Computability (08/11/2021)

* Diagonalisation

idea: $x_i \quad i \in I$

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad \dots$

$\hookrightarrow \quad x \neq x_i \quad \forall i \in I$ — x differs from x_i "at position i"

Cantor: $\forall X \quad |X| < |2^X|$

powerset $2^X = \{Y \mid Y \subseteq X\}$

$X = \{0, 1, 2\}$

$2^X = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, X\}$

$|2^X| = 2^{|X|}$

$|X| = 3 \qquad |2^X| = 8$

Example: $|\mathbb{N}| < |2^{\mathbb{N}}|$

proof

by contradiction, $\boxed{|\mathbb{N}| \geq |2^{\mathbb{N}}|}$ i.e. $2^{\mathbb{N}}$ countable

$$\overbrace{X_0 \quad X_1 \quad X_2 \quad X_3 \quad \dots}^{2^{\mathbb{N}}}$$

$X_0 = \{2, 3\}$

	X_0	X_1	X_2	X_3	...
0	~~NO~~ YES	NO			
1	NO	~~YES~~ NO			
2	YES	YES	~~NO~~ YES		
3	YES	NO			
⋮	⋮	⋮			

$D = \{ i \mid i \notin X_i \} \subseteq \mathbb{N}$

$\Rightarrow \exists k$ s.t. $D = X_k$

$k \in D$?

YES	$k \notin X_k = D$	ABSURD
NO	$k \in X_k = D$	"

$\Rightarrow \quad |\mathbb{N}| < |2^{\mathbb{N}}|$

possibly partial

* EXAMPLE : $\mathcal{F} = \{ f \mid f : \mathbb{N} \to \mathbb{N} \}$

$|\mathcal{F}| > |\mathbb{N}|$

(1 APPROACH) $\mathcal{F}_2 = \{ f \mid f : \mathbb{N} \to \mathbb{N} \text{ total}, \ \forall x \ f(x) \in \{0, 1\} \}$
$\subseteq \mathcal{F}$

$\mathcal{F}_2 \xleftrightarrow[\text{bijective}]{} 2^{\mathbb{N}} \qquad f \in \mathcal{F}_2 \rightsquigarrow X_f = \{ x \in \mathbb{N} \mid f(x) = 1 \}$

$$\boxed{\begin{array}{c} |\mathcal{F}_2| \\ \wedge\!\!| \\ |\mathcal{F}| \end{array}} = |2^{\mathbb{N}}| > |\mathbb{N}|$$

$\mathcal{F}_2 \subseteq \mathcal{F} \Rightarrow |\mathcal{F}_2| \le |\mathcal{F}|$

$i : \mathcal{F}_2 \to \mathcal{F}$
$f \mapsto f$

(2nd approach) $|\mathcal{F}| > |\mathbb{N}|$
$\uparrow$

$$\begin{array}{c|cccccc} & f_0 & f_1 & f_2 & f_3 \ldots & \ldots & f_i \in \mathcal{F} \\ 0 & f_0(0) & f_1(0) & f_2(0) & & & \\ 1 & f_0(1) & f_1(1) & f_2(1) & & & \\ 2 & f_0(2) & f_1(2) & f_2(2) & & & \\ & \vdots & \vdots & \vdots & & & \end{array}$$

$$f(i) = \begin{cases} \uparrow & f_i(i)\downarrow \\ 0 & f_i(i)\uparrow \end{cases}$$

$\forall i \quad f_i \neq f$
$(f_i(i) \neq f(i))$

no enumeration of functions in $\mathcal{G}$ can can contain all $\mathcal{G}$

$\Rightarrow$ $\mathcal{G}$ is not countable.

OBSERVATION: There is a $\boxed{\text{total}}$ non-computable function

$$f : \mathbb{N} \to \mathbb{N}$$

$$f(n) = \begin{cases} \varphi_n(n) + 1 & \text{if } \varphi_n(n)\downarrow \\ 0 & \text{if } \varphi_n(n)\uparrow \end{cases}$$

f is not computable because $\forall n$ $\quad \varphi_n \neq f$

$$\varphi_n(n) \neq f(n)$$

in fact

if $\varphi_n(n)\downarrow \Rightarrow f(n) = \varphi_n(n) + 1 \neq \varphi_n(n)$

if $\varphi_n(n)\uparrow \Rightarrow f(n) = 0 \neq \varphi_n(n)$

	φ_0	φ_1	φ_2	...
0	$\varphi_0(0)$	$\varphi_1(0)$	$\varphi_2(0)$	
1	$\varphi_0(1)$	$\varphi_1(1)$	$\varphi_2(1)$	
2	$\varphi_0(2)$	$\varphi_1(2)$	$\varphi_2(2)$	
⋮				

$$f(n) = \begin{cases} \varphi_n(n)+1 & \varphi_n(n)\downarrow \\ 0 & \varphi_n(n)\uparrow \end{cases}$$

EXERCISE: let $f : \mathbb{N} \to \mathbb{N}$ function, $m \in \mathbb{N}$

show that there exists a non computable function $g : \mathbb{N} \to \mathbb{N}$

such that $\quad g(n) = f(n) \quad \forall\, n < m$

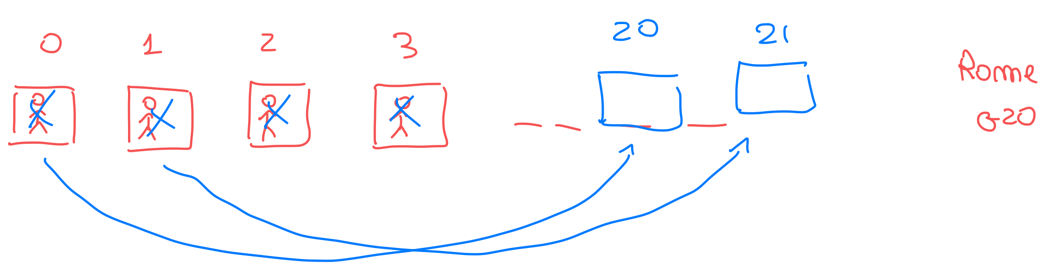
$$
\begin{array}{c|cccc}
 & \varphi_0 & \varphi_1 & \varphi_2 & \cdots \\
0 & \varphi_0(0) & \varphi_1(0) & \varphi_2(0) & \\
1 & \varphi_0(1) & \varphi_1(1) & \varphi_2(1) & \\
2 & \varphi_0(2) & \varphi_1(2) & \varphi_2(2) & \\
3 & & & & \\
\vdots & & & & \\
m-1 & & & & \\
\boxed{m} & \bullet & & & \\
m+1 & & \bullet & & \\
m+2 & & & \bullet & \\
\end{array}
$$

$$
g(n) = \begin{cases} f(n) & \text{if } n < m \\ \varphi_{n-m}(n)+1 & \text{if } n \geq m,\ \varphi_{n-m}(n)\downarrow \\ 0 & \text{if } n \geq m,\ \varphi_{n-m}(n)\uparrow \end{cases}
$$

g not computable since $\forall n$

$\varphi_n \neq g$

$\varphi_n(n+m) \neq g(n+m)$



another approach

$$
g(n) = \begin{cases} f(n) & n < m \\ \varphi_n(n)+1 & n \geq m,\ \varphi_n(n)\downarrow \\ 0 & n \geq m,\ \varphi_n(n)\uparrow \end{cases}
$$

$\forall n \geq m \quad g \neq \varphi_n \quad \overset{?}{\Longrightarrow} \quad g$ not computable

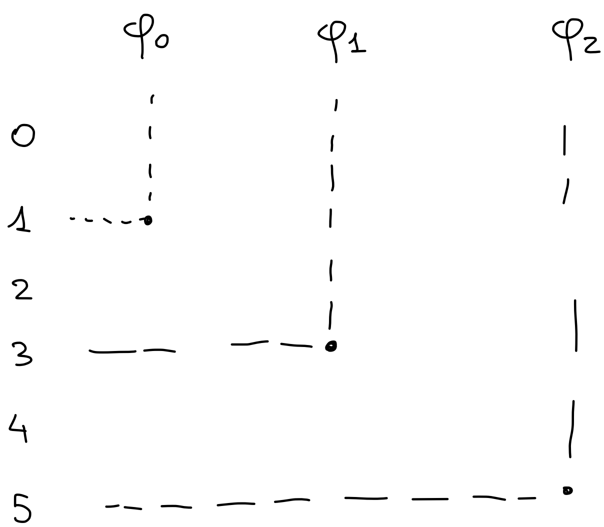
$\varphi_0 \quad \varphi_1 \quad \varphi_2 \; - \; - \; | \; -$

$\boxed{\forall n} \quad \exists\, n' \geq n \qquad \varphi_n = \varphi_{n'}$

$\Rightarrow \quad g \neq \varphi_{n'} = \varphi_n \qquad \Rightarrow \quad \boxed{g \neq \varphi_n}$

EXERCISE : there exists $g : \mathbb{N} \to \mathbb{N}$ total non-computable

$\forall n$ even $\qquad g(n) = 0$



$$g(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \varphi_{\frac{n-1}{2}}(n) + 1 & \text{if } n \text{ is odd} \\ & \quad \varphi_{\frac{n-1}{2}}(n)\downarrow \\ 0 & \text{if } n \text{ is odd} \\ & \quad \varphi_{\frac{n-1}{2}}(n)\uparrow \end{cases}$$

total
not computable $\qquad \forall n \qquad g(n) \neq \varphi_n(2n+1)$

EXERCISE : $f_0, f_1, \ldots \qquad (f_i)_{i \in \mathbb{N}}$ collection of functions

construct f s.t. $\operatorname{dom}(f) \neq \operatorname{dom}(f_i) \quad \forall i \in \mathbb{N}$

Parametrisation (smn) theorem

Let $f: \mathbb{N}^2 \to \mathbb{N}$ computable function

there exists $e \in \mathbb{N}$ s.t. $f = \varphi_e^{(2)}$ (P_e)

$f(x,y) = \varphi_e^{(2)}(x,y)$

Let $x \in \mathbb{N}$ be fixed

$f_x : \mathbb{N} \to \mathbb{N}$ computable :

$f_x(y) = f(x,y) = \varphi_e^{(2)}(x,y)$

def P_e $(\overset{1}{x}, y)$: (hard code fixed param.)

$\vdots$

$\cancel{x}$ 1

e.g. $f(x,y) = y^x$

$f_0(y) = y^0 = 1$

$f_1(y) = y^1 = y$

$f_2(y) = y^2$

$\vdots$

since all f_x are computable $\forall x \; \exists d$ s.t.

$f_x = \varphi_d$

depends on e, x

I can consider a function $S: \mathbb{N}^2 \to \mathbb{N}$ total

$S(e,x) = d$ computable

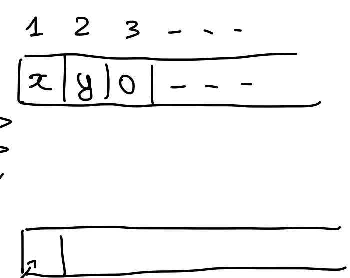
$P_e(x,y)$

$\boxed{\text{idea}}$

we have e and x

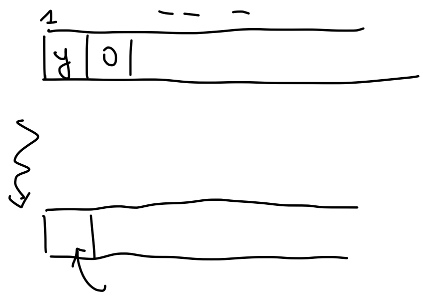
$P_e = \gamma^{-1}(e)$

P_e



$\varphi_e^{(2)}(x,y) = f(x,y)$

I want a program P' which does the following



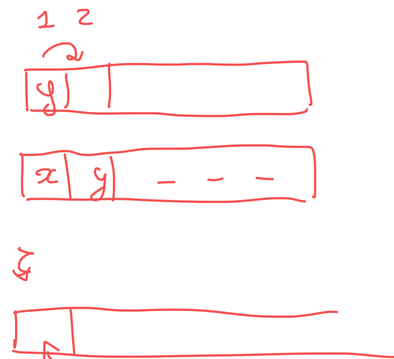
$f_x(y) = f(x,y) = \varphi_e^{(2)}(x,y)$

P'

→ move y to R_2

→ write x on R_1

→ P_e



$f(x,y) = f_x(y)$

$$S(e, x) = \gamma\begin{pmatrix} \text{move } y \text{ to } R_2 \\ \text{write } x \text{ on } R_1 \\ P_e = \gamma^{-1}(e) \end{pmatrix}$$