Computability (10/01/2022)

(8.7) $A = \{ x \mid W_x \subseteq E_x \}$

$\uparrow \quad \uparrow$

* A is saturated

$A = \{ x \mid \varphi_x \in \mathcal{A} \} \qquad \mathcal{A} = \{ f \mid dom(f) \subseteq cod(f) \}$

* A is not r.e.

$\mathbb{1} \notin \mathcal{A} \qquad dom(\mathbb{1}) = \mathbb{N} \not\subseteq \{1\} = cod(\mathbb{1})$

$\vartheta = \phi \subseteq \mathbb{1} \qquad dom(\vartheta) = \phi \subseteq \phi = cod(\vartheta) \qquad \Rightarrow \vartheta \in \mathcal{A}$

$\Rightarrow$ A is not r.e. (by Rice-Shapiro)

* $\bar{A}$ is not r.e.

$pred(x) = x \dot{-} 1 \qquad dom(pred) = \mathbb{N} \subseteq cod(pred) = \mathbb{N}$

$\hookrightarrow pred \in \mathcal{A} \Rightarrow pred \notin \bar{\mathcal{A}}$

$$\vartheta = \begin{cases} 0 & x \leq 1 \\ \uparrow & \text{otherwise} \end{cases} = \begin{cases} x \dot{-} 1 & x \leq 1 \\ \uparrow & \text{otherwise} \end{cases} \qquad \vartheta \subseteq pred$$

finite

$dom(\vartheta) = \{0,1\} \not\subseteq \{0\} = cod(pred)$

$\vartheta \notin \mathcal{A} \Rightarrow \vartheta \in \bar{\mathcal{A}}$

$\Rightarrow \bar{A}$ is not r.e.

Hence $A, \bar{A}$ are neither r.e. $\square$

(we could have considered

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x = 1 \\ \uparrow & \text{otherwise} \end{cases}$$

$$\vartheta(x) = \begin{cases} 1 & x = 0 \\ \uparrow & \text{otherwise} \end{cases}$$

$f \notin \bar{\mathcal{A}} \qquad \vartheta \subseteq f \qquad \vartheta \in \bar{\mathcal{A}}$)

$g(x,y)$ computable

by $\underbrace{\text{smn theorem}}_{\text{corollary}}$ $\exists$ total computable $s: \mathbb{N} \to \mathbb{N}$ such that

$$\varphi_{s(x)}(y) = g(x,y)$$

$\boxed{8.32}$ Define $f: \mathbb{N} \to \mathbb{N}$ injective if $\forall x, y \in dom(f)$ $\quad f(x) = f(y) \Rightarrow x = y$

$$\uparrow \qquad \uparrow$$

$A = \{ x \mid \varphi_x \text{ is injective} \}$

conjecture $\left.\begin{array}{l}\bar{A} \text{ is r.e.} \\ \text{not recursive}\end{array}\right\} \Rightarrow A$ not r.e. ($\Rightarrow A$ is not recursive)

- $\bar{A}$ is r.e.

$$sc_{\bar{A}}(x) = \text{" find } y, z \quad y \neq z \quad f(y) = f(z) \text{ "}$$

$$\stackrel{!}{=} \mathbb{1}\left(\mu_{\substack{(y,z,v,t) \\ y \neq z}} \; S(x, y, v, t) \wedge S(x, z, v, t) \right)$$

$$= \mathbb{1}\left(\mu w. \; S(x, (w)_1, (w)_3, (w)_4) \wedge S(x, (w)_2, (w)_3, (w)_4) \wedge ((w)_1 \neq (w)_2) \right)$$

$$(w)_1 = y \quad (w)_2 = z \quad (w)_3 = v \quad (w)_4 = t$$

$$= \mathbb{1}\left(\mu w. \; \underset{\uparrow \text{ decidable}}{S(x, (w)_1, (w)_3, (w)_4)} \wedge S(x, (w)_1 + 1 + (w)_2, (w)_3, (w)_4) \right)$$

computable $\Rightarrow \bar{A}$ is r.e.

- $\bar{A}$ is not recursive

① $K \leq_m \bar{A}$

② use Rice's theorem (BEST!)

① define $g : \mathbb{N}^2 \to \mathbb{N}$

$$g(x,y) = \begin{cases} 1 & x \in K \\ \uparrow & x \notin K \end{cases} = sc_K(x)$$

computable

by smn $\exists\, s : \mathbb{N} \to \mathbb{N}$ total computable such that

$$\varphi_{s(x)}(y) = g(x,y) \quad \forall x, y$$

s is the reduction function for $K \leq \bar{A}$

- $x \in K$ then $\varphi_{s(x)}(y) = g(x,y) = 1 \quad \forall y \Rightarrow \varphi_{s(x)}$ not injective

$\Rightarrow s(x) \in \bar{A}$

- $x \notin K$ then $\varphi_{s(x)}(y) = g(x,y) \uparrow \quad \forall y \Rightarrow \varphi_{s(x)}$ injective

$\Rightarrow s(x) \notin \bar{A}$

$\Rightarrow$ hence $K \leq_m \bar{A}$, K not recursive $\Rightarrow \bar{A}$ not recursive

② use Rice's theorem

note $A = \{ x \mid \varphi_x \in \mathcal{A} \}$ $\quad \mathcal{A} = \{ f \mid f \text{ is injective} \}$

$\Rightarrow A$ saturated

- $id \in \mathcal{A} \Rightarrow A \neq \emptyset$
- $\mathbb{1} \notin \mathcal{A} \Rightarrow A \neq \mathbb{N}$

$\Rightarrow$ A not recursive

$\bar{A}$ " "

much simpler

Exercise

Define PR. Let $f : \mathbb{N}^2 \to \mathbb{N}$ $\quad f(x,y) = 2^y x$

show $f \in PR$ by using only the definition of PR

$$\begin{cases} f(x,0) = 2^0 \cdot x = x \\ f(x,y+1) = 2^{y+1} \cdot x = 2\, 2^y x \end{cases}$$

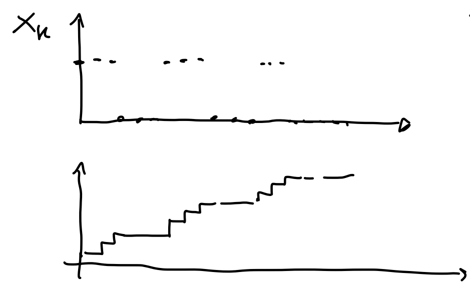
$$= 2\, f(x,y) = \text{twice}(f(x,y))$$

$$\begin{cases} \text{twice}(0) = 0 \\ \text{twice}(y+1) = \text{twice}(y) + 2 \end{cases}$$

$$= \text{succ}(\text{succ}(\text{twice}(y)))$$

Exercise: Is there a total non computable function $f: \mathbb{N} \to \mathbb{N}$ such that $g(x) = \sum_{y<x} f(y)$ is computable?

$f(y) = \chi_K(y) \cdot y$



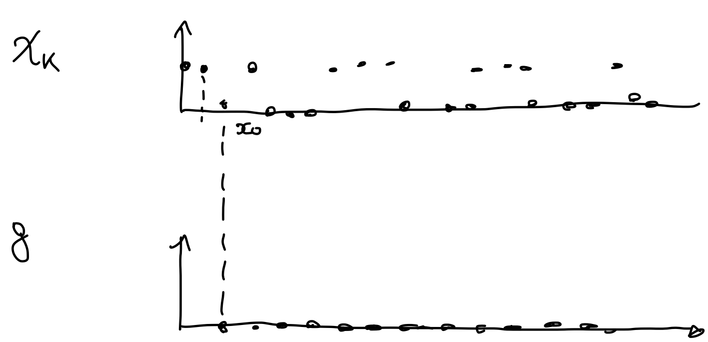
NO $\quad f(x) = g(x+1) \dot{-} g(x) \quad \forall x$

$$= \sum_{y<x+1} f(y) \dot{-} \sum_{y<x} f(y)$$

$$= f(x)$$

Exercise: Is there a total non computable function $f: \mathbb{N} \to \mathbb{N}$ such that $g(x) = \prod_{y<x} f(y)$ is computable?

$\boxed{\text{yes}}$ $\quad f = \chi_K$



$$x_0 = \min\{x \mid \chi_K(x) = 0\}$$

$$g(x) = \begin{cases} 1 & x < x_0 \\ 0 & \text{otherwise} \end{cases} = sg(x_0 \dot{-} x)$$