

Principles of Abstract Interpretation

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Language

Abinop \ni aop $::= + \mid - \mid * \mid \backslash$

Aexp \ni e $::= x \mid n \mid -e \mid e_1 \text{ aop } e_2$

Bbinop \ni bop $::= = \mid \neq \mid < \mid \leq \mid > \mid \geq$

Bexp \ni b $::= \text{true} \mid \text{false} \mid e_1 \text{ bop } e_2 \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \mid \neg b$

While \ni S $::= x := e \mid \text{skip} \mid S_1; S_2 \mid \text{if } b \text{ then } S_1 \text{ else } S_2$
 $\mid \text{while } b \text{ do } S$

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Abstract Interpreter

$$Al_{A,\mathbb{S}} : \mathbf{While} \rightarrow bool \rightarrow \mathbb{S} \rightarrow \mathbb{S} \times List(\mathbb{S})$$

$$Al_{A,\mathbb{S}}(x := e)(w)(s^\sharp) := (s^\sharp[x \mapsto \mathcal{A}^\sharp(e)(s^\sharp)], [])$$

$$Al_{A,\mathbb{S}}(\mathbf{skip})(w)(s^\sharp) := (s^\sharp, [])$$

$$Al_{A,\mathbb{S}}(S_1; S_2)(w)(s^\sharp) := (u^\sharp, invs_1 +_{List(\mathbb{S})} invs_2)$$

$$\text{with } (u^\sharp, invs_2) := Al_{A,\mathbb{S}}(S_2)(w)(t^\sharp)$$

$$(t^\sharp, invs_1) := Al_{A,\mathbb{S}}(S_1)(w)(s^\sharp)$$

Abstract Interpreter

$$\begin{aligned}
 Al_{A,\mathbb{S}}(\text{if } b \text{ then } S_1 \text{ else } S_2)(w)(s^\sharp) &:= (t^\sharp \vee_{\mathbb{S}} u^\sharp, invs_1 +_{List(\mathbb{S})} invs_2) \\
 &\text{with } (t^\sharp, invs_1) := (Al_{A,\mathbb{S}}(S_1)(w) \circ \mathcal{B}^\sharp(b))(s^\sharp) \\
 &\quad (u^\sharp, invs_2) := (Al_{A,\mathbb{S}}(S_2)(w) \circ \mathcal{B}^\sharp(\neg b))(s^\sharp) \\
 Al_{A,\mathbb{S}}(\text{while } b \text{ do } S)(w)(s^\sharp) &:= (\mathcal{B}^\sharp(\neg b)(t^\sharp), invs) \\
 &\text{with } (t^\sharp, invs) := \text{ab-lfp}(Al_{A,\mathbb{S}}(S)(w))(b)(t^\sharp)(w)
 \end{aligned}$$

Abstract Semantics of arithmetic expressions

$$\mathcal{A}^\sharp : \mathbf{Aexp} \rightarrow \mathbb{S} \rightarrow A$$

$$\mathcal{A}^\sharp(n)(s^\sharp) := \alpha_{\text{singleton}}(n)$$

$$\mathcal{A}^\sharp(x)(s^\sharp) := \text{lookup}_{\mathbb{S}}(s^\sharp)(x)$$

$$\mathcal{A}^\sharp(e_1 \text{ aop } e_2)(s^\sharp) := \mathcal{A}^\sharp(e_1)(s^\sharp) \text{ aop}^A \mathcal{A}^\sharp(e_2)(s^\sharp)$$

$$\mathcal{A}^\sharp(-e)(s^\sharp) := -^A \mathcal{A}^\sharp(e)(s^\sharp)$$

Abstract Semantics of boolean expressions

$$\mathcal{B}^\sharp : \mathbf{Bexp} \rightarrow \mathbb{S} \rightarrow \mathbb{S}$$

$$\mathcal{B}^\sharp(\text{true})(s^\sharp) := s^\sharp$$

$$\mathcal{B}^\sharp(\text{false})(s^\sharp) := \perp_{\mathbb{S}}$$

$$\mathcal{B}^\sharp(e_1 \text{ bop } e_2)(s^\sharp) \text{ depends on the domain}$$

$$\mathcal{B}^\sharp(b_1 \wedge b_2)(s^\sharp) := (\mathcal{B}^\sharp(b_2) \circ \mathcal{B}^\sharp(b_1))(s^\sharp)$$

$$\mathcal{B}^\sharp(b_1 \vee b_2)(s^\sharp) := \mathcal{B}^\sharp(b_1)(s^\sharp) \vee_{\mathbb{S}} \mathcal{B}^\sharp(b_2)(s^\sharp)$$

Step

$$\begin{aligned} \text{step} : (\mathbb{S} \rightarrow \mathbb{S} \times \text{List}(\mathbb{S})) &\rightarrow \text{bool} \rightarrow \mathbb{S} \rightarrow \mathbb{S} \rightarrow \mathbb{S} \times \text{List}(\mathbb{S}) \\ \text{step}(f)(b)(s^\sharp)(t^\sharp) &:= (s^\sharp \vee_{\mathbb{S}} u^\sharp, \text{invs}) \\ \text{with } (u^\sharp, \text{invs}) &:= f(\mathcal{B}^\sharp(b)(t^\sharp)) \end{aligned}$$

Invariant Check

$$\begin{aligned} \text{is-inv} &: (\mathbb{S} \rightarrow \mathbb{S} \times \text{List}(\mathbb{S})) \rightarrow \mathbb{S} \rightarrow \mathbb{S} \rightarrow \text{bool} \\ \text{is-inv}(f)(s^\sharp)(t^\sharp) &:= t^\sharp \sqsubseteq_{\mathbb{S}} u^\sharp \\ &\text{with } u^\sharp := \pi_1(\text{step}(f)(b)(s^\sharp)(t^\sharp)) \end{aligned}$$

Steps

$\text{steps} : (\mathbb{S} \rightarrow \mathbb{S} \times \text{List}(\mathbb{S})) \rightarrow \text{bool} \rightarrow \mathbb{S} \rightarrow \mathbb{S} \rightarrow \mathbb{S} \times \text{List}(\mathbb{S})$

$$\text{steps}(f)(b)(s^\sharp)(t^\sharp) := \begin{cases} (t^\sharp, [t^\sharp]) & \text{if is-inv}(f)(s^\sharp)(t^\sharp) \\ (v^\sharp, \text{invs}_1 +_{\text{List}(\mathbb{S})} \text{invs}_2) & \text{otherwise} \end{cases}$$

with $(u^\sharp, \text{invs}_1) := \text{step}(f)(b)(s^\sharp)(t^\sharp)$

$(v^\sharp, \text{invs}_2) := \text{steps}(f)(b)(s^\sharp)(u^\sharp)$

Widening

$\text{wid} : (\mathbb{S} \rightarrow \mathbb{S} \times \text{List}(\mathbb{S})) \rightarrow \text{bool} \rightarrow \mathbb{S} \rightarrow \mathbb{S} \rightarrow \mathbb{S}$

$$\text{wid}(f)(b)(s^\sharp)(t^\sharp) := \begin{cases} t^\sharp & \text{if is-inv}(f)(s^\sharp)(t^\sharp) \\ \text{wid}(f)(b)(s^\sharp)(t^\sharp \nabla_{\mathbb{S}} u^\sharp) & \text{otherwise} \end{cases}$$

with $u^\sharp := \pi_1(\text{step}(f)(b)(s^\sharp)(t^\sharp))$

Narrowing

$\text{nar} : (\mathbb{S} \rightarrow \mathbb{S} \times \text{List}(\mathbb{S})) \rightarrow \text{bool} \rightarrow \mathbb{S} \rightarrow \mathbb{S} \rightarrow \mathbb{S} \times \text{List}(\mathbb{S})$

$$\text{nar}(f)(b)(s^\sharp)(t^\sharp) := \begin{cases} (v^\sharp, [v^\sharp]) & \text{if } \text{is-inv}(f)(s^\sharp)(v^\sharp) \\ (z^\sharp, \text{invs}_1 +_{\text{List}(\mathbb{S})} \text{invs}_2) & \text{otherwise} \end{cases}$$

with $u^\sharp := \pi_1(\text{step}(f)(b)(s^\sharp)(t^\sharp))$

$v^\sharp := t^\sharp \Delta_{\mathbb{S}} u^\sharp$

$(w^\sharp, \text{invs}_1) := \text{step}(f)(b)(s^\sharp)(v^\sharp)$

$(z^\sharp, \text{invs}_2) := \text{nar}(f)(b)(s^\sharp)(v^\sharp \Delta_{\mathbb{S}} w^\sharp)$

Abstract Least Fixed Point

$$\text{ab-lfp} : (\mathbb{S} \rightarrow \mathbb{S} \times \text{List}(\mathbb{S})) \rightarrow \text{bool} \rightarrow \mathbb{S} \rightarrow \text{bool} \rightarrow \mathbb{S} \times \text{List}(\mathbb{S})$$

$$\text{ab-lfp}(f)(b)(s^\sharp)(w) := \begin{cases} \text{nar}(f)(b)(s^\sharp)(t^\sharp) & \text{if } w \\ \text{steps}(f)(b)(s^\sharp)(s^\sharp) & \text{otherwise} \end{cases}$$

with $t^\sharp := \text{wid}(f)(b)(s^\sharp)(s^\sharp)$.

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Abstract State Update

$s^\# [x \mapsto a]$ defined by

$$\left\{ \begin{array}{l} \perp_{\mathbb{S}} [x \mapsto a] := [(x, a)] \\ \top_{\mathbb{S}} [x \mapsto a] := [(x, a)] \\ ((y, a') :: ts^\#) [x \mapsto a] := \begin{cases} (y, a') :: ts^\# & \text{if } x = y \\ (y, a') :: ts^\# [x \mapsto a] & \text{otherwise} \end{cases} \end{array} \right.$$

Abstract State Join

$s^\# \vee_{\mathbb{S}} t^\#$ defined by

$$\begin{cases} \perp_{\mathbb{S}} \vee_{\mathbb{S}} t^\# := t^\# \\ \top_{\mathbb{S}} \vee_{\mathbb{S}} t^\# := \top_{\mathbb{S}} \\ ((x, a) :: ts^\#) \vee_{\mathbb{S}} t^\# := (ts^\# \vee_{\mathbb{S}} t^\#)[x \mapsto a \vee_A \text{lookup}(t^\#)(x)] \end{cases}$$

Abstract State Lookup

$\text{lookup}_{\mathbb{S}}(s^{\sharp})(x)$ defined by

$$\begin{cases} \text{lookup}_{\mathbb{S}}(\perp_{\mathbb{S}})(x) := \perp_A \\ \text{lookup}_{\mathbb{S}}(\top_{\mathbb{S}})(x) := \top_A \\ \text{lookup}_{\mathbb{S}}((y, a) :: ts^{\sharp})(x) := \begin{cases} a & \text{if } x = y \\ \text{lookup}_{\mathbb{S}}(ts^{\sharp})(x) & \text{otherwise} \end{cases} \end{cases}$$

Partial order

$$a_1 \leq_A a_2 := (a_1 \vee_A a_2) = a_2$$

$$s^\# \sqsubseteq_{\mathbb{S}} t^\# := (s^\# = \perp_{\mathbb{S}}) \vee \forall x, \text{lookup}(s^\#)(x) \leq_A \text{lookup}(t^\#)(x)$$

State Widening

$$s^\sharp \nabla_{\mathbb{S}} t^\sharp := \begin{cases} t^\sharp & \text{if } s^\sharp = \perp_{\mathbb{S}} \\ \text{map}(f_{t^\sharp})(s^\sharp) & \text{otherwise} \end{cases}$$

with $f_{t^\sharp}(x, a) := (x, a \nabla \text{lookup}(t^\sharp)(x))$

State Narrowing

$$s^\# \Delta_{\mathbb{S}} t^\# := \begin{cases} \perp_{\mathbb{S}} & \text{if } s^\# = \perp_{\mathbb{S}} \\ \text{map}(f_{t^\#})(s^\#) & \text{otherwise} \end{cases}$$

with $f_{t^\#}(x, a) := (x, a \Delta \text{lookup}(t^\#)(x))$

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Extended Sign

$$A := \text{ExtSign}$$
$$\mathbb{S} := \text{List}(\text{String} \times \text{ExtSign}) \cup \{\star\}$$
$$\perp_{\mathbb{S}} := \star$$
$$\top_{\mathbb{S}} := []$$
$$\text{ExtSign} \ni a ::= \perp \mid < 0 \mid = 0 \mid > 0 \mid \leq 0 \mid \neq 0 \mid \geq 0 \mid \top$$

α on Singletons

$$\alpha_{\text{singleton}}(n) := \begin{cases} = 0 & \text{if } n = 0 \\ < 0 & \text{if } n < 0 \\ > 0 & \text{otherwise} \end{cases}$$

Opposite

$-$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
	\perp	> 0	$= 0$	< 0	≥ 0	$\neq 0$	≤ 0	\top

Addition

$+$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	\perp							
< 0	\perp	< 0						
$= 0$	\perp	< 0	$= 0$					
> 0	\perp	\top	> 0	> 0				
≤ 0	\perp	< 0	≤ 0	\top	≤ 0			
$\neq 0$	\perp	\top	$\neq 0$	\top	\top	\top		
≥ 0	\perp	\top	≥ 0	> 0	\top	\top	≥ 0	
\top	\perp	\top	\top	\top	\top	\top	\top	\top

Subtraction

$-$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	\perp	\perp	\perp	\perp	\perp	\perp	\perp	\perp
< 0	\perp	\top	> 0	> 0	\top	\top	> 0	\top
$= 0$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
> 0	\perp	< 0	< 0	\top	< 0	\top	\top	\top
≤ 0	\perp	\top	≥ 0	> 0	\top	\top	≥ 0	\top
$\neq 0$	\perp	\top	$\neq 0$	\top	\top	\top	\top	\top
≥ 0	\perp	< 0	≤ 0	\top	≤ 0	\top	\top	\top
\top	\perp	\top	\top	\top	\top	\top	\top	\top

Multiplication

*	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	\perp							
< 0	\perp	> 0						
$= 0$	\perp	$= 0$	$= 0$					
> 0	\perp	< 0	$= 0$	> 0				
≤ 0	\perp	≥ 0	$= 0$	≤ 0	≥ 0			
$\neq 0$	\perp	$\neq 0$	$= 0$	$\neq 0$	\top	$\neq 0$		
≥ 0	\perp	≤ 0	$= 0$	≥ 0	≤ 0	\top	≥ 0	
\top	\perp	\top	$= 0$	\top	\top	\top	\top	\top

Division

$/$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	\perp	\perp	\perp	\perp	\perp	\perp	\perp	\perp
< 0	\perp	> 0	$= 0$	< 0	≥ 0	$\neq 0$	≤ 0	\top
$= 0$	\perp	\perp	\perp	\perp	\perp	\perp	\perp	\perp
> 0	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
≤ 0	\perp	> 0	$= 0$	< 0	≥ 0	$\neq 0$	≤ 0	\top
$\neq 0$	\perp	$\neq 0$	$= 0$	$\neq 0$	\top	$\neq 0$	\top	\top
≥ 0	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\top	\perp	$\neq 0$	$= 0$	$\neq 0$	\top	$\neq 0$	\top	\top

Equal

$e_1 = e_2$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	$\perp_{\mathbb{S}}$							
< 0	$\perp_{\mathbb{S}}$	$s^{\#}$						
$= 0$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}$					
> 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}$				
≤ 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$\perp_{\mathbb{S}}$	$s^{\#}$			
$\neq 0$	$\perp_{\mathbb{S}}$	$s^{\#}$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$		
≥ 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	
\top	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Equal

$x = e$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	\perp_s	\perp_s	\perp_s	\perp_s	\perp_s	\perp_s	\perp_s	\perp_s
< 0	\perp_s	$s^\#$	\perp_s	\perp_s	$s^\#[x \mapsto < 0]$	$s^\#[x \mapsto < 0]$	\perp_s	$s^\#[x \mapsto < 0]$
$= 0$	\perp_s	\perp_s	$s^\#$	\perp_s	$s^\#[x \mapsto = 0]$	\perp_s	$s^\#[x \mapsto = 0]$	$s^\#[x \mapsto = 0]$
> 0	\perp_s	\perp_s	\perp_s	$s^\#$	\perp_s	$s^\#[x \mapsto > 0]$	$s^\#[x \mapsto > 0]$	$s^\#[x \mapsto > 0]$
≤ 0	\perp_s	$s^\#$	$s^\#$	\perp_s	$s^\#$	$s^\#[x \mapsto < 0]$	$s^\#[x \mapsto = 0]$	$s^\#[x \mapsto \leq 0]$
$\neq 0$	\perp_s	$s^\#$	\perp_s	$s^\#$	$s^\#[x \mapsto < 0]$	$s^\#$	$s^\#[x \mapsto > 0]$	$s^\#[x \mapsto \neq 0]$
≥ 0	\perp_s	\perp_s	$s^\#$	$s^\#$	$s^\#[x \mapsto = 0]$	$s^\#[x \mapsto > 0]$	$s^\#$	$s^\#[x \mapsto \geq 0]$
\top	\perp_s	$s^\#$	$s^\#$	$s^\#$	$s^\#$	$s^\#$	$s^\#$	$s^\#$

Not Equal

$e_1 \neq e_2$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	$\perp_{\mathbb{S}}$							
< 0	$\perp_{\mathbb{S}}$	$s^{\#}$						
$= 0$	$\perp_{\mathbb{S}}$	$s^{\#}$	$\perp_{\mathbb{S}}$					
> 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$				
≤ 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$			
$\neq 0$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$		
≥ 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	
\top	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Not Equal

$x \neq e$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$
< 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$= 0$	$\perp_{\mathcal{S}}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}[x \mapsto < 0]$	$s^{\#}$	$s^{\#}[x \mapsto > 0]$	$s^{\#}[x \mapsto \neq 0]$
> 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
≤ 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$\neq 0$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
≥ 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
\top	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Less Than

$e_1 < e_2$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$
< 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$s^{\#}$
$= 0$	$\perp_{\mathcal{S}}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$s^{\#}$
> 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
≤ 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$s^{\#}$
$\neq 0$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
≥ 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
\top	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Less Than

$x < e$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
< 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}[x \mapsto < 0]$	$s^{\#}[x \mapsto < 0]$	$\perp_{\mathbb{S}}$	$s^{\#}[x \mapsto < 0]$
$= 0$	$\perp_{\mathbb{S}}$	$s^{\#}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}[x \mapsto < 0]$	$s^{\#}[x \mapsto < 0]$	$\perp_{\mathbb{S}}$	$s^{\#}[x \mapsto < 0]$
> 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
≤ 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}[x \mapsto < 0]$	$s^{\#}[x \mapsto < 0]$	$\perp_{\mathbb{S}}$	$s^{\#}[x \mapsto < 0]$
$\neq 0$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
≥ 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
\top	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Greater Than

$e_1 > e_2$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
< 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$= 0$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
> 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
≤ 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$\neq 0$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
≥ 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
\top	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Greater Than

$x > e$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$
< 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$= 0$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$s^{\#}[x \mapsto > 0]$	$s^{\#}[x \mapsto > 0]$	$s^{\#}[x \mapsto > 0]$
> 0	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$s^{\#}[x \mapsto > 0]$	$s^{\#}[x \mapsto > 0]$	$s^{\#}[x \mapsto > 0]$
≤ 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$\neq 0$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
≥ 0	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$s^{\#}[x \mapsto > 0]$	$s^{\#}[x \mapsto > 0]$	$s^{\#}[x \mapsto > 0]$
\top	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Less Than or Equal

$e_1 \leq e_2$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$
< 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$s^{\#}$
$= 0$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
> 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
≤ 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$\neq 0$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
≥ 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
\top	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Less Than or Equal

$x \leq e$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	\perp_s	\perp_s	\perp_s	\perp_s	\perp_s	\perp_s	\perp_s	\perp_s
< 0	\perp_s	$s^\#$	\perp_s	\perp_s	$s^\#[x \mapsto < 0]$	$s^\#[x \mapsto < 0]$	\perp_s	$s^\#[x \mapsto < 0]$
$= 0$	\perp_s	$s^\#$	$s^\#$	\perp_s	$s^\#$	$s^\#[x \mapsto < 0]$	$s^\#[x \mapsto = 0]$	$s^\#[x \mapsto \leq 0]$
> 0	\perp_s	$s^\#$	$s^\#$	$s^\#$	$s^\#$	$s^\#$	$s^\#$	$s^\#$
≤ 0	\perp_s	$s^\#$	$s^\#$	\perp_s	$s^\#$	$s^\#[x \mapsto < 0]$	$s^\#[x \mapsto = 0]$	$s^\#[x \mapsto \leq 0]$
$\neq 0$	\perp_s	$s^\#$	$s^\#$	$s^\#$	$s^\#$	$s^\#$	$s^\#$	$s^\#$
≥ 0	\perp_s	$s^\#$	$s^\#$	$s^\#$	$s^\#$	$s^\#$	$s^\#$	$s^\#$
\top	\perp_s	$s^\#$	$s^\#$	$s^\#$	$s^\#$	$s^\#$	$s^\#$	$s^\#$

Greater Than or Equal

$e_1 \geq e_2$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$
< 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$= 0$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
> 0	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
≤ 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$\neq 0$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
≥ 0	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
\top	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Greater Than or Equal

$x \geq e$	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$
< 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$= 0$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}[x \mapsto = 0]$	$s^{\#}[x \mapsto > 0]$	$s^{\#}$	$s^{\#}[x \mapsto \geq 0]$
> 0	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$\perp_{\mathcal{S}}$	$s^{\#}[x \mapsto > 0]$	$s^{\#}[x \mapsto > 0]$	$s^{\#}[x \mapsto > 0]$
≤ 0	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$\neq 0$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
≥ 0	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}[x \mapsto = 0]$	$s^{\#}[x \mapsto > 0]$	$s^{\#}$	$s^{\#}[x \mapsto \geq 0]$
\top	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Join

\vee	\perp	< 0	$= 0$	> 0	≤ 0	$\neq 0$	≥ 0	\top
\perp	\perp							
< 0	< 0	< 0						
$= 0$	$= 0$	≤ 0	$= 0$					
> 0	> 0	≥ 0	≥ 0	> 0				
≤ 0	≤ 0	≤ 0	≤ 0	\top	≤ 0			
$\neq 0$	$\neq 0$	$\neq 0$	\top	$\neq 0$	\top	$\neq 0$		
≥ 0	≥ 0	\top	≥ 0	≥ 0	\top	\top	≥ 0	
\top	\top	\top	\top	\top	\top	\top	\top	\top

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Intervals

$$A := \text{Int}$$
$$\mathbb{S} := \text{List}(\text{String} \times \text{Int}) \cup \{\star\}$$
$$\perp_{\mathbb{S}} := \star$$
$$\top_{\mathbb{S}} := []$$
$$\text{Int} \ni c ::= \perp \mid (-\infty, b] \mid [a, b] \mid [a, +\infty) \mid \top$$

α on Singletons

$$\alpha_{\text{singleton}}(n) := [n, n]$$

Opposite

$-$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
	\perp	$[-b, +\infty)$	$[-b, -a]$	$(-\infty, -a]$	\top

Addition

$+$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	\perp				
$(-\infty, d]$	\perp	$(-\infty, b + d]$			
$[c, d]$	\perp	$(-\infty, b + d]$	$[a + c, b + d]$		
$[c, +\infty)$	\perp	\top	$[a + c, +\infty)$	$[a + c, +\infty)$	
\top	\perp	\top	\top	\top	\top

Subtraction

$-$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	\perp	\perp	\perp	\perp	\perp
$(-\infty, d]$	\perp	\top	$[a - d, +\infty)$	$[a - d, +\infty)$	\top
$[c, d]$	\perp	$(-\infty, b - c]$	$[a - d, b - c]$	$[a - d, +\infty)$	\top
$[c, +\infty)$	\perp	$(-\infty, b - c]$	$(-\infty, b - c]$	\top	\top
\top	\perp	\top	\top	\top	\top

Multiplication

$*$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	\perp		\perp	\perp	\perp
$(-\infty, d]$	\perp	$b > 0 \vee d > 0 \quad \top$ $b \leq 0 \wedge d \leq 0 \quad [bd, +\infty)$	$(a < 0 \wedge b > 0) \vee (a > 0 \wedge b < 0) \quad \top$ $(a \leq 0 \wedge b \leq 0) \wedge (a \neq 0 \vee b \neq 0) \quad [\min(ad, bd), +\infty)$ $(a \geq 0 \wedge b \geq 0) \wedge (a \neq 0 \vee b \neq 0) \quad (-\infty, \max(ad, bd)]$ $(a = 0 \wedge b = 0) \quad [0, 0]$	$d > 0 \vee a < 0 \quad \top$ $d \leq 0 \wedge a \geq 0 \quad (-\infty, ad]$	\top
$[c, d]$	\perp	$(c < 0 \wedge d > 0) \vee (c > 0 \wedge d < 0) \quad \top$ $(c \leq 0 \wedge d \leq 0) \wedge (c \neq 0 \vee d \neq 0) \quad [\min(bc, bd), +\infty)$ $(c \geq 0 \wedge d \geq 0) \wedge (c \neq 0 \vee d \neq 0) \quad (-\infty, \max(bc, bd)]$ $(c = 0 \wedge d = 0) \quad [0, 0]$	$[\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$	$(c < 0 \wedge d > 0) \vee (c > 0 \wedge d < 0) \quad \top$ $(c \leq 0 \wedge d \leq 0) \wedge (c \neq 0 \vee d \neq 0) \quad (-\infty, \max(ac, ad)]$ $(c \geq 0 \wedge d \geq 0) \wedge (c \neq 0 \vee d \neq 0) \quad [\min(ac, ad), +\infty)$ $(c = 0 \wedge d = 0) \quad [0, 0]$	$c = d = 0 \quad [0, 0]$ $c \neq 0 \vee d \neq 0 \quad \top$
$[c, +\infty)$	\perp	$b > 0 \vee c < 0 \quad \top$ $b \leq 0 \wedge c \geq 0 \quad (-\infty, bc]$	$(a < 0 \wedge b > 0) \vee (a > 0 \wedge b < 0) \quad \top$ $(a \leq 0 \wedge b \leq 0) \wedge (a \neq 0 \vee b \neq 0) \quad (-\infty, \max(ac, bc)]$ $(a \geq 0 \wedge b \geq 0) \wedge (a \neq 0 \vee b \neq 0) \quad [\min(ac, bc), +\infty)$ $(a = 0 \wedge b = 0) \quad [0, 0]$	$a < 0 \vee c < 0 \quad \top$ $a \geq 0 \wedge c \geq 0 \quad (-\infty, ac]$	\top
\top	\perp	\top	$a = b = 0 \quad [0, 0]$ $a \neq 0 \vee b \neq 0 \quad \top$	\top	\top

Division

/	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	\perp	\perp	\perp	\perp	\perp
$(-\infty, d]$	\perp	$d < 0$ $[\min(0, b/d), +\infty)$ $d = 0$ $(-\infty, b]/(-\infty, -1]$ otherwise \top	$d < 0$ $[\min(0, b/d), \max(0, a/d)]$ $d = 0$ $[a, b]/(-\infty, -1]$ otherwise $[a, b]/(-\infty, -1] \vee [a, b]/[1, d]$	$d < 0$ $(-\infty, \max(a/d, 0))$ $d = 0$ $[a, +\infty)/(-\infty, -1]$ otherwise \top	\top
$[c, d]$	\perp	$c = d = 0$ \perp $0 < c \leq d$ $(-\infty, \max(b/c, b/d))$ $0 = c < d$ $(-\infty, b]/[1, d]$ $c \leq d < 0$ $[\min(b/c, b/d), +\infty)$ $c < d = 0$ $(-\infty, b]/[c, -1]$ otherwise \top	$c = d = 0$ \perp $0 < c \leq d \vee c \leq d < 0$ $[\min(a/c, a/d, b/c, b/d), \max(a/c, a/d, b/c, b/d)]$ $0 = c < d$ $[a, b]/[1, d]$ $c < d = 0$ $[a, b]/[c, -1]$ otherwise $[a, b]/[c, -1] \vee [a, b]/[1, d]$	$c = d = 0$ \perp $0 < c \leq d$ $[\min(a/c, a/d), +\infty)$ $0 = c < d$ $[a, +\infty)/[1, d]$ $c \leq d < 0$ $(-\infty, \max(a/c, a/d))$ $c < d = 0$ $[a, +\infty)/[c, -1]$ otherwise \top	$c = d = 0$ \perp otherwise \top
$[c, +\infty)$	\perp	$c > 0$ $(-\infty, \max(0, b/c))$ $c = 0$ $(-\infty, b]/[1, +\infty)$ otherwise \top	$c > 0$ $[\min(0, a/c), \max(0, b/c)]$ $c = 0$ $[a, b]/[1, +\infty)$ otherwise $[a, b]/[c, -1] \vee [a, b]/[1, +\infty)$	$c > 0$ $(\min(a/c, 0), +\infty)$ $c = 0$ $[a, +\infty)/[1, +\infty)$ otherwise \top	\top
\top	\perp	\top	$[a, b]/(-\infty, -1] \vee [a, b]/[1, +\infty)$	\top	\top

Equal

$e_1 = e_2$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
$(-\infty, d]$	$\perp_{\mathbb{S}}$	$s^{\#}$	if $a > d$ then $\perp_{\mathbb{S}}$ else $s^{\#}$	if $a > d$ then $\perp_{\mathbb{S}}$ else $s^{\#}$	$s^{\#}$
$[c, d]$	$\perp_{\mathbb{S}}$	if $b < c$ then $\perp_{\mathbb{S}}$ else $s^{\#}$	if $a > d$ or $b < c$ then $\perp_{\mathbb{S}}$ else $s^{\#}$	if $a > d$ then $\perp_{\mathbb{S}}$ else $s^{\#}$	$s^{\#}$
$[c, +\infty)$	$\perp_{\mathbb{S}}$	if $b < c$ then $\perp_{\mathbb{S}}$ else $s^{\#}$	if $b < c$ then $\perp_{\mathbb{S}}$ else $s^{\#}$	$s^{\#}$	$s^{\#}$
\top	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Equal

$x = e$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	\perp_S	\perp_S	\perp_S	\perp_S	\perp_S
$(-\infty, d]$	\perp_S	$b \leq d \quad s^\#$ $b > d \quad s^\#[x \mapsto (-\infty, d]]$	$a > d \quad \perp_S$ $a \leq d \wedge b > d \quad s^\#[x \mapsto [a, d]]$ $a \leq d \wedge b \leq d \quad s^\#$	$a > d \quad \perp_S$ $a < d \quad s^\#[x \mapsto [a, d]]$	$s^\#[x \mapsto (-\infty, d]]$
$[c, d]$	\perp_S	$b < c \quad \perp_S$ $c \leq b \leq d \quad s^\#[x \mapsto [c, b]]$ $b > d \quad s^\#[x \mapsto [c, d]]$	$b < c \vee a > d \quad \perp_S$ $b > d \wedge a < c \quad s^\#[x \mapsto [c, d]]$ $b > d \wedge a \geq c \quad s^\#[x \mapsto [a, d]]$ $c \leq b \leq d \wedge a < c \quad s^\#[x \mapsto [c, b]]$ $c \leq b \leq d \wedge c \leq a \leq d \quad s^\#$	$a > d \quad \perp_S$ $a < c \quad s^\#[x \mapsto [c, d]]$ $c \leq a \leq d \quad s^\#[x \mapsto [a, d]]$	$s^\#[x \mapsto [c, d]]$
$[c, +\infty)$	\perp_S	$b < c \quad \perp_S$ $b \geq c \quad s^\#[x \mapsto [c, b]]$	$b < c \quad \perp_S$ $b \geq c \wedge a < c \quad s^\#[x \mapsto [c, b]]$ $a \geq c \quad s^\#$	$a < c \quad s^\#[x \mapsto [c, +\infty)]$ $a \geq c \quad s^\#$	$s^\#[x \mapsto [c, +\infty)]$
\top	\perp_S	$s^\#$	$s^\#$	$s^\#$	$s^\#$

Not Equal

$e_1 \neq e_2$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
$(-\infty, d]$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$[c, d]$	$\perp_{\mathbb{S}}$	$s^{\#}$	$a = b = c = d \quad \perp_{\mathbb{S}}$ otherwise $s^{\#}$	$s^{\#}$	$s^{\#}$
$[c, +\infty)$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
\top	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Not Equal

$x \neq e$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	\perp_S	\perp_S	\perp_S	\perp_S	\perp_S
$(-\infty, d]$	\perp_S	$s^\#$	$s^\#$	$s^\#$	$s^\#$
$[c, d]$	\perp_S	$b = c = d \quad s^\#[x \mapsto (-\infty, b - 1]]$ otherwise $s^\#$	$a = c = d \wedge a \neq b \quad s^\#[x \mapsto [a + 1, b]]$ $b = c = d \wedge a \neq b \quad s^\#[x \mapsto [a, b - 1]]$ $a = b = c = d \quad \perp_S$ otherwise $s^\#$	$a = c = d \quad s^\#[x \mapsto [a + 1, +\infty))$ otherwise $s^\#$	$s^\#$
$[c, +\infty)$	\perp_S	$s^\#$	$s^\#$	$s^\#$	$s^\#$
\top	\perp_S	$s^\#$	$s^\#$	$s^\#$	$s^\#$

Less Than

$e_1 < e_2$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
$(-\infty, d]$	$\perp_{\mathbb{S}}$	$s^{\#}$	$a \geq d \quad \perp_{\mathbb{S}}$ $a < d \quad s^{\#}$	$a \geq d \quad \perp_{\mathbb{S}}$ $a < d \quad s^{\#}$	$s^{\#}$
$[c, d]$	$\perp_{\mathbb{S}}$	$s^{\#}$	$a \geq d \quad \perp_{\mathbb{S}}$ $a < d \quad s^{\#}$	$a \geq d \quad \perp_{\mathbb{S}}$ $a < d \quad s^{\#}$	$s^{\#}$
$[c, +\infty)$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
\top	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Less Than

$x < e$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
$(-\infty, d]$	$\perp_{\mathbb{S}}$	$b \geq d \quad s^{\#}[x \mapsto (-\infty, d-1]]$ $b < d \quad s^{\#}$	$a \geq d \quad \perp_{\mathbb{S}}$ $a < d \wedge b \geq d \quad s^{\#}[x \mapsto [a, d-1]]$ $a < d \wedge b < d \quad s^{\#}$	$a \geq d \quad \perp_{\mathbb{S}}$ $a < d \quad s^{\#}[x \mapsto [a, d-1]]$	$s^{\#}[x \mapsto (-\infty, d-1]]$
$[c, d]$	$\perp_{\mathbb{S}}$	$b \geq d \quad s^{\#}[x \mapsto (-\infty, d-1]]$ $b < d \quad s^{\#}$	$a \geq d \quad \perp_{\mathbb{S}}$ $a < d \wedge b \geq d \quad s^{\#}[x \mapsto [a, d-1]]$ $a < d \wedge b < d \quad s^{\#}$	$a \geq d \quad \perp_{\mathbb{S}}$ $a < d \quad s^{\#}[x \mapsto [a, d-1]]$	$s^{\#}[x \mapsto (-\infty, d-1]]$
$[c, +\infty)$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
\top	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Less Than or Equal

$e_1 \leq e_2$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
$(-\infty, d]$	$\perp_{\mathbb{S}}$	$s^{\#}$	$a > d \quad \perp_{\mathbb{S}}$ $a \leq d \quad s^{\#}$	$a > d \quad \perp_{\mathbb{S}}$ $a \leq d \quad s^{\#}$	$s^{\#}$
$[c, d]$	$\perp_{\mathbb{S}}$	$s^{\#}$	$a > d \quad \perp_{\mathbb{S}}$ $a \leq d \quad s^{\#}$	$a > d \quad \perp_{\mathbb{S}}$ $a \leq d \quad s^{\#}$	$s^{\#}$
$[c, +\infty)$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
\top	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Less Than or Equal

$x \leq e$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$
$(-\infty, d]$	$\perp_{\mathcal{S}}$	$b > d \quad s^{\#}[x \mapsto (-\infty, d]]$ $b \leq d \quad s^{\#}$	$a > d \quad \perp_{\mathcal{S}}$ $a \leq d \wedge b > d \quad s^{\#}[x \mapsto [a, d]]$ $a \leq d \wedge b \leq d \quad s^{\#}$	$a > d \quad \perp_{\mathcal{S}}$ $a \leq d \quad s^{\#}[x \mapsto [a, d]]$	$s^{\#}[x \mapsto (-\infty, d]]$
$[c, d]$	$\perp_{\mathcal{S}}$	$b > d \quad s^{\#}[x \mapsto (-\infty, d]]$ $b \leq d \quad s^{\#}$	$a > d \quad \perp_{\mathcal{S}}$ $a \leq d \wedge b > d \quad s^{\#}[x \mapsto [a, d]]$ $a \leq d \wedge b \leq d \quad s^{\#}$	$a > d \quad \perp_{\mathcal{S}}$ $a \leq d \quad s^{\#}[x \mapsto [a, d]]$	$s^{\#}[x \mapsto (-\infty, d]]$
$[c, +\infty)$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
\top	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Greater Than

$e_1 > e_2$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
$(-\infty, d]$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$[c, d]$	$\perp_{\mathbb{S}}$	$b \leq c \quad \perp_{\mathbb{S}}$ $b > c \quad s^{\#}$	$b \leq c \quad \perp_{\mathbb{S}}$ $b > c \quad s^{\#}$	$s^{\#}$	$s^{\#}$
$[c, +\infty)$	$\perp_{\mathbb{S}}$	$b \leq c \quad \perp_{\mathbb{S}}$ $b > c \quad s^{\#}$	$b \leq c \quad \perp_{\mathbb{S}}$ $b > c \quad s^{\#}$	$s^{\#}$	$s^{\#}$
\top	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Greater Than

$x > e$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$
$(-\infty, d]$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$[c, d]$	$\perp_{\mathcal{S}}$	$b \leq c \quad \perp_{\mathcal{S}}$ $b > c \quad s^{\#}[x \mapsto [c+1, b]]$	$b \leq c \quad \perp_{\mathcal{S}}$ $b > c \wedge a \leq c \quad s^{\#}[x \mapsto [c+1, b]]$ $b > c \wedge a > c \quad s^{\#}$	$a \leq c \quad s^{\#}[x \mapsto [c+1, +\infty)]$ $a > c \quad s^{\#}$	$s^{\#}[x \mapsto [c+1, +\infty)]$
$[c, +\infty)$	$\perp_{\mathcal{S}}$	$b \leq c \quad \perp_{\mathcal{S}}$ $b > c \quad s^{\#}[x \mapsto [c+1, b]]$	$b \leq c \quad \perp_{\mathcal{S}}$ $b > c \wedge a \leq c \quad s^{\#}[x \mapsto [c+1, b]]$ $b > c \wedge a > c \quad s^{\#}$	$a \leq c \quad s^{\#}[x \mapsto [c+1, +\infty)]$ $a > c \quad s^{\#}$	$s^{\#}[x \mapsto [c+1, +\infty)]$
\top	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Greater Than or Equal

$e_1 \geq e_2$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
$(-\infty, d]$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$[c, d]$	$\perp_{\mathbb{S}}$	$b < c \quad \perp_{\mathbb{S}}$ $b \geq c \quad s^{\#}$	$b < c \quad \perp_{\mathbb{S}}$ $b \geq c \quad s^{\#}$	$s^{\#}$	$s^{\#}$
$[c, +\infty)$	$\perp_{\mathbb{S}}$	$b < c \quad \perp_{\mathbb{S}}$ $b \geq c \quad s^{\#}$	$b < c \quad \perp_{\mathbb{S}}$ $b \geq c \quad s^{\#}$	$s^{\#}$	$s^{\#}$
\top	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Greater Than or Equal

$x \geq e$	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$	$\perp_{\mathcal{S}}$
$(-\infty, d]$	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
$[c, d]$	$\perp_{\mathcal{S}}$	$b < c \quad \perp_{\mathcal{S}}$ $b \geq c \quad s^{\#}[x \mapsto [c, b]]$	$b < c \quad \perp_{\mathcal{S}}$ $b \geq c \wedge a < c \quad s^{\#}[x \mapsto [c, b]]$ $b \geq c \wedge a \geq c \quad s^{\#}$	$a < c \quad s^{\#}[x \mapsto [c, +\infty))$ $a \geq c \quad s^{\#}$	$s^{\#}[x \mapsto [c, +\infty))$
$[c, +\infty)$	$\perp_{\mathcal{S}}$	$b < c \quad \perp_{\mathcal{S}}$ $b \geq c \quad s^{\#}[x \mapsto [c, b]]$	$b < c \quad \perp_{\mathcal{S}}$ $b \geq c \wedge a < c \quad s^{\#}[x \mapsto [c, b]]$ $b \geq c \wedge a \geq c \quad s^{\#}$	$a < c \quad s^{\#}[x \mapsto [c, +\infty))$ $a \geq c \quad s^{\#}$	$s^{\#}[x \mapsto [c, +\infty))$
\top	$\perp_{\mathcal{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$

Join

\top	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
$(-\infty, d]$	$(-\infty, d]$	$(-\infty, \max(b, d)]$	$(-\infty, \max(b, d)]$	\top	\top
$[c, d]$	$[c, d]$	$(-\infty, \max(b, d)]$	$[\min(a, c), \max(b, d)]$	$[\min(a, c), +\infty)$	\top
$[c, +\infty)$	$[c, +\infty)$	\top	$[\min(a, c), +\infty)$	$[\min(a, c), +\infty)$	\top
\top	\top	\top	\top	\top	\top

Widen

∇	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
$(-\infty, d]$	$(-\infty, d]$	$d \leq b \quad (-\infty, b]$ otherwise \top	$d \leq b \quad (-\infty, b]$ otherwise \top	\top	\top
$[c, d]$	$[c, d]$	$d \leq b \quad (-\infty, b]$ otherwise \top	$a \leq c \wedge d \leq b \quad [a, b]$ $a \leq c \wedge d > b \quad [a, +\infty)$ $a > c \wedge d \leq b \quad (-\infty, b]$ otherwise \top	$a \leq c \quad [a, +\infty)$ otherwise \top	\top \top
$[c, +\infty)$	$[c, +\infty)$	\top	$a \leq c \quad [a, +\infty)$ otherwise \top	$a \leq c \quad [a, +\infty)$ otherwise \top	\top
\top	\top	\top	\top	\top	\top

Narrow

Δ	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top
\perp	\perp	\perp	\perp	\perp	\perp
$(-\infty, d]$	\perp	$(-\infty, b]$	$[a, b]$	$[a, d]$	$(-\infty, d]$
$[c, d]$	\perp	$[c, b]$	$[a, b]$	$[a, d]$	$[c, d]$
$[c, +\infty)$	\perp	$[c, b]$	$[a, b]$	$[a, +\infty)$	$[c, +\infty)$
\top	\perp	$(-\infty, b]$	$[a, b]$	$[a, +\infty)$	\top

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Example 1

$P := \text{while } x \neq 0 \text{ do } x := x + 1$

$AI_{ExtSign}(P)(false)([(x, < 0)]) = ([(x, = 0)], [(x, \top)])$

$AI_{ExtSign}(P)(false)([(x, = 0)]) = ([(x, = 0)], [(x, = 0)])$

$AI_{ExtSign}(P)(false)([(x, > 0)]) = (\perp_{\mathbb{S}}, [(x, > 0)])$

Example 2

$$P := x := x + y; y := y + 1$$
$$AI_{ExtSign}(P)(false)([(x, \leq 0), (y, < 0)]) =([(x, < 0), (y, \top)], [])$$

Example 3

$P := x := 40; \textbf{while } x \neq 0 \textbf{ do } x := x - 1$

$AI_{ExtSign}(P)(false)(\top_{\mathbb{S}}) = ([x, = 0], [\top_{\mathbb{S}}])$ in 1 iteration

$AI_{Int}(P)(false)(\top_{\mathbb{S}}) = ([x, [0, 0]], [[x, [0, 40]]])$ in 40 iterations

$AI_{Int}(P)(true)(\top_{\mathbb{S}}) = ([x, [0, 0]], [[x, (-\infty, 40]]])$ in $1 + 1$ iterations

Example 4

$P := \text{while } x \geq 0 \text{ do } (x := x - 1; y := y + 1)$

$AI_{Int}(P)(false)((x, [10, 10]), (y, [0, 0]))$ loops

$AI_{Int}(P)(true)((x, [10, 10]), (y, [0, 0])) =$
 $((x, [-1, -1]), (y, [0, +\infty))), [(x, [-1, 10]), (y, [0, +\infty))])$
 in $1 + 1$ iterations

Example 5

$P := \text{while } x < 10 \text{ do } x := x + 1$

$AI_{Int}(P)(false)((x, [0, 0])) =$
 $((x, [10, 10]), [(x, [0, 10])])$ in 10 iterations

$AI_{Int}(P)(true)((x, [0, 0])) =$
 $((x, [10, 10]), [(x, [0, 10])])$ in $1 + 1$ iterations

Example 6

$P := \text{while } x \leq 100 \text{ do } x := x + 1$

$AI_{Int}(P)(false)((x, [1, 1])) =$
 $((x, [101, 101]), [(x, [1, 101])])$ in 101 iterations

$AI_{Int}(P)(true)((x, [1, 1])) =$
 $((x, [101, 101]), [(x, [1, 101])])$ in $1 + 1$ iterations

Example 7

$P := x := 0; \textbf{while } x < 40 \textbf{ do } x := x + 1$

$AI_{Int}(P)(false)(\top_{\mathbb{S}}) = ([(x, [40, 40])], [(x, [0, 40])])$ in 40 iterations

$AI_{Int}(P)(true)(\top_{\mathbb{S}}) = ([(x, [40, 40])], [(x, [0, 40])])$ in $1 + 1$ iterations

Example 8

```

P := x := 0;
    while 1 = 1 do
        (if y = 0 then
            (x := x + 1; if x < 40 then x := 0 else skip)
        else skip)

```

$AI_{Int}(P)(false)([(y, [0, 1])]) = (\perp_{\mathbb{S}}, [[(x, [0, 40])]])$ in 40 iterations
 $AI_{Int}(P)(true)([(y, [0, 1])]) = (\perp_{\mathbb{S}}, [[(x, [0, +\infty))]])$ in $1 + 1$ iterations

Example 9

```

P := i := 1;
      while i ≤ 3 do
        (j := 1;
          while j ≤ i do j := j + 1;
          i := i + 1)
AIInt(P)(false)( $\top_{\mathbb{S}}$ ) = ([(i, [4, 4])],
  [[(i, [1, 1]), (j, [1, 2])], [(i, [1, 2]), (j, [1, 3])],
  [(i, [1, 3]), (j, [1, 4])], [(i, [1, 4])]])
AIInt(P)(true)( $\top_{\mathbb{S}}$ ) = ([(i, [4, 4])], [(i, [1, 4])])

```

Example 10

```
 $P :=$   $i := 1;$   
  while  $i \leq 4$  do  
     $(j := 0;$   
      while  $j \leq 3$  do  
         $(k := 0;$   
          while  $k \leq 5$  do  $(z := i * j * k; k := k + 1);$   
           $j := j + 1)$   
         $i := i + 1)$ 
```

$$AI_{Int}(P)(false)(\top_{\mathbb{S}}) = ([(i, [5, 5])], [..., [(i, [1, 5])]])$$

$$AI_{Int}(P)(true)(\top_{\mathbb{S}}) = ([(i, [5, 5])], [(i, [1, 5])])$$

Example 11

$P := x := 1/0; \mathbf{while} \ x \leq 5 \ \mathbf{do} \ \mathbf{skip}$

$AI_{Int}(P)(false)(\top_{\mathbb{S}}) = (\perp_{\mathbb{S}}, [(x, \perp)])$

Example 12

$P := \text{while } 1/0 < 1 \text{ do skip}$
 $AI_{Int}(P)(false)(\top_S) = (\perp_S, [])$