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UNIVERSITÀ di VERONA
Dipartimento di INFORMATICA

Principles and Applications of Abstract Interpretation

Abstract Domains

Michele Pasqua, PhD

Verona, IT

October 2024

PhD Course @ UniVR 2024/2025

Almost all domains are for trace properties

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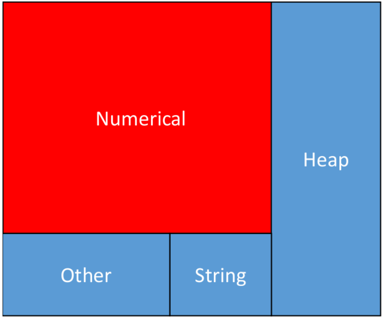

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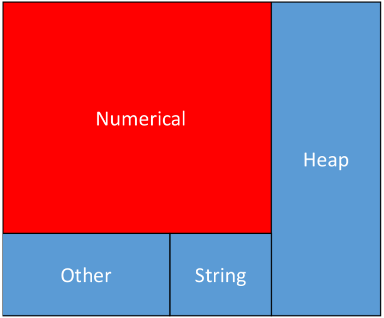

Buffer overflow
Division by zero
Integer overflow
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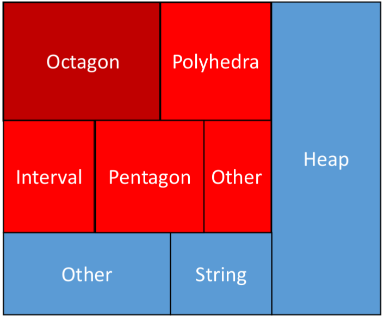

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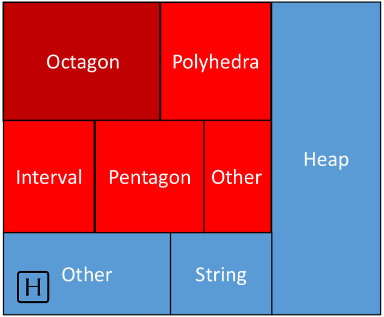
Almost all domains are for trace properties ... almost all



Buffer overflow
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Information flows
Data races

Numerical Abstractions

We aim to abstract elements in $\wp(\mathbb{M}) = \wp(\mathbb{L} \to \mathbb{V})$ e.g., $\mathbb{V} = \mathbb{Z}$

- Find an invariant for all values of all variables in $X \in \wp(\mathbb{M})$

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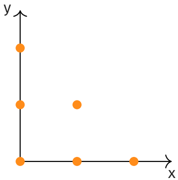

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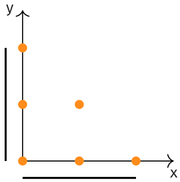

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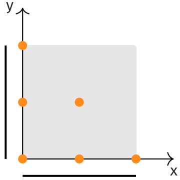

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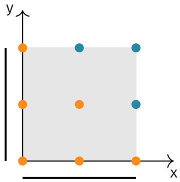
non-relational



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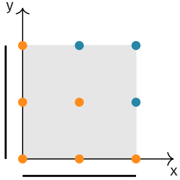

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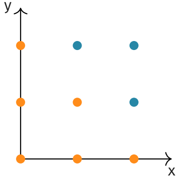

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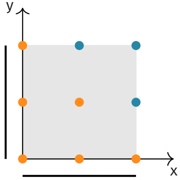

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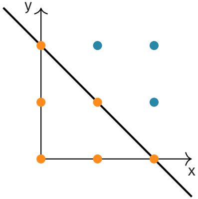

linear inequalities

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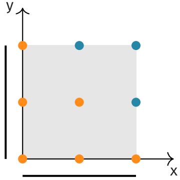

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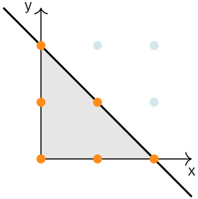

linear inequalities

relational

Forget relations between variables by applying the cartesian abstraction

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Upper closure operator on $\wp(\mathbb{M})$:

$$\eta_{nr} \triangleq \gamma_{nr} \circ \alpha_{nr} = \lambda X \,.\, \{\mathbb{m} \in \mathbb{M} \mid \forall \mathtt{x} \in \mathbb{X}\, \exists \mathbb{m}' \in X \,.\, \mathbb{m}(\mathtt{x}) = \mathbb{m}'(\mathtt{x})\}$$

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A domain is non-relational if $\eta_{nr} \circ \gamma = \gamma$

- The domain cannot distinguish between X and X' when $\eta_{nr}(X) = \eta_{nr}(X')$

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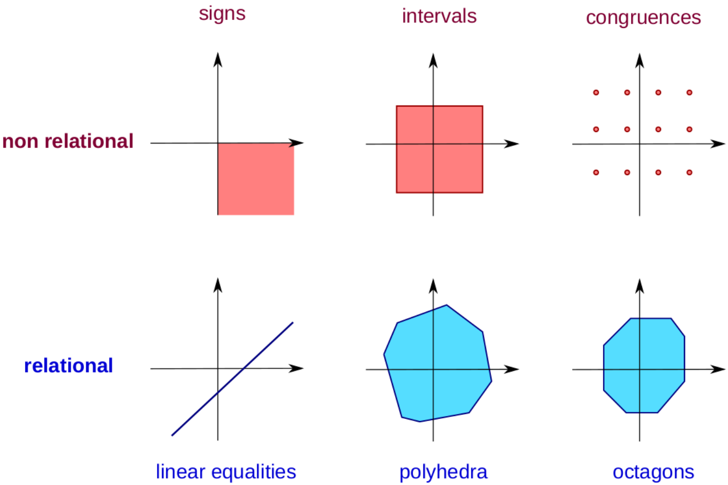
All domains seen so far (sign, intervals, constants) are non-relational

Numerical domain examples



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Prove relational invariants, of course

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Prove some complex non-relational invariants with relational loop invariants

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Prove some complex non-relational invariants with relational loop invariants

```
x = 0;
i = 1;
while (i < 5000) {
   if (rand(0,1) == 1)
      { x = x + 1 }
      { x = x - 1 };
   i = i + 1
}
```

The need for relational abstractions

Prove relational invariants, of course

Prove some complex non-relational invariants with relational loop invariants

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    7 i = i + 1
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A non-relational analysis at 8 finds: $\mathtt{i} = 5000 \wedge \mathtt{x} \in \mathbb{Z}$

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The best invariant at 8 is: $\mathtt{i} = 5000 \wedge \mathtt{x} \in [-4999, 4999] \wedge \mathtt{x} \equiv_2 1$

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- Find the relational loop invariant $\text{-}\mathtt{i} < \mathtt{x} < \mathtt{i} \wedge \mathtt{x} + \mathtt{i} \equiv_2 1$ at 4
- Filter out by applying the negation of the guard at 3

Relational Numerical Abstractions

We look for invariants of the form:

$$\bigwedge_{j} \left(\sum_{i=1}^{n} c_{ij} \cdot \mathsf{x}_i \geq d_j \right) \qquad c, d \in \mathbb{V}$$

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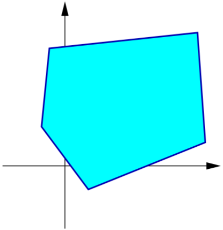
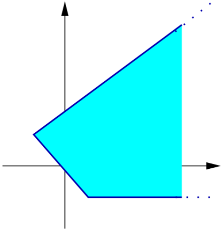
Polyhedra domain $\mathbb{M}^\sharp \triangleq \{\text{closed convex polyhedra of } \mathbb{X} \to \mathbb{V}\}$





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Invariants are systems of affine inequalities, that can be expressed in matrix form $\langle \mathbf{M}, \vec{C} \rangle$

$$\gamma(\langle \mathbf{M}, \vec{C} \rangle) = \{\vec{X} \in \mathbb{V}^n \mid \mathbf{M} \times \vec{X} \geq \vec{C}\} \simeq \left\{\sum_{i=1}^{n} c_{ij} \cdot \mathsf{x}_i \geq d_j\right\}$$

given $\mathbf{M} \in \mathbb{V}^{m \times n}$ and $\vec{C} \in \mathbb{V}^m$

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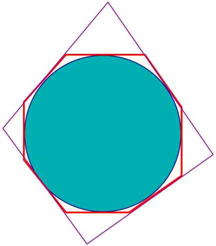
given $\mathbf{M} \in \mathbb{V}^{m \times n}$ and $\vec{C} \in \mathbb{V}^m$

- No best abstraction α (a disc has infinite approximations)
- No bound on the representation (exponential in practice)



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Topological closure of the convex hull of $\gamma(\mathbb{m}_1^\sharp) \cup \gamma(\mathbb{m}_2^\sharp)$

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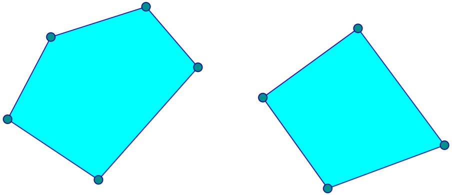

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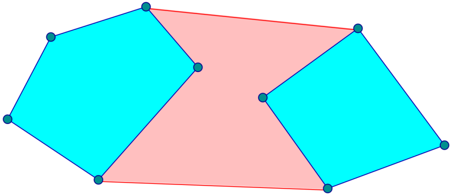

A convex polyhedron containing the polyhedra to join

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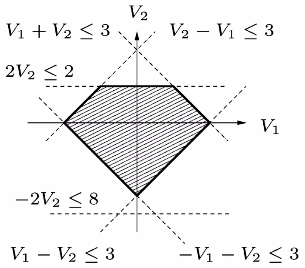
Octagons domain: special case of polyhedra domain (symmetric and bounded)



Simpler matrix representation $\mathbf{M}$

$$\gamma(\mathbf{M}) = \{\vec{X} \in \mathbb{V}^n \mid \forall i,j \in [1,n] \,.\, \vec{X}_j - \vec{X}_i \leq \mathbf{M}_{i,j}\} \simeq \{\pm \mathrm{x}_i \pm \mathrm{x}_j \leq c_k\}$$

given $\mathbf{M} \in (\mathbb{V} \cup \{\infty\})^{2n \times 2n}$

difference bound matrix

we assume $\mathbb{V} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$

Simpler matrix representation **M**

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- Best abstraction α does exist
- Quadratic memory cost / Cubic time cost

Element-wise maximum

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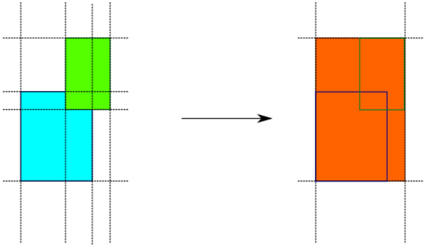
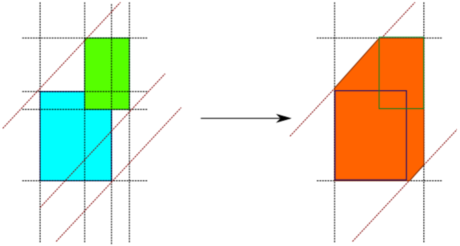


No better approximation than intervals

Element-wise maximum with closure (we add new constraints to increase precision)



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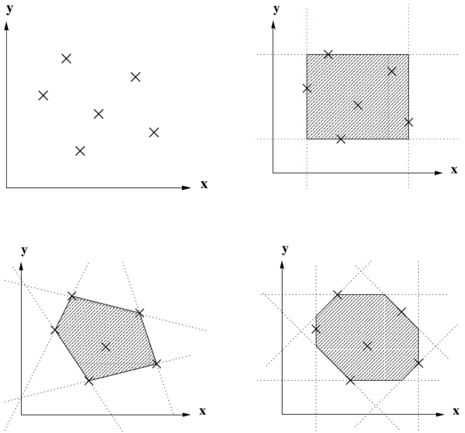
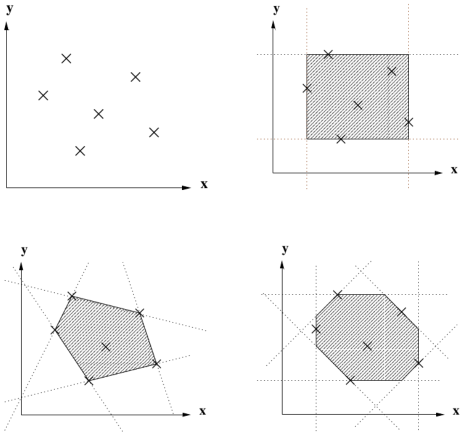
Better approximation than intervals

Summary





Intervals: *non-relational*

Complexity: *linear*

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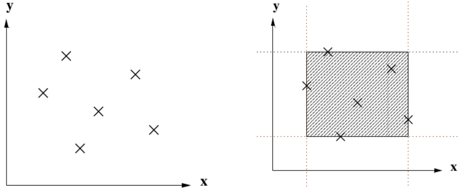
Summary



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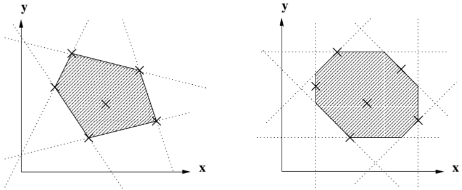
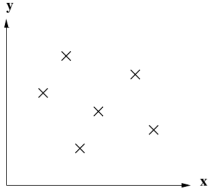
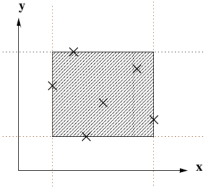


Octagons: *weakly relational*

Complexity: *quadratic/cubic*



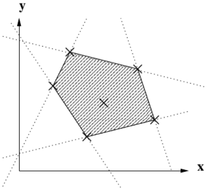


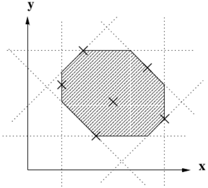
Intervals: *non-relational*

Complexity: *linear*

Polyhedra: *relational*

Complexity: *exponential*





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Thanks for the attention!

LaTeX is the way