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UNIVERSITÀ di VERONA
Dipartimento di INFORMATICA

Principles and Applications of Abstract Interpretation

Abstract Interpretation Primer

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Verona, IT

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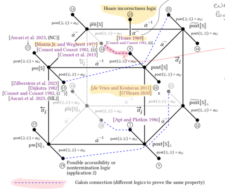
A gentle introduction...

Assertional Postcondition Transformer

- $\text{post}[S] = \lambda P. \{\sigma' \mid \exists \sigma \in P. \langle \sigma, \sigma' \rangle \in S\}$
- Preserves joins $\text{Post}[S] \bigcup_i P_i = \bigcup_i \text{Post}[S] P_i$
- Isomorphism
  $T = \text{post}[R]$
  $\Leftrightarrow R = \{\langle \sigma, \sigma' \rangle \mid \sigma' \in T(\{\sigma\})\}$
  $\triangleq \text{post}^{-1}(T)$
- $\langle \wp(\Sigma \times \Sigma_\perp), \subseteq \rangle \underset{\text{post}}{\overset{\text{post}^{-1}}{\leftrightarrows}} \langle \wp(\Sigma) \overset{\cup}{\rightarrow} \wp(\Sigma_\perp), \dot{\subseteq} \rangle$

The subhierarchy of assertional logics



$\alpha^{\neg}$ can be added everywhere (32 logics)

equivalent logics (a proof by one logic can be transformed into a proof by the other logic)

iteration

symbolic rules ↓ — Aczel semantics of the rules

$$\sigma \vdash W \overset{i}{\Rightarrow} \sigma \qquad \text{i.e.} \quad \frac{\emptyset}{\langle \sigma, \sigma \rangle} \quad \sigma \in \Sigma \text{ axioms}$$

$$\frac{\mathcal{B}[\![B]\!]\sigma, \quad \sigma \vdash S \overset{e}{\Rightarrow} \sigma', \quad \sigma' \vdash W \overset{i}{\Rightarrow} \sigma''}{\sigma \vdash W \overset{i}{\Rightarrow} \sigma''} \qquad \text{i.e.} \quad \frac{\{\langle \sigma', \sigma'' \rangle\}}{\langle \sigma, \sigma'' \rangle} \quad \mathcal{B}[\![B]\!]\sigma \; \langle \sigma, \sigma' \rangle \in [\![S]\!]^e$$

structural induction ↑

Consequence operator

$$F^e(X) = \{\langle \sigma, \sigma \rangle \mid \sigma \in \Sigma\} \cup \{\langle \sigma, \sigma'' \rangle \mid \mathcal{B}[\![B]\!]\sigma \wedge \langle \sigma, \sigma' \rangle \in [\![S]\!]^e \wedge \langle \sigma', \sigma'' \rangle \in X\}$$

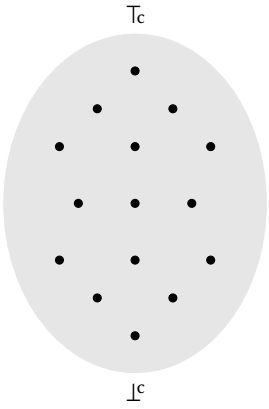
$$= \text{id} \cup ([\![B]\!] \,⨟\, [\![S]\!]^e \,⨟\, (X \setminus \Sigma \times \{\perp\})),$$

this is the termination case

defines $\text{lfp}^{\subseteq} F^e = ([\![B]\!] \,⨟\, [\![S]\!]^e)^*$

The Abstraction Process

Concrete domain $\langle C, \sqsubseteq^c \rangle$



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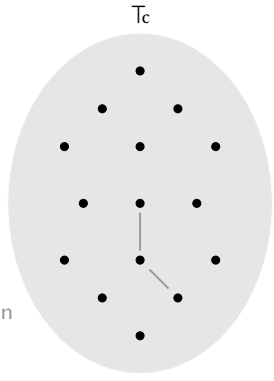
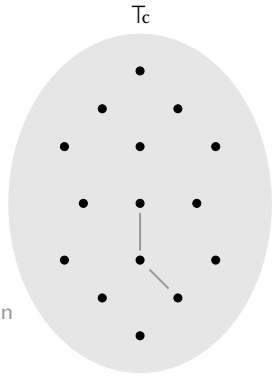
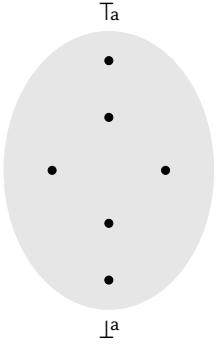


Concrete domain $\langle C, \sqsubseteq^c \rangle$ $\langle A, \sqsubseteq^a \rangle$ Abstract domain





$\sqsubseteq^c$ models relative precision

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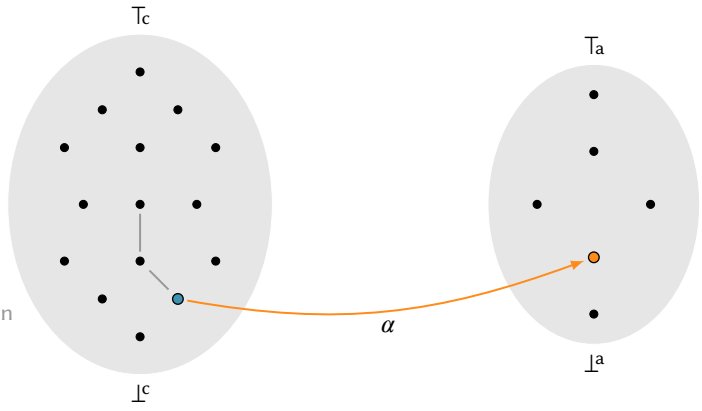
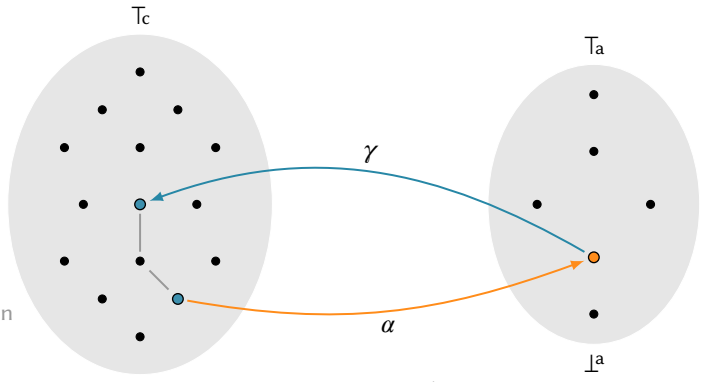


$\sqsubseteq^c$ models relative precision

What is abstraction? (formally)

Concrete domain $\langle C, \sqsubseteq^c \rangle$ $\langle A, \sqsubseteq^a \rangle$ Abstract domain



$\sqsubseteq^c$ models relative precision

$\gamma \circ \alpha$ extensive

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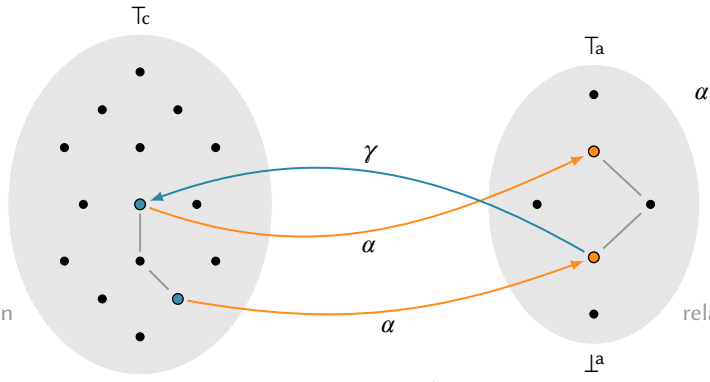

α monotonic

$\sqsubseteq^c$ models relative precision

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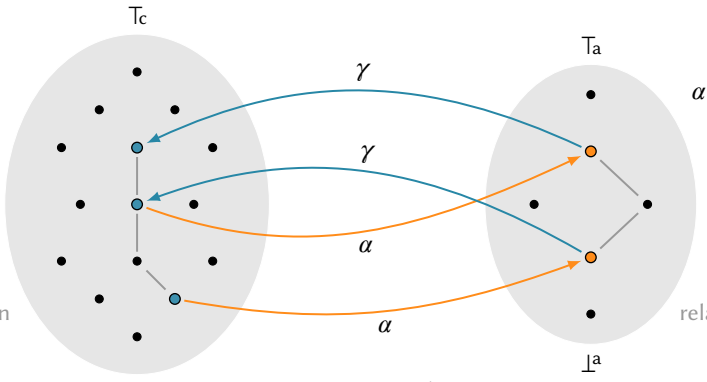


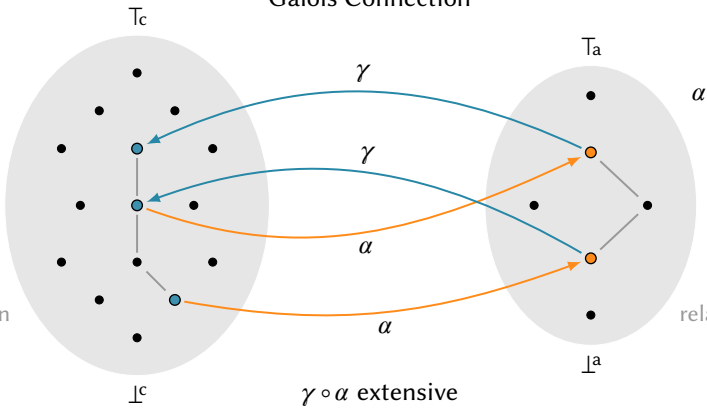
What is abstraction? (formally)

Concrete domain

$\langle C, \sqsubseteq^c \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle A, \sqsubseteq^a \rangle$

Abstract domain

Galois Connection



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$\gamma \circ \alpha$ extensive

Isolate the subsets of integers $\mathbb{Z}$ we are interested in

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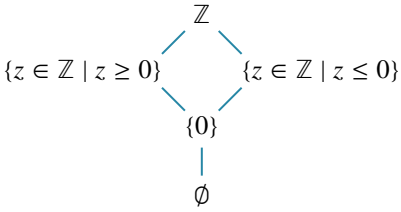

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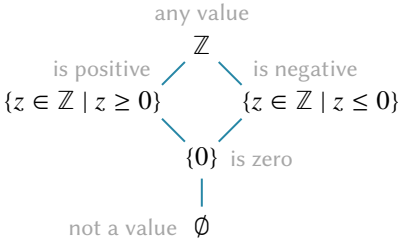
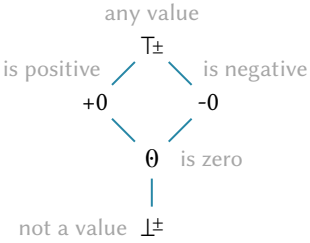


Isolate the subsets of integers $\mathbb{Z}$ we are interested in

Encode subsets into a machine-representable domain $\mathbb{Z}^{\pm}$

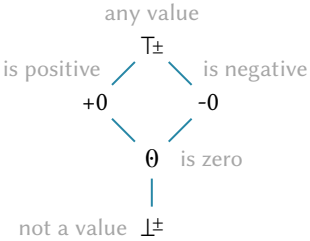


Isolate the subsets of integers $\mathbb{Z}$ we are interested in

Encode subsets into a machine-representable domain $\mathbb{Z}^{\pm}$

Define an abstraction (encoding) function $\alpha : \wp(\mathbb{Z}) \to \mathbb{Z}^{\pm}$

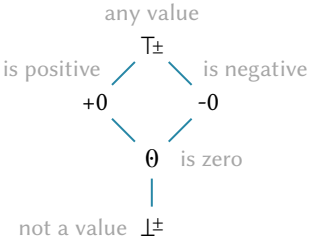
$$\alpha(X) \triangleq \begin{cases} \bot^{\pm} & \text{if } X = \emptyset \\ 0 & \text{if } X = \{0\} \\ +0 & \text{if } X \subseteq \{z \in \mathbb{Z} \mid z \geq 0\} \wedge X \neq \emptyset \\ -0 & \text{if } X \subseteq \{z \in \mathbb{Z} \mid z \leq 0\} \wedge X \neq \emptyset \\ \top^{\pm} & \text{otherwise} \end{cases}$$



Isolate the subsets of integers $\mathbb{Z}$ we are interested in

Encode subsets into a machine-representable domain $\mathbb{Z}^{\pm}$

Define a concretization (decoding) function $\gamma : \mathbb{Z}^{\pm} \to \wp(\mathbb{Z})$



$$\gamma(z^{\pm}) \triangleq \begin{cases} \emptyset & \text{if } z^{\pm} = \bot^{\pm} \\ \{0\} & \text{if } z^{\pm} = 0 \\ \{z \in \mathbb{Z} \mid z \geq 0\} & \text{if } z^{\pm} = +0 \\ \{z \in \mathbb{Z} \mid z \leq 0\} & \text{if } z^{\pm} = -0 \\ \mathbb{Z} & \text{if } z^{\pm} = \top^{\pm} \end{cases}$$

Isolate the subsets of integers $\mathbb{Z}$ we are interested in

Encode subsets into a machine-representable domain $\mathbb{Z}^{\pm}$

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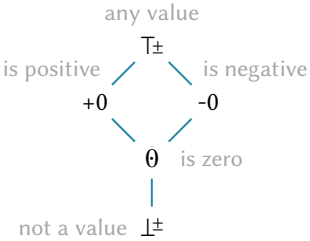

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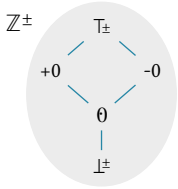
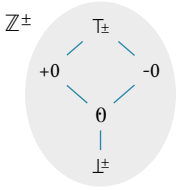
Extensivity: $\{1,4,7\} \subseteq \gamma\alpha(\{1,4,7\})$
$\subseteq \gamma(+0)$
$\subseteq \{z \in \mathbb{Z} \mid z \geq 0\}$

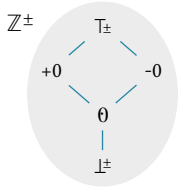

$\mathbb{Z}^{\pm}$
$\mathbb{T}_{\pm}$
+0
-0
θ
$\perp^{\pm}$

$\times^{\sharp}$	$\bot^{\pm}$	θ	+0	-0	$\top^{\pm}$
$\bot^{\pm}$	$\bot^{\pm}$	$\bot^{\pm}$	$\bot^{\pm}$	$\bot^{\pm}$	$\bot^{\pm}$
θ	$\bot^{\pm}$	θ	θ	θ	θ
+0	$\bot^{\pm}$	θ	+0	-0	$\top^{\pm}$
-0	$\bot^{\pm}$	θ	-0	+0	$\top^{\pm}$
$\top^{\pm}$	$\bot^{\pm}$	θ	$\top^{\pm}$	$\top^{\pm}$	$\top^{\pm}$

1×2

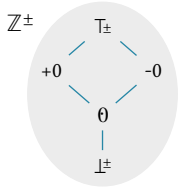


$\times^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	θ	θ	θ
$+0$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
-0	$\bot^\pm$	θ	-0	$+0$	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$



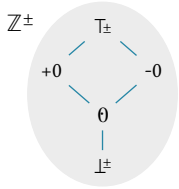
1×2 $\quad\quad$ $\alpha(1) \times^\sharp \alpha(2)$

$\times^\sharp$	$\bot^\pm$	θ	+0	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	θ	θ	θ
+0	$\bot^\pm$	θ	+0	-0	$\top^\pm$
-0	$\bot^\pm$	θ	-0	+0	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$



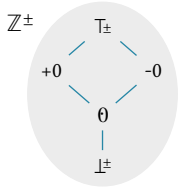
$1 \times 2 \qquad \alpha(1) \times^\sharp \alpha(2) = +0 \times^\sharp +0$

$\times^\sharp$	$\bot^\pm$	θ	+0	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	θ	θ	θ
+0	$\bot^\pm$	θ	+0	-0	$\top^\pm$
-0	$\bot^\pm$	θ	-0	+0	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$



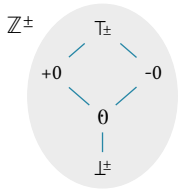
$1 \times 2 \quad +0 = \alpha(1) \times^\sharp \alpha(2) = +0 \times^\sharp +0$

$\times^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	θ	θ	θ
$+0$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
-0	$\bot^\pm$	θ	-0	$+0$	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$



$$\alpha(1 \times 2) = +0 = \alpha(1) \times^\sharp \alpha(2) = +0 \times^\sharp +0$$

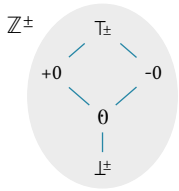
$\times^\sharp$	$\bot^\pm$	θ	+0	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	θ	θ	θ
+0	$\bot^\pm$	θ	+0	-0	$\top^\pm$
-0	$\bot^\pm$	θ	-0	+0	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$



$$\alpha(1 \times 2) = +0 = \alpha(1) \times^\sharp \alpha(2) = +0 \times^\sharp +0$$

$$\alpha(-1 \times 0) = \theta = \alpha(-1) \times^\sharp \alpha(0) = -0 \times^\sharp \theta$$

$\times^\sharp$	$\bot^\pm$	θ	+0	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	θ	θ	θ
+0	$\bot^\pm$	θ	+0	-0	$\top^\pm$
-0	$\bot^\pm$	θ	-0	+0	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$



$$\alpha(1 \times 2) = +0 = \alpha(1) \times^\sharp \alpha(2) = +0 \times^\sharp +0$$
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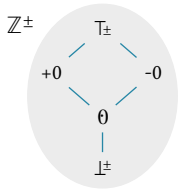
$\times^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	θ	θ	θ
$+0$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
-0	$\bot^\pm$	θ	-0	$+0$	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$

$+^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$+0$	$\bot^\pm$	$+0$	$+0$	$\top^\pm$	$\top^\pm$
-0	$\bot^\pm$	-0	$\top^\pm$	-0	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$



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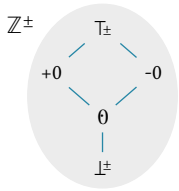
$$\alpha(-1 \times 0) = \theta = \alpha(-1) \times^\sharp \alpha(0) = -0 \times^\sharp \theta$$
$$\alpha(-1 \times 2) = -0 = \alpha(-1) \times^\sharp \alpha(2) = -0 \times^\sharp +0$$

$\times^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	θ	θ	θ
$+0$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
-0	$\bot^\pm$	θ	-0	$+0$	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$

$+^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$+0$	$\bot^\pm$	$+0$	$+0$	$\top^\pm$	$\top^\pm$
-0	$\bot^\pm$	-0	$\top^\pm$	-0	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$



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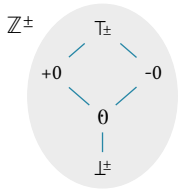
$\alpha(-1 \times 2) = -0 = \alpha(-1) \times^\sharp \alpha(2) = -0 \times^\sharp +0$

$\alpha(1 + 2) = +0 = \alpha(1) +^\sharp \alpha(2) = +0 +^\sharp +0$

$\times^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	θ	θ	θ
$+0$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
-0	$\bot^\pm$	θ	-0	$+0$	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$

$+^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$+0$	$\bot^\pm$	$+0$	$+0$	$\top^\pm$	$\top^\pm$
-0	$\bot^\pm$	-0	$\top^\pm$	-0	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$



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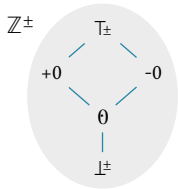
$$\alpha(-1 \times 2) = -0 = \alpha(-1) \times^\sharp \alpha(2) = -0 \times^\sharp +0$$

$$\alpha(1 + 2) = +0 = \alpha(1) +^\sharp \alpha(2) = +0 +^\sharp +0$$
$$\alpha(-1 + 0) = -0 = \alpha(-1) +^\sharp \alpha(0) = -0 +^\sharp \theta$$

$\times^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	θ	θ	θ
$+0$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
-0	$\bot^\pm$	θ	-0	$+0$	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$

$+^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$+0$	$\bot^\pm$	$+0$	$+0$	$\top^\pm$	$\top^\pm$
-0	$\bot^\pm$	-0	$\top^\pm$	-0	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$



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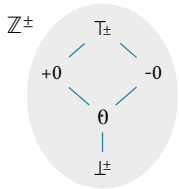
$$\alpha(-1 + 0) = -0 = \alpha(-1) +^\sharp \alpha(0) = -0 +^\sharp \theta$$
$$-1 + 2$$

$\times^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	θ	θ	θ
$+0$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
-0	$\bot^\pm$	θ	-0	$+0$	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$

$+^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$+0$	$\bot^\pm$	$+0$	$+0$	$\top^\pm$	$\top^\pm$
-0	$\bot^\pm$	-0	$\top^\pm$	-0	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$



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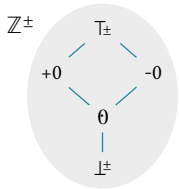
$$\alpha(1 + 2) = +0 = \alpha(1) +^\sharp \alpha(2) = +0 +^\sharp +0$$
$$\alpha(-1 + 0) = -0 = \alpha(-1) +^\sharp \alpha(0) = -0 +^\sharp \theta$$
$$-1 + 2 \qquad \alpha(-1) +^\sharp \alpha(2)$$

$\times^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	θ	θ	θ
$+0$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
-0	$\bot^\pm$	θ	-0	$+0$	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$

$+^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$+0$	$\bot^\pm$	$+0$	$+0$	$\top^\pm$	$\top^\pm$
-0	$\bot^\pm$	-0	$\top^\pm$	-0	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$



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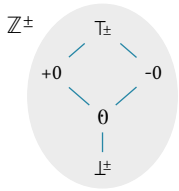
$\alpha(1 + 2) = +0 = \alpha(1) +^\sharp \alpha(2) = +0 +^\sharp +0$

$\alpha(-1 + 0) = -0 = \alpha(-1) +^\sharp \alpha(0) = -0 +^\sharp \theta$

$-1 + 2 \qquad \alpha(-1) +^\sharp \alpha(2) = -0 +^\sharp +0$

$\times^\sharp$	$\bot^\pm$	θ	+0	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	θ	θ	θ
+0	$\bot^\pm$	θ	+0	-0	$\top^\pm$
-0	$\bot^\pm$	θ	-0	+0	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$

$+^\sharp$	$\bot^\pm$	θ	+0	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	+0	-0	$\top^\pm$
+0	$\bot^\pm$	+0	+0	$\top^\pm$	$\top^\pm$
-0	$\bot^\pm$	-0	$\top^\pm$	-0	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$



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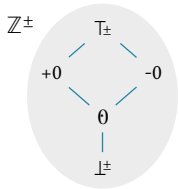
$\alpha(-1 + 0) = -0 = \alpha(-1) +^\sharp \alpha(0) = -0 +^\sharp \theta$

$-1 + 2 \quad \top^\pm = \alpha(-1) +^\sharp \alpha(2) = -0 +^\sharp +0$

$\times^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	θ	θ	θ
$+0$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
-0	$\bot^\pm$	θ	-0	$+0$	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$

$+^\sharp$	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
θ	$\bot^\pm$	θ	$+0$	-0	$\top^\pm$
$+0$	$\bot^\pm$	$+0$	$+0$	$\top^\pm$	$\top^\pm$
-0	$\bot^\pm$	-0	$\top^\pm$	-0	$\top^\pm$
$\top^\pm$	$\bot^\pm$	θ	$\top^\pm$	$\top^\pm$	$\top^\pm$



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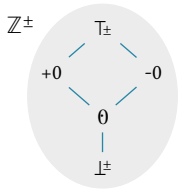
$$\alpha(1 + 2) = +0 = \alpha(1) +^\sharp \alpha(2) = +0 +^\sharp +0$$
$$\alpha(-1 + 0) = -0 = \alpha(-1) +^\sharp \alpha(0) = -0 +^\sharp \theta$$
$$\alpha(-1 + 2) \neq \top^\pm = \alpha(-1) +^\sharp \alpha(2) = -0 +^\sharp +0$$

$\times^\sharp$	$\bot^\pm$	0	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
0	$\bot^\pm$	0	0	0	0
$+0$	$\bot^\pm$	0	$+0$	-0	$\top^\pm$
-0	$\bot^\pm$	0	-0	$+0$	$\top^\pm$
$\top^\pm$	$\bot^\pm$	0	$\top^\pm$	$\top^\pm$	$\top^\pm$

$+^\sharp$	$\bot^\pm$	0	$+0$	-0	$\top^\pm$
$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$	$\bot^\pm$
0	$\bot^\pm$	0	$+0$	-0	$\top^\pm$
$+0$	$\bot^\pm$	$+0$	$+0$	$\top^\pm$	$\top^\pm$
-0	$\bot^\pm$	-0	$\top^\pm$	-0	$\top^\pm$
$\top^\pm$	$\bot^\pm$	0	$\top^\pm$	$\top^\pm$	$\top^\pm$



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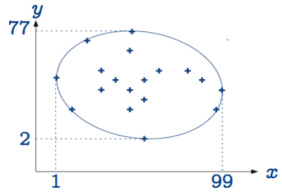
$$\alpha(1 + 2) = +0 = \alpha(1) +^\sharp \alpha(2) = +0 +^\sharp +0$$
$$\alpha(-1 + 0) = -0 = \alpha(-1) +^\sharp \alpha(0) = -0 +^\sharp 0$$
$$\alpha(-1 + 2) \neq \top^\pm = \alpha(-1) +^\sharp \alpha(2) = -0 +^\sharp +0$$

Abstract operations may lose precision during computation!

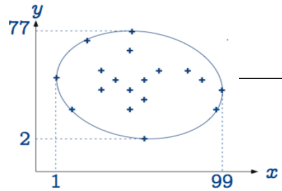
Abstraction: graphical example

Encode a set of points in $\mathbb{R}^2$ into an (integer) interval



Abstraction: graphical example

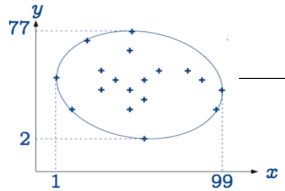
Encode a set of points in $\mathbb{R}^2$ into an (integer) interval



$\xrightarrow{\alpha}$ $\{ x \mapsto [1, 99],\ y \mapsto [2, 77] \}$

Abstraction: graphical example

Encode a set of points in $\mathbb{R}^2$ into an (integer) interval



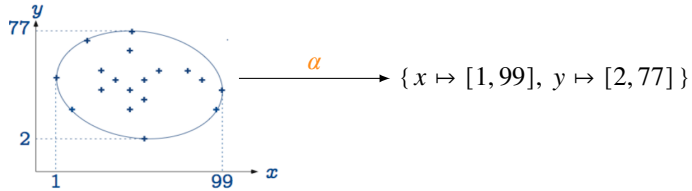
$\xrightarrow{\alpha}$ $\{ x \mapsto [1, 99],\ y \mapsto [2, 77] \}$

Decode an (integer) interval into a set of points in $\mathbb{R}^2$

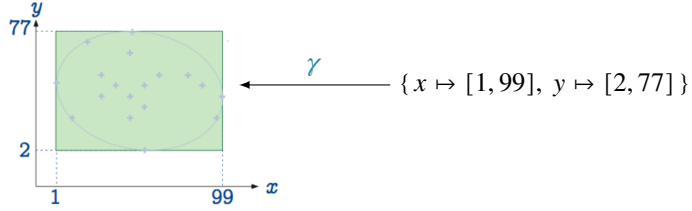
$\{ x \mapsto [1, 99],\ y \mapsto [2, 77] \}$

Abstraction: graphical example

Encode a set of points in $\mathbb{R}^2$ into an (integer) interval

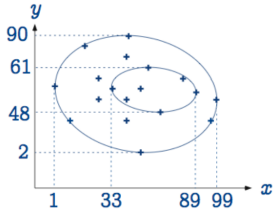


Decode an (integer) interval into a set of points in $\mathbb{R}^2$



Abstraction: graphical example (Cont'd)

The abstraction α is monotone: $S_1 \subseteq S_2 \Rightarrow \alpha(S_1) \sqsubseteq \alpha(S_2)$



Abstraction: graphical example (Cont'd)

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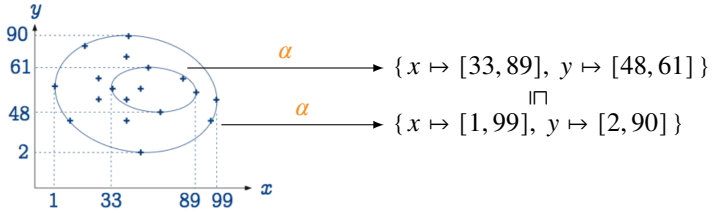
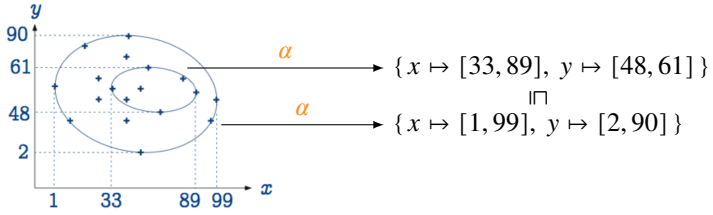


Abstraction: graphical example (Cont'd)

The abstraction α is monotone: $S_1 \subseteq S_2 \Rightarrow \alpha(S_1) \sqsubseteq \alpha(S_2)$

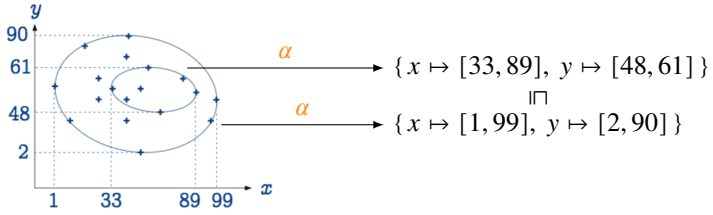


The concretization γ is monotone: $I_1 \sqsubseteq I_2 \Rightarrow \gamma(I_1) \subseteq \gamma(I_2)$

$$\{ x \mapsto [33, 89],\ y \mapsto [48, 61] \}$$
$$\sqsubseteq$$
$$\{ x \mapsto [1, 99],\ y \mapsto [2, 90] \}$$

Abstraction: graphical example (Cont'd)

The abstraction α is monotone: $S_1 \subseteq S_2 \Rightarrow \alpha(S_1) \sqsubseteq \alpha(S_2)$



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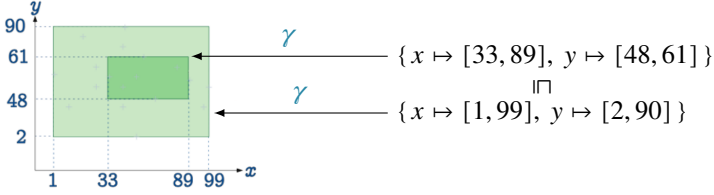
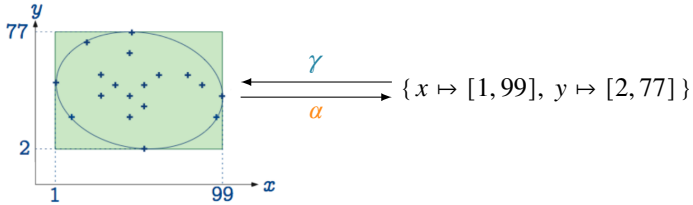
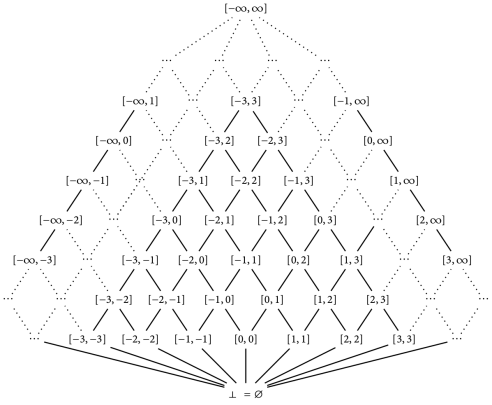


Abstraction: graphical example (Cont'd)

The composition $\gamma \circ \alpha$ is extensive: $S \subseteq \gamma\,\alpha(S)$

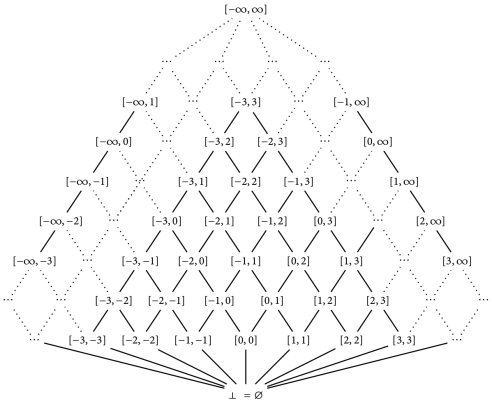


Bounded integer intervals $\langle (\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{\infty\}), \sqsubseteq, \sqcup, \sqcap, \bot, \top \rangle$



Bounded integer intervals $\langle (\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{\infty\}), \sqsubseteq, \sqcup, \sqcap, \bot, \top \rangle$

- $[a,b] \sqsubseteq [a',b'] \triangleq a \geq a' \wedge b \leq b'$
- $[a,b] \sqcap [a',b'] \triangleq \left(\max\{a,a'\} \leq \min\{b,b'\} \ ? \ [\max\{a,a'\}, \min\{b,b'\}] \ : \bot \right)$
- $[a,b] \sqcup [a',b'] \triangleq [\min\{a,a'\}, \max\{b,b'\}]$
- $\bot \triangleq [\infty, -\infty]$ and $\top \triangleq [-\infty, \infty]$



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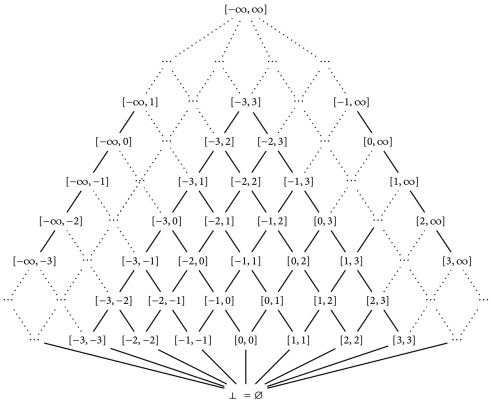
- $[a,b] \sqcup [a',b'] \triangleq [\min\{a,a'\}, \max\{b,b'\}]$
- $\bot \triangleq [\infty, -\infty]$ and $\top \triangleq [-\infty, \infty]$

Abstraction $\alpha \triangleq \lambda X \,.\, \left(X = \emptyset \; ? \; \bot \; : \; [\min X, \max X] \right)$



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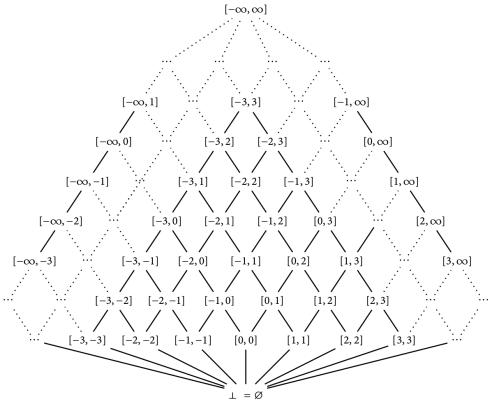

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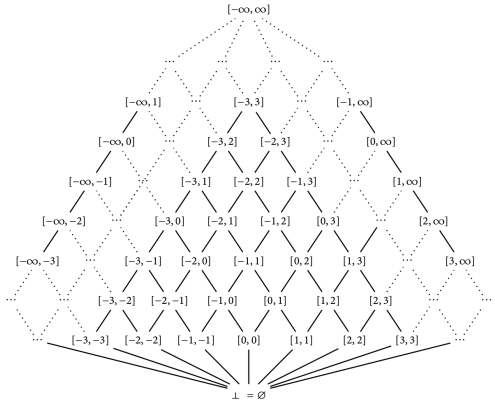

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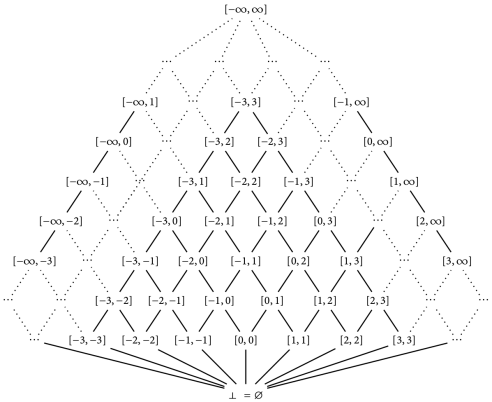

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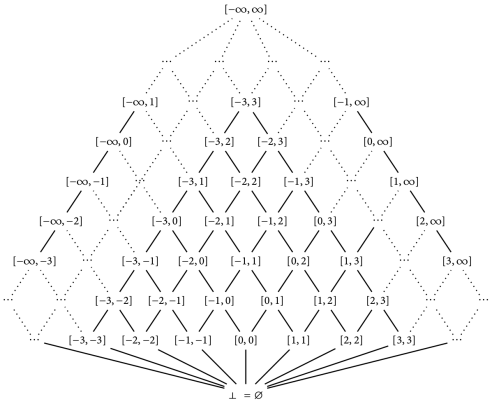

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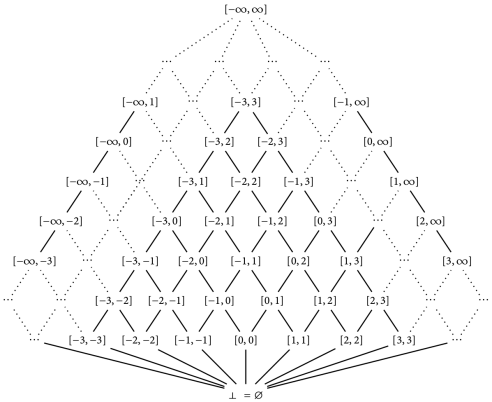

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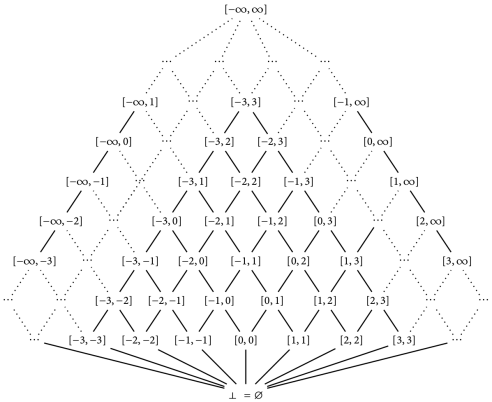

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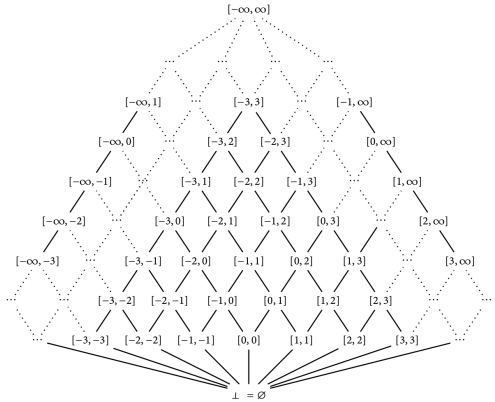

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We say that A is more precise than B when

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Prove that $\langle \{\top_c\}, = \rangle$ is the less precise abstraction of C

Abstract Computation

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Note: we usually aim at computing the fixpoints of f (and $f^\sharp$)

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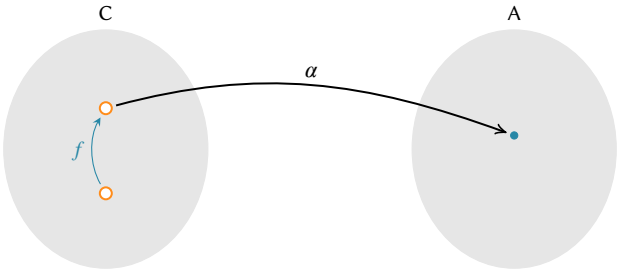

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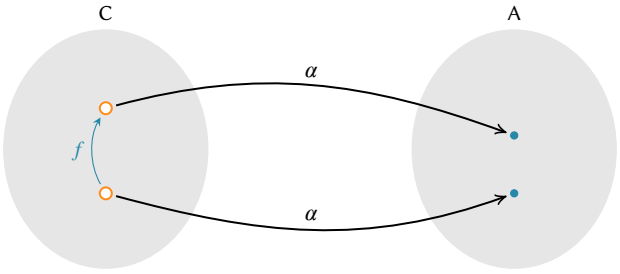

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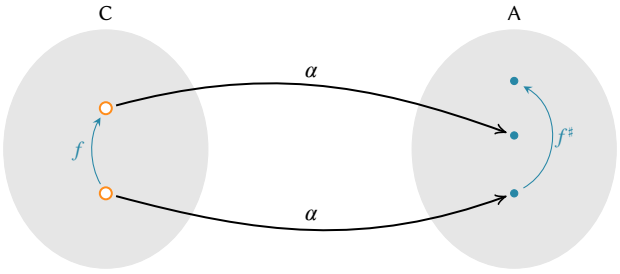

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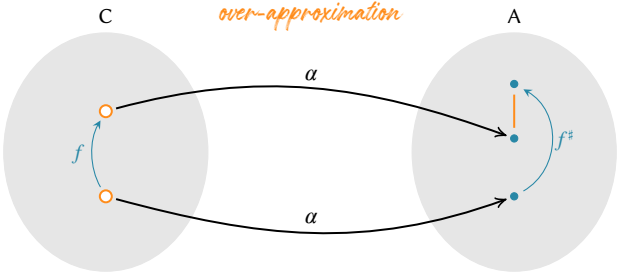

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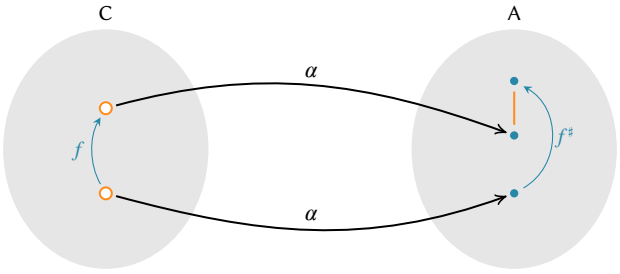

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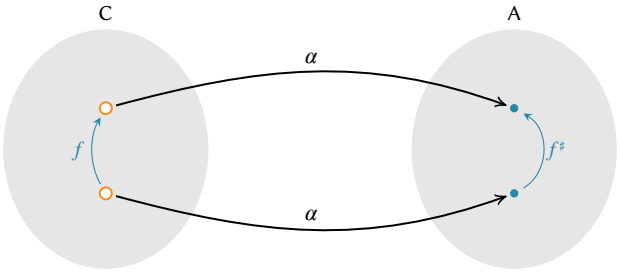

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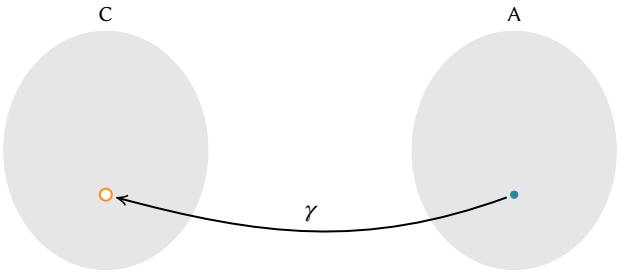

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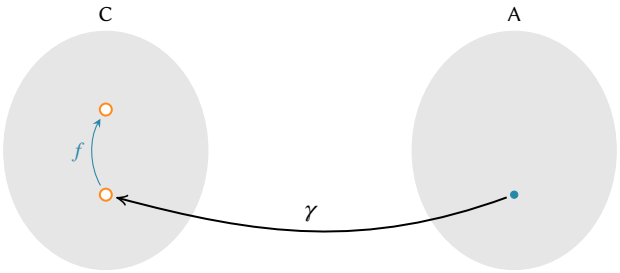

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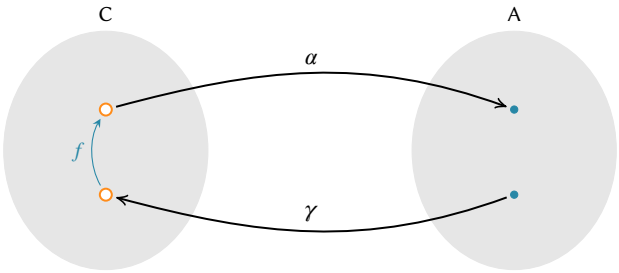

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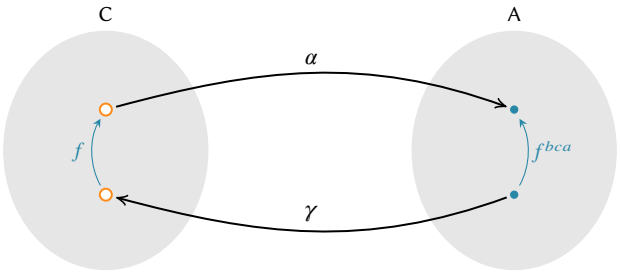

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The (fixpoint) abstraction must be sound: $\alpha(\mathrm{lfp} f) \sqsubseteq^a \mathrm{lfp} f^\sharp$ $\qquad \mathrm{lfp} f \sqsubseteq^c \gamma(\mathrm{lfp} f^\sharp)$

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- Define an abstract version $f^\sharp : A \to A$ of f
- Compute $\mathrm{lfp} f^\sharp$

The (fixpoint) abstraction must be sound: $\alpha(\mathrm{lfp} f) \sqsubseteq^a \mathrm{lfp} f^\sharp$ $\qquad \mathrm{lfp} f \sqsubseteq^c \gamma(\mathrm{lfp} f^\sharp)$

Sometimes the abstraction is exact (or complete): $\alpha(\mathrm{lfp} f) = \mathrm{lfp} f^\sharp$

THEOREM :: Point-wise Fixpoint Over-approximation

Given two monotonic functions $f, \overline{f} : \mathscr{S} \to \mathscr{S}$ on a complete lattice $\langle \mathscr{S}, \sqsubseteq, \sqcup, \sqcap, \bot, \top \rangle$, if $f \dot{\sqsubseteq} \overline{f}$ then $\mathrm{lfp}\, f \sqsubseteq \mathrm{lfp}\, \overline{f}$.

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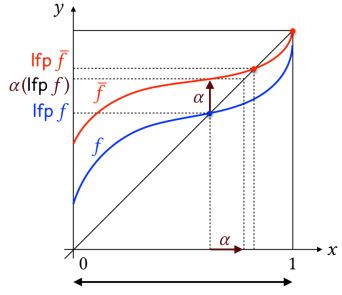

THEOREM :: Bca Fixpoint Over-approximation

Consider a monotonic function $f : \mathscr{S} \to \mathscr{S}$ on a complete lattice $\langle \mathscr{S}, \sqsubseteq \rangle$ and a Galois Connection $\langle \mathscr{S}, \sqsubseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \mathscr{L}, \preccurlyeq \rangle$, with $\langle \mathscr{L}, \preccurlyeq \rangle$ complete lattice. Then $\mathrm{lfp}^{\sqsubseteq} f \sqsubseteq \gamma(\mathrm{lfp}^{\preccurlyeq}\ \alpha \circ f \circ \gamma)$.

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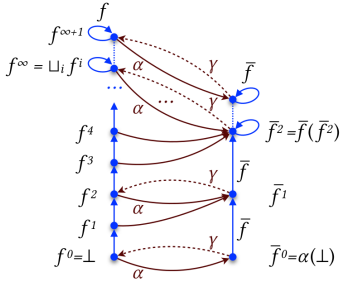
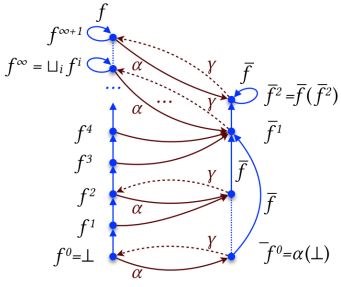


THEOREM :: Correct Fixpoint Over-approximation

Consider two monotonic functions $f : \mathscr{S} \to \mathscr{S}$ and $\overline{f} : \mathscr{L} \to \mathscr{L}$ on the complete lattices $\langle \mathscr{S}, \sqsubseteq \rangle$ and $\langle \mathscr{L}, \preccurlyeq \rangle$ such that $\langle \mathscr{S}, \sqsubseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \mathscr{L}, \preccurlyeq \rangle$ and $\alpha \circ f \circ \gamma \mathrel{\dot{\preccurlyeq}} \overline{f}$. Then $\mathrm{lfp}^{\sqsubseteq} f \sqsubseteq \gamma(\mathrm{lfp}^{\preccurlyeq} \overline{f})$.



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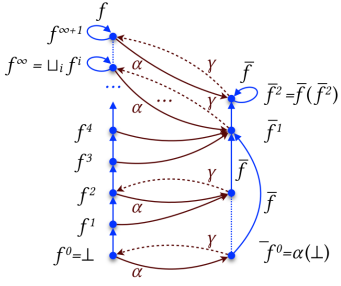

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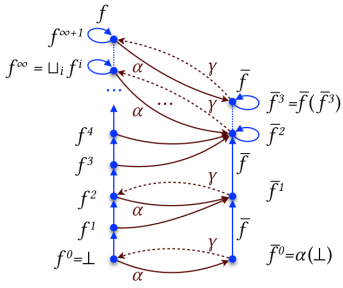

Fixpoint Acceleration

Computing fixpoints may involve an infinite number of iterations: non-termination

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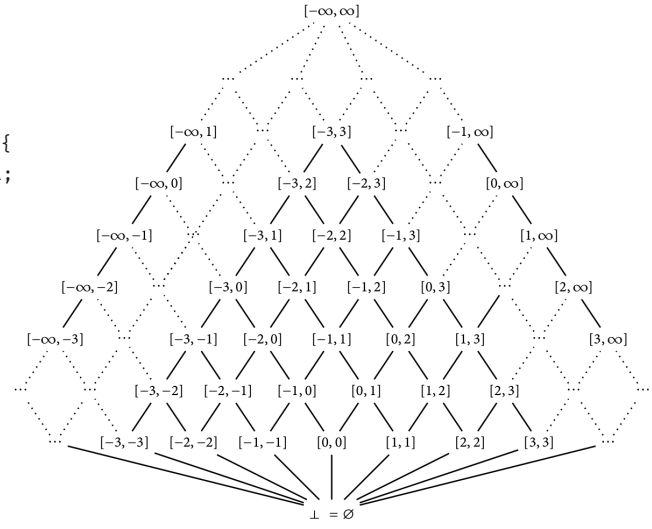
Idea: exploit fixpoint extrapolation

- To force iteration termination
- To speed-up iteration computation

Infinite iterations: example


```
1 x = 0;
2 while (true) {
3   x = x + 1;
4 }
```



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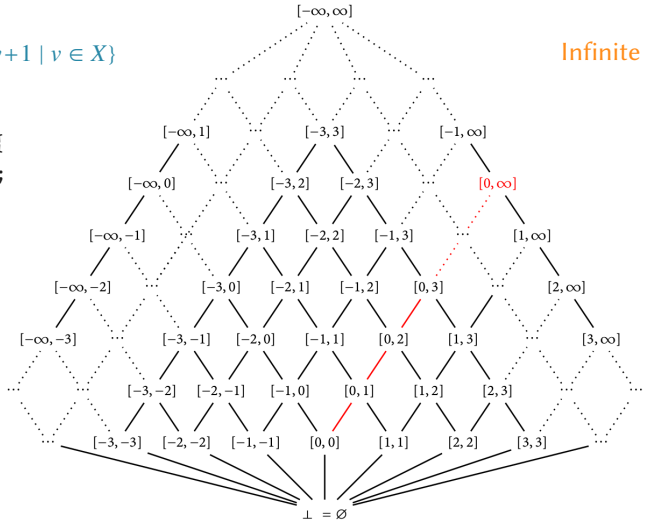

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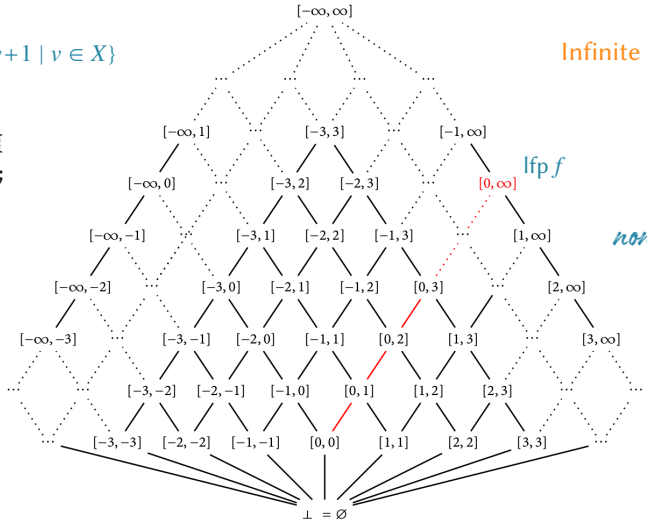

Infinite ascending chain

lfp f

non-termination

Idea: extrapolate from subsequent iterates f^n and f^{n+1} to an upper bound $f^n \nabla f^{n+1}$

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Rationale: a sound although imprecise answer is better than no answer at all

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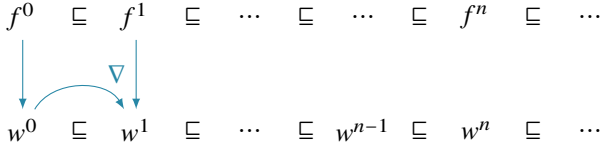

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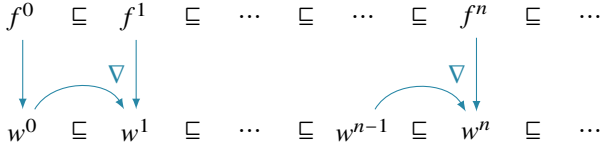

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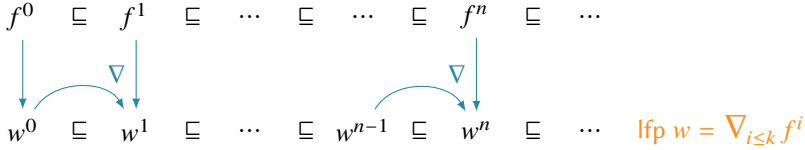

Extrapolate unstable bounds to infinity:

$$\bot \nabla [l, h] \triangleq [l, h] \nabla \bot \triangleq [l, h]$$
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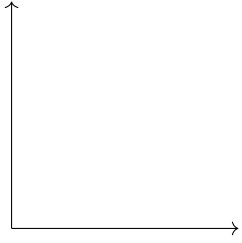

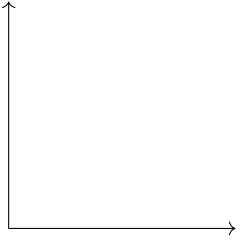

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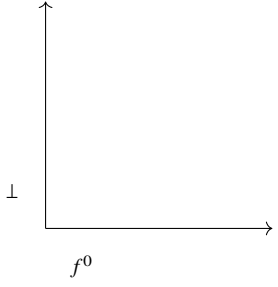

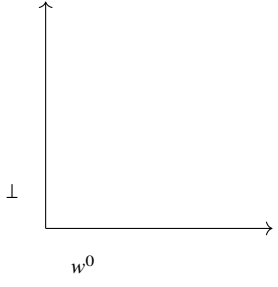

$w^0 = f^0 = \bot$

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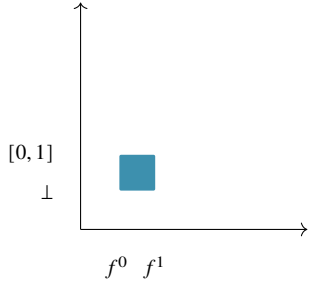

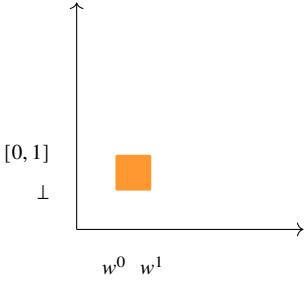

$$w^1 = w^0 \nabla f^1 =$$
$$= \bot \nabla [0,1] = [0,1]$$

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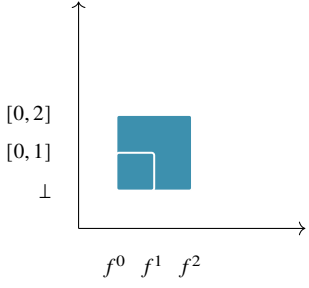
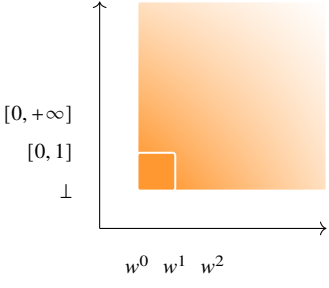
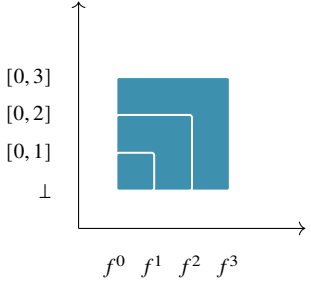


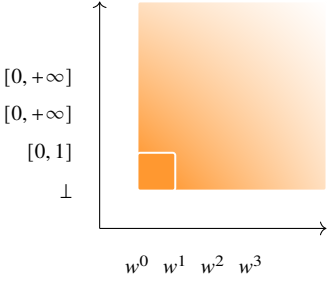


$$w^2 = w^1 \nabla f^2 =$$
$$= [0,1] \nabla [0,2] = [0, +\infty]$$

Extrapolate unstable bounds to infinity:

$$\bot \nabla [l,h] \triangleq [l,h] \nabla \bot \triangleq [l,h]$$
$$[l_1,h_1] \nabla [l_2,h_2] \triangleq [(\!| l_2 < l_1 \,?\, -\infty : l_1 |\!), (\!| h_2 > h_1 \,?\, \infty : h_1 |\!)]$$





$$w^3 = w^2 \nabla f^3 =$$
$$= [0,+\infty] \nabla [0,3] = [0,+\infty]$$

Widening computation is correct (sound)

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- Iterations sequences with widening are increasing and stationary after finitely many steps
- The limit of the iteration sequence with widening $\nabla_{i \leq k} f^i$ is a post-fixpoint of f

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- Iterations sequences with widening are increasing and stationary after finitely many steps
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Hence, $\nabla_{i \leq k} f^i$ is an over-approximation of the least fixpoint of f:

$$\mathrm{lfp}\, f \sqsubseteq \nabla_{i \leq k} f^i$$

computable in finite time

Thanks for the attention!

LaTeX is the way

Additional Slides

Further reading

CC-LOGCOM-1992 "Abstract Interpretation Frameworks", P. Cousot and R. Cousot, In: *Journal of logic and computation* (1992)