Real Number Definitions, Equivalence Classes, and Embeddings (No Countable Choice)

This document lists the first twenty-three constructive definitions of real numbers, partitions them into equivalence classes, and records the canonical embeddings. Note that while many equivalences hold in standard constructive mathematics (assuming Countable Choice, AC_ω), in plain Cubical Agda without AC_ω, the hierarchy is more fractured (e.g., Cauchy reals are not provably Dedekind reals). General surveys and formalization resources include [95, 109, 5].

1. Definitions (1–33)

1. $\mathbb{R}_\mathrm{D}$: Dedekind reals (located cuts of $\mathbb{Q}$) [34, 21, 22, 28, 65, 108, 122, 75, 101, 84, 50, 35].
2. $\mathbb{R}_\mathrm{C}$: Cauchy reals (modulated Cauchy sequences of rationals, quotiented) [21, 22, 63, 76, 73, 71, 79].
3. $\mathbb{R}_\mathrm{E}$: Eudoxus reals (almost-homomorphisms $\mathbb{Z} \to \mathbb{Z}$) [9, 98, 102, 93, 45, 56, 74].
4. $\mathbb{R}_\mathrm{FC}$ / $\mathbb{R}_\mathrm{I}$: fast Cauchy reals / interval reals (Cauchy sequences with explicit moduli or nested rational intervals) [31, 3, 114, 119].
5. $\mathbb{R}_\mathrm{CF}$: continued fraction reals (streams of partial quotients) [30, 118, 112].
6. $\mathbb{R}_b$: coinductive base-b reals (digit streams, e.g., binary/decimal) [115].
7. $\mathbb{R}_\mathrm{SD}$: signed-digit reals (streams over $\{-1, 0, 1\}$) [8, 42, 115, 17, 77, 54, 59, 24, 66, 27, 99, 20, 19, 41, 120].
8. $\mathbb{R}_\mathrm{ID}$: interval domain reals (maximal elements of the interval domain) [113, 87, 53, 13, 15, 38, 37, 36, 106, 39, 14].
9. $\mathbb{R}_\mathrm{L}$: lower reals (rounded lower sets of $\mathbb{Q}$) [83, 23, 33].
10. $\mathbb{R}_\mathrm{U}$: upper reals (rounded upper sets of $\mathbb{Q}$) [83, 23, 33].
11. $\mathbb{R}_\mathrm{M}$: MacNeille reals (double-negation closed cuts) [68, 82, 46, 100].
12. $\mathbb{R}_\mathrm{H}$: HIT/HoTT-book reals (higher inductive type with universal property) [107, 81, 26, 25, 12, 92].
13. $\mathbb{R}_\mathrm{ES}$: Escardó–Simpson reals (least Cauchy-complete subobject of $\mathbb{R}_\mathrm{D}$ containing $\mathbb{Q}$) [40, 25].
14. $\mathbb{R}_\mathrm{formal}$: formal/locale reals (points of the locale of reals) [67, 51, 52, 91, 49, 84].
15. $\mathbb{R}_\mathrm{init}$: initial sequentially modulated Cauchy-complete Archimedean ordered field [40, 25, 78].
16. $\mathbb{R}_\mathrm{term}$: terminal Archimedean ordered field [40, 67, 78]. Items 34 and 35 are distinct definitions with different construction proofs, but they result in the same object if the category is well-behaved.

17. $\mathbb{R}_{\text{DedComp}}$: Dedekind-complete ordered field (axiomatic characterization) [67, 52, 11].

18. $\mathbb{R}_{\text{CauComp}}$: Cauchy-complete ordered field (axiomatic characterization of the Cauchy completion) [22, 109].

19. $\mathbb{R}_{\text{Tarski}}$: Archimedean Tarski group reals (characterization via Tarski's axioms) [104, 35].

20. $[0,1]_{\text{coalg}}$: unit interval as a terminal coalgebra [40, 4, 80, 89].

21. $\mathbb{R}^{+}_{\text{coalg}}$: positive reals as a terminal coalgebra [40, 4, 80].

22. Sheaf-theoretic reals: the internal real numbers object in a topos [67, 52, 101, 49, 84].

23. Real numbers object (RNO) in a topos [67, 52, 101, 49, 84]. Items 41 and 42 are essentially the same mathematical object described from two different points of view (internal language vs. category theory).

24. $\mathbb{R}_{\text{SDG}}$: Smooth Reals (synthetic differential geometry). In Synthetic Differential Geometry, the reals are defined to include "nilpotent" infinitesimals (elements d where $d^2 = 0$ but $d \neq 0$). These are distinct from standard Dedekind/Cauchy reals because they violate the field axiom $x \neq 0 \implies x$ is invertible (nilpotents are not invertible). They are a distinct mathematical object internal to a smooth topos, not isomorphic to the usual Cauchy/Dedekind reals [58, 60, 72, 85].

25. $^*\mathbb{R}$: Hyperreals (non-standard analysis). These include infinite and infinitesimal numbers. While usually constructed classically (using ultrafilters), there are constructive approaches (e.g., Palmgren's constructive non-standard analysis) that result in a structure distinct from $\mathbb{R}_{\text{D}}$ or $\mathbb{R}_{\text{C}}$. They are strict extensions of the ordinary reals [96, 55, 86].

26. Predicative Reals: In systems stricter than Agda (like those prohibiting impredicativity), Dedekind cuts must be restricted (e.g., to "generalized" or "weak" cuts) to avoid circular definitions. The document hints at this with lower/upper reals (items 9, 10), but specific predicative formalizations often stand alone [44, 29, 88].

27. **No**: Surreal Numbers (Conway's construction). While the Surreals contain the Reals, the "Real subset" of the Surreals is a valid constructive definition of the reals. Inside **No** there is a canonical embedded copy of $\mathbb{R}$; this embedding can be used as yet another definition of the real line [32, 48, 69, 57, 121].

28. Geometric Reals: Defined synthetically in Euclidean Geometry (e.g., Tarski's axioms for geometry, or Hilbert's axioms). Defined as "points on a line" rather than arithmetically. Constructively, relating "points on a line" to "Dedekind cuts" is a non-trivial project (requires the Cantor-Dedekind axiom) [104, 105, 16, 90, 116].

29. Computable Reals (Turing): Specifically defined as "Turing machines that output digits". This is distinct from $\mathbb{R}_{\text{C}}$ because $\mathbb{R}_{\text{C}}$ allows *any* function, whereas Computable Reals restrict the functions to computable ones. In strongly normalizing type theories, every *definable* function is computable (meta-theoretically), so formalising computable reals inside such a system is natural. But this does not by itself make $\mathbb{R}_{\text{C}}$ "the same" as the usual Cauchy reals object; you still have to choose a semantic setting (e.g. an effective topos) where every function in the space is interpreted computably [110, 1, 114, 47].

30. Decimal / Base-10 Cauchy Reals: Reals as equivalence classes of decimal expansions; classically standard, but constructively they are just another representation type akin to digit-based reals [103, 10, 117].

31. Apartness / Located Reals (Bishop Style): Reals as located, rounded lower cuts (or Cauchy sequences with an apartness relation). Bishop's "Constructive Analysis" uses a specific flavor of Cauchy reals (regular sequences with a fixed modulus of convergence, usually $1/n$). While often isomorphic to standard Cauchy reals, in a strict intensional type theory, the specific choice of modulus makes the type definition distinct [21, 22, 29, 94].

32. Filter-based Completions: Reals as equivalence classes of Cauchy filters (or regular Cauchy filters) on $\mathbb{Q}$; conceptually close to Dedekind/Cauchy but more topological [43].

33. Locale-of-Reals Variants: Several flavours exist internally: lower reals, upper reals, rounded reals, etc. Some topos texts distinguish a few different "real objects" as default [111, 46, 51, 84].

2. Equivalence Classes (Provable without Countable Choice)

Each class below consists of definitions that are often equivalent in constructive mathematics with AC_ω. In plain Cubical Agda, equivalences between classes (e.g., A and B) may fail.

A. Dedekind-Type Completions

$\mathbb{R}_\mathrm{D}$, $\mathbb{R}_\mathrm{formal}$, Sheaf-theoretic reals, and the real numbers object in a topos all present the Dedekind completion of $\mathbb{Q}$ via localized/topos-theoretic perspectives [67, 51, 52, 84, 49].

B. Cauchy/HIT-Type Completions

$\mathbb{R}_\mathrm{C}$, $\mathbb{R}_\mathrm{FC}$, $\mathbb{R}_\mathrm{I}$, $\mathbb{R}_\mathrm{H}$, $\mathbb{R}_\mathrm{init}$, $\mathbb{R}_\mathrm{ES}$, and (axiomatically) $\mathbb{R}_\mathrm{CauComp}$ represent the Cauchy completion, differing only in presentation (explicit modulus, higher inductive, universal property, or internal closure of $\mathbb{Q}$ within $\mathbb{R}_\mathrm{D}$) [107, 26, 25, 76, 99].

C. Representation (Digit/Continued Fraction) Pre-Reals

$\mathbb{R}_\mathrm{CF}$, $\mathbb{R}_b$, $\mathbb{R}_\mathrm{SD}$, Decimal/Base-10 reals give concrete digit- or fraction-based streams. These are not literally "the reals" until quotiented; they are "presentations of $\mathbb{R}$". Raw types are not fields because of non-unique encodings, but their quotients by the appropriate equivalence relation coincide with Class B [114, 115, 70, 18, 17, 77, 54, 59, 24, 66, 27, 99, 20, 19].

D. Coalgebraic Subspaces

$[0,1]_\mathrm{coalg}$ and $\mathbb{R}^+_\mathrm{coalg}$ describe the unit interval and positive reals as terminal coalgebras. Constructively they model subspaces of $\mathbb{R}_\mathrm{D}$ but do not deliver the entire field without additional principles [40, 4, 80, 97, 41].

E. Generalized Cuts

$\mathbb{R}_\mathrm{L}$, $\mathbb{R}_\mathrm{U}$, and $\mathbb{R}_\mathrm{M}$ relax locatedness/density requirements. They contain $\mathbb{R}$ as a canonical subobject (or as maximal elements) but are bigger structures and not isomorphic to $\mathbb{R}$ as an ordered field [111, 68, 23, 46, 100].

F. Domain-Theoretic

$\mathbb{R}_\mathrm{ID}$ sits in domain/locale theory. Its equivalence to Dedekind reals is not provable in plain Cubical Agda, so it remains a separate class. Note that in frameworks like Abstract Stone Duality or general Topos Theory, these often collapse into Class A (Dedekind-type) via duality results, but internally to Agda without extra axioms, the distinction is maintained [2, 38, 113, 87, 13, 36, 106].

G. Axiomatic/Universal Characterizations

$\mathbb{R}_{\text{term}}$, $\mathbb{R}_{\text{DedComp}}$, and $\mathbb{R}_{\text{Tarski}}$ capture Dedekind-like structures via universal properties or axioms. These are abstract characterizations; one still needs to show they are realized by some concrete construction. They coincide with the usual reals only once classical principles (e.g., choice) are assumed [67, 52, 11].

H. Isolated/Unresolved

$\mathbb{R}_{\text{E}}$ (Eudoxus reals) currently lacks a constructive proof of equivalence with either Dedekind or Cauchy completions. We therefore mark it as isolated [9, 98, 93, 45, 56].

2b. Diagrams

Legend

- Solid arrows: provable embeddings in plain Cubical Agda (no countable choice, no LEM).
- Dashed arrows: relationships that typically require countable choice or classical principles.
- "quotient" labels: representations coincide with Cauchy reals after quotienting the appropriate equivalence.

Diagram: Plain Cubical Agda (no CC/LEM)

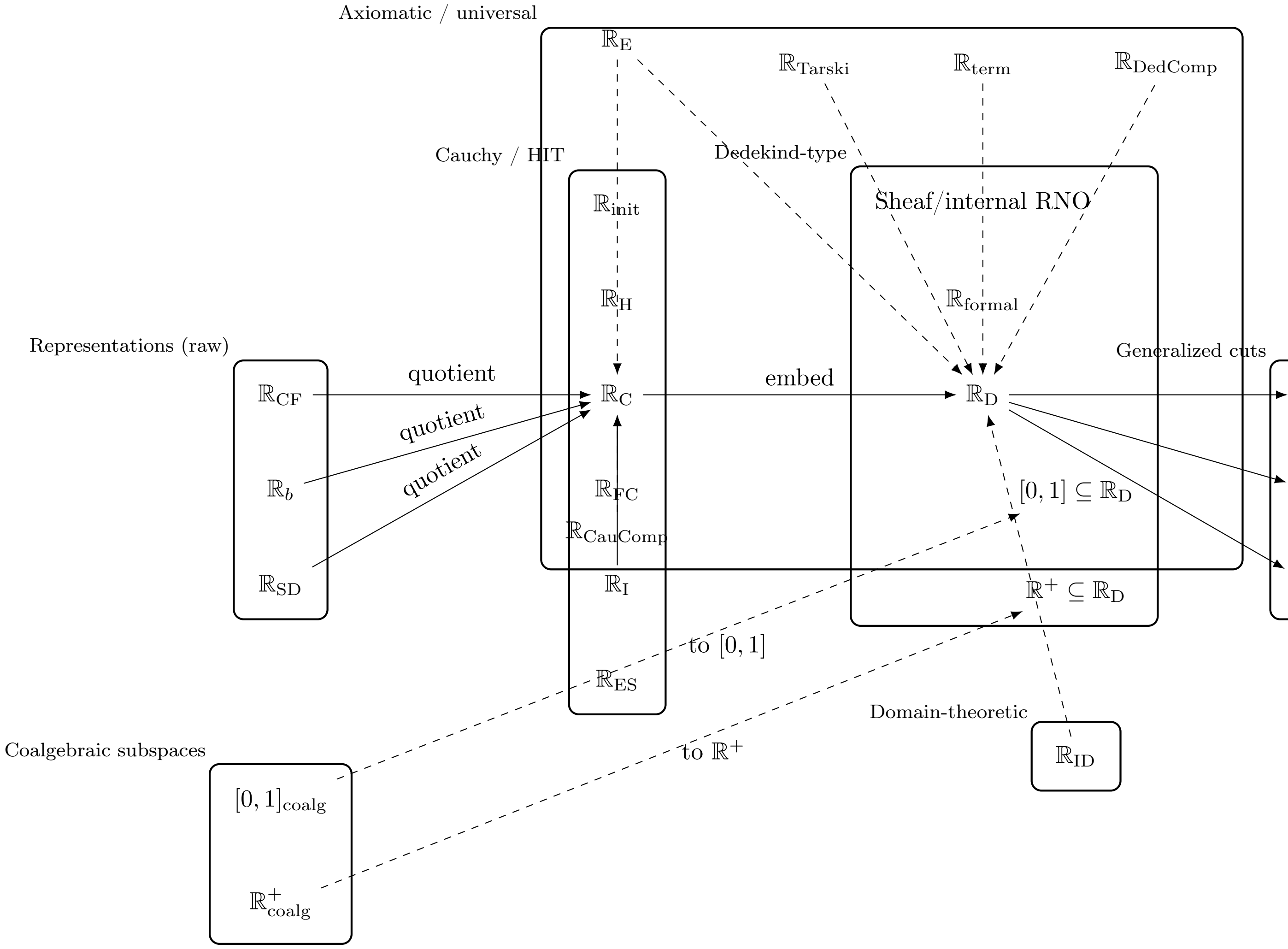
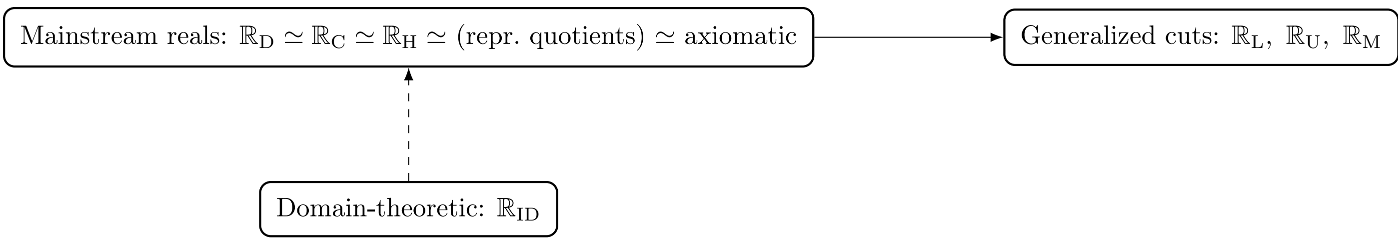


Diagram: With Countable Choice (illustrative collapse)



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Representations (raw) --quotient--> RC --embed--> RD --embed--> RL, RU, RM
     |                              \
     |                               \-- (subspaces) [0,1], R+ (via coalgebras; caveats)
     +-- R_CF, R_b, R_SD

Cauchy/HIT-type: RC, RFC, RI, [RH, R_init (distinct from RD constructively)], RES

Axiomatic: R_term, R_DedComp, R_Tarski, R_CauComp  ... (classically collapse to mainstream)

Domain-theoretic: R_ID  ... (related to RD; not provably equivalent)
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Eudoxus: R_E  ... (isolated; classically related to Cauchy/Dedekind)
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2c. Corrections and Cautions

- Embeddings go $\mathbb{R}_{\mathrm{D}} \to \mathbb{R}_{\mathrm{L}}, \mathbb{R}_{\mathrm{U}}$ by taking lower/upper cuts; the reverse is not constructive.
- Place $\mathbb{R}_{\mathrm{ES}}$ with Cauchy-type (the Cauchy closure inside $\mathbb{R}_{\mathrm{D}}$), not with Dedekind-type.
- Keep $\mathbb{R}_{\mathrm{H}}$ and $\mathbb{R}_{\mathrm{init}}$ together and distinct from $\mathbb{R}_{\mathrm{D}}$ (they coincide only with extra principles).
- Treat $\mathbb{R}_{\mathrm{ID}}$ as domain-theoretic and distinct in plain Cubical Agda; do not assert equivalence to $\mathbb{R}_{\mathrm{D}}$ without additional structure.
- Signed-digit/base-b/continued-fraction representations coincide with Cauchy only after quotienting and with suitable moduli.

3. Canonical Embeddings (No Countable Choice)

- Cauchy into Dedekind: there is a canonical field embedding $\mathbb{R}_{\mathrm{C}} \hookrightarrow \mathbb{R}_{\mathrm{D}}$ that sends each Cauchy real to the located cut defined by its values [22, 63, 64, 73].
- Cauchy/HIT equivalences: $\mathbb{R}_{\mathrm{FC}}$, $\mathbb{R}_{\mathrm{I}}$, $\mathbb{R}_{\mathrm{H}}$, $\mathbb{R}_{\mathrm{init}}$, $\mathbb{R}_{\mathrm{ES}}$, and $\mathbb{R}_{\mathrm{CauComp}}$ are inter-definable and embed into $\mathbb{R}_{\mathrm{C}}$ (hence into $\mathbb{R}_{\mathrm{D}}$) [107, 26, 25, 40].
- Representation quotients: the quotient of $\mathbb{R}_{\mathrm{CF}}$, $\mathbb{R}_b$, or $\mathbb{R}_{\mathrm{SD}}$ by the digit-equivalence relation embeds into $\mathbb{R}_{\mathrm{C}}$. Without quotienting there is a surjection obstruction due to non-unique encodings (Booij [27] shows locators provide a choice-free signed-digit conversion for Dedekind reals with extra structure) [114, 18, 70, 115, 17, 77, 54, 59, 24, 66, 99, 20, 19].
- Dedekind to generalized cuts: taking lower (resp. upper) shadows gives embeddings $\mathbb{R}_{\mathrm{D}} \to \mathbb{R}_{\mathrm{L}}$ and $\mathbb{R}_{\mathrm{D}} \to \mathbb{R}_{\mathrm{U}}$. Composing with double-negation closure embeds into $\mathbb{R}_{\mathrm{M}}$ [111, 83, 82, 68].
- Coalgebraic subspaces: maps $[0,1]_{\mathrm{coalg}} \to [0,1] \subseteq \mathbb{R}_{\mathrm{D}}$ and $\mathbb{R}^{+}_{\mathrm{coalg}} \to \mathbb{R}^{+} \subseteq \mathbb{R}_{\mathrm{D}}$ exist, but surjectivity constructs require additional principles [40, 4, 80].
- Axiomatic to Dedekind/Cauchy: the objects $\mathbb{R}_{\mathrm{term}}$, $\mathbb{R}_{\mathrm{DedComp}}$, and $\mathbb{R}_{\mathrm{Tarski}}$ admit maps into $\mathbb{R}_{\mathrm{D}}$ matching their universal properties, yet converses rely on classical axioms and are not derivable constructively [67, 52, 11, 35].
- Domain to Dedekind: $\mathbb{R}_{\mathrm{ID}}$ maps to $\mathbb{R}_{\mathrm{D}}$ via evaluation at maximal elements, but equivalence is unproven constructively [2, 38, 13, 37].
- Eudoxus: natural maps from $\mathbb{R}_{\mathrm{E}}$ into either $\mathbb{R}_{\mathrm{C}}$ or $\mathbb{R}_{\mathrm{D}}$ require countable choice for surjectivity and therefore remain dashed (non-provable) in this setting [9, 93, 45, 56, 74].

4. Higher Coinductive Types (HCITs) for Signed-Digit Reals

Following Altenkirch [7], we explore defining the signed-digit reals as a *Higher Coinductive Type* (HCIT)—the coinductive dual of HITs/QIITs. An HCIT has constructors (operations) and path constructors (equations built into the type), with its universal property being *terminality* rather than initiality.

4.1 Signature

An HCIT for the signed-digit interval $\mathbb{I}$ is specified by:

- **Operations:** $\mathsf{cons} : \text{Digit} \to \mathbb{I} \to \mathbb{I}, \quad \mathsf{inc}, \mathsf{dec} : \mathbb{I} \to \mathbb{I}$
- **Inc/dec equations** (carry/borrow propagation):

$$\begin{aligned} \mathsf{inc}(\mathsf{cons}\ (-1)\ x) &\equiv \mathsf{cons}\ 0\ (\mathsf{inc}\ x) \\ \mathsf{inc}(\mathsf{cons}\ 0\ x) &\equiv \mathsf{cons}\ (+1)\ (\mathsf{cons}\ 0\ x) \\ \mathsf{inc}(\mathsf{cons}\ (+1)\ x) &\equiv \mathsf{cons}\ (+1)\ (\mathsf{inc}\ x) \end{aligned}$$

  (symmetric for dec)

- **Generation:** $\forall y.\ \exists d\, x.\ y \equiv \mathsf{cons}\ d\ x$
- **Completeness:** $\mathsf{cons}\ 0\ x \equiv \mathsf{inc}\ y \ \Rightarrow\ \mathsf{cons}\ (-1)\ x \equiv \mathsf{cons}\ 0\ y$
- **Separation:** $\mathsf{cons}\ (-1)\ x \equiv \mathsf{cons}\ 0\ y \ \Rightarrow\ \mathsf{cons}\ 0\ x \equiv \mathsf{inc}\ y$
- **Set truncation:** $\mathsf{isSet}(\mathbb{I})$

The carry/borrow equations $\mathsf{cons}\ (+1)\ (\mathsf{cons}\ (-1)\ x) \equiv \mathsf{cons}\ 0\ (\mathsf{inc}\ x)$ are derivable from completeness + separation.

Note: The "no-confusion" axiom $\mathsf{cons}\ (-1)\ x \equiv \mathsf{cons}\ (+1)\ y \to \bot$ from Altenkirch's first attempt (slide 10) is *false* in the quotient model: $[-1 :: 1^\omega]$ and $[+1 :: (-1)^\omega]$ both represent 0.

4.2 Type-theoretic rules

Formation. Given an HCIT signature $\Sigma = (\text{Ops}, \text{Eqs})$:

$$\frac{\Sigma : \mathsf{HCIT\text{-}Sig}}{\mathsf{HCIT}(\Sigma) : \mathsf{Type}}$$

Introduction. Operations become point constructors; equations become path constructors:

$$\frac{d : \text{Digit} \quad x : \mathsf{HCIT}(\Sigma)}{\mathsf{cons}\ d\ x : \mathsf{HCIT}(\Sigma)} \qquad \frac{}{\mathsf{carry} : \mathsf{cons}\ (+1)\ (\mathsf{cons}\ (-1)\ x) \equiv \mathsf{cons}\ 0\ (\mathsf{inc}\ x)}$$

Elimination (terminality). For any Σ-algebra A, there is a unique morphism into $\mathsf{HCIT}(\Sigma)$:

$$\frac{A : \Sigma\text{-Alg}}{\mathsf{corec}_A : A.\text{Carrier} \to \mathsf{HCIT}(\Sigma)}$$

Computation (β). $\mathsf{corec}_A(\mathsf{cons}_A\ d\ x) \equiv \mathsf{cons}\ d\ (\mathsf{corec}_A\ x)$

Uniqueness (η). $f : \Sigma\text{-Hom}\ A\ \mathsf{HCIT}(\Sigma) \ \Rightarrow\ f = \mathsf{corec}_A$

4.3 Encoding in Cubical Agda

Native HCITs are unavailable in Cubical Agda. The quotient-of-codata encoding $\mathbb{I}_{\text{sd}} = \not\models^{\mathbb{N}} / \approx_{\text{sd}}$ recovers the equations but the corecursion principle (terminality) requires lifting through the quotient, which needs AC_ω (countable dependent choice). This matches the obstruction for $\mathsf{lim}_{\mathbb{I}_{\text{sd}}}$ documented in the codebase.

4.4 Comparison with existing approaches

Approach	**Type theory**	**Equalities**	**Corecursion**	**AC_ω-free**
Quotient of codata	Cubical Agda	via quotient	blocked	No
Native HCIT	Cubical + primitive	native	native	Yes
Coconditions	dTT/Narya	via coconditions	via comatching	Yes
Greatest HITs	Clocked Cubical	guarded rec + HITs	guarded rec	Yes

References: Altenkirch [7] (HCIT proposal); Lorenzen–Shulman [62] (coconditions); Kristensen–Møgelberg [61] (Greatest HITs); Ahrens–Capriotti–Spadotti [6] (M-types in HoTT); Shulman et al. (Narya, narya.readthedocs.io) (displayed coinductive types).

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