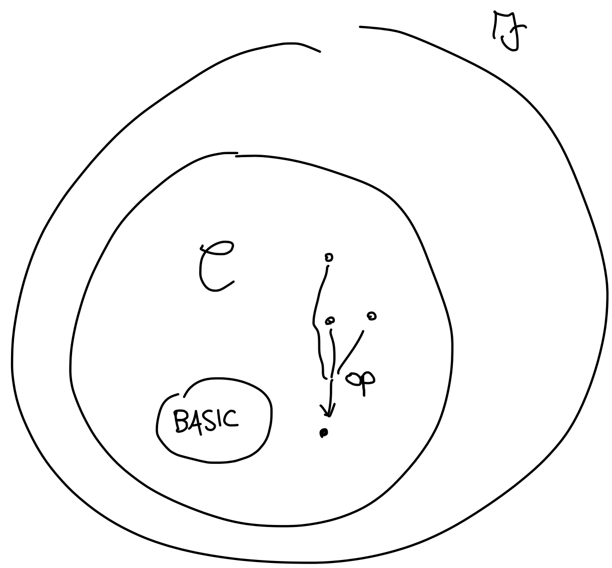
Computability (18/10/2021)

$\mathcal{C}$ class of URM- computable functions



→ it contains the basic functions

(1) zero $\quad z : \mathbb{N}^k \to \mathbb{N}$
$\qquad z(\vec{x}) = 0$

(2) successor $\quad s : \mathbb{N} \to \mathbb{N}$
$\qquad s(x) = x + 1$

(3) projections $\quad U_j^k : \mathbb{N}^k \to \mathbb{N}$
$\qquad U_j^k(\vec{x}) = x_j$

→ it is closed under the following operations

(a) generalised composition (substitution) ←

(b) primitive recursion

(c) unbounded minimalisation

* PRIMITIVE RECURSION

$$\begin{cases} 0! = 1 \\ (n+1)! = (n+1) \cdot n! \end{cases}$$

$$\begin{cases} fib(0) = 1 \\ fib(1) = 1 \\ fib(n+2) = fib(n) + fib(n+1) \end{cases}$$

⋮

Def. : Given $f : \mathbb{N}^k \to \mathbb{N}$

$g : \mathbb{N}^{k+2} \to \mathbb{N}$

we define $h : \mathbb{N}^{k+1} \to \mathbb{N}$ by PRIMITIVE RECURSION

$$\begin{cases} h(\vec{x}, 0) = f(\vec{x}) \\ h(\vec{x}, y+1) = g(\vec{x}, y, h(\vec{x}, y)) \end{cases}$$

$x \in \mathbb{R}$

$$\boxed{\frac{\sqrt{x}}{\log x} = x^2 + \frac{1}{e^x}}$$

- any solution ?
- unique ?

<u>Example</u> : $h : \mathbb{N}^2 \to \mathbb{N}$

$h(x,y) = x+y$

$x+0 = x$

$x+(y+1) = \underbrace{(x+y)} + 1$

$f : \mathbb{N} \to \mathbb{N} \quad f(x) = x$

$g : \mathbb{N}^3 \to \mathbb{N} \quad g(x,y,z) = z+1$

$h : \mathbb{N}^2 \to \mathbb{N}$

$h(x,y) = x * y$

$x * 0 = 0$

$x * (y+1) = \underbrace{(x*y)} + x$

$f(x) = 0$

$g(x,y,z) = z + x$

⋮

<u>Proposition</u> : $\mathcal{C}$ is <u>closed</u> under primitive recursion

<u>proof</u>

Let $f : \mathbb{N}^k \to \mathbb{N}$, $g : \mathbb{N}^{k+2} \to \mathbb{N}$ be computable functions $(\in \mathcal{C})$

we want to prove that $h : \mathbb{N}^{k+1} \to \mathbb{N}$ $\in \mathcal{C}$

$h(\vec{x},0) = f(\vec{x})$

$h(\vec{x},y+1) = g(\vec{x},y,h(\vec{x},y))$

Let F, G be programs for f, g in standard form

We want a program H for h

1 … k	k+1		m+1 … m+k	m+k+1	m+k+2	m+k+3
$x_1 \dots x_k$	y		$\vec{x}$	i	$h(\vec{x},i)$	y

in order to compute $h(\vec{x},y)$ we compute $h(\vec{x},i)$ $i=0,1,2,\dots,y$

$i=0$ $\quad h(\vec{x},0) = f(\vec{x})$ use F

$i=1$ $\quad h(\vec{x},1) = g(\vec{x},0,h(\vec{x},0))$ use G

$\vdots$

$i \to i+1$ $\quad h(\vec{x},i+1) = g(\vec{x}, i, h(\vec{x},i))$

$\vdots$

stop if $i=y$

Program H is :

let $m = \max\{\rho(F), \rho(G), k+2\}$

$T(1, m+1)$
$\vdots$
$T(k, m+k)$
$T(k+1, m+k+3)$
$F[m+1,\dots, m+k \to m+k+2]$ // $h(\vec{x},0)$

LOOP : $J(m+k+1, m+k+3, \text{END})$ // $i=y$? yes $\to$ DONE, no $\to$ continue
$G[m+1,\dots, m+k+2 \to m+k+2]$ // $h(\vec{x},i+1) = g(\vec{x},i,h(\vec{x},i))$
$S(m+k+1)$ // $i = i+1$

$J(1,1,\text{LOOP})$

END: $T(m+k+2, 1)$

* **OBSERVATION** : The following functions are computable

* sum $\quad x+y$ $\quad x+0 = x$
$x+(y+1) = (x+y)+1$

* product $\quad x*y$ $\quad x*0 = 0$
$x*(y+1) = (x*y)+x$

* exponential $\quad x^y$ $\quad x^0 = 1$
$x^{y+1} = (x^y)*x$

* predecessor $y \dot- 1 = \begin{cases} 0 & \text{if } y = 0 \\ y-1 & \text{if } y > 0 \end{cases}$

$$0 \dot- 1 = 0$$
$$(y+1) \dot- 1 = y$$

* $x \dot- y = \begin{cases} 0 & x \leq y \\ x-y & \text{otherwise} \end{cases}$

$$x \dot- 0 = x$$
$$x \dot- (y+1) = (x \dot- y) \dot- 1$$

* sign $sg(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0 \end{cases}$

$$sg(0) = 0$$
$$sg(x+1) = 1$$

* sign, complemented

$$\overline{sg}(x) = \begin{cases} 1 & x = 0 \\ 0 & x > 0 \end{cases}$$

* $\min(x,y) = \overbrace{x \dot- \overbrace{(x \dot- y)}^{0}}^{x}$

$x - \underbrace{(x-y)}_{} = y$ if $x \geq y$ if $x < y$

* $\max(x,y)$

* $rm(x,y)$ = remainder of y divided by x

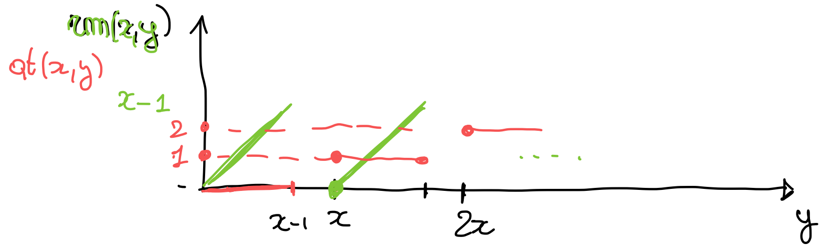
$$= \begin{cases} y \bmod x & \text{if } x \neq 0 \\ y & \text{if } x = 0 \end{cases}$$

$$rm(x,0) = 0$$

$$rm(x, y+1) = \begin{cases} 0 & \text{if } rm(x,y) + 1 = x \\ rm(x,y) + 1 & \text{if } rm(x,y)+1 < x \end{cases}$$

$$= (rm(x,y) + 1) * sg(x \dot- (rm(x,y)+1))$$

$\uparrow 1$ $\nwarrow 0$



computable

$*\ qt(x,y) = y \text{ div } x \qquad (\text{convention } qt(0,y) = y)$

$$qt(x,0) = 0$$

$$qt(x,y+1) = \begin{cases} qt(x,y)+1 & \text{if } rm(x,y)+1 = x \\ qt(x,y) & \text{if } \underbrace{rm(x,y)+1 < x} \end{cases}$$

computable

$$= qt(x,y) + \overline{sg}\left(x \dot{-} (rm(x,y)+1)\right)$$

$$*\ div(x,y) = \begin{cases} 1 & \text{if } rm(x,y) = 0 \\ 0 & \text{otherwise} \end{cases} = 1 \dot{-} sg(rm(x,y))$$

$$= \overline{sg}(rm(x,y))$$

<u>OBSERVATION</u> :

Let $f_1, \dots, f_n : \mathbb{N}^k \to \mathbb{N}$ computable TOTAL

$Q_1(\vec{x}), \dots, Q_n(\vec{x}) \subseteq \mathbb{N}^k$ predicates decidable
«mutually disjunctive» : $\forall \vec{x} \in \mathbb{N}^k$
$\exists!\, j$ s.t. $Q_j(\vec{x})$ holds

Then $f : \mathbb{N}^k \to \mathbb{N}$

$$f(\vec{x}) = \begin{cases} f_1(\vec{x}) & \text{if } Q_1(\vec{x}) \\ f_2(\vec{x}) & \text{if } Q_2(\vec{x}) \\ \vdots & \\ f_n(\vec{x}) & \text{if } Q_n(\vec{x}) \end{cases}$$

is computable
TOTAL

<u>proof</u>

$$f(\vec{x}) = f_1(\vec{x}) \cdot \chi_{Q_1}(\vec{x}) + f_2(\vec{x})\, \chi_{Q_2}(\vec{x}) + \dots + f_n(\vec{x}) \cdot \chi_{Q_n}(\vec{x})$$

computable computable

NOTE: $f_1(x) = x \qquad f_2(x)\uparrow \quad \forall x$

$Q_1(x) \equiv \text{true} \qquad Q_2(x) \equiv \text{false}$

$$f(x) = \begin{cases} f_1(x) & \text{if } Q_1(x) \\ f_2(x) & \text{if } Q_2(x) \end{cases} = f_1(x)$$

$$g(x) = f_1(x)\cdot \chi_{Q_1}(x) + f_2(x)\cdot \chi_{Q_2}(x) \quad \uparrow \quad \forall x$$

0

* **Algebra of decidability**

Let $Q_1(\vec{x}), Q_2(\vec{x})$ decidable predicates. Then

1. $\neg Q_1(\vec{x})$
2. $Q_1(\vec{x}) \wedge Q_2(\vec{x})$ are decidable
3. $Q_1(\vec{x}) \vee Q_2(\vec{x})$

proof

1. $$\chi_{\neg Q_1}(\vec{x}) = \begin{cases} 1 & \text{if } \neg Q_1(\vec{x}) \\ 0 & \text{if } Q_1(\vec{x}) \end{cases} = \overline{sg}(\chi_{Q_1}(\vec{x}))$$
computable

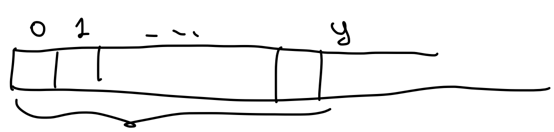
2. $$\chi_{Q_1 \wedge Q_2}(\vec{x}) = \chi_{Q_1}(\vec{x}) * \chi_{Q_2}(\vec{x})$$

3. $$\chi_{Q_1 \vee Q_2}(\vec{x}) = \max\{\chi_{Q_1}(\vec{x}), \chi_{Q_2}(\vec{x})\}$$
$$= sg(\chi_{Q_1}(\vec{x}) + \chi_{Q_2}(\vec{x}))$$

* **Bounded sums & products**

$f(\vec{x}, z) \qquad f: \mathbb{N}^{k+1} \to \mathbb{N}$ total computable

$h: \mathbb{N}^{k+1} \to \mathbb{N}$



$$h(\vec{x}, y) = \sum_{z<y} f(\vec{x}, z) = f(\vec{x}, 0) + f(\vec{x}, 1) + \dots + f(\vec{x}, y-1)$$

$$h(\vec{x}, 0) = 0$$

computable

$$h(\vec{x}, y+1) = h(\vec{x}, y) + \underbrace{f(\vec{x}, y)}$$

* **product** : $\prod_{z<y} f(\vec{x}, z)$

$$\prod_{z<0} f(\vec{x}, z) = 1$$

$$\prod_{z<y+1} f(\vec{x}, z) = \left(\prod_{z<y} f(\vec{x}, z)\right) \times f(\vec{x}, y)$$

computable

* **Bounded Quantification**

$Q(\vec{x}, z)$ decidable

1. $\forall z < y.\ Q(\vec{x}, z)$
2. $\exists z < y.\ Q(\vec{x}, z)$

are decidable

[exercise]