A Cubical Path from Algebra to Analysis

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Abstract

We present contributions to the `agda/cubical` library for constructive analysis. Starting from *pseudolattices* and *ordered commutative rings*—algebraic structures adapted from the HoTT Book's ordered Heyting fields—we build a unified framework that applies uniformly to integers, dyadic rationals, rationals, and reals. We also outline ongoing work on *premetric spaces* following Gilbert's approach, generalizing the closeness relation to use the positive cone of an ordered commutative ring rather than just the positive rationals.

1 Introduction

Constructive analysis in type theory requires careful attention to the algebraic and order-theoretic structures underlying the real numbers. Different constructions of the reals (Cauchy sequences [5], Dedekind cuts [9, 3], Eudoxus reals [2, 11], or HIITs [6]) share common properties: any reasonable definition should be an Archimedean field where all Cauchy sequences of rationals converge [7]. Cubical Agda provides computational univalence and higher inductive types, enabling formalizations where equality has computational content.

We contribute to the `agda/cubical` library [23] with two main developments: (1) algebraic structures for ordered commutative rings and pseudolattices [17, 16, 15], providing a unified framework for reasoning about ordered number types; and (2) ongoing work on premetric spaces following Gilbert [12], with a generalization that allows closeness relations valued in arbitrary ordered commutative rings rather than just the positive rationals. In particular, the HIIT construction of the reals yields Cauchy-completeness directly, unlike naive Cauchy sequence constructions which may fail to be complete [14].

2 Ordered Algebraic Structures

We begin with order structure. A *pseudolattice* combines a poset with binary meet ($\wedge$) and join ($\vee$) operations. (The `agda/cubical` library [23] reserves "lattice" for bounded pseudolattices.) We define `IsPseudolattice` with a `Properties` module establishing standard lattice laws and connection lemmas such as $(a \leq b) \simeq (a = a \wedge b)$.

Building on this, an *ordered commutative ring* (OCR) combines commutative ring structure with compatible order structure, adapting the HoTT Book's ordered Heyting fields [19] by dropping invertibility axioms. The record `IsOrderedCommRing` bundles commutative ring and pseudolattice structures with a compatible strict ordering $<$ satisfying weak linearity ($x < z \to (x < y) \vee (y < z)$), mixed transitivity, monotonicity of operations, and positivity conditions including $0 < 1$ and closure of positives under multiplication. These axioms are not all independent [7]; when an OCR additionally forms a Heyting field with apartness, we obtain an *ordered Heyting field* as in the HoTT Book.

We use rings rather than fields for generality: the same structure applies to integers, dyadic rationals [13], rationals, and reals.

Finally, an *Archimedean ring* is an OCR with a cancellability property and the Archimedean property: for all x, y such that $0 < x$ and $0 < y$, there exists $n : \mathbb{N}^+$ such that $x < n \cdot y$.

3 Premetric Spaces

With the algebraic foundations in place, we turn to analysis. Following Gilbert [12], a *premetric space* is a type A equipped with a closeness relation $\approx : \mathbb{Q}^+ \to A \to A \to \mathsf{Prop}$, interpreted as "distance$(x, y) < \varepsilon$", satisfying reflexivity, symmetry, separatedness, triangularity, and roundedness. The HIIT (higher inductive-inductive type) completion $\mathcal{C}(T)$ of a premetric space T mutually defines a closeness relation on $\mathcal{C}(T)$, with point constructors $\eta : T \to \mathcal{C}(T)$ (embedding) and lim (limits of Cauchy approximations), and a path constructor identifying arbitrarily close points. This forms an idempotent monad on premetric spaces with Lipschitz functions, yielding the reals as completion of rationals or dyadics.

Our algebraic structures enable a generalization: $\mathbb{Q}^+$ can be replaced with the positive cone of an arbitrary OCR. We show that any OCR in which $2 := 1 + 1$ has a multiplicative inverse admits a premetric structure valued in its own positive cone. In particular, $\mathbb{Q}$ is a standard premetric space. Moreover, any Archimedean ring can be turned into a premetric space: for such a ring, we would like to define a closeness relation by $|x - y| < \varepsilon$ (where $|z| := z \vee -z$). However, since the rationals do not embed into every Archimedean ring, we multiply both sides by the denominator of $\varepsilon = \frac{a}{b}$, obtaining $b \cdot |x - y| < a$, which can be expressed using only the Archimedean structure.

In parallel with this generalization, a comprehensive development of Cauchy reals following HoTT Book Chapter 11.3 [22] includes the Riemann integral, the fundamental theorem of calculus, the mean value theorem, nth roots, and trigonometric functions.

Current work includes adapting the Closeness module [22] to the generalized framework and proving that the HoTT Book reals are initial among Cauchy-complete Archimedean ordered fields [19, 10, 4].

4 Conclusion and Future Work

From pseudolattices through ordered commutative rings and Archimedean rings to premetric spaces, we trace a path from algebra to analysis. Unlike approaches that rely on setoids [8, 20, 18], we leverage computational univalence and higher inductive types to work directly with identity types.

For the development of the theory, we follow Bishop [5] and Booij [7], adapting the proofs for arbitrary Archimedean ordered fields with appropriate completeness conditions.

Future work includes implementing the premetric generalization, proving monadicity of the HIIT completion, and comparison with the Agda-UniMath library's approach to metric spaces [1]. Our long-term goal is to formalize a constructive version of the Picard-Lindelöf theorem, establishing existence and uniqueness of solutions to ordinary differential equations, adapting the proof in Booij's thesis [7, 21].

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