Computability (06/12/2021)

* Recursively enumerable sets and reducibility

Given $A, B \subseteq \mathbb{N}$ e $A \leq_m B$ then

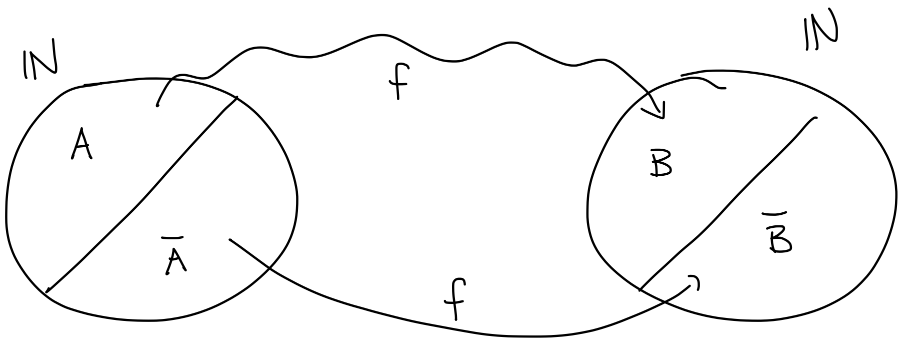
(i) if B is r.e. then A is r.e.

(ii) if A is not r.e. then B is not r.e.

proof

Let $A \leq_m B$ i.e. there is $f: \mathbb{N} \to \mathbb{N}$ total computable

$\forall x \quad x \in A$ iff $f(x) \in B$



(i) let B r.e.

$$SC_B(x) = \begin{cases} 1 & x \in B \\ \uparrow & \text{otherwise} \end{cases} \quad \text{is computable}$$

we want A r.e.

$$SC_A(x) = \begin{cases} 1 & x \in A \\ \uparrow & \text{otherwise} \end{cases} = SC_B(f(x))$$

is computable by composition.

(ii) equivalent.

* Recursively enumerable

enumerable / countable $\quad |A| \leq |\mathbb{N}|$

i.e. there is $f: \mathbb{N} \to A$ surjective

$$\underbrace{f(0) \quad f(1) \quad f(2) \quad \ldots}_{\text{enumeration of } A}$$

recursively enumerable $\stackrel{?}{\equiv}$ enumerable via a computable f

<u>Proposition</u>: Let $A \subseteq \mathbb{N}$ be a set

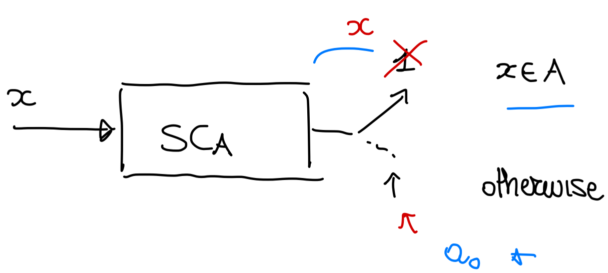
A r.e. iff $A = \emptyset$ or ($A = img(f)$ $f : \mathbb{N} \to \mathbb{N}$ total computable)

$\Rightarrow$

<u>proof</u>

($\Rightarrow$) Let A be r.e.

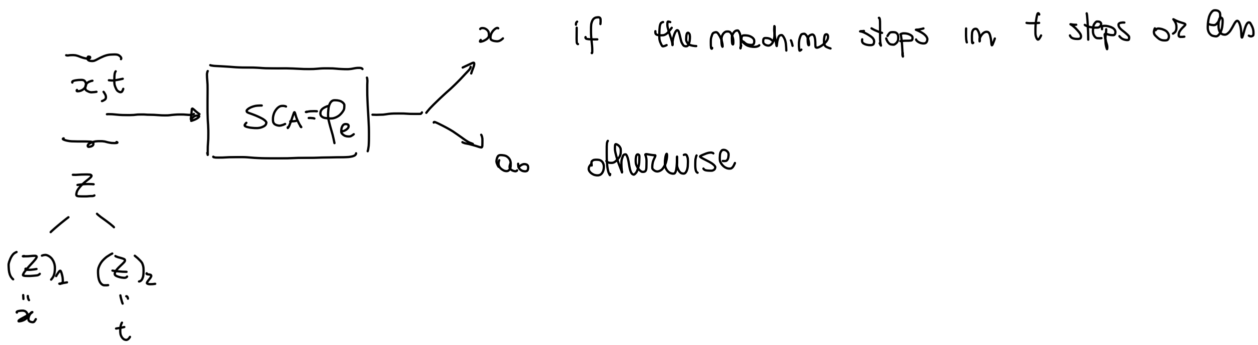
$$\Rightarrow \quad SC_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases} \qquad \text{computable}$$



$f(x) = x \cdot SC_A(x)$ computable

$img(f) = A$ NOT TOTAL

Assume $A \neq \emptyset$ and let $a_0 \in A$, let $e \in \mathbb{N}$ s.t. $SC_A = \varphi_e$



$$f(z) = \begin{cases} (z)_1 & \text{if } H(e, (z)_1, (z)_2) \\ a_0 & \text{otherwise} \end{cases}$$

* computable

$$f(z) = (z)_1 \cdot \chi_H(e, (z)_1, (z)_2) + a_0 \quad \overline{sg}(\chi_H(e, (z)_1, (z)_2))$$

computable

* total (by definition)

* $img(f) = A$

$(img(f) \subseteq A)$ let $x \in img(f) \leadsto \exists z$ s.t. $f(z) = x$

two possibilities

① $x = f(z) = (z)_1 \Rightarrow H(e, (z)_1, (z)_2)$

i.e. $P_e((z)_1)\downarrow$ in $(z)_2$ steps. Hence

$SC_A((z)_1) = 1 \Rightarrow x = (z)_1 \in A$

② $x = f(z) = a_0 \in A$

$(A \subseteq img(f))$

let $x \in A \Rightarrow SC_A(x) = 1 \downarrow \Rightarrow \exists t$ s.t. $H(e, x, t)$

if we take z s.t. $(z)_1 = x \quad (z)_2 = t$

$f(z) = (z)_1 = x$

$\Rightarrow x \in img(f)$

$(\Leftarrow)$ ① if $A = \emptyset \Rightarrow A$ r.e. since $SC_A(x)\uparrow \ \forall x$

computable <u>ok</u>

② if $A = \mathrm{img}(f)$ $\quad f$ total computable

$x \in A$ iff $\exists z \in \mathbb{N}$ st. $f(z) = x$

$SC_A(x) = \mathbb{1}\left(\mu z. \, |f(z) - x| \right)$ computable

$\Rightarrow$ A is r.e.

$\square$

<u>Proposition</u>: Let $A \subseteq \mathbb{N}$

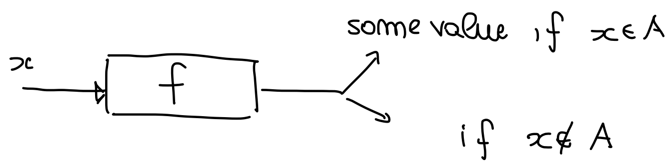
A r.e. iff $A = \mathrm{dom}(f)$, f computable

<u>proof</u>

$(\Rightarrow)$ let A be r.e. so $SC_A(x) = \begin{cases} 1 & x \in A \\ \uparrow & \text{otherwise} \end{cases}$ computable

$A = \mathrm{dom}(SC_A)$ as desired

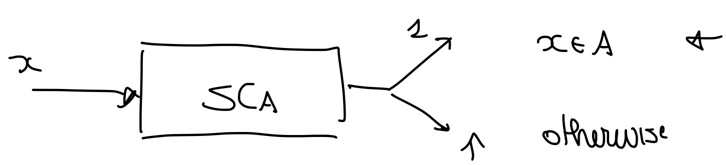
$(\Leftarrow)$ let $A = \mathrm{dom}(f)$ $\quad f$ computable



we want to show that

$SC_A(x) = \begin{cases} 1 & x \in A \\ \uparrow & \text{otherwise} \end{cases} = \mathbb{1}(f(x))$

computable



$\Downarrow$

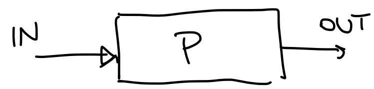
A is r.e.

$\square$

<u>EXERCISE</u>: A r.e. $\Leftrightarrow$ $A = \mathrm{img}(f)$ computable function

* Rice - Shapiro Theorem

The only properties that can be semi-decidable about I/O behaviour of programs are the "finitary properties"

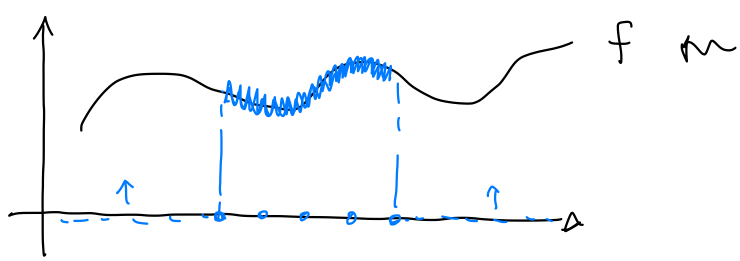


it depends only on the behaviour of P on a finite number of inputs

Examples :

- the program on input 0 provides 1 as output finitary
- the program is defined at least on two inputs finitary
- the program always stops not finitary

how to formalise the notion of a finitary property ?



→ finite function

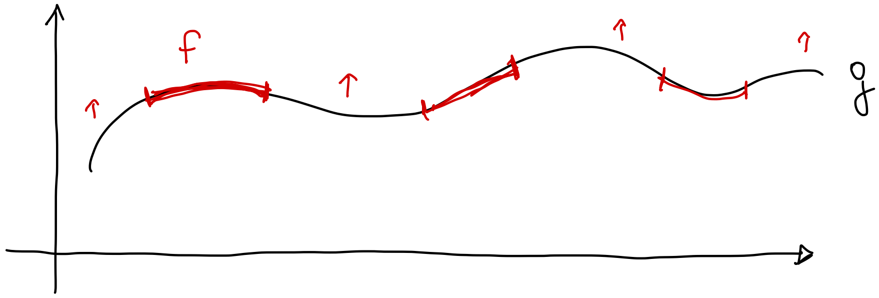
$\vartheta : \mathbb{N} \to \mathbb{N}$ is a finite function if dom(ϑ) finite

$$\vartheta(x) = \begin{cases} y_1 & \text{if } x = x_1 \\ y_2 & \text{if } x = x_2 \\ \vdots & \vdots \\ y_n & \text{if } x = x_n \\ \uparrow & \text{otherwise} \end{cases}$$

→ subfunction

we say that f is a subfunction of g written $f \subseteq g$

if $\forall x \in \mathbb{N}$ if $f(x)\downarrow$ then $g(x)\downarrow$ and $f(x) = g(x)$



Theorem (Rice - Shapiro)

Let $\boxed{\mathcal{A} \subseteq \mathcal{C} \text{ be a set of computable functions}}$

and let $A = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{A} \}$

If $\boxed{A \text{ is r.e.}}$ then $\Downarrow$ $\cancel{\Uparrow}$

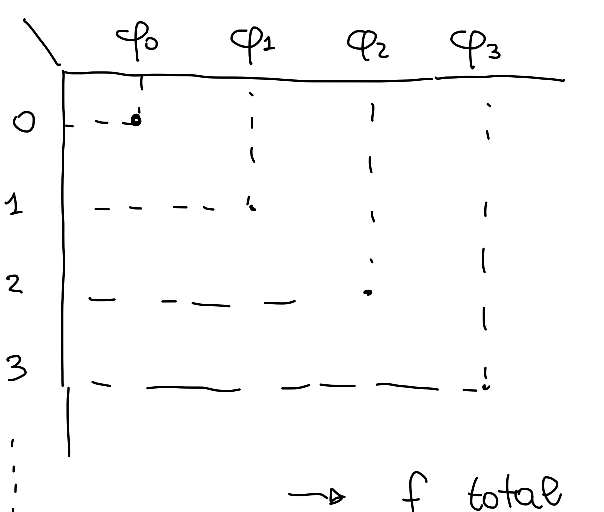
$$\boxed{\forall f \quad (f \in \mathcal{A} \iff \exists \vartheta \subseteq f \quad \vartheta \text{ finite such that } \vartheta \in \mathcal{A})}$$

proof (tomorrow)

* **EXERCISE** :

Find (if it exists) a total non-computable function $f : \mathbb{N} \to \mathbb{N}$

such that $\operatorname{img}(f) = \{ 2^n \mid n \in \mathbb{N} \}$



$$f(x) = \begin{cases} 2^{\varphi_x(x)} & \text{if } \varphi_x(x)\downarrow \\ 1 & \text{if } \varphi_x(x)\uparrow \end{cases}$$

$\varphi_x(x)$

→ f total

→ f not computable $\quad \forall x \quad f \neq \varphi_x$

$\rightarrow \quad \mathrm{img}(f) = \{ 2^m \mid m \in \mathbb{N} \}$

$(\subseteq) \quad \mathrm{img}(f) \subseteq \{ 2^m \mid m \in \mathbb{N} \}$

$$\forall x \qquad f(x) \begin{cases} 2^{\varphi_x(x)} \\ 1 = 2^0 \end{cases}$$

$(\supseteq) \quad$ let $m \in \mathbb{N}$ $\quad$ show $\quad 2^m \in \mathrm{img}(f)$

i.e. $\exists\, x$ s.t. $\quad f(x) = 2^m$

$\uparrow$

x s.t. $\quad \varphi_x(z) = m \quad \forall z$

$f(x) = 2^m \quad$ since

$\varphi_x(x) = m \downarrow$

$\square$

<u>Exercise</u> : there exists a function $s : \mathbb{N} \to \mathbb{N}$ total computable

$|W_{s(x)}| = 2^x \qquad |E_x| = x+1$

Define

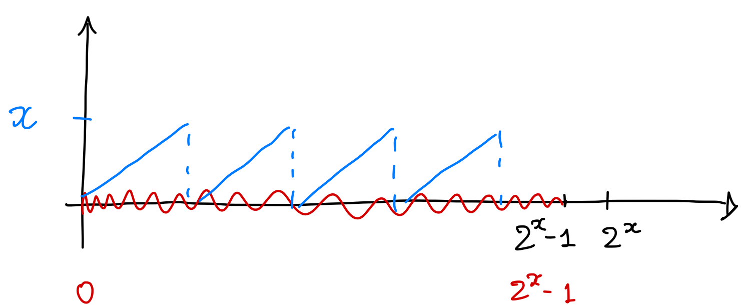
$g : \mathbb{N}^2 \to \mathbb{N}$

$$g(x, y) = \begin{cases} rm(x+1, y) & y < 2^x \\ \uparrow & \text{otherwise} \end{cases}$$

$$= rm(x+1, y) + \mu w.\ \overline{sg}(\underbrace{2^x \dot{-} y})$$

$\llcorner \to 0 \quad$ if $\; y < 2^w$

$\llcorner \to \uparrow \quad$ otherwise



computable

By smn there exists $s : \mathbb{N} \to \mathbb{N}$ such that

$$\varphi_{s(x)}(y) = g(x,y) \quad \forall x,y$$

Conclude by showing that s is the desired function

① $|W_{s(x)}| = 2^x$

② $|E_{s(x)}| = x+1$

① $W_{s(x)} = \{ y \mid \varphi_{s(x)}(y)\downarrow \}$

$= \{ y \mid g(x,y)\downarrow \} = \{ y \mid y < 2^x \} = [0, 2^x)$

$\Rightarrow |W_{s(x)}| = 2^x$

② $E_{s(x)} = \{ \varphi_{s(x)}(y) \mid y \in W_{s(x)} \}$

$= \{ rm(x+1, y) \mid y \in [0, 2^x) \}$

$= [0, x+1)$

$\Rightarrow |E_{s(x)}| = |[0, x+1)| = x+1$