

Real Number Definitions, Equivalence Classes, and Embeddings (No Countable Choice)

This document lists the first twenty-three constructive definitions of real numbers, partitions them into equivalence classes, and records the canonical embeddings. Note that while many equivalences hold in standard constructive mathematics (assuming Countable Choice, \mathbf{AC}_ω), in plain Cubical Agda without \mathbf{AC}_ω , the hierarchy is more fractured (e.g., Cauchy reals are not provably Dedekind reals). General surveys and formalization resources include [80, 93, 5].

1. Definitions (1–33)

1. \mathbb{R}_D : Dedekind reals (located cuts of \mathbb{Q}) [27, 16, 17, 21, 52, 92, 106, 61, 85, 69, 41, 28].
2. \mathbb{R}_C : Cauchy reals (modulated Cauchy sequences of rationals, quotiented) [16, 17, 50, 62, 59, 57, 64].
3. \mathbb{R}_E : Eudoxus reals (almost-homomorphisms $\mathbb{Z} \rightarrow \mathbb{Z}$) [6, 83, 86, 78, 36, 46, 60].
4. \mathbb{R}_{FC} / \mathbb{R}_I : fast Cauchy reals / interval reals (Cauchy sequences with explicit moduli or nested rational intervals) [24, 3, 15, 98, 104].
5. \mathbb{R}_{CF} : continued fraction reals (streams of partial quotients) [23, 103, 96].
6. \mathbb{R}_b : coinductive base- b reals (digit streams, e.g., binary/decimal) [15, 56, 99].
7. \mathbb{R}_{SD} : signed-digit reals (streams over $\{-1, 0, 1\}$) [100, 12, 32, 7].
8. \mathbb{R}_{ID} : interval domain reals (maximal elements of the interval domain) [97, 72, 44, 11, 13, 31, 30, 29, 90].
9. \mathbb{R}_L : lower reals (rounded lower sets of \mathbb{Q}) [68, 18, 26].
10. \mathbb{R}_U : upper reals (rounded upper sets of \mathbb{Q}) [68, 18, 26].
11. \mathbb{R}_M : MacNeille reals (double-negation closed cuts) [54, 67, 37, 84].
12. \mathbb{R}_H : HIT/HoTT-book reals (higher inductive type with universal property) [91, 66, 20, 19, 10, 77].
13. \mathbb{R}_{ES} : Escardó–Simpson reals (least Cauchy-complete subobject of \mathbb{R}_D containing \mathbb{Q}) [33, 19].
14. $\mathbb{R}_{\text{formal}}$: formal/locale reals (points of the locale of reals) [53, 42, 43, 76, 40, 69].
15. \mathbb{R}_{init} : initial sequentially modulated Cauchy-complete Archimedean ordered field [33, 19, 63].
16. \mathbb{R}_{term} : terminal Archimedean ordered field [33, 53, 63]. Items 34 and 35 are distinct definitions with different construction proofs, but they result in the same object if the category is well-behaved.
17. $\mathbb{R}_{\text{DedComp}}$: Dedekind-complete ordered field (axiomatic characterization) [53, 43, 9].

18. $\mathbb{R}_{\text{CauComp}}$: Cauchy-complete ordered field (axiomatic characterization of the Cauchy completion) [17, 93].
19. $\mathbb{R}_{\text{Tarski}}$: Archimedean Tarski group reals (characterization via Tarski’s axioms) [88, 28].
20. $[0, 1]_{\text{coalg}}$: unit interval as a terminal coalgebra [33, 4, 65, 74].
21. $\mathbb{R}_{\text{coalg}}^+$: positive reals as a terminal coalgebra [33, 4, 65].
22. Sheaf-theoretic reals: the internal real numbers object in a topos [53, 43, 85, 40, 69].
23. Real numbers object (RNO) in a topos [53, 43, 85, 40, 69]. Items 41 and 42 are essentially the same mathematical object described from two different points of view (internal language vs. category theory).
24. \mathbb{R}_{SDG} : Smooth Reals (synthetic differential geometry). In Synthetic Differential Geometry, the reals are defined to include “nilpotent” infinitesimals (elements d where $d^2 = 0$ but $d \neq 0$). These are distinct from standard Dedekind/Cauchy reals because they violate the field axiom $x \neq 0 \implies x$ is invertible (nilpotents are not invertible). They are a distinct mathematical object internal to a smooth topos, not isomorphic to the usual Cauchy/Dedekind reals [48, 49, 58, 70].
25. $^*\mathbb{R}$: Hyperreals (non-standard analysis). These include infinite and infinitesimal numbers. While usually constructed classically (using ultrafilters), there are constructive approaches (e.g., Palmgren’s constructive non-standard analysis) that result in a structure distinct from \mathbb{R}_{D} or \mathbb{R}_{C} . They are strict extensions of the ordinary reals [81, 45, 71].
26. Predicative Reals: In systems stricter than Agda (like those prohibiting impredicativity), Dedekind cuts must be restricted (e.g., to “generalized” or “weak” cuts) to avoid circular definitions. The document hints at this with lower/upper reals (items 9, 10), but specific predicative formalizations often stand alone [35, 22, 73].
27. **No**: Surreal Numbers (Conway’s construction). While the Surreals contain the Reals, the “Real subset” of the Surreals is a valid constructive definition of the reals. Inside **No** there is a canonical embedded copy of \mathbb{R} ; this embedding can be used as yet another definition of the real line [25, 39, 55, 47, 105].
28. Geometric Reals: Defined synthetically in Euclidean Geometry (e.g., Tarski’s axioms for geometry, or Hilbert’s axioms). Defined as “points on a line” rather than arithmetically. Constructively, relating “points on a line” to “Dedekind cuts” is a non-trivial project (requires the Cantor-Dedekind axiom) [88, 89, 14, 75, 101].
29. Computable Reals (Turing): Specifically defined as “Turing machines that output digits”. This is distinct from \mathbb{R}_{C} because \mathbb{R}_{C} allows *any* function, whereas Computable Reals restrict the functions to computable ones. In strongly normalizing type theories, every *definable* function is computable (meta-theoretically), so formalising computable reals inside such a system is natural. But this does not by itself make \mathbb{R}_{C} “the same” as the usual Cauchy reals object; you still have to choose a semantic setting (e.g. an effective topos) where every function in the space is interpreted computably [94, 1, 98, 38].
30. Decimal / Base-10 Cauchy Reals: Reals as equivalence classes of decimal expansions; classically standard, but constructively they are just another representation type akin to digit-based reals [87, 8, 102].

31. Apartness / Located Reals (Bishop Style): Reals as located, rounded lower cuts (or Cauchy sequences with an apartness relation). Bishop’s “Constructive Analysis” uses a specific flavor of Cauchy reals (regular sequences with a fixed modulus of convergence, usually $1/n$). While often isomorphic to standard Cauchy reals, in a strict intensional type theory, the specific choice of modulus makes the type definition distinct [16, 17, 22, 79].
32. Filter-based Completions: Reals as equivalence classes of Cauchy filters (or regular Cauchy filters) on \mathbb{Q} ; conceptually close to Dedekind/Cauchy but more topological [34].
33. Locale-of-Reals Variants: Several flavours exist internally: lower reals, upper reals, rounded reals, etc. Some topos texts distinguish a few different “real objects” as default [95, 37, 42, 69].

2. Equivalence Classes (Provable without Countable Choice)

Each class below consists of definitions that are often equivalent in constructive mathematics with AC_ω . In plain Cubical Agda, equivalences between classes (e.g., A and B) may fail.

A. Dedekind-Type Completions

\mathbb{R}_D , $\mathbb{R}_{\text{formal}}$, Sheaf-theoretic reals, and the real numbers object in a topos all present the Dedekind completion of \mathbb{Q} via localized/topos-theoretic perspectives [53, 42, 43, 69, 40].

B. Cauchy/HIT-Type Completions

\mathbb{R}_C , \mathbb{R}_{FC} , \mathbb{R}_I , \mathbb{R}_H , \mathbb{R}_{init} , \mathbb{R}_{ES} , and (axiomatically) $\mathbb{R}_{\text{CauComp}}$ represent the Cauchy completion, differing only in presentation (explicit modulus, higher inductive, universal property, or internal closure of \mathbb{Q} within \mathbb{R}_D) [91, 20, 19, 62].

C. Representation (Digit/Continued Fraction) Pre-Reals

\mathbb{R}_{CF} , \mathbb{R}_b , \mathbb{R}_{SD} , Decimal/Base-10 reals give concrete digit- or fraction-based streams. These are not literally “the reals” until quotiented; they are “presentations of \mathbb{R} ”. Raw types are not fields because of non-unique encodings, but their quotients by the appropriate equivalence relation coincide with Class B [98, 15, 100, 56].

D. Coalgebraic Subspaces

$[0, 1]_{\text{coalg}}$ and $\mathbb{R}_{\text{coalg}}^+$ describe the unit interval and positive reals as terminal coalgebras. Constructively they model subspaces of \mathbb{R}_D but do not deliver the entire field without additional principles [33, 4, 65, 82].

E. Generalized Cuts

\mathbb{R}_L , \mathbb{R}_U , and \mathbb{R}_M relax locatedness/density requirements. They contain \mathbb{R} as a canonical subobject (or as maximal elements) but are bigger structures and not isomorphic to \mathbb{R} as an ordered field [95, 54, 18, 37, 84].

F. Domain-Theoretic

\mathbb{R}_D sits in domain/locale theory. Its equivalence to Dedekind reals is not provable in plain Cubical Agda, so it remains a separate class. Note that in frameworks like Abstract Stone Duality or general Topos Theory, these often collapse into Class A (Dedekind-type) via duality

results, but internally to Agda without extra axioms, the distinction is maintained [2, 31, 97, 72, 11, 29, 90].

G. Axiomatic/Universal Characterizations

\mathbb{R}_{term} , $\mathbb{R}_{\text{DedComp}}$, and $\mathbb{R}_{\text{Tarski}}$ capture Dedekind-like structures via universal properties or axioms. These are abstract characterizations; one still needs to show they are realized by some concrete construction. They coincide with the usual reals only once classical principles (e.g., choice) are assumed [53, 43, 9].

H. Isolated/Unresolved

\mathbb{R}_{E} (Eudoxus reals) currently lacks a constructive proof of equivalence with either Dedekind or Cauchy completions. We therefore mark it as isolated [6, 83, 78, 36, 46].

3. Canonical Embeddings (No Countable Choice)

- Cauchy into Dedekind: there is a canonical field embedding $\mathbb{R}_{\text{C}} \hookrightarrow \mathbb{R}_{\text{D}}$ that sends each Cauchy real to the located cut defined by its values [17, 50, 51, 59].
- Cauchy/HIT equivalences: \mathbb{R}_{FC} , \mathbb{R}_{I} , \mathbb{R}_{H} , \mathbb{R}_{init} , \mathbb{R}_{ES} , and $\mathbb{R}_{\text{CauComp}}$ are inter-definable and embed into \mathbb{R}_{C} (hence into \mathbb{R}_{D}) [91, 20, 19, 33].
- Representation quotients: the quotient of \mathbb{R}_{CF} , \mathbb{R}_b , or \mathbb{R}_{SD} by the digit-equivalence relation embeds into \mathbb{R}_{C} . Without quotienting there is a surjection obstruction due to non-unique encodings [98, 15, 56, 100].
- Dedekind to generalized cuts: taking lower (resp. upper) shadows gives embeddings $\mathbb{R}_{\text{D}} \rightarrow \mathbb{R}_{\text{L}}$ and $\mathbb{R}_{\text{D}} \rightarrow \mathbb{R}_{\text{U}}$. Composing with double-negation closure embeds into \mathbb{R}_{M} [95, 68, 67, 54].
- Coalgebraic subspaces: maps $[0, 1]_{\text{coalg}} \rightarrow [0, 1] \subseteq \mathbb{R}_{\text{D}}$ and $\mathbb{R}_{\text{coalg}}^+ \rightarrow \mathbb{R}^+ \subseteq \mathbb{R}_{\text{D}}$ exist, but surjectivity constructs require additional principles [33, 4, 65].
- Axiomatic to Dedekind/Cauchy: the objects \mathbb{R}_{term} , $\mathbb{R}_{\text{DedComp}}$, and $\mathbb{R}_{\text{Tarski}}$ admit maps into \mathbb{R}_{D} matching their universal properties, yet converses rely on classical axioms and are not derivable constructively [53, 43, 9, 28].
- Domain to Dedekind: \mathbb{R}_{ID} maps to \mathbb{R}_{D} via evaluation at maximal elements, but equivalence is unproven constructively [2, 31, 11, 30].
- Eudoxus: natural maps from \mathbb{R}_{E} into either \mathbb{R}_{C} or \mathbb{R}_{D} require countable choice for surjectivity and therefore remain dashed (non-provable) in this setting [6, 78, 36, 46, 60].

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