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UNIVERSITÀ di VERONA

Dipartimento di INFORMATICA

Principles and Applications of Abstract Interpretation

Static Analysis

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Verona, IT

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UNIVERSITÀ di VERONA
Dipartimento di INFORMATICA

Static Analysis by Abstract Interpretation

Ingredients for a sound static analysis

Start from the denotation of program executions (e.g., sequences of states)

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Terminating Static Analysis

Automated proof

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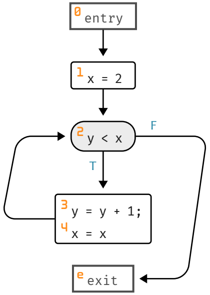
Compute an invariant on program variables at a program point

```
1 x = 2;
2 while (y < x) {
    3 y = y + 1;
    4 x = x
  }
```



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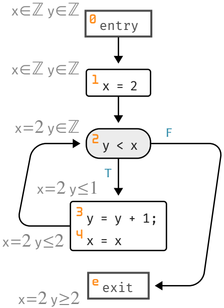

Compute an invariant on program variables at a program point

- To check the (un)reachability of unsafe states

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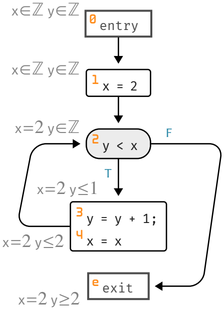
```

$@\{x{:}\{2\}, y{:}\{z \in \mathbb{Z} \mid z \geq 2\}\}$



Compute an invariant on program variables at a program point

- To check the (un)reachability of unsafe states
- To optimize the code

```
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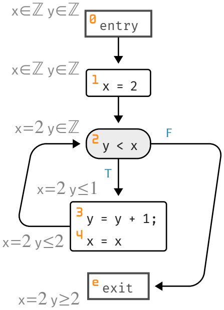
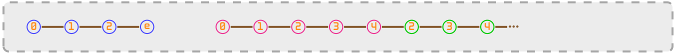


$$[\![P]\!] \in \wp(\mathbb{S}^* \cup \mathbb{S}^\infty)$$



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$$\Big\downarrow \alpha_\propto$$

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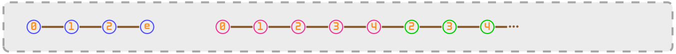

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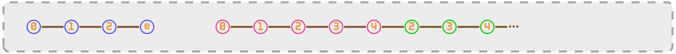
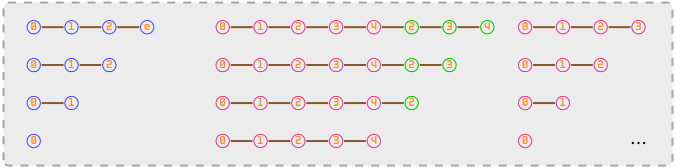


Trace prefixes



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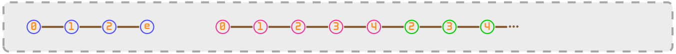

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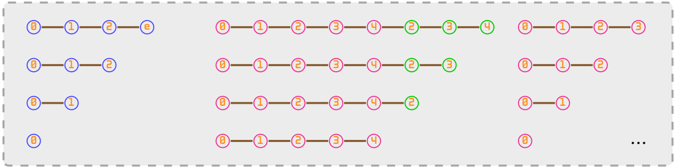

$$\Big\downarrow \alpha_\mathrm{r}$$

$$[\![P]\!]^\mathrm{r} \in \wp(\mathbb{S})$$

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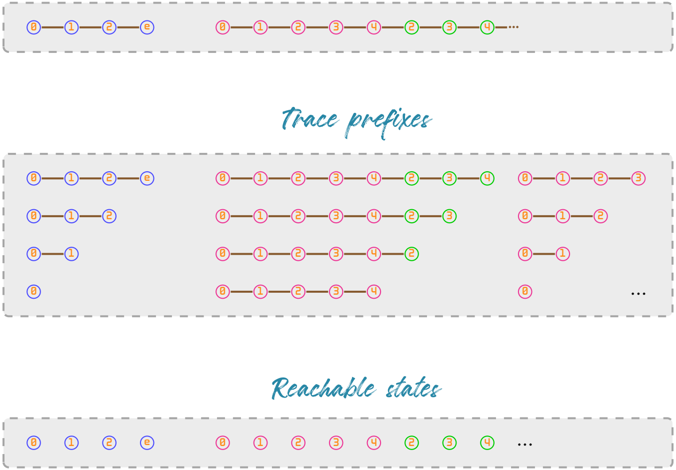

Reachability semantics: alternative definition

We can compute $[\![P]\!]^{\mathrm{r}}$ as least fixpoint of $\mathrm{f}_r : \wp(\mathbb{S}) \to \wp(\mathbb{S})$ inductively defined on P syntax

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Indeed, we have the Galois Correspondence

$$\langle \wp(\mathbb{L} \times \mathbb{M}), \subseteq \rangle \underset{\alpha_\ell}{\overset{\gamma_\ell}{\leftrightarrows}} \langle \mathbb{L} \to \wp(\mathbb{M}), \dot{\subseteq} \rangle$$

where

$$\alpha_\ell \triangleq \lambda X\,.\,\lambda \ell\,.\,\{\mathbb{m} \in \mathbb{M} \mid (\ell, \mathbb{m}) \in X\} \qquad \gamma_\ell \triangleq \lambda f\,.\,\{(\ell, \mathbb{m}) \in \mathbb{L} \times \mathbb{M} \mid \mathbb{m} \in f(\ell)\}$$

Reachability semantics: alternative definition (Cont'd)

In static analysis, we often use the alternative (yet isomorphic) representation $\alpha_\ell(\llbracket P \rrbracket^{\mathsf{r}})$

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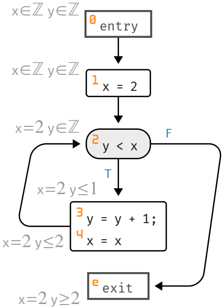
Change representation of reachability semantics: attach memory invariants to program points



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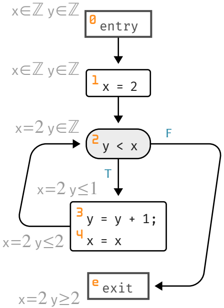
- $\alpha_\ell(\llbracket P \rrbracket^{\mathrm{r}})(\ell)$ is the most precise (memory) invariant at point ℓ



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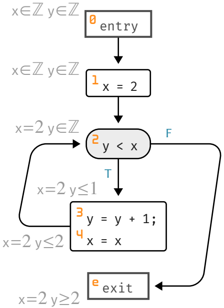

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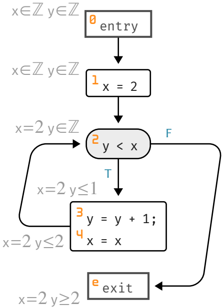
- $\alpha_\ell(\llbracket P \rrbracket^{\mathrm{r}})(\ell)$ is the most precise (memory) invariant at point ℓ
- Standard settings of many static analyses (e.g., data-flow)
- Usually computed solving a system of equations:

$$(X_\ell = \mathbf{F}_\ell(X_0, \ldots, X_\ell, \ldots, X_e))_{\ell \in \{0,1,2,3,4,e\}}$$



Define transfer functions for conditions and code blocks

Conditions $\llparenthesis\, {}^{\ell} bexp \,\rrparenthesis : \wp(\mathbb{M}) \to \wp(\mathbb{M})$

$\llparenthesis\, {}^{\ell} \mathtt{x} = exp \,\rrparenthesis : \wp(\mathbb{M}) \to \wp(\mathbb{M})$ Code blocks

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- $(\!|\,{}^{\ell}\mathtt{x} = exp\,|\!)(X) \triangleq \{\mathbb{m}[\mathtt{x} \hookleftarrow \mathbb{v}] \mid \mathbb{m} \in X \wedge \mathbb{v} = \text{EVAL}[exp](\mathbb{m})\}$

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$$\llparenthesis\, ^{\ell}\texttt{x=y+2} \,\rrparenthesis(X) \triangleq \{\mathbb{m}[\texttt{x} \hookleftarrow \mathbb{v}] \mid \mathbb{m} \in X \wedge \mathbb{v} = \textsc{Eval}[\texttt{y+2}](\mathbb{m})\}$$

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The reachability semantics is the least solution of the system of equations

$$\begin{cases} X_{\mathtt{0}} \triangleq \mathbb{M} \\ X_{\ell} \triangleq \bigcup_{\ell' \in \mathbb{L}} (\!| {}^{\ell'} stmt \,|\!) X_{\ell'} \quad \text{if next}\left({}^{\ell'} stmt\right) = \ell \end{cases}$$

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Each $X_{\ell} \mapsto \bigcup_{\ell' \in \mathbb{L}} \llparenthesis\, {}^{\ell'}stmt \,\rrparenthesis X_{\ell'}$ is monotonic on the complete lattice $\langle \wp(\mathbb{M}), \subseteq, \cup, \cap, \emptyset, \mathbb{M} \rangle$

Equational reachability semantics (Cont'd)

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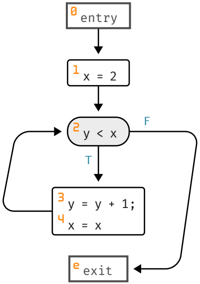
- The least fixpoint does exist
- The solution can be computed by increasing iterations

$$\begin{cases} X_{\mathtt{0}}^0 \triangleq \mathbb{M} \\ X_{\ell}^0 \triangleq \emptyset \end{cases} \qquad \begin{cases} X_{\mathtt{0}}^{n+1} \triangleq \mathbb{M} \\ X_{\ell}^{n+1} \triangleq \bigcup_{\ell' \in \mathbb{L}} (\!| {}^{\ell'}stmt |\!) X_{\ell'}^{n} \end{cases}$$

Equational reachability semantics: example



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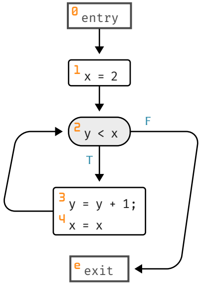

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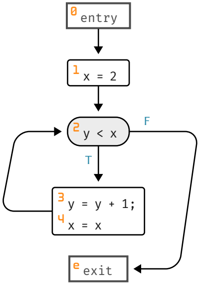

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X_0	X_1	X_2	X_3	X_4	X_e

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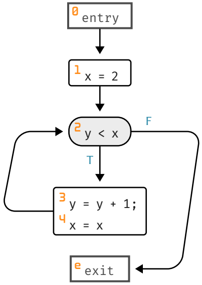

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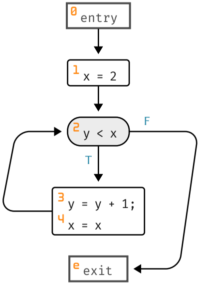

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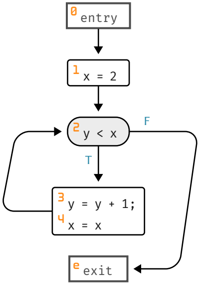
$$X_{4} = \llparenthesis\, {}^{3}\texttt{y = y + 1} \,\rrparenthesis X_{3}$$

$$X_{e} = \llparenthesis\, {}^{2}\texttt{y >= x} \,\rrparenthesis X_{2}$$

X_{0}	X_{1}	X_{2}	X_{3}	X_{4}	X_{e}
$(\mathbb{Z}, \mathbb{Z})$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(\mathbb{Z}, \mathbb{Z})$	$(\mathbb{Z}, \mathbb{Z})$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Equational reachability semantics: example



$$\begin{cases} X_0 \triangleq \mathbb{M} \\ X_\ell \triangleq \bigcup_{\ell' \in \mathbb{L}} (\!| {}^{\ell'} stmt |\!) X_{\ell'} \quad \text{if next}\left({}^{\ell'} stmt\right) = \ell \end{cases}$$

$$X_0 = \mathbb{M}$$
$$X_1 = (\!| {}^{0}\texttt{entry} |\!) X_0$$
$$X_2 = (\!| {}^{1}\texttt{x = 2} |\!) X_1 \cup (\!| {}^{4}\texttt{x = x} |\!) X_4$$
$$X_3 = (\!| {}^{2}\texttt{y < x} |\!) X_2$$
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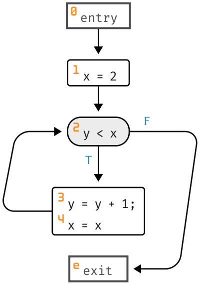
X_0	X_1	X_2	X_3	X_4	X_e
$(\mathbb{Z}, \mathbb{Z})$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(\mathbb{Z}, \mathbb{Z})$	$(\mathbb{Z}, \mathbb{Z})$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(\mathbb{Z}, \mathbb{Z})$	$(\mathbb{Z}, \mathbb{Z})$	$(\{2\}, \mathbb{Z})$	$\emptyset$	$\emptyset$	$\emptyset$

Equational reachability semantics: example

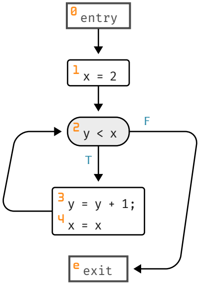


$$
\begin{cases}
X_0 \triangleq \mathbb{M} \\
X_\ell \triangleq \bigcup_{\ell' \in \mathbb{L}} \llparenthesis {}^{\ell'}stmt \rrparenthesis X_{\ell'} \quad \text{if next}\left({}^{\ell'}stmt\right) = \ell
\end{cases}
$$

$$
\begin{aligned}
X_0 &= \mathbb{M} \\
X_1 &= \llparenthesis {}^{0}\texttt{entry} \rrparenthesis X_0 \\
X_2 &= \llparenthesis {}^{1}\texttt{x = 2} \rrparenthesis X_1 \cup \llparenthesis {}^{4}\texttt{x = x} \rrparenthesis X_4 \\
X_3 &= \llparenthesis {}^{2}\texttt{y < x} \rrparenthesis X_2 \\
X_4 &= \llparenthesis {}^{3}\texttt{y = y + 1} \rrparenthesis X_3 \\
X_e &= \llparenthesis {}^{2}\texttt{y >= x} \rrparenthesis X_2
\end{aligned}
$$

X_0	X_1	X_2	X_3	X_4	X_e
$(\mathbb{Z},\mathbb{Z})$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(\mathbb{Z},\mathbb{Z})$	$(\mathbb{Z},\mathbb{Z})$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(\mathbb{Z},\mathbb{Z})$	$(\mathbb{Z},\mathbb{Z})$	$(\{2\},\mathbb{Z})$	$\emptyset$	$\emptyset$	$\emptyset$
$(\mathbb{Z},\mathbb{Z})$	$(\mathbb{Z},\mathbb{Z})$	$(\{2\},\mathbb{Z})$	$(\{2\},\mathbb{Z}^{<2})$	$\emptyset$	$\emptyset$

Equational reachability semantics: example



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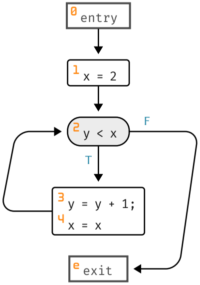
X_0	X_1	X_2	X_3	X_4	X_e
$(\mathbb{Z}, \mathbb{Z})$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(\mathbb{Z}, \mathbb{Z})$	$(\mathbb{Z}, \mathbb{Z})$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(\mathbb{Z}, \mathbb{Z})$	$(\mathbb{Z}, \mathbb{Z})$	$(\{2\}, \mathbb{Z})$	$\emptyset$	$\emptyset$	$\emptyset$
$(\mathbb{Z}, \mathbb{Z})$	$(\mathbb{Z}, \mathbb{Z})$	$(\{2\}, \mathbb{Z})$	$(\{2\}, \mathbb{Z}^{<2})$	$\emptyset$	$\emptyset$
$(\mathbb{Z}, \mathbb{Z})$	$(\mathbb{Z}, \mathbb{Z})$	$(\{2\}, \mathbb{Z})$	$(\{2\}, \mathbb{Z}^{<2})$	$(\{2\}, \mathbb{Z}^{<3})$	$\emptyset$

Equational reachability semantics: example



$$\begin{cases} X_0 \triangleq \mathbb{M} \\ X_\ell \triangleq \bigcup_{\ell' \in \mathbb{L}} \llparenthesis\, {}^{\ell'}stmt \,\rrparenthesis X_{\ell'} \quad \text{if } \mathrm{next}\left({}^{\ell'}stmt\right) = \ell \end{cases}$$

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\end{aligned}$$

X_0	X_1	X_2	X_3	X_4	X_e
$(\mathbb{Z}, \mathbb{Z})$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(\mathbb{Z}, \mathbb{Z})$	$(\mathbb{Z}, \mathbb{Z})$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$(\mathbb{Z}, \mathbb{Z})$	$(\mathbb{Z}, \mathbb{Z})$	$(\{2\}, \mathbb{Z})$	$\emptyset$	$\emptyset$	$\emptyset$
$(\mathbb{Z}, \mathbb{Z})$	$(\mathbb{Z}, \mathbb{Z})$	$(\{2\}, \mathbb{Z})$	$(\{2\}, \mathbb{Z}^{<2})$	$\emptyset$	$\emptyset$
$(\mathbb{Z}, \mathbb{Z})$	$(\mathbb{Z}, \mathbb{Z})$	$(\{2\}, \mathbb{Z})$	$(\{2\}, \mathbb{Z}^{<2})$	$(\{2\}, \mathbb{Z}^{<3})$	$\emptyset$
$(\mathbb{Z}, \mathbb{Z})$	$(\mathbb{Z}, \mathbb{Z})$	$(\{2\}, \mathbb{Z})$	$(\{2\}, \mathbb{Z}^{<2})$	$(\{2\}, \mathbb{Z}^{<3})$	$(\{2\}, \mathbb{Z}^{\geq 2})$

Compute the invariant on the last control point only

Compute the invariant on the last control point only

$$\langle \wp(\mathbb{S}), \subseteq \rangle \underset{\alpha_{\mathsf{e}}}{\overset{\gamma_{\mathsf{e}}}{\leftrightarrows}} \langle \wp(\mathbb{M}), \subseteq \rangle$$

$$\alpha_{\mathsf{e}} \triangleq \lambda X \,.\, \{\mathtt{m} \in \mathbb{M} \mid (\mathsf{e}, \mathtt{m}) \in X\} \qquad \gamma_{\mathsf{e}} \triangleq \lambda X \,.\, \{(\mathsf{e}, \mathtt{m}) \in \mathbb{L} \times \mathbb{M} \mid \mathtt{m} \in X\} \cup (\mathbb{L} \setminus \{\mathsf{e}\}) \times \mathbb{M}$$

Compute the invariant on the last control point only *// reduce memory space by a LoC factor*

$$\langle \wp(\mathbb{S}), \subseteq \rangle \underset{\alpha_{\mathsf{e}}}{\overset{\gamma_{\mathsf{e}}}{\leftrightarrows}} \langle \wp(\mathbb{M}), \subseteq \rangle$$

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Post-condition semantics $[\![P]\!]^{\mathrm{c}} = \alpha_{\mathsf{e}}([\![P]\!]^{\mathrm{r}})$

Compute the invariant on the last control point only // *reduce memory space by a LoC factor*

$$\langle \wp(\mathbb{S}), \subseteq \rangle \xleftrightarrow[\alpha_{\mathsf{e}}]{\gamma_{\mathsf{e}}} \langle \wp(\mathbb{M}), \subseteq \rangle$$

$\alpha_{\mathsf{e}} \triangleq \lambda X . \{ \mathbb{m} \in \mathbb{M} \mid (\mathsf{e}, \mathbb{m}) \in X \}$ $\gamma_{\mathsf{e}} \triangleq \lambda X . \{ (\mathsf{e}, \mathbb{m}) \in \mathbb{L} \times \mathbb{M} \mid \mathbb{m} \in X \} \cup (\mathbb{L} \setminus \{\mathsf{e}\}) \times \mathbb{M}$

Post-condition semantics $[\![P]\!]^{\mathrm{c}} = \alpha_{\mathsf{e}}([\![P]\!]^{\mathrm{r}})$

- Inductively computed on program syntax $[\![P]\!]^{\mathrm{c}} \triangleq \mathbf{S}^{\mathrm{c}}_{stmt_n} \circ \ldots \circ \mathbf{S}^{\mathrm{c}}_{stmt_1}(\mathfrak{I})$ // $\mathfrak{I} \subseteq \mathbb{M}$
- Given a transfer function $\mathbf{S}^{\mathrm{c}} : \wp(\mathbb{M}) \rightarrow \wp(\mathbb{M})$ for all statements $stmt_1, \ldots, stmt_n$ of P

Compute the invariant on the last control point only // *reduce memory space by a LoC factor*

$$\langle \wp(\mathbb{S}), \subseteq \rangle \underset{\alpha_{e}}{\overset{\gamma_{e}}{\leftrightarrows}} \langle \wp(\mathbb{M}), \subseteq \rangle$$

$\alpha_{e} \triangleq \lambda X.\ \{\mathsf{m} \in \mathbb{M} \mid (e, \mathsf{m}) \in X\}$ $\gamma_{e} \triangleq \lambda X.\ \{(e, \mathsf{m}) \in \mathbb{L} \times \mathbb{M} \mid \mathsf{m} \in X\} \cup (\mathbb{L} \setminus \{e\}) \times \mathbb{M}$

Define $\langle \mathbb{L} \to \wp(\mathbb{M}), \dot{\subseteq} \rangle \underset{\bar{\alpha}_{e}}{\overset{\bar{\gamma}_{e}}{\leftrightarrows}} \langle \wp(\mathbb{M}), \subseteq \rangle$

Post-condition semantics $[\![P]\!]^{c} = \alpha_{e}([\![P]\!]^{r})$

- Inductively computed on program syntax $[\![P]\!]^{c} \triangleq \mathbf{S}^{c}_{stmt_n} \circ \ldots \circ \mathbf{S}^{c}_{stmt_1}(\mathfrak{I})$ // $\mathfrak{I} \subseteq \mathbb{M}$
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Boolean expressions $\mathbf{S}^{c}_{bexp} : \wp(\mathbb{M}) \rightarrow \wp(\mathbb{M})$

Programs $\mathbf{S}^{c}_{P} : \wp(\mathbb{M}) \rightarrow \wp(\mathbb{M})$

Boolean expressions $\mathbf{S}^{c}_{bexp} : \wp(\mathbb{M}) \to \wp(\mathbb{M})$

- $\mathbf{S}^{c}_{bexp}(X) \triangleq \{\mathfrak{m} \in X \mid \textsc{Eval}[bexp](\mathfrak{m}) = \mathsf{true}\}$

Programs $\mathbf{S}^{c}_{P} : \wp(\mathbb{M}) \to \wp(\mathbb{M})$

Post-condition semantics (Cont'd)

Boolean expressions $\mathbf{S}^{c}_{bexp} : \wp(\mathbb{M}) \to \wp(\mathbb{M})$

- $\mathbf{S}^{c}_{bexp}(X) \triangleq \{\mathfrak{m} \in X \mid \textsc{Eval}[bexp](\mathfrak{m}) = \mathsf{true}\}$

$$\mathbf{S}^{c}_{\mathsf{x<y}}(X) \triangleq \{\mathfrak{m} \in X \mid \textsc{Eval}[\mathsf{x<y}](\mathfrak{m}) = \mathsf{true}\}$$

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Post-condition semantics (Cont'd)

Boolean expressions $\mathbf{S}^{c}_{bexp} : \wp(\mathbb{M}) \to \wp(\mathbb{M})$

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Post-condition semantics (Cont'd)

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Programs $\mathbf{S}^{c}_{P} : \wp(\mathbb{M}) \rightarrow \wp(\mathbb{M})$

- $\mathbf{S}^{c}_{\mathtt{x}=exp}(X) \triangleq \{\mathbb{m}[\mathtt{x} \leftarrow \mathbb{v}] \mid \mathbb{m} \in X \wedge \mathbb{v} = \textsc{Eval}[exp](\mathbb{m})\}$

Post-condition semantics (Cont'd)

Boolean expressions $\mathbf{S}^{c}_{bexp} : \wp(\mathbb{M}) \to \wp(\mathbb{M})$

- $\mathbf{S}^{c}_{bexp}(X) \triangleq \{\mathbb{m} \in X \mid \textsc{Eval}[bexp](\mathbb{m}) = \mathsf{true}\}$

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$$\mathbf{S}^{c}_{\mathtt{x=y+2}}(X) \triangleq \{\mathbb{m}[\mathtt{x} \leftarrow \mathbb{v}] \mid \mathbb{m} \in X \wedge \mathbb{v} = \textsc{Eval}[\mathtt{y+2}](\mathbb{m})\}$$

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$$\mathbf{S}^{c}_{\mathtt{x=y+2}}(X) \triangleq \{\mathbb{m}[\mathtt{x} \leftarrow \mathbb{v}] \mid \mathbb{m} \in X \wedge \mathbb{v} = \textsc{Eval}[\mathtt{y+2}](\mathbb{m})\} = \{\mathbb{m}[\mathtt{x} \leftarrow \mathbb{v}] \mid \mathbb{m} \in X \wedge \mathbb{v} = \mathbb{m}(\mathtt{y}) + 2\}$$

Boolean expressions $\mathbf{S}^{c}_{bexp} : \wp(\mathbb{M}) \to \wp(\mathbb{M})$

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- $\mathbf{S}^{c}_{stmt_1\,;\,stmt_2}(X) \triangleq \mathbf{S}^{c}_{stmt_2} \circ \mathbf{S}^{c}_{stmt_1}(X)$

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- $\mathbf{S}^c_{bexp}(X) \triangleq \{\mathfrak{m} \in X \mid \textsc{Eval}[bexp](\mathfrak{m}) = \mathsf{true}\}$

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Programs $\mathbf{S}^c_P : \wp(\mathbb{M}) \to \wp(\mathbb{M})$

- $\mathbf{S}^c_{\mathtt{x=}exp}(X) \triangleq \{\mathfrak{m}[\mathtt{x} \hookleftarrow \mathbb{v}] \mid \mathfrak{m} \in X \wedge \mathbb{v} = \textsc{Eval}[exp](\mathfrak{m})\}$

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Post-condition semantics (Cont'd)

Boolean expressions $\mathbf{S}^c_{bexp} : \wp(\mathbb{M}) \rightarrow \wp(\mathbb{M})$

- $\mathbf{S}^c_{bexp}(X) \triangleq \{\mathfrak{m} \in X \mid \text{EVAL}[bexp](\mathfrak{m}) = \text{true}\}$

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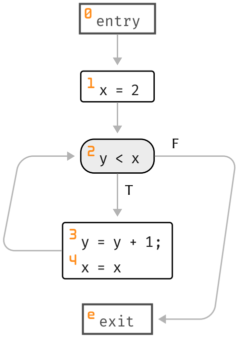
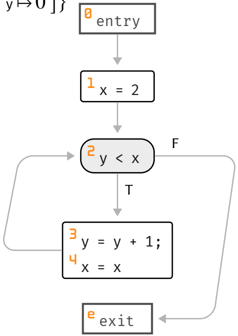
- $\mathbf{S}^c_{stmt_1\,;\,stmt_2}(X) \triangleq \mathbf{S}^c_{stmt_2} \circ \mathbf{S}^c_{stmt_1}(X)$
- $\mathbf{S}^c_{\mathtt{if}(bexp)\{stmt_1\}\mathtt{else}\{stmt_2\}}(X) \triangleq \mathbf{S}^c_{stmt_1} \circ \mathbf{S}^c_{bexp}(X) \cup \mathbf{S}^c_{stmt_2} \circ \mathbf{S}^c_{\neg bexp}(X)$
- $\mathbf{S}^c_{\mathtt{while}(bexp)\{stmt\}}(X) \triangleq \mathbf{S}^c_{\neg bexp} \circ (\text{lfp}^{\subseteq}\, \lambda Y\,.\, X \cup \mathbf{S}^c_{stmt} \circ \mathbf{S}^c_{bexp}(Y))$

Post-condition semantics: example



Post-condition semantics: example

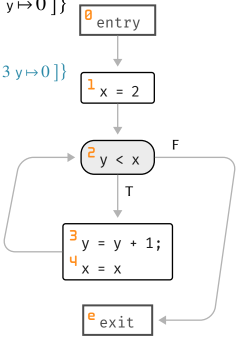
Input: $\{[\,x \mapsto 2\ y \mapsto 3\,], [\,x \mapsto 3\ y \mapsto 0\,]\}$



Post-condition semantics: example

Input: $\{[\,x \mapsto 2 \;\, y \mapsto 3\,], [\,x \mapsto 3 \;\, y \mapsto 0\,]\}$

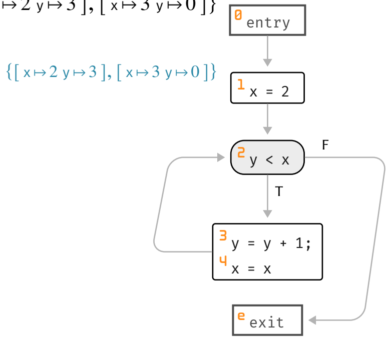
$\{[\,x \mapsto 2 \;\, y \mapsto 3\,], [\,x \mapsto 3 \;\, y \mapsto 0\,]\}$



Post-condition semantics: example

$$\mathbf{S}^{c}_{\texttt{x=2}}(X) = \{\mathbb{m}[\texttt{x} \hookleftarrow 2] \mid \mathbb{m} \in X\}$$

Input: $\{[\,\texttt{x} \mapsto 2\ \texttt{y} \mapsto 3\,], [\,\texttt{x} \mapsto 3\ \texttt{y} \mapsto 0\,]\}$



Post-condition semantics: example

$$\mathbf{S}^{c}_{\mathtt{x=2}}(X) = \{\mathbb{m}[\mathtt{x} \hookleftarrow 2] \mid \mathbb{m} \in X\} = \{[\,\mathtt{x}\mapsto 2\ \mathtt{y}\mapsto 3\,], [\,\mathtt{x}\mapsto 2\ \mathtt{y}\mapsto 0\,]\}$$

Input: $\{[\,\mathtt{x}\mapsto 2\ \mathtt{y}\mapsto 3\,], [\,\mathtt{x}\mapsto 3\ \mathtt{y}\mapsto 0\,]\}$

$\{[\,\mathtt{x}\mapsto 2\ \mathtt{y}\mapsto 3\,], [\,\mathtt{x}\mapsto 3\ \mathtt{y}\mapsto 0\,]\}$

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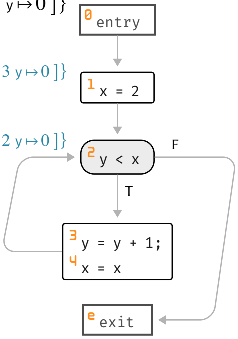

Post-condition semantics: example

$$\mathbf{S}^c_{\texttt{while(y<x)\{y=y+1;x=x\}}}(X) = \mathbf{S}^c_{\texttt{y>=x}}(\mathsf{lfp}^{\subseteq}\, \lambda Y\,.\, X \cup \mathbf{S}^c_{\texttt{y=y+1;x=x}} \circ \mathbf{S}^c_{\texttt{y<x}}(Y))$$

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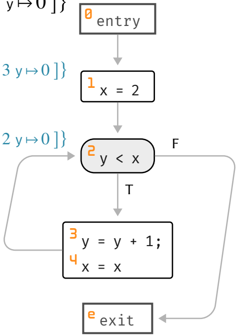

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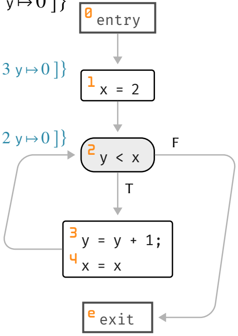

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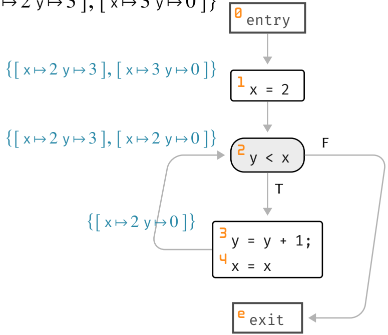

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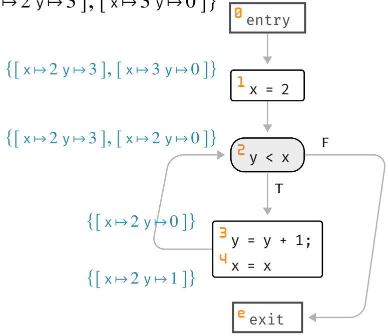

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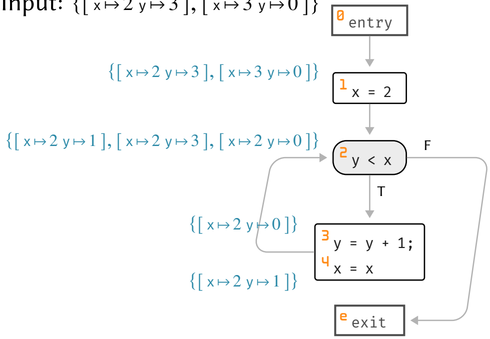

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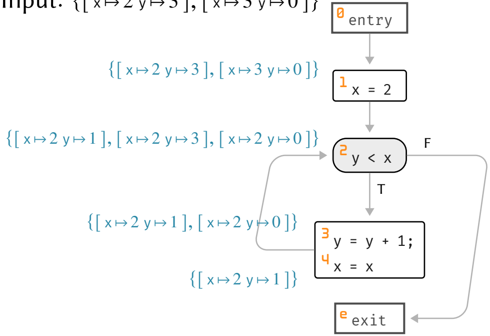

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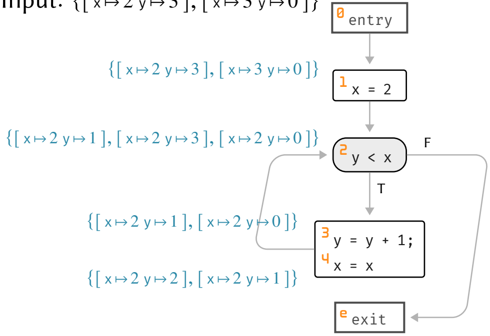

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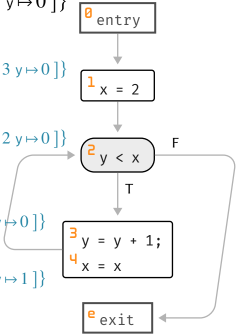

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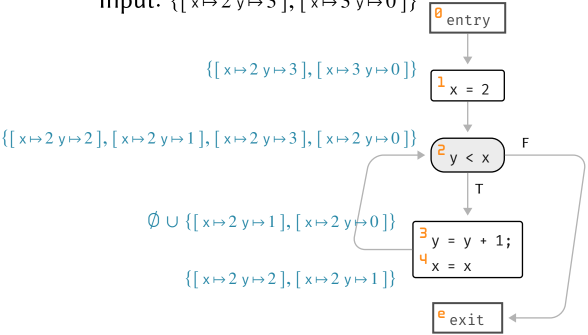

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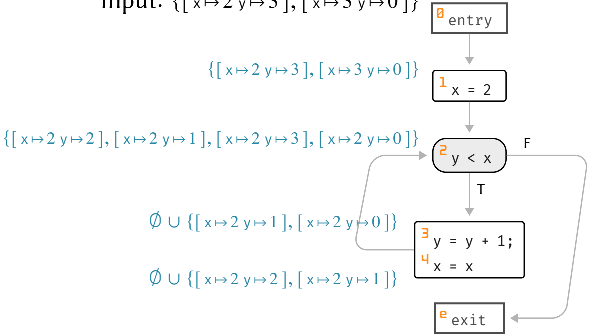

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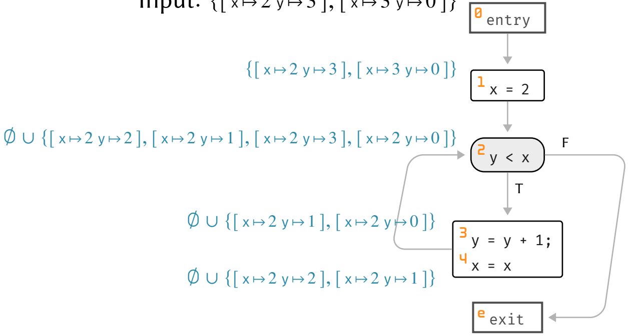

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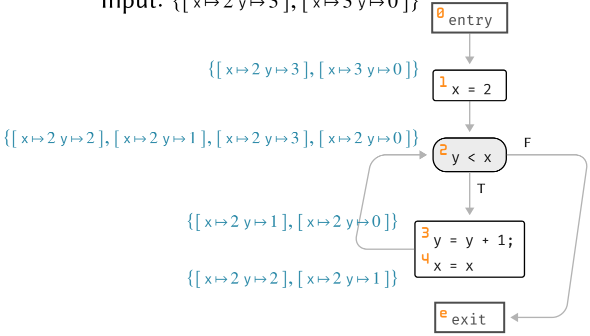

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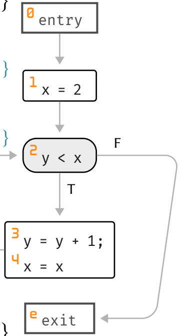

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Computing Abstract Invariants

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- Elements in $\wp(\mathbb{M})$ are not computer-representable
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Note: being $\mathbb{V}$ finite is not particularly helpful from a practical point of view

- Representing elements in $\wp(\mathbb{X} \to \mathbb{V})$ in extension is expensive
- Explicitly computing $\llparenthesis\, {}^{\ell'}stmt \,\rrparenthesis$ and $\mathbf{S}^{c}_{stmt}$ is expensive
- The lattice $\langle \wp(\mathbb{X} \to \mathbb{V}), \subseteq \rangle$ has large height, hence iterations are expensive

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$$\langle \wp(\mathbb{M}), \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \mathbb{M}^\sharp, \subseteq^\sharp \rangle$$

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Sound abstract transfer functions $(\!|\ {}^{\ell'} stmt\ |\!)^\sharp$ and $\mathbf{S}^\sharp_{stmt}$ on $\mathbb{M}^\sharp \to \mathbb{M}^\sharp$

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Sound abstract transfer functions $\llparenthesis\, {}^{\ell'} stmt \,\rrparenthesis^\sharp$ and $\mathbf{S}^\sharp_{stmt}$ on $\mathbb{M}^\sharp \rightarrow \mathbb{M}^\sharp$

Abstract transfer functions can be systematically derived from soundness proofs!

Define abstract transfer functions for conditions and code blocks, given $\langle \wp(\mathbb{M}), \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \mathbb{M}^\sharp, \subseteq^\sharp \rangle$

Conditions $(\!|\, {}^{\ell} bexp \,|\!)^\sharp : \mathbb{M}^\sharp \to \mathbb{M}^\sharp$

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Soundness

$$\forall \mathfrak{m}^\sharp \in \mathbb{M}^\sharp$$

$$\llparenthesis\, {}^{\ell} bexp \,\rrparenthesis \circ \gamma(\mathfrak{m}^\sharp) \subseteq \gamma \circ \llparenthesis\, {}^{\ell} bexp \,\rrparenthesis^\sharp(\mathfrak{m}^\sharp)$$

$$\llparenthesis\, {}^{\ell} \mathtt{x} = exp \,\rrparenthesis \circ \gamma(\mathfrak{m}^\sharp) \subseteq \gamma \circ \llparenthesis\, {}^{\ell} \mathtt{x} = exp \,\rrparenthesis^\sharp(\mathfrak{m}^\sharp)$$

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Soundness

$$\forall X \in \wp(\mathbb{M})$$

$$\alpha \circ (\!|\, {}^{\ell} bexp \,|\!) X \subseteq^\sharp (\!|\, {}^{\ell} bexp \,|\!)^\sharp \circ \alpha(X)$$

$$\alpha \circ (\!|\, {}^{\ell} \mathtt{x} = exp \,|\!) X \subseteq^\sharp (\!|\, {}^{\ell} \mathtt{x} = exp \,|\!)^\sharp \circ \alpha(X)$$

The abstract reachability semantics is the least solution of the system of equations

$$\begin{cases} X_{\mathtt{0}} \triangleq \mathbb{m}^{\sharp}_{\top} \\ X_{\ell} \triangleq \bigcup^{\sharp}_{\ell' \in \mathbb{L}} (\!| {}^{\ell'} stmt \,|\!)^{\sharp} X_{\ell'} \quad \text{if next}\left({}^{\ell'} stmt\right) = \ell \end{cases}$$

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Each $X_\ell \mapsto \bigcup^\sharp_{\ell' \in \mathbb{L}} \llparenthesis\, {}^{\ell'}stmt \,\rrparenthesis^\sharp X_{\ell'}$ is monotonic on the complete lattice $\langle \mathbb{M}^\sharp, \subseteq^\sharp, \cup^\sharp, \cap^\sharp, \mathbb{m}^\sharp_\bot, \mathbb{m}^\sharp_\top \rangle$

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- The least fixpoint does exist

The abstract reachability semantics is the least solution of the system of equations

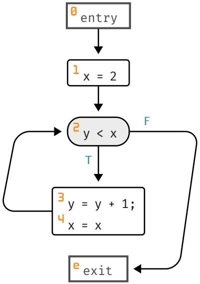
$$\begin{cases} X_{\mathtt{0}} \triangleq \mathbb{m}^\sharp_\top \\ X_\ell \triangleq \bigcup^\sharp_{\ell' \in \mathbb{L}} (\!| {}^{\ell'} stmt |\!)^\sharp X_{\ell'} \quad \text{if next}\left({}^{\ell'} stmt\right) = \ell \end{cases}$$

Each $X_\ell \mapsto \bigcup^\sharp_{\ell' \in \mathbb{L}} (\!| {}^{\ell'} stmt |\!)^\sharp X_{\ell'}$ is monotonic on the complete lattice $\langle \mathbb{M}^\sharp, \subseteq^\sharp, \cup^\sharp, \cap^\sharp, \mathbb{m}^\sharp_\bot, \mathbb{m}^\sharp_\top \rangle$

- The least fixpoint does exist
- The solution can be computed by increasing iterations

$$\begin{cases} X^0_{\mathtt{0}} \triangleq \mathbb{m}^\sharp_\top \\ X^0_\ell \triangleq \mathbb{m}^\sharp_\bot \end{cases} \qquad \begin{cases} X^{n+1}_{\mathtt{0}} \triangleq \mathbb{m}^\sharp_\top \\ X^{n+1}_\ell \triangleq \bigcup^\sharp_{\ell' \in \mathbb{L}} (\!| {}^{\ell'} stmt |\!)^\sharp X^n_{\ell'} \end{cases}$$

Equational abstract reachability semantics: example



$$\begin{cases} X_{0} \triangleq \mathbb{m}^{\sharp}_{\top} \\ X_{\ell} \triangleq \bigcup^{\sharp}_{\ell' \in \mathbb{L}} (\!| \, {}^{\ell'}stmt \, |\!)^{\sharp} X_{\ell'} \quad \text{if next}\left({}^{\ell'}stmt\right) = \ell \end{cases}$$

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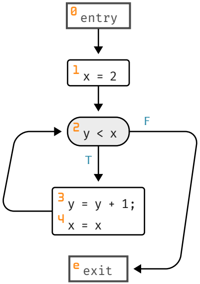
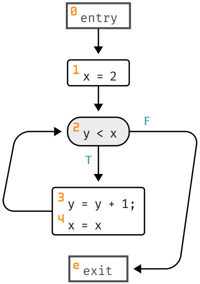


$$\begin{cases} X_{0} \triangleq \mathfrak{m}^{\sharp}_{\top} \\ X_{\ell} \triangleq \bigcup^{\sharp}_{\ell' \in \mathbb{L}} (\!|\, {}^{\ell'}stmt \,|\!)^{\sharp} X_{\ell'} & \text{if next}\left({}^{\ell'}stmt\right) = \ell \end{cases}$$

$$\begin{aligned}
X_{0} &= \mathfrak{m}^{\sharp}_{\top} \\
X_{1} &= (\!|\, {}^{0}\texttt{entry} \,|\!)^{\sharp} X_{0} \\
X_{2} &= (\!|\, {}^{1}\texttt{x = 2} \,|\!)^{\sharp} X_{1} \cup^{\sharp} (\!|\, {}^{4}\texttt{x = x} \,|\!)^{\sharp} X_{4} \\
X_{3} &= (\!|\, {}^{2}\texttt{y < x} \,|\!)^{\sharp} X_{2} \\
X_{4} &= (\!|\, {}^{3}\texttt{y = y + 1} \,|\!)^{\sharp} X_{3} \\
X_{e} &= (\!|\, {}^{2}\texttt{y >= x} \,|\!)^{\sharp} X_{2}
\end{aligned}$$

Equational abstract reachability semantics: example



$$
\begin{cases}
X_0 \triangleq \mathfrak{m}^\sharp_\top \\
X_\ell \triangleq \bigcup^\sharp_{\ell' \in \mathbb{L}} \llparenthesis {}^{\ell'}stmt \rrparenthesis^\sharp X_{\ell'} \quad \text{if next}\left({}^{\ell'}stmt\right) = \ell
\end{cases}
$$

$$
\begin{aligned}
X_0 &= \mathfrak{m}^\sharp_\top \\
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X_2 &= \llparenthesis {}^{1}\texttt{x = 2} \rrparenthesis^\sharp X_1 \cup^\sharp \llparenthesis {}^{4}\texttt{x = x} \rrparenthesis^\sharp X_4 \\
X_3 &= \llparenthesis {}^{2}\texttt{y < x} \rrparenthesis^\sharp X_2 \\
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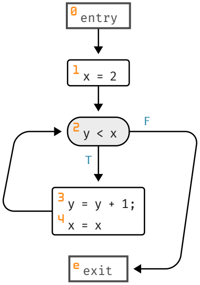
\end{aligned}
$$

X_0	X_1	X_2	X_3	X_4	X_e

Equational abstract reachability semantics: example



$$\begin{cases} X_0 \triangleq \mathfrak{m}^\sharp_\top \\ X_\ell \triangleq \bigcup^\sharp_{\ell' \in \mathbb{L}} (\!|\, {}^{\ell'}stmt \,|\!)^\sharp X_{\ell'} \quad \text{if next}\left({}^{\ell'}stmt\right) = \ell \end{cases}$$

X_0	X_1	X_2	X_3	X_4	X_e
$(\top^\pm, \top^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$

$$X_0 = \mathfrak{m}^\sharp_\top$$
$$X_1 = (\!|\, {}^{0}\texttt{entry} \,|\!)^\sharp X_0$$
$$X_2 = (\!|\, {}^{1}\texttt{x = 2} \,|\!)^\sharp X_1 \cup^\sharp (\!|\, {}^{4}\texttt{x = x} \,|\!)^\sharp X_4$$
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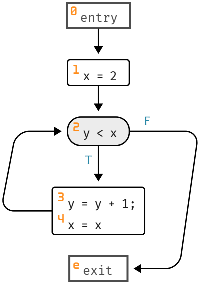
Equational abstract reachability semantics: example



$$\begin{cases} X_{0} \triangleq \mathfrak{m}^{\sharp}_{\top} \\ X_{\ell} \triangleq \bigcup^{\sharp}_{\ell' \in \mathbb{L}} \llparenthesis\, {}^{\ell'}stmt \,\rrparenthesis^{\sharp} X_{\ell'} \quad \text{if next}\left({}^{\ell'}stmt\right) = \ell \end{cases}$$

X_0	X_1	X_2	X_3	X_4	X_e
$(\top^{\pm}, \top^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$
$(\top^{\pm}, \top^{\pm})$	$(\top^{\pm}, \top^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$

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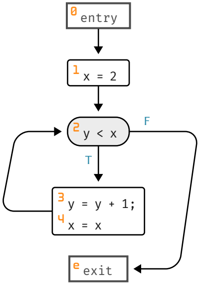
Equational abstract reachability semantics: example



$$\begin{cases} X_0 \triangleq \mathfrak{m}^\sharp_\top \\ X_\ell \triangleq \bigcup^\sharp_{\ell' \in \mathbb{L}} (\!| {}^{\ell'} stmt |\!)^\sharp X_{\ell'} \quad \text{if next}\left({}^{\ell'} stmt\right) = \ell \end{cases}$$

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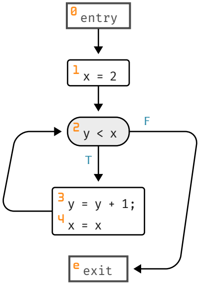
X_0	X_1	X_2	X_3	X_4	X_e
$(\top^\pm, \top^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$
$(\top^\pm, \top^\pm)$	$(\top^\pm, \top^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$
$(\top^\pm, \top^\pm)$	$(\top^\pm, \top^\pm)$	$(+0, \top^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$

Equational abstract reachability semantics: example



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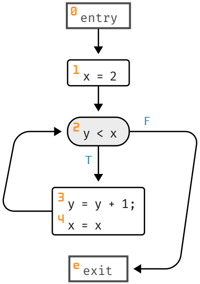
X_0	X_1	X_2	X_3	X_4	X_e
$(\top^\pm, \top^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$
$(\top^\pm, \top^\pm)$	$(\top^\pm, \top^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$
$(\top^\pm, \top^\pm)$	$(\top^\pm, \top^\pm)$	$(+0, \top^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$
$(\top^\pm, \top^\pm)$	$(\top^\pm, \top^\pm)$	$(+0, \top^\pm)$	$(+0, \top^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$

Equational abstract reachability semantics: example



$$\begin{cases} X_{0} \triangleq \mathfrak{m}^{\sharp}_{\top} \\ X_{\ell} \triangleq \bigcup^{\sharp}_{\ell' \in \mathbb{L}} (\!| {}^{\ell'}stmt |\!)^{\sharp} X_{\ell'} \quad \text{if next}\left({}^{\ell'}stmt\right) = \ell \end{cases}$$

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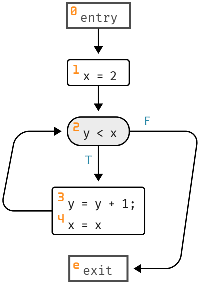
$(\top^{\pm}, \top^{\pm})$	$(\top^{\pm}, \top^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$
$(\top^{\pm}, \top^{\pm})$	$(\top^{\pm}, \top^{\pm})$	$(+0, \top^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$
$(\top^{\pm}, \top^{\pm})$	$(\top^{\pm}, \top^{\pm})$	$(+0, \top^{\pm})$	$(+0, \top^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$
$(\top^{\pm}, \top^{\pm})$	$(\top^{\pm}, \top^{\pm})$	$(+0, \top^{\pm})$	$(+0, \top^{\pm})$	$(+0, \top^{\pm})$	$(\bot^{\pm}, \bot^{\pm})$

Equational abstract reachability semantics: example



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$(\top^\pm, \top^\pm)$	$(\top^\pm, \top^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$
$(\top^\pm, \top^\pm)$	$(\top^\pm, \top^\pm)$	$(+0, \top^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$
$(\top^\pm, \top^\pm)$	$(\top^\pm, \top^\pm)$	$(+0, \top^\pm)$	$(+0, \top^\pm)$	$(\bot^\pm, \bot^\pm)$	$(\bot^\pm, \bot^\pm)$
$(\top^\pm, \top^\pm)$	$(\top^\pm, \top^\pm)$	$(+0, \top^\pm)$	$(+0, \top^\pm)$	$(+0, \top^\pm)$	$(\bot^\pm, \bot^\pm)$
$(\top^\pm, \top^\pm)$	$(\top^\pm, \top^\pm)$	$(+0, \top^\pm)$	$(+0, \top^\pm)$	$(+0, \top^\pm)$	$(+0, +0)$

Abstract post-condition semantics $\llbracket P \rrbracket^\sharp$, given $\langle \wp(\mathbb{M}), \subseteq \rangle \xleftrightarrow[\alpha]{\gamma} \langle \mathbb{M}^\sharp, \subseteq^\sharp \rangle$

Abstract post-condition semantics $[\![P]\!]^\sharp$, given $\langle \wp(\mathbb{M}), \subseteq\rangle \xleftarrow[\alpha]{\gamma} \langle \mathbb{M}^\sharp, \subseteq^\sharp\rangle$

- Inductively computed on program syntax $[\![P]\!]^\sharp \triangleq \mathbf{S}^\sharp_{stmt_n} \circ \ldots \circ \mathbf{S}^\sharp_{stmt_1}(\mathfrak{I}^\sharp)$ // $\mathfrak{I}^\sharp \triangleq \alpha(\mathfrak{I} \subseteq \mathbb{M})$
- Given an abstract transfer function $\mathbf{S}^\sharp : \mathbb{M}^\sharp \to \mathbb{M}^\sharp$ for all statements $stmt_1, \ldots, stmt_n$ of P

Abstract post-condition semantics $[\![P]\!]^\sharp$, given $\langle \wp(\mathbb{M}), \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \mathbb{M}^\sharp, \subseteq^\sharp \rangle$

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Boolean expressions $\mathbf{S}^\sharp_{bexp} : \mathbb{M}^\sharp \to \mathbb{M}^\sharp$

$\mathbf{S}^\sharp_{\mathtt{x}=exp} : \mathbb{M}^\sharp \to \mathbb{M}^\sharp$ Assignments

Abstract post-condition semantics $[\![P]\!]^\sharp$, given $\langle \wp(\mathbb{M}), \subseteq \rangle \xleftrightarrow[\alpha]{\gamma} \langle \mathbb{M}^\sharp, \subseteq^\sharp \rangle$

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Boolean expressions $\mathbf{S}^\sharp_{bexp} : \mathbb{M}^\sharp \to \mathbb{M}^\sharp$

$\mathbf{S}^\sharp_{\mathtt{x}=exp} : \mathbb{M}^\sharp \to \mathbb{M}^\sharp$ Assignments

Soundness

$$\forall \mathfrak{m}^\sharp \in \mathbb{M}^\sharp$$

$$\mathbf{S}^c_{bexp} \circ \gamma(\mathfrak{m}^\sharp) \subseteq \gamma \circ \mathbf{S}^\sharp_{bexp}(\mathfrak{m}^\sharp)$$

$$\mathbf{S}^c_{bexp} \circ \gamma(\mathfrak{m}^\sharp) \subseteq \gamma \circ \mathbf{S}^\sharp_{bexp}$$

Abstract post-condition semantics $[\![P]\!]^\sharp$, given $\langle \wp(\mathbb{M}), \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \mathbb{M}^\sharp, \subseteq^\sharp \rangle$

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Boolean expressions $\mathbf{S}^\sharp_{bexp} : \mathbb{M}^\sharp \to \mathbb{M}^\sharp$

$\mathbf{S}^\sharp_{\mathtt{X}=exp} : \mathbb{M}^\sharp \to \mathbb{M}^\sharp$ Assignments

Soundness

$$\forall X \in \wp(\mathbb{M})$$

$$\alpha \circ \mathbf{S}^c_{\mathtt{X}=exp}(X) \subseteq^\sharp \mathbf{S}^\sharp_{\mathtt{X}=exp} \circ \alpha(X)$$

$$\alpha \circ \mathbf{S}^c_{\mathtt{X}=exp}(X) \subseteq^\sharp \mathbf{S}^\sharp_{\mathtt{X}=exp} \circ \alpha(X)$$

Programs $\mathbf{S}^{\sharp}_{P} : \mathbb{M}^{\sharp} \rightarrow \mathbb{M}^{\sharp}$

Inductive cases

Abstract Post-condition semantics (Cont'd)

Programs $\mathbf{S}^\sharp_P : \mathbb{M}^\sharp \to \mathbb{M}^\sharp$

Inductive cases

- $\mathbf{S}^\sharp_{stmt_1\,;\,stmt_2}(\mathfrak{m}^\sharp) \triangleq \mathbf{S}^\sharp_{stmt_2} \circ \mathbf{S}^\sharp_{stmt_1}(\mathfrak{m}^\sharp)$
- $\mathbf{S}^\sharp_{\mathtt{if}(bexp)\{stmt_1\}\mathtt{else}\{stmt_2\}}(\mathfrak{m}^\sharp) \triangleq \mathbf{S}^\sharp_{stmt_1} \circ \mathbf{S}^\sharp_{bexp}(\mathfrak{m}^\sharp) \cup^\sharp \mathbf{S}^\sharp_{stmt_2} \circ \mathbf{S}^\sharp_{\neg bexp}(\mathfrak{m}^\sharp)$

Programs $\mathbf{S}^\sharp_P : \mathbb{M}^\sharp \to \mathbb{M}^\sharp$

Inductive cases

- $\mathbf{S}^\sharp_{stmt_1\,;\,stmt_2}(\mathfrak{m}^\sharp) \triangleq \mathbf{S}^\sharp_{stmt_2} \circ \mathbf{S}^\sharp_{stmt_1}(\mathfrak{m}^\sharp)$
- $\mathbf{S}^\sharp_{\texttt{if}(bexp)\{stmt_1\}\texttt{else}\{stmt_2\}}(\mathfrak{m}^\sharp) \triangleq \mathbf{S}^\sharp_{stmt_1} \circ \mathbf{S}^\sharp_{bexp}(\mathfrak{m}^\sharp) \cup^\sharp \mathbf{S}^\sharp_{stmt_2} \circ \mathbf{S}^\sharp_{\neg bexp}(\mathfrak{m}^\sharp)$
- $\mathbf{S}^\sharp_{\texttt{while}(bexp)\{stmt\}}(\mathfrak{m}^\sharp) \triangleq \mathbf{S}^\sharp_{\neg bexp} \circ \left(\mathsf{lfp}^{\subseteq^\sharp}\ \lambda\overline{\mathfrak{m}}^\sharp .\ \mathfrak{m}^\sharp \cup^\sharp \mathbf{S}^\sharp_{stmt} \circ \mathbf{S}^c_{bexp}(\overline{\mathfrak{m}}^\sharp)\right)$

Programs $\mathbf{S}^\sharp_P : \mathbb{M}^\sharp \to \mathbb{M}^\sharp$

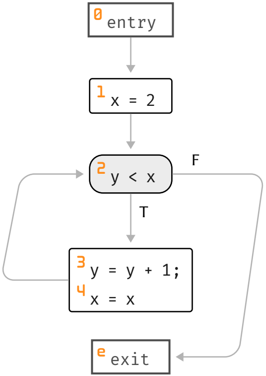
Inductive cases

- $\mathbf{S}^\sharp_{stmt_1\,;\,stmt_2}(\mathfrak{m}^\sharp) \triangleq \mathbf{S}^\sharp_{stmt_2} \circ \mathbf{S}^\sharp_{stmt_1}(\mathfrak{m}^\sharp)$
- $\mathbf{S}^\sharp_{\texttt{if}(bexp)\{stmt_1\}\texttt{else}\{stmt_2\}}(\mathfrak{m}^\sharp) \triangleq \mathbf{S}^\sharp_{stmt_1} \circ \mathbf{S}^\sharp_{bexp}(\mathfrak{m}^\sharp) \cup^\sharp \mathbf{S}^\sharp_{stmt_2} \circ \mathbf{S}^\sharp_{\neg bexp}(\mathfrak{m}^\sharp)$
- $\mathbf{S}^\sharp_{\texttt{while}(bexp)\{stmt\}}(\mathfrak{m}^\sharp) \triangleq \mathbf{S}^\sharp_{\neg bexp} \circ (\mathsf{lfp}^{\subseteq^\sharp}\ \lambda\overline{\mathfrak{m}}^\sharp\,.\ \mathfrak{m}^\sharp \cup^\sharp \mathbf{S}^\sharp_{stmt} \circ \mathbf{S}^{c}_{bexp}(\overline{\mathfrak{m}}^\sharp))$

Soundness

$$[\![P]\!]^c \subseteq \gamma([\![P]\!]^\sharp) \qquad\qquad \alpha([\![P]\!]^c) \subseteq^\sharp [\![P]\!]^\sharp$$

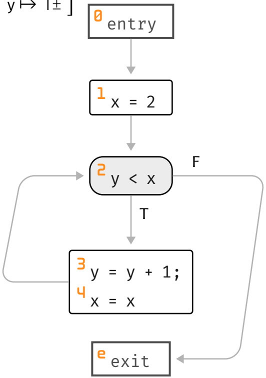
Abstract Post-condition semantics: example



Abstract memories: $\mathbb{M}^\sharp = \mathbb{X} \to \mathbb{Z}^\pm$

Abstract Post-condition semantics: example

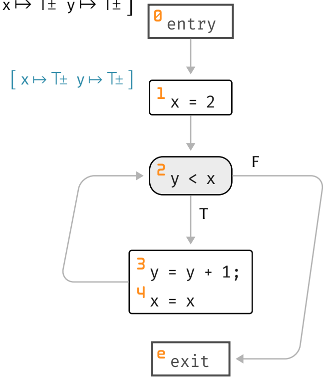
Input: $[\, x \mapsto \top^{\pm} \;\; y \mapsto \top^{\pm} \,]$



Abstract memories: $\mathbb{M}^{\sharp} = \mathbb{X} \to \mathbb{Z}^{\pm}$

Abstract Post-condition semantics: example

Input: $[\, x \mapsto \top_{\pm} \;\; y \mapsto \top_{\pm} \,]$

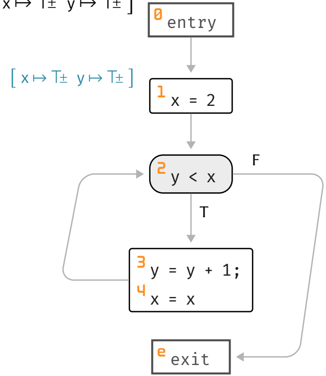


Abstract memories: $\mathbb{M}^{\sharp} = \mathbb{X} \to \mathbb{Z}^{\pm}$

Abstract Post-condition semantics: example

$$\mathbf{S}^{\sharp}_{\mathtt{x=2}}(\mathfrak{m}^{\sharp}) \triangleq \mathfrak{m}^{\sharp}[\mathtt{x} \leftarrow \alpha(2)]$$

Input: $[\, \mathtt{x} \mapsto \top_{\pm} \;\; \mathtt{y} \mapsto \top_{\pm} \,]$

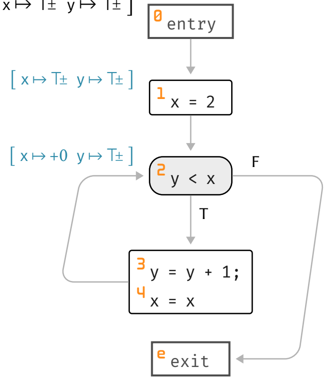


Abstract memories: $\mathbb{M}^{\sharp} = \mathbb{X} \to \mathbb{Z}^{\pm}$

Abstract Post-condition semantics: example

$$\mathbf{S}^{\sharp}_{\mathtt{x=2}}(\mathbb{m}^{\sharp}) \triangleq \mathbb{m}^{\sharp}[\mathtt{x} \leftarrow \alpha(2)] = [\, \mathtt{x} \mapsto +0 \;\; \mathtt{y} \mapsto \top\pm \,]$$

Input: $[\, \mathtt{x} \mapsto \top\pm \;\; \mathtt{y} \mapsto \top\pm \,]$

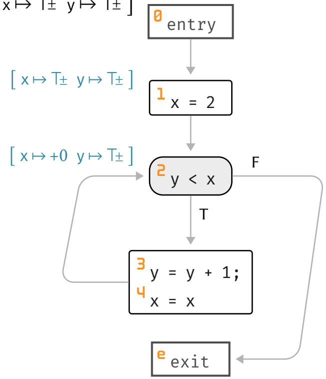


Abstract memories: $\mathbb{M}^{\sharp} = \mathbb{X} \to \mathbb{Z}^{\pm}$

Abstract Post-condition semantics: example

$$\mathbf{S}^\sharp_{\texttt{while(x<y)\{y=y+1;x=x\}}}(\mathbb{m}^\sharp) \triangleq \mathbf{S}^\sharp_{\texttt{x>=y}}\left(\mathrm{lfp}^{\subseteq^\sharp}\ \lambda\overline{\mathbb{m}}^\sharp\ .\ \mathbb{m}^\sharp \cup^\sharp \mathbf{S}^\sharp_{\texttt{y=y+1;x=x}} \circ \mathbf{S}^\sharp_{\texttt{x<y}}(\overline{\mathbb{m}}^\sharp)\right)$$

Input: $[\ x \mapsto \top\pm\ \ y \mapsto \top\pm\ ]$



Abstract memories: $\mathbb{M}^\sharp = \mathbb{X} \to \mathbb{Z}^\pm$

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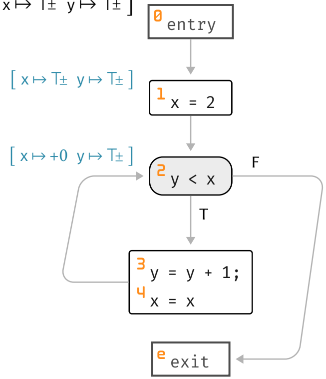

$X^0 = [\ x \mapsto +0\ \ y \mapsto \top_\pm\ ]$

Abstract memories: $\mathbb{M}^\sharp = \mathbb{X} \to \mathbb{Z}^\pm$

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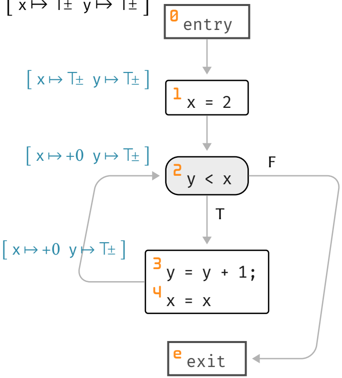
Input: $[\,\texttt{x} \mapsto \top^{\pm}\ \ \texttt{y} \mapsto \top^{\pm}\,]$



$$X^0 = [\,\texttt{x} \mapsto +0\ \ \texttt{y} \mapsto \top^{\pm}\,]$$

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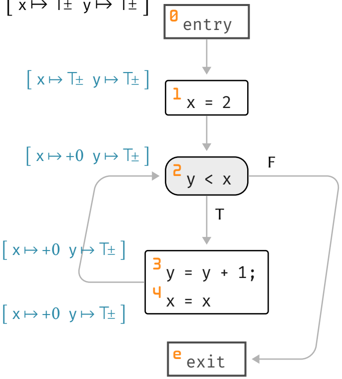

$X^0 = [\,x \mapsto +0\ \ y \mapsto \top_\pm\,]$

Abstract memories: $\mathbb{M}^\sharp = \mathbb{X} \to \mathbb{Z}^\pm$

Abstract Post-condition semantics: example

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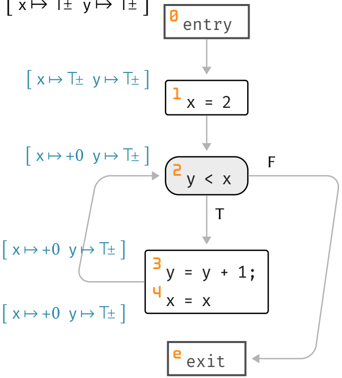
Input: $[\,\mathtt{x} \mapsto \top_\pm \ \ \mathtt{y} \mapsto \top_\pm\,]$



$$X^0 = [\,\mathtt{x} \mapsto +0 \ \ \mathtt{y} \mapsto \top_\pm\,]$$

$$X^1 = [\,\mathtt{x} \mapsto +0 \ \ \mathtt{y} \mapsto \top_\pm\,]$$

Abstract memories: $\mathbb{M}^\sharp = \mathbb{X} \to \mathbb{Z}^\pm$

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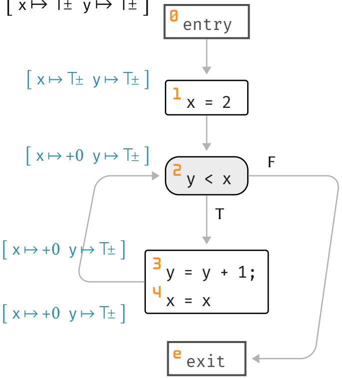

$X^0 = [\, x \mapsto +0 \ \ y \mapsto \top_\pm \,]$

$\mathrm{lfp} = [\, x \mapsto +0 \ \ y \mapsto \top_\pm \,]$

Abstract memories: $\mathbb{M}^\sharp = \mathbb{X} \to \mathbb{Z}^\pm$

Abstract Post-condition semantics: example

$$\mathbf{S}^\sharp_{\texttt{while(x<y)\{y=y+1;x=x\}}}(\mathbb{m}^\sharp) \triangleq \mathbf{S}^\sharp_{\texttt{x>=y}}\left(\mathrm{lfp}^{\subseteq^\sharp}\ \lambda \overline{\mathbb{m}}^\sharp\,.\ \mathbb{m}^\sharp \cup^\sharp \mathbf{S}^\sharp_{\texttt{y=y+1;x=x}} \circ \mathbf{S}^\sharp_{\texttt{x<y}}(\overline{\mathbb{m}}^\sharp)\right)$$

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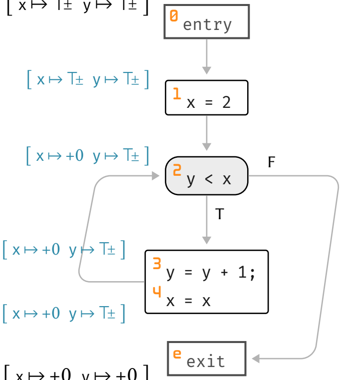

Output: $[\,x \mapsto +0 \;\; y \mapsto +0\,]$

$X^0 = [\,x \mapsto +0 \;\; y \mapsto \top_\pm\,]$

$\mathrm{lfp} = [\,x \mapsto +0 \;\; y \mapsto \top_\pm\,]$

Output: $[\,x \mapsto +0 \;\; y \mapsto +0\,]$

Abstract memories: $\mathbb{M}^\sharp = \mathbb{X} \to \mathbb{Z}^\pm$

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Non-relational Abstraction

Forget all relations between variables

$$\langle \wp(\mathbb{X} \to \mathbb{V}), \subseteq \rangle \underset{\alpha_{nr}}{\overset{\gamma_{nr}}{\leftrightarrows}} \langle \mathbb{X} \to \wp(\mathbb{V}), \dot{\subseteq} \rangle$$

where
$$\alpha_{nr} \triangleq \lambda X \, . \, \lambda \mathbb{x} \, . \, \{ \mathbb{m}(\mathbb{x}) \in \mathbb{V} \mid \mathbb{m} \in X \} \qquad \gamma_{nr} \triangleq \lambda \overline{\mathbb{m}} \, . \, \{ \mathbb{m} \in \mathbb{M} \mid \forall \mathbb{x} \in \mathbb{X} \, . \, \mathbb{m}(\mathbb{x}) \in \overline{\mathbb{m}}(\mathbb{x}) \}$$

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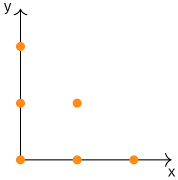

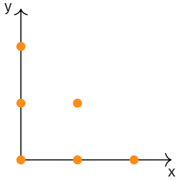

$\{\mathbb{m} \in \mathbb{M} \mid \mathbb{m}(\mathrm{x}), \mathbb{m}(\mathrm{y}) \in [0, 2] \wedge \mathrm{x} + \mathrm{y} \leq 2\}$

Forget all relations between variables

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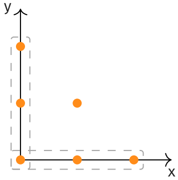
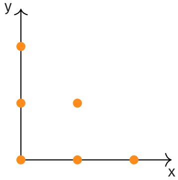
where $\alpha_{nr} \triangleq \lambda X.\ \lambda \mathbb{x}.\ \{\mathbb{m}(\mathbb{x}) \in \mathbb{V} \mid \mathbb{m} \in X\}$ $\qquad$ $\gamma_{nr} \triangleq \lambda \overline{\mathbb{m}}.\ \{\mathbb{m} \in \mathbb{M} \mid \forall \mathbb{x} \in \mathbb{X}.\ \mathbb{m}(\mathbb{x}) \in \overline{\mathbb{m}}(\mathbb{x})\}$





$\{\mathbb{m} \in \mathbb{M} \mid \mathbb{m}(\mathsf{x}), \mathbb{m}(\mathsf{y}) \in [0, 2] \wedge \mathsf{x} + \mathsf{y} \leq 2\}$

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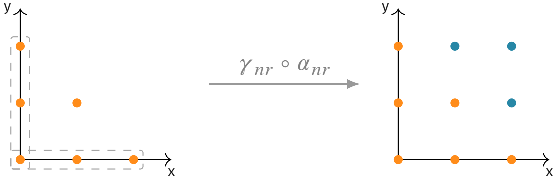

$\{\mathbb{m} \in \mathbb{M} \mid \mathbb{m}(\mathsf{x}), \mathbb{m}(\mathsf{y}) \in [0, 2] \land \mathsf{x} + \mathsf{y} \leq 2\}$ $\quad \{\mathbb{m} \in \mathbb{M} \mid \mathbb{m}(\mathsf{x}) \in [0, 2] \land \mathbb{m}(\mathsf{y}) \in [0, 2]\}$

Abstraction $\mathbb{V}^\sharp$ of values $\wp(\mathbb{V})$

on single variables

Abstraction $\mathbb{V}^\sharp$ of values $\wp(\mathbb{V})$ — on single variables

Ingredients

$\mathbb{V}^\sharp$	machine-representable abstract values
$\alpha^v : \wp(\mathbb{V}) \to \mathbb{V}^\sharp$	abstraction (encoding)
$\gamma^v : \mathbb{V}^\sharp \to \wp(\mathbb{V})$	concretization (decoding)
$\subseteq_v$	computable abstract partial order
$\perp^v$ and $\top^v$	abstract representation of $\emptyset$ and $\mathbb{V}$
$\cup^v$ and $\cap^v$	abstract version of $\cup$ and $\cap$
∇^v and Δ^v	extrapolation and interpolation operators

Define a sound version of arithmetic operations in $\mathbb{V}^\sharp$

$$
\begin{aligned}
\{\text{-}\mathbb{v} \mid \mathbb{v} \in \gamma^{v}(\mathbb{v}^\sharp)\} &\subseteq \gamma^{v}(\text{-}_{v}\mathbb{v}^\sharp) \\
\{\mathbb{v}_1 + \mathbb{v}_2 \mid \mathbb{v}_1 \in \gamma^{v}(\mathbb{v}_1^\sharp) \wedge \mathbb{v}_2 \in \gamma^{v}(\mathbb{v}_2^\sharp)\} &\subseteq \gamma^{v}(\mathbb{v}_1^\sharp +_{v} \mathbb{v}_2^\sharp) \\
&\dots
\end{aligned}
$$

Define a sound version of arithmetic operations in $\mathbb{V}^\sharp$

$$\begin{aligned}\{\text{-}\mathbb{v} \mid \mathbb{v} \in \gamma^{v}(\mathbb{v}^\sharp)\} &\subseteq \gamma^{v}(\text{-}_v \mathbb{v}^\sharp)\\ \{\mathbb{v}_1 + \mathbb{v}_2 \mid \mathbb{v}_1 \in \gamma^{v}(\mathbb{v}_1^\sharp) \wedge \mathbb{v}_2 \in \gamma^{v}(\mathbb{v}_2^\sharp)\} &\subseteq \gamma^{v}(\mathbb{v}_1^\sharp +_v \mathbb{v}_2^\sharp)\\ &\dots\end{aligned}$$

Approximate the best correct abstract operations in the case of Galois Connections

$$\begin{aligned}\text{-}_v \mathbb{v}^\sharp &\triangleq \alpha_v(\{\text{-}\mathbb{v} \mid \mathbb{v} \in \gamma^{v}(\mathbb{v}^\sharp)\})\\ \mathbb{v}_1^\sharp +_v \mathbb{v}_2^\sharp &\triangleq \alpha_v(\{\mathbb{v}_1 + \mathbb{v}_2 \mid \mathbb{v}_1 \in \gamma^{v}(\mathbb{v}_1^\sharp) \wedge \mathbb{v}_2 \in \gamma^{v}(\mathbb{v}_2^\sharp)\})\\ &\dots\end{aligned}$$

Lift to non-relational program memories

$$\langle \wp(\mathbb{V}), \subseteq \rangle \underset{\alpha^{\nu}}{\overset{\gamma^{\nu}}{\leftrightarrows}} \langle \mathbb{V}^{\sharp}, \subseteq_{\nu} \rangle$$

Lift to non-relational program memories

$$\langle \wp(\mathbb{V}), \subseteq \rangle \underset{\alpha^{v}}{\overset{\gamma^{v}}{\leftrightarrows}} \langle \mathbb{V}^{\sharp}, \subseteq_{v} \rangle \quad \text{lift to set } \mathbb{X} \quad \langle \mathbb{X} \to \wp(\mathbb{V}), \dot{\subseteq} \rangle \underset{\lambda f\,.\,\alpha^{v} \circ f}{\overset{\lambda f\,.\,\gamma^{v} \circ f}{\leftrightarrows}} \langle \mathbb{X} \to \mathbb{V}^{\sharp}, \dot{\subseteq}^{\sharp}_{v} \rangle$$

Lift to non-relational program memories

Associate an abstract value to each variable

$$\langle \wp(\mathbb{V}), \subseteq \rangle \underset{\alpha^{v}}{\overset{\gamma^{v}}{\leftrightarrows}} \langle \mathbb{V}^{\sharp}, \subseteq_{v} \rangle \quad \text{lift to set } \mathbb{X} \quad \langle \mathbb{X} \to \wp(\mathbb{V}), \dot{\subseteq} \rangle \underset{\lambda \overline{\mathbb{m}} .\ \alpha^{v} \circ \overline{\mathbb{m}}}{\overset{\lambda \overline{\mathbb{m}} .\ \gamma^{v} \circ \overline{\mathbb{m}}}{\leftrightarrows}} \langle \mathbb{X} \to \mathbb{V}^{\sharp}, \dot{\subseteq}^{\sharp}_{v} \rangle$$

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By composition we have with $\mathbb{M}^{\sharp} \triangleq \mathbb{X} \to \mathbb{V}^{\sharp}$ and $\subseteq^{\sharp} \triangleq \dot{\subseteq}_{v}$

$$\langle \wp(\mathbb{M}), \subseteq \rangle \underset{\alpha^{v}_{nr}}{\overset{\gamma^{v}_{nr}}{\leftrightarrows}} \langle \mathbb{M}^{\sharp}, \subseteq^{\sharp} \rangle$$

Associate an abstract value to each variable

$$\langle \wp(\mathbb{V}), \subseteq \rangle \underset{\alpha^{v}}{\overset{\gamma^{v}}{\leftrightarrows}} \langle \mathbb{V}^{\sharp}, \subseteq_{v} \rangle \quad \text{lift to set } \mathbb{X} \quad \langle \mathbb{X} \to \wp(\mathbb{V}), \dot{\subseteq} \rangle \underset{\lambda \overline{\mathrm{m}}\,.\, \alpha^{v} \circ \overline{\mathrm{m}}}{\overset{\lambda \overline{\mathrm{m}}\,.\, \gamma^{v} \circ \overline{\mathrm{m}}}{\leftrightarrows}} \langle \mathbb{X} \to \mathbb{V}^{\sharp}, \dot{\subseteq}^{\sharp}_{v} \rangle$$

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$$\langle \wp(\mathbb{M}), \subseteq \rangle \underset{\alpha^{v}_{nr}}{\overset{\gamma^{v}_{nr}}{\leftrightarrows}} \langle \mathbb{M}^{\sharp}, \subseteq^{\sharp} \rangle$$

$\bot^{\sharp} \triangleq \lambda \mathrm{x}\,.\, \bot^{v}$ and $\top^{\sharp} \triangleq \lambda \mathrm{x}\,.\, \top^{v}$

Associate an abstract value to each variable

$$\langle \wp(\mathbb{V}), \subseteq \rangle \underset{\alpha^{v}}{\overset{\gamma^{v}}{\leftrightarrows}} \langle \mathbb{V}^{\sharp}, \subseteq_{v} \rangle \quad \text{lift to set } \mathbb{X} \quad \langle \mathbb{X} \to \wp(\mathbb{V}), \dot{\subseteq} \rangle \underset{\lambda \overline{\mathbb{m}}\,.\, \alpha^{v} \circ \overline{\mathbb{m}}}{\overset{\lambda \overline{\mathbb{m}}\,.\, \gamma^{v} \circ \overline{\mathbb{m}}}{\leftrightarrows}} \langle \mathbb{X} \to \mathbb{V}^{\sharp}, \dot{\subseteq}^{\sharp}_{v} \rangle$$

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$\bot^{\sharp} \triangleq \lambda \mathrm{x}\,.\, \bot^{v}$ and $\top^{\sharp} \triangleq \lambda \mathrm{x}\,.\, \top^{v}$

$\alpha^{v}_{nr} \triangleq \lambda X\,.\, \llbracket X = \emptyset \,?\, \bot^{\sharp} \,:\, \lambda \mathrm{x}\,.\, \alpha^{v}(\{\mathbb{m}(\mathrm{x}) \in \mathbb{V} \mid \mathbb{m} \in X\}) \rrbracket$

Associate an abstract value to each variable

$$\langle \wp(\mathbb{V}), \subseteq \rangle \underset{\alpha^v}{\overset{\gamma^v}{\leftrightarrows}} \langle \mathbb{V}^\sharp, \subseteq_v \rangle \quad \text{lift to set } \mathbb{X} \quad \langle \mathbb{X} \to \wp(\mathbb{V}), \dot{\subseteq} \rangle \underset{\lambda \overline{\mathbb{m}}\,.\, \alpha^v \circ \overline{\mathbb{m}}}{\overset{\lambda \overline{\mathbb{m}}\,.\, \gamma^v \circ \overline{\mathbb{m}}}{\leftrightarrows}} \langle \mathbb{X} \to \mathbb{V}^\sharp, \dot{\subseteq}^\sharp_v \rangle$$

By composition we have with $\mathbb{M}^\sharp \triangleq \mathbb{X} \to \mathbb{V}^\sharp$ and $\subseteq^\sharp \triangleq \dot{\subseteq}_v$

$$\langle \wp(\mathbb{M}), \subseteq \rangle \underset{\alpha^v_{nr}}{\overset{\gamma^v_{nr}}{\leftrightarrows}} \langle \mathbb{M}^\sharp, \subseteq^\sharp \rangle$$

$\bot^\sharp \triangleq \lambda \mathsf{x}\,.\, \bot^v$ and $\top^\sharp \triangleq \lambda \mathsf{x}\,.\, \top^v$

$\alpha^v_{nr} \triangleq \lambda X\,.\, \llparenthesis X = \emptyset \ ?\ \bot^\sharp \ :\ \lambda \mathsf{x}\,.\, \alpha^v(\{\mathbb{m}(\mathsf{x}) \in \mathbb{V} \mid \mathbb{m} \in X\}) \rrparenthesis$

$\gamma^v_{nr} \triangleq \lambda \mathbb{m}^\sharp\,.\, \llparenthesis \mathbb{m}^\sharp = \bot^\sharp \ ?\ \emptyset \ :\ \{\mathbb{m} \in \mathbb{M} \mid \forall \mathbb{x} \in \mathbb{X}\,.\, \mathbb{m}(\mathbb{x}) \in \gamma^v(\mathbb{m}^\sharp(\mathbb{x}))\} \rrparenthesis$

Abstract partial order

$$\mathbb{m}_1^\sharp \subseteq^\sharp \mathbb{m}_2^\sharp \triangleq \forall x \in \mathbb{X} \,.\, \mathbb{m}_1^\sharp(x) \mathrel{\dot{\subseteq}_{\mathcal{V}}} \mathbb{m}_2^\sharp(x)$$

Abstract partial order

$$\mathbb{m}_1^\sharp \subseteq^\sharp \mathbb{m}_2^\sharp \triangleq \forall \mathrm{x} \in \mathbb{X} \,.\, \mathbb{m}_1^\sharp(\mathrm{x}) \mathrel{\dot{\subseteq}_\nu} \mathbb{m}_2^\sharp(\mathrm{x})$$

Abstract join and meet

$$\mathbb{m}_1^\sharp \cup^\sharp \mathbb{m}_2^\sharp \triangleq \lambda \mathrm{x} \,.\, \mathbb{m}_1^\sharp(\mathrm{x}) \cup^\nu \mathbb{m}_2^\sharp(\mathrm{x})$$

$$\mathbb{m}_1^\sharp \cap^\sharp \mathbb{m}_2^\sharp \triangleq \lambda \mathrm{x} \,.\, \mathbb{m}_1^\sharp(\mathrm{x}) \cap^\nu \mathbb{m}_2^\sharp(\mathrm{x})$$

Abstract partial order

$$\mathbb{m}_1^\sharp \subseteq^\sharp \mathbb{m}_2^\sharp \triangleq \forall \mathtt{x} \in \mathbb{X} \,.\, \mathbb{m}_1^\sharp(\mathtt{x}) \,\dot{\subseteq}_v\, \mathbb{m}_2^\sharp(\mathtt{x})$$

Abstract join and meet

$$\mathbb{m}_1^\sharp \cup^\sharp \mathbb{m}_2^\sharp \triangleq \lambda \mathtt{x} \,.\, \mathbb{m}_1^\sharp(\mathtt{x}) \cup^v \mathbb{m}_2^\sharp(\mathtt{x})$$
$$\mathbb{m}_1^\sharp \cap^\sharp \mathbb{m}_2^\sharp \triangleq \lambda \mathtt{x} \,.\, \mathbb{m}_1^\sharp(\mathtt{x}) \cap^v \mathbb{m}_2^\sharp(\mathtt{x})$$

Abstract transfer functions $\llparenthesis\, ^{\ell'} stmt \,\rrparenthesis^\sharp$ and $\mathbf{S}^\sharp_{stmt}$ on $\mathbb{M}^\sharp \to \mathbb{M}^\sharp$ derived from abstract arithmetic operations $-_v$, $+_v$, ...

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Abstract assignments for relational domains

$$\llparenthesis\, {}^{\ell}\mathtt{x}=exp\, \rrparenthesis^{\sharp}\mathfrak{m}^{\sharp} \triangleq \mathbf{S}^{\sharp}_{\mathtt{x}=exp}(\mathfrak{m}^{\sharp}) \triangleq \mathfrak{m}^{\sharp}[\mathtt{x} \hookleftarrow \textsc{AbsEval}[exp](\mathfrak{m}^{\sharp})]$$

Abstract assignments for relational domains

$$(\!|\ {}^{\ell}\mathtt{x}{=}exp\ |\!)^{\sharp}\mathbb{m}^{\sharp} \triangleq \mathbf{S}^{\sharp}_{\mathtt{x}=exp}(\mathbb{m}^{\sharp}) \triangleq \mathbb{m}^{\sharp}[\mathtt{x} \hookleftarrow \textsc{AbsEval}[exp](\mathbb{m}^{\sharp})]$$

Abstract evaluation for expressions $\textsc{AbsEval}[exp] : \mathbb{M}^{\sharp} \to \mathbb{M}^{\sharp}$

$$\begin{aligned}
\textsc{AbsEval}[exp]\bot^{\sharp} &\triangleq \bot^{v} \\
\textsc{AbsEval}[\mathtt{v}]\mathbb{m}^{\sharp} &\triangleq \alpha^{v}(\{v\}) \\
\textsc{AbsEval}[\mathtt{x}]\mathbb{m}^{\sharp} &\triangleq \mathbb{m}^{\sharp}(\mathtt{x}) \\
\textsc{AbsEval}[\text{-}\,exp]\mathbb{m}^{\sharp} &\triangleq \text{-}_{v}\,\textsc{AbsEval}[exp]\mathbb{m}^{\sharp} \\
\textsc{AbsEval}[exp_1 + exp_2]\mathbb{m}^{\sharp} &\triangleq \textsc{AbsEval}[exp_1]\mathbb{m}^{\sharp} +_{v} \textsc{AbsEval}[exp_2]\mathbb{m}^{\sharp} \\
&\ldots
\end{aligned}$$

Abstract assignments for relational domains

$$(\!|\,^{\ell}\texttt{x=}exp\,|\!)^{\sharp}\mathfrak{m}^{\sharp} \triangleq \mathbf{S}^{\sharp}_{\texttt{x=}exp}(\mathfrak{m}^{\sharp}) \triangleq \mathfrak{m}^{\sharp}[\texttt{x} \hookleftarrow \textsc{AbsEval}[exp](\mathfrak{m}^{\sharp})]$$

Abstract evaluation for expressions $\textsc{AbsEval}[exp] : \mathbb{M}^{\sharp} \to \mathbb{M}^{\sharp}$

$$\begin{aligned}
\textsc{AbsEval}[exp]\bot^{\sharp} &\triangleq \bot^{v} \\
\textsc{AbsEval}[\texttt{v}]\mathfrak{m}^{\sharp} &\triangleq \alpha^{v}(\{v\}) \\
\textsc{AbsEval}[\texttt{x}]\mathfrak{m}^{\sharp} &\triangleq \mathfrak{m}^{\sharp}(\texttt{x}) \\
\textsc{AbsEval}[\texttt{-}\,exp]\mathfrak{m}^{\sharp} &\triangleq \texttt{-}_{v}\,\textsc{AbsEval}[exp]\mathfrak{m}^{\sharp} \\
\textsc{AbsEval}[exp_1 + exp_2]\mathfrak{m}^{\sharp} &\triangleq \textsc{AbsEval}[exp_1]\mathfrak{m}^{\sharp} +_{v} \textsc{AbsEval}[exp_2]\mathfrak{m}^{\sharp} \\
&\ldots
\end{aligned}$$

Soundness: $\forall \mathfrak{m}^{\sharp} \in \mathbb{M}^{\sharp}\,.\,\{z \in \mathbb{Z} \mid z = \textsc{Eval}[exp](\mathfrak{m}) \wedge \mathfrak{m} \in \gamma^{v}_{nr}(\mathfrak{m}^{\sharp})\} \subseteq \gamma^{v}(\textsc{AbsEval}[exp](\mathfrak{m}^{\sharp}))$

Abstract transfer functions for conditions/boolean expressions $(\!|\,{}^{\ell'}bexp\,|\!)^\sharp$ and $\mathbf{S}^\sharp_{bexp}$ on $\mathbb{M}^\sharp \to \mathbb{M}^\sharp$

Abstract transfer functions for conditions/boolean expressions $(\!|\ {}^{\ell'}bexp\ |\!)^\sharp$ and $\mathbf{S}^\sharp_{bexp}$ on $\mathbb{M}^\sharp \to \mathbb{M}^\sharp$

- Abstract domain dependent (depend on $\mathbb{V}^\sharp$)

Abstract transfer functions for conditions/boolean expressions $(\!|\, {}^{\ell'} bexp \,|\!)^\sharp$ and $\mathbf{S}^\sharp_{bexp}$ on $\mathbb{M}^\sharp \to \mathbb{M}^\sharp$

- Abstract domain dependent (depend on $\mathbb{V}^\sharp$)
- The identity is a coarse but sound solution:

$$(\!|\, {}^{\ell'} bexp \,|\!)^\sharp \mathfrak{m}^\sharp \triangleq \mathbf{S}^\sharp_{bexp}(\mathfrak{m}^\sharp) \triangleq \mathfrak{m}^\sharp$$

Abstract transfer functions for conditions/boolean expressions $(\!|\, {}^{\ell'}bexp \,|\!)^\sharp$ and $\mathbf{S}^\sharp_{bexp}$ on $\mathbb{M}^\sharp \to \mathbb{M}^\sharp$

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Iteration strategies and inductive cases are abstract domain independent

Abstract transfer functions for conditions/boolean expressions $\llparenthesis\,^{\ell'} bexp\,\rrparenthesis^\sharp$ and $\mathbf{S}^\sharp_{bexp}$ on $\mathbb{M}^\sharp \to \mathbb{M}^\sharp$

- Abstract domain dependent (depend on $\mathbb{V}^\sharp$)
- The identity is a coarse but sound solution:

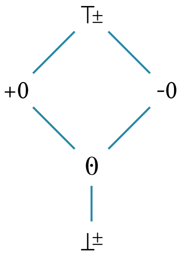
$$\llparenthesis\,^{\ell'} bexp\,\rrparenthesis^\sharp \mathfrak{m}^\sharp \triangleq \mathbf{S}^\sharp_{bexp}(\mathfrak{m}^\sharp) \triangleq \mathfrak{m}^\sharp$$

Iteration strategies and inductive cases are abstract domain independent

- $X_\ell \triangleq \bigcup^\sharp_{\ell' \in \mathbb{L}} \llparenthesis\,^{\ell'} stmt\,\rrparenthesis^\sharp X_{\ell'}$ such that $X_\ell \in \mathbb{M}^\sharp$
- $\mathbf{S}^\sharp_{\texttt{if}(bexp)\{stmt_1\}\texttt{else}\{stmt_2\}}(\mathfrak{m}^\sharp) \triangleq \mathbf{S}^\sharp_{stmt_1} \circ \mathbf{S}^\sharp_{bexp}(\mathfrak{m}^\sharp) \cup^\sharp \mathbf{S}^\sharp_{stmt_2} \circ \mathbf{S}^\sharp_{\neg bexp}(\mathfrak{m}^\sharp)$

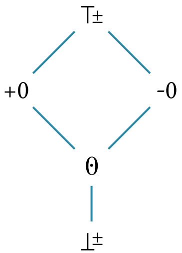
Non-relational Numerical Analyses

Complete lattice $\langle \mathbb{Z}^{\pm}, \subseteq^{\pm}, \cup^{\pm}, \cap^{\pm}, \bot^{\pm}, \top^{\pm} \rangle$ with $\langle \wp(\mathbb{Z}), \subseteq \rangle \xleftrightarrow[\alpha^{\pm}]{\gamma^{\pm}} \langle \mathbb{Z}^{\pm}, \subseteq^{\pm} \rangle$



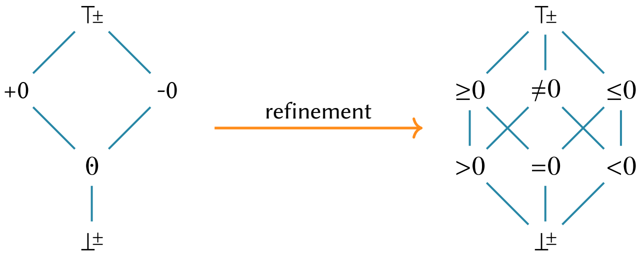
Sign abstract domain

Complete lattice $\langle \mathbb{Z}^{\pm}, \subseteq^{\pm}, \cup^{\pm}, \cap^{\pm}, \perp^{\pm}, \top^{\pm} \rangle$ with $\langle \wp(\mathbb{Z}), \subseteq \rangle \underset{\alpha^{\pm}}{\overset{\gamma^{\pm}}{\leftrightarrows}} \langle \mathbb{Z}^{\pm}, \subseteq^{\pm} \rangle$



Abstraction $\alpha^{\pm}$ and concretization $\gamma^{\pm}$ defined as in the previous class

Complete lattice $\langle \mathbb{Z}^{\pm}, \subseteq^{\pm}, \cup^{\pm}, \cap^{\pm}, \bot^{\pm}, \top^{\pm} \rangle$ with $\langle \wp(\mathbb{Z}), \subseteq \rangle \underset{\alpha^{\pm}}{\overset{\gamma^{\pm}}{\leftrightarrows}} \langle \mathbb{Z}^{\pm}, \subseteq^{\pm} \rangle$



Abstraction $\alpha^{\pm}$ and concretization $\gamma^{\pm}$ defined as in the previous class

Abstract arithmetic operations

Abstract arithmetic operations

$$z^\sharp \triangleq \alpha^\pm(\{z\}) = \begin{cases} 0 & \text{if } z = 0 \\ \text{-}0 & \text{if } z < 0 \\ +0 & \text{if } z > 0 \end{cases}$$

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$$\text{-}^\sharp\, \mathbb{v}^\pm \triangleq \alpha^\pm(\{\text{-}z \mid z \in \gamma^\pm(\mathbb{v}^\pm)\})$$

$$= \begin{cases} \bot^\pm & \text{if } \mathbb{v}^\pm = \bot^\pm \\ 0 & \text{if } \mathbb{v}^\pm = 0 \\ \text{-}0 & \text{if } \mathbb{v}^\pm = +0 \\ +0 & \text{if } \mathbb{v}^\pm = \text{-}0 \\ \top^\pm & \text{otherwise} \end{cases}$$

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$$\mathbb{v}_1^\pm +^\sharp \mathbb{v}_2^\pm \triangleq \alpha^\pm(\{z_1 + z_2 \mid z_1 \in \gamma^\pm(\mathbb{v}_1^\pm) \wedge z_2 \in \gamma^\pm(\mathbb{v}_2^\pm)\})$$

$$= \begin{cases} \bot^\pm & \text{if } \mathbb{v}_1^\pm = \bot^\pm \vee \mathbb{v}_2^\pm = \bot^\pm \\ 0 & \text{if } \mathbb{v}_1^\pm, \mathbb{v}_2^\pm \in \{0\} \\ \text{-}0 & \text{if } \mathbb{v}_1^\pm, \mathbb{v}_2^\pm \in \{0, \text{-}0\} \\ +0 & \text{if } \mathbb{v}_1^\pm, \mathbb{v}_2^\pm \in \{0, +0\} \\ \top^\pm & \text{otherwise} \end{cases}$$

Abstract conditions/boolean expressions for $\mathbb{Z}^{\pm}$

Abstract conditions/boolean expressions for $\mathbb{Z}^{\pm}$

$$(\!|\, {}^{\ell}\texttt{x <= 0} \,|\!)^{\sharp}\mathfrak{m}^{\sharp} \triangleq \mathbf{S}^{\sharp}_{\texttt{x<=0}}(\mathfrak{m}^{\sharp}) \triangleq \begin{cases} \mathfrak{m}^{\sharp}[\mathtt{x} \hookleftarrow 0] & \text{if } \mathfrak{m}^{\sharp}(\mathtt{x}) \in \{0, +0\} \\ \mathfrak{m}^{\sharp}[\mathtt{x} \hookleftarrow -0] & \text{if } \mathfrak{m}^{\sharp}(\mathtt{x}) \in \{-0, \top^{\pm}\} \\ \bot^{\pm} & \text{otherwise} \end{cases}$$

Abstract conditions/boolean expressions for $\mathbb{Z}^{\pm}$

$$
(\!|\, {}^{\ell}\texttt{x <= 0} \,|\!)^{\sharp}\mathfrak{m}^{\sharp} \triangleq \mathbf{S}^{\sharp}_{\texttt{x<=0}}(\mathfrak{m}^{\sharp}) \triangleq \begin{cases} \mathfrak{m}^{\sharp}[\texttt{x} \leftarrow \mathbb{0}] & \text{if } \mathfrak{m}^{\sharp}(\texttt{x}) \in \{\mathbb{0}, +0\} \\ \mathfrak{m}^{\sharp}[\texttt{x} \leftarrow -0] & \text{if } \mathfrak{m}^{\sharp}(\texttt{x}) \in \{-0, \top\pm\} \\ \bot^{\pm} & \text{otherwise} \end{cases}
$$

$$
(\!|\, {}^{\ell}\texttt{x <= } z \,|\!)^{\sharp}\mathfrak{m}^{\sharp} \triangleq \mathbf{S}^{\sharp}_{\texttt{x<=}z}(\mathfrak{m}^{\sharp}) \triangleq \begin{cases} (\!|\, {}^{\ell}\texttt{x <= 0} \,|\!)^{\sharp}\mathfrak{m}^{\sharp} = \mathbf{S}^{\sharp}_{\texttt{x<=0}}(\mathfrak{m}^{\sharp}) & \text{if } z \leq 0 \\ \mathfrak{m}^{\sharp} & \text{otherwise} \end{cases}
$$

Abstract conditions/boolean expressions for $\mathbb{Z}^{\pm}$

$$(\!|\, {}^{\ell}\mathtt{x\ <=\ 0}\, |\!)^{\sharp}\mathfrak{m}^{\sharp} \triangleq \mathbf{S}^{\sharp}_{\mathtt{x<=0}}(\mathfrak{m}^{\sharp}) \triangleq \begin{cases} \mathfrak{m}^{\sharp}[\mathtt{x} \hookleftarrow 0] & \text{if } \mathfrak{m}^{\sharp}(\mathtt{x}) \in \{0, +0\} \\ \mathfrak{m}^{\sharp}[\mathtt{x} \hookleftarrow -0] & \text{if } \mathfrak{m}^{\sharp}(\mathtt{x}) \in \{-0, \top\pm\} \\ \bot^{\pm} & \text{otherwise} \end{cases}$$

$$(\!|\, {}^{\ell}\mathtt{x\ <=\ } z\, |\!)^{\sharp}\mathfrak{m}^{\sharp} \triangleq \mathbf{S}^{\sharp}_{\mathtt{x<=}z}(\mathfrak{m}^{\sharp}) \triangleq \begin{cases} (\!|\, {}^{\ell}\mathtt{x\ <=\ 0}\, |\!)^{\sharp}\mathfrak{m}^{\sharp} = \mathbf{S}^{\sharp}_{\mathtt{x<=0}}(\mathfrak{m}^{\sharp}) & \text{if } z \leq 0 \\ \mathfrak{m}^{\sharp} & \text{otherwise} \end{cases}$$

$$\begin{aligned} (\!|\, {}^{\ell}\mathtt{x\ <=\ y}\, |\!)^{\sharp}\mathfrak{m}^{\sharp} \triangleq \mathbf{S}^{\sharp}_{\mathtt{x<=y}}(\mathfrak{m}^{\sharp}) \triangleq\ & (\!|\, \mathfrak{m}^{\sharp}(\mathtt{y}) \in \{0, -0\} \mathrel{?} (\!|\, {}^{\ell}\mathtt{x\ <=\ 0}\, |\!)^{\sharp}\mathfrak{m}^{\sharp} = \mathbf{S}^{\sharp}_{\mathtt{x<=0}}(\mathfrak{m}^{\sharp}) \mathbin{\text{⁝}} \mathfrak{m}^{\sharp} \,|\!) \\ & \cap^{\sharp} \\ & (\!|\, \mathfrak{m}^{\sharp}(\mathtt{x}) \in \{0, +0\} \mathrel{?} (\!|\, {}^{\ell}\mathtt{y\ >=\ 0}\, |\!)^{\sharp}\mathfrak{m}^{\sharp} = \mathbf{S}^{\sharp}_{\mathtt{y>=0}}(\mathfrak{m}^{\sharp}) \mathbin{\text{⁝}} \mathfrak{m}^{\sharp} \,|\!) \end{aligned}$$

Abstract conditions/boolean expressions for $\mathbb{Z}^{\pm}$ identity for the other cases

$$(\!|\, {}^{\ell}\mathtt{x} \;\mathtt{<=}\; \mathtt{0} \,|\!)^{\sharp}\mathfrak{m}^{\sharp} \triangleq \mathbf{S}^{\sharp}_{\mathtt{x<=0}}(\mathfrak{m}^{\sharp}) \triangleq \begin{cases} \mathfrak{m}^{\sharp}[\mathtt{x} \leftarrow 0] & \text{if } \mathfrak{m}^{\sharp}(\mathtt{x}) \in \{0, +0\} \\ \mathfrak{m}^{\sharp}[\mathtt{x} \leftarrow \text{-}0] & \text{if } \mathfrak{m}^{\sharp}(\mathtt{x}) \in \{\text{-}0, \top^{\pm}\} \\ \bot^{\pm} & \text{otherwise} \end{cases}$$

$$(\!|\, {}^{\ell}\mathtt{x} \;\mathtt{<=}\; z \,|\!)^{\sharp}\mathfrak{m}^{\sharp} \triangleq \mathbf{S}^{\sharp}_{\mathtt{x<=}z}(\mathfrak{m}^{\sharp}) \triangleq \begin{cases} (\!|\, {}^{\ell}\mathtt{x} \;\mathtt{<=}\; \mathtt{0} \,|\!)^{\sharp}\mathfrak{m}^{\sharp} = \mathbf{S}^{\sharp}_{\mathtt{x<=0}}(\mathfrak{m}^{\sharp}) & \text{if } z \leq 0 \\ \mathfrak{m}^{\sharp} & \text{otherwise} \end{cases}$$

$$\begin{aligned} (\!|\, {}^{\ell}\mathtt{x} \;\mathtt{<=}\; \mathtt{y} \,|\!)^{\sharp}\mathfrak{m}^{\sharp} \triangleq \mathbf{S}^{\sharp}_{\mathtt{x<=y}}(\mathfrak{m}^{\sharp}) \triangleq\; & [\![\, \mathfrak{m}^{\sharp}(\mathtt{y}) \in \{0, \text{-}0\} \;?\; (\!|\, {}^{\ell}\mathtt{x} \;\mathtt{<=}\; \mathtt{0} \,|\!)^{\sharp}\mathfrak{m}^{\sharp} = \mathbf{S}^{\sharp}_{\mathtt{x<=0}}(\mathfrak{m}^{\sharp}) \;\text{:}\; \mathfrak{m}^{\sharp} \,]\!] \\ & \cap^{\sharp} \\ & [\![\, \mathfrak{m}^{\sharp}(\mathtt{x}) \in \{0, +0\} \;?\; (\!|\, {}^{\ell}\mathtt{y} \;\mathtt{>=}\; \mathtt{0} \,|\!)^{\sharp}\mathfrak{m}^{\sharp} = \mathbf{S}^{\sharp}_{\mathtt{y>=0}}(\mathfrak{m}^{\sharp}) \;\text{:}\; \mathfrak{m}^{\sharp} \,]\!] \end{aligned}$$

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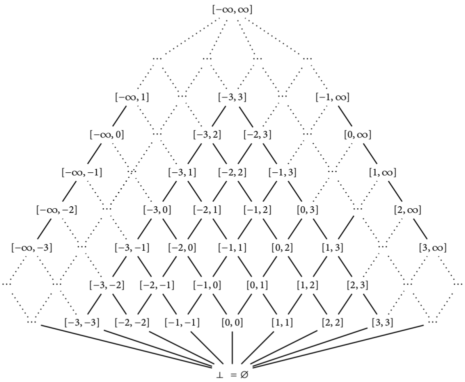
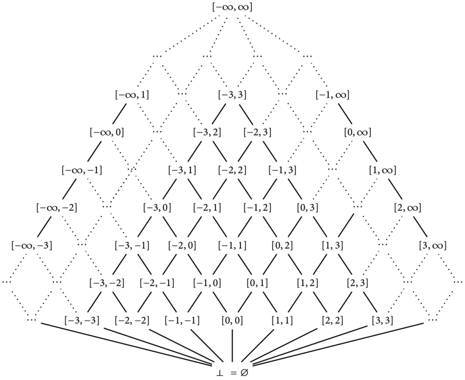
Complete lattice $\langle \mathbb{Z}^\imath, \subseteq^\imath, \cup^\imath, \cap^\imath, \perp^\imath, \top^\imath \rangle$ with $\langle \wp(\mathbb{Z}), \subseteq \rangle \underset{\alpha^\imath}{\overset{\gamma^\imath}{\leftrightarrows}} \langle \mathbb{Z}^\imath, \subseteq^\imath \rangle$



Complete lattice $\langle \mathbb{Z}^\imath, \subseteq^\imath, \cup^\imath, \cap^\imath, \perp^\imath, \top^\imath \rangle$ with $\langle \wp(\mathbb{Z}), \subseteq \rangle \underset{\alpha^\imath}{\overset{\gamma^\imath}{\leftrightarrows}} \langle \mathbb{Z}^\imath, \subseteq^\imath \rangle$



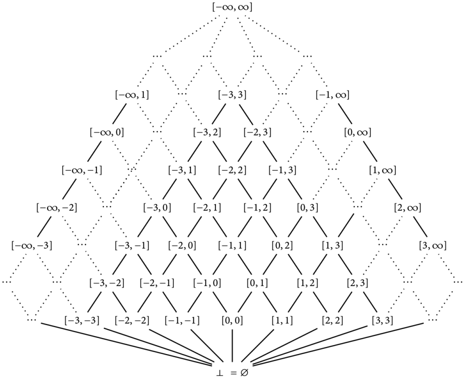
$$[a,b] \subseteq^\imath [a',b'] \triangleq a \geq a' \wedge b \leq b'$$

$$[a,b] \cup^\imath [a',b'] \triangleq [a \curlyvee a', b \curlywedge b']$$

$$[a,b] \cap^\imath [a',b'] \triangleq \begin{cases} [a \curlywedge a', b \curlyvee b'] & \text{if } a \curlywedge a' \leq b \curlyvee b' \\ \perp^\imath & \text{otherwise} \end{cases}$$

$\perp^\imath \triangleq [\infty, -\infty]$ and $\top^\imath \triangleq [-\infty, \infty]$

Complete lattice $\langle \mathbb{Z}^\imath, \subseteq^\imath, \cup^\imath, \cap^\imath, \bot^\imath, \top^\imath \rangle$ with $\langle \wp(\mathbb{Z}), \subseteq \rangle \underset{\alpha^\imath}{\overset{\gamma^\imath}{\leftrightarrows}} \langle \mathbb{Z}^\imath, \subseteq^\imath \rangle$



$$[a,b] \subseteq^\imath [a',b'] \triangleq a \geq a' \wedge b \leq b'$$

$$[a,b] \cup^\imath [a',b'] \triangleq [a \vee a', b \wedge b']$$

$$[a,b] \cap^\imath [a',b'] \triangleq \begin{cases} [a \wedge a', b \vee b'] & \text{if } a \wedge a' \leq b \vee b' \\ \bot^\imath & \text{otherwise} \end{cases}$$

$\bot^\imath \triangleq [\infty, -\infty]$ and $\top^\imath \triangleq [-\infty, \infty]$

Abstraction $\alpha^\imath$ and concretization $\gamma^\imath$ defined as in the previous class

Abstract arithmetic operations

Abstract arithmetic operations

$$z^\sharp \triangleq \alpha^\imath(\{z\}) = [z, z]$$

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Abstract arithmetic operations

$$z^\sharp \triangleq \alpha^\imath(\{z\}) = [z, z]$$

$$-^\sharp [a, a'] \triangleq \alpha^\imath(\{-z \mid z \in \gamma^\imath([a, a'])\}) = [-a', -a]$$

Abstract arithmetic operations

$$z^\sharp \triangleq \alpha^\imath(\{z\}) = [z, z]$$

$$-^\sharp [a, a'] \triangleq \alpha^\imath(\{-z \mid z \in \gamma^\imath([a, a'])\}) = [-a', -a]$$

$$\begin{aligned}
[a, a'] +^\sharp [b, b'] &\triangleq \alpha^\imath(\{z_1 + z_2 \mid z_1 \in \gamma^\imath([a, a']) \wedge z_2 \in \gamma^\imath([b, b'])\}) = [a + b, a' + b'] \\
[a, a'] -^\sharp [b, b'] &\triangleq \alpha^\imath(\{z_1 - z_2 \mid z_1 \in \gamma^\imath([a, a']) \wedge z_2 \in \gamma^\imath([b, b'])\}) = [a - b', a' - b] \\
[a, a'] \times^\sharp [b, b'] &\triangleq \alpha^\imath(\{z_1 \cdot z_2 \mid z_1 \in \gamma^\imath([a, a']) \wedge z_2 \in \gamma^\imath([b, b'])\}) \\
&= [\bigvee\{a \cdot b,\ a \cdot b',\ a' \cdot b,\ a' \cdot b'\}, \bigwedge\{a \cdot b,\ a \cdot b',\ a' \cdot b,\ a' \cdot b'\}]
\end{aligned}$$

Abstract arithmetic operations

$z^\sharp \triangleq \alpha^\imath(\{z\}) = [z, z]$

Operations are strict: $\text{-}^\sharp \bot^\imath \triangleq \bot^\imath$, $[a, a'] +^\sharp \bot^\imath \triangleq \bot^\imath$, ...

$\text{-}^\sharp\, [a, a'] \triangleq \alpha^\imath(\{\text{-}\,z \mid z \in \gamma^\imath([a, a'])\}) = [\text{-}a', \text{-}a]$

$$
\begin{aligned}
[a, a'] +^\sharp [b, b'] &\triangleq \alpha^\imath(\{z_1 + z_2 \mid z_1 \in \gamma^\imath([a, a']) \wedge z_2 \in \gamma^\imath([b, b'])\}) = [a + b, a' + b'] \\
[a, a'] -^\sharp [b, b'] &\triangleq \alpha^\imath(\{z_1 - z_2 \mid z_1 \in \gamma^\imath([a, a']) \wedge z_2 \in \gamma^\imath([b, b'])\}) = [a - b', a' - b] \\
[a, a'] \times^\sharp [b, b'] &\triangleq \alpha^\imath(\{z_1 \cdot z_2 \mid z_1 \in \gamma^\imath([a, a']) \wedge z_2 \in \gamma^\imath([b, b'])\}) \\
&= [\textstyle\bigvee\{a \cdot b,\ a \cdot b',\ a' \cdot b,\ a' \cdot b'\}, \bigwedge\{a \cdot b,\ a \cdot b',\ a' \cdot b,\ a' \cdot b'\}]
\end{aligned}
$$

Abstract conditions/boolean expressions for $\mathbb{Z}^\imath$

Abstract conditions/boolean expressions for $\mathbb{Z}^\imath$

Assuming $\mathfrak{m}^\sharp(\mathtt{x}) = [a, a']$

$$(\!|\, {}^{\ell}\mathtt{x} \;\; \texttt{<=}\; z \,|\!)^\sharp \mathfrak{m}^\sharp \triangleq \mathbf{S}^\sharp_{\mathtt{x} \texttt{<=} z}(\mathfrak{m}^\sharp) \triangleq \begin{cases} \mathfrak{m}^\sharp_\perp & \text{if } a > z \\ \mathfrak{m}^\sharp[\mathtt{x} \hookleftarrow [a, a' \curlywedge z]] & \text{otherwise} \end{cases}$$

Abstract conditions/boolean expressions for $\mathbb{Z}^\imath$

Assuming $\mathfrak{m}^\sharp(\mathtt{x}) = [a, a']$

$$(\!|\, {}^{\ell}\mathtt{x} \;\texttt{<=}\; z \,|\!)^\sharp \mathfrak{m}^\sharp \triangleq \mathbf{S}^\sharp_{\mathtt{x<=}z}(\mathfrak{m}^\sharp) \triangleq \begin{cases} \mathfrak{m}^\sharp_\perp & \text{if } a > z \\ \mathfrak{m}^\sharp[\mathtt{x} \leftarrow [a, a' \curlywedge z]] & \text{otherwise} \end{cases}$$

Assuming $\mathfrak{m}^\sharp(\mathtt{x}) = [a, a']$ and $\mathfrak{m}^\sharp(\mathtt{y}) = [b, b']$

$$(\!|\, {}^{\ell}\mathtt{x} \;\texttt{<=}\; \mathtt{y} \,|\!)^\sharp \mathfrak{m}^\sharp \triangleq \mathbf{S}^\sharp_{\mathtt{x<=y}}(\mathfrak{m}^\sharp) \triangleq \begin{cases} \mathfrak{m}^\sharp_\perp & \text{if } a > b' \\ \mathfrak{m}^\sharp[\mathtt{x} \leftarrow [a, a' \curlywedge b']\ \mathtt{y} \leftarrow [a \curlyvee b, b']] & \text{otherwise} \end{cases}$$

Abstract conditions/boolean expressions for $\mathbb{Z}^\imath$ identity for the other cases

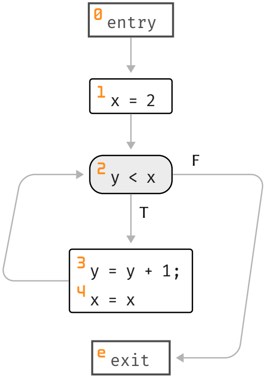
Assuming $\mathbb{m}^\sharp(\mathtt{x}) = [a, a']$

$$(\!|\, {}^{\ell}\mathtt{x}\ \texttt{<=}\ z \,|\!)^\sharp \mathbb{m}^\sharp \triangleq \mathbf{S}^\sharp_{\mathtt{x}\texttt{<=}z}(\mathbb{m}^\sharp) \triangleq \begin{cases} \mathbb{m}^\sharp_\bot & \text{if } a > z \\ \mathbb{m}^\sharp[\mathtt{x} \hookleftarrow [a, a' \curlyvee z]] & \text{otherwise} \end{cases}$$

Assuming $\mathbb{m}^\sharp(\mathtt{x}) = [a, a']$ and $\mathbb{m}^\sharp(\mathtt{y}) = [b, b']$

$$(\!|\, {}^{\ell}\mathtt{x}\ \texttt{<=}\ \mathtt{y} \,|\!)^\sharp \mathbb{m}^\sharp \triangleq \mathbf{S}^\sharp_{\mathtt{x}\texttt{<=}\mathtt{y}}(\mathbb{m}^\sharp) \triangleq \begin{cases} \mathbb{m}^\sharp_\bot & \text{if } a > b' \\ \mathbb{m}^\sharp[\mathtt{x} \hookleftarrow [a, a' \curlyvee b']\ \mathtt{y} \hookleftarrow [a \curlywedge b, b']] & \text{otherwise} \end{cases}$$

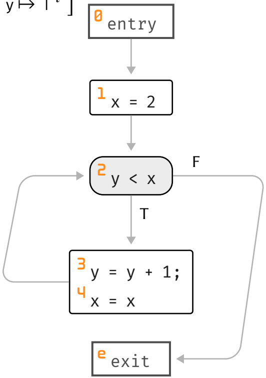
Intervals analysis: example



Abstract memories: $\mathbb{M}^\sharp = \mathbb{X} \to \mathbb{Z}^\imath$

Intervals analysis: example

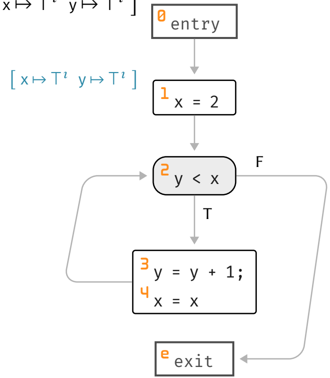
Input: $[\,x \mapsto \top^{\imath} \quad y \mapsto \top^{\imath}\,]$



Abstract memories: $\mathbb{M}^\sharp = \mathbb{X} \to \mathbb{Z}^{\imath}$

Intervals analysis: example

Input: $[\, x \mapsto \top^{\imath} \;\; y \mapsto \top^{\imath} \,]$

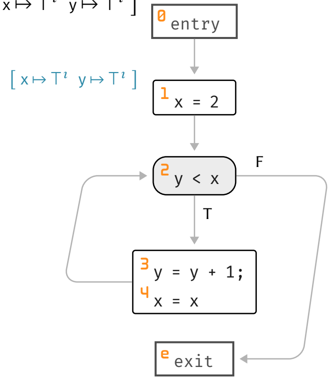


Abstract memories: $\mathbb{M}^{\sharp} = \mathbb{X} \to \mathbb{Z}^{\imath}$

Intervals analysis: example

$\mathbf{S}^\sharp_{\texttt{x=2}}(\mathfrak{m}^\sharp) \triangleq \mathfrak{m}^\sharp[\texttt{x} \hookleftarrow \alpha^\imath(2)]$

Input: $[\,\texttt{x} \mapsto \top^\imath \;\; \texttt{y} \mapsto \top^\imath\,]$

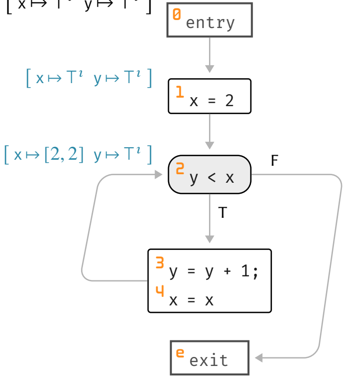


Abstract memories: $\mathbb{M}^\sharp = \mathbb{X} \to \mathbb{Z}^\imath$

Intervals analysis: example

$$\mathbf{S}^{\sharp}_{\mathtt{x=2}}(\mathfrak{m}^{\sharp}) \triangleq \mathfrak{m}^{\sharp}[\mathtt{x} \leftarrow \alpha^{\imath}(2)] = [\,\mathtt{x} \mapsto [2,2] \;\; \mathtt{y} \mapsto \top^{\imath}\,]$$

Input: $[\,\mathtt{x} \mapsto \top^{\imath} \;\; \mathtt{y} \mapsto \top^{\imath}\,]$

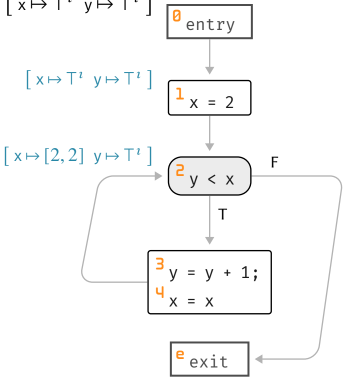


Abstract memories: $\mathbb{M}^{\sharp} = \mathbb{X} \to \mathbb{Z}^{\imath}$

Intervals analysis: example

$$\mathbf{S}^\sharp_{\texttt{while(x<y)\{y=y+1;x=x\}}}(\mathfrak{m}^\sharp) \triangleq \mathbf{S}^\sharp_{\texttt{x>=y}}\left(\mathrm{lfp}^{\subseteq^\sharp}\ \lambda \overline{\mathfrak{m}}^\sharp .\ \mathfrak{m}^\sharp \cup^\sharp \mathbf{S}^\sharp_{\texttt{y=y+1;x=x}} \circ \mathbf{S}^\sharp_{\texttt{x<y}}(\overline{\mathfrak{m}}^\sharp)\right)$$

Input: $[\, x \mapsto \top^\imath \quad y \mapsto \top^\imath \,]$

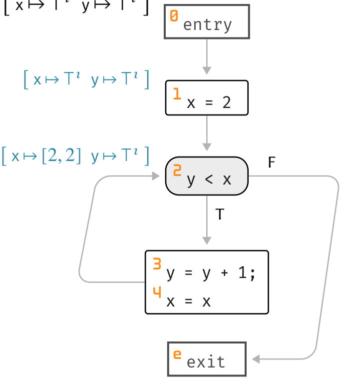


Abstract memories: $\mathbb{M}^\sharp = \mathbb{X} \to \mathbb{Z}^\imath$

Intervals analysis: example

$$\mathbf{S}^\sharp_{\texttt{while(x<y)\{y=y+1;x=x\}}}(\mathfrak{m}^\sharp) \triangleq \mathbf{S}^\sharp_{\texttt{x>=y}}(\mathrm{lfp}^{\subseteq^\sharp}\ \lambda\overline{\mathfrak{m}}^\sharp.\ \mathfrak{m}^\sharp \cup^\sharp \mathbf{S}^\sharp_{\texttt{y=y+1;x=x}} \circ \mathbf{S}^\sharp_{\texttt{x<y}}(\overline{\mathfrak{m}}^\sharp))$$

Input: $[\,\texttt{x} \mapsto \top^\imath\ \ \texttt{y} \mapsto \top^\imath\,]$



$X^0 = [\,\texttt{x} \mapsto [2,2]\ \ \texttt{y} \mapsto \top^\imath\,]$

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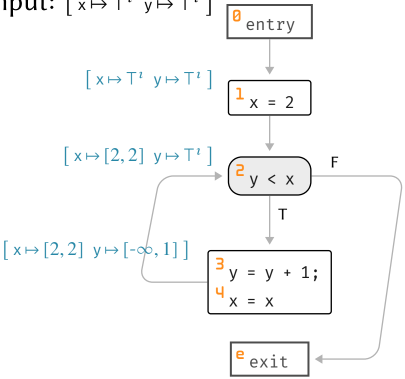

$$X^0 = [\,x \mapsto [2,2] \;\; y \mapsto \top^\imath\,]$$

Abstract memories: $\mathbb{M}^\sharp = \mathbb{X} \to \mathbb{Z}^\imath$

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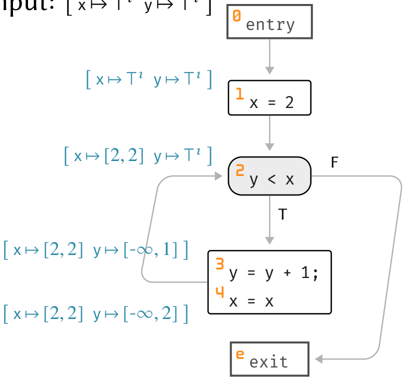

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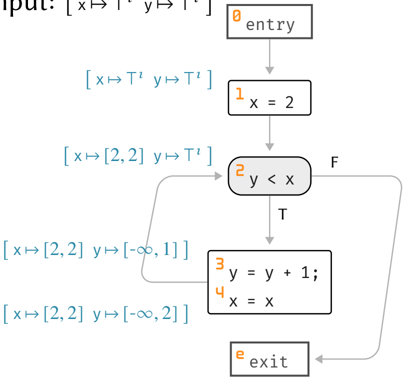

$$X^0 = [\,\mathtt{x} \mapsto [2,2]\ \ \mathtt{y} \mapsto \top^\imath\,]$$
$$X^1 = [\,\mathtt{x} \mapsto [2,2]\ \ \mathtt{y} \mapsto \top^\imath\,]$$

Abstract memories: $\mathbb{M}^\sharp = \mathbb{X} \to \mathbb{Z}^\imath$

Intervals analysis: example

$$\mathbf{S}^\sharp_{\texttt{while(x<y)\{y=y+1;x=x\}}}(\mathfrak{m}^\sharp) \triangleq \mathbf{S}^\sharp_{\texttt{x>=y}}\left(\mathrm{lfp}^{\subseteq^\sharp}\ \lambda \overline{\mathfrak{m}}^\sharp .\ \mathfrak{m}^\sharp \cup^\sharp \mathbf{S}^\sharp_{\texttt{y=y+1;x=x}} \circ \mathbf{S}^\sharp_{\texttt{x<y}}(\overline{\mathfrak{m}}^\sharp)\right)$$

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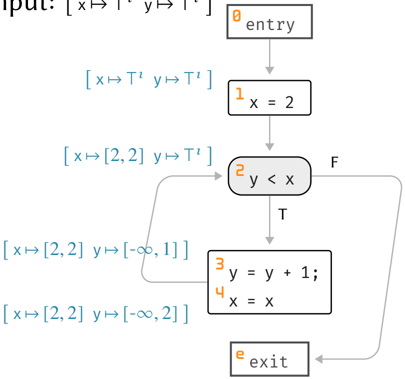

$$X^0 = [\,\texttt{x} \mapsto [2,2]\ \ \texttt{y} \mapsto \top^\imath\,]$$

$$\mathrm{lfp} = [\,\texttt{x} \mapsto [2,2]\ \ \texttt{y} \mapsto \top^\imath\,]$$

Abstract memories: $\mathbb{M}^\sharp = \mathbb{X} \to \mathbb{Z}^\imath$

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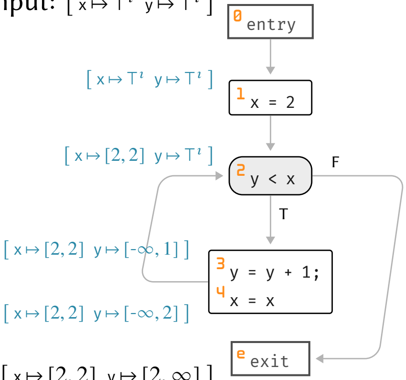

Output: $[\, \texttt{x} \mapsto [2,2] \ \ \texttt{y} \mapsto [2,\infty] \,]$

$X^0 = [\, \texttt{x} \mapsto [2,2] \ \ \texttt{y} \mapsto \top^\imath \,]$

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Output: $[\, \texttt{x} \mapsto [2,2] \ \ \texttt{y} \mapsto [2,\infty] \,]$

Abstract memories: $\mathbb{M}^\sharp = \mathbb{X} \rightarrow \mathbb{Z}^\imath$

The abstract Post-conditions semantics for intervals is sound:

$$[\![P]\!]^{\mathsf{c}} \circ \gamma^{\imath}_{nr}(\mathbb{m}^{\sharp}) \subseteq \gamma^{\imath}_{nr} \circ [\![P]\!]^{\sharp}\mathbb{m}^{\sharp} \qquad \mathbb{m}^{\sharp} \triangleq \alpha^{\imath}_{nr}(\mathfrak{I} \subseteq \mathbb{M}) \qquad \alpha^{\imath}_{nr} \circ [\![P]\!]^{\mathsf{c}}\mathfrak{I} \subseteq^{\sharp} [\![P]\!]^{\sharp} \circ \alpha^{\imath}_{nr}(\mathfrak{I})$$

The abstract Post-conditions semantics for intervals is sound:

$$\llbracket P \rrbracket^{c} \circ \gamma^{\imath}_{nr}(\mathbb{m}^{\sharp}) \subseteq \gamma^{\imath}_{nr} \circ \llbracket P \rrbracket^{\sharp}\mathbb{m}^{\sharp} \qquad \mathbb{m}^{\sharp} \triangleq \alpha^{\imath}_{nr}(\mathfrak{I} \subseteq \mathbb{M}) \qquad \alpha^{\imath}_{nr} \circ \llbracket P \rrbracket^{c}\mathfrak{I} \subseteq^{\sharp} \llbracket P \rrbracket^{\sharp} \circ \alpha^{\imath}_{nr}(\mathfrak{I})$$

(proof)

Prove $\alpha^{\imath}_{nr} \circ \llbracket P \rrbracket^{c} \circ \gamma^{\imath}_{nr} \subseteq^{\sharp} \llbracket P \rrbracket^{\sharp}$ by structural induction on $P = stmt_1 \,;\, \ldots \,;\, stmt_n$

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Prove $\alpha^{\imath}_{nr} \circ [\![P]\!]^{c} \circ \gamma^{\imath}_{nr} \subseteq^{\sharp} [\![P]\!]^{\sharp}$ by structural induction on $P = stmt_1 \,;\, \dots \,;\, stmt_n$

Recall that

- $[\![P]\!]^{c}\mathfrak{I} = \mathbf{S}^{c}_{stmt_n} \circ \dots \circ \mathbf{S}^{c}_{stmt_1}(\mathfrak{I})$ and $[\![P]\!]^{\sharp}\mathbb{m}^{\sharp} = \mathbf{S}^{\sharp}_{stmt_n} \circ \dots \circ \mathbf{S}^{\sharp}_{stmt_1}(\mathbb{m}^{\sharp})$

The abstract Post-conditions semantics for intervals is sound:

$$[\![P]\!]^{c} \circ \gamma^{\imath}_{nr}(\mathbb{m}^{\sharp}) \subseteq \gamma^{\imath}_{nr} \circ [\![P]\!]^{\sharp}\mathbb{m}^{\sharp} \qquad \mathbb{m}^{\sharp} \triangleq \alpha^{\imath}_{nr}(\mathfrak{I} \subseteq \mathbb{M}) \qquad \alpha^{\imath}_{nr} \circ [\![P]\!]^{c}\mathfrak{I} \subseteq^{\sharp} [\![P]\!]^{\sharp} \circ \alpha^{\imath}_{nr}(\mathfrak{I})$$

(proof)

Prove $\alpha^{\imath}_{nr} \circ [\![P]\!]^{c} \circ \gamma^{\imath}_{nr} \subseteq^{\sharp} [\![P]\!]^{\sharp}$ by structural induction on $P = stmt_1 ; \ldots ; stmt_n$

Recall that

- $[\![P]\!]^{c}\mathfrak{I} = \mathbf{S}^{c}_{stmt_n} \circ \ldots \circ \mathbf{S}^{c}_{stmt_1}(\mathfrak{I})$ and $[\![P]\!]^{\sharp}\mathbb{m}^{\sharp} = \mathbf{S}^{\sharp}_{stmt_n} \circ \ldots \circ \mathbf{S}^{\sharp}_{stmt_1}(\mathbb{m}^{\sharp})$
- We assume soundness of arithmetic operations, and hence of AbsEval$[exp]$

$$\forall \mathbb{m}^{\sharp} \in \mathbb{M}^{\sharp} \,.\, \{z \in \mathbb{Z} \mid z = \textsc{Eval}[exp](\mathbb{m}) \wedge \mathbb{m} \in \gamma^{\imath}_{nr}(\mathbb{m}^{\sharp})\} \subseteq \gamma^{\imath}(\textsc{AbsEval}[exp](\mathbb{m}^{\sharp}))$$

(proof) Base case of assignments

$\alpha^{\imath}_{nr} \circ \mathbf{S}^{c}_{\mathtt{x}=exp} \circ \gamma^{\imath}_{nr}(\mathfrak{m}^{\sharp})$

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$= $ by definition of $\mathbf{S}^{c}_{\cdot}$

$$\alpha^{\imath}_{nr}(\{\mathbb{m}[\mathtt{x} \hookleftarrow z] \mid \mathbb{m} \in \gamma^{\imath}_{nr}(\mathbb{m}^{\sharp}) \wedge z = \textsc{Eval}[exp](\mathbb{m})\})$$

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$\alpha^{\imath}_{nr}(\{\mathfrak{m}[\mathtt{x} \hookleftarrow z] \mid \mathfrak{m} \in \gamma^{\imath}_{nr}(\mathfrak{m}^{\sharp}) \wedge z = \textsc{Eval}[exp](\mathfrak{m})\})$

$=$ by $\alpha^{\imath}_{nr} = \dot{\alpha}^{\imath} \circ \alpha_{nr}$ and definition of α_{nr}

$\dot{\alpha}^{\imath} \circ (\lambda \mathtt{y}\,.\, \{\mathfrak{m}(\mathtt{y}) \mid \mathfrak{m} \in \{\mathfrak{m}[\mathtt{x} \hookleftarrow z] \mid \mathfrak{m} \in \gamma^{\imath}_{nr}(\mathfrak{m}^{\sharp}) \wedge z = \textsc{Eval}[exp](\mathfrak{m})\}\})$

(proof) Base case of assignments

$\alpha^{\imath}_{nr} \circ \mathbf{S}^{c}_{\mathtt{x}=exp} \circ \gamma^{\imath}_{nr}(\mathbb{m}^{\sharp})$

$=$ by definition of $\mathbf{S}^{c}_{.}$

$\alpha^{\imath}_{nr}(\{\mathbb{m}[\mathtt{x} \hookleftarrow z] \mid \mathbb{m} \in \gamma^{\imath}_{nr}(\mathbb{m}^{\sharp}) \wedge z = \text{EVAL}[exp](\mathbb{m})\})$

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$=$

$\dot{\alpha}^{\imath} \circ (\lambda \mathtt{y} .\ \llbracket \mathtt{y} = \mathtt{x}\ ?\ \{z \mid \exists \mathbb{m} \in \gamma^{\imath}_{nr}(\mathbb{m}^{\sharp}) .\, z = \text{EVAL}[exp](\mathbb{m})\} : \{\mathbb{m}(\mathtt{y}) \mid \mathbb{m} \in \gamma^{\imath}_{nr}(\mathbb{m}^{\sharp})\} \rrbracket)$

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$\alpha^{\imath}_{nr} \circ \mathbf{S}^{c}_{\mathtt{x}=exp} \circ \gamma^{\imath}_{nr}(\mathtt{m}^{\sharp})$

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$=$

$\dot{\alpha}^{\imath} \circ (\lambda \mathtt{y}\,.\, (\!\vert\, \mathtt{y} = \mathtt{x} \,?\, \{z \mid \exists \mathtt{m} \in \gamma^{\imath}_{nr}(\mathtt{m}^{\sharp})\,.\, z = \textsc{Eval}[exp](\mathtt{m})\} \,{\circ\atop\circ}\, \{\mathtt{m}(\mathtt{y}) \mid \mathtt{m} \in \gamma^{\imath}_{nr}(\mathtt{m}^{\sharp})\} \,\vert\!))$

$\subseteq^{\sharp}$ by soundness of $\textsc{AbsEval}[exp]$

$\dot{\alpha}^{\imath} \circ (\lambda \mathtt{y}\,.\, (\!\vert\, \mathtt{y} = \mathtt{x} \,?\, \gamma^{\imath} \circ \textsc{AbsEval}[exp](\mathtt{m}^{\sharp}) \,{\circ\atop\circ}\, \{\mathtt{m}(\mathtt{y}) \mid \mathtt{m} \in \gamma^{\imath}_{nr}(\mathtt{m}^{\sharp})\} \,\vert\!))$

(proof) Cont'd

$$\dot{\alpha}^{\imath} \circ (\lambda y\,.\, (\!| y = x \mathrel{?} \gamma^{\imath} \circ \textsc{AbsEval}[exp](\mathfrak{m}^{\sharp}) \mathrel{\mathring{\circ}} \{\mathfrak{m}(y) \mid \mathfrak{m} \in \gamma^{\imath}_{nr}(\mathfrak{m}^{\sharp})\} |\!))$$

(proof) Cont'd

$$\dot{\alpha}^{\imath} \circ (\lambda \mathsf{y}\,.\, [\![\, \mathsf{y} = \mathsf{x}\ ?\ \gamma^{\imath} \circ \textsc{AbsEval}[exp](\mathbb{m}^{\sharp}) : \{\mathbb{m}(\mathsf{y}) \mid \mathbb{m} \in \gamma^{\imath}_{nr}(\mathbb{m}^{\sharp})\}\,]\!])$$

$=$ by definition of $\dot{\alpha}^{\imath}$

$$\lambda \mathsf{y}\,.\, [\![\, \mathsf{y} = \mathsf{x}\ ?\ \alpha^{\imath} \circ \gamma^{\imath} \circ \textsc{AbsEval}[exp](\mathbb{m}^{\sharp}) : \alpha^{\imath}(\{\mathbb{m}(\mathsf{y}) \mid \mathbb{m} \in \gamma^{\imath}_{nr}(\mathbb{m}^{\sharp})\})\,]\!]$$

(proof) Cont'd

$$\dot{\alpha}^{\imath} \circ (\lambda \mathsf{y}\,.\,\llbracket\, \mathsf{y} = \mathsf{x} \;?\; \gamma^{\imath} \circ \textsc{AbsEval}[exp](\mathbb{m}^{\sharp}) \;\text{⸬}\; \{\mathbb{m}(\mathsf{y}) \mid \mathbb{m} \in \gamma^{\imath}_{nr}(\mathbb{m}^{\sharp})\} \,\rrbracket)$$

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$=$ by definition of $\gamma^{\imath}_{nr}$

$$\lambda \mathsf{y}\,.\,\llbracket\, \mathsf{y} = \mathsf{x} \;?\; \alpha^{\imath} \circ \gamma^{\imath} \circ \textsc{AbsEval}[exp](\mathbb{m}^{\sharp}) \;\text{⸬}\; \alpha^{\imath} \circ \gamma^{\imath}(\mathbb{m}^{\sharp}(\mathsf{y})) \,\rrbracket$$

(proof) Cont'd

$\dot{\alpha}^{\imath} \circ (\lambda y.\ (\!|\ y = x\ ?\ \gamma^{\imath} \circ \textsc{AbsEval}[exp](\mathrm{m}^\sharp) : \{\mathrm{m}(y) \mid \mathrm{m} \in \gamma^{\imath}_{nr}(\mathrm{m}^\sharp)\}\ |\!)\ )$

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$=$ by definition of $\gamma^{\imath}_{nr}$

$\lambda y.\ (\!|\ y = x\ ?\ \alpha^{\imath} \circ \gamma^{\imath} \circ \textsc{AbsEval}[exp](\mathrm{m}^\sharp) : \alpha^{\imath} \circ \gamma^{\imath}(\mathrm{m}^\sharp(y))\ |\!)$

$\subseteq^\sharp$ by reductivity of $\alpha^{\imath} \circ \gamma^{\imath}$

$\lambda y.\ (\!|\ y = x\ ?\ \textsc{AbsEval}[exp](\mathrm{m}^\sharp) : \mathrm{m}^\sharp(y)\ |\!)$

(proof) Cont'd

$$\dot{\alpha}^{\imath} \circ (\lambda y\,.\, (\!|\, y = x \mathbin{?} \gamma^{\imath} \circ \textsc{AbsEval}[exp](m^{\sharp}) \mathbin{⁏} \{m(y) \mid m \in \gamma^{\imath}_{nr}(m^{\sharp})\} \,|\!))$$

$=$ by definition of $\dot{\alpha}^{\imath}$

$$\lambda y\,.\, (\!|\, y = x \mathbin{?} \alpha^{\imath} \circ \gamma^{\imath} \circ \textsc{AbsEval}[exp](m^{\sharp}) \mathbin{⁏} \alpha^{\imath}(\{m(y) \mid m \in \gamma^{\imath}_{nr}(m^{\sharp})\}) \,|\!)$$

$=$ by definition of $\gamma^{\imath}_{nr}$

$$\lambda y\,.\, (\!|\, y = x \mathbin{?} \alpha^{\imath} \circ \gamma^{\imath} \circ \textsc{AbsEval}[exp](m^{\sharp}) \mathbin{⁏} \alpha^{\imath} \circ \gamma^{\imath}(m^{\sharp}(y)) \,|\!)$$

$\subseteq^{\sharp}$ by reductivity of $\alpha^{\imath} \circ \gamma^{\imath}$

$$\lambda y\,.\, (\!|\, y = x \mathbin{?} \textsc{AbsEval}[exp](m^{\sharp}) \mathbin{⁏} m^{\sharp}(y) \,|\!)$$

$=$

$$m^{\sharp}[x \leftarrow \textsc{AbsEval}[exp](m^{\sharp})]$$

(proof) Cont'd

$$\dot{\alpha}^{\imath} \circ (\lambda \mathtt{y}\,.\, (\!| \mathtt{y} = \mathtt{x} \,?\, \gamma^{\imath} \circ \textsc{AbsEval}[exp](\mathfrak{m}^\sharp) \,:\, \{\mathfrak{m}(\mathtt{y}) \mid \mathfrak{m} \in \gamma^{\imath}_{nr}(\mathfrak{m}^\sharp)\} |\!))$$

$=$ by definition of $\dot{\alpha}^{\imath}$

$$\lambda \mathtt{y}\,.\, (\!| \mathtt{y} = \mathtt{x} \,?\, \alpha^{\imath} \circ \gamma^{\imath} \circ \textsc{AbsEval}[exp](\mathfrak{m}^\sharp) \,:\, \alpha^{\imath}(\{\mathfrak{m}(\mathtt{y}) \mid \mathfrak{m} \in \gamma^{\imath}_{nr}(\mathfrak{m}^\sharp)\}) |\!)$$

$=$ by definition of $\gamma^{\imath}_{nr}$

$$\lambda \mathtt{y}\,.\, (\!| \mathtt{y} = \mathtt{x} \,?\, \alpha^{\imath} \circ \gamma^{\imath} \circ \textsc{AbsEval}[exp](\mathfrak{m}^\sharp) \,:\, \alpha^{\imath} \circ \gamma^{\imath}(\mathfrak{m}^\sharp(\mathtt{y})) |\!)$$

$\sqsubseteq^\sharp$ by reductivity of $\alpha^{\imath} \circ \gamma^{\imath}$

$$\lambda \mathtt{y}\,.\, (\!| \mathtt{y} = \mathtt{x} \,?\, \textsc{AbsEval}[exp](\mathfrak{m}^\sharp) \,:\, \mathfrak{m}^\sharp(\mathtt{y}) |\!)$$

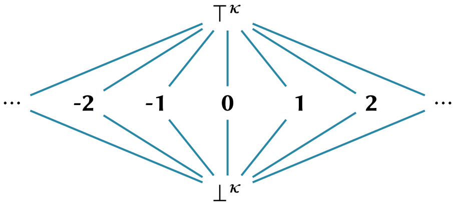
$=$

$$\mathfrak{m}^\sharp[\mathtt{x} \hookleftarrow \textsc{AbsEval}[exp](\mathfrak{m}^\sharp)]$$

$=$ by definition of $\mathbf{S}^\sharp$

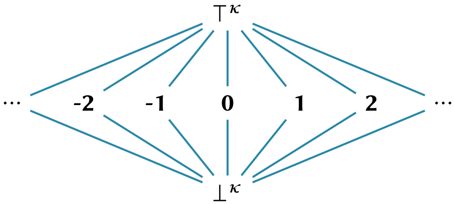
$$\mathbf{S}^\sharp_{\mathtt{x}=exp}(\mathfrak{m}^\sharp)$$

Complete lattice $\langle \mathbb{Z}^{\kappa}, \subseteq^{\kappa}, \cup^{\kappa}, \cap^{\kappa}, \bot^{\kappa}, \top^{\kappa} \rangle$ with $\langle \wp(\mathbb{Z}), \subseteq \rangle \underset{\alpha^{\kappa}}{\overset{\gamma^{\kappa}}{\leftrightarrows}} \langle \mathbb{Z}^{\kappa}, \subseteq^{\kappa} \rangle$



Constants abstract domain

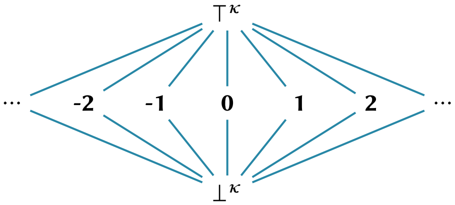
Complete lattice $\langle \mathbb{Z}^\kappa, \subseteq^\kappa, \cup^\kappa, \cap^\kappa, \bot^\kappa, \top^\kappa \rangle$ with $\langle \wp(\mathbb{Z}), \subseteq \rangle \underset{\alpha^\kappa}{\overset{\gamma^\kappa}{\leftrightarrows}} \langle \mathbb{Z}^\kappa, \subseteq^\kappa \rangle$



$\subseteq^\kappa, \cup^\kappa, \cap^\kappa$ trivially defined

Constants abstract domain

Complete lattice $\langle \mathbb{Z}^{\kappa}, \subseteq^{\kappa}, \cup^{\kappa}, \cap^{\kappa}, \perp^{\kappa}, \top^{\kappa} \rangle$ with $\langle \wp(\mathbb{Z}), \subseteq \rangle \underset{\alpha^{\kappa}}{\overset{\gamma^{\kappa}}{\leftrightarrows}} \langle \mathbb{Z}^{\kappa}, \subseteq^{\kappa} \rangle$



$\subseteq^{\kappa}, \cup^{\kappa}, \cap^{\kappa}$ trivially defined

Abstraction $\alpha^{\kappa} \triangleq \lambda X . \begin{cases} \perp^{\kappa} & \text{if } X = \emptyset \\ \mathbf{z} & \text{if } X = \{z\} \\ \top^{\kappa} & \text{otherwise} \end{cases}$ and concretization $\gamma^{\kappa} \triangleq \lambda \mathrm{v}^{\kappa} . \begin{cases} \emptyset & \text{if } \mathrm{v}^{\kappa} = \perp^{\kappa} \\ \{z\} & \text{if } \mathrm{v}^{\kappa} = \mathbf{z} \\ \mathbb{Z} & \text{otherwise} \end{cases}$

Abstract arithmetic operations

UNIVERSITÀ di VERONA
Dipartimento di INFORMATICA

Abstract arithmetic operations

$$z^\sharp \triangleq \alpha^\kappa(\{z\}) = \mathbf{z}$$

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$$\begin{aligned} \mathbb{v}_1^\kappa +^\sharp \mathbb{v}_2^\kappa &\triangleq \alpha^\kappa(\{z_1 + z_2 \mid z_1 \in \gamma^\kappa(\mathbb{v}_1^\kappa) \land z_2 \in \gamma^\kappa(\mathbb{v}_2^\kappa)\}) \\ &= \begin{cases} \bot^\kappa & \text{if } \mathbb{v}_1^\kappa = \bot^\kappa \lor \mathbb{v}_2^\kappa = \bot^\kappa \\ \top^\kappa & \text{if } \mathbb{v}_1^\kappa = \top^\kappa \lor \mathbb{v}_2^\kappa \in \top^\kappa \\ \mathbf{z} & \text{if } \mathbb{v}_1^\kappa = \mathbf{z}_1 \land \mathbb{v}_2^\kappa = \mathbf{z}_2 \land z = z_1 + z_2 \end{cases} \end{aligned}$$

Abstract arithmetic operations

$$z^\sharp \triangleq \alpha^\kappa(\{z\}) = \mathbf{z}$$

$$\mathbb{v}_1^\kappa +^\sharp \mathbb{v}_2^\kappa \triangleq \alpha^\kappa(\{z_1 + z_2 \mid z_1 \in \gamma^\kappa(\mathbb{v}_1^\kappa) \wedge z_2 \in \gamma^\kappa(\mathbb{v}_2^\kappa)\})$$
$$= \begin{cases} \bot^\kappa & \text{if } \mathbb{v}_1^\kappa = \bot^\kappa \vee \mathbb{v}_2^\kappa = \bot^\kappa \\ \top^\kappa & \text{if } \mathbb{v}_1^\kappa = \top^\kappa \vee \mathbb{v}_2^\kappa \in \top^\kappa \\ \mathbf{z} & \text{if } \mathbb{v}_1^\kappa = \mathbf{z}_1 \wedge \mathbb{v}_2^\kappa = \mathbf{z}_2 \wedge z = z_1 + z_2 \end{cases}$$

$$\mathbb{v}_1^\kappa \times^\sharp \mathbb{v}_2^\kappa \triangleq \alpha^\kappa(\{z_1 \cdot z_2 \mid z_1 \in \gamma^\kappa(\mathbb{v}_1^\kappa) \wedge z_2 \in \gamma^\kappa(\mathbb{v}_2^\kappa)\})$$
$$= \begin{cases} \bot^\kappa & \text{if } \mathbb{v}_1^\kappa = \bot^\kappa \vee \mathbb{v}_2^\kappa = \bot^\kappa \\ \mathbf{0} & \text{if } \mathbb{v}_1^\kappa = \mathbf{0} \vee \mathbb{v}_2^\kappa = \mathbf{0} \\ \top^\kappa & \text{if } \mathbb{v}_1^\kappa = \top^\kappa \vee \mathbb{v}_2^\kappa \in \top^\kappa \\ \mathbf{z} & \text{if } \mathbb{v}_1^\kappa = \mathbf{z}_1 \wedge \mathbb{v}_2^\kappa = \mathbf{z}_2 \wedge z = z_1 \cdot z_2 \end{cases}$$

Abstract conditions/boolean expressions for $\mathbb{Z}^\kappa$

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$$(\!|\,{}^{\ell}\mathtt{x} == z\,|\!)^{\sharp}\mathfrak{m}^{\sharp} \triangleq \mathbf{S}^{\sharp}_{\mathtt{x}==z}(\mathfrak{m}^{\sharp}) \triangleq \begin{cases} \mathfrak{m}^{\sharp}_{\perp} & \text{if } \mathfrak{m}^{\sharp}(\mathtt{x}) \notin \{\mathbf{z}, \top^{\kappa}\} \\ \mathfrak{m}^{\sharp}[\mathtt{x} \hookleftarrow \mathbf{z}] & \text{otherwise} \end{cases}$$

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$$(\!|\, {}^{\ell}\mathtt{x} == \mathtt{y} + z \,|\!)^\sharp \mathfrak{m}^\sharp \triangleq \mathbf{S}^\sharp_{\mathtt{x}==\mathtt{y}+z}(\mathfrak{m}^\sharp) \triangleq$$

$$\left(\begin{cases} (\!|\, {}^{\ell}\mathtt{x} == w + z \,|\!)^\sharp \mathfrak{m}^\sharp = \mathbf{S}^\sharp_{\mathtt{x}==w+z}(\mathfrak{m}^\sharp) & \text{if } \mathfrak{m}^\sharp(\mathtt{y}) = \mathbf{w} \\ \mathfrak{m}^\sharp & \text{otherwise} \end{cases} \right)$$

$$\cap^\kappa$$

$$\left(\begin{cases} (\!|\, {}^{\ell}\mathtt{y} == w - z \,|\!)^\sharp \mathfrak{m}^\sharp = \mathbf{S}^\sharp_{\mathtt{y}==w-z}(\mathfrak{m}^\sharp) & \text{if } \mathfrak{m}^\sharp(\mathtt{x}) = \mathbf{w} \\ \mathfrak{m}^\sharp & \text{otherwise} \end{cases} \right)$$

Constants analysis: example

$\mathbb{Z}^\kappa$ has finite height: convergence guaranteed in finite time

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1 x = 0;
2 y = 10;
3 while (x < 100) {
    4 y = y - 3;
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The analysis can find constants that do not appear syntactically in the program!

Non-relational Extrapolation

$\mathbb{Z}^\imath$ has infinite height, so does $\mathbb{M}^\sharp$

- Abstract computation may not converge in finite time
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Soundness: $\gamma^\imath_{nr}(\mathfrak{m}^\sharp_1) \cup \gamma^\imath_{nr}(\mathfrak{m}^\sharp_2) \subseteq \gamma^\imath_{nr}(\mathfrak{m}^\sharp_1 \;\nabla\; \mathfrak{m}^\sharp_2)$

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Termination: for every chain $\left(\mathfrak{m}^\sharp_i\right)_{i>0}$, the increasing chain $\left(\overline{\mathfrak{m}}^\sharp_i\right)_{i>0}$ defined as

$$\begin{cases} \overline{\mathfrak{m}}^\sharp_0 & \triangleq \quad \mathfrak{m}^\sharp_0 \\ \overline{\mathfrak{m}}^\sharp_{n+1} & \triangleq \quad \overline{\mathfrak{m}}^\sharp_n \;\nabla\; \mathfrak{m}^\sharp_{n+1} \end{cases}$$

is stationary, i.e., there exists $k \in \mathbb{N}$ such that $\overline{\mathfrak{m}}^\sharp_{k+1} = \overline{\mathfrak{m}}^\sharp_k$

Intervals widening $\nabla^{\imath} : \mathbb{Z}^{\imath} \times \mathbb{Z}^{\imath} \to \mathbb{Z}^{\imath}$

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unstable bounds set to infinity

$$\bot^\imath \ \nabla^\imath \ [a, b] \triangleq [a, b] \ \nabla^\imath \ \bot^\imath \triangleq [a, b]$$

$$[a, a'] \ \nabla^\imath \ [b, b'] \triangleq [(\!| b < a \ ? \ \text{-}\infty \ ⁏ \ a |\!), (\!| b' > a' \ ? \ \infty \ ⁏ \ a' |\!)]$$

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Point-wise lift of intervals widening to (abstract) memories $\nabla : \mathbb{M}^\sharp \times \mathbb{M}^\sharp \to \mathbb{M}^\sharp$

$$\mathtt{m}^\sharp_1 \ \nabla \ \mathtt{m}^\sharp_2 \triangleq \lambda \mathtt{x} \, . \, \mathtt{m}^\sharp_1(\mathtt{x}) \ \nabla^\imath \ \mathtt{m}^\sharp_2(\mathtt{x})$$

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The construction works for generic non-relational abstractions

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Widening with thresholds

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- Useful when it is easy to find a good T — e.g., array bound checking
- Useful when bound over-approximation suffices — e.g., arithmetic overflow checking

Thanks for the attention!

LaTeX is the way