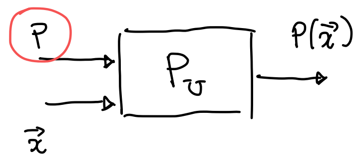
Computability (02/11/2021)



* Emumeration of URM programs

set X countable if $|X| \leq |\mathbb{N}|$

i.e. we have $f : \mathbb{N} \to X$ surjective

$$\underbrace{f(0) \quad f(1) \quad f(2) \quad \dots}_{\text{enumeration of } X}$$

if f is injective then bijective enumeration (without repetitions)

f is effective

Lemma: there are bijective enumerations of effective

1. $\mathbb{N}^2$
2. $\mathbb{N}^3$
3. $\bigcup_{k \geq 1} \mathbb{N}^k$

(1) $\pi : \mathbb{N}^2 \to \mathbb{N}$

$\pi(x,y) = 2^x(2y+1) - 1$ [computable]

$\pi^{-1} : \mathbb{N} \to \mathbb{N}^2$

$\pi^{-1}(n) = (\pi_1(n), \pi_2(n))$

$\pi_1, \pi_2 : \mathbb{N} \to \mathbb{N}$

$\pi_1(n) = (n+1)_1$

$\pi_2(n) = \left(\left(\frac{n+1}{2^{\pi_1(n)}}\right) - 1\right) / 2$

[computable]

(2) $\nu : \mathbb{N}^3 \to \mathbb{N}$

$\nu(x,y,z) = \pi(\pi(x,y), z)$

$\nu^{-1} : \mathbb{N} \to \mathbb{N}^3$

$\nu^{-1}(n) = \left(\nu_1(n), \nu_2(n), \nu_3(n)\right)$

$$\left.\begin{array}{l} \nu_1(n) = \pi_1(\pi_1(n)) \\ \nu_2 \ \text{----} \\ \nu_3 \ \text{----} \end{array}\right\} \text{[computable]}$$

(3) $\tau : \bigcup_{k \geq 1} \mathbb{N}^k \to \mathbb{N}$

$$\tau(x_1, \ldots, x_k) = \prod_{i=1}^{k} p_i^{x_i} - 1$$

NO

$$\left.\begin{array}{lll} (1,0) & \rightsquigarrow & p_1^1 \cdot p_2^0 - 1 = 1 \\ (1,0,0) & \rightsquigarrow & p_1^1 \, p_2^0 \, p_3^0 - 1 = 1 \\ (1) & \rightsquigarrow & p_1^1 - 1 = 1 \end{array}\right\} \text{not injective}$$

$$\tau(x_1, \ldots, x_k) = \left(\prod_{i=1}^{k-1} p_i^{x_i}\right) p_k^{x_k+1} - 2$$

$\tau^{-1} : \mathbb{N} \to \bigcup_k \mathbb{N}^k$

$a : \mathbb{N}^2 \to \mathbb{N}$

$$\tau^{-1}(n) = ? = \underbrace{(a(n,1) \ \ldots\ldots \ a(n,k))}_{\ell(n)}$$

$\ell : \mathbb{N} \to \mathbb{N}$

$$\begin{cases} \ell(n) = \max k \text{ such that } p_k \text{ divides } n+2 \\ = \text{exercise: write this as a bounded minimalisation} \\ a(n,i) = \begin{cases} (n+2)_i & 1 \leq i < \ell(n) \\ (n+2)_i \dot{-} 1 & i = \ell(n) \end{cases} \end{cases}$$

* **OBSERVATION**: Let $\mathcal{P}$ the set of all URM programs

There exists an effective bijective enumeration of $\mathcal{P}$

$$\gamma : \mathcal{P} \to \mathbb{N}$$

proof

let $\mathcal{Y}$ set of all URM instructions

0	1	2	3	4	5	6	7	8	9	...
$Z(1)$	S	T	J	$Z(2)$	S	T	J	$Z(3)$	S	...

$$\beta : \mathcal{Y} \to \mathbb{N}$$

$$\begin{cases} \beta(Z(n)) = 4 * (n-1) \\ \beta(S(n)) = 4 * (n-1) + 1 \\ \beta(T(m,n)) = 4 * \pi(m-1, n-1) + 2 \\ \beta(J(m,n,t)) = 4 * \nu(m-1, n-1, t-1) + 3 \end{cases}$$

$$\beta^{-1} : \mathbb{N} \to \mathcal{Y}$$

let
$r = rm(4, x)$
$q = qt(4, x)$

$$\beta^{-1}(x) = \begin{cases} Z(q+1) & \text{if } r=0 \\ S(q+1) & \text{if } r=1 \\ T(\pi_1(q)+1, \pi_2(q)+1) & \text{if } r=2 \\ J(\nu_1(q)+1, \nu_2(q)+1, \nu_3(q)+1) & \text{if } r=3 \end{cases}$$

Now $\gamma : \mathcal{P} \to \mathbb{N}$ can be defined as follows:

if $P = \begin{matrix} I_1 \\ \vdots \\ I_s \end{matrix}$ $\qquad \gamma(P) = \tau(\beta(I_1) \dots \beta(I_s))$

Inverse:

$$\gamma^{-1}(x) = P = \begin{matrix} I_1 \\ \vdots \\ I_{\ell(x)} \end{matrix} \qquad I_i = \beta^{-1}(a(x,i))$$

* γ fixed enumeration of URM programs

$\gamma(P)$ Gödel number of P

given n we write $P_n = \gamma^{-1}(n)$

* Examples

$P \begin{cases} T(1,2) \\ S(2) \\ T(2,1) \end{cases}$

$T(1,2) \overset{\beta}{\rightsquigarrow} 4 * \pi(1-1, 2-1) + 2 = 4 \times \underbrace{\pi(0,1)}_{2^0(2\cdot 1+1)-1 \,=\, 2} + 2 = 10$

$S(2) \rightsquigarrow 4 * (2-1) + 1 = 5$

$T(2,1) \rightsquigarrow = 6$

$\gamma(P) = \tau(10, 5, 6) = p_1^{10} \cdot p_2^{5} \cdot p_3^{6+1} - 2 = 2^{10} \cdot 3^5 \cdot 5^7 - 2$

$= 19.439.999.998$

$P_{19433999998} = \begin{matrix} T(1,2) \\ S(2) \\ T(2,1) \end{matrix}$

$P' \{ S(1)$

$\gamma(P') = \tau(\beta(S(1))) = \tau(4 * (1-1) + 1) = \tau(1) = p_1^{1+1} - 2$

$= 2^2 - 2 = 2$

$\gamma^{-1}(100) = ?$

$$100 + 2 = \left(\prod_{i=1}^{\ell(100)-1} p_i^{\beta(I_i)} \right) p_{\ell(100)}^{\beta(I_{\ell(100)})+1}$$

$\| $

$102 = 2^1 \cdot 3^1 \cdot 17^1$

$= p_1^1 \; p_2^1 \; p_3^0 \; p_4^0 \; p_5^0 \; p_6^0 \; p_7^1$

$\ell(100) = 7$

I_1	$\beta^{-1}(1)$	$S(1)$
I_2	$\beta^{-1}(1)$	$S(1)$
I_3	$\beta^{-1}(0)$	$Z(1)$
I_4	$\vdots$	$Z(1)$
I_5		$Z(1)$
I_6	$\beta^{-1}(0)$	$Z(1)$
I_7	$\beta^{-1}(0)$	$Z(1)$

$*$ Fixed $\gamma : \mathcal{P} \to \mathbb{N}$

this induces an enumeration of the <u>computable functions</u>

$$\boxed{\varphi_n^{(k)} : \mathbb{N}^k \to \mathbb{N}}$$

function of k-arguments computed by $P_n = \gamma^{-1}(n)$

$(\quad f_{P_n}^{(k)} \quad)$

$$\boxed{W_n^{(k)}} = \mathrm{dom}\left(\varphi_n^{(k)}\right) = \{ \vec{x} \in \mathbb{N}^k : \varphi_n^{(k)}(\vec{x})\downarrow \}$$

$$\subseteq \mathbb{N}^k$$

$$\boxed{E_n^{(k)}} = \mathrm{cod}\left(\varphi_n^{(k)}\right) = \{ y \mid \exists \vec{x} \in W_n^{(k)} \quad \varphi_n^{(k)}(\vec{x}) = y \}$$

if $k = 1$ we omit it

φ_n for $\varphi_n^{(1)}$

$\varphi_{100} : \mathbb{N} \to \mathbb{N} \qquad \varphi_{100}(x) = 0 \quad \forall x$

$P' : S(1) \qquad \gamma(P') = 2$

$\varphi_2(x) = x + 1 \qquad W_2 = \mathbb{N} \qquad E_2 = \mathbb{N} \setminus \{0\}$

$\varphi_2(x) = x + 1$

$\varphi_{194399999998}(x) = x + 1$

<u>Note</u>

$\varphi_0 \quad \varphi_1 \quad \varphi_2 \quad \varphi_3 \quad \varphi_4 \quad \cdots \quad \varphi_{19433999998}$

enumeration of all computable (unary) functions

$\Downarrow$

$|\mathcal{C}^{(1)}| \leq |\mathbb{N}|$

more generally

$|\mathcal{C}^{(k)}| \leq |\mathbb{N}| \qquad \forall k$

$\hookrightarrow \quad \mathcal{C} = \bigcup_{k \geq 1} \mathcal{C}^{(k)}$

$|\mathcal{C}| \leq |\mathbb{N}|$ countable (countable union of countable sets)

$=$

$f_c(x) = c \quad \forall x$

by induction on c, they are all computable ("exercise")

<u>Exercise</u> : $\mathcal{R}$ partial recursive functions

least class of functions { including base functions; closed under (a) composition (b) primitive recursion (c) minimalisation }

Original definition (Gödel - Kleene) $\mathcal{R}_0$

least class of functions { including base functions; closed under (a) composition (b) primitive recursion (c) minimalisation restricted to produce total functions }

$\mathcal{R}_0 \subseteq \mathcal{R} \cap \mathrm{Tot}$

$\supseteq$?

<u>Note</u> : you can obtain total functions by minimising partial functions

$f : \mathbb{N}^2 \to \mathbb{N} \qquad f(x,y) = \begin{cases} 1 & y < x \\ 0 & y = x \\ \uparrow & \text{otherwise} \end{cases}$

$h(x) = \mu y.\ f(x,y) = x$