* Computability (21/12/21)

* Exercise : let $(f_i)_{i\in\mathbb{N}}$ with $f_i : \mathbb{N} \to \mathbb{N}$ be functions

Define $f : \mathbb{N} \to \mathbb{N}$ such that $dom(f) \neq dom(f_i) \ \forall i \in \mathbb{N}$

$\boxed{\text{solution}}$

	f_0	f_1	f_2	f_3 ----
0	$\downarrow$	$\downarrow$	$\uparrow$	
1	$\uparrow$	$\downarrow$	$\uparrow$	
2	$\uparrow$	$\downarrow$	$\uparrow$	
3	$\downarrow$	$\uparrow$	$\vdots$	
$\vdots$		$\vdots$		

$$f(x) = \begin{cases} \uparrow & \text{if } f_x(x)\downarrow \\ 0 & \text{if } f_x(x)\uparrow \end{cases}$$

$\forall x \qquad dom(f) \neq dom(f_x)$

two cases

- if $x \in dom(f_x) \Rightarrow f_x(x)\downarrow \Rightarrow f(x)\uparrow$
  $\Rightarrow x \notin dom(f)$
- if $x \notin dom(f_x) \Rightarrow f_x(x)\uparrow \Rightarrow f(x) = 0\downarrow$
  $\Rightarrow x \in dom(f)$

$\square$

* Exercise

1. state the 2nd recursion theorem
2. use it for showing that

$$\exists\, n \in \mathbb{N} \qquad \varphi_n \text{ is total} \qquad E_n = \{n * y \mid y \in \mathbb{N}\}$$

solution

1. given $f : \mathbb{N} \to \mathbb{N}$ total and computable $\exists\, e \in \mathbb{N}$ s.t. $\varphi_e = \varphi_{f(e)}$

2. First define

$$g : \mathbb{N}^2 \to \mathbb{N}$$

$$g(x, y) = x * y$$

$\forall x \in \mathbb{N}$ $\nearrow$ $g(x, \_)$ total

$\text{cod}(g(x, \_)) = \{x * y \mid y \in \mathbb{N}\}$

Note that g is computable, hence by smn theorem to get $s : \mathbb{N} \to \mathbb{N}$ total computable such that

$$g(x, y) = \varphi_{s(x)}(y) \qquad \forall x, y$$

$$\boxed{\begin{array}{l} \varphi_{s(x)} \text{ total} \\ E_{s(x)} = \{x * y \mid y \in \mathbb{N}\} \end{array}} \qquad \begin{array}{l} \varphi_n \text{ is total} \\ E_n = \{n * y \mid y \in \mathbb{N}\} \end{array}$$

since s is total and computable, by the 2nd recursion theorem $\exists\, n$ s.t. $\varphi_n = \varphi_{s(n)}$

$$\varphi_n = \varphi_{s(n)} \text{ total}$$

$$E_n = E_{s(n)} = \{n * y \mid y \in \mathbb{N}\}$$

$\square$

Exercise : Show there exists $n, m \in \mathbb{N}$ such that

(i) $\varphi_n = \varphi_{n+1}$

(ii) $\varphi_m \neq \varphi_{m+1}$

solution

(i) since $s(x) = x+1$ is computable total, by the II rec. th. there exists n $\varphi_n = \varphi_{s(n)} = \varphi_{n+1}$

(ii) if it were $\forall m$

$$\varphi_m = \varphi_{m+1} \qquad \varphi_0 = \varphi_1 = \varphi_2 = \varphi_3 - \cdots$$

all computable functions would coincide, while this is not the case (e.g. $id \neq \mathbb{1}$)

Exercise : Consider

$$A = \{x \mid |W_x| \geq 2\}$$

conjecture : A r.e, not recursive $\Rightarrow$ $\bar{A}$ not r.e. (hence not recursive)

NOTE : A is saturated

$$A = \{x \mid \varphi_x \in \mathcal{A}\} \qquad \mathcal{A} = \{f \mid |dom(f)| \geq 2\}$$

- $\boxed{A \text{ is not recursive}}$ (by Rice's theorem)

$A \neq \emptyset$ e.g. if e_1 is s.t. $\varphi_{e_1} = id$ then $|W_{e_1}| = |\mathbb{N}| \geq 2$
$\Rightarrow e_1 \in A$

$A \neq \mathbb{N}$ e.g. if e_0 is s.t. $\varphi_{e_0} = \emptyset$ then $|W_{e_0}| = |\emptyset| = 0 \not\geq 2$
$\Rightarrow e_0 \notin A$

Since A is saturated, by Rice's theorem, it is not recursive.

- $\boxed{A \text{ is r.e.}}$

try all inputs for all number of steps

$\boxed{x}$

idea: to establish if $x \in A$

search (y, t_1), (z, t_2) s.t. $H(x, y, t_1)$

$H(x, z, t_2)$

$$SC_A(x) = \mathbb{1}\left(\mu w. \quad H(x, (w)_1, (w)_2) \wedge H(x, (w)_3, (w)_4)\right)$$
$$\wedge \ (w)_1 \neq (w)_3$$

we can use $(w)_2$

$$\begin{array}{cccc} (w)_1 & (w)_2 & (w)_3 & (w)_4 \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ y & t_1 & z & t_2 \end{array}$$

$$y = (w)_1$$

$$z = (w)_1 + 1 + (w)_3$$

equivalently

$$= \mathbb{1}\left(\mu w. H(x, (w)_1, (w)_2) \wedge H(x, (w)_1 + (w)_3 + 1, (w)_2)\right)$$

computable $\Rightarrow$ A r.e.

A r.e., not recursive $\Rightarrow$ $\bar{A}$ is not r.e., hence not recursive $\square$

* **Exercise** :

(1) $A = \{ x \mid \varphi_x(x) = x^2 \}$

(2) $B = \{ x \mid \varphi_x(y) = y^2$ for infinitely many inputs $\}$

Question :

→ Characterise A, B w.r.t. recursiveness

→ Are A, B saturated ?

(1) $A = \{ x \mid \varphi_x(x) = x^2 \}$

conjecture :

A r.e., not recursive $\Rightarrow$ $\bar{A}$ not r.e. (not recursive)

* **A r.e.**

$$SC_A(x) = \mathbb{1}\big(\mu w. \underbrace{|\varphi_x(x) - x^2|}\big)$$

$0 \rightsquigarrow 0$

$\neq 0, \uparrow \rightsquigarrow \uparrow$

$$= \mathbb{1}\big(\mu w. \ |\Psi_U(x,x) - x^2|\big) \quad \text{computable}$$

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def P(y)
  P_x(x)
  return y^2
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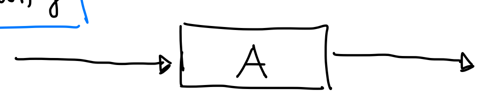
* **A not recursive**

we prove $K \leq_m A$



we need a reduction function

$P_x(x)\downarrow \iff P$ on its own code provides as an output the square of the code

Define $g : \mathbb{N}^2 \to \mathbb{N}$

$$g(x,y) = \mathbb{1}(\varphi_x(x)) \cdot y^2 = \begin{cases} y^2 & \text{if } \varphi_x(x)\downarrow \\ \uparrow & \text{if } \varphi_x(x)\uparrow \end{cases}$$

$$= \mathbb{1}(\Psi_U(x,x)) \times y^2$$

computable.

Hence by smn theorem we get $s : \mathbb{N} \to \mathbb{N}$ total computable such that

$\forall x, y$

$$g(x,y) = \underset{\sim\sim}{\varphi_{s(x)}}(y) = \begin{cases} y^2 & \text{if } \varphi_x(x)\downarrow \\ \uparrow & \text{if } \varphi_x(x)\uparrow \end{cases}$$

We claim that s is the reduction function for $K \leq_m A$

$*$ $\boxed{\text{if } x \in K \rightsquigarrow s(x) \in A}$

if $x \in K \Rightarrow \varphi_x(x)\downarrow \Rightarrow \forall y \quad \varphi_{s(x)}(y) = g(x,y) = y^2$

$\Rightarrow \varphi_{s(x)}(s(x)) = s(x)^2 \Rightarrow s(x) \in A$

$*$ $\boxed{\text{if } x \notin K \rightsquigarrow s(x) \notin A}$

if $x \notin K$ then $\varphi_x(x)\uparrow \Rightarrow \forall y \quad \varphi_{s(x)}(y) = g(x,y)\uparrow$

$\Rightarrow \varphi_{s(x)}(s(x))\uparrow \neq s(x)^2$

$\Rightarrow s(x) \notin A$

Hence $K \leq_m A$, since K not recursive we conclude A not recursive

Recall A r.e., not recursive hence $\bar{A}$ not r.e. hence not recursive.

$*$ $\boxed{\text{Is } A \text{ saturated?}}$

conjecture: it is not

i.e. $\boxed{e \in A}$

$e' \notin A \quad \varphi_e = \varphi_{e'}$

idea: find e s.t.

$$\varphi_e(y) = \begin{cases} e^2 & y = e \\ \uparrow & \text{otherwise} \end{cases}$$

define

$$g(x,y) = \begin{cases} x^2 & y = x \\ \uparrow & \text{otherwise} \end{cases} = \underbrace{\mu z. |y-x|}_{\substack{0 \text{ if } x=y \\ \uparrow \text{ otherwise}}} + x^2$$

computable

by smn theorem there exists $s : \mathbb{N} \to \mathbb{N}$ total computable such that

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} x^2 & y = x \\ \uparrow & \text{otherwise} \end{cases}$$

By II rec. theorem $\exists e$ s.t. $\varphi_e = \varphi_{s(e)}$

$$\varphi_e(y) = \varphi_{s(e)}(y) = \begin{cases} e^2 & \text{if } y = e \\ \uparrow & \text{otherwise} \end{cases}$$

$\Rightarrow$ $\boxed{e \in A}$

Note that since a computable function is computed by infinitely many programs there is $\boxed{e' \neq e}$ s.t.

$$\boxed{\varphi_{e'} = \varphi_e}$$

and $\boxed{e' \notin A}$ since

$$\varphi_{e'}(e') = \varphi_e(e') \uparrow \neq e'^2$$

$\Rightarrow$ A is not saturated

(2) $B = \{ x \mid \varphi_x(y) = y^2$ for infinitely many y's $\}$

Conjecture: $B, \bar{B}$ not r.e.

* B is saturated

$B = \{ x \mid \varphi_x \in \mathcal{B} \}$ $\quad\quad$ $\mathcal{B} = \{ f \mid f(y) = y^2$ on infinitely many y's $\}$

* Rice-Shapiro ?

(i) B is not r.e.

$f(y) = y^2 \ \forall y$ $\quad\quad$ $f \in \mathcal{B}$

$\forall \vartheta \subseteq f$ $\quad$ ϑ finite $\quad\quad$ $\{ y \mid \vartheta(y) = y^2 \} \subseteq dom(\vartheta)$ finite

hence $\vartheta \notin \mathcal{B}$

$\Rightarrow$ B not r.e. by Rice-Shapiro

(ii) $\bar{B}$ is not r.e.

$\left. \begin{array}{l} f \text{ (defined above) } \notin \bar{\mathcal{B}} \\ \vartheta = \emptyset \subseteq f \quad \vartheta \notin \mathcal{B} \text{ i.e. } \vartheta \in \bar{\mathcal{B}} \end{array} \right\} \Rightarrow \bar{B}$ is not r.e. by Rice-Shapiro

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