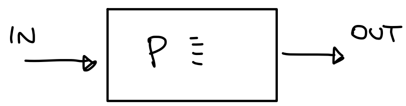
# Computability (07/12/2021)

## Rice-Shapiro's theorem




program properties concerning I/O

properties of computable functions $\mathcal{A} \subseteq \mathcal{C}$

$\mathcal{T} = \{ f \in \mathcal{C} \mid f \text{ is total} \}$

$ONE = \{ \mathbb{1} \}$

⋮

program properties (extensional) as sets of programs

$T = \{ x \mid \varphi_x \in \mathcal{T} \}$

$P_{ONE} = \{ x \mid \varphi_x \in ONE \} = \{ x \mid \varphi_x = \mathbb{1} \}$

→ Rice's theorem : NO (meaningful) I/O program property is decidable

→ Rice-Shapiro's theorem : a property of programs (extensional) can be semidecidable only if it is finitary (it talks about the behaviour of the program on a finite set of inputs)

## Rice-Shapiro's theorem

Let $\mathcal{A} \subseteq \mathcal{C}$ be a set of computable functions,

let $A = \{x \mid \varphi_x \in \mathcal{A}\}$

If $\boxed{A \text{ is r.e.}}$ (*) then

$$\boxed{\forall f \quad ( f \in \mathcal{A} \iff \exists \vartheta \subseteq f \quad \vartheta \text{ finite} \quad \vartheta \in \mathcal{A} )} \ (**)$$

proof

for proving (*) $\Rightarrow$ (**)

we prove

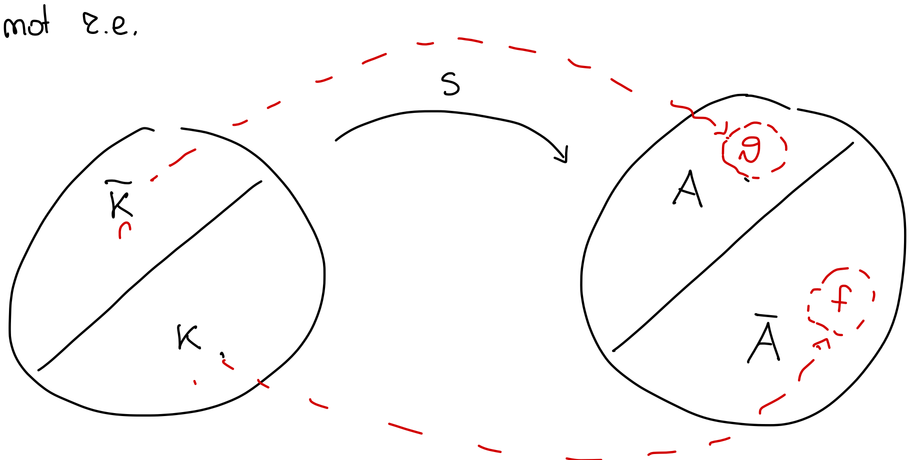
$\neg$ (**) $\Rightarrow$ $\neg$ (*)

(**) can be false in "two ways"

1. $\exists f \quad f \notin \mathcal{A}$ and $\exists \vartheta \subseteq f \quad \vartheta$ finite $\vartheta \in \mathcal{A} \Rightarrow A$ not r.e.
2. $\exists f \quad f \in \mathcal{A}$ and $\forall \vartheta \subseteq f \quad \vartheta$ finite $\vartheta \notin \mathcal{A} \Rightarrow A$ not r.e.

(1) let $f \notin \mathcal{A}$ and let $\vartheta \subseteq f$ finite $\vartheta \in \mathcal{A}$

$$\underbrace{\bar{K} = \{x \mid x \notin W_x\}}_{\text{not r.e.}} \leq_m A$$




Define

$$g(x,y) = \begin{cases} \boxed{\vartheta(y) \quad \text{if } x \in \bar{K}} \\ f(y) \quad \text{if } x \in K \end{cases}$$

$$= \begin{cases} \overset{f}{\cancel{\vartheta}}(y) & \text{if } x \in \bar{K} \text{ and } y \in dom(\vartheta) \\ \uparrow & \text{if } x \in \bar{K} \text{ and } y \notin dom(\vartheta) \\ f(y) & \text{if } x \in K \end{cases}$$

since $\vartheta \subseteq f$

$$= \begin{cases} f(y) & \text{if } \underbrace{x \in K \text{ or } y \in dom(\vartheta)} \\ \uparrow & \text{otherwise} \end{cases}$$

$$Q(x,y) \equiv \text{"} \underbrace{\underset{\text{semi-decidable}}{x \in K} \vee \overset{\text{decidable}}{\overbrace{y \in \underset{\text{finite}}{dom(\vartheta)}}}}_{\text{semi-decidable}} \text{"}$$

$$\hookrightarrow sc_Q(x,y) = \begin{cases} 1 & \text{if } Q(x,y) \\ \uparrow & \text{otherwise} \end{cases}$$

computable

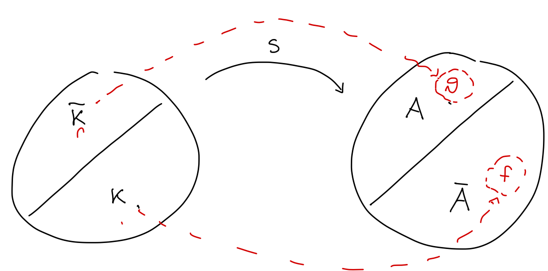
$$= \underset{\text{computable}}{f(y) \cdot sc_Q(x,y)}$$

$g$ is computable

By smn theorem there is total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $\forall x, y$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} \vartheta(y) & \text{if } x \in \bar{K} \\ f(y) & \text{if } x \in K \end{cases}$$

We show that $s$ is the reduction function for $\underset{\text{not r.e.}}{\bar{K}} \leq A$




* if $x \in \bar{K} \rightsquigarrow s(x) \in A$

let $x \in \bar{K}$ then $\forall y$ $\varphi_{s(x)}(y) = g(x,y) = \vartheta(y)$

hence $\varphi_{s(x)} = \vartheta \in A \Rightarrow s(x) \in A$

* if $x \notin \bar{K} \rightsquigarrow s(x) \in \bar{A}$

let $x \notin \bar{K}$ i.e. $x \in K$ then $\varphi_{s(x)}(y) = g(x,y) = f(y)$ $\forall y$

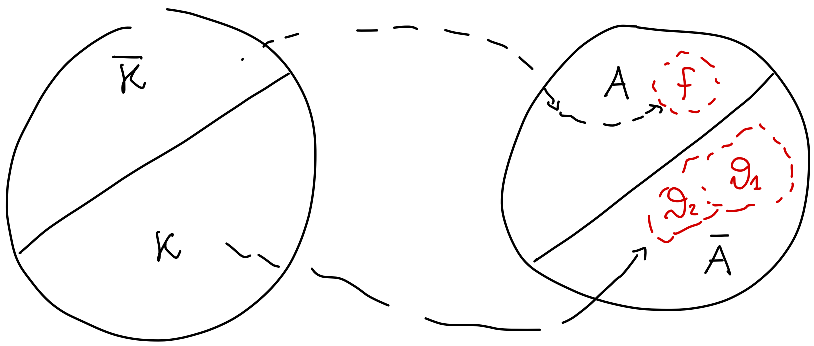
hence $\varphi_{s(x)} = f \notin A$ and thus $s(x) \notin A$

Since $\overline{K} \le_m A$ and $\overline{K}$ not r.e. we conclude $A$ not r.e.

(2) if there is $f \in \mathcal{A}$ such that $\forall \vartheta \subseteq f$ $\vartheta$ finite $\vartheta \notin \mathcal{A}$ $\Rightarrow$ $A$ not r.e.

let $f \in \mathcal{A}$ such $\forall \vartheta \subseteq f$ finite $\vartheta \notin \mathcal{A}$

$\overline{K} \le_m A$




$$g(x,y) = \text{"}\begin{cases} f(y) & \text{if } x \in \overline{K} \\ \vartheta(y) & \text{if } x \in K \end{cases}\text{"}$$

$P_x(x)\uparrow$ ← infinite

$P_x(x)\downarrow$ ← finite

intuitive

some $\vartheta \subseteq f$ finite

$$g(x,y) = \begin{cases} f(y) & \text{if } \neg H(x,x,y) \\ \uparrow & \text{if } H(x,x,y) \end{cases}$$

$$= f(y) + \mu z.\ \chi_H(x,x,y)$$

↳ if $H(x,x,y)$ $\quad \chi_H(x,x,y) = 1 \Rightarrow \uparrow$

↳ if $\neg H(x,x,y)$ $\quad \chi_H(x,x,y) = 0 \Rightarrow 0$

computable

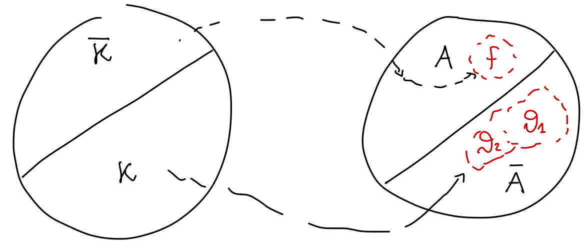
Hence by smn theorem $\exists s : \mathbb{N} \to \mathbb{N}$ total computable such that

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} f(y) & \text{if } \neg H(x,x,y) \\ \uparrow & \text{if } H(x,x,y) \end{cases}$$

We show that $s$

is the reduction function for $\overline{K} \le_m A$

* $x \in \overline{K}$ $\overset{?}{\rightsquigarrow}$ $s(x) \in A$




if $x \in \overline{K} \Rightarrow \varphi_x(x)\uparrow$

$\Rightarrow P_x(x)\uparrow \Rightarrow \forall y \quad \neg H(x,x,y)$

$\Rightarrow \forall y \quad \varphi_{s(x)}(y) = g(x,y) = f(y)$

$\Rightarrow \varphi_{s(x)} = f \in \mathcal{A} \Rightarrow s(x) \in A$

* $x \notin \overline{K} \overset{?}{\rightsquigarrow} s(x) \in \overline{A}$

if $x \notin \overline{K} \Rightarrow x \in K \Rightarrow P_x(x)\downarrow$ i.e.

$\exists y_0 \quad \forall y < y_0 \quad \neg H(x,x,y) \qquad \forall y \geq y_0 \quad H(x,x,y)$

then

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} f(y) & \text{if } \neg H(x,x,y) \\ \uparrow & \text{if } H(x,x,y) \end{cases}$$

$$= \begin{cases} f(y) & y < y_0 \\ \uparrow & \text{otherwise} \end{cases}$$

$\varphi_{s(x)} \subseteq f \qquad \text{dom}(\varphi_{s(x)}) \subseteq [0, y_0) \qquad \varphi_{s(x)}$ finite

$\Rightarrow \varphi_{s(x)} \in \overline{\mathcal{A}} \Rightarrow s(x) \in \overline{A}$

Hence $\overline{K} \leq_m A$ and $\overline{K}$ not r.e. $\Rightarrow$ $A$ not r.e. $\square$

* How do we use it? We use it to show that sets are not r.e.

1. $\exists f \quad f \notin \mathcal{A}$ and $\exists \vartheta \subseteq f$ $\vartheta$ finite $\vartheta \in \mathcal{A} \Rightarrow A$ not r.e
2. $\exists f \quad f \in \mathcal{A}$ and $\forall \vartheta \subseteq f$ $\vartheta$ finite $\vartheta \notin \mathcal{A} \Rightarrow A$ not r.e.

Example: $\quad \mathcal{T} = \{ f \mid f \text{ is total} \}$

$T = \{ x \mid x \in \mathcal{T} \} = \{ x \mid \varphi_x \text{ is total} \}$

$*$ $T$ is not r.e.

$id \in \mathcal{T}$ $\quad dom(id) = \mathbb{N}$

$\left.\forall \vartheta \subseteq id \quad \vartheta \text{ finite} \quad dom(\vartheta) \text{ finite} \Rightarrow \vartheta \notin \mathcal{T}\right\}$

$\Rightarrow$ $T$ is not r.e. (by Rice - Shapiro)

$*$ $\overline{T}$ is not r.e.

$id \notin \overline{\mathcal{T}}$ $\quad$ and $\quad \underset{\text{finite}}{\vartheta = \emptyset} \quad \emptyset(x)\uparrow \ \forall x \quad \vartheta \subseteq id \quad \vartheta \in \overline{\mathcal{T}}$

$\Rightarrow$ $\overline{T}$ is not r.e.

$*$ <u>Example</u> : $\quad ONE = \{ x \mid \varphi_x = \mathbb{1} \}$

$= \{ x \mid \varphi_x \in \underbrace{\{ \mathbb{1} \}}_{\mathcal{A} = \{ \mathbb{1} \}} \}$

$\rightarrow$ ONE is not r.e.

$\mathbb{1} \in \mathcal{A}$ $\quad$ and $\quad \forall \vartheta \subseteq \mathbb{1} \quad \vartheta \text{ finite} \quad \vartheta \notin \mathcal{A}$ $\quad \Rightarrow$ Rice-Shapiro ONE not r.e.

$\rightarrow$ $\overline{ONE}$ is not r.e.

$\mathbb{1} \notin \overline{\mathcal{A}}$ $\quad \vartheta = \emptyset \subseteq \mathbb{1}$ $\quad$ finite $\quad \vartheta \in \overline{\mathcal{A}}$ $\quad \Rightarrow$ $\overline{ONE}$ not r.e. by Rice-Shapiro

$\boxed{\text{OBSERVATION}}$ : The converse implication

$\mathcal{A} \subseteq \mathcal{C} \qquad A = \{ x \mid \varphi_x \in \mathcal{A} \}$

$\forall f \quad ( \ f \in \mathcal{A} \Leftrightarrow \exists \vartheta \subseteq f \ \ \vartheta \text{ finite} \ \ \vartheta \in \mathcal{A} )$

$\cancel{\Downarrow}$ false

$A$ r.e.

counterexample $\quad \mathcal{A} \subseteq \mathcal{C}$ such that

(1) $\forall f \; ( f \in \mathcal{A} \iff \underline{\exists \vartheta \subseteq f \; \vartheta \text{ finite } \vartheta \in \mathcal{A}} )$

(2) $A = \{ x \mid \varphi_x \in \mathcal{A} \}$ not r.e.

$$\mathcal{A} = \{ f \mid dom(f) \cap \overline{K} \neq \emptyset \}$$

(1) let $f$ be a function

* $f \in \mathcal{A} \Rightarrow dom(f) \cap \overline{K} \neq \emptyset$ $\quad$ let $x_0 \in dom(f) \cap \overline{K}$

and $\vartheta(x) = \begin{cases} f(x) & x = x_0 \\ \uparrow & \text{otherwise} \end{cases}$

$\vartheta \subseteq f \quad \vartheta$ finite $\quad dom(\vartheta) = \{x_0\} \cap \overline{K} = \{x_0\} \neq \emptyset$

$\Rightarrow \vartheta \in \mathcal{A}$

* $f$ such that $\exists \vartheta \subseteq f$ finite $\vartheta \in \mathcal{A} \quad \overset{?}{\rightsquigarrow} \quad \underline{f \in \mathcal{A}}$

since $\vartheta \in \mathcal{A} \quad dom(f) \cap \overline{K} \supseteq \underset{\underset{dom(f)}{\cap\!\!\mid}}{dom(\vartheta)} \cap \overline{K} \neq \emptyset$

$\Rightarrow dom(f) \cap \overline{K} \neq \emptyset$

$\Rightarrow f \in \mathcal{A}$

(2) $A = \{ x \mid \varphi_x \in \mathcal{A} \} = \{ x \mid dom(\varphi_x) \cap \overline{K} \neq \emptyset \}$ not r.e.

<u>intuition</u>: given $x \in \mathbb{N}$

in order to check $x \in \overline{K}$

```
def P'(y)
    if y = x
        then 0
    else loop
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defined only on $x$

$dom(P') = \{x\}$

$\boxed{dom(P') \cap \overline{K} \neq \emptyset \iff x \in \overline{K}}$

$$g(x,y) = \begin{cases} 0 & y = x \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \mu z.\ |y - x| \qquad \text{computable}$$

by smn theorem $\quad s : \mathbb{N} \to \mathbb{N}\quad$ total comp.

$$= \varphi_{s(x)}(y)$$

$s$ is the reduction function for $\overline{K} \leq A$

$$x \in \overline{K} \iff \underbrace{dom(\varphi_{s(x)})}_{=\{x\}} \cap \overline{K} \neq \emptyset \iff s(x) \in A$$

since $\overline{K}$ not r.e. then $A$ not r.e.

$\square$