# **Principles of Abstract Interpretation**

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#### **Table of Contents**

#### Language

## Language

```
Abinop \ni aop ::= + | - | * | \
Aexp \ni e ::= x \mid n \mid -e \mid e_1 aop e_2
Bbinop \ni bop ::= = | \neq | < | \leq | > | \geq
Bexp \ni b ::= true \mid false \mid e_1 bop e_2 \mid b_1 \land b_2 \mid b_1 \lor b_2 \mid \neg b
While \ni S ::= x := e \mid skip \mid S_1; S_2 \mid if \ b then S_1 else S_2 \mid b while b do S
```

#### **Table of Contents**

#### Abstract Interpreter

### **Abstract Interpreter**

$$AI_{A,\mathbb{S}}: \mathbf{While} o bool o \mathbb{S} o \mathbb{S} imes List(\mathbb{S}) \ AI_{A,\mathbb{S}}(x:=e)(w)(s^{\sharp}) := (s^{\sharp}[x \mapsto \mathcal{A}^{\sharp}(e)(s^{\sharp})], []) \ AI_{A,\mathbb{S}}(\mathbf{skip})(w)(s^{\sharp}) := (s^{\sharp}, []) \ AI_{A,\mathbb{S}}(S_1; S_2)(w)(s^{\sharp}) := (u^{\sharp}, invs_1 +_{List(\mathbb{S})} invs_2) \ \text{with } (u^{\sharp}, invs_2) := AI_{A,\mathbb{S}}(S_2)(w)(t^{\sharp}) \ (t^{\sharp}, invs_1) := AI_{A,\mathbb{S}}(S_1)(w)(s^{\sharp})$$

### **Abstract Interpreter**

$$AI_{A,\mathbb{S}}(\textbf{if }b \textbf{ then } S_1 \textbf{ else } S_2)(w)(s^{\sharp}) := (t^{\sharp} \vee_{\mathbb{S}} u^{\sharp}, invs_1 +_{List(\mathbb{S})} invs_2)$$

$$\text{with } (t^{\sharp}, invs_1) := (AI_{A,\mathbb{S}}(S_1)(w) \circ \mathcal{B}^{\sharp}(b))(s^{\sharp})$$

$$(u^{\sharp}, invs_2) := (AI_{A,\mathbb{S}}(S_2)(w) \circ \mathcal{B}^{\sharp}(\neg b))(s^{\sharp})$$

$$AI_{A,\mathbb{S}}(\textbf{while }b \textbf{ do }S)(w)(s^{\sharp}) := (\mathcal{B}^{\sharp}(\neg b)(t^{\sharp}), invs)$$

$$\text{with } (t^{\sharp}, invs) := \text{ab-lfp}(AI_{A,\mathbb{S}}(S)(w))(b)(t^{\sharp})(w)$$

## **Abstract Semantics of arithmetic expressions**

$$egin{aligned} \mathcal{A}^{\sharp}: \mathbf{Aexp} &
ightarrow \mathbb{S} 
ightarrow A \ \mathcal{A}^{\sharp}(n)(s^{\sharp}) &:= lpha_{singleton}(n) \ \mathcal{A}^{\sharp}(x)(s^{\sharp}) &:= \operatorname{lookup}_{\mathbb{S}}(s^{\sharp})(x) \ \mathcal{A}^{\sharp}(e_1 ext{ aop } e_2)(s^{\sharp}) &:= \mathcal{A}^{\sharp}(e_1)(s^{\sharp}) ext{ aop}^A \mathcal{A}^{\sharp}(e_2)(s^{\sharp}) \ \mathcal{A}^{\sharp}(-e)(s^{\sharp}) &:= -^A \mathcal{A}^{\sharp}(e)(s^{\sharp}) \end{aligned}$$

## **Abstract Semantics of boolean expressions**

$$egin{aligned} \mathcal{B}^{\sharp}: \mathbf{Bexp} &
ightarrow \mathbb{S} 
ightarrow \mathbb{S} \ \mathcal{B}^{\sharp}(true)(s^{\sharp}) := s^{\sharp} \ \mathcal{B}^{\sharp}(false)(s^{\sharp}) := ot_{\mathbb{S}} \ \mathcal{B}^{\sharp}(e_1 ext{ bop } e_2)(s^{\sharp}) ext{ depends on the domain} \ \mathcal{B}^{\sharp}(b_1 \wedge b_2)(s^{\sharp}) := (\mathcal{B}^{\sharp}(b_2) \circ \mathcal{B}^{\sharp}(b_1))(s^{\sharp}) \ \mathcal{B}^{\sharp}(b_1 ee b_2)(s^{\sharp}) := \mathcal{B}^{\sharp}(b_1)(s^{\sharp}) ee_{\mathbb{S}} \mathcal{B}^{\sharp}(b_2)(s^{\sharp}) \end{aligned}$$

## Step

$$egin{aligned} \mathsf{step} : (\mathbb{S} o \mathbb{S} imes \mathit{List}(\mathbb{S})) o \mathit{bool} o \mathbb{S} o \mathbb{S} o \mathbb{S} imes \mathit{List}(\mathbb{S}) \ & \mathsf{step}(f)(b)(s^\sharp)(t^\sharp) := (s^\sharp ee_{\mathbb{S}} \ \mathit{u}^\sharp, \mathit{invs}) \ & \mathsf{with} \ (\mathit{u}^\sharp, \mathit{invs}) := f(\mathcal{B}^\sharp(b)(t^\sharp)) \end{aligned}$$

#### **Invariant Check**

$$\begin{array}{l} \text{is-inv}: (\mathbb{S} \to \mathbb{S} \times \textit{List}(\mathbb{S})) \to \mathbb{S} \to \mathbb{S} \to \textit{bool} \\ \\ \text{is-inv}(f)(s^{\sharp})(t^{\sharp}) := t^{\sharp} \sqsubseteq_{\mathbb{S}} u^{\sharp} \\ \\ \text{with } u^{\sharp} := \pi_{1}(\textit{step}(f)(b)(s^{\sharp})(t^{\sharp})) \end{array}$$

### **Steps**

$$\mathsf{steps} : (\mathbb{S} o \mathbb{S} imes \mathsf{List}(\mathbb{S})) o bool o \mathbb{S} o \mathbb{S} o \mathbb{S} imes \mathsf{List}(\mathbb{S})$$
  $\mathsf{steps}(f)(b)(s^\sharp)(t^\sharp) := egin{cases} (t^\sharp, [t^\sharp]) & \mathsf{if} \; \mathsf{is-inv}(f)(s^\sharp)(t^\sharp) \\ (v^\sharp, \mathsf{invs}_1 +_{\mathsf{List}(\mathbb{S})} \mathsf{invs}_2) & \mathsf{otherwise} \end{cases}$   $\mathsf{with}\; (u^\sharp, \mathsf{invs}_1) := \mathsf{step}(f)(b)(s^\sharp)(t^\sharp) \\ (v^\sharp, \mathsf{invs}_2) := \mathsf{steps}(f)(b)(s^\sharp)(u^\sharp) \end{cases}$ 

## Widening

$$egin{aligned} \mathsf{wid} : (\mathbb{S} o \mathbb{S} imes \mathsf{List}(\mathbb{S})) o bool o \mathbb{S} o \mathbb{S} \ & \mathsf{wid}(f)(b)(s^\sharp)(t^\sharp) := egin{cases} t^\sharp & \mathsf{if is\text{-}inv}(f)(s^\sharp)(t^\sharp) \ & \mathsf{wid}(f)(b)(s^\sharp)(t^\sharp) \ & \mathsf{otherwise} \end{cases} \ & \mathsf{with} \ u^\sharp := \pi_1(step(f)(b)(s^\sharp)(t^\sharp)) \end{aligned}$$

## **Narrowing**

$$egin{aligned} \mathsf{nar} : (\mathbb{S} o \mathbb{S} imes \mathit{List}(\mathbb{S})) o bool o \mathbb{S} o \mathbb{S} o \mathbb{S} imes \mathit{List}(\mathbb{S}) \ &\mathsf{nar}(f)(b)(s^\sharp)(t^\sharp) := egin{cases} (v^\sharp, [v^\sharp]) & \mathsf{if} \; \mathsf{is-inv}(f)(s^\sharp)(v^\sharp) \ (z^\sharp, \mathit{invs}_1 +_{\mathit{List}(\mathbb{S})} \; \mathit{invs}_2) & \mathsf{otherwise} \end{cases} \ &\mathsf{with} \; u^\sharp := \pi_1(\mathsf{step}(f)(b)(s^\sharp)(t^\sharp)) \ &v^\sharp := t^\sharp \; \Delta_{\mathbb{S}} \; u^\sharp \ &(w^\sharp, \mathit{invs}_1) := \mathsf{step}(f)(b)(s^\sharp)(v^\sharp) \ &(z^\sharp, \mathit{invs}_2) := \mathsf{nar}(f)(b)(s^\sharp)(v^\sharp\Delta_{\mathbb{S}} w^\sharp) \end{aligned}$$

#### **Abstract Least Fixed Point**

$$\begin{split} \mathsf{ab\text{-}lfp} : (\mathbb{S} \to \mathbb{S} \times \mathit{List}(\mathbb{S})) \to \mathit{bool} \to \mathbb{S} \to \mathit{bool} \to \mathbb{S} \times \mathit{List}(\mathbb{S}) \\ \mathsf{ab\text{-}lfp}(f)(b)(s^\sharp)(w) := \begin{cases} \mathsf{nar}(f)(b)(s^\sharp)(t^\sharp) & \text{if } w \\ \mathsf{steps}(f)(b)(s^\sharp)(s^\sharp) & \text{otherwise} \end{cases} \\ \mathsf{with} \ t^\sharp := \mathsf{wid}(f)(b)(s^\sharp)(s^\sharp). \end{aligned}$$

List Functions

#### **Table of Contents**

Language

Abstract Interpreter

List Functions

Extended Sign

Intervals

Examples

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## **Abstract State Update**

$$s^{\sharp}[x \mapsto a]$$
 defined by

$$\begin{cases} \bot_{\mathbb{S}}[x\mapsto a] := [(x,a)] \\ \top_{\mathbb{S}}[x\mapsto a] := [(x,a)] \\ ((y,a') :: ts^{\sharp})[x\mapsto a] := \begin{cases} (y,a) :: ts^{\sharp} & \text{if } x=y \\ (y,a') :: ts^{\sharp}[x\mapsto a] & \text{otherwise} \end{cases}$$

#### **Abstract State Join**

 $s^{\sharp} \vee_{\mathbb{S}} t^{\sharp}$  defined by

$$egin{cases} egin{aligned} egin{$$

### **Abstract State Lookup**

 $lookup_{\mathbb{S}}(s^{\sharp})(x)$  defined by

$$egin{cases} \mathsf{lookup}_{\mathbb{S}}(ot_{\mathbb{S}})(x) := ot_{\mathcal{A}} \ \mathsf{lookup}_{\mathbb{S}}(ot_{\mathbb{S}})(x) := ot_{\mathcal{A}} \ \mathsf{lookup}_{\mathbb{S}}((y,a) :: ts^{\sharp})(x) := egin{cases} a & \text{if } x = y \ \mathsf{lookup}_{\mathbb{S}}(ts^{\sharp})(x) & \text{otherwise} \end{cases}$$

#### Partial order

$$egin{aligned} a_1 &\leq_A a_2 := & (a_1 \lor_A a_2) = a_2 \ s^\sharp \sqsubseteq_\mathbb{S} t^\sharp := & (s^\sharp = \bot_\mathbb{S}) \lor orall x, \mathsf{lookup}(s^\sharp)(x) \leq_A \mathsf{lookup}(t^\sharp)(x) \end{aligned}$$

## **State Widening**

$$s^{\sharp} \; 
abla_{\mathbb{S}} \; t^{\sharp} := egin{cases} t^{\sharp} & ext{if} \; s^{\sharp} = ot_{\mathbb{S}} \ map(f_{t^{\sharp}})(s^{\sharp}) & ext{otherwise} \end{cases}$$
 with  $f_{t^{\sharp}}(x,a) := (x,a \; 
abla \; ext{lookup}(t^{\sharp})(x))$ 

## **State Narrowing**

$$egin{aligned} s^{\sharp} \; \Delta_{\mathbb{S}} \; t^{\sharp} &:= egin{cases} igsplus_{\mathbb{S}} & ext{if } s^{\sharp} = ot_{\mathbb{S}} \ map(f_{t^{\sharp}})(s^{\sharp}) & ext{otherwise} \end{cases} \ & ext{with } f_{t^{\sharp}}(x,a) := (x,a \; \Delta \; ext{lookup}(t^{\sharp})(x)) \end{aligned}$$

Extended Sign

#### **Table of Contents**

Language

Abstract Interpreter

List Functions

Extended Sign

Intervals

Examples

## **Extended Sign**

```
A := ExtSign
\mathbb{S} := List(String \times ExtSign) \cup \{\star\}
\perp_{\mathbb{S}} := \star
\top_{\mathbb{S}} := []
ExtSign \ni a ::= \bot \mid <0 \mid = 0 \mid > 0 \mid \leq 0 \mid \neq 0 \mid \geq 0 \mid \top
```

## $\alpha$ on Singletons

$$lpha_{singleton}(n) := egin{cases} = 0 & ext{if } n = 0 \ < 0 & ext{if } n < 0 \ > 0 & ext{otherwise} \end{cases}$$

## **Opposite**

### **Addition**

+	$\perp$	< 0	= 0	> 0	<b>S</b> 0	<b>=</b> 0	≥ 0	Т
$\perp$	丄							
< 0	丄	< 0						
= 0	丄	< 0	= 0					
> 0	丄	Т	> 0	> 0				
<u>≤</u> 0	丄	< 0	<b>S</b> 0	Т	<b>S</b> 0			
<b>7 0</b>	上	Т	<i>≠</i> 0	Т	Т	Т		
≥ 0	丄	Т	≥ 0	> 0	Т	Т	≥ 0	
Т	T	Т	Т	Т	Т	Т	Т	Т

### **Subtraction**

_	$\perp$	< 0	= 0	> 0	<b>S</b> 0	$\neq 0$	≥ 0	Τ
上	$\perp$	上	上	上	上	上	上	上
< 0	丄	Т	> 0	> 0	Т	Т	> 0	T
= 0		< 0	= 0	> 0	<b>S</b> 0	<i>≠</i> 0	≥ 0	T
> 0		< 0	< 0	Т	< 0	Т	Т	Т
<u>≤</u> 0	$\perp$	Т	≥ 0	> 0	Т	Т	≥ 0	Т
<b>=</b> 0		Т	<i>≠</i> 0	Т	Т	Т	Т	T
≥ 0	丄	< 0	<b>S</b> 0	Т	<b>S</b> 0	Т	Т	Т
Т	$\perp$	Т	Т	Т	Т	Т	Т	Т

## Multiplication

*	上	< 0	= 0	> 0	<b>S</b> 0	$\neq 0$	≥ 0	Т
上	丄							
< 0	丄	> 0						
= 0	丄	= 0	= 0					
> 0	丄	< 0	= 0	> 0				
<u>≤</u> 0	丄	≥ 0	= 0	<b>S</b> 0	≥ 0			
<b>7 0</b>	上	<i>≠</i> 0	= 0	<i>≠</i> 0	Т	<i>≠</i> 0		
≥ 0	上	<b>≤</b> 0	= 0	≥ 0	<b>S</b> 0	Т	≥ 0	
Т	上	Т	= 0	Т	Т	Т	Т	Т

### **Division**

/	Т	< 0	= 0	> 0	<b>≤</b> 0	<b>=</b> 0	≥ 0	Т
$\perp$	丄	上	$\perp$	上	上	上	上	上
< 0	上	> 0	= 0	< 0	≥ 0	<i>≠</i> 0	<b>≤</b> 0	Т
= 0	丄	上		上	上	1		工
> 0	丄	< 0	= 0	> 0	<b>S</b> 0	<i>≠</i> 0	≥ 0	Т
<u>≤</u> 0	丄	> 0	= 0	< 0	≥ 0	<i>≠</i> 0	<u>≤</u> 0	Т
<b>7 0</b>	上	<b>=</b> 0	= 0	<i>≠</i> 0	Т	<i>≠</i> 0	Т	Т
≥ 0	上	< 0	= 0	> 0	<b>S</b> 0	<i>≠</i> 0	≥ 0	Т
Т	上	<i>≠</i> 0	= 0	<i>≠</i> 0	Т	<i>≠</i> 0	Т	Т

# **Equal**

$e_1=e_2$		< 0	= 0	> 0	<u> </u>	<b>=</b> 0	≥ 0	Т
上	$\perp_{\mathbb{S}}$							
< 0	Ls	$s^{\#}$						
= 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}$					
> 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>				
<b>≤</b> 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>			
<b>=</b> 0	Ls	$s^{\#}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>		
≥ 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	
Т	$\perp_{\mathbb{S}}$	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>

# **Equal**

x = e		< 0	= 0	> 0	<b>≤</b> 0	<b>≠</b> 0	≥ 0	Т
	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
< 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}[x\mapsto <0]$	$s^{\#}[x\mapsto <0]$	$\perp_{\mathbb{S}}$	$s^{\#}[x\mapsto <0]$
= 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}$	$\perp_{\mathbb{S}}$	$s^{\#}[x\mapsto=0]$	$\perp_{\mathbb{S}}$	$s^{\#}[x\mapsto=0]$	$s^{\#}[x\mapsto=0]$
> 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$\perp_{\mathbb{S}}$	$s^{\#}[x\mapsto>0]$	$s^{\#}[x\mapsto>0]$	$s^{\#}[x\mapsto>0]$
≤ 0	$\perp_{\mathbb{S}}$	s#	s#	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}[x\mapsto < 0]$	$s^{\#}[x\mapsto=0]$	$s^{\#}[x\mapsto\leq 0]$
<b>≠</b> 0	$\perp_{\mathbb{S}}$	s#	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}[x\mapsto <0]$	s#	$s^{\#}[x\mapsto>0]$	$s^{\#}[x\mapsto\neq 0]$
≥ 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s#	s <sup>#</sup>	$s^{\#}[x\mapsto=0]$	$s^{\#}[x\mapsto>0]$	s <sup>#</sup>	$s^{\#}[x\mapsto\geq 0]$
Т	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>

## **Not Equal**

$e_1  eq e_2$		< 0	= 0	> 0	<b>≤</b> 0	<b>=</b> 0	≥ 0	Τ
	$\perp_{\mathbb{S}}$							
< 0	Ls	$s^{\#}$						
= 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$\perp_{\mathbb{S}}$					
> 0	$\perp_{\mathbb{S}}$	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>				
<b>≤</b> 0	$\perp_{\mathbb{S}}$	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>			
<b>=</b> 0	Ls	$s^{\#}$	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>		
≥ 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	
Т	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	$s^{\#}$

## **Not Equal**

			_		< 0	/ 0	> 0	_
$x \neq e$		< 0	=0	> 0	$\leq 0$	$\neq 0$	$\geq 0$	
1	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	⊥ <sub>S</sub>	⊥s
< 0	Δs	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>
= 0	Δs	s <sup>#</sup>	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$s^{\#}[x\mapsto < 0]$	s <sup>#</sup>	$s^{\#}[x \mapsto > 0]$	$s^{\#}[x\mapsto\neq 0]$
> 0	Δs	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s#	s <sup>#</sup>	s <sup>#</sup>
<b>≤</b> 0	Δs	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s#	s <sup>#</sup>
<b>≠</b> 0	Δs	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s#	s <sup>#</sup>
≥ 0	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s#	s <sup>#</sup>	s#
Т	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s#	s <sup>#</sup>	s#

### **Less Than**

$e_1 < e_2$		< 0	= 0	> 0	<b>≤</b> 0	<b>=</b> 0	≥ 0	Т
上	$\perp_{\mathbb{S}}$							
< 0	LS	$s^{\#}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	$\perp_{\mathbb{S}}$	$s^{\#}$
= 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	$\perp_{\mathbb{S}}$	$s^{\#}$
> 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	s <sup>#</sup>				
<u>≤</u> 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	$\perp_{\mathbb{S}}$	s <sup>#</sup>
<b>=</b> 0	LS	$s^{\#}$	$s^{\#}$	s <sup>#</sup>				
≥ 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	s <sup>#</sup>				
Τ	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	s <sup>#</sup>				

### **Less Than**

x < e	Τ	< 0	= 0	> 0	<b>≤</b> 0	$\neq 0$	≥ 0	Т
Τ	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
< 0	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}[x\mapsto <0]$	$s^{\#}[x\mapsto <0]$	$\perp_{\mathbb{S}}$	$s^{\#}[x\mapsto <0]$
= 0	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}[x\mapsto <0]$	$s^{\#}[x\mapsto <0]$	$\perp_{\mathbb{S}}$	$s^{\#}[x\mapsto <0]$
> 0	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s#	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>
≤ 0	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}[x\mapsto <0]$	$s^{\#}[x\mapsto <0]$	$\perp_{\mathbb{S}}$	$s^{\#}[x\mapsto <0]$
$\neq 0$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s#	s <sup>#</sup>	s <sup>#</sup>	$s^{\#}$	s <sup>#</sup>	$s^{\#}$
$\geq 0$	$\perp_{\mathbb{S}}$	s#	s#	s#	s <sup>#</sup>	$s^{\#}$	s#	$s^{\#}$
Τ	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	s#	s <sup>#</sup>	$s^{\#}$	s#	$s^{\#}$

### **Greater Than**

$e_1 > e_2$	$\perp$	< 0	= 0	> 0	<b>≤</b> 0	<b>=</b> 0	≥ 0	Т
	$\perp_{\mathbb{S}}$							
< 0	Ls	$s^{\#}$						
= 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
> 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	$s^{\#}$
<b>≤</b> 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	$s^{\#}$	$s^{\#}$
<b>=</b> 0	Ls	$s^{\#}$	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	$s^{\#}$	$s^{\#}$
≥ 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$
Т	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	$s^{\#}$

### **Greater Than**

x > e		< 0	= 0	> 0	< 0	<b>≠</b> 0	≥ 0	Т
	1-		-				_	1 -
	⊥s	⊥s	⊥s	⊥s	⊥s	⊥s	⊥s	⊥s
< 0	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>
= 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$\perp_{\mathbb{S}}$	$s^{\#}[x\mapsto>0]$	$s^{\#}[x\mapsto>0]$	$s^{\#}[x \mapsto > 0]$
> 0	Δs	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$\perp_{\mathbb{S}}$	$s^{\#}[x\mapsto>0]$	$s^{\#}[x\mapsto>0]$	$s^{\#}[x \mapsto > 0]$
≤ 0	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>
$\neq 0$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s#	s#	s#	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>
≥ 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s#	$\perp_{\mathbb{S}}$	$s^{\#}[x\mapsto>0]$	$s^{\#}[x\mapsto>0]$	$s^{\#}[x\mapsto>0]$
Т	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s#	s#	s#	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>

### **Less Than or Equal**

$e_1 \leq e_2$		< 0	= 0	> 0	<u>≤</u> 0	<b>=</b> 0	≥ 0	Т
上	$\perp_{\mathbb{S}}$							
< 0	LS	$s^{\#}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	$\perp_{\mathbb{S}}$	$s^{\#}$
= 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	$s^{\#}$
> 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	s <sup>#</sup>				
<u>≤</u> 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>
<b>=</b> 0	LS	$s^{\#}$	$s^{\#}$	s <sup>#</sup>				
≥ 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	s <sup>#</sup>				
Т	$\perp_{\mathbb{S}}$	s <sup>#</sup>						

### **Less Than or Equal**

$x \le e$	1	< 0	= 0	> 0	$\leq 0$	<b>≠</b> 0	≥ 0	Т
上	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	Ls	⊥s	$\perp_{\mathbb{S}}$
< 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}[x\mapsto <0]$	$s^{\#}[x\mapsto < 0]$	⊥s	$s^{\#}[x\mapsto <0]$
= 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}[x\mapsto < 0]$	$s^{\#}[x \mapsto = 0]$	$s^{\#}[x\mapsto\leq 0]$
> 0	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s#	s <sup>#</sup>	s <sup>#</sup>
<b>≤</b> 0	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$s^{\#}[x\mapsto < 0]$	$s^{\#}[x \mapsto = 0]$	$s^{\#}[x\mapsto\leq 0]$
<b>≠</b> 0	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s#	s <sup>#</sup>	s <sup>#</sup>
≥ 0	$\perp_{\mathbb{S}}$	s#	s#	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>
Т	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>

### **Greater Than or Equal**

$e_1 \geq e_2$		< 0	= 0	> 0	<u>≤</u> 0	<b>=</b> 0	≥ 0	Т
	$\perp_{\mathbb{S}}$							
< 0	LS	$s^{\#}$	$s^{\#}$	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>	$s^{\#}$	$s^{\#}$
= 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>	$s^{\#}$
> 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	$s^{\#}$
<u>≤</u> 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	s <sup>#</sup>				
<b>=</b> 0	LS	$s^{\#}$	$s^{\#}$	s <sup>#</sup>				
≥ 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}$	s <sup>#</sup>				
Т	$\perp_{\mathbb{S}}$	s <sup>#</sup>						

### **Greater Than or Equal**

$x \ge e$		< 0	= 0	> 0	$\leq 0$	<b>≠</b> 0	≥ 0	Т
上	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	Ls	⊥s	$\perp_{\mathbb{S}}$
< 0	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>
= 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$	$s^{\#}[x\mapsto=0]$	$s^{\#}[x\mapsto>0]$	s <sup>#</sup>	$s^{\#}[x\mapsto\geq 0]$
> 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$\perp_{\mathbb{S}}$	$s^{\#}[x \mapsto > 0]$	$s^{\#}[x \mapsto > 0]$	$s^{\#}[x\mapsto>0]$
<b>≤</b> 0	$\perp_{\mathbb{S}}$	s#	s#	s <sup>#</sup>	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>	s#
<i>≠</i> 0	$\perp_{\mathbb{S}}$	s#	s#	s <sup>#</sup>	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>	s#
≥ 0	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	s#	s <sup>#</sup>	$s^{\#}[x\mapsto=0]$	$s^{\#}[x\mapsto>0]$	s <sup>#</sup>	$s^{\#}[x\mapsto\geq 0]$
Т	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	s <sup>#</sup>				

# Join

V		< 0	= 0	> 0	<b>≤</b> 0	<b>=</b> 0	≥ 0	Т
上	上							
< 0	< 0	< 0						
= 0	= 0	<b>S</b> 0	= 0					
> 0	> 0	≥ 0	≥ 0	> 0				
<b>S</b> 0	<b>S</b> 0	<b>S</b> 0	<b>S</b> 0	Т	<b>S</b> 0			
<b>=</b> 0	<i>≠</i> 0	<i>≠</i> 0	Τ	<i>≠</i> 0	Т	<i>≠</i> 0		
≥ 0	≥ 0	Т	≥ 0	≥ 0	T	Т	≥ 0	
Т	Т	Т	Т	Т	T	T	Т	Т

#### **Table of Contents**

Language

Abstract Interpreter

List Functions

Extended Sign

Intervals

Examples

October 22, 2024

#### **Intervals**

$$A := Int$$

$$\mathbb{S} := List(String \times Int) \cup \{\star\}$$

$$\perp_{\mathbb{S}} := \star$$

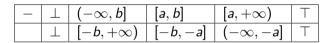
$$\top_{\mathbb{S}} := []$$

$$Int \ni c ::= \bot \mid (-\infty, b] \mid [a, b] \mid [a, +\infty) \mid \top$$

# $\alpha$ on Singletons

$$\alpha_{singleton}(n) := [n, n]$$

## **Opposite**



### **Addition**

+	上	$(-\infty,b]$	[a, b]	$[a,+\infty)$	Т
	上				
$[-\infty,d]$	上	$(-\infty, b+d]$			
[c, d]	上	$(-\infty, b+d]$	[a+c,b+d]		
$[c,+\infty)$	上	Т	$[a+c,+\infty)$	$[a+c,+\infty)$	
Т	上	Т	Т	Т	Т

### **Subtraction**

_	上	$(-\infty,b]$	[a, b]	$[a,+\infty)$	T
	上	上		$\perp$	Н
$[-\infty,d]$	上	Т	$[a-d,+\infty)$	$[a-d,+\infty)$	Η
[c, d]	丄	$(-\infty, b-c]$	[a-d,b-c]	$[a-d,+\infty)$	$\vdash$
$[c,+\infty)$	上	$(-\infty, b-c]$	$(-\infty, b-c]$	$\top$	$\vdash$
Т	上	Т	Т	Т	$\vdash$

### Multiplication

*	Τ	$(-\infty, b]$		[a, b]		[a, +∞)		Т	
Τ.	Τ.	1		1		1		1	
/0		$b > 0 \lor d > 0$	Т	$(a < 0 \land b > 0) \lor (a > 0 \land b < 0)$ $(a \le 0 \land b \le 0) \land (a \ne 0 \lor b \ne 0)$	T $[min(ad, bd), +\infty)$	$d > 0 \lor a < 0$	Т	-	
$(-\infty, d]$	_	$b \leq 0 \land d \leq 0$	$[bd, +\infty)$	$(a \ge 0 \land b \ge 0) \land (a \ne 0 \lor b \ne 0)$ $(a = 0 \land b = 0)$	$(-\infty, max(ad, bd)]$ [0, 0]	$d \le 0 \land a \ge 0$	$(-\infty, ad]$	ļ '	
[c, d]	Τ	$(c < 0 \land d > 0) \lor (c > 0 \land d < 0)$ $(c \le 0 \land d \le 0) \land (c \ne 0 \lor d \ne 0)$ $(c \ge 0 \land d \ge 0) \land (c \ne 0 \lor d \ne 0)$ $(c \ge 0 \land d \ge 0) \land (c \ne 0 \lor d \ne 0)$	T $[min(bc, bd), +\infty)$ $(-\infty, max(bc, bd)]$ [0, 0]	[min(ac, ad, bc, bd), max(ac, ad, bc	-, bd)]	$(c < 0 \land d > 0) \lor (c > 0 \land d < 0)$ $(c \le 0 \land d \le 0) \land (c \ne 0 \lor d \ne 0)$ $(c \ge 0 \land d \ge 0) \land (c \ne 0 \lor d \ne 0)$ $(c = 0 \land d = 0)$	$(-\infty, max(ac, ad)]$ $[min(ac, ad), +\infty)$ [0, 0]	c = d = 0 $c \neq 0 \lor d \neq 0$	[0, 0] T
$[c, +\infty)$	Τ	$b > 0 \lor c < 0$ $b \le 0 \land c \ge 0$	⊤ (-∞, bc]	$(a < 0 \land b > 0) \lor (a > 0 \land b < 0)$ $(a \le 0 \land b \le 0) \land (a \ne 0 \lor b \ne 0)$ $(a \ge 0 \land b \ge 0) \land (a \ne 0 \lor b \ne 0)$ $(a = 0 \land b = 0)$	$(-\infty, max(ac, bc)]$ $[min(ac, bc), +\infty)$ [0, 0]	$a < 0 \lor c < 0$ $a \ge 0 \land c \ge 0$	⊤ (-∞, ac]	т	
Т	Τ	Т		a = b = 0 $a \neq 0 \lor b \neq 0$	[0,0] T	Т		Т	

### **Division**

		$(-\infty, b]$		[a, b]		$[a, +\infty)$		7
/	-	$(-\infty, b]$		[a, b]		$[a, +\infty)$		
1	1	1		1		1	1	
	Т	d < 0	$[min(0, b/d), +\infty)$	d < 0	[min(0, b/d), max(0, a/d)]	d < 0	$(-\infty, max(a/d, 0)]$	
$(-\infty, d]$	1	d = 0	$(-\infty, b]/(-\infty, -1]$	d = 0	$[a, b]/(-\infty, -1]$	d = 0	$[a, +\infty)/(-\infty, -1]$	T
	1	otherwise	T	otherwise	$[a, b]/(-\infty, -1] \vee [a, b]/[1, d]$	otherwise	T	
	$\overline{}$	c = d = 0	1	c = d = 0	1	c = d = 0	1	
	1	$0 < c \le d$		$0 < c \le d \lor c \le d < 0$	[min(a/c, a/d, b/c, b/d), max(a/c, a/d, b/c, b/d)]		$[min(a/c, a/d), +\infty)$	$c = d = 0$ $\perp$
[c, d]	Ι.	0 = c < d	$(-\infty, b]/[1, d]$	0 = c < d	[a, b]/[1, d]	0 = c < d	$[a, +\infty)/[1, d]$	
[c, a]	1.	$c \le d < 0$	$[min(b/c, b/d), +\infty)$	c < d = 0	[a, b]/[c, -1]	$c \le d < 0$		
	1	c < d = 0	$(-\infty, b]/[c, -1]$	c < 0 < d	$[a, b]/[c, -1] \vee [a, b]/[1, d]$	c < d = 0	$[a, +\infty)/[c, -1]$	otherwise T
		otherwise	T			otherwise	T	
	т	c > 0	$(-\infty, max(0, b/c)]$	c > 0	[min(0, a/c), max(0, b/c)]	c > 0	$[min(a/c, 0), +\infty)$	
$[c, +\infty)$	1	c = 0	$(-\infty, b]/[1, +\infty)$	c = 0	$[a, b]/[1, +\infty)$	c = 0	$[a, +\infty)/[1, +\infty)$	T
		otherwise	T	otherwise	$[a, b]/[c, -1] \vee [a, b]/[1, +\infty)$	otherwise	T	
Т	1	Т		$[a, b]/(-\infty, -1] \vee [a, b]/$	$(1, +\infty)$	Т		Т

# **Equal**

$e_1=e_2$	1	$(-\infty,b]$	[a, b]	$[a,+\infty)$	Т
1	$\perp_{\mathbb{S}}$	⊥s	⊥s	⊥s	$\perp_{\mathbb{S}}$
$(-\infty, d]$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	if $a>d$ then $\perp_{\mathbb{S}}$ else $s^{\#}$	if $a>d$ then $\perp_{\mathbb{S}}$ else $s^{\#}$	s#
[c, d]	$\perp_{\mathbb{S}}$	if $b < c$ then $\perp_{\mathbb{S}}$ else $s^{\#}$	if $a>d$ or $b< c$ then $\perp_{\mathbb{S}}$ else $s^{\#}$	if $a>d$ then $\perp_{\mathbb{S}}$ else $s^{\#}$	s#
$[c, +\infty)$	$\perp_{\mathbb{S}}$	if $b < c$ then $\perp_{\mathbb{S}}$ else $s^{\#}$	if $b < c$ then $\perp_{\mathbb{S}}$ else $s^{\#}$	s <sup>#</sup>	s#
Т	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s#	s <sup>#</sup>	s#

# **Equal**

x = e	1	$(-\infty, b]$		[a, b]		$[a, +\infty)$		Т
Τ	Δg	1s		⊥g		⊥s		⊥s
		$b \le d$	s#	a > d	⊥s	a > d	⊥g	
$(-\infty, d]$	⊥s	b > d	$s^\#[x\mapsto (-\infty,d]]$	$ \begin{aligned} a &\leq d \land b > d \\ a &\leq d \land b \leq d \end{aligned} $	$s^{\#}[x \mapsto [a, d]]$ $s^{\#}$	$a \leq d$	$s^{\#}[x\mapsto [a,d]]$	$s^{\#}[x \mapsto (-\infty, d]]$
		b < c	⊥s	$b < c \lor a > d$	⊥s	a > d	⊥s	
[c, d]	⊥s	$c \leq b \leq d$	$s^{\#}[x\mapsto [c,b]]$	$b > d \land a < c$ $b > d \land a \ge c$	$s^{\#}[x \mapsto [c, d]]$ $s^{\#}[x \mapsto [a, d]]$	a < c	$s^{\#}[x\mapsto [c,d]]$	$s^{\#}[x \mapsto [c, d]]$
		b > d	$s^{\#}[x\mapsto [c,d]]$	$c \le b \le d \land a < c$ $c \le b \le d \land c \le a \le d$	$s^{\#}[x \mapsto [c, b]]$ $s^{\#}$	$c \leq a \leq d$	$s^{\#}[x\mapsto [a,d]]$	
		b < c	⊥s	b < c	$\perp_{\mathrm{S}}$	a < c	$s^{\#}[x \mapsto [c, +\infty)]$	_
$[c, +\infty)$	Δs	$b \ge c$	$s^{\#}[x\mapsto [c,b]]$	$b \ge c \land a < c$ $a \ge c$	$s^{\#}[x \mapsto [c, b]]$ $s^{\#}$	$a \ge c$	s#	$s^{\#}[x \mapsto [c, +\infty)]$
Т	Δg	s#		s#		s#		s#

# **Not Equal**

$e_1 \neq e_2$	上	$(-\infty,b]$	[a, b]		$[a,+\infty)$	Τ
上	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$		$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
$(-\infty,d]$	$\perp_{\mathbb{S}}$	$s^{\#}$	s <sup>#</sup>		$s^{\#}$	$s^\#$
[c,d]	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$a = b = c = d$ $\bot$ otherwise $s$	L <sub>S</sub>	s <sup>#</sup>	s <sup>#</sup>
$[c,+\infty)$	Ls	s <sup>#</sup>	s <sup>#</sup>		s <sup>#</sup>	$s^{\#}$
Т	$\perp_{\mathbb{S}}$	$s^{\#}$	s <sup>#</sup>		$s^{\#}$	$s^{\#}$

# **Not Equal**

$x \neq e$	Τ	$(-\infty,b]$	[a, b]	$[a, +\infty)$	Т
1	Δs	⊥s	⊥s	⊥s	Δg
$(-\infty, d]$	Δs		s <sup>#</sup>	$s^{\#}$	s#
[c, d]	1-	$b=c=d$ $s^{\#}[x\mapsto (-\infty,b-1]]$	$a = c = d \land a \neq b$ $s^{\#}[x \mapsto [a+1, b]]$ $b = c = d \land a \neq b$ $s^{\#}[x \mapsto [a, b-1]]$	$a=c=d$ $s^{\#}[x\mapsto [a+1,+\infty)]$	c#
[c, a]	±s	otherwise $s^{\#}$	$a=b=c=d$ $\perp_{\mathbb{S}}$ otherwise $s^{\#}$	otherwise $s^{\#}$	5"
$[c,+\infty)$	⊥s	s#	s#	s#	s#
Т	Δs	$s^{\#}$	s#	s <sup>#</sup>	s#

### **Less Than**

$e_1 < e_2$	上	$(-\infty,b]$	[a, b]		$[a, +\infty]$	)	Τ
上	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$		$\perp_{\mathbb{S}}$		$\perp_{\mathbb{S}}$
( ~ dl	La	s <sup>#</sup>	$a \ge d$	$\perp_{\mathbb{S}}$	$a \ge d$	$\perp_{\mathbb{S}}$	s#
$[-\infty,d]$	$\perp_{\mathbb{S}}$	3"	a < d	$s^{\#}$	a < d	$s^{\#}$	3
[c, d]	1	s <sup>#</sup>	$a \ge d$	$\perp_{\mathbb{S}}$	$a \ge d$	Lς	s#
[c, a]	⊥s	3"	a < d	$s^{\#}$	a < d	$s^{\#}$	3
$[c,+\infty)$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$		s <sup>#</sup>		s <sup>#</sup>
Т	$\perp_{\mathbb{S}}$	$s^{\#}$	s <sup>#</sup>		s <sup>#</sup>		s <sup>#</sup>

### **Less Than**

x < e	Τ	$(-\infty, b]$	[a, b]	$[a, +\infty)$	Т
1	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	⊥s	$\perp_{\mathbb{S}}$
		$b \ge d$ $s^{\#}[x \mapsto (-\infty, d-1]]$	$a \ge d$ $\perp_{\mathbb{S}}$	$a \ge d$ $\perp_{\mathbb{S}}$	
$(-\infty, d]$	Δs	$b < d s^{\#}$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$a < d$ $s^{\#}[x \mapsto [a d - 1]]$	$s^{\#}[x \mapsto (-\infty, d-1]]$
			0 0 10 0	5 (A 7 [8, 0 1])	
		$b \ge d$ $s^{\#}[x \mapsto (-\infty, d-1]]$	$a \ge d$ $\perp_{\mathbb{S}}$	$a \ge d$ $\perp_{\mathbb{S}}$	
[c,d]	Δs	$b < d s^{\#}$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$a < d$ $s^{\#}[x \mapsto [a d - 1]]$	$s^{\#}[x \mapsto (-\infty, d-1]]$
		2 3	$a < d \land b < d  s^{\#}$	8 ( 0 3 [x · / [a, 0 1]]	
$[c,+\infty)$	Δs	$s^{\#}$	$s^{\#}$	s <sup>#</sup>	s <sup>#</sup>
Т	$\perp_{\mathbb{S}}$	s#	s#	s#	s#

# **Less Than or Equal**

$e_1 \leq e_2$		$(-\infty,b]$	[a, b]		$[a, +\infty]$	)	Т
	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$		$\perp_{\mathbb{S}}$		$\perp_{\mathbb{S}}$
$[-\infty,d]$	1	s <sup>#</sup>	a > d	$\perp_{\mathbb{S}}$	a > d	$\perp_{\mathbb{S}}$	s#
	⊥ <sub>S</sub>	5"	$a \leq d$	$s^\#$	$a \leq d$	$s^{\#}$	5"
[]	1_	s <sup>#</sup>	a > d	$\perp_{\mathbb{S}}$	a > d	Ls	s#
[c,d]	$\perp_{\mathbb{S}}$	5"	$a \leq d$	$s^{\#}$	$a \leq d$	$s^\#$	5"
$[c,+\infty)$	$\perp_{\mathbb{S}}$	$s^{\#}$	$s^{\#}$		s <sup>#</sup>		s <sup>#</sup>
Т	$\perp_{\mathbb{S}}$	$s^{\#}$	s <sup>#</sup>		s <sup>#</sup>		s <sup>#</sup>

## **Less Than or Equal**

$x \leq e$	1	$(-\infty, b]$	[a, b]	$[a, +\infty)$	Т
Τ.	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
		$b > d$ $s^{\#}[x \mapsto (-\infty, d]]$	$a>d$ $\perp_{\mathbb{S}}$	a > d ⊥s	
$(-\infty,d]$	Δg	$b \le d  s^\#$	$a \le d \land b > d$ $s^{\#}[x \mapsto [a, d]]$ $a \le d \land b \le d$ $s^{\#}$	$a \leq d  s^{\#}[x \mapsto [a,d]]$	$s^{\#}[x \mapsto (-\infty, d]]$
		$b > d$ $s^{\#}[x \mapsto (-\infty, d]]$		a > d ⊥s	
[c, d]	Δs	$b \le d$ $s^\#$	$a \le d \land b > d$ $s^{\#}[x \mapsto [a, d]]$ $a \le d \land b \le d$ $s^{\#}$	$a \le d$ $s^{\#}[x \mapsto [a, d]]$	$s^{\#}[x \mapsto (-\infty, d]]$
$[c,+\infty)$	Δs	s#	s#	s <sup>#</sup>	s <sup>#</sup>
Т	$\perp_{\mathbb{S}}$	s#	s#	s <sup>#</sup>	s#

### **Greater Than**

$e_1 > e_2$	上	$(-\infty,b]$	[a, b]	$[a,+\infty)$	Τ
上	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
$(-\infty,d]$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	$s^{\#}$	s <sup>#</sup>
[c, d]	Ls	$b \leq c \perp_{\mathbb{S}}$	$b \leq c$ $\perp_{\mathbb{S}}$	s <sup>#</sup>	s#
		$b>c$ $s^{\#}$	$b>c$ $s^{\#}$	5"	5"
$[c,+\infty)$	$\perp_{\mathbb{S}}$	$b \leq c$ $\perp_{\mathbb{S}}$	$b \leq c$ $\perp_{\mathbb{S}}$	s <sup>#</sup>	s#
		$b > c$ $s^{\#}$	$b>c$ $s^{\#}$	3"	3"
Т	$\perp_{\mathbb{S}}$	s <sup>#</sup>	s <sup>#</sup>	$s^{\#}$	s <sup>#</sup>

#### **Greater Than**

x > e	1	$(-\infty, b]$	[a, b]	$[a, +\infty)$	Т
1	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	⊥s	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
$(-\infty, d]$	Δg	s <sup>#</sup>	s#	s#	s <sup>#</sup>
		$b \le c$ $\perp_{\mathbb{S}}$	$b \le c$ $\perp_{\mathbb{S}}$	$a \le c$ $s^{\#}[x \mapsto [c+1, +\infty)]$	
[c, d]	⊥s	$b>c$ $s^{\#}[x\mapsto [c+1,b]]$	$b > c \land a \le c$ $s^{\#}[x \mapsto [c+1,b]]$ $b > c \land a > c$ $s^{\#}$	$a>c$ $s^{\#}$	$s^{\#}[x \mapsto [c+1,+\infty)]$
		$b \le c \perp_{\mathbb{S}}$	$b \le c$ $\perp_{\mathbb{S}}$	$a \le c$ $s^{\#}[x \mapsto [c+1, +\infty)]$	
$[c,+\infty)$	Δg	$b>c$ $s^{\#}[x\mapsto [c+1,b]]$	$b > c \land a \le c$ $s^{\#}[x \mapsto [c+1,b]]$ $b > c \land a > c$ $s^{\#}$	$a>c$ $s^{\#}$	$s^{\#}[x \mapsto [c+1,+\infty)]$
Т	$\perp_{\mathbb{S}}$	s#	s <sup>#</sup>	s#	s <sup>#</sup>

# **Greater Than or Equal**

$e_1 \geq e_2$	上	$(-\infty,b]$	[a, b]	$[a,+\infty)$	$\top$
	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$	$\perp_{\mathbb{S}}$
$(-\infty,d]$	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$s^{\#}$	$s^{\#}$	s <sup>#</sup>
[c, d]	$\perp_{\mathbb{S}}$	$egin{array}{ccc} b < c & ot_{\mathbb{S}} \ b \geq c & s^\# \end{array}$	$b < c$ $\perp_{\mathbb{S}}$	s <sup>#</sup>	s#
		$b \geq c - s^{\#}$	$b \geq c - s^{\#}$	5"	5"
$[c,+\infty)$	$\perp_{\mathbb{S}}$	$egin{array}{ccc} b < c & ot_{\mathbb{S}} \ b \geq c & s^\# \end{array}$	$b < c$ $\perp_{\mathbb{S}}$	s#	s#
		$b \geq c  s^{\#}$	$b \geq c - s^{\#}$	3"	3"
Т	$\perp_{\mathbb{S}}$	s <sup>#</sup>	$s^{\#}$	$s^{\#}$	s <sup>#</sup>

# **Greater Than or Equal**

$x \ge e$	1	$(-\infty,b]$	[a, b]	$[a, +\infty)$	Т
Τ.	Δs	$\perp_{\mathbb{S}}$	⊥s	⊥s	$\perp_{\mathbb{S}}$
$(-\infty, d]$	Δg	s <sup>#</sup>	s#	s <sup>#</sup>	s#
		$b < c$ $\perp_{\mathbb{S}}$	$b < c$ $\perp_{\mathbb{S}}$	$a < c$ $s^{\#}[x \mapsto [c, +\infty)]$	
[c,d]	Δs	h > c e <sup>#</sup> [v → [c h]]	$b \ge c \land a < c \qquad s^{\#}[x \mapsto [c, b]]$ $b \ge c \land a \ge c \qquad s^{\#}$	3 3 6 8#	$s^{\#}[x \mapsto [c, +\infty)]$
		<i>b</i> ≥ <i>c</i> 3 [x → [c, b]]	$b \ge c \land a \ge c  s^\#$	a <u>&gt;</u> c 3	
		$b < c \perp_{\mathbb{S}}$	$b < c$ $\perp_{\mathbb{S}}$	$a < c$ $s^{\#}[x \mapsto [c, +\infty)]$	
$[c,+\infty)$	⊥s	$b \ge c$ $s^{\#}[x \mapsto [c, b]]$	$b \ge c \land a < c  s^{\#}[x \mapsto [c,b]]$	$a \ge c - s^\#$	$s^{\#}[x \mapsto [c, +\infty)]$
		<i>b</i> ≥ <i>c</i> 3 [x → [c, <i>b</i> ]]	$b \ge c \land a \ge c  s^\#$	a ≥ c 3	
Т	Δs	s#	s#	s#	s#

## Join

V	Τ	$[-\infty,b]$	[a, b]	$[a,+\infty)$	Т
$\perp$	1	$[-\infty,b]$	[a,b]	$[a,+\infty)$	$\top$
$[-\infty,d]$	$(-\infty,d]$	$(-\infty, max(b, d)]$	$(-\infty, max(b, d)]$	$\vdash$	Т
[c, d]	[c, d]	$(-\infty, max(b, d)]$	[min(a,c), max(b,d)]	$[min(a,c),+\infty)$	Т
$[c,+\infty)$	$[c,+\infty)$	Т	$[min(a,c),+\infty)$	$[min(a,c),+\infty)$	Т
Т	Т	Т	Т	Τ	Т

### Widen

						,		
$\nabla$	⊥	$(-\infty,b]$		[a,b]		$[a, +\infty)$		T
	1	$(-\infty,b]$		[a, b]		$[a,+\infty)$		$\vdash$
$(-\infty,d]$	$(-\infty,d]$	$d \leq b$	$(-\infty,b]$	$d \leq b$	$(-\infty,b]$	т		_
		otherwise	Т	otherwise	Т	'		'
		d < b	$(-\infty,b]$	$a \le c \land d \le b$	[a,b]	a ≤ c	$[a, +\infty)$	_
[c, d]	[6 4]	$ u  \leq b$	$(-\infty, \nu]$	$a \le c \land d > b$	$[a,+\infty)$	a <u>&gt;</u> c	$[a, +\infty)$	'
[c, a]	[c,d]	otherwise	_	$a > c \wedge d \leq b$	$(-\infty,b]$	otherwise	_	_
		Otherwise	1	otherwise	Τ	Otherwise	'	'
[0.100]	[0.100]	_		$a \le c$	$[a,+\infty)$	a ≤ c	$[a, +\infty)$	_
$[c,+\infty)$	$[c, +\infty)$	'		otherwise	Т	otherwise	Τ	'
Т	Τ	Т		Т		Т		T

### **Narrow**

Δ	上	$(-\infty,b]$	[a, b]	$[a,+\infty)$	Τ
$\perp$	丄		上	上	$\perp$
$(-\infty,d]$	丄	$(-\infty,b]$	[a, b]	[a, d]	$(-\infty,d]$
[c,d]	丄	[c,b]	[a, b]	[a, d]	[c,d]
$[c,+\infty)$	上	[c,b]	[a, b]	$[a,+\infty)$	$[c,+\infty)$
Τ		$(-\infty,b]$	[a, b]	$[a,+\infty)$	Т

#### **Table of Contents**

Language

Abstract Interpreter

List Functions

Extended Sign

Intervals

$$P :=$$
 while  $x \neq 0$  do  $x := x + 1$   $AI_{ExtSign}(P)(false)([(x, < 0)]) = ([(x, = 0)], [[(x,  $\top)]])$   $AI_{ExtSign}(P)(false)([(x, = 0)]) = ([(x, = 0)], [[(x, = 0)]])$   $AI_{ExtSign}(P)(false)([(x, > 0)]) = (\bot_{\mathbb{S}}, [[(x, > 0)]])$$ 

$$P := x := x + y; y := y + 1$$
  
 $AI_{ExtSign}(P)(false)([(x, \le 0), (y, < 0)]) = ([(x, < 0), (y, \top)], [])$ 

```
P := x := 40; while x \neq 0 do x := x - 1

AI_{ExtSign}(P)(false)(\top_{\mathbb{S}}) = ([(x, = 0)], [\top_{\mathbb{S}}]) in 1 iteration

AI_{Int}(P)(false)(\top_{\mathbb{S}}) = ([(x, [0, 0])], [[(x, [0, 40])]]) in 40 iterations

AI_{Int}(P)(true)(\top_{\mathbb{S}}) = ([(x, [0, 0])], [[(x, (-\infty, 40])]]) in 1 + 1 iterations
```

$$\begin{split} P := & \text{ while } x \geq 0 \text{ do } (x := x - 1; y := y + 1) \\ & Al_{Int}(P)(false)([(x, [10, 10]), (y, [0, 0])]) \text{ loops} \\ & Al_{Int}(P)(true)([(x, [10, 10]), (y, [0, 0])]) = \\ & ([(x, [-1, -1]), (y, [0, +\infty))], [[(x, [-1, 10]), (y, [0, +\infty))]]) \\ & \text{ in } 1 + 1 \text{ iterations} \end{split}$$

```
\begin{split} P := & \text{ while } x < 10 \text{ do } x := x + 1 \\ & Al_{Int}(P)(false)([(x,[0,0])]) = \\ & ([(x,[10,10])],[[(x,[0,10])]]) \text{ in } 10 \text{ iterations} \\ & Al_{Int}(P)(true)([(x,[0,0])]) = \\ & ([(x,[10,10])],[[(x,[0,10])]]) \text{ in } 1 + 1 \text{ iterations} \end{split}
```

```
\begin{split} P := & \text{ while } x \leq 100 \text{ do } x := x+1 \\ & Al_{Int}(P)(false)([(x,[1,1])]) = \\ & ([(x,[101,101])],[[(x,[1,101])]]) \text{ in } 101 \text{ iterations} \\ & Al_{Int}(P)(true)([(x,[1,1])]) = \\ & ([(x,[101,101])],[[(x,[1,101])]]) \text{ in } 1+1 \text{ iterations} \end{split}
```

$$P := x := 0$$
; **while**  $x < 40$  **do**  $x := x + 1$   $AI_{Int}(P)(false)(\top_{\mathbb{S}}) = ([(x, [40, 40])], [[(x, [0, 40])]])$  in 40 iterations  $AI_{Int}(P)(true)(\top_{\mathbb{S}}) = ([(x, [40, 40])], [[(x, [0, 40])]])$  in  $1 + 1$  iterations

```
\begin{split} P := & \ x := 0; \\ & \ \text{while } 1 = 1 \ \text{do} \\ & \ (\text{if } y = 0 \ \text{then} \\ & \ (x := x + 1; \text{if } x < 40 \ \text{then } x := 0 \ \text{else skip}) \\ & \ \text{else skip}) \\ & \ Al_{Int}(P)(false)([(y,[0,1])]) = (\bot_{\mathbb{S}}, [[(x,[0,40])]]) \ \text{in } 40 \ \text{iterations} \\ & \ Al_{Int}(P)(true)([(y,[0,1])]) = (\bot_{\mathbb{S}}, [[(x,[0,+\infty))]]) \ \text{in } 1 + 1 \ \text{iterations} \end{split}
```

```
P := i := 1:
         while i < 3 do
            (i := 1;
            while i < i do i := i + 1:
            i := i + 1
AI_{Int}(P)(false)(\top_{\mathbb{S}}) = ([(i, [4, 4])],
            [[(i, [1, 1]), (i, [1, 2])], [(i, [1, 2]), (i, [1, 3])],
            [(i, [1, 3]), (j, [1, 4])], [(i, [1, 4])])
AI_{Int}(P)(true)(\top_{\mathbb{S}}) = ([(i, [4, 4])], [[(i, [1, 4])]])
```

```
P := i := 1:
        while i \le 4 do
           (i := 0)
           while i < 3 do
                 (k := 0)
                 while k < 5 do (z := i * i * k; k := k + 1);
                 i := i + 1
           i := i + 1
AI_{Int}(P)(false)(\top_{\mathbb{S}}) = ([(i, [5, 5])], [..., [(i, [1, 5])]])
AI_{Int}(P)(true)(\top_{\mathbb{S}}) = ([(i, [5, 5])], [[(i, [1, 5])]])
```

$$P := x := 1/0$$
; while  $x \le 5$  do skip  $AI_{Int}(P)(false)(\top_{\mathbb{S}}) = (\bot_{\mathbb{S}}, [(x, \bot)])$ 

$$P :=$$
 while  $1/0 < 1$  do skip  $AI_{Int}(P)(false)(\top_{\mathbb{S}}) = (\bot_{\mathbb{S}}, [])$