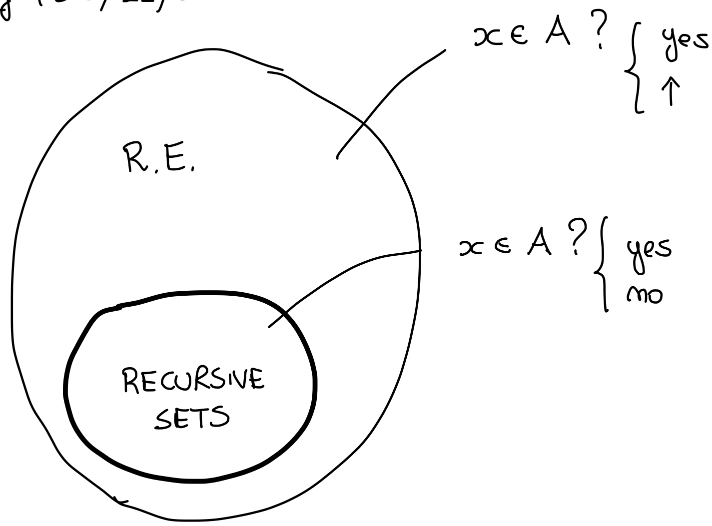
Computability (30/11/2021



Recursively enumerable sets

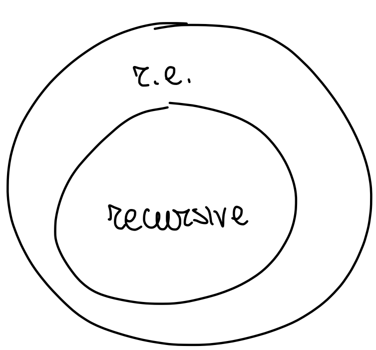
Def (r.e. set) : A set $A \subseteq \mathbb{N}$ is recursively enumerable (r.e.) if its semi-characteristic function

$$SC_A : \mathbb{N} \to \mathbb{N}$$

$$SC_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}$$ is computable

A property (predicate) $Q(x) \subseteq \mathbb{N}$ is semi-decidable if $\{x \in \mathbb{N} \mid Q(x)\}$ is r.e.

$$\left.\begin{array}{ll} \text{subsets} & A \subseteq \mathbb{N}^k \\ \text{predicates} & Q \subseteq \mathbb{N}^k \end{array}\right\}$$ easily generalisable but "useless" (conceptually)



OBSERVATION: Let $A \subseteq \mathbb{N}$

$$\underbrace{A \text{ recursive}}_{\text{yes/no}} \iff \underbrace{A}_{\text{yes}}, \underbrace{\bar{A}}_{\text{yes}} \text{ r.e.}$$

proof

($\Rightarrow$) let $A \subseteq \mathbb{N}$ be recursive

$$\chi_A : \mathbb{N} \to \mathbb{N}$$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad \text{is computable}$$

we want to prove that A r.e. i.e.

$$SC_A : \mathbb{N} \to \mathbb{N}$$

$$SC_A(x) = \begin{cases} 1 & x \in A \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \mathbb{1}\left(\underbrace{\mu w . \overbrace{|\chi_A(x) - 1|}^{\begin{cases} 0 & \text{if } x \in A \\ 1 & \text{otherwise} \end{cases}}}_{\begin{cases} 0 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}}\right)$$

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def P_SCA (x):
    if P_χA (x) = 1
        return 1
    else
        loop
```

computable, as desired.

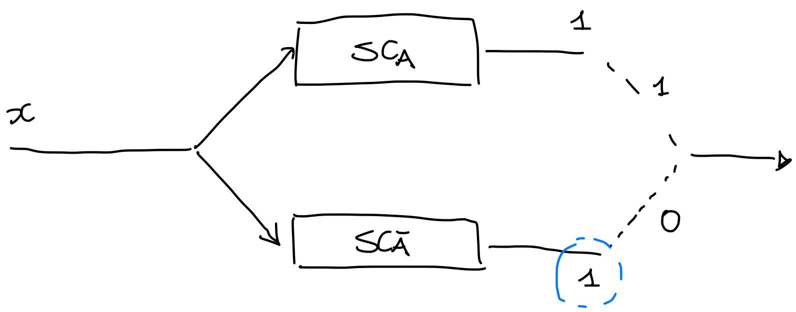
* since A is recursive $\Rightarrow$ $\bar{A}$ recursive $\underset{\text{by first part}}{\Longrightarrow}$ $\bar{A}$ r.e.

($\Leftarrow$) if $A, \bar{A}$ are r.e. then A recursive

we assume $A, \bar{A}$ are r.e.

$$SC_A(x) = \begin{cases} 1 & x \in A \\ \uparrow & \text{otherwise} \end{cases}$$

$$\underbrace{1 \dot{-} SC_{\bar{A}}(x)}_{} = \begin{cases} \cancel{1}\ 0 & x \in \bar{A} \\ \uparrow & \text{otherwise} \end{cases}$$



let $e_1, e_0 \in \mathbb{N}$ such that $SC_A = \varphi_{e_1}$

$1 \dot{-} SC_{\bar{A}} = \varphi_{e_0}$

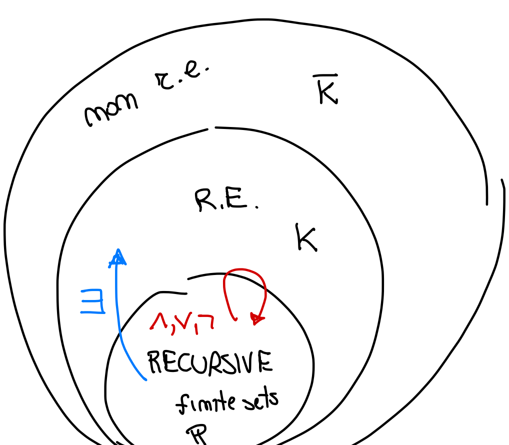
$$\text{“}\mu(t,y).\ S(e_1, x, y, t) \vee S(e_0, x, y, t)\text{”}$$

$$\chi_A(x) = \big(\mu w.\ S(e_1, x, (w)_2, (w)_1) \vee S(e_0, x, (w)_2, (w)_1)\big)_2$$

idea $(w)_1 = t,\ (w)_2 = y$

$$= \big(\mu w.\ \overline{sg}\big(\max\big(\chi_S(e_1, x, (w)_2, (w)_1),\ \chi_S(e_0, x, (w)_2, (w)_1)\big)\big)\big)_2$$

computable $\Rightarrow$ A recursive. $\square$



- K is r.e. (not recursive)

$$SC_K(x) = \begin{cases} 1 & x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

$$= 1\!\!1(\varphi_x(x)) = 1\!\!1(\varphi_U(x,x))$$

$\bar{K} = \{x \mid \varphi_x(x)\uparrow\}$

not r.e. (otherwise K would be recursive)

$Q(t, \vec{x})$ decidable

$\exists t.\ Q(t, \vec{x})$

Proposition: (STRUCTURE THEOREM) Let $P(\vec{x}) \subseteq \mathbb{N}^k$ predicate

$P(\vec{x})$ semi decidable $\iff$ there is $Q(t, \vec{x}) \subseteq \mathbb{N}^{k+1}$ decidable s.t. $P(\vec{x}) \equiv \exists t.\ Q(t, \vec{x})$

proof

($\Longrightarrow$) let $P(\vec{x})$ semi-decidable, i.e.

$$sc_P(\vec{x}) = \begin{cases} 1 & \text{if } P(\vec{x}) \\ \uparrow & \text{otherwise} \end{cases} \quad \text{is computable}$$

i.e. $\exists\ e \in \mathbb{N}$ s.t. $sc_P = \varphi_e^{(k)}$

Observe

$$P(\vec{x}) \equiv \text{“} sc_P(\vec{x}) = 1 \text{”} \equiv \text{“} sc_P(\vec{x})\downarrow \text{”}$$

$$\equiv \text{“} \varphi_e(\vec{x})\downarrow \text{”} \equiv \exists t.\ \underbrace{H(e, \vec{x}, t)}$$

$Q(t, \vec{x}) \equiv H(e, \vec{x}, t)$
decidable since H is so

$$\equiv \exists t.\ Q(t, \vec{x})$$

($\Longleftarrow$) let $P(\vec{x}) \equiv \exists t.\ Q(t, \vec{x})$ with $Q(t, \vec{x})$ decidable

observe that

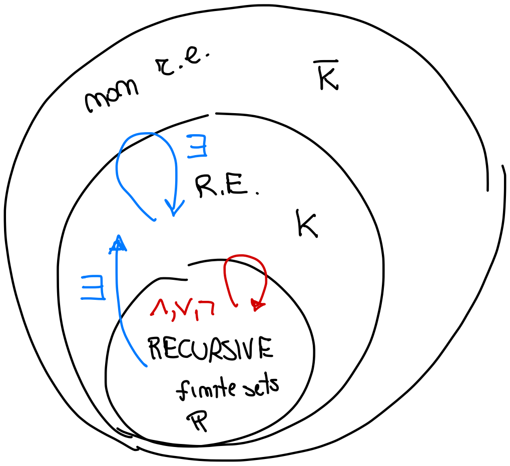
$$P(\vec{x}) \equiv \exists t.\ \chi_Q(t, \vec{x}) = 1$$

$$sc_P(\vec{x}) = \mathbb{1}\left(\mu t.\ |\chi_Q(t, \vec{x}) - 1|\right) \begin{cases} \nearrow 1 & \text{if } \exists t.\ Q(t, \vec{x}) \\ \searrow \uparrow & \text{otherwise.} \end{cases}$$

computable

$\Rightarrow$ P semi-decidable

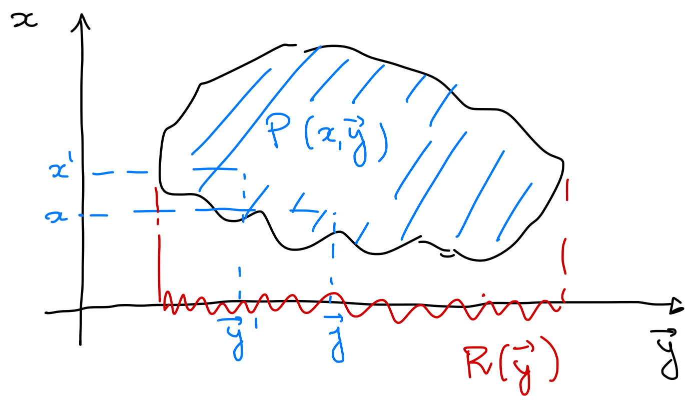
$\square$



Projection theorem:

Let $P(x, \vec{y}) \subseteq \mathbb{N}^{k+1}$ semi-decidable

Then
$$R(\vec{y}) \equiv \exists x.\ P(x, \vec{y})$$
semidecidable



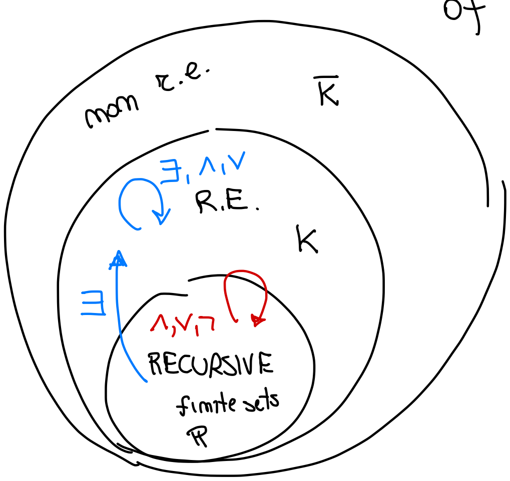
proof

Let $P(x, \vec{y})$ semi-decidable. Then there is $Q(t, x, \vec{y})$ decidable s.t.
$$P(x, \vec{y}) \equiv \exists t.\ Q(t, x, \vec{y})$$

Thus
$$\begin{aligned} R(\vec{y}) &\equiv \exists x.\ P(x, \vec{y}) \equiv \\ &\equiv \exists x.\ \exists t.\ Q(t, x, \vec{y}) \\ &\equiv \exists \omega.\ \underbrace{Q((\omega)_1, (\omega)_2, \vec{y})}_{\text{decidable}} \end{aligned}$$

$(\omega)_1 = t$
$(\omega)_2 = x$

semidecidable, since it is the existential quantification of a decidable predicate. $\square$



Closure of semidecidable predicates w.r.t. $\wedge$ and $\vee$

Given $P_1(\vec{x}), P_2(\vec{x}) \subseteq \mathbb{N}^k$ semi-decidable. Then also

① $P_1(\vec{x}) \wedge P_2(\vec{x})$

② $P_1(\vec{x}) \vee P_2(\vec{x})$

are semi decidable

proof

Since $P_1(\vec{x}), P_2(\vec{x})$ are semi-decidable

$$P_1(\vec{x}) \equiv \exists t.\ Q_1(t, \vec{x})$$
$$P_2(\vec{x}) \equiv \exists t.\ Q_2(t, \vec{x})$$

with Q_1, Q_2 decidable

Hence

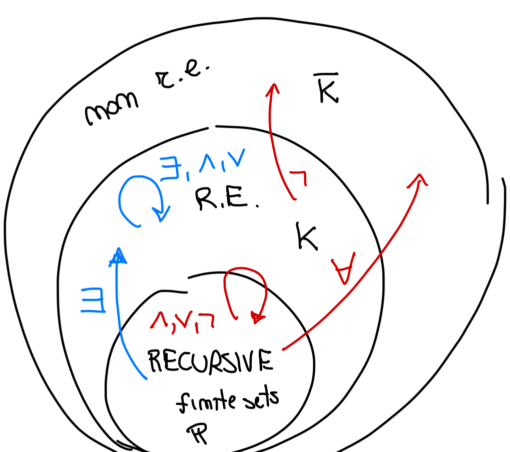
① $$P_1(\vec{x}) \wedge P_2(\vec{x}) \equiv \exists t.\ Q_1(t, \vec{x}) \wedge \exists t.\ Q_2(t, \vec{x})$$
$$\equiv \exists \omega.\ \underbrace{\left(Q_1((\omega)_1, \vec{x}) \wedge Q_2((\omega)_2, \vec{x})\right)}_{\text{decidable}}$$

$\downarrow$ semi-decidable by structure theorem

② $$P_1(\vec{x}) \vee P_2(\vec{x}) \equiv \exists t.\ Q_1(t, \vec{x}) \vee \exists t.\ Q_2(t, \vec{x})$$
$$\equiv \exists t.\ \left(\underbrace{Q_1(t, \vec{x}) \vee Q_2(t, \vec{x})}_{\text{decidable}} \right)$$

$\downarrow$ semi-decidable by structure theorem

$\square$



$\boxed{\text{negation}}$

$P(x) \equiv$ "$x \in K$" $\equiv$ "$x \in W_x$" $\equiv$ "$\varphi_x(x)\downarrow$"

semi decidable

$Q(x) \equiv \neg P(x) \equiv$ "$x \notin W_x$" $\equiv$ "$\varphi_x(x)\uparrow$"

is not semi-decidable

* universal quantification

$Q(t,x) \equiv \neg H(x,x,t)$ decidable

$P(x) \equiv \forall t.\ Q(t,x) \equiv$ "$\varphi_x(x)\uparrow$" $\equiv$ "$x \in \bar{K}$"

not semidecidable

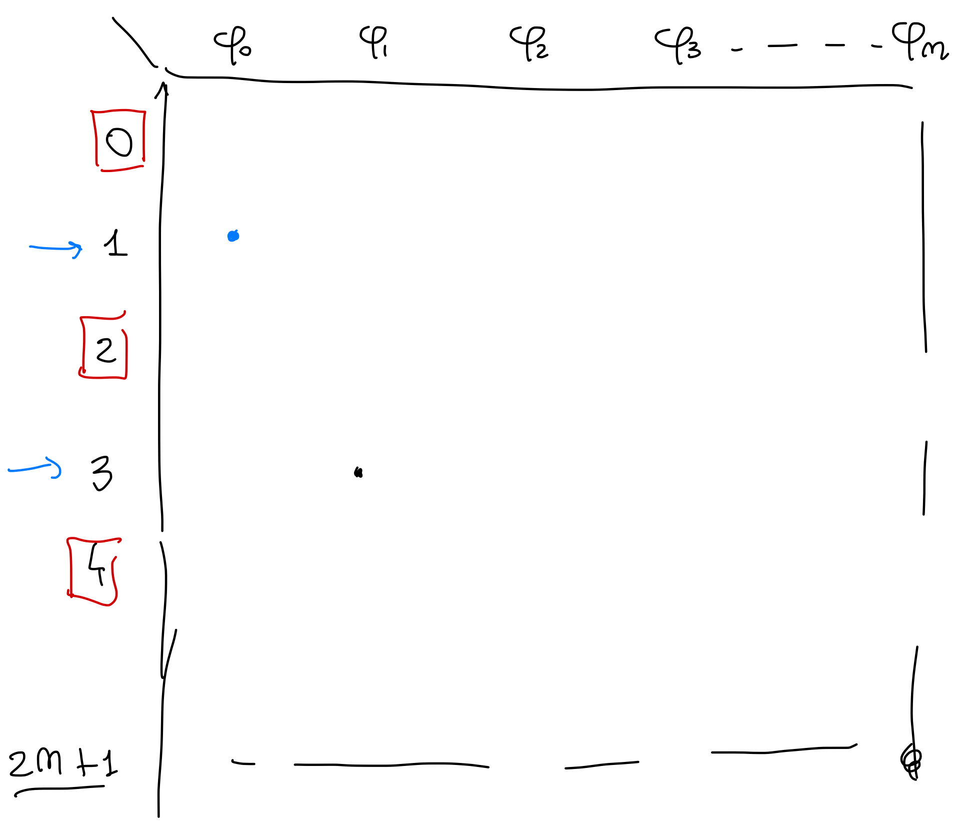
Exercise :

Define a function $f: \mathbb{N} \to \mathbb{N}$ total not computable $f(x) = x$ for infinitely many $x \in \mathbb{N}$

$$f(x) = \begin{cases} x & \text{if } x \text{ is even} \\ \varphi_{\frac{x-1}{2}}(x)+1 & \text{if } x \text{ is odd and } \varphi_{\frac{x-1}{2}}(x)\downarrow \\ 0 & \text{if } x \text{ is odd and } \varphi_{\frac{x-1}{2}}(x)\uparrow \end{cases}$$

$x = 2m+1$

$m = \frac{x-1}{2}$



→ $f(x) = x$
on infinitely many $x \in \mathbb{N}$

$f(2m) = 2m \quad \forall m$

→ f is total
immediate from definition

→ f is not computable
$\forall m$ if φ_m is total
then $f(2m+1) = \varphi_m(2m+1)+1$

hence $\neq \varphi_n(2n+1)$

$f \neq \varphi_n$

$\Downarrow$

f not computable

second possible solution

$$f(n) = \begin{cases} \varphi_n(n)+1 & \text{if } \underline{\varphi_n(n)\downarrow} \quad (n \in K) \\ n & \text{otherwise} \quad (n \notin K) \end{cases}$$

→ total

→ not computable: $\forall n.\ f \neq \varphi_n$

if $\varphi_n(n)\downarrow \Rightarrow f(n) = \varphi_n(n)+1$
$\neq \varphi_n(n)$

if $\varphi_n(n)\uparrow \Rightarrow f(n) = n \neq \varphi_n(n)\uparrow$

→ $f(n) = n \quad \forall n \notin K$
$n \in \overline{K}$

and $\boxed{\overline{K} \text{ is infinite}}$
(because it is not recursive)