APPM 5360, Spring 2023 - Written Homework 7

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1. We are interested in finding the Lagrangian dual of the problem P', where P' is given by

$$\begin{aligned} \min_{X,z} \quad & \frac{1}{2}\|X - Y\|_F^2 \\ \text{subject to} \quad & g(z) \leq \tau, \\ & L(X) = z. \end{aligned}$$

Here, $X \in \mathbb{R}^{n_1 \times n_2}$, $z \in \mathbb{R}^{n_1 n_2 \times 2}$, and L is the discrete gradient operator as defined in the problem set.

The Lagrangian of P' is given by

$$\begin{aligned} \mathcal{L}(X, z; \nu) &= \frac{1}{2}\|X - Y\|_F^2 + \langle \text{Vec}(\nu), \text{Vec}(L(X) - z)\rangle, \\ &= \frac{1}{2}\|\text{Vec}(X - Y)\|_2^2 + \langle \text{Vec}(\nu), \text{Vec}(L(X))\rangle - \langle \text{Vec}(\nu), \text{Vec}(z)\rangle \end{aligned}$$

where the operator $\text{Vec}(\cdot)$ is the operator that vectorizes its input in column-major order. The Lagrangian dual is then given by

$$\begin{aligned} g(\nu) &= \inf_{X,z,g(z)\leq\tau} \left(\frac{1}{2}\|\text{Vec}(X - Y)\|_2^2 + \langle \text{Vec}(\nu), \text{Vec}(L(X))\rangle - \langle \text{Vec}(\nu), \text{Vec}(z)\rangle\right) \\ &= \inf_X \left(\frac{1}{2}\|\text{Vec}(X - Y)\|_2^2 + \langle \text{Vec}(\nu), \text{Vec}(L(X))\rangle\right) + \inf_{z,g(z)\leq\tau} (-\langle \text{Vec}(\nu), \text{Vec}(z)\rangle) \\ &= \inf_X \left(\frac{1}{2}\|\text{Vec}(X - Y)\|_2^2 + \langle \text{Vec}(\nu), \text{Vec}(L(X))\rangle\right) - \sup_{z,g(z)\leq\tau} (\langle \text{Vec}(\nu), \text{Vec}(z)\rangle), \end{aligned}$$

where we split the infimum of the Lagrangian as the sum of the following:

(a) the infimum over X of $\frac{1}{2}\|\text{Vec}(X - Y)\|_2^2 + \langle \text{Vec}(\nu), \text{Vec}(L(X))\rangle$, and

(b) the negative supremum over z of $\langle \text{Vec}(\nu), \text{Vec}(z)\rangle$ such that $g(z) \leq \tau$.

We will find first the negative supremum of the second component subject to the constraint that $g(z) \leq \tau$. We note first that

$$|\langle \text{Vec}(\nu), \text{Vec}(z)\rangle| \leq \|\text{Vec}(\nu)\|_\infty \|\text{Vec}(z)\|_1$$

by Hölder's inequality. To make the above inequality tight, we take $\text{Vec}(z)$ to be such that $(\text{Vec}(z))_i = \tau$, where i is the index such that $|(\text{Vec}(\nu))_i| = \|\text{Vec}(\nu)\|_\infty$, and $(\text{Vec}(z))_j = 0$ if $j \neq i$. Thus,

$$-\sup_{z,g(z)\leq\tau} \langle \text{Vec}(\nu), \text{Vec}(z)\rangle = -\tau\|\text{Vec}(\nu)\|_\infty.$$

Next, we will find the infimum over X of $\frac{1}{2}\|\text{Vec}(X - Y)\|_2^2 + \langle \text{Vec}(\nu), \text{Vec}(L(X))\rangle$.

We note first that $\langle \text{Vec}\,(\nu)\,, \text{Vec}\,(L\,(X))\rangle = \langle \text{Vec}\,(L^*\,(\nu))\,, \text{Vec}\,(X)\rangle$, where $L^* = L_h^* + L_v^*$ (**Note**: depending on how L is implemented, the dimensions of the explicit representation of L will vary.)

We also note that the above function of X is differentiable with respect to X, and taking the gradient with respect to X and setting equal to 0, we get the following:

$$\text{Vec}\,(X) - \text{Vec}\,(Y) + \text{Vec}\,(L^*\,(\nu)) = 0 \implies \text{Vec}\,(X) = \text{Vec}\,(Y) - \text{Vec}\,(L^*\,(\nu))\,.$$

Thus, we have that

$$g\,(\nu) = \frac{1}{2}\|\text{Vec}\,(L^*\,(\nu))\|_2^2 + \langle \text{Vec}\,(\nu)\,, \text{Vec}\,(Y) - \text{Vec}\,(L^*\,(\nu))\rangle - \tau\|\text{Vec}\,(\nu)\|_\infty.$$