Computability (22/11/2021)

* URM$^-$ instruction

$Z(n)$

$T(m,n)$

$J(m,n,t)$

$\cancel{S(n)}$ $P(n)$ $r_n \leftarrow r_n \dot{-} 1$

$\rightsquigarrow \mathcal{C}^-$

$$\mathcal{C}^- \quad \subseteq \quad \supseteq \quad \mathcal{C}$$

$(\mathcal{C}^- \subseteq \mathcal{C})$

given a URM- program (standard form)

$\vdots$

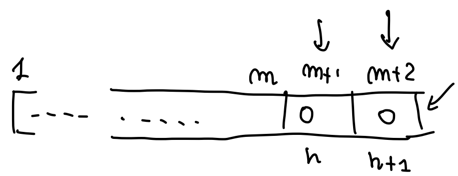
t : $\cancel{P(n)}$ — $J(1,1,SUB)$

$t+1$: $\vdots$

$P(n)$

$\vdots$

$\rightsquigarrow$



SUB : $J(n, m+1, t+1)$
$S(m+2)$

LOOP : $J(n, m+2, ENDSUB)$
$S(m+1)$
$S(m+2)$
$J(1,1,LOOP)$

ENDSUB: $T(m+1, n)$
$J(1,1,t+1)$

FORMAL PROOF :

Let $f \in \mathcal{C}^-$ $f: \mathbb{N}^k \to \mathbb{N}$

there exists a program P URM$^-$ such that $f_P^{(k)} = f$

program P using Z, S, P, T, J

Show that for every ~~URM$^-$ program P~~ there exists a URM program P' such that

$$f_P^{(k)} = f_{P'}^{(k)}$$

By induction on the number h of instructions $P(n)$ in P

$(\mathcal{C} \not\subseteq \mathcal{C}^-)$

URM$^-$

| 1 | | 5 |

Given a program P of URM$^-$ and $\vec{x} \in \mathbb{N}^k$

the largest content of the memory during the computation of

$P(x_1, \dots, x_k) \leq \max_i x_i$

Let's prove this by induction on number h of computation steps of $P(x_1, \dots, x_k)$

$(h = 0)$

$r_1 \quad \dots \quad r_k \quad r_{k+1} \dots$

| x_1 | $\dots$ | x_k | 0 | $\dots$ |

$$\max_{i \geq 1} r_i = \max_{i=1..k} x_i$$

$(h \to h+1)$

| x_1 | $\dots$ | x_k | $\dots$ |

h steps

| r_1' | r_2' | $\dots$ |

$\max_i r_i' \leq \max_i x_i$ (inductive hyp.)

$(h+1)$th step

| r_1 | r_2 | $\dots$ |

several cases depending on last instruction

$Z(n)$

$T(m, n)$

$J(m, n, t)$

$P(n)$

$$\max_i r_i \leq \max_i r_i' \leq \max_i x_i$$

$\Downarrow$

successor is not computable

1 2

| 0 | 0 | 0 | $\dots$ |

| 1 |

impossible. $\square$

* Show that there is a total computable function

$K : \mathbb{N} \to \mathbb{N}$

such that $\boxed{E_{K(x)} = W_x}$

$P_x \rightsquigarrow P_{K(x)}$ produces in output exactly the inputs where P_x terminates

K

def $P_{K(x)}(y)$:

$P_x(y)$
return y

$\rightsquigarrow$

$\forall x$
$\mathbb{1}(x) = 1$

define

$$f(x, y) = \begin{cases} y & \text{if } \varphi_x(y)\downarrow \\ \uparrow & \text{otherwise} \end{cases} = \underbrace{\mathbb{1}(\varphi_x(y))}_{\begin{cases} 1 & \text{if } \varphi_x(y)\downarrow \\ \uparrow & \text{otherwise} \end{cases}} \cdot y$$

$$= \mathbb{1}(\Psi_U(x,y)) \cdot y$$

computable (by composition)

use smn theorem to get $K : \mathbb{N} \to \mathbb{N}$ total computable

such that $\forall x, y \quad f(x,y) = \varphi_{K(x)}(y)$

K is the desired function : $E_{K(x)} = W_x$

$(E_{K(x)} \subseteq W_x)$ let $y \in E_{K(x)} \rightsquigarrow \exists z$ s.t. $\varphi_{K(x)}(z) = y$

$$\varphi_{K(x)}(z) = f(x, z) = \begin{cases} z & \text{if } \varphi_x(z)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

$\Rightarrow z = y \quad \varphi_x(y)\downarrow$

$\Rightarrow y \in W_x$

$(W_x \subseteq E_{k(x)})$ let $y \in W_x$ then $\varphi_x(y)\downarrow$

therefore $\varphi_{k(x)}(y) = f(x,y) = y \Rightarrow y \in E_{k(x)}$

$$f(x,y) = \begin{cases} y & \text{if } \varphi_x(y)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

(3.4) there is $k: \mathbb{N} \to \mathbb{N}$ total computable s.t.

$E_{k(x)} = \{ y \in \mathbb{N} \mid y \geq x \}$

$W_x = \mathbb{P}$ (even numbers)

define
$$g(x,y) = \begin{cases} x + y/2 & y \text{ is even} \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \begin{cases} x + qt(2,y) & \text{if } rm(2,y) = 0 \\ \uparrow & \text{otherwise} \end{cases}$$

$$= x + qt(2,y) + \underbrace{\mu w.\ sg(rm(2,y))}_{\substack{0 \text{ if } rm(2,y)=0 \\ \uparrow \text{ otherwise}}}$$

$g(y,w) = sg(rm(2,y))$

computable

By smn we get $k: \mathbb{N} \to \mathbb{N}$ total computable such that

$\forall x,y \quad g(x,y) = \varphi_{k(x)}(y)$

- $W_{k(x)} = \mathbb{P}$ (even numbers) $\quad \varphi_{k(x)}(y) = g(x,y)\downarrow$ iff $y \in \mathbb{P}$
- $E_{k(x)} \overset{?}{=} \{ y \mid y \geq x \}$

$E_{k(x)} = \{ \varphi_{k(x)}(y) \mid y \in W_x \}$

$= \{ \varphi_{k(x)}(y) \mid y \in \mathbb{P} \} = \{ x + qt(2,y) \mid y = 2n \ \ n \in \mathbb{N} \}$

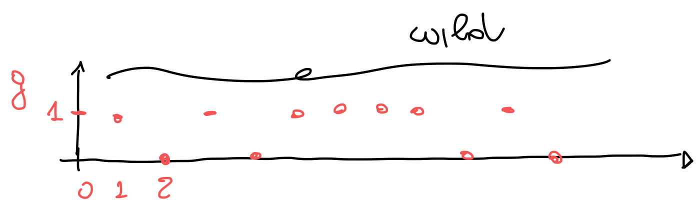
$= \{ x + qt(2, 2n) \mid n \in \mathbb{N} \} = \{ x + n \mid n \in \mathbb{N} \} = \{ y \mid y \geq x \}$ □

* Are there f, g $\quad f$ computable, g not computable and $f \circ g$ computable? YES

$$\left.\begin{array}{l} f(x) = 0 \quad \forall x \\ g \text{ any total non-computable function} \end{array}\right\} \Rightarrow \underset{\text{computable}}{f \circ g = f}$$

e.g. $g(x) = \begin{cases} 1 & \text{if } x \in W_x \\ 0 & \text{otherwise} \end{cases}$

* Are there f, g not-computable such that $f \circ g$ is computable? YES



$f \circ g$ will be always 0

ex.: $g(x) = \begin{cases} 1 & x \in W_x \\ 0 & \text{otherwise} \end{cases}$ non computable

$$f(x) = \begin{cases} 0 & x \leq 1 \\ \varphi_x(x)+1 & \text{if } x \in W_x \\ 0 & \text{otherwise} \end{cases}$$ non computable

$f \neq \varphi_x \quad \forall x \geq 2$

$\Rightarrow f \neq \varphi_y \quad \forall y \in \mathbb{N}$

$\Rightarrow f$ not computable

f is not computable (more details):

$\to \forall y \quad f \neq \varphi_y$: different from all computable functions $\Rightarrow$ not computable

in fact since every function is computed by infinitely many programs there are infinitely many x's such that $\varphi_y = \varphi_x$

$\Rightarrow \exists\, x \geq 2$ such that $\varphi_x = \varphi_y$

$f \neq \varphi_x$ 2 cases ① $x \in W_x$ $\quad f(x) = \varphi_x(x)+1 \neq \varphi_x(x)$

$\varphi_x = \varphi_y$

② $x \notin W_x$ $\quad f(x) = 0 \neq \varphi_x(x)\uparrow$

NOTE : $f \circ g\,(x) = f(\underbrace{g(x)}_{\geq 1}) = 0 \quad \forall x$ computable

* Define PR { least class of functions including base functions { zero, successors, projections
closed under composition, primitive recursion

By using only the definition show that $pow_2 : \mathbb{N} \to \mathbb{N}$, $pow_2(y) = 2^y$ is in PR

$x+y$

$x + 0 = x$ ← Id (special projection)

$x + (y+1) = (x+y)+1$ ← successor

$x * y$

$x * 0 = 0$

$x * (y+1) = (x*y) + x$

x^y

$x^0 = 1$

$x^{y+1} = x * x^y$

$pow2(y) = 2^y$

or alternatively

$$pow2(0) = 2^0 = 1$$

$$pow2(y+1) = 2^{y+1} = 2^y \cdot 2 = \underbrace{2y}_{} + \underbrace{2y}_{}$$

$$x + 0 = x$$
$$x + (y+1) = (x+y) + 1$$

* show $\chi_{\mathbb{P}} \in \mathcal{PR}$

$$\chi_{\mathbb{P}}(x) = \begin{cases} 1 & x \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

$$\chi_{\mathbb{P}}(x) = \overline{sg}\,(rm\,(2, y))$$

difficult

more directly

$$\begin{cases} \chi_{\mathbb{P}}(0) = 1 \\ \chi_{\mathbb{P}}(y+1) = \overline{sg}\,(\chi_{\mathbb{P}}(y)) \end{cases}$$

$$\begin{cases} \overline{sg}\,(0) = 1 \\ \overline{sg}\,(y+1) = 0 \end{cases}$$