Computability (26/10/2021)

* Primitive Recursive Functions

$\mathcal{PR}$ = class of functions obtainable from the BASIC functions using
1. composition
2. primitive recursion ← for-loop
3. ~~minimalisation~~ forbidden! ← while-loop

$$\mathcal{PR} \overset{?}{\subsetneq} \mathcal{R} \cap Tot$$

$$\mathcal{PR} = \mathcal{C}_{for} \not\ni \psi \qquad \mathcal{R}\cap Tot = \mathcal{C}_{for,while} \ni \psi$$

Ackermann's functions

$h(\vec{x}, y+1) \leftarrow h(\vec{x}, y)$

$\psi : \mathbb{N}^2 \to \mathbb{N}$

$$\begin{cases} \psi(0,y) = y+1 \\ \psi(x+1, 0) = \psi(x, 1) \\ \psi(x+1, y+1) = \psi(x, \psi(x+1, y)) \end{cases}$$

$(x+1, 0) >_{lex} (x, 1)$

$(x+1, y+1) >_{lex} (x+1, y)$

$>_{lex} (x, -)$

$(\mathbb{N}^2, \leq_{lex})$ $\quad (x,y) \leq_{lex} (x', y')$ if $x < x'$

or $x = x'$ and $y \leq y'$

$(1000, 1000000) \leq_{lex} (1001, 0)$

$(1000, 100000) >_{lex} (1000, 0)$

$\vdots$
3
2
1
0
-1
-2
$\vdots$

$\mathbb{Z}$ ↑ $f(-1)$

$$f(x) = \begin{cases} 0 & x \geq 0 \\ f(x-1) & x < 0 \end{cases}$$

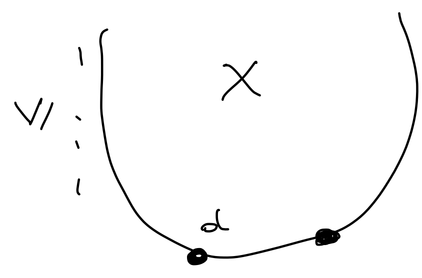
* <u>partially ordered sets</u> (poset)

$(D, \leq)$

$\leq$ reflexive $\quad x \leq x$
antisymmetric $\quad x \leq y$ and $y \leq x$ then $x = y$
transitive $\quad x \leq y$ and $y \leq z$ then $x \leq z$

* <u>well founded posets</u>

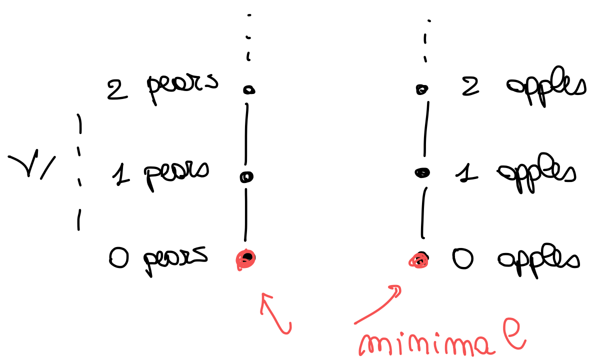
$(D, \leq)$ is well-founded if $\forall\, X \subseteq D \quad X \neq \emptyset$ has a minimal element



$d \in X$

$\forall d' \in X$ if $d' \leq d$
$\Rightarrow d' = d$

$D = \{ (\text{pears}, n), (\text{apples}, n) \mid n \in \mathbb{N} \}$



$(x, y) \leq (x', y')$ if
$x = x'$ and $y \leq y'$

$\mathbb{Z}$ is not well-founded, $\mathbb{N}$ is

NOTE: $(D, \leq)$ is well-founded iff there is no infinite descending chain
$d_0 > d_1 > d_2 > \ldots\ldots$

[exercise]

* $(\mathbb{N}^2, \leq_{lex})$ is well-founded

let $X \subseteq \mathbb{N}^2 \quad X \neq \emptyset$

$x_0 = \min \{ x \mid \exists y \in \mathbb{N}.\ (x, y) \in X \}$

$y_0 = \min \{ y \mid (x_0, y) \in X \}$

$\rightsquigarrow (x_0, y_0) = \min X$

* INDUCTION $\quad P(n) \quad n \in \mathbb{N}$

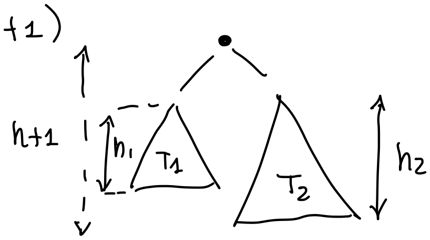
$P(0)$ and (assuming $P(n) \rightsquigarrow$ prove $P(n+1)$)

* $\underbrace{\text{A binary tree with heigth } h \text{ has at most } 2^{h+1}-1 \text{ nodes}}_{P(h)}$

$(h=0)$ $\quad\bullet$

number of nodes $= 1 \leq 2^{0+1} - 1 = 2-1 = 1$

$(h \to h+1)$



either h_1 or h_2 is h

$h_1 = h$

$h_2 = ?$

* $\boxed{\text{Complete induction}}$: to prove that $\forall n.\ P(n)$

prove : assuming $\forall n' < n \quad P(n')$

you deduce $P(n)$

* Well-founded Induction :

$(D, \leq)$ well-founded partial order

$P(x)$ property of elements of D

if for all $d \in D$, assuming $P(d')$ for $d' < d$

I can conclude that $P(d)$

$\Downarrow$

$\forall d \in D \quad P(d)$

1. ψ is total

$\forall (x,y) \in \mathbb{N}^2 \quad \psi(x,y)\downarrow$

we proceed by well-founded induction on $(\mathbb{N}^2, \leq_{lex})$

proof

let $(x,y) \in \mathbb{N}$, assume $\forall (x',y') <_{lex} (x,y) \quad \psi(x',y')\downarrow$

we want to prove $\psi(x,y)\downarrow$

we have 3 cases

$(x=0) \qquad \psi(0,y) = y+1 \downarrow$

$(x>0, y=0) \quad \psi(x,0) = \underbrace{\psi(x-1,1)}\downarrow$

$(x-1,1) <_{lex} (x,0)$

$\Rightarrow \psi(x-1,1)\downarrow$ by ind. hyp

$(x>0, y>0) \quad \psi(x,y) = \psi(x-1, \underbrace{\psi(x,y-1)})$

$(x,y-1) <_{lex} (x,y)$

$\Rightarrow \psi(x,y-1)\downarrow$ by ind. hyp. $= u$

$= \psi(x-1,u)\downarrow$ by ind hyp

$(x-1,u) <_{lex} (x,y)$

$\square$

$(\mathbb{N}^2, \leq_{lex})$

$\underbrace{(0,0)\ (0,1)\ (0,2) \ldots}_{\mathbb{N}} \quad \underbrace{(1,0)\ (1,1)\ (1,2) \ldots}_{\mathbb{N}} \quad \underbrace{(2,0)\ (2,1)\ (2,2) \ldots}_{\mathbb{N}} \quad \underbrace{(3,0)\ (3,1)\ (3,2) \ldots}_{\mathbb{N}}$

2. $\psi \in \mathcal{R} = \mathcal{C}$

$\psi(1,1) = \psi(0, \underbrace{\psi(1,0)}) = \psi(0,2)$

$\psi(1,0) = \psi(0,1) = 2$

$\psi(0,2) = 3$

$(\underline{1, 1}, 3) \quad (1, 0, 2) \quad (0, 1, 2) \quad (0, 2, 3)$

$(x, y, \varphi(x, y))$

valid set of triples $\quad S \subseteq \mathbb{N}^3$

$(x, y, z) \in S$

- $z = \varphi(x, y)$
- S contains also all the triples needed to compute $\varphi(x, y)$

formally: $\quad S \subseteq \mathbb{N}^3$ is <u>valid</u>

1. $(0, y, z) \in S \Rightarrow z = y+1$
2. $(x+1, 0, z) \in S \Rightarrow (x, 1, z) \in S$
3. $(x+1, y+1, z) \in S \Rightarrow \exists u \quad (x+1, y, u) \in S$, $(x, u, z) \in S$

you can prove that $\quad \forall (x, y, z) \in \mathbb{N}^3$

$\varphi(x, y) = z$ iff there exist a valid <u>finite</u> set of triples $S \subseteq \mathbb{N}^3$ such that $(x, y, z) \in S$

<u>idea</u>:

$$\varphi(x, y) = \text{“} \underset{\substack{\uparrow \\ \text{number}}}{\mu (S, z)} \underbrace{\left(\left(S \text{ finite valid set of triples} \right) \wedge (x, y, z) \in S \right)}\text{”}$$

$S = \{(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_m, y_m, z_m)\}$

$\{\pi(\pi(x_1, y_1), z_1), \quad \ldots \quad \pi(\pi(x_m, y_m), z_m)\}$

$k_1 \quad \ldots \quad k_m$

$$\prod_{i=1}^{m} p_i^{k_i}$$

$\rightsquigarrow \quad \varphi \in \mathcal{R} = \mathcal{C}$

By now ① ψ is total
② $\psi \in \mathcal{R} = \mathcal{C}$
③ $\psi \notin \mathcal{PR}$?

③ ψ is not primitive recursive

$x+y$
$$x+0 = x$$
$$x+(y+1) = x+y+1$$

$x*y$
$$x*0 = 0$$
$$x*(y+1) = (x*y)+x$$

x^y
$$x^0 = 1$$
$$x^{y+1} = x^y * x$$

⋮

$$\psi(x,y) = \psi_x(y)$$

$$\begin{cases} \psi(0,y) = y+1 \\ \psi(x+1,0) = \psi(x,1) \\ \psi(x+1,y+1) = \psi(x, \psi(x+1,y)) \end{cases}$$

$$\boxed{\psi_{x+1}(y)} = \psi_x(\psi_{x+1}(y-1))$$
$$= \psi_x(\psi_x(\psi_{x+1}(y-2)))$$
$$= \overbrace{\psi_x \, \psi_x \, \psi_x \cdots}^{y} \underbrace{\psi_{x+1}(0)}_{\psi_x(1)} = \boxed{\psi_x^{y+1}(1)}$$

$$\psi_0(y) = y+1$$
$$\psi_1(y) = \psi_0^{y+1}(1) = y+2$$
$$\psi_2(y) = \psi_1^{y+1}(1) = 2(y+1)+1 = 2y+3 \approx 2y$$
$$\psi_3(y) = \psi_2^{y+1}(1) \cong 2^y$$
$$\psi_4(y) = \psi_3^{y+1}(1) \cong \underbrace{2^{2^{2^{2}}}}_{y}$$

es: $\varphi_0(1) = 2$

$\varphi_2(1) = 5$

$\varphi_3(1) = 13$

$\varphi_4(1) \cong 2^{16}$

$\varphi_4(2) \cong 2^{2^{16}} \cong 10^{6400}$

ONE CAN PROVE: Given a function $f: \mathbb{N}^m \to \mathbb{N} \in PR$ and a program P computing f using only "for-loops" (primitive recursion) if J is the maximum level of nesting of for-loops

$P(x_1, \dots, x_m) \downarrow$ in a number of steps $< \varphi_{J+1}(\max\{x_i\})$

$\hookrightarrow f(\vec{x}) < \varphi_{J+1}(\max\{x_i\})$

Now, assume $\varphi \in PR$, let J be the level of nesting of for-loops (of primitive recursive defs) for computing φ

$\forall (x,y)$

$\varphi(x,y) < \varphi_{J+1}(\max\{x,y\})$

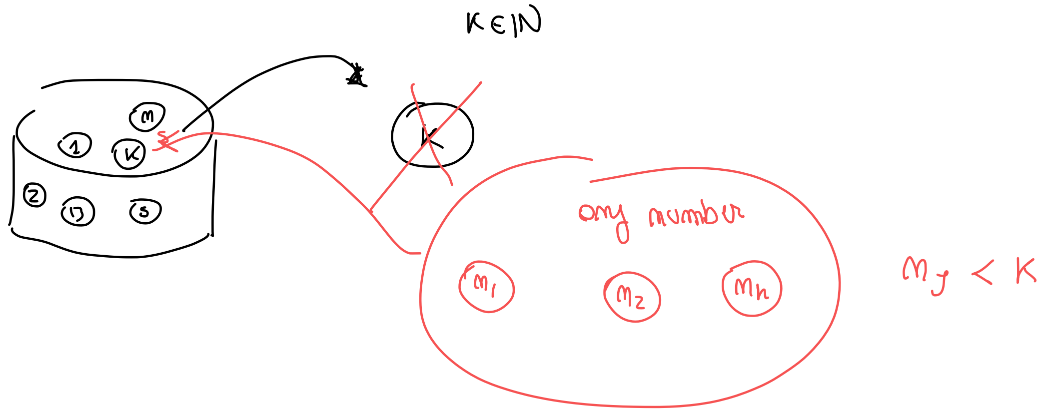
let $x = y = J+1$

$\varphi(J+1, J+1) < \varphi_{J+1}(J+1) = \varphi(J+1, J+1)$

contradiction

$\Rightarrow \varphi \notin PR$

EXERCISE



→ Does the process terminate? ←

→ Why?