Equivalence Classes of Real Numbers (Constructive, Cubical Agda)

Legend

- Solid arrows: provable embeddings in plain Cubical Agda (no countable choice, no LEM).
- Dashed arrows: relationships that typically require countable choice or classical principles.
- "quotient" labels: representations coincide with Cauchy reals after quotienting the appropriate equivalence.

Diagram: Plain Cubical Agda (no CC/LEM)

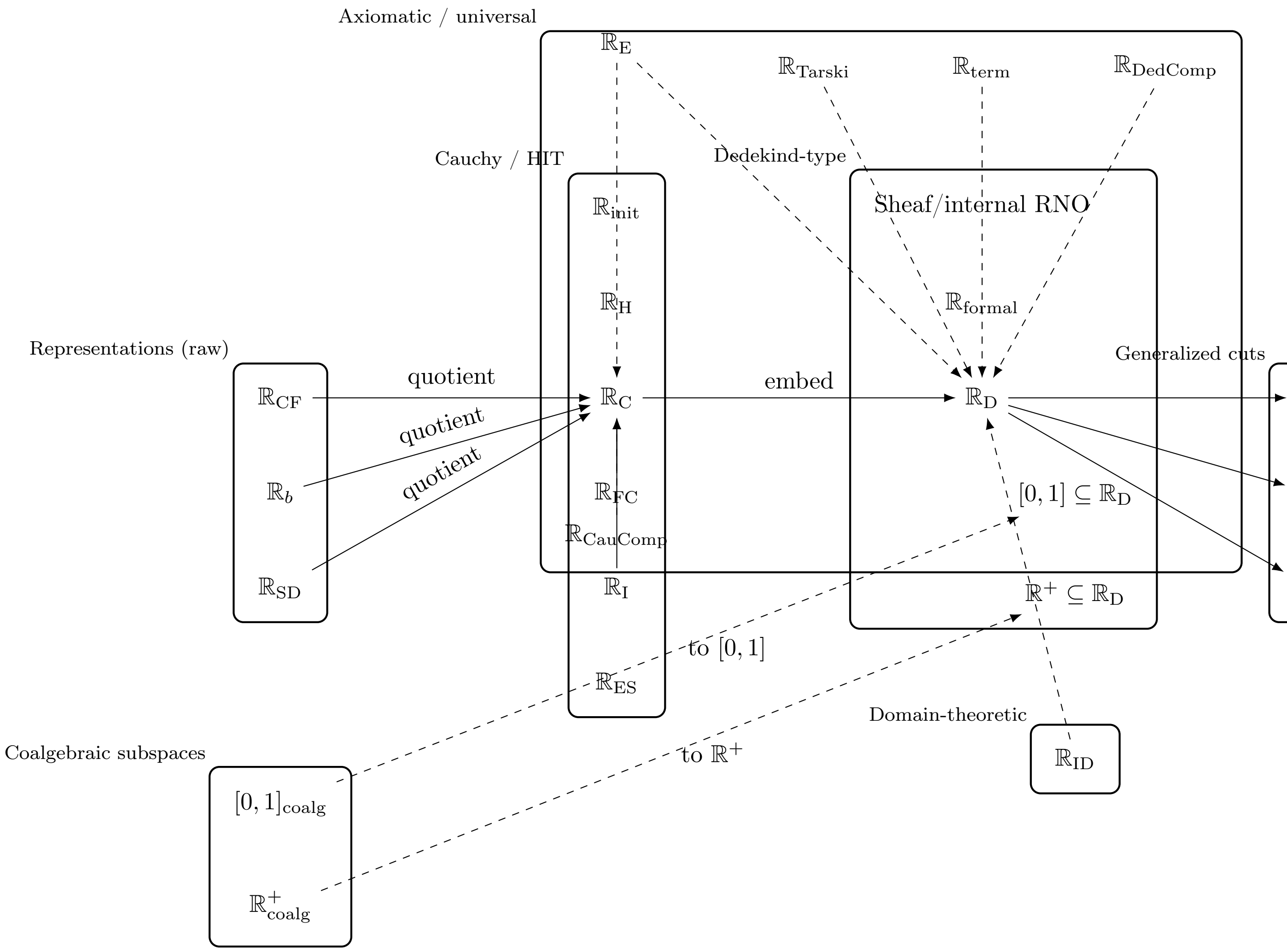

Diagram: With Countable Choice (illustrative collapse)

$$
\begin{tikzcd}
\text{Mainstream reals: } \mathbb{R}_{\mathrm{D}} \simeq \mathbb{R}_{\mathrm{C}} \simeq \mathbb{R}_{\mathrm{H}} \simeq \text{(repr. quotients)} \simeq \text{axiomatic} \arrow[r] & \text{Generalized cuts: } \mathbb{R}_{\mathrm{L}},\ \mathbb{R}_{\mathrm{U}},\ \mathbb{R}_{\mathrm{M}} \\
\text{Domain-theoretic: } \mathbb{R}_{\mathrm{ID}} \arrow[u, dashed] &
\end{tikzcd}
$$

Summary: Agreements

- Canonical embedding $\mathbb{R}_{\mathrm{C}} \to \mathbb{R}_{\mathrm{D}}$ is provable; the reverse is not in Cubical Agda without countable choice.
- Representation streams (continued fractions, base-b, signed digits) require quotienting to address non-unique encodings; their quotients align with Cauchy-style reals.
- Lower/Upper/MacNeille reals are weaker objects (not fields) and sit below the mainstream reals.

Summary: Corrections / Cautions

- Embeddings go $\mathbb{R}_{\mathrm{D}} \to \mathbb{R}_{\mathrm{L}},\ \mathbb{R}_{\mathrm{U}}$ by taking lower/upper cuts; the reverse is not constructive.
- Place $\mathbb{R}_{\mathrm{ES}}$ with Cauchy-type (the Cauchy closure inside $\mathbb{R}_{\mathrm{D}}$), not with Dedekind-type.
- Keep $\mathbb{R}_{\mathrm{H}}$ and $\mathbb{R}_{\mathrm{init}}$ together and distinct from $\mathbb{R}_{\mathrm{D}}$ (they coincide only with extra principles).
- Treat $\mathbb{R}_{\mathrm{ID}}$ as domain-theoretic and distinct in plain Cubical Agda; do not assert equivalence to $\mathbb{R}_{\mathrm{D}}$ without additional structure.
- Signed-digit/base-b/continued-fraction representations coincide with Cauchy only after quotienting and with suitable moduli.

Recommended Partition (plain Cubical Agda)

- Dedekind-type: $\mathbb{R}_{\mathrm{D}}$, $\mathbb{R}_{\mathrm{formal}}$, Sheaf/internal RNO.
- Cauchy/HIT-type: $\mathbb{R}_{\mathrm{C}}$, $\mathbb{R}_{\mathrm{FC}}$, $\mathbb{R}_{\mathrm{I}}$, $\mathbb{R}_{\mathrm{H}}$, $\mathbb{R}_{\mathrm{init}}$, and (axiomatization) $\mathbb{R}_{\mathrm{CauComp}}$.
- Escardó–Simpson: $\mathbb{R}_{\mathrm{ES}}$ (Cauchy closure inside $\mathbb{R}_{\mathrm{D}}$; group with Cauchy-type).
- Representations (raw, quotient to Cauchy): $\mathbb{R}_{\mathrm{CF}}$, $\mathbb{R}_{b}$, $\mathbb{R}_{\mathrm{SD}}$.
- Coalgebraic subspaces: $[0,1]_{\mathrm{coalg}}$, $\mathbb{R}^{+}_{\mathrm{coalg}}$ (subspaces of $\mathbb{R}_{\mathrm{D}}$ with constructive caveats).
- Generalized cuts: $\mathbb{R}_{\mathrm{L}}$, $\mathbb{R}_{\mathrm{U}}$, $\mathbb{R}_{\mathrm{M}}$ (weaker, not fields).
- Domain-theoretic: $\mathbb{R}_{\mathrm{ID}}$ (related but not provably equivalent to $\mathbb{R}_{\mathrm{D}}$).
- Axiomatic/universal: $\mathbb{R}_{\mathrm{term}}$, $\mathbb{R}_{\mathrm{DedComp}}$, $\mathbb{R}_{\mathrm{Tarski}}$ (and $\mathbb{R}_{\mathrm{CauComp}}$ if not grouped above) — classically collapse with mainstream reals.
- Isolated/uncertain: $\mathbb{R}_{\mathrm{E}}$ (Eudoxus), pending additional principles for equivalence.

ASCII Fallback

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Representations (raw) --quotient--> RC --embed--> RD --embed--> RL, RU, RM
     |                               \
     |                                \-- (subspaces) [0,1], R+ (via coalgebras; caveats)
     +-- R_CF, R_b, R_SD

Cauchy/HIT-type: RC, RFC, RI, [RH, R_init (distinct from RD constructively)], RES

Axiomatic: R_term, R_DedComp, R_Tarski, R_CauComp  ... (classically collapse to mainstream)

Domain-theoretic: R_ID  ... (related to RD; not provably equivalent)

Eudoxus: R_E  ... (isolated; classically related to Cauchy/Dedekind)
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