Real Number Definitions, Equivalence Classes, and Embeddings (No Countable Choice)

This document lists the first twenty-three constructive definitions of real numbers, partitions them into equivalence classes, and records the canonical embeddings. Note that while many equivalences hold in standard constructive mathematics (assuming Countable Choice, AC_ω), in plain Cubical Agda without AC_ω, the hierarchy is more fractured (e.g., Cauchy reals are not provably Dedekind reals). General surveys and formalization resources include [81, 94, 5].

1. Definitions (1–33)

1. $\mathbb{R}_{\mathrm{D}}$: Dedekind reals (located cuts of $\mathbb{Q}$) [27, 16, 17, 21, 53, 93, 107, 62, 86, 70, 42, 28].
2. $\mathbb{R}_{\mathrm{C}}$: Cauchy reals (modulated Cauchy sequences of rationals, quotiented) [16, 17, 51, 63, 60, 58, 65].
3. $\mathbb{R}_{\mathrm{E}}$: Eudoxus reals (almost-homomorphisms $\mathbb{Z} \to \mathbb{Z}$) [7, 84, 87, 79, 37, 47, 61].
4. $\mathbb{R}_{\mathrm{FC}}$ / $\mathbb{R}_{\mathrm{I}}$: fast Cauchy reals / interval reals (Cauchy sequences with explicit moduli or nested rational intervals) [24, 3, 99, 104].
5. $\mathbb{R}_{\mathrm{CF}}$: continued fraction reals (streams of partial quotients) [23, 103, 97].
6. $\mathbb{R}_b$: coinductive base-b reals (digit streams, e.g., binary/decimal) [100].
7. $\mathbb{R}_{\mathrm{SD}}$: signed-digit reals (streams over $\{-1, 0, 1\}$) [6, 34, 100, 105].
8. $\mathbb{R}_{\mathrm{ID}}$: interval domain reals (maximal elements of the interval domain) [98, 73, 45, 11, 13, 31, 30, 29, 91, 32, 12].
9. $\mathbb{R}_{\mathrm{L}}$: lower reals (rounded lower sets of $\mathbb{Q}$) [69, 18, 26].
10. $\mathbb{R}_{\mathrm{U}}$: upper reals (rounded upper sets of $\mathbb{Q}$) [69, 18, 26].
11. $\mathbb{R}_{\mathrm{M}}$: MacNeille reals (double-negation closed cuts) [55, 68, 38, 85].
12. $\mathbb{R}_{\mathrm{H}}$: HIT/HoTT-book reals (higher inductive type with universal property) [92, 67, 20, 19, 10, 78].
13. $\mathbb{R}_{\mathrm{ES}}$: Escardó–Simpson reals (least Cauchy-complete subobject of $\mathbb{R}_{\mathrm{D}}$ containing $\mathbb{Q}$) [33, 19].
14. $\mathbb{R}_{\mathrm{formal}}$: formal/locale reals (points of the locale of reals) [54, 43, 44, 77, 41, 70].
15. $\mathbb{R}_{\mathrm{init}}$: initial sequentially modulated Cauchy-complete Archimedean ordered field [33, 19, 64].
16. $\mathbb{R}_{\mathrm{term}}$: terminal Archimedean ordered field [33, 54, 64]. Items 34 and 35 are distinct definitions with different construction proofs, but they result in the same object if the category is well-behaved.
17. $\mathbb{R}_{\mathrm{DedComp}}$: Dedekind-complete ordered field (axiomatic characterization) [54, 44, 9].

18. $\mathbb{R}_{\text{CauComp}}$: Cauchy-complete ordered field (axiomatic characterization of the Cauchy completion) [17, 94].

19. $\mathbb{R}_{\text{Tarski}}$: Archimedean Tarski group reals (characterization via Tarski's axioms) [89, 28].

20. $[0,1]_{\text{coalg}}$: unit interval as a terminal coalgebra [33, 4, 66, 75].

21. $\mathbb{R}^{+}_{\text{coalg}}$: positive reals as a terminal coalgebra [33, 4, 66].

22. Sheaf-theoretic reals: the internal real numbers object in a topos [54, 44, 86, 41, 70].

23. Real numbers object (RNO) in a topos [54, 44, 86, 41, 70]. Items 41 and 42 are essentially the same mathematical object described from two different points of view (internal language vs. category theory).

24. $\mathbb{R}_{\text{SDG}}$: Smooth Reals (synthetic differential geometry). In Synthetic Differential Geometry, the reals are defined to include "nilpotent" infinitesimals (elements d where $d^2 = 0$ but $d \neq 0$). These are distinct from standard Dedekind/Cauchy reals because they violate the field axiom $x \neq 0 \implies x$ is invertible (nilpotents are not invertible). They are a distinct mathematical object internal to a smooth topos, not isomorphic to the usual Cauchy/Dedekind reals [49, 50, 59, 71].

25. $^*\mathbb{R}$: Hyperreals (non-standard analysis). These include infinite and infinitesimal numbers. While usually constructed classically (using ultrafilters), there are constructive approaches (e.g., Palmgren's constructive non-standard analysis) that result in a structure distinct from $\mathbb{R}_{\text{D}}$ or $\mathbb{R}_{\text{C}}$. They are strict extensions of the ordinary reals [82, 46, 72].

26. Predicative Reals: In systems stricter than Agda (like those prohibiting impredicativity), Dedekind cuts must be restricted (e.g., to "generalized" or "weak" cuts) to avoid circular definitions. The document hints at this with lower/upper reals (items 9, 10), but specific predicative formalizations often stand alone [36, 22, 74].

27. **No**: Surreal Numbers (Conway's construction). While the Surreals contain the Reals, the "Real subset" of the Surreals is a valid constructive definition of the reals. Inside **No** there is a canonical embedded copy of $\mathbb{R}$; this embedding can be used as yet another definition of the real line [25, 40, 56, 48, 106].

28. Geometric Reals: Defined synthetically in Euclidean Geometry (e.g., Tarski's axioms for geometry, or Hilbert's axioms). Defined as "points on a line" rather than arithmetically. Constructively, relating "points on a line" to "Dedekind cuts" is a non-trivial project (requires the Cantor-Dedekind axiom) [89, 90, 14, 76, 101].

29. Computable Reals (Turing): Specifically defined as "Turing machines that output digits". This is distinct from $\mathbb{R}_{\text{C}}$ because $\mathbb{R}_{\text{C}}$ allows *any* function, whereas Computable Reals restrict the functions to computable ones. In strongly normalizing type theories, every *definable* function is computable (meta-theoretically), so formalising computable reals inside such a system is natural. But this does not by itself make $\mathbb{R}_{\text{C}}$ "the same" as the usual Cauchy reals object; you still have to choose a semantic setting (e.g. an effective topos) where every function in the space is interpreted computably [95, 1, 99, 39].

30. Decimal / Base-10 Cauchy Reals: Reals as equivalence classes of decimal expansions; classically standard, but constructively they are just another representation type akin to digit-based reals [88, 8, 102].

31. Apartness / Located Reals (Bishop Style): Reals as located, rounded lower cuts (or Cauchy sequences with an apartness relation). Bishop's "Constructive Analysis" uses a specific flavor of Cauchy reals (regular sequences with a fixed modulus of convergence, usually $1/n$). While often isomorphic to standard Cauchy reals, in a strict intensional type theory, the specific choice of modulus makes the type definition distinct [16, 17, 22, 80].

32. Filter-based Completions: Reals as equivalence classes of Cauchy filters (or regular Cauchy filters) on $\mathbb{Q}$; conceptually close to Dedekind/Cauchy but more topological [35].

33. Locale-of-Reals Variants: Several flavours exist internally: lower reals, upper reals, rounded reals, etc. Some topos texts distinguish a few different "real objects" as default [96, 38, 43, 70].

2. Equivalence Classes (Provable without Countable Choice)

Each class below consists of definitions that are often equivalent in constructive mathematics with AC_ω. In plain Cubical Agda, equivalences between classes (e.g., A and B) may fail.

A. Dedekind-Type Completions

$\mathbb{R}_{\mathrm{D}}$, $\mathbb{R}_{\mathrm{formal}}$, Sheaf-theoretic reals, and the real numbers object in a topos all present the Dedekind completion of $\mathbb{Q}$ via localized/topos-theoretic perspectives [54, 43, 44, 70, 41].

B. Cauchy/HIT-Type Completions

$\mathbb{R}_{\mathrm{C}}$, $\mathbb{R}_{\mathrm{FC}}$, $\mathbb{R}_{\mathrm{I}}$, $\mathbb{R}_{\mathrm{H}}$, $\mathbb{R}_{\mathrm{init}}$, $\mathbb{R}_{\mathrm{ES}}$, and (axiomatically) $\mathbb{R}_{\mathrm{CauComp}}$ represent the Cauchy completion, differing only in presentation (explicit modulus, higher inductive, universal property, or internal closure of $\mathbb{Q}$ within $\mathbb{R}_{\mathrm{D}}$) [92, 20, 19, 63].

C. Representation (Digit/Continued Fraction) Pre-Reals

$\mathbb{R}_{\mathrm{CF}}$, $\mathbb{R}_b$, $\mathbb{R}_{\mathrm{SD}}$, Decimal/Base-10 reals give concrete digit- or fraction-based streams. These are not literally "the reals" until quotiented; they are "presentations of $\mathbb{R}$". Raw types are not fields because of non-unique encodings, but their quotients by the appropriate equivalence relation coincide with Class B [99, 100, 57, 15].

D. Coalgebraic Subspaces

$[0,1]_{\mathrm{coalg}}$ and $\mathbb{R}^+_{\mathrm{coalg}}$ describe the unit interval and positive reals as terminal coalgebras. Constructively they model subspaces of $\mathbb{R}_{\mathrm{D}}$ but do not deliver the entire field without additional principles [33, 4, 66, 83].

E. Generalized Cuts

$\mathbb{R}_{\mathrm{L}}$, $\mathbb{R}_{\mathrm{U}}$, and $\mathbb{R}_{\mathrm{M}}$ relax locatedness/density requirements. They contain $\mathbb{R}$ as a canonical subobject (or as maximal elements) but are bigger structures and not isomorphic to $\mathbb{R}$ as an ordered field [96, 55, 18, 38, 85].

F. Domain-Theoretic

$\mathbb{R}_{\mathrm{ID}}$ sits in domain/locale theory. Its equivalence to Dedekind reals is not provable in plain Cubical Agda, so it remains a separate class. Note that in frameworks like Abstract Stone Duality or general Topos Theory, these often collapse into Class A (Dedekind-type) via duality results, but internally to Agda without extra axioms, the distinction is maintained [2, 31, 98, 73, 11, 29, 91].

G. Axiomatic/Universal Characterizations

$\mathbb{R}_{\text{term}}$, $\mathbb{R}_{\text{DedComp}}$, and $\mathbb{R}_{\text{Tarski}}$ capture Dedekind-like structures via universal properties or axioms. These are abstract characterizations; one still needs to show they are realized by some concrete construction. They coincide with the usual reals only once classical principles (e.g., choice) are assumed [54, 44, 9].

H. Isolated/Unresolved

$\mathbb{R}_{\text{E}}$ (Eudoxus reals) currently lacks a constructive proof of equivalence with either Dedekind or Cauchy completions. We therefore mark it as isolated [7, 84, 79, 37, 47].

3. Canonical Embeddings (No Countable Choice)

- Cauchy into Dedekind: there is a canonical field embedding $\mathbb{R}_{\text{C}} \hookrightarrow \mathbb{R}_{\text{D}}$ that sends each Cauchy real to the located cut defined by its values [17, 51, 52, 60].
- Cauchy/HIT equivalences: $\mathbb{R}_{\text{FC}}$, $\mathbb{R}_{\text{I}}$, $\mathbb{R}_{\text{H}}$, $\mathbb{R}_{\text{init}}$, $\mathbb{R}_{\text{ES}}$, and $\mathbb{R}_{\text{CauComp}}$ are inter-definable and embed into $\mathbb{R}_{\text{C}}$ (hence into $\mathbb{R}_{\text{D}}$) [92, 20, 19, 33].
- Representation quotients: the quotient of $\mathbb{R}_{\text{CF}}$, $\mathbb{R}_{b}$, or $\mathbb{R}_{\text{SD}}$ by the digit-equivalence relation embeds into $\mathbb{R}_{\text{C}}$. Without quotienting there is a surjection obstruction due to non-unique encodings [99, 15, 57, 100].
- Dedekind to generalized cuts: taking lower (resp. upper) shadows gives embeddings $\mathbb{R}_{\text{D}} \to \mathbb{R}_{\text{L}}$ and $\mathbb{R}_{\text{D}} \to \mathbb{R}_{\text{U}}$. Composing with double-negation closure embeds into $\mathbb{R}_{\text{M}}$ [96, 69, 68, 55].
- Coalgebraic subspaces: maps $[0,1]_{\text{coalg}} \to [0,1] \subseteq \mathbb{R}_{\text{D}}$ and $\mathbb{R}^{+}_{\text{coalg}} \to \mathbb{R}^{+} \subseteq \mathbb{R}_{\text{D}}$ exist, but surjectivity constructs require additional principles [33, 4, 66].
- Axiomatic to Dedekind/Cauchy: the objects $\mathbb{R}_{\text{term}}$, $\mathbb{R}_{\text{DedComp}}$, and $\mathbb{R}_{\text{Tarski}}$ admit maps into $\mathbb{R}_{\text{D}}$ matching their universal properties, yet converses rely on classical axioms and are not derivable constructively [54, 44, 9, 28].
- Domain to Dedekind: $\mathbb{R}_{\text{ID}}$ maps to $\mathbb{R}_{\text{D}}$ via evaluation at maximal elements, but equivalence is unproven constructively [2, 31, 11, 30].
- Eudoxus: natural maps from $\mathbb{R}_{\text{E}}$ into either $\mathbb{R}_{\text{C}}$ or $\mathbb{R}_{\text{D}}$ require countable choice for surjectivity and therefore remain dashed (non-provable) in this setting [7, 79, 37, 47, 61].

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