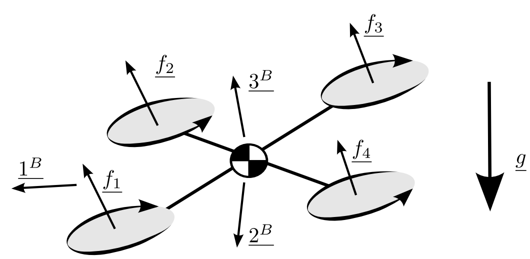
5. Quadcopter dynamics

We are now ready to describe the dynamics of a quadcopter, which we can then use to control such a vehicle. Our quadcopter will be as shown below.



We define a body frame B, where $\underline{3}^B$ is perpendicular to the plane of the propellers, and $\underline{1}^B$ bisects two arms. The quadcopter is assumed to be rotationally symmetric, so that the arms are perpendicular to one another, and the propellers are all equally far from the vehicle's geometric centre (which is also taken to coincide with its centre of mass) B. We define an earth-fixed frame E, which we take to be inertial, and wherein $\underline{1}^E$ is a horizontal direction, and $\underline{3}^E$ points opposite to gravity (upwards).

The propeller P_i is displaced from the vehicle's centre of mass by $\underline{s_{P_iB}}$, and produces a force $\underline{f_{P_i}}$, and a moment about its rotation axis $\underline{n_{P_i}}$.

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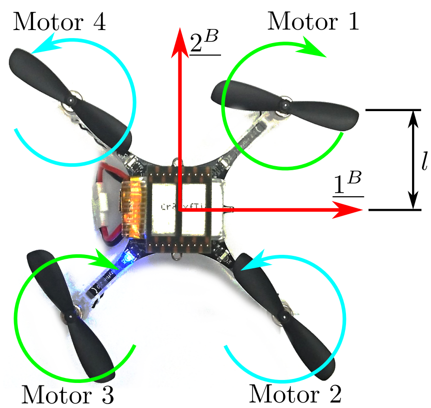

Figure 5.1: The convention that we will use for the propeller numbering, frame, and size.

The displacements are defined as below, introducing a scalar length l,

$$\underline{s_{P_1B}} = l\left(+\underline{1}^B + \underline{2}^B\right) \tag{5.1a}$$
$$\underline{s_{P_2B}} = l\left(+\underline{1}^B - \underline{2}^B\right) \tag{5.1b}$$
$$\underline{s_{P_3B}} = l\left(-\underline{1}^B - \underline{2}^B\right) \tag{5.1c}$$
$$\underline{s_{P_4B}} = l\left(-\underline{1}^B + \underline{2}^B\right) \tag{5.1d}$$

Note that thus the propellers are a distance $\sqrt{2}l$ from the centre of mass.

We assume that the propellers have negligible mass (and thus neglect their mass moment of inertia and angular momentum). The vehicle has an inertia tensor $\underline{J_B^B}$, and has mass m^B. Because of its symmetry, it follows that the inertia, when expressed in the body-fixed B coordinate system, is diagonal, with only two unique entries:

$$\left[\underline{J_B^B}\right]^B = \begin{bmatrix} J_{xx} & 0 & 0 \\ 0 & J_{xx} & 0 \\ 0 & 0 & J_{zz} \end{bmatrix} \tag{5.2}$$

The orientation of the quadcopter with respect to the earth is given by the rotation tensor $\underline{R^{BE}}$, expressed using the transformation matrix $[T]^{EB}$. We fix a point on the Earth, for example where the vehicle took off, and call it T, so that the vehicle's displacement from that point is given by $\underline{s_{BT}}$.

5.1 Propeller forces and torques

Using the simple propeller model, the force of each propeller is parallel to the body-fixed $\underline{3^B}$ axis: $\underline{f_{P_i}} = c_{P_i}\,\underline{3^B}$ where c_{P_i} is a scalar. The torque around the thrust axis is assumed to be proportional to the thrust, with the sign determined by the propeller's handedness (and conveniently expressed as $(-1)^{i+1}$), and having proportionality constant κ:

$$\underline{n_{P_i}} = (-1)^{i+1}\kappa\,\underline{f_{P_i}} = (-1)^{i+1}\kappa c_{P_I}\,\underline{3^B} \quad \text{for} \quad i \in \{1,2,3,4\} \tag{5.3}$$

Finally, the vehicle is acted upon by gravity $\underline{g}$, of magnitude $\left\|\underline{g}\right\| \approx 9.81\,\mathrm{m\,s^{-2}}$. Our choice of E frame means that $\underline{g} = -\left\|\underline{g}\right\|\,\underline{3^E}$, or $\left[\underline{g}\right]^E \approx (0,0,-9.81)\,\mathrm{m/s}$.

Now we can write down Newton's law, and use the simplification that the vehicle's mass is constant, and choosing the B point as the vehicle's centre of mass:

$$D^E\,\underline{p_B^E} = m^B D^E\,\underline{v_B^E} = \underline{f} = \sum_{i=1}^{4}\underline{f_{P_i}} + m^B\,\underline{g} \tag{5.4a}$$

$$\Leftrightarrow m^B\,\underline{a_B^E} = \sum_{i=1}^{4}\underline{f_{P_i}} + m^B\,\underline{g} \tag{5.4b}$$

$$= \left(\sum_{i=1}^{4} c_{P_i}\right)\underline{3^B} + m^B\,\underline{g} \tag{5.4c}$$

Similarly, we can write down Euler's law, where we directly use the form for easy coordination in the body-fixed frame

$$\underline{J_B^B}\,D^B\,\underline{\omega^{BE}} + \underline{\Omega^{BE}}\,\underline{J_B^B}\,\underline{\omega^{BE}} = \underline{n_B} \tag{5.5a}$$

$$\underline{n_B} = \sum_{i=1}^{4}\left(\underline{S_{P_iB}}\,c_{P_i}\,\underline{3^B} + (-1)^{i+1}\kappa c_{P_I}\,\underline{3^B}\right) \tag{5.5b}$$

$$= \sum_{i=1}^{4}\left(c_{P_i}\left(\underline{S_{P_iB}}\,\underline{3^B} + (-1)^{i+1}\kappa\,\underline{3^B}\right)\right) \tag{5.5c}$$

where we've noted that the force acting at a distance also produces a moment. Substituting for $\underline{S_{P_iB}}$, and noting the simple form of the cross product, we get

$$\begin{aligned}\underline{n_B} = &c_{P_1}\left(+l\,\underline{1^B} - l\,\underline{2^B} + \kappa\,\underline{3^B}\right)\\ &+c_{P_2}\left(-l\,\underline{1^B} - l\,\underline{2^B} - \kappa\,\underline{3^B}\right)\\ &+c_{P_3}\left(-l\,\underline{1^B} + l\,\underline{2^B} + \kappa\,\underline{3^B}\right)\\ &+c_{P_4}\left(+l\,\underline{1^B} + l\,\underline{2^B} - \kappa\,\underline{3^B}\right)\end{aligned} \tag{5.6}$$

We now coordinate in B, and will look at it term-by-term. On the left-hand side of (5.5a) we have

$$\left[\underline{J_B^B}\right]^B D^B \left[\underline{\omega^{BE}}\right]^B = \begin{bmatrix} J_{xx}\dot{p} \\ J_{xx}\dot{q} \\ J_{zz}\dot{r} \end{bmatrix} \tag{5.7a}$$

$$\left[\underline{\Omega^{BE}}\right]^B \left[\underline{J_B^B}\right]^B \left[\underline{\omega^{BE}}\right]^B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} J_{xx}p \\ J_{xx}q \\ J_{zz}r \end{bmatrix} = \begin{bmatrix} (J_{zz} - J_{xx})\, qr \\ (J_{xx} - J_{zz})\, pr \\ 0 \end{bmatrix} \tag{5.7b}$$

The right-hand side of (5.5a), coordinated in B, solves to the below, where we've introduced the three scalars for convenience $\left[\underline{n_B}\right]^B =: (n_1, n_2, n_2)$.

$$\left[\underline{n_B}\right]^B = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = c_{P_1} \begin{bmatrix} l \\ -l \\ \kappa \end{bmatrix} + c_{P_2} \begin{bmatrix} -l \\ -l \\ -\kappa \end{bmatrix} + c_{P_3} \begin{bmatrix} -l \\ l \\ \kappa \end{bmatrix} + c_{P_4} \begin{bmatrix} l \\ l \\ -\kappa \end{bmatrix} \tag{5.8a}$$

$$= \begin{bmatrix} l & -l & -l & l \\ -l & -l & l & l \\ \kappa & -\kappa & \kappa & -\kappa \end{bmatrix} \begin{bmatrix} c_{P_1} \\ c_{P_2} \\ c_{P_3} \\ c_{P_4} \end{bmatrix} \tag{5.8b}$$

We note that we can achieve any moment by a suitable choice of forces c_{P_i}.

Combining (5.7a)-(5.8b) and simplifying gives

$$\begin{bmatrix} l & -l & -l & l \\ -l & -l & l & l \\ -\kappa & \kappa & -\kappa & \kappa \end{bmatrix} \begin{bmatrix} c_{P_1} \\ c_{P_2} \\ c_{P_3} \\ c_{P_4} \end{bmatrix} = \begin{bmatrix} J_{xx}\dot{p} \\ J_{xx}\dot{q} \\ J_{zz}\dot{r} \end{bmatrix} + \begin{bmatrix} (J_{zz} - J_{xx})\, qr \\ (J_{xx} - J_{zz})\, pr \\ 0 \end{bmatrix} \tag{5.9}$$

We note that this is a coupled, nonlinear, differential equation for p, q, and r.

Next, we look at the translational dynamics, coordinating (5.4c) in the Earth-fixed frame:

$$m^B \left[\underline{a_B^E}\right]^E = \left(\sum_{i=1}^{4} c_{P_i}\right) \left[\underline{3^B}\right]^E + m^B \left[\underline{g}\right]^E \tag{5.10a}$$

$$= \left(\sum_{i=1}^{4} c_{P_i}\right) [T]^{EB} \left[\underline{3^B}\right]^B + m^B \left[\underline{g}\right]^E \tag{5.10b}$$

Defining $c_\Sigma := \sum_{i=1}^{4} c_{P_i}$; and $\left[\underline{a_B^E}\right]^E =: (a_1, a_2, a_3)$, we have

$$m^B \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c_\Sigma \, [T]^{EB} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + m^B \begin{bmatrix} 0 \\ 0 \\ -\|\underline{g}\| \end{bmatrix} \tag{5.11}$$

5.1.1 Transformation of the inputs

We notice that the inputs (the quantities we can choose instantaneously) are c_{P_i}, but enter the dynamics in a very specific way: the translational acceleration is a function *only* of their sum, while the angular dynamics map the four-dimensional inputs in a linear way to three-dimensional torque vector. For this reason it is usually more convenient to use the torques $\left[\underline{n_B}\right]^B$ from (5.8b) and the total thrust c_Σ from (5.11) as inputs. The transformation between the two forms is linear:

$$\begin{bmatrix} c_\Sigma \\ n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ l & -l & -l & l \\ -l & -l & l & l \\ \kappa & -\kappa & \kappa & -\kappa \end{bmatrix} \begin{bmatrix} c_{P_1} \\ c_{P_2} \\ c_{P_3} \\ c_{P_4} \end{bmatrix} \tag{5.12}$$

Alternatively, we can solve the forces as

$$\begin{bmatrix} c_{P_1} \\ c_{P_2} \\ c_{P_3} \\ c_{P_4} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & l^{-1} & -l^{-1} & \kappa^{-1} \\ 1 & -l^{-1} & -l^{-1} & -\kappa^{-1} \\ 1 & -l^{-1} & l^{-1} & \kappa^{-1} \\ 1 & l^{-1} & l^{-1} & -\kappa^{-1} \end{bmatrix} \begin{bmatrix} c_\Sigma \\ n_1 \\ n_2 \\ n_3 \end{bmatrix}. \tag{5.13}$$

Often, the logical component that implements (5.13) is called the *mixer*, which “mixes” the desired force and torque to compute the individual motor forces.

5.1.2 State-space description

We are now in a position to write the quadcopter’s dynamics as a set of first-order differential equations, for control. We will define the state to consist of:

position of the quadcopter’s centre of mass as expressed in the earth-fixed coordinate system, $\left[\underline{s_{BT}}\right]^E =: (s_1, s_2, s_3)$ (recall that T was a fixed “target” point, fixed with respect to the earth);

velocity of the quadcopter's centre of mass relative to the earth-fixed frame, expressed in the earth-fixed coordinate system, $\left[\underline{v_B^E}\right]^E =: (v_1, v_2, v_3)$;

orientation of the body-fixed frame with respect to the earth-fixed frame, as encoded in the transformation matrix

$$[T]^{EB} =: \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \tag{5.14}$$

and

angular velocity of the body-fixed frame with respect to the earth-fixed frame, expressed in the body-fixed coordinate system, $\left[\underline{\omega^{BE}}\right]^B = (p, q, r)$.

We will use the simplified input of the torque vector and the total force, noting that this can be converted back to motor forces using (5.13).

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{5.15a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \frac{c_\Sigma}{m^B}\begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\parallel \underline{g} \parallel \end{bmatrix} \tag{5.15b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}\begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \tag{5.15c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} n_1/J_{xx} \\ n_2/J_{xx} \\ n_3/J_{zz} \end{bmatrix} + \frac{J_{xx} - J_{zz}}{J_{xx}}\begin{bmatrix} qr \\ -pr \\ 0 \end{bmatrix} \tag{5.15d}$$

Given a set of initial conditions, and inputs as a function of time, we can use (5.15a)-(5.15d) to solve for the full state trajectory of the quadcopter*. Notice how we went from tensor modelling to matrix coding.

*Note that you would not implement the matrix differential equation of (5.15c), as it is much easier to employ the Euler Symmetric Parameters (AKA unit quaternion). For analysis, though, we prefer the transformation matrix form.

5.1.3 Equilibrium and linearization about hover

For small deviations from hover, where the velocity, the orientation, and the angular velocity are all small, we can compute a simpler set of differential equations. We will first compute the equilibrium state of the quadcopter, that is, a state where it is at rest. The formal method for doing this is to set all the derivatives in (5.15a)-(5.15d) to zero. We denote by an asterisk the value of the states at equilibrium, and solve

From (5.15a) follows that $v_1^* = v_2^* = v_3^* = 0$, from (5.15c) that $p^* = q^* = r^* = 0$. Substituting the rates into (5.15d) we can conclude that the input torque must be zero, $m_1^* = m_2^* = m_3^* = 0$. We note that the right-hand side column of the transformation matrix is specified by (5.15b), so that $t_{13} = t_{23} = 0$ and (because the columns are unit vectors) $t_{33} = 1$. Since the rows are also unit vectors, we have $t_{31} = t_{32} = 0$. Finally, we can conclude that the total thrust must be the weight of the vehicle, $c_\Sigma^* = m^B \left\| \underline{g} \right\|$.

For the linearization, it will be more convenient to express the transformation matrix using the Euler yaw-pitch-roll (ψ-θ-ϕ) sequence. The equilibrium values can be computed by substituting our knowledge for t_{ij}^*:

$$\begin{bmatrix} t_{11}^* & t_{12}^* & 0 \\ t_{21}^* & t_{22}^* & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{c}\psi^*\mathrm{c}\theta^* & \mathrm{c}\psi^*\mathrm{s}\theta^*\mathrm{s}\phi^* - \mathrm{s}\psi^*\mathrm{c}\phi^* & \mathrm{c}\psi^*\mathrm{s}\theta^*\mathrm{c}\phi^* - \mathrm{s}\psi^*\mathrm{s}\phi^* \\ \mathrm{s}\psi^*\mathrm{c}\theta^* & \mathrm{s}\psi^*\mathrm{s}\theta^*\mathrm{s}\phi^* + \mathrm{c}\psi^*\mathrm{c}\phi^* & \mathrm{s}\psi^*\mathrm{s}\theta^*\mathrm{c}\phi^* - \mathrm{c}\psi^*\mathrm{s}\phi^* \\ -\mathrm{s}\theta^* & \mathrm{c}\theta^*\mathrm{s}\phi^* & \mathrm{c}\theta^*\mathrm{c}\phi^* \end{bmatrix} \tag{5.16}$$

From this we can immediately conclude that $\theta^* = 0$ and $\phi^* = 0$ (which also matches our intuition).

We note that the equilibrium does not specify a position s_i^*, or the yaw angle ψ^*. Intuitively, this makes sense: the quadcopter can hover anywhere in the world, and at any yaw angle. For simplicity, we will now make the additional restriction that $\psi^* = 0$, i.e. we'll look at hovering with small yaw angles. Note that this is without loss of generality – if you wanted to hover at a different angle, simply create a new world-fixed coordinate system wherein that yaw angle is zero.

We now linearize the dynamics (5.15a)-(5.15d) at these values, and use the Euler angle dynamics for (5.15c). We introduce the input deviation Δc_Σ, so that $c_\Sigma =: c_\Sigma^* + \Delta c_\Sigma$ (since all other quantities are zero at the equilibrium, we don't need to explicitly identify deviations).

For a small angles we make the substitution that $\sin\theta \approx \theta$, $\cos\theta \approx 1$, and that a product of any two deviations is zero, e.g. $\phi\theta \approx 0$. We can now compute the small-angles transformation matrix:

$$[T]^{EB} = \begin{bmatrix} \mathrm{c}\psi\mathrm{c}\theta & \mathrm{c}\psi\mathrm{s}\theta\mathrm{s}\phi - \mathrm{s}\psi\mathrm{c}\phi & \mathrm{c}\psi\mathrm{s}\theta\mathrm{c}\phi - \mathrm{s}\psi\mathrm{s}\phi \\ \mathrm{s}\psi\mathrm{c}\theta & \mathrm{s}\psi\mathrm{s}\theta\mathrm{s}\phi + \mathrm{c}\psi\mathrm{c}\phi & \mathrm{s}\psi\mathrm{s}\theta\mathrm{c}\phi - \mathrm{c}\psi\mathrm{s}\phi \\ -\mathrm{s}\theta & \mathrm{c}\theta\mathrm{s}\phi & \mathrm{c}\theta\mathrm{c}\phi \end{bmatrix} \tag{5.17a}$$

$$\approx \begin{bmatrix} 1 & -\psi & \theta \\ \psi & 1 & -\phi \\ -\theta & \phi & 1 \end{bmatrix} \tag{5.17b}$$

For the derivative of the velocity, (5.15b), we note that we need the product $[T]^{EB} \left[\underline{3^B} \right]^B$:

$$\begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 1 & -\psi & \theta \\ \psi & 1 & -\phi \\ -\theta & \phi & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \theta \\ -\phi \\ 1 \end{bmatrix} \tag{5.18}$$

and we substitute $c_\Sigma = m^B \left\| \underline{g} \right\| + \Delta c_\Sigma$

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \approx \frac{m^B \left\| \underline{g} \right\| + \Delta c_\Sigma}{m^B} \begin{bmatrix} \theta \\ -\phi \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\left\| \underline{g} \right\| \end{bmatrix} \approx \begin{bmatrix} \theta \left\| \underline{g} \right\| \\ -\phi \left\| \underline{g} \right\| \\ \frac{\Delta c_\Sigma}{m^B} \end{bmatrix} \tag{5.19}$$

We can make the same substitutions and simplifications for the kinematics of the Euler angles:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi / \cos\theta & \cos\phi / \cos\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{5.20a}$$

$$\approx \begin{bmatrix} 1 & 0 & \theta \\ 0 & 1 & -\phi \\ 0 & \phi & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \approx \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{5.20b}$$

Since the angular velocity is small, we neglect the quadratic term in the angular velocity dynamics (the terms with pq), and we can make all the substitutions and write the dynamics of the quadcopter as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{5.21a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \approx \begin{bmatrix} \theta \left\| \underline{g} \right\| \\ -\phi \left\| \underline{g} \right\| \\ \frac{\Delta c_\Sigma}{m^B} \end{bmatrix} \tag{5.21b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \approx \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{5.21c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p \\ q \\ r \end{bmatrix} \approx \begin{bmatrix} n_1/J_{xx} \\ n_2/J_{xx} \\ n_3/J_{zz} \end{bmatrix} \tag{5.21d}$$

This is extremely convenient, and we notice that it decouples into four simpler systems: one for each horizontal direction, one for the height, and one for the yaw angle. Each system has an own scalar input, furthermore.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} s_1 \\ v_1 \\ \theta \\ q \end{bmatrix} = \begin{bmatrix} v_1 \\ \theta \left\| \underline{g} \right\| \\ q \\ n_2/J_{xx} \end{bmatrix}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} s_2 \\ v_2 \\ \phi \\ p \end{bmatrix} = \begin{bmatrix} v_2 \\ -\phi \left\| \underline{g} \right\| \\ p \\ n_1/J_{xx} \end{bmatrix}, \tag{5.22a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} s_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_3 \\ \Delta c_\Sigma / m^B \end{bmatrix}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \psi \\ r \end{bmatrix} = \begin{bmatrix} r \\ n_3/J_{zz} \end{bmatrix} \tag{5.22b}$$

References

[1] R. Siegwart, I. R. Nourbakhsh, and D. Scaramuzza, *Introduction to autonomous mobile robots*. MIT press, 2011.