SET THEORY

SOME IMPORTANT DEFINITION

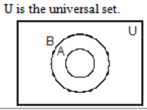
- Set: A well-defined collection of objects, different from each other, and which one can see or think of, is called a set.
- Members of a Set: The objects which belong to a set are called its elements or members
- Empty set: A set which does not contain any element
- Finite set: A set that contains definite number of elements
- Infinite set: A set that contains indefinite number of elements
- **Subset :** If A and B are two sets given in such a way that every element of A is in B, then A is a **subset** of set B and it is written as $A \subset B$ (read as 'A is contained in B').
- Power set: The collection of all subsets of a set A is called the power set of A. It is denoted by P(A). In P(A), every element is a set
- Equal sets: Two sets A and B are said to be equal, if every element of A is in B and every element of B is in A. It is written as A = B
- **Equivalent sets:** Two finite sets A and B are said to be equivalent, if $n(A) = n(B)$, where $n(A)$ or $n(B)$ is the number of elements in set A or B. It is written as $B \leftrightarrow B$. Equal sets are always equivalent. But, equivalent sets need not be equal.
- **Disjoint sets:** Two sets are said to be **disjoint,** if they have no element in common.
- **Super set:** If A is subset of B, then we say that B is a **superset** set A and is written as $B \supset A$
- **Proper subset:** If A is a subset of B but $A \neq B$, then A is called a **proper subset** of B and is written as $A \subsetneq B$.
- **Universal set:** If there are some sets under consideration, then there happens to be a set which is a superset of each one of the given sets such a set is called the **universal set** and it is denoted by $\cup$ or ξ.
- **Complement of a set:** If U is the universal set and A is a subset of U, then the complement of A is the set of all elements of U which are not the elements of A. Denoted by A'.

REPRESENTATION OF A SET

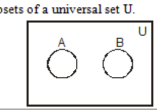
- **Tabular form or roster form:** Here all the elements of a set are **actually written** down, separated by commas and enclosed within curly bracket.
  e.g; V, the set of vowels in the English alphabet = $\{a, e, i, o, u\}$,
- **The set builder form:** In this form, we list the property or properties satisfied by the elements of the set.
  e.g; $A = \{6, 7, 8, 9, 10, 11\} = \{x \mid x \in N, 5 < x < 12\}$
  Where $\in$ stands for '*belongs to*' and '|' stands for '*Such That*'.

VENN DIAGRAMS - Most of the relationships between sets can be represented by means of diagrams. Figures representing sets in the form of enclosed region in the plane are called Venn diagrams

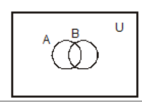
- $A \subset B \subset U$, where U is the universal set.



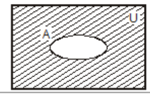
- Two disjoint subsets of a universal set U.



- Two intersecting subsets of a universal set U.

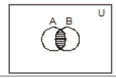


- $A \subset U$, where U is the universal set.

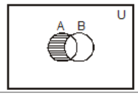


OPERATION OF SETS

- **Union of two sets:** The union of two sets A and B, denoted by $A \cup B$ (read as A union B), is the set of all those elements which are either in A or in B or in both. Symbolically:
  $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- **Intersections of sets:** The intersection of two sets A and B, denoted by $A \cap B$ (read an A intersection B), is the set of all elements common to both A and B. Symbolically



- **Difference of two sets:** The difference of two sets A and B, denoted by $A - B$ (read as A minus B), is the set of all those elements of A which are not in B. Symbolically,
  $A - B = \{x : x \in A \text{ and } x \notin B\}$.



LAWS OF SET OPERATIONS

- Commutative Laws: For any two sets A and B, we have
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$.
- Associative Laws: For any three sets A, B and C, we have
  - $(A \cup B) \cup X = A \cup (B \cup X)$,
  - $(A \cap B) \cap X = A \cap (B \cap X)$
- Distributive Laws: For any three sets A, B and C, we have
  - $A \cup (B \cap X) = (A \cup B) \cap (A \cup X)$
  - $A \cap (B \cup X) = (A \cap B) \cup (A \cap X)$
- Idempotent Laws: If A is any set, then
  - $A \cap A = A$
  - $A \cup A = A$
- **Identity Laws:** If A is any subset of a universal set U and ϕ is the null set, then
  - $A \cap Y = A$, $A \cap \phi = \phi$
  - $A \cup \phi = A$, $A \cup Y = Y$.

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Quadratic Equations

Standard Form

- $ax^2 + bx + c = 0, \quad a, b, c \in R, \quad a \neq 0$
- Roots of the equation, $\alpha, \beta = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $\alpha + \beta = -b/a$, $\alpha\beta = c/a$
- Factors of $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

Nature of Roots

- $D = b^2 - 4ac$ is called the discriminant.
  - If $D > 0$, roots are real and unequal
  - $D = 0$, roots are real and equal
  - $D < 0$, roots are imaginary and unequal

100/100 Marks

Common roots

$D = 0$, roots are real and equal

- If $D > 0$, roots are real and unequal.
- If **both roots are common**, the two equations must essentially be the same. Hence, $\dfrac{a}{d} = \dfrac{b}{e} = \dfrac{c}{f}$.
- If **only one root is common**, and the common root is α,

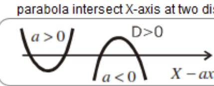
  then $a\alpha^2 + b\alpha + c = 0$ and $d\alpha^2 + e\alpha + f = 0$

  $$\frac{\alpha^2}{bf - ce} = \frac{-\alpha}{af - cd} = \frac{1}{ae - bd} \quad \text{or} \quad (cd - af)^2 = (bf - ce)(ae - bd)$$

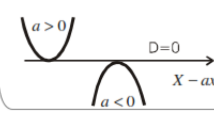
Graphs of Quardratic Polynomial

Let $f(x) = ax^2 + bx + c$, $(a \neq 0)$ be a quadratic polynomial.

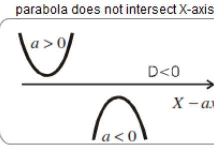
- **Shape:** The shape of quadratic polynomial is always a parabola.
- **Opening:** If $a > 0$, parabola opens upwards.
  If $a < 0$, parabola opens downwards.
- **Intersection with X-axis**
  - $D > 0$: parabola intersect X-axis at two distinct points



  - $D = 0$: parabola intersect X-axis at a single points



  - $D < 0$: parabola does not intersect X-axis



Important Results

- **Remainder Theorem**
  If a polynomial $f(x)$ is divided by $(x - \alpha)$ the remainder obtained is $f(\alpha)$.
- **Factor Theorem**
  A polynomial $f(x)$ is divisible by $(x - \alpha)$ if $f(\alpha) = 0$
  If $f(\alpha) = 0$ and $f'(\alpha) = 0$, then $(x - \alpha)^2$ is the factor of $f(x) = 0$ and α is known as repeated root of $f(x) = 0$.
  Also, $f(x) = ax^2 + bx + c = a(x - \alpha)^2$ where $\alpha = -b/2a$
- If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$, then $\alpha + \beta + \gamma = -b/a$, $\alpha\beta + \alpha\gamma + \beta\gamma = c/a$, $\alpha\beta\gamma = -d/a$
- If α, β, γ are roots of $f(x) = 0$, then
  - the equation whose roots are $-\alpha, -\beta, -\gamma$ is $f(-x) = 0$
  - the equation whose roots are $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$ is $f\left(\dfrac{1}{x}\right) = 0$
  - the equation whose roots are $k\alpha, k\beta, k\gamma$ is $f\left(\dfrac{x}{k}\right) = 0$
- If the sum of coefficients of a polynomial equation f(x) = 0 is zero then 1 is a root of the equation f(x) = 0.

MATRIX: A rectangular array of **mn** numbers in the form of **m** horizontal lines (called rows) and **n** vertical lines (called columns) is a matrix of order **m** by **n**. An element occurring in the i^{th} rows and j^{th} column of a matrix **A** is called (i, j) th element of A and is denoted by a_{ij}

Types of Matrices

- **Row Matrix :** It is a matrix having only one row
- **Column Matrix :** It is a matrix having only one column
- **Null Matrix:** It is a matrix each of whose element is zero.
- **Square Matrix:** It is a matrix having the same numbers of rows and columns. If the number of rows and columns is n, then the matrix is a square matrix of order n.

Diagonal Elements of a Matrix: The elements a_{ij} for which $i = j$ are called the diagonal elements of square matrix A.

Diagonal Matrix: A square matrix in which every non diagonal elements is zero is called a diagonal matrix.

e.g $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Scalar Matrix: A square matrix in which every non diagonal element is zero and all diagonal elements are equal, is known as a scalar matrix.

e.g $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

Unit Matrix : A square matrix in which every non diagonal element is zero and every diagonal element is 1, is called a unit matrix or an identity matrix. A unit matrix of order n is represented as I_n.

Comparable matrices: Two matrices A and B are said to be comparable if they have same order.

100/100 Marks

Equal Matrices: Two matrices are said to be equal if they are comparable and their corresponding elements are equal.

Singular Matrix : A square matrix is said to be singular if $|A| = 0$. A square matrix is non singular if $|A| \neq 0$

Upper triangular Matrix: A square matrix $A = (a_{ij})_{n \times n}$ is an upper triangular matrix if $a_{ij} = 0$ for $i > j$

Lower triangular Matrix: A square matrix $A(a_{ij})_{n \times n}$ is a lower triangular matrix if $a_{ij} = 0$ for $i < j$.

Involutory Matrix: A square matrix is said to be Inovolutory Matrix if $A^2 = I_n$

Addition of Matrices: Let A and B be two comparable matrices each of order mxn, then their sum (A+B) is a matrix of order (mxn), obtained by adding the corresponding elements of A and B

If $A = \begin{bmatrix} 5 & 1 & 3 \\ 0 & 4 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 6 \\ 8 & 7 & 4 \end{bmatrix}$ then $A + B = \begin{bmatrix} 5+2 & 1+3 & 3+6 \\ 0+8 & 4+7 & 9+4 \end{bmatrix} = \begin{bmatrix} 7 & 4 & 9 \\ 8 & 11 & 13 \end{bmatrix}$

Scalar Multiplication: Let A be any matrix of order mxn and K be a number. Then, the matrix obtained by multiplying each element of A by K is called the scalar multiple of A by K and is denoted by KA.

$\therefore KA = (Ka_{ij})_{m \times n}$

Matrices

Transpose of Matrix: Let A be an mxn matrix. Then nxm matrix obtained by interchanging the rows and columns of A is called the transpose of A and is denoted by A'

- If A and B are two matrices of the same order, then $(A + B)' = A' + B'$
- if A is any matrix and K is a scalar then $(KA)' = KA'$

Symmetric Matrix: A Square matrix A is said to be symmetric if $A' = A$

Skew Symmetric Matrix : A Square matrix A is said to be skew symmetric if $A' = -A$

- Sum of two symmetric matrices is symmetric
- Sum of two skew symmetric matrices is a skew symmetric matrix.
- if A is any square matrix, then $A + A'$ is symmetric and $A - A'$ is skew symmetric.
- Every diagonal element of a skew symmetric matrix is zero
- Every square matrix A can be uniquely expressed as the sum of a symmetric and a skew symmetric matrix. $A = \frac{A+A'}{2} + \frac{A-A'}{2} = P + Q$ (Here, $P' = P$ and $Q' = -Q$)
- Determinant of a skew symmetric matrix of odd order is zero and of even order is a non zero perfect square.

Product of Matrices: Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{nxp}$ be two matrices, Their product AB is an mxp matrix, given by $AB = [c_{ik}]_{m \times p}$ where $c_{ik} = \sum_{j=1}^{n} a_{ij} \; b_{jk}$

- If A and B are two matrices, then AB may exist and BA may not exist, i.e matrix multiplication is not commutative.
- If A and B are two matrices such that AB is defined, then $(AB)' = B'A'$.

Adjoint of a Matrix.

- adj A is the transpose of the matrix of corresponding cofactors of elements of **A**
- If A is a square matrix of order n, then $A \cdot (adj\ A) = (adj\ A) \cdot A = |A| \cdot I_n$.
- If A and B are invertible square matrices of the same order, then $adj(AB) = (adj\,B)(adj\,A)$
- If A is an invertible square matrix, then $(adj\ A)' = (adj\ A')$

Inverse of a Matrix

A non-zero square matrix A of order n is said to be invertible if there exists a square matrix B of order n, such that $AB = BA = I_n$ where $B = A^{-1} = \frac{1}{|A|} adj \quad A$

- A square matrix A is invertible if and only if A is non-singular i.e. $|A| \neq 0$.
- Let A, B, C be three square matrices each of order n such that AB = AC. Then, if A is non singular, then B = C
- If A and B are invertible square matrices of the same order, then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$
- $|adj\ A| = |A|^{n-1}$

Rank of a Matrix: A non zero matrix A is said to have a rank r if there exists at least one minor of order r which is non singular and every minor of order $r+1$ is singular.

Solving a system of Nonhomogeneous Equations : Consider the following system of n linear equations in n unknowns :

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1;$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2;$

$\cdots \quad \cdots \quad \cdots \quad \cdots$

$\cdots \quad \cdots \quad \cdots \quad \cdots$

$a_{n1}x_1 + a_{n2}x_2 + ... a_{nn}x_n = b_n$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_n \end{bmatrix}$.

The above system can be written in matrix form as $AX = B \Rightarrow A^{-1}AX = A^{-1}B$ or $X = A^{-1}B$ $X = A^{-1}B$ is the solution of the above system of equations.

Criterion for Consistency of a System of Linear Equations

Let $AX = B$ be a system of n linear equations in n unknowns. Then, we have the following conditions:

- If $|A| \neq 0$, then the system is consistent and has a unique solution, given by $X = A^{-1}B$.
- If $|A| = 0$ and $(adj\,A)B \neq 0$, then the system is inconsistent.
- If $|A| = 0$ and $(adj\,A)B = 0$, then the system is consistent and has infinitely many solutions.

Homogeneous System of Linear Equations

Consider the system of equations :

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0;$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0;$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0;$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

So, the given system in matrix form is, $AX = 0$

If $|A| \neq 0$, then A^{-1} exists. So, $AX = 0$ $\Rightarrow A^{-1}(AX) = A^{-1}.0$ $\Rightarrow (A^{-1}A)X = 0$ $\therefore X = 0$

Thus, in this case $x_1 = 0, x_2 = 0, x_3 = 0$ is the only solution.

This solution is known as trivial solution.

For the system $AX = 0$ to have nontrivial solution, we must have $|A| = 0$

Limits

Definition :

The statement $\lim_{x\to a} f(x)=l$ is read as 'the limit of $f(x)$ as 'x' approaches 'a' is 'l'

Left hand and Right hand limits

If 'x' approaches 'a' from the left, i.e.from smaller values of 'x' than 'a', the limit of 'f' is called the left hand limit and is written as

$\lim_{x\to a-0} f(x)$ or $f(a^-)$ or $\lim_{h\to 0} f(a-h)$

If 'x' approaches 'a' from the right, i.e. from larger values of 'x' than 'a', the limit of 'f' is called the right hand limit and is written as

$\lim_{x\to a+0} f(x)$ or $f(a^+)$ or $\lim_{h\to 0} f(a+h)$

A limit exists if LHL = RHL

Properties of Limits

$$\lim_{x\to a}\{f(x)\pm g(x)\}=\lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\{f(x)g(x)\}=\lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

$$\lim_{x\to a}\{f(x)\}^{g(x)}=\left\{\lim_{x\to a} f(x)\right\}^{\lim_{x\to a} g(x)}$$

$$\lim_{x\to a} f\{g(x)\}= f\left(\lim_{x\to a} g(x)\right) \text{ if } f(x) \text{ is continuous}$$

100/100 Marks

Some Standard Limits

$$\lim_{x\to\infty}\frac{1}{x}=0$$

$$\lim_{x\to 0}|x|^n=0, \text{ where } n>0$$

$$\lim_{n\to\infty}|x|^n=\infty \quad \text{if} |x|>1$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$$

$$\lim_{x\to 0}(1+ax)^{1/x}=e^a$$

$$\lim_{x\to 0}\frac{\log_e(1+x)}{x}=1$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x=e^a$$

$$\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$$

$$\lim_{x\to 0}\frac{\tan x}{x}=1$$

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$$

$$\lim_{n\to\infty}x^n=0 \quad \text{if } |x|<1$$

$$\lim_{x\to 0}\frac{1}{|x|}=\infty$$

L' Hospital Rule

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \quad \{\text{if } f(a)=g(a)=0 \text{ or } \infty\}$$

$$=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)} \quad \{\text{if } f'(a)=g'(a)=0 \text{ or } \infty\}$$

L' Hospital rule is applicable only when the function is in 0/0 form or ∞/∞ form.

Evaluation of Limits

Evaluate $\lim_{x\to a}\frac{f(x)}{g(x)}$

Cancel the common factors in $f(x)$ and $g(x)$ and then put $x=a$ to solve.

Method of substitution : Evaluate $\lim_{x\to a}\frac{f(x)}{g(x)}$

Put x = a + h When $x\to a$, $h\to 0$, Substitute x = a + h in Numerator and Denominator, simplify and cancel common factor involving h. Now put h = 0 to solve

CONTINUITY & DIFFERENTIABILITY

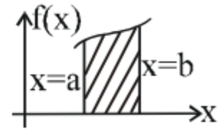
- **Continuity**
  - **Definition**
    - A function f(x) is said to be continuous at 'a' if $f(a^-) = f(a^+) = f(a)$
      i.e. $\lim_{x \to a} f(x) = f(a)$ or $\lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(a-h) = f(a)$
    - We say that Left hand limit = Right hand limit = Value at $(x = a)$
      If any two of the above three are unequal, then $f(x)$ is discontinuous at $(x = a)$
  - If $f(x)$ and $g(x)$ are two functions both continuous functions at $x = a$, then $cf(x)$, $f(x) \pm g(x)$, $f(x)g(x)$, $f(x)/g(x)$ are all continuous at $x = a$
  - Hence, if in a continuous function, a kink is observed, the function is not differentiable at the kink.
-
- **Differentiability**
  - **Definition:** The derivative of function y=f(x) is defined as the instantaneous rate of change of $y\{or \quad f(x)\}$ with respect to the change in 'x'.
    Derivative = $\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $\quad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$
  - **Existence of derivative at a point**
    - $f(x)$ is continuous at $x = a$
    - Left hand derivative = Right hand derivative at $(x = a)$ i.e.,
      $\lim_{h \to 0} \frac{f(a) - f(a-h)}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$
    - In other words, a continuous function $f(x)$ at a is differentiable if slope at a^- = slope at a^+
    - Hence, if in a continuous function, a kink is observed, the function is not differentiable at the kink.

- Binomial Theorem
  - Theorem
    - If x and a are real numbers, then for all $n \in N$, we have $(x+a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r$
  - General Term in the expansion of
    - $(x+a)^n$ is given by $T_{r+1} = {}^nC_r x^{n-r} a^r$
    - $(x-a)^n$ is given by $T_{r+1} = (-1)^r \, {}^nC_r x^{n-r} a^r$
    - $(x+a)^n + (x-a)^n = 2\left\{ {}^nC_0 x^n a^0 + {}^nC_2 x^{n-2} a^2 + \ldots\ldots \right\}$
    - $(x+a)^n - (x-a)^n = 2\left\{ {}^nC_1 x^{n-1} a^1 + {}^nC_3 x^{n-3} a^3 + \ldots\ldots \right\}$
    - If n is odd, then $\{(x+a)^n + (x-a)^n\}$ and $\{(x+a)^n - (x-a)\}$ both have $\left(\frac{n+1}{2}\right)$ terms.
    - If n is even, then $\{(x+a)^n + (x-a)^n\}$ has $\left(\frac{n}{2}+1\right)$ terms while $\{(x+a)^n - (x-a)^n\}$ has $\left(\frac{n}{2}\right)$.
  - O and E denote respectively the sums of odd terms and even terms in the expansion of $(x+a)^n$
    - $(x+a)^n = O + E$ and $(x-a)^n = O - E$
    - $(x^2 - a^2)^n = O^2 - E^2$
    - $4OE = (x-a)^{2n} - (n-a)^{2n}$
    - $(x+a)^{2n} + (x-a)^{2n} = 2(O^2 + E^2)$
  - If n is even, then $\left(\frac{n}{2}+1\right)^{th}$ is the middle term. If n is odd, then $\left(\frac{n+1}{2}\right)$ and $\left(\frac{n+3}{2}\right)$ are middle terms

Area Under The Curve

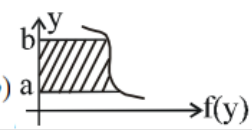
Let f(x) be a continous function in (a, b). Then the area bounded by the curve $y = f(x)$, x-axis, lines $x = a$ and $x = b$ is given by

$$A = \left| \int_a^b f(x)dx \right| \quad \text{provided} \quad f(x) \geq 0 \text{ or } f(x) \leq 0 \;\; \forall x \in (a,b)$$



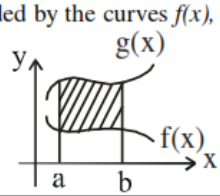
The area between $x = f(y)$, y-axis, lines $y = a$ and $y = b$ is given by

$$A = \left| \int_a^b f(y)dy \right| \quad \text{provided} \quad f(y) \geq 0 \quad \text{or} \quad f(y) \leq 0 \;\; \forall x \in (a,b)$$



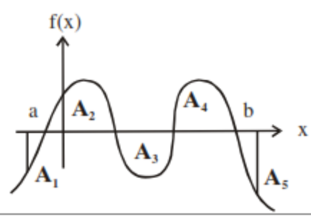
Let $f(x)$ and $g(x)$ be two continous functions in (a,b), then the area bounded by the curves $f(x)$, $g(x)$, lines $x = a$ and $x = b$ is given by

$$A = \int_a^b [g(x) - f(x)]\, dx \qquad \text{provided} \quad g(x) \geq f(x) \;\; \forall x \in (a,b)$$



If the curve lies completely above x-axis, the area is positive and if it lies completely below x-axis, the area is negative. Since, only the magnitude of area is considered, we need to calculate areas above the x-axis separately and below the x-axis separately.

The total area between the curve $f(x)$, x-axis, the lines $x = a$ and $x = b$ in the sketch is given by $A = |A_1| + |A_2| + |A_3| + |A_4| + |A_5|$



CURVE TRACING

Symmetry: If the curve remains unaltered by replacing x by $-x$, then it is symmetrical about y-axis. If the curve remains unaltered by replacing y by $-y$, then it is symmetrical about x-axis.

Intercepts on axes: These are the points where the curve crosses coordinate axes. To find these points, we substitute $x = 0$ or $y = 0$ to find y intercept and x intercept respectively.

Extent of the curve: The domain and range of the curve gives the extent of the area in which the curve lies.

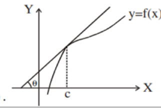
Asymptotes: Observe how y behaves as $x \to \pm\infty$ Also, observe how x behaves as $y \to \pm\infty$

Local Maxima and Minima: By putting $\frac{dy}{dx} = 0$, we can also find the points of local maxima and minima.

Periodicity: By checking the periodicity if any, the rough sketch of the curve can be conviniently drawn.

Application of derivatives

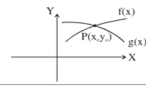
Equation of tangents and normal



Let the curve by $y = f(x)$ and let a point on the curve be (x_0, y_0).

- Slope of tangent at (x_0, y_0) is $f'(x_0)$
- Slope of normal at (x_0, y_0) is $-\dfrac{1}{f'(x_0)}$
- Equation of tangent is $\dfrac{y-y_o}{x-x_o} = f'(x_o)$
- Equation of normal is $\dfrac{y-y_o}{x-x_o} = -\dfrac{1}{f'(x_o)}$
- If tangent is parallel to x-axis, $dy/dx = 0$
- If tangent is parallel to y-axis, $dy/dx \to \infty$

Angle of intersection of two curves



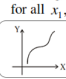
Let $y = f(x)$ and $y = g(x)$ be two curves intersecting at point $P(x_0, y_0)$.
The slopes of the tangents of the two curves at P are $m_1 = f'(x_0)$ and $m_2 = g'(x_0)$

- Then, angle between the two curves θ, is given by
  $\tan\theta = \pm\dfrac{m_1 - m_2}{1+m_1m_2} = \pm\dfrac{f'(x_o) - g'(x_o)}{1+f'(x_o)g'(x_o)}$
- If $f'(x_o) = g'(x_o)$, $\theta = 0^\circ \Rightarrow$ the two curves touch each other
  If $f'(x_o)g'(x_o) = -1$, $\theta = 90^\circ \Rightarrow$ the two curves intersect orthogonally.

Increasing and decreasing Functions

Increasing Function

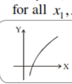
- If $x_1 < x_2$ in interval (a, b)
- $\Rightarrow f(x_1) \le f(x_2)$ for all x_1, x_2 in (a, b)



- $f'(x) \ge 0$ for all $x \in (a,b)$

Strictly Increasing Function

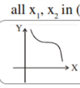
- If $x_1 < x_2$ in interval (a, b)
- $f(x_1) < f(x_2)$ for all x_1, x_2 in (a, b) for



- $f'(x) > 0$ for all $x \in (a,b)$

Decreasing Function

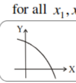
- If $x_1 < x_2$ in interval (a, b)
- $f(x_1) \ge f(x_2)$ all x_1, x_2 in (a, b)



- $f'(x) \le 0$ for all $x \in (a,b)$

Strictly Decreasing Function

- If $x_1 < x_2$ in interval (a, b)
- $f(x_1) > f(x_2)$ for all x_1, x_2 in (a, b)



- $f'(x) < 0$ for all $x \in (a,b)$

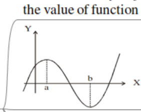
Monotonic Function

Monotonic Function : A function f is said to be monotonic in an interval (a, b) if it is either strictly increasing or strictly decreasing in the interval (a, b).

Maximum and minimum values of a function

Local maximum and Local minimum

For a function $y = f(x)$, $x = a$ a is a point of local maximum since in the local region around a, at $x = a$ the value of function $f(a)$ is maximum.

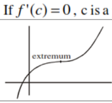


Similarly $x = b$ is a point of local minimum since in the local region around b, at $x = b$ the value of function $f(b)$ is minimum

At the point of local maximum and local minimum,
i.e. at $x = a$ and $x = b$, the tangents are parallel to the x-axis.
Hence, $f'(x) = 0$ at $x = a, b$
Therefore, if $f'(c) = 0$, $f(x)$ has a local maximum or local minimum at $x = c$

Check for Local maximum, Local Minimum or Exremum

If $f'(c) = 0$, c is a point of local maximum or local minimum or extremum

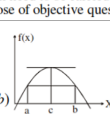


If $f''(c) > 0$, c is a point of local minimum
< 0, c is a point of local maximum
$= 0$, c is a point of extremum

If $f''(c) = 0$, c may still be a point of local maximum/minimum/extremum
for which further check need to be carried out.
However, for the purpose of objective questions, above mentioned criterion for $f''(c) = 0$ should suffice.

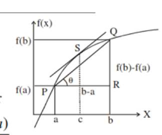
Rolle's Theorem

If a function $f(x)$ defined on $[a,b]$ is
continuous on $[a,b]$
differentiable on (a,b)
$f(a) = f(b)$
then, there exists at least one real number $c \in (a,b)$
such that $f'(c) = 0$



Langrange's Mean Value Theorem

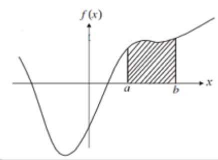
If a function $f(x)$ defined on $[a,b]$ is
continuous on $[a,b]$
differentiable on (a,b)
then, there exists at least one real number
$c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$



Definite Intigral

GEOMETRICAL INTERPRETATION OF DEFINITE INTEGRATION

Let $f(x)$ be a function defined on a closed interval $[a,b]$. Then, $\int_a^b f(x)dx$ respresents the algebraic sum of all areas of the regions bounded by the curve $f(x)$, the x-axis and the straight lines $x = a$ and $x = b$.



- The areas above the x-axis are +ve in sign
- The areas below the x-axis are -ve in sign.

PROPERTIES OF DEFINITE INTEGRAL

- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^b f(t)dt$
- $\int_o^a f(x)dx = \int_o^a f(a-x)dx$
- $\int_o^{2a} f(x)dx = \int_o^a f(x)dx + \int_o^a f(2a-x)dx$
- If the functions f(x) and g(x) are defined on [a, b] and differentiable at all points in the interval, $\frac{d}{dx}\int_{f(x)}^{g(x)} h(t)dt = h[g(x)]g'(x) - h[f(x)]f'(x)$
- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
- $\int_{-a}^a f(x)\,dx = \int_0^a [f(x)+f(-x)]dx = 2\int_0^a f(x)\,dx$
- $\int_{-a}^a f(x)\,dx = \int_0^a [f(x)+f(-x)]dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is even, i.e. $f(-x) = f(x)$
  $= 0$ if $f(x)$ is odd i.e. $f(-x) = -f(x)$
- $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$, where $f(x)$ is a periodic function with period T and n is an integer
- $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$
- If $g(x) \ge f(x) \ge h(x)$ for all $x \in [a,b]$ then, $\int_a^b g(x)dx \ge \int_a^b f(x)dx \ge \int_a^b h(x)dx$
- If m and M are respectively the global minimum and global maximum values of f(x) in [a, b], then, $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

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DEFINITION An ellipse is the locus of a point which moves in a plane such that the ratio of its distance from the fixed point (Focus) to its distance from the fixed line (Directrix) is always constant and equal to a quantity less than 1.

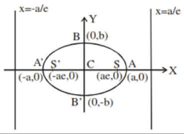
Ellipse

Equation of ellipse in various forms

General form

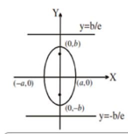
$ax^2+2hxy+by^2+2gx+2fy+c=0$ where $\Delta = abc+2fgh-af^2-bg^2-ch^2 \neq 0$ and $h^2-ab<0$

Most standard form



Equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, a>b

- Focus $S(ae,0)$ and $S'(-ae,0)$
- Vertex $A(a,0)$ and $A'(-a,0)$
- Directrix $x=a/e$ and $x=-a/e$
- Centre $C(0,0)$
- Principal Axis - Major Axis and Minor Axis
  Major Axis - $AA'=2a$ Minor Axis- $BB'=2b$
- Semi Major Axis = a Semi Minor Axis = b
- $b^2=a^2(1-e^2)$, $e^2=1-\dfrac{b^2}{a^2}$



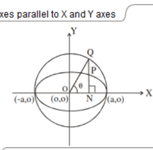
Equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where $b>a$

- Focus $(0,be)$ and $(0,-be)$
- Directrix $y=b/e$ and $y=-b/e$
- Semi Major Axis = b, Semi Minor Axis = a
- $a^2=b^2(1-e^2)$, $e^2=1-\dfrac{a^2}{b^2}$

Centre at (h, k) and axes parallel to X and Y axes

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Parametric Form



$x=a\cos\theta$ and $y=b\sin\theta$

The circle described on the focal axis with major axis as diameter is called Auxiliary Circle

Let Q be a point on the Auxiliary circle and P be a point on the Ellipse as shown in the figure

Then $\angle QON=\theta$ is called as Eccentric Angle of the point P on the Ellipse and $0\le\theta<2\pi$

Note, that $Q(\theta)=(a\cos\theta, a\sin\theta)$ and $P(\theta)=(a\cos\theta, b\sin\theta)$

100/100 Marks

Location of a point with respect to ellipse

Let the ellipse be $S\equiv\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$ and the point be $P(x_1,y_1)$, then $S_1\equiv\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1$

- If $S_1<0$, P is inside the ellipse
- If $S_1=0$, P is on the ellipse
- If $S_1>0$, P is outside the ellipse

Tangents and Normals

If the ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, then the tangent at point $P(x_1,y_1)$ is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$

In parametric form, $x=a\cos\theta$ and $y=b\sin\theta$

Slope of tangent is $\dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{b\cos\theta}{-a\sin\theta}=-\dfrac{b\cos\theta}{a\sin\theta}$ $\Rightarrow$ Slope of normal is $\dfrac{a\sin\theta}{b\cos\theta}$

Eqn of tangent is $\dfrac{y-b\sin\theta}{x-a\cos\theta}=-\dfrac{b\cos\theta}{a\sin\theta}$ $\Rightarrow$ $\dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=1$

Eqn of normal is $\dfrac{y-b\sin\theta}{x-a\cos\theta}=\dfrac{a\sin\theta}{b\cos\theta}$ $\Rightarrow$ $\dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta}=a^2-b^2$

The equation of tangent in terms of its slope m at the point of contact $\left(-\dfrac{a^2m}{c},\dfrac{b^2}{c}\right)$ is

$y=mx+c$ where $c=\pm\sqrt{a^2m^2+b^2}$

The equation of pair of tangents from the point $P(x_1,y_1)$ to the Ellipse

$S\equiv\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$ is $S.S_1=T_1^2 \Rightarrow \left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1\right)\left(\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1\right)=\left(\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}-1\right)^2$

Chord of contact

The equation of chord of contact of tangents drawn from the point (x_1,y_1) to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$

Equation of chord with its mid point

If (x_1,y_1) is the modpoint of the chord to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$, then the equation of the chord is

$T_1=S_1$ or $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}-1=\dfrac{x_1^2}{a^2}+\dfrac{y^2}{b^2}-1$ or $\dfrac{x_1}{a^2}x+\dfrac{y_1}{b^2}y-\left(\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}\right)=0$

Properties of Ellipse

The sum of the focal distances of a point on an ellipse is always constant and is equal to the major axis.

Tangents at the ends of a focal chord intersect on the directrix.

Permutation and combination

- **Principle of Counting**
  - If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is m x n
- **Permutations**
  - permutations of n different things taken r at a time ,repetition not allowed: $^nP_r = \frac{n!}{(n-r)!}$
  - permutations of n different things taken r at a time where repetition is allowed: n^r
  - p_1 objects are of one kind, p_2 are of second kind, p_k are of kth kind: $\frac{n!}{p_1!p_2!\ldots\ldots p_k!}$
- **Combinations**
  - The number of combinations of n different things taken r at a time: $^nC_r = \frac{n!}{r!(n-r)!}$, where $0 \le r \le n$
  - Theorem 1: $^nP_r = {}^nC_r \times r!\ ,\ 0 \le r \le n.$
  - Theorem 2: $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
  - Theorem 3: $^nC_r = {}^nC_{n-r}$
  - Theorem 4: $^nC_{r1} = {}^nC_{r2}$ then either $r_1 = r_2$ or $r_1 + r_2 = n$
  - Theorem 5: $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$

Probability

DEFINITION OF PROBABILITY

If S is the sample space of an experiment, then the probability of occurance of an event A is defined as

$$P(A)=\frac{n(A)}{n(S)}=\frac{\text{no. of outcomes favourable to A}}{\text{no. of possible outcomes}}$$

ODDS IN FAVOUR AND ODDS AGAINST AN EVENT

If in an experiment, the number of outcomes favourable to an event A is 'm' and number of outcomes not favourable to event A is 'n', then

$$\text{Odds in favour of A}=\frac{\text{no. of outcomes favourable}}{\text{no. of outcomes unfavourable}}=\frac{m}{n}$$

$$\text{Odds against A}=\frac{\text{no. of unfavourable outcomes}}{\text{no. of favourable outcomes}}=\frac{n}{m}$$

If odds in favour of an event are m:n, then the probability of the occurence of that event is $\frac{m}{m+n}$ and the probability of non occurence of that event is $\frac{n}{m+n}$

ADDITION THEORY (TOTAL PROBABILITY THEOREM)

If A and B are any events in S, then, $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

If A and B are mutually exclusive, then, $P(A\cap B)=0$ and $P(A\cup B)=P(A)+P(B)$

The probability that one of several mutually exclusive events $A_1, A_2,\ldots\ldots A_n$ will happen is sum of the probabilities of the separate events

$$P(A_1\cup A_2\cup\ldots\ldots\ldots A_n)=P(A_1)+P(A_2)+\ldots\ldots\ldots P(A_n)$$

CONDITIONAL PROBABILITY THEOREM

Probability of occurrence of event A given that event B has already occurred is known as conditional probability of A w.r.t. B. and is denoted as P(A/B)

Since, event B has already occurred, it implies that the outcomes favourable to B become the total outcomes and outcomes favourable to P(A/B) are outcomes common to A and B

$$\Rightarrow P(A/B)=P(A\ \ given\ \ B)=\frac{\text{Total no. of favourable cases}}{\text{Total no. of cases}}=\frac{n(A\cap B)}{n(B)}=\frac{P(A\cap B)}{P(B)}$$

Similarly, $P(B/A)=\frac{P(A\cap B)}{P(A)}$

INDEPENDENT EVENTS

If the occurrence of event A does not depend on the occurrence or the non occurrence of the event B, then A and B are said to be independent events.

If A and B are independent events, then $P(A\cap B)=P(A)\ \ P(B)$

$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{P(A)P(B)}{P(B)}=P(A)$ and $P(B/A)=P(B))$

MATHEMATICAL EXPECTATION

Let X be a discrete random variable which assumes the value $x_1, x_2\ldots\ldots x_n$ with the corresponding probabilities $p_1, p_2\ldots\ldots\ldots p_n$.

Then the expected values of X is known as mathematical expectation of X which is denoted by E(x) and given as

$$E(X)=\sum_{i=1}^{n}x_i p_i \qquad \text{where} \quad \sum_{i=1}^{n}p_i=1$$

PROBABILITY DISTRIBUTION

Let X be a random variable which can take the value $X=x_1, x_2, x_3\ldots\ldots x_n$

We can assign corresponding probabilities to each value $p_i=p(X=x_i),\quad 1\le i\le n$

The possible values of X and their corresponding probabilities are known as the probability distribution of X.

BINOMIAL PROBABILITY DISTRIBUTION

An experiment may be repeated n no. of times. Suppose these trials are independent of one another and each trial gives only one of the two possible outcomes say

'success' denoted by 'p' and

'failure' denoted by 'q'

We can assign corresponding probabilities to each value $p_i=p(X=x_i),\quad 1\le i\le n$

PROGRESSION

ARITHMETIC PROGRESSION (AP)

- a = first term, d = common difference, $T_n = a + (n-1)d$
- $S_n = \frac{n}{2}(a+l) = \frac{n}{2}[2a+(n-1)d]$ $\{l = \text{last term}\}$
- AM between two numbers a and b is $= \frac{a+b}{2}$
- n AMs between two numbers a and b, denoted by $A_1, A_2, A_3, .. A_n$ form an AP given by $a, A_1, A_2, A_3, .. A_n, b$ and $A_k = a + \frac{k(b-a)}{n+1}$
- If a, b, c are in AP, then ak, bk, ck are also in AP ; $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}$ are also in AP $a \pm k, b \pm k, c \pm k$ are also in AP
- Arithmetic mean of n positive numbers $a_1, a_2, a_3.....a_n$ is $A = \frac{1}{n}(a_1 + a_2 + ...a_n)$
- Sum of first and last terms of an AP is equal to the sum of any two terms equidistant from the first and the last terms.

GEOMETRIC PROGRESSION (GP)

- $T_n = ar^{n-1}$ $\{a = \text{first term}, r = \text{common ratio}\}$
- $S_n = \frac{a(r^n-1)}{r-1} = \frac{a(1-r^n)}{1-r}$
- $S_\infty = \frac{a}{1-r}$ if $|r| < 1$
- GM between two numbers a and b, $G = \sqrt{ab}$
- n GMs between numbers a and b denoted by $G_1, G_2, G_3,......G_n$ form a GP given by
- $a, G_1, G_2, G_3,......G_n, b$ and $G_k = a\left(\frac{b}{a}\right)^{\frac{k}{n+1}}$
- If a, b, c are in G.P., then ak, bk, ck, are also in GP $(k \neq 0)$ $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}$ are also in GP $(k \neq 0)$
- For solving problems, three terms in G.P. are usually taken to be $ar, a, \frac{a}{r}$, 4 terms may be taken $ar^3, ar, \frac{a}{r}, \frac{a}{r^3}$
- Geometric mean of n positive numbers $a_1, a_2, ...a_n$ is $G = (a_1, a_2, ..., a_n)^{1/n}$
- If $a_1, a_2, ..., a_n$ is an G.P. with common ratio r, then $\log a_1, \log a_2, ..., \log a_n$ is an A.P. with common difference $\log r$

HARMONIC PROGRESSION (HP)

- The reciprocals of two consecutive terms increase /decrease by a fixed quantity.
- $a_1, a_2, a_3, ..., a_n$ are said to be in HP if $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ..., \frac{1}{a_n}$ are in A.P.
- If H is HM of a and b, then $\frac{1}{a} - \frac{1}{H} = \frac{1}{H} - \frac{1}{b}$ $\Rightarrow$ $H = \frac{2ab}{a+b}$
- n HMs between two numbers a and b denoted by $H_1, H_2, H_3, ..., H_n$ form an AP given by $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, ..., \frac{1}{H_n}, \frac{1}{b}$ and $\frac{1}{H_k} = \frac{1}{a} + \frac{k(a-b)}{ab(n+1)}$
- If a, b, c are in HP, then ak, bk, ck are also in HP $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}$ are also in HP, $\frac{1}{a \pm k}, \frac{1}{b \pm k}, \frac{1}{c \pm k}$ are in AP
- If Harmonic mean of n positive numbers $a_1, a_2, ..., a_n$ is H, then $\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + ... + \frac{1}{a_n}\right)$

RELATION AMONG A, G, H

If a and b are two numbers, then $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $H = \frac{2ab}{a+b}$, $AH = G^2$, $A > G > H$

SUMMATION OF n NATURAL NUMBERS

- $\Sigma n = 1 + 2 + 3 + ...n = \frac{n(n+1)}{2}$,
- $n^2 = 1^2 + 2^2 + 3^2 + ...n^2 = \frac{n(n+1)(2n+1)}{6}$
- $\Sigma n^3 = 1^3 + 2^3 + 3^3 + ...n^3 = \left[\frac{n(n+1)}{2}\right]^2$

ARITHEMETIC-GEOMETRIC SERIES

- This series is a combination of AP and GP in the manner $a, (a + d)r, (a + 2d)r^2, (a + 3d)r^3, ...$ $\{a = \text{first term}, d = \text{common difference}, r = \text{common ratio}\}$
- $T_n = [a + (n-1)d]r^{n-1}$
- $S_n = \frac{a}{(1-r)} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{(1-r)}$
- $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$ if $|r| < 1$

SUMMATION OF TRIGONOMETRIC SERIES

If $a_1, a_2, ... a_n$ are in A.P. with common difference 'd', then

- $\sin a_1 + \sin a_2 + ... + \sin a_n = \frac{\sin\left(\frac{a_1+a_n}{2}\right)\sin\left(\frac{nd}{2}\right)}{\sin(d/2)}$
- $\cos a_1 + \cos a_2 + ... + \cos a_n = \frac{\cos\left(\frac{a_1+a_n}{2}\right)\sin\left(\frac{nd}{2}\right)}{\sin(d/2)}$

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Straight line

The distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is:

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The coordinates (x, y) of the point P which divides the line joining the points A(x1, y1) and B(x2, y2) in the ratio m1: m2 are $x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \quad y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$

100/100 Marks

Centroid of a triangle : If A(x1, y1), B(x2, y2) and C(x3, y3) are the vertices of a triangle, then coordinates of its centroid are: $x = \frac{x_1 + x_2 + x_3}{3}, \quad y = \frac{y_1 + y_2 + y_3}{3}$

Area of a triangle : If A(x1, y1), B(x2, y2) and C(x3, y3) are the vertices of ∆ABC, then its area

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

Equation of line : The equation of a line passing through two points P (x1, y1) and Q (x2, y2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

If a straight line is inclined at an angle θ with the positive direction of x-axis, where $0° \le \theta \le 180°$, $\theta \ne 90°$ then the slope of the line is defined as $m = \tan\theta$

- For $\theta = 0$, $m = 0$ and the line is parallel to x–axis.
- For $\theta = 90°$, the line is parallel to y-axis.

Collinearity of three points : If three points A(x1, y1), B(x2, y2) and C(x3, y3) are collinear, then the points lie on a straight line. The area of the triangle formed by these points is, thus, zero i.e.

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0 \Rightarrow \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_3}{x_2 - x_3} \text{ or } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Angle between two straight lines : $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

Length of the perpendicular from a point on a line : The length of a perpendicular from a point $P(x', y')$ on a line $ax + by + c = 0$ is: $p = \frac{ax' + by' + c}{\sqrt{a^2 + b^2}}$

Parallel lines : Two parallel lines have the same slopes. Any line parallel to ax + by + c = 0 is ax + by + k = 0 where k is determined by the given conditions in the problem.

If the length of the perpendicular from origin on the line is p and α the angle which this perpendicular makes with positive direction of x-axis, then the equation of line is: $x\cos\alpha + y\sin\alpha = p$

General form : The equation of a straight line in general form is ax + by + c = 0

100/100 Marks

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Three Dimensional Geometry

Direction cosines of a vector (line)

Suppose a vector makes an angle α with x-axis, β with y-axis and γ with z-axis, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are its direction cosines and are denoted by ℓ, m, n respectively.

$\ell=\cos\alpha,\ m=\cos\beta$ and $n=\cos\gamma$

$\ell^2+m^2+n^2=1$

Direction Ratios of a vector (line)

If $\vec{r}=a\hat{i}+b\hat{j}+c\hat{k}$, then a, b, c are the direction ratios of $\vec{r}$.

Direction cosines of $\vec{r}$ are given by $\ell=\dfrac{a}{\sqrt{a^2+b^2+c^2}},\ m=\dfrac{b}{\sqrt{a^2+b^2+c^2}},\ n=\dfrac{c}{\sqrt{a^2+b^2+c^2}}$

Angle between Two vectors (line)

If θ is the angle between two vectors with direction cosines (ℓ_1,m_1,n_1) and (ℓ_2,m_2,n_2), then $\cos\theta=\ell_1\ell_2+m_1m_2+n_1n_2$

If two vectors are perpendicular, then $l_1l_2+m_1m_2+n_1n_2=0$

Parallel vectors have same direction ratios

Equation of a straight line

Vector form: The equation of a straight line passing through a given point with position vector $\vec{r}_1$ and parallel to a given vector $\vec{m}$ is $\vec{r}=\vec{r}_1+\lambda\vec{m}$

Cartesian form : The equation of a straight line with direction ratios a, b, c and passing through a fixed point (x_1,y_1,z_1) is $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}$

Line passing through two points

Vector form : The equation of a line passing through two points with position vectors $\vec{r}_1$ and $\vec{r}_2$ is $\vec{r}=\vec{r}_1+\lambda(\vec{r}_2-\vec{r}_1)$ or $\vec{r}=(1-\lambda)\vec{r}_1+\lambda\vec{r}_2$

Cartesian form : The equation of a line passing through two given points (x_1,y_1,z_1) and (x_2,y_2,z_2) is given by $\dfrac{x-x_1}{x_2-x_1}=\dfrac{y-y_1}{y_2-y_1}=\dfrac{z-z_1}{z_2-z_1}$

Angle between two lines

Vector form: Let two lines be $\vec{r}=\vec{r}_1+\lambda\vec{m}_1$ and $\vec{r}=\vec{r}_2+\lambda\vec{m}_2$ then, $\cos\theta=\dfrac{\vec{m}_1.\vec{m}_2}{|\vec{m}_1|\,|\vec{m}_2|}$

Cartesian form: If θ is the angle between two lines whose equations are $\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$ and $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$, then $\cos\theta=\dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

Skew Lines : If two lines are not parallel and do not intersect, then they are known as skew lines.

Shortest Distance (S.D) between two skew lines

S.D is length of line segement which is $\perp$ to two given lines

Vector form : If two lines are $\vec{r}=\vec{r}_1+\lambda\vec{\mu}$ and $\vec{r}=\vec{r}_2+\lambda\vec{v}$,

then S.D. $=\left|\dfrac{(\vec{r}_2-\vec{r}_1)\cdot(\vec{u}\times\vec{v})}{|\vec{u}\times\vec{v}|}\right|$ {S.D = projection of $(\vec{r}_2-\vec{r}_1)$ on $(\vec{u}\times\vec{v})$}

Cartesian form: The shortest distance between the lines $\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$ and $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$ is $S.D=\dfrac{\begin{vmatrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(a_2c_1-a_1c_2)^2+(a_1b_2-a_2b_1)^2}}$

The two lines will intersect if S.D = 0

Equation of a plane in normal form

Vector form: If $\vec{n}$ is a vector normal to a given plane, then the equation of the plane is $\vec{r}\cdot\vec{n}=q$ where q is some scalar.

Cartesian form: If ℓ, m, n are the direction cosines of the normal to a given plane and p is the length of the perpendicular form the origin to the plane, then the equation of the plane is $\ell x+my+n=p$

Equation of a plane passing through a given point

Vector form : The vector equation of a plane through a given point $\vec{r}_1$ and perpendicular to $\hat{n}$ is $(\vec{r}-\vec{r}_1)\cdot\hat{n}=0$

Cartesian form : The eqn. of a plane passing through a given point (x_1,y_1,z_1) and normal to the plane having direction ratios a, b, c is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

Eqn. of a plane through two given points and parallel to a given vector

Vector form : The equation of the plane through two given points with position vectors $\vec{r}_1$ and $\vec{r}_2$ and parallel to a given vector $\vec{m}$ is $(\vec{r}-\vec{r}_1)\cdot[(\vec{r}_2-\vec{r}_1)\times\vec{m}]=0$

Cartesian form : The equation of a plane passing through given points (x_1,y_1,z_1) and (x_2,y_2,z_2) and parallel to a line having direction ratios a, b, c, is $\begin{vmatrix}x-x_1 & y-y_1 & z-z_1\\ x-x_2 & y-y_2 & z-z_2\\ a & b & c\end{vmatrix}=0$

Equation of a plane in intercept form

If a plane makes intercepts of lengths a, b, c with x-axis, y-axis and z-axis respectively, then the equation of the plane is given by $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$

Angle between two Planes

Vector form : The angle 'θ' between the planes. $\vec{r}\cdot\vec{n}_1=q_1$ and $\vec{r}\cdot\vec{n}_2=q_2$ is given by $\cos\theta=\dfrac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

Cartesian form : The angle θ between the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is given by $\cos\theta=\dfrac{a_1a_2+b_1b_2+c_1c_2}{\left(\sqrt{a_1^2+b_1^2+c_1^2}\right)\left(\sqrt{a_2^2+b_2^2+c_2^2}\right)}$

Diametric form of the equation of a sphere

Vector form : The equation of a sphere, the position vectors of the extremities of whose diameter are $\vec{a}$ and $\vec{b}$ is $(\vec{r}-\vec{a})\cdot(\vec{r}-\vec{b})=0$

Cartesian form: The eqn. of a sphere, the coordinates of the extremities of whose diameter are (x_1,y_1,z_1) and (x_2,y_2,z_2) is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$

Equation of a sphere

Vector form : The equation of a sphere of radius a and center having position vector $\vec{c}$, is $|\vec{r}-\vec{c}|=a$

Cartesian form :

a) The equation of a sphere with center (α,β,γ) and radius 'a' is $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=a^2$

b) The general equation of a sphere is $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

c) Its center is $(-u,-v,-w)$ and radius is $\sqrt{u^2+v^2+w^2-d}$

Plane passing through intersection of two planes

Vector form : The equation of a plane passing through the intersection of two planes $\vec{r}\cdot\vec{n}_1=q_1$ and $\vec{r}\cdot\vec{n}_2=q_2$ is $\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=q_1+\lambda q_2$

Cartesian Form : Any plane through the intersection of two planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is a plane of the form $(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0$

Planes bisecting angles between two planes

The equations of the planes bisecting the angles between the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are $\left(\dfrac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}\right)=\pm\left(\dfrac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}\right)$

Angle between a line and a Plane

Vector form : If θ is the angle between the line $\vec{r}=\vec{r}_1+\lambda\vec{m}$ and the plane $\vec{r}\cdot\vec{n}=q$, then $\sin\theta=\dfrac{\vec{m}\cdot\vec{n}}{|\vec{m}||\vec{n}|}$

Cartesian form : If θ is the angle between the line $\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$ and the plane $a_2x+b_2y+c_2z+d=0$, then $\sin\theta=\dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

Distance of a point from a Plane

Vector form : The length of perpendicular from a point $\vec{r}_1$ to the plane $\vec{r}\cdot\vec{n}=q$ is given by $p=\dfrac{|q-\vec{r}_1\cdot\vec{n}|}{|\vec{n}|}$

Cartesian Form : The length of perpendicular from a point $A(x_1,y_1z_1)$ to the plane $ax+by+cz+d=0$ is given by $p=\left|\dfrac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$

Trigonometric Functions

- **Ratios of allied angles**
  - T-RATIOS FOR $2n\pi + \theta$
    - $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, $\tan(2n\pi + \theta) = \tan\theta$
  - **T–RATIOS FOR** $\frac{\pi}{2} \pm \theta$
    - $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ $\quad$ $\sin\left(\frac{\pi}{2} + \theta\right) = +\cos\theta$
    - $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ $\quad$ $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$
    - $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$ $\quad$ $\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$
  - T-RATIOS FOR $\pi \pm \theta$
    - $\sin(\pi - \theta) = +\sin\theta$ $\quad$ $\sin(\pi + \theta) = -\sin\theta$
    - $\cos(\pi - \theta) = -\cos\theta$ $\quad$ $\cos(\pi + \theta) = -\cos\theta$
    - $\tan(\pi - \theta) = -\tan\theta$ $\quad$ $\tan(\pi + \theta) = +\tan\theta$
  - T-RATIOS FOR $\pm\theta$
    - $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = +\cos\theta$, $\tan(-\theta) = -\tan\theta$
- **STANDARD RESULTS**
  - Sum and difference of angles
    - $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$
    - $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
    - $\sin(A + B) + \sin(A - B) = 2\sin A\cos B$.
    - $\cos(A + B) + \cos(A - B) = 2\cos A\cos B$.
    - $\cos(A + B) - \cos(A - B) = -2\sin A\sin B$.
    - $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
    - $\cot(A \pm B) = \dfrac{\cot A\cot B \mp 1}{\cot A \pm \cot B}$
    - $\tan(A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$
    - $\cot(A + B + C) = \dfrac{\cot A + \cot B + \cot C - \cot A\cot B\cot C}{1 - \cot A\cot B - \cot B\cot C - \cot C\cot A}$
  - Half angles
    - $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1 + \tan^2\theta}$
    - $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$
    - $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$; $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.
    - $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
    - $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$.
    - $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$
    - $\tan\left(\frac{\pi}{4} \pm x\right) = \dfrac{\tan\frac{\pi}{4} \pm \tan x}{1 \mp \tan\frac{\pi}{4}\tan x} = \dfrac{1 \pm \tan x}{1 \mp \tan x}$
    - $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$
    - $1 - \cos 2\theta = 2\sin^2\theta$
    - $1 + \cos 2\theta = 2\cos^2\theta$
    - $\dfrac{1 - \cos 2\theta}{1 + \cos 2\theta} = \tan^2\theta$
- **Some useful results**
  - $\tan A \pm \tan B = \dfrac{\sin(A \pm B)}{\cos A\cos B}$
  - $\cot A \pm \cot B = \dfrac{\sin(B \pm A)}{\sin A\sin B}$
  - **If A + B + C = π**
    - $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$
    - $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$
    - $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
    - $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
    - $\tan A + \tan B + \tan C = \tan A\tan B\tan C$
    - $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$
    - $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$
    - $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

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100/100 Marks

VECTORS

- **Vector addition**: $\vec{R} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$ and $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$
- **Unit vector**: $\hat{A} = (\vec{A}/A)$
- **Magnitude**: $A = \sqrt{(A_x^2 + A_y^2 + A_z^2)}$
- **Direction cosines**: $\cos\alpha = (A_x/A)$, $\cos\beta = (A_y/A)$, $\cos\gamma = (A_z/A)$
- Projection :
  - Component of $\vec{A}$ along $\vec{B} = \vec{A}.\hat{B}$
  - Component of $\vec{B}$ along $\vec{A} = \hat{A}.\vec{B}$
  - If $\vec{A} = A_x\hat{i} + A_y\hat{j}$, then its angle with the x–axis is $\theta = \tan^{-1}(A_y/A_x)$
- Dot product :
  - $\vec{A}.\vec{B} = AB\cos\theta$
  - $\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$
- Cross product :
  - $\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$;
  - $\vec{A} \times \vec{A} = 0$
  - $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$
- **Area of a parallelogram**: Area = $|\vec{A} \times \vec{B}|$
- **Area of a triangle**: Area = $\frac{1}{2}|\vec{A} \times \vec{B}|$
- **Gradient operator**: $\overline{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$
- **Volume of a parallelopiped**: $V = \vec{A}.\left(\vec{B} \times \vec{C}\right)$