Analytic framework

The analytic framework includes equations from the atmospheric fluid dynamics and wind farm fluid dynamics. The equations from atmospheric fluid dynamics concern steady-state, horizontal, large-scale flows under neutral conditions above an infinite homogeneous surface with roughness z_0 [1][2][3]. First, we recall the governing equations of the geostrophic flow, i.e., the momentum equations [4]:

$$-fV_g = -\frac{1}{\rho}\frac{\partial p}{\partial x} , \qquad fU_g = -\frac{1}{\rho}\frac{\partial p}{\partial y} , \tag{1}$$

where U_g and V_g are the geostrophic wind components (G is the modulus), p the pressure, ρ the density, and f the Coriolis parameter. The Coriolis parameter is given by $2\Omega \sin\varphi$, where Ω is the rotation rate of the Earth ($7.2921 \cdot 10^{-5}$ rad/s), and φ the latitude. In the Ekman layer, mechanical turbulence becomes important, and the resulting momentum equations then become [5]:

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial\left(\overline{u'w'}\right)}{\partial z} , \qquad fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{\partial\left(\overline{v'w'}\right)}{\partial z} , \tag{2}$$

where u and v are the mean wind components, and $\overline{u'w'}$ and $\overline{v'w'}$ are the turbulent fluxes (also called Reynolds stresses). Substituting the pressure-gradient terms of Eq. 2 with the Coriolis force terms from Eq. 1, the equations of motion can be written as [5]:

$$f\left(v - V_g\right) = \frac{\partial\left(\overline{u'w'}\right)}{\partial z} , \qquad f\left(u - U_g\right) = -\frac{\partial\left(\overline{v'w'}\right)}{\partial z} . \tag{3}$$

From these governing equations, Blackadar and Tennekes [1] derived, for the case of large Rossby numbers, $Ro = G/(fz_0)$, two kinds of self-similar solutions, one valid only in the Ekman layer well outside the surface layer and another valid inside the surface layer [3]. By matching those solutions in a region of overlap, they derived the following expressions:

$$\frac{U_g}{u_*} = \frac{1}{\kappa}\ln\left(\frac{u_*}{f z_0}\right) - 4\,, \qquad \frac{V_g}{u_*} = -12\,, \tag{4}$$

$$\frac{G}{u_*} = \sqrt{\left(\frac{U_g}{u_*}\right)^2 + \left(\frac{V_g}{u_*}\right)^2} = \sqrt{\left(\frac{1}{\kappa}\ln\left(\frac{u_*}{f z_0}\right) - 4\right)^2 + 12^2}\,, \tag{5}$$

where u_* is the surface friction velocity and κ the Von Kármán constant (≈ 0.4). The resistant constants (4 and 12) on the right-hand side of Eq. 4 are derived in Refs. [1][3]. These equations relate the large-scale flow (geostrophic flow and Ekman layer) with the small-scale flow (surface layer, where turbines are located).

The equations from wind farm fluid dynamics concern instead steady-state, horizontal flows under neutral conditions within and over an infinite wind farm [6][7]. In this case, the derivation neglects any effect of the Coriolis force and focuses therefore on smaller spatial scales. This analysis provides an equivalent surface roughness, $z_{0,wf}$, for the effect of a large wind farm on the overlying atmospheric boundary layer [6]:

$$z_{0,wf} = z_H\left(1 + \frac{D}{2z_H}\right)^{\nu_w^*/1+\nu_w^*} \exp\left\{-\left\{\frac{c_{ft}}{2\kappa^2} + \left\{\ln\left[\frac{z_H}{z_0}\left(1 - \frac{D}{2z_H}\right)^{\nu_w^*/1+\nu_w^*}\right]\right\}^{-2}\right\}^{-0.5}\right\}, \tag{6}$$

where D and z_H are the turbine diameter and huh height, respectively, z_0 the actual surface roughness of the underlying terrain, and

$$\nu_w^* = \frac{\sqrt{0.5c_{ft}}U_H D}{\kappa u_{*,wf} z_H}, \tag{7}$$

$$c_{ft} = \frac{\pi C_T}{4 s_x s_y}, \tag{8}$$

where U_H is the hub-height wind speed, $u_{*,wf}$ the equivalent surface friction velocity generated by the wind farm, C_T is the thrust coefficient provided by the manufacturer (see Supplementary

Fig. 1), and s_x and s_y are the horizontal nondimensional spacings. For the case of aligned layout and installed capacity density of 9 W/m^2, $s_x = s_y = 1000/D = 6.1$, whereas for the case of staggered layout and installed capacity density of 4.5 W/m^2, $s_x = s_y = \sqrt{2} \cdot 1000/D = 8.6$. The hub-height wind speed is calculated according to [6]:

$$U_H = \frac{u_{*,wf}}{\kappa} \ln \left[\frac{z_H}{z_{0,wf}} \left(1 + \frac{D}{2z_H}\right)^{\nu_w^* / 1+\nu_w^*} \right]. \tag{9}$$

Here, we combine Eq. 5 with Eqs. 6-9 to provide a closed-form system that relates the influence of geostrophic wind and Coriolis parameter (latitude-dependent) with the wind farm power density. z_0 and u_* in Eq. 5 are replaced with $z_{0,wf}$ and $u_{*,wf}$ to account for the effect of the wind farm on the Ekman layer. The system represented by Eqs. 5-9, with unknowns $u_{*,wf}$, $z_{0,wf}$, ν_w^*, c_{ft} and U_H, can be solved iteratively and provides a value for U_H, with which we calculate the power produced according to the power curve given by the manufacturer.

References

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