# Computability (16/11/2021)

* Exercise: Let $Q(x)$ be a decidable predicate

$f_1, f_2 : \mathbb{N} \to \mathbb{N}$ computable

define

$$f(x) = \begin{cases} f_1(x) & \text{if } Q(x) \\ f_2(x) & \text{otherwise} \end{cases}$$

Then $f$ is computable

proof

since $f_1, f_2$ are computable there are $e_1, e_2 \in \mathbb{N}$ s.t. $f_1 = \varphi_{e_1}$

$f_2 = \varphi_{e_2}$

$$f(x) \cancel{=} f_1(x) \cdot \chi_Q(x) + f_2(x) \cdot \chi_{\neg Q}(x)$$

$$f(x) \overset{''''}{=} \Big(\mu(y,t). \big((S(e_1, x, y, t) \wedge Q(x)) \vee (S(e_2, x, y, t) \wedge \neg Q(x))\big)\Big)_y \ ''''$$

$$= \Big(\underset{\uparrow}{\mu w}. \Big(\underbrace{(S(e_1, x, (w)_2, (w)_1) \wedge Q(x)) \vee (S(e_2, x, (w)_2, (w)_1) \wedge \neg Q(x))}_{\text{decidable predicate}}\Big)\Big)_2$$

$(w)_1 = t$

$(w)_2 = y$

$\boxed{\underline{\text{computable}}}$

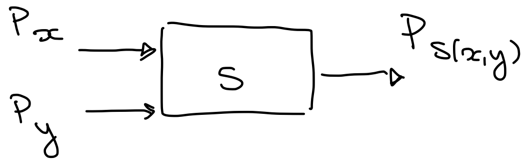
where by

$$\mu x. \underset{\text{decidable}}{\underset{\uparrow}{P(x)}} \quad \text{we mean} \quad \mu x. \underbrace{|\chi_P(x) - 1|}_{\text{computable}}$$

## * Effective operations on computable functions

① there exists a total computable function $s : \mathbb{N}^2 \to \mathbb{N}$

$\forall x, y \in \mathbb{N} \qquad \varphi_{s(x,y)}(z) = \varphi_x(z) * \varphi_y(z)$




def $P_{s(x,y)}(z)$:
  $v_x = P_x(z)$
  $v_y = P_y(z)$
  return $v_x * v_y$

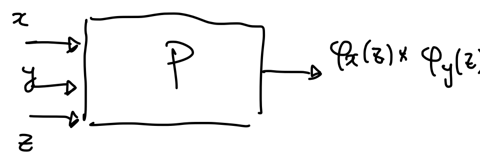
$g : \mathbb{N}^3 \to \mathbb{N}$

$g(x, y, z) = \varphi_x(z) * \varphi_y(z)$

$= \psi_U(x, z) * \psi(y, z)$

computable since it arises as composition of comp. funs

computable ?




S is the function which takes P and hardcodes in P the values of $x$ and $y$

By the (corollary of the) smn theorem

there exists $s : \mathbb{N}^2 \to \mathbb{N}$ total computable s.t.

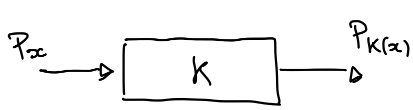
$\forall x, y, z$

$$\varphi_{s(x,y)}(z) = g(x, y, z) \underset{\text{def.}}{=} \varphi_x(z) * \varphi_y(z)$$

## * Effectiveness of the inverse function

there exists a total computable function such that

$\forall x$ if $\varphi_x$ is injective then $\varphi_{k(x)} = (\varphi_x)^{-1}$




$(\varphi_x^{-1})(y)$

$z = 0 \quad \varphi_x(0) = y$?
$z = 1$
$z = 2$
$\vdots$

$$g(x,y) = (\varphi_x)^{-1}(y) = \begin{cases} z & \text{unique s.t. } \varphi_x(z) = y \\ \uparrow & \text{if there is no such } z \end{cases}$$

$$= \text{"}\left(\mu(z,t).\quad S(x,z,y,t)\right)_z\text{"}$$

$$= \left(\mu w.\quad S(x,(w)_1,y,(w)_2)\right)_1$$

$$= \left(\mu w.\quad |\chi_S(x,(w)_1,y,(w)_2) - 1|\right)_1$$

computable by minimalisation

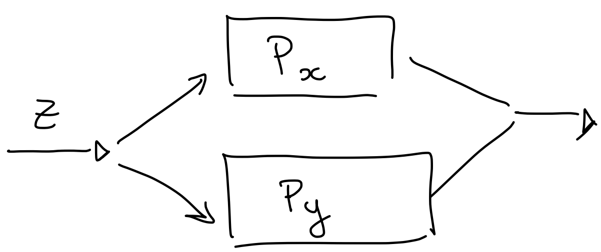
Hence by smn theorem there is $k$ total, computable

s.t. $\forall x,y$ $\qquad \varphi_{k(x)}(y) = g(x,y) = (\varphi_x)^{-1}(y)$ ∎

(3) there is a total computable function $s: \mathbb{N}^2 \to \mathbb{N}$ such that

$\forall x,y \qquad W_{s(x,y)} = W_x \cup W_y$

$$\varphi_{s(x,y)}(z)\downarrow \quad \text{iff} \quad \varphi_x(z)\downarrow \quad \text{or} \quad \varphi_y(z)\downarrow$$




$H(e,x,t) \equiv P_e(x)\downarrow$ in $t$ or less steps

$g: \mathbb{N}^3 \to \mathbb{N}$

$$g(x,y,z) = \begin{cases} 1 & \text{if } \varphi_x(z)\downarrow \text{ or } \varphi_y(z)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

$\mathbb{1}: \mathbb{N} \to \mathbb{N}$
$x \mapsto 1$

$$= \mathbb{1}\left(\mu t.\left(\ H(x,z,t) \vee H(y,z,t)\right)\right)$$

computable

Hence by smn theorem $\exists\, s : \mathbb{N}^2 \to \mathbb{N}$ total computable such that

$$\varphi_{s(x,y)}(z) = g(x,y,z)$$

$s$ is the desired function i.e. $W_{s(x,y)} = W_x \cup W_y$

$$z \in W_{s(x,y)} \quad \text{iff} \quad \underset{\displaystyle g(x,y,z)}{\varphi_{s(x,y)}(z)\downarrow} \quad \text{iff} \quad \varphi_x(z)\downarrow \text{ or } \varphi_y(z)\downarrow$$

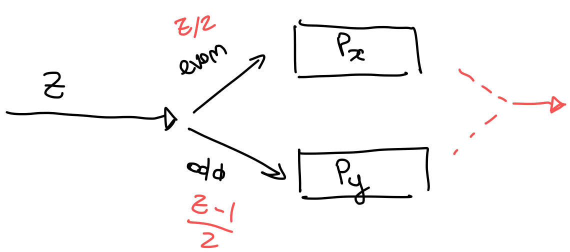
$$\text{iff} \quad z \in W_x \text{ or } z \in W_y$$

$$\text{iff} \quad z \in W_x \cup W_y$$

④ there exists a total computable function $s : \mathbb{N}^2 \to \mathbb{N}$ such that

$$E_{s(x,y)} = E_x \cup E_y$$

$P_{s(x,y)}$ produces in output all values produced by $P_x$ and $P_y$




| | m 0 | 1 | 2 | |
|---|---|---|---|---|
| $P_x$ | 0 | 1 | 2 | 7 ... |
| $P_y$ | 2 | 1 | 3 | 5 ... |

$$\boxed{g(x,y,z) = \begin{cases} \varphi_x(z/2) & \text{if } z \text{ is even} \\ \varphi_y\left(\frac{z-1}{2}\right) & \text{if } z \text{ is odd} \end{cases}}$$

$$= \cancel{\Psi_U(x, qt(2,z)) \cdot \overline{sg}(rm(2,z)) +}$$

$$\cancel{\Psi_U(y, qt(2,z)) \cdot rm(2,z)}$$

$$= \big(\mu(y,t) . (S(x, qt(2,z), y, t) \wedge z \text{ even}) \vee$$

$$(S(y, qt(2,z), y, t) \wedge z \text{ odd})\big)_y$$

$$= \left( \mu w. \ \overline{sg} \begin{pmatrix} \chi_s(x, qt(2,z), (w)_1, (w)_2) \cdot \overline{sg}(rm(2,z)) + \\ \chi_s(y, qt(2,z), (w)_1, (w)_2) \cdot sg(rm(2,z)) \end{pmatrix} \right)_1$$

computable

By smn theorem $\exists\, s : \mathbb{N}^2 \to \mathbb{N}$ total computable s.t. $\forall x,y,z \quad \varphi_{s(x,y)}(z) = g(x,y,z)$

$$\varphi_{s(x,y)}(z) = g(x,y,z) = \begin{cases} \varphi_x(z/2) & \text{if } z \text{ is even} \\ \varphi_y\left(\frac{z-1}{2}\right) & \text{if } z \text{ is odd} \end{cases}$$

$s$ is the desired function : $\quad E_{s(x,y)} = E_x \cup E_y$

$(\subseteq)$ $\quad v \in E_{s(x,y)}$

$\Rightarrow \exists z$ s.t. $\varphi_{s(x,y)}(z) = v$

$\Rightarrow$ 2 possibilities either $\quad v = \varphi_x(z/2) \Rightarrow v \in E_x$

or $\quad v = \varphi_y\left(\frac{z-1}{2}\right) \Rightarrow v \in E_y$

$\Rightarrow v \in E_x \cup E_y$

$(\supseteq)$ $\quad v \in E_x \cup E_y \leadsto v \in E_{s(x,y)}$

i.e. $\quad \boxed{v \in E_x \leadsto v \in E_{s(x,y)}}$

and $\quad v \in E_y \leadsto v \in E_{s(x,y)}$

Let $v \in E_x$. $\leadsto$ there is $z$ s.t. $v = \varphi_x(z)$

$\Rightarrow \varphi_{s(x,y)}(2z) = \varphi_x(\cancel{2}z/\cancel{2}) = \varphi_x(z) = v$

$\Rightarrow v \in E_{s(x,y)} \quad \square$

* Exercise : $URM_P$ machine

$Z(n)$
$\cancel{S(n)}$ $\quad P(n) \quad r_n \leftarrow r_n \dot{-} 1$
$T(m,n)$
$J(m,n,t)$

$\mathcal{C}_P$ = class of $URM_P$ computable functions

$$\mathcal{C}_P \; ?\overset{\subseteq}{\underset{\supseteq}{=}}? \; \mathcal{C}$$

* Exercise : $f, g : \mathbb{N} \to \mathbb{N}$

1. $f$ computable, $g$ not computable and $f \circ g$ computable
2. $f$ not computable, $g$ not computable $f \circ g$ computable