Computability (11/10/2021)

EXERCISE: URM^s

$\cancel{T(m,n)}$ $\quad T_s(m,n)$ $\quad r_m \leftrightarrow r_n$

$\mathcal{C}^s = \mathcal{C}$?

dim

$(\mathcal{C} \subseteq \mathcal{C}^s)$ Let $f: \mathbb{N}^k \to \mathbb{N}$ $\quad f \in \mathcal{C} \overset{?}{\dashrightarrow} f \in \mathcal{C}^s$

There exists P URM program such $f_P^{(k)} = f$

By last exercise there exists P' such that P' does not use $T(m,n)$

and $f_{P'}^{(k)} = f_P^{(k)}$. Just notice that P' is a URM^s program

$\Rightarrow f_{P'}^{(k)} = f_P^{(k)} = f \in \mathcal{C}^s$

$(\mathcal{C}^s \subseteq \mathcal{C})$ Given $f: \mathbb{N}^k \to \mathbb{N}$ $\quad f \in \mathcal{C}^s \overset{?}{\dashrightarrow} f \in \mathcal{C}$

There is P URM^s program such that $f = f_P^{(k)}$

We want to prove that P can be transformed in P' URM program such that $f_{P'}^{(k)} = f_P^{(k)}$

$T_s(m,n) \rightsquigarrow$ $\quad T(m,i)$
$\quad T(n,m)$
$\quad T(i,n)$

We prove that P URM^s program can be transformed into P' URM progr. such that $f_P^{(k)} = f_{P'}^{(k)}$ by induction on $h = \#$ of $T^s(m,n)$ in P

$(h=0)$ P is already a URM program $\Rightarrow$ we can take $P' = P$

$(h \to h+1)$ Let P has $h+1$ $T_s(m,n)$ instructions

P is of the shape URM^S

$$P \left\{ \begin{array}{ll} I_1 & \\ \vdots & \\ I_t & T_s(m,n) \\ \vdots & \\ I_s & \end{array} \right. \quad \rightsquigarrow \quad P'' \quad \begin{array}{ll} I_1 & \\ \vdots & \\ I_t & J(1,1,SUB) \\ & \\ I_s & \\ I_{s+1} & J(1,1,END) \\ SUB: & T(m,i) \\ & T(n,m) \\ & T(i,n) \\ & J(1,1,t+1) \\ END: & \end{array}$$

we assume that P terminates at $s+1$

take $i = \max(\{n \mid R_n \text{ is used in } P\} \cup \{k\}) + 1$

Note that $f_{P''}^{(k)} = f_P^{(k)}$ and # instructions T_s in P'' is h

By inductive hyp. on P'' there is P' URM program such that

$f_{P'}^{(k)} = f_{P''}^{(k)} = f_P^{(k)}$ $\square$

NOT WORKING!
P'' NOT a URM^S progr.
it uses both T and T_s

SOLUTION: The proof goes through by showing that more generally:

" given any program P using both T and T_s there is P' URM program such that $f_P^{(k)} = f_{P'}^{(k)}$ "

* EXAMPLE: All natural numbers are equal [BE CAREFUL WITH INDUCTION]

$\forall X \subseteq \mathbb{N} \quad |X| \geq 1 \quad \forall n, m \in X \quad n = m$

proof

induction on $|X|$

$(|X| = 1) \quad \forall m, n \in X \quad n = m$

$(l(x) = h \Rightarrow |x| = h+1)$

Let $|X| = h+1$ $\qquad X = \{m_1, m_2, \dots, m_h, m_{h+1}\}$

$$X = \underbrace{\{m_1, \dots, m_h\}}_{X_1} \cup \underbrace{\{m_2, \dots, m_h, m_{h+1}\}}_{X_2}$$

$|X_1| = h$ $\qquad \forall m, n \in X_1 \quad m = n$

$|X_2| = h$ $\qquad \forall m, n \in X_2 \quad m = n$

$\forall m_i, m_j \in X$

3 possibilities

- $m_i, m_j \in X_1 \Rightarrow m_i = m_j$
- $m_i, m_j \in X_2 \Rightarrow m_i = m_j$
- $m_i \in X_1, \; m_j \in X_2$

  $m_i = m_h = m_j \Rightarrow m_i = m_j$

NOT WORKING FOR $|X| = 2$!!

EXERCISE: $\quad URM^=$ without $J(m, n, t)$

$\mathcal{C}^= \overset{?}{=} \mathcal{C}$

$\mathcal{C}^= \subseteq \mathcal{C}$

$\not\supseteq$

$URM^=$ program $\qquad\qquad \ell(P) = \#$ of instructions in P

I_1
I_2
$\vdots$
$I_{\ell(P)}$

the computation of P terminates after $\ell(P)$ steps

$\hookrightarrow$ $f_P^{(k)}$ is always total

hence $\quad f: \mathbb{N} \to \mathbb{N}$ $\qquad f \in URM \quad f = f_P$ with P $J(1,1,1)$

$f(x)\uparrow \;\forall x$ $\qquad f \notin URM^=$

functions in $\mathcal{C}^=$ are of the shape

$$f(x) = c \qquad \forall x \in \mathbb{N}$$

or

$$f(x) = x + c \qquad \forall x \in \mathbb{N}$$

We prove by induction on the number k of steps that after k steps in register R_1 the value

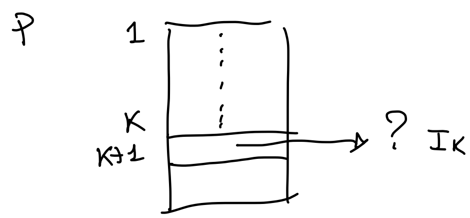
$r_1(x,k)$ = content of R_1 after k steps of P starting from $\overset{1}{\boxed{x}}\boxed{0}\boxed{0}\cdots$

$$\begin{cases} = x + c \\ = c \end{cases}$$

$(k=0)$ $\quad r_1(x,0) = x + 0$

$(k \to k+1)$ By inductive hyp.

$$r_1(x,k) = \begin{cases} c \\ x+c \end{cases} \qquad \text{for some } c \text{ constant}$$



I_k can be

- $Z(n)$
  two subcases
  $(n = 1)$ $\quad r_1(x, k+1) = 0$
  $(n \neq 1)$ $\quad r_1(x, k+1) = r_1(x,k)$ $\quad$ ok by ind. hyp.

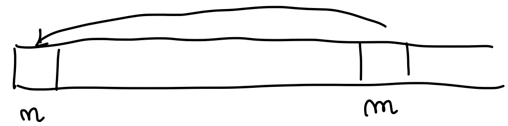
- $S(n)$
  two subcases
  $(n = 1)$ $\quad r_1(x, k+1) = \overbrace{\underbrace{r_1(x,k)}_{c \;/\; x+c} + 1}^{c+1 \;/\; x+(c+1)}$ $\quad$ ok by ind. hyp.
  $(n \neq 1)$ $\quad r_1(x, k+1) = r_1(x,k)$ $\quad$ " "

- $T(m,n)$

$(n \neq 1$ or $(n = 1$ and $m = 1))$

$r_1(x, k+1) = r_1(x, k)$ ok by inductive hyp.

$(n = 1$ and $m \neq 1)$



VERY BAD

* try to show that if

$r_J(x, k) =$ content of register J after k steps of computation starting from $\boxed{x}\boxed{0}$ - - -

$$r_J(x, k) = \begin{cases} c & c \in \mathbb{N} \\ x + c & c \in \mathbb{N} \end{cases}$$

by induction on k

$(k = 0)$ $r_J(x, 0) = ?$

if $J = 1$ $r_1(x, 0) = x + 0$

if $J > 1$ $r_1(x, 0) = 0$

$(k \to k+1)$ depending on what is I_{k+1} :

- $Z(n)$ $n = J$ $r_J(x, k+1) = 0$

  $n \neq J$ $r_J(x, k+1) = r_J(x, k)$ ok by ind. hyp

- $S(n)$ $n = J$ $r_J(x, k+1) = r_J(x, k) + 1$ ......

  $n \neq J$ $r_J(x, k+1) = r_J(x, k)$ ......

- $T(m,n)$ $\cancel{n \neq J \text{ or } ((n = J) \text{ and } (m = J))}$

  $\cancel{r_J(x, k+1) = r_J(x, k)}$ ......

  $\cancel{n = J, m \neq J}$

  $r_J(x, k+1) = r_m(x, k)$ ......

CASE DISTINCTION NOT NEEDED