Computability (20/12/2021)

* Exercise : for $f : \mathbb{N} \to \mathbb{N}$ fixed

$$B_f = \{ e \mid \varphi_e = f \}$$
$$= \{ e \mid \varphi_e \in \mathcal{B}_f \} \qquad \mathcal{B}_f = \{f\} \qquad \text{saturated}$$

- f not computable $\Rightarrow$ $B_f = \emptyset$ recursive
  $\overline{B_f} = \mathbb{N}$

- f computable

  * B_f is not r.e.

  → if $f = \emptyset$ $\quad f(x)\uparrow \ \forall x$

  let $g \neq f$ $\quad g \notin \mathcal{B}_f \qquad \underset{\text{finite}}{\underset{\uparrow}{f}} \subseteq g \quad f \in \mathcal{B}_f$

  $\Rightarrow$ By Rice-Shapiro B_f is not r.e.

  → more generally, if $f = \vartheta$ finite

  take any total g such that $\vartheta \subseteq g$

  $g \notin \mathcal{B}_f \quad \underset{\text{finite}}{f = \vartheta} \subseteq g \qquad \vartheta \in \mathcal{B}_f \quad \Rightarrow$ Rice-Shapiro
  B_f is not r.e.

  → if f is infinite

  $f \in \mathcal{B}_f = \{f\} \qquad \forall \vartheta \subseteq f \ \ \vartheta$ finite $\quad \vartheta \notin \mathcal{B}_f$
  $\vartheta \neq f$

  $\Rightarrow$ B_f is not r.e.

* $\overline{B_f}$?

→ if $f = \emptyset$ then $\overline{B_f}$ is r.e. (see last lemma)

→ if $f \neq \emptyset$ then $f \notin \overline{B_f} = \overline{\{f\}}$

$\vartheta = \emptyset \subseteq f \qquad \vartheta \in \overline{B_f}$

↳ by Rice-Shapiro $\overline{B_f}$ is not r.e.

* <u>EXERCISE</u>

Show that

$gcd : \mathbb{N}^2 \to \mathbb{N}$

$gcd(x,y)$ = greatest common divisor of x and y

is computable

$$gcd(x,y) = \max \underset{\substack{\uparrow \\ \text{bound} \\ min(x,y)}}{z} . \underbrace{z \text{ divides } x}_{rm(z,x)=0} \text{ and } \underset{rm(z,y)=0}{z \text{ divides } y}$$

$$= \max z \leq min(x,y) \,.\, rm(z,x) + rm(z,y)$$

$$= min(x,y) - \min w \leq min(x,y) \,.\, (z = min(x,y) - w \,\wedge\, rm(z,x) + rm(z,y) = 0)$$

$$= min(x,y) \dot{-} \mu w \leq min(x,y) \,.\, (rm(min(x,y) \dot{-} w, x) + rm(min(x,y) \dot{-} w, y))$$

computable (more precisely, primitive recursive)

* EXERCISE: Classify the following set according to recursiveness

$$A = \{ x \mid W_x \setminus E_x \text{ infinite} \}$$

where $X \setminus Y = \{ x \mid x \in X \wedge x \notin Y \}$

Solution

A is saturated

$A = \{ x \mid \varphi_x \in \mathcal{A} \}$ $\qquad$ $\mathcal{A} = \{ f \mid dom(f) \setminus cod(f) \text{ infinite} \}$

$\boxed{\text{idea}}$ $A, \bar{A}$ not r.e.

- A not r.e.

  $\mathbb{1} \in \mathcal{A}$ $\qquad$ $dom(\mathbb{1}) = \mathbb{N}$, $cod(\mathbb{1}) = \{1\}$ $\qquad$ $dom(f) \setminus cod(f) = \mathbb{N} \setminus \{1\}$ infinite

  $\forall \vartheta \subseteq \mathbb{1}$ finite $\qquad$ $\vartheta \notin \mathcal{A}$ $\qquad$ $\underbrace{dom(\vartheta)}_{\text{finite}} \setminus \underbrace{cod(\vartheta)}_{\text{finite}}$ finite

  by Rice-Shapiro the set A is not r.e.

- $\bar{A}$ not r.e.

  $f \notin \bar{\mathcal{A}}$ and $\vartheta \subseteq f$ $\qquad$ $\vartheta \in \bar{\mathcal{A}}$

  $\mathbb{1} \notin \bar{\mathcal{A}}$ and $\vartheta = \emptyset \subseteq \mathbb{1}$ $\qquad$ $\underbrace{dom(\vartheta)}_{\emptyset} \setminus cod(\vartheta) = \emptyset$

  $\vartheta \in \bar{\mathcal{A}}$

  $\leadsto$ by Rice-Shapiro $\bar{A}$ is not r.e.

* **Exercise**: Define what it means $A \leq_m B$ for $A, B \subseteq \mathbb{N}$

and show that if $A \leq_m B$ and B is recursive then A is recursive

SOLUTION

$A \leq_m B$ if there exists a total computable function $f: \mathbb{N} \to \mathbb{N}$

such that $\forall x \in \mathbb{N}$

$$x \in A \quad \text{iff} \quad f(x) \in B$$

Assume that $A \leq_m B$ and B is recursive

Let $f: \mathbb{N} \to \mathbb{N}$ be the reduction function. i.e.

f total computable s.t. $\forall x \quad x \in A$ iff $f(x) \in B$

Since B is recursive then

$$\chi_B(x) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases} \quad \text{is computable}$$

Then

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} = f(\chi_B(x))$$

hence χ_A is computable (composition of computable functions)

and thus A is recursive.

* Is it the case that if $A \leq_m B$ then $B \leq_m A$. ?

counter example:

$$\boxed{\underset{A \text{ (recursive)}}{\mathbb{N}} \leq_m \underset{B}{K}}$$

$f: \mathbb{N} \to \mathbb{N}$ total computable

$\forall x \quad x \in A$ iff $f(x) \in B$

$$\underbrace{x \in \mathbb{N}}_{\text{true}} \quad \text{iff} \quad f(x) \in K \qquad (*)$$

Let e_0 be such that $e_0 \in K$ (e.g. $\varphi_{e_0} = \emptyset$)

then

$$f(x) = e_0 \qquad \forall x$$

this verifies the requirements.

(total computable (constant function))

$$\forall x \quad \underbrace{x \in \mathbb{N}}_{\text{true}} \quad \text{iff} \quad \underbrace{f(x) = e_0 \in K}_{\text{true}}$$

Clearly

$$K \not\leq_m \mathbb{N}$$

because K is not recursive, while $\mathbb{N}$ is. $\square$

* <u>Exercise</u> :

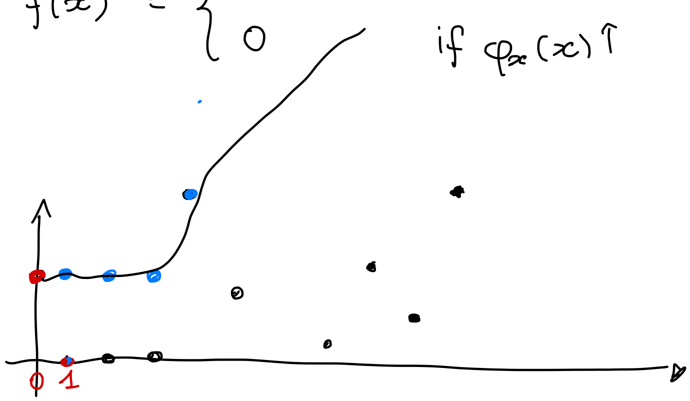
Define $f : \mathbb{N} \to \mathbb{N}$ monotone (increasing)

if f is total and $\forall x, y \quad x \leq y$ then $f(x) \leq f(y)$

Is there a monotone non computable function?

<u>solution</u> : consider

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{if } \varphi_x(x)\uparrow \end{cases}$$



now define

$$g(x) = \sum_{z \leq x} f(x)$$

$\rightarrow$ g is <u>total</u>, or f is total $\rightsquigarrow$ g is total

$\rightarrow$ g is <u>not computable</u>

$$f(x) = \begin{cases} \varphi_x(x)+1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{if } \varphi_x(x)\uparrow \end{cases}$$

$\forall x \quad g(x) \neq \varphi_x(x)$

- if $\varphi_x(x)\downarrow$

$$g(x) = \sum_{z \leq x} f(z) \geq f(x) = \varphi_x(x)+1 \neq \varphi_x(x)$$

- if $\varphi_x(x)\uparrow$

$$g(x)\downarrow \neq \varphi_x(x)$$

$\Rightarrow$ g not computable

* g is <u>monotone</u> $\quad \forall x,y \quad x \leq y$ then $g(x) \leq g(y)$

$$g(x) = \sum_{z \leq x} f(z) \leq \sum_{z \leq x} f(z) + \sum_{x+1 \leq z \leq y} f(z)$$

$$= \sum_{z \leq y} f(z) = g(y)$$

Alternative solution

$g : \mathbb{N} \rightarrow \mathbb{N}$

$$g(x) = \begin{cases} x+1 & \text{if } \varphi_x(x)\downarrow \text{ and } \varphi_x(x) \neq x+1 \\ x & \text{otherwise} \end{cases}$$

- <u>total</u>
- <u>monotone</u> $\quad g(x) \leq x+1 \leq g(x+1) \quad \forall x$

- <u>not computable</u>

$\forall x.\quad g \neq \varphi_x$

$\forall x$ if $\varphi_x(x)\downarrow$

$\to \varphi_x(x) = x+1$

$g(x) = x \neq x+1 = \varphi_x(x)$

$\to \varphi_x(x) \neq x+1$

$g(x) = x+1 \neq \varphi_x(x)$

if $\varphi_x(x)\uparrow$

$g(x)\downarrow \neq \varphi_x(x)$

Alternative solution, again

$g : \mathbb{N} \to \mathbb{N}$

$$g(x) = \begin{cases} x+1 & \text{if } \varphi_x(x)\downarrow \\ x & \text{otherwise} \end{cases}$$

→ <u>total</u> (as before)

→ <u>monotone</u> (as before)

→ <u>non computable</u> ? yes

if g were computable then

$h(x) = g(x) \dot{-} x$ is computable

$$= \begin{cases} 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

$= \chi_K(x)$

$\square$