Computability (12/10/2021)

* Decidable predicate

$div(x,y) \equiv$ "x is a divisor of y"

$\hookrightarrow \quad div \subseteq \mathbb{N} \times \mathbb{N}$

$div = \{ (n, n * k) \mid n, k \in \mathbb{N} \}$

$div : \mathbb{N} \times \mathbb{N} \to \{ \text{true}, \text{false} \}$

$$div(x,y) = \begin{cases} \text{true} & \text{if } x \text{ is a divisor of } y \\ \text{false} & \text{otherwise} \end{cases}$$

Predicates K-ary

$Q(x_1, \dots x_K) \subseteq \mathbb{N}^K$

$$Q : \mathbb{N}^K \to \{ \underset{\substack{\uparrow \\ 1}}{\text{true}}, \underset{\substack{\uparrow \\ 0}}{\text{false}} \}$$

$\chi_Q : \mathbb{N}^K \to \mathbb{N}$

$$\chi_Q(x_1, \dots x_K) = \begin{cases} 1 & \text{if } Q(x_1, \dots x_K) \\ 0 & \text{otherwise} \end{cases}$$

Def: $Q(x_1, \dots x_K) \subseteq \mathbb{N}^K$ is decidable if $\chi_Q : \mathbb{N}^K \to \mathbb{N}$ is computable (URM)

Example:

* $Q(x_1, x_2) \subseteq \mathbb{N}^2$

$Q(x_1, x_2) \equiv$ "$x_1 = x_2$" decidable

$\chi_Q : \mathbb{N}^2 \to \mathbb{N}$ computable

R_1	R_2	R_3	
x_1	x_2	0	...

$\rightsquigarrow$

R_1
1/0

$J(1,2,YES)$

NO: $J(1,1,END)$

YES: $S(3)$

END: $T(3,1)$

Example : $Q(x) \equiv$ " x even "

R_1	R_2	R_3
x	0	0

↑
1
2
3
4
⋮

EVEN : $J(1, 2, \text{YES})$
$S(2)$
ODD : $J(1, 2, \text{NO})$
$S(2)$
$J(1, 1, \text{EVEN})$
YES : $S(3)$
NO : $T(3, 1)$

* Computability on other domains

D countable

$\alpha : D \to \mathbb{N}$ bijective "effective"
(α^{-1} effective)

$\boxed{A^*, \ \mathbb{Q}, \ \mathbb{Z}, \ ----}$

$\mathbb{R}$

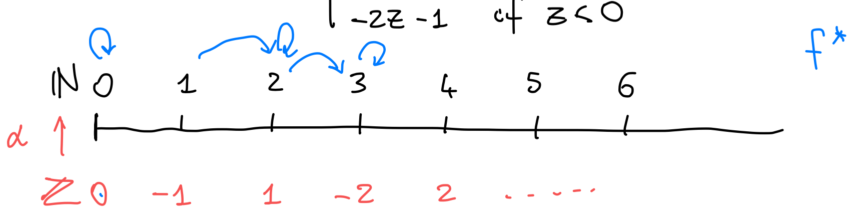
Given $f : D \to D$ function. We say that it is computable if $f^* = \alpha \circ f \circ \alpha^{-1} : \mathbb{N} \to \mathbb{N}$ is URM-computable

$$\begin{tikzcd} D \arrow[r, "f"] & D \arrow[d, dashed, "\alpha"] \\ \mathbb{N} \arrow[u, dashed, "\alpha^{-1}"] \arrow[r, "f^*"] & \mathbb{N} \end{tikzcd}$$

Example : $D = \mathbb{Z}$

$\alpha : \mathbb{Z} \to \mathbb{N}$

$$\alpha(z) = \begin{cases} 2z & \text{if } z \geq 0 \\ -2z-1 & \text{if } z < 0 \end{cases}$$



$\alpha^{-1} : \mathbb{N} \to \mathbb{Z}$

$$\alpha^{-1}(m) = \begin{cases} m/2 & m \text{ even} \\ -\frac{m+1}{2} & m \text{ odd} \end{cases}$$

$f: D \to D$

$f: \mathbb{Z} \to \mathbb{Z}$

$f(z) = |z|$

computable if $f^* = \alpha \circ f \cdot \alpha^{-1} : \mathbb{N} \to \mathbb{N}$ is URM-computable

$$f^*(m) = \alpha f \alpha^{-1}(m)$$

$$= \begin{cases} m \text{ even} & \alpha f\left(\frac{m}{2}\right) = \alpha\left(\frac{m}{2}\right) = 2\frac{m}{2} = m \\ m \text{ odd} & \alpha f\left(-\frac{m+1}{2}\right) = \alpha\left(\frac{m+1}{2}\right) = m+1 \end{cases}$$

$$= \begin{cases} m & \text{is } m \text{ is even} \\ m+1 & \text{is } m \text{ is odd} \end{cases}$$

↑ URM-computable

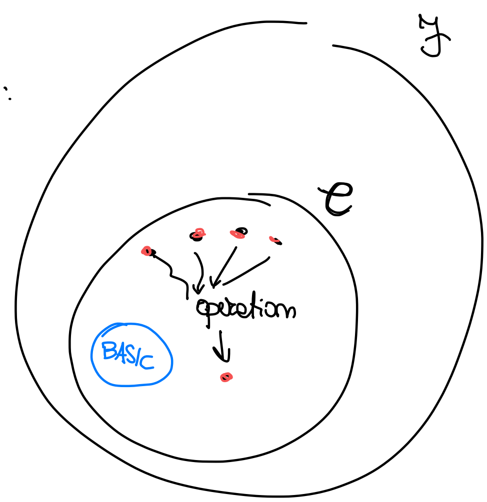
* Generation of computable functions

$\mathcal{C}$ is closed under the following operations:

→ composition (generalised)

→ primitive recursion

→ unbounded minimalisation



* BASIC FUNCTIONS

1. zero constant $z : \mathbb{N}^K \to \mathbb{N}$ $z(x_1, \dots, x_K) = 0$
2. successor $S: \mathbb{N} \to \mathbb{N}$ $S(x) = x + 1$
3. projections $U_i^K : \mathbb{N}^K \to \mathbb{N}$ $U_i^K(x_1, \dots, x_K) = x_i$

They are computable

① program $Z(1)$

② " $S(1)$

③ " $T(i,1)$

* Notation

given a program P

→ $\rho(P) = \max\{m \mid R_m \text{ is used in } P\}$

→ $\ell(P)$ = number of instructions in P

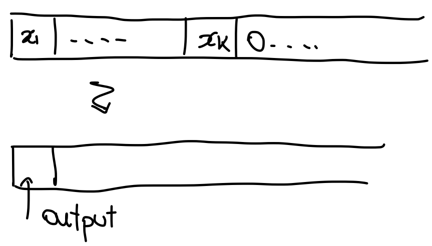
→ P is in std form if whenever it terminates it does it in $\ell(P)+1$

* concatenation : given P, Q in standard form

P $\quad\quad$ P

Q $\quad\quad$ Q' $\quad$ Q' is obtained by Q by replacing each $J(m,n,t)$
with $J(m,n,t+\ell(P))$

* given program P



we want a program

$P[r_{i_1} .. r_{i_k} \to r_e]$

that takes input from r_{i_1} .. r_{i_k}

puts output in r_e

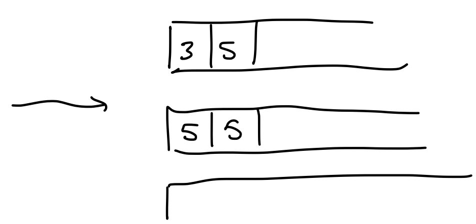
without assuming the remaining registers set to $\emptyset$

$\boxed{P[r_{i_1} \_\_ r_{i_k} \to r_e]}$ is as follows:

$T(i_1, 1)$
$\vdots$
$T(i_k, k)$
$Z(k+1)$
$\vdots$
$Z(\rho(P))$
P
$T(1, e)$

$P[2,1 \to 1]$

~~$T(2,1)$~~
~~$T(1,2)$~~
~~P~~
~~$T(1,1)$~~



* (Generalised) composition

Given $f : \mathbb{N}^k \to \mathbb{N}$

$g_1, \dots, g_k : \mathbb{N}^m \to \mathbb{N}$

we define $h : \mathbb{N}^m \to \mathbb{N}$

$$h(x_1, \dots, x_m) = \begin{cases} f(g_1(x_1, \dots, x_m), \dots, g_k(x_1, \dots, x_m)) \\ \quad \text{if } g_1(x_1, \dots, x_m)\downarrow, \dots, g_k(x_1, \dots, x_m)\downarrow \\ \quad \text{and } f(g_1(x_1, \dots, x_m), \dots, g_k(x_1, \dots, x_m))\downarrow \\ \uparrow \quad \text{otherwise} \end{cases}$$

$z(x) = 0 \quad \forall x$

$\emptyset(x)\uparrow \quad \forall x$

$z(\emptyset(x))\uparrow$

$U_1^2(x_1, x_2) = x_1$

$U_1^2(x_1, \emptyset(x_2))\uparrow$

Proposition : $\mathcal{C}$ is closed under generalised composition.

proof

let $f : \mathbb{N}^k \to \mathbb{N}$
$g_1, \dots, g_k : \mathbb{N}^m \to \mathbb{N}$

$f, g_1, \dots, g_m \in \mathcal{C}$

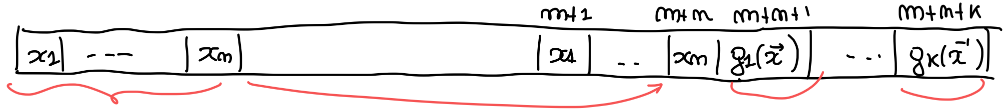
then $h : \mathbb{N}^m \to \mathbb{N}$

$h(\vec{x}) = f(g_1(\vec{x}), \dots, g_k(\vec{x}))$

$h \in \mathcal{C}$

since $f, g_1, \dots, g_k \in \mathcal{C}$ we can take $F, G_1, \dots, G_k$ programs in std form for $f, g_1, \dots, g_k$

The program for h can be as follows:



let $m = \max\{\rho(F), \rho(G_1), \ldots, \rho(G_k), n, k\}$

$T(1, m+1)$
$\vdots$
$T(n, m+n)$

$G_1[m+1, \ldots, m+n \to m+n+1]$ — program for h
$\vdots$
$G_k[m+1 \ldots m+n \to m+n+k]$

$F[m+n+1 \ldots m+n+k \to 1]$

$\square$

Example

$f(x_1, x_2) = x_1 + x_2$ computable

$g : \mathbb{N}^3 \to \mathbb{N}$

$g(x_1, x_2, x_3) = x_1 + x_2 + x_3$

$= \boxed{f(f(x_1, x_2), x_3)}$

$\uparrow$
$f : \mathbb{N}^2 \to \mathbb{N}$

$$f\left(f\left(U_1^3(x_1, x_2, x_3), U_2^3(x_1, x_2, x_3)\right), U_3^3(x_1, x_2, x_3)\right)$$

example : let $f : \mathbb{N} \to \mathbb{N}$ total computable function

$Q_f(x, y) \equiv$ " $f(x) = y$ " decidable

$$\chi_{Q_f}(x, y) = \begin{cases} 1 & \text{if } f(x) = y \\ 0 & \text{otherwise} \end{cases}$$ is computable

remember

$$\chi_{Eq}(x, y) = \begin{cases} 1 & x = y \\ 0 & \text{otherwise} \end{cases}$$ computable

so $\chi_{Q_f}(x, y) = \chi_{Eq}(f(x), y)$
$\uparrow$ computable

$\Rightarrow$ χ_{Q_f} is computable by composition