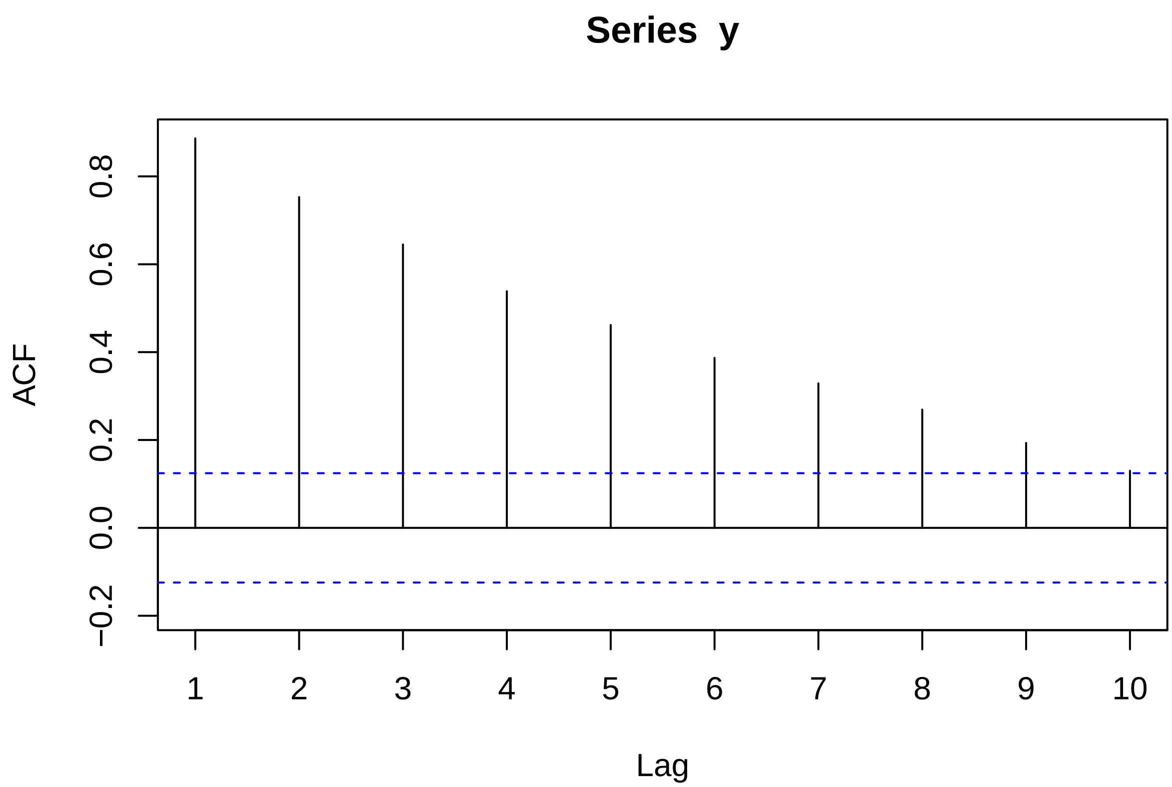
```r
# Calculate theorertical autocorrelations
(theo_acf = ARMAacf(ar = c(1.1, -0.18), ma = 0, lag.max = 10))
```

```
##         0         1         2         3         4         5         6         7
## 1.0000000 0.9322034 0.8454237 0.7621695 0.6862102 0.6176407 0.5558869 0.5003003
##         8         9        10
## 0.4502707 0.4052437 0.3647193
```

```r
# Plot them
plot(theo_acf, type = 'h', main = 'Theoretical ACF', ylab = 'ACF', xlab = 'Lags')
```




```r
# Plot sample ACF
Acf(y, lag.max = 10)
```

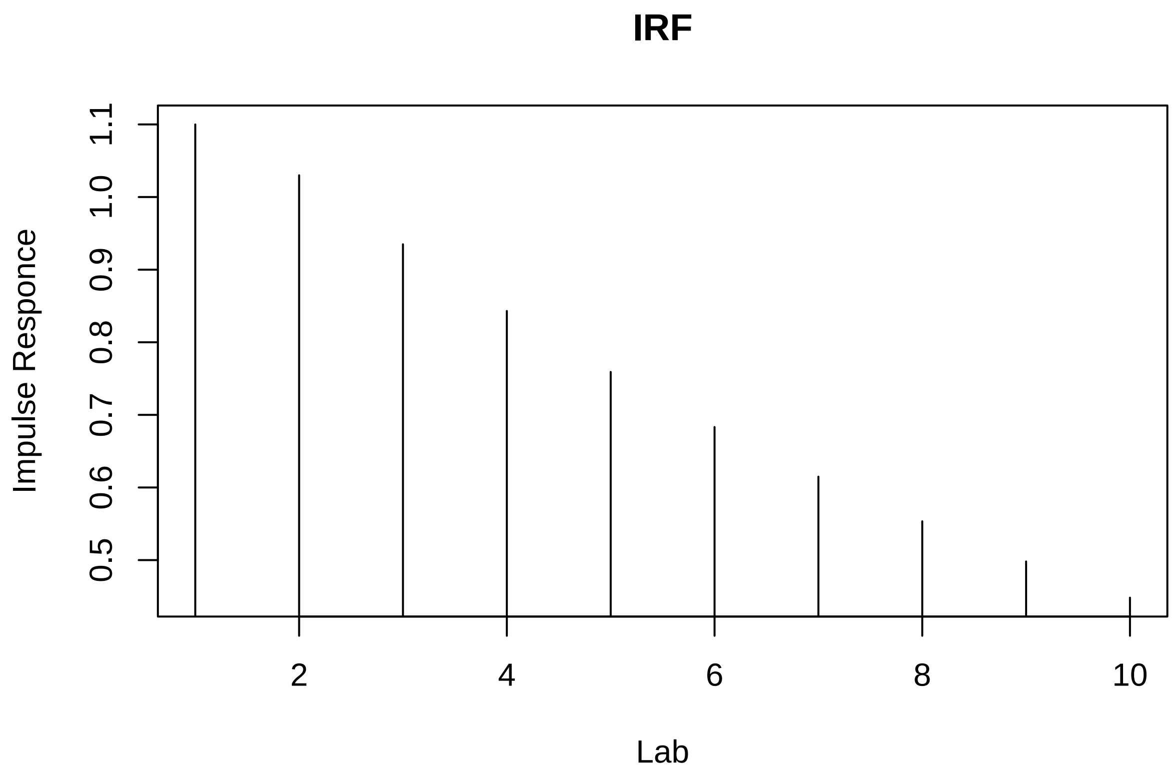


5)

```r
# Convert to moving average
(irf_y = ARMAtoMA(ar = c(1.1, -0.18), ma = 0, lag.max = 10))
```

```
##  [1] 1.1000000 1.0300000 0.9350000 0.8431000 0.7591100 0.6832630 0.6149495
##  [8] 0.5534571 0.4981119 0.4483008
```

```r
# Plot IRF
plot(irf_y, type = 'h', main = 'IRF', ylab = 'Impulse Responce', xlab = 'Lab')
```



The IRF is very similar to the ACF, but the differ by a bit.

6. $y_t = z_t \sigma_t \qquad z_t \sim iid\ N(0,1)$

$\sigma^2 = w + \alpha y_{t-1}^2 \qquad w > 0 \qquad 0 \le \alpha < 1$

1.) $I_t = \{y_t, y_{t-1}, \dots\}$

show $\{y_t, I_t\}$ is MDS

To be an MDS

$E\{y_t \mid I_{t-1}\} = 0$

$E\{y_t \mid I_{t-1}\} = E\{z_t \sigma_t \mid y_{t-1}, y_{t-2}, \dots\}$

$= E\{z_t \sqrt{w + \alpha y_{t-1}} \mid y_{t-1}, y_{t-2}, \dots\}$

y_{t-1} is in info set, so

$= E\{z_t\}$

$= 0$

So its an MDS

2) $Var(y_t \mid I_{t-1})$

$E\{y_t^2 \mid I_{t-1}\}$

$= E\{(z_t \sigma_t)^2 \mid y_{t-1}, y_{t-2}, \dots\}$

$= E\{z_t^2 (w + \alpha y_{t-1}^2) \mid y_{t-1}, y_{t-2}, \dots\}$

$= \alpha y_{t-1}^2 + w E\{z_t^2 \mid y_{t-1}, y_{t-2}, \dots\}$

$= w + \alpha y_{t-1}^2$

$\Rightarrow \operatorname{Var}(y_t | I_{t-1}) = w + \alpha y_{t-1}^2 = \sigma_t^2$

3) $E[y_t] = E[E[y_t | I_{t-1}]]$

$= E[0]$

$= 0$

$\operatorname{Var}(y_t) = E[y_t^2]$

$= E[E[y_t^2 | I_{t-1}]]$

$= E[\sigma_t^2]$

4) $E[\sigma_t^2] = E[w + \alpha y_{t-1}^2]$

$= w + \alpha E[y_{t-1}^2]$

$= w + \alpha E[y_t^2]$

$= w + \alpha E[\sigma_t^2]$

$(1-\alpha) E[\sigma_t^2] = w$

$E[\sigma_t^2] = \frac{w}{1-\alpha}$

5)

$\sigma_t^2 = w + \alpha y_{t-1}^2$

$\sigma_t^2 + y_t^2 = w + \alpha y_{t-1}^2 + y_t^2$

$y_t^2 = w + \alpha y_{t-1}^2 + y_t^2 - \sigma_t^2$

$y_t^2 = w + \alpha y_{t-1}^2 + \eta_t$