APPM 5600 - Homework 7

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October 15, 2021

1. We want to construct a rational function of the form

$$R(z) = \frac{\alpha + \beta z}{1 + \gamma z}$$

that interpolates the data (z_1, f_1), (z_2, f_2), and (z_3, f_3) at distinct points z_1, z_2, z_3. In other words, we see α, β, and γ such that

$$R(z_j) = f_j, \quad j = 1, 2, 3.$$

Show how you can determine α, β, and γ by setting a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ for the unknown vector $\mathbf{x} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$.

(Just write the system down - you do not need to solve it.)

For the rational function to interpolate f at the three data points, it must be the case that

$$\begin{aligned}
\frac{\alpha + \beta z_1}{1 + \gamma z_1} = f_1 &\implies \alpha + \beta z_1 = f_1 + f_1 \gamma z_1 \implies \alpha + \beta z_1 - f_1 z_1 \gamma = f_1, \\
\frac{\alpha + \beta z_2}{1 + \gamma z_2} = f_2 &\implies \alpha + \beta z_2 = f_2 + f_2 \gamma z_2 \implies \alpha + \beta z_2 - f_2 z_2 \gamma = f_2, \\
\frac{\alpha + \beta z_3}{1 + \gamma z_3} = f_3 &\implies \alpha + \beta z_3 = f_3 + f_3 \gamma z_3 \implies \alpha + \beta z_3 - f_3 z_3 \gamma = f_3.
\end{aligned}$$

Thus, to determine α, β, and γ, we can solve the following linear system

$$\begin{bmatrix} 1 & z_1 & -f_1 z_1 \\ 1 & z_2 & -f_2 z_2 \\ 1 & z_3 & -f_3 z_3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

2. We studied in class interpolation of functions defined in $1D$. We can adapt the technique to higher dimensions. For instance, let

$$f(x, y) = e^x \sin(y).$$

We want to construct a polynomial of the form

$$p(x, y) = c_0 + c_1 x + c_2 y + c_3 xy + c_4 x^2 + c_5 y^2$$

that interpolates f at the points (x_i, y_i):

$$p(x_i, y_i) = f(x_i, y_i), \quad 0 \leq i \leq 5.$$

(a) Set up a linear system $\mathbf{A}\mathbf{c} = \mathbf{f}$ to determine the coefficients $c_0, \ldots, c_5$.

We can set up the following linear system to determine the coefficients $c_0, \ldots c_5$:

$$\begin{bmatrix} 1 & x_0 & y_0 & x_0y_0 & x_0^2 & y_0^2 \\ 1 & x_1 & y_1 & x_1y_1 & x_1^2 & y_1^2 \\ 1 & x_2 & y_2 & x_2y_2 & x_2^2 & y_2^2 \\ 1 & x_3 & y_3 & x_3y_3 & x_3^2 & y_3^2 \\ 1 & x_4 & y_4 & x_4y_4 & x_4^2 & y_4^2 \\ 1 & x_5 & y_5 & x_5y_5 & x_5^2 & y_5^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} f(x_0, y_0) \\ f(x_1, y_1) \\ f(x_2, y_2) \\ f(x_3, y_3) \\ f(x_4, y_4) \\ f(x_5, y_5) \end{bmatrix}.$$

(b) Write a MATLAB code to determine $\mathbf{c}$ when the data points are

$$(0,0), (0,2), (1,0), (1,2), (2,1), (2,3).$$

Report your value for $\mathbf{c}$.

Using the above data points, we get that

$$\mathbf{c} = \begin{bmatrix} 0 \\ -0.949163105223492 \\ 5.059193000070850 \\ 0.781214622589569 \\ 0.949163105223492 \\ -2.302272143329005 \end{bmatrix}.$$

(c) Plot your polynomial p over $x \in [-1, 3]$, $y \in [-1, 3]$ using MATLAB's `surf` command (or the corresponding command in your preferred coding language). Compare this plot to a similar plot for f.

See attached code.

We see that the second-order multivariate polynomial interpolates f at the provided notes, which is expected, but does not match f elsewhere. This is also expected because the nodes are not clustered near the boundaries of the meshgrid, and thus, the interpolant does a poor job of approximating other than at the specified nodes.

3. Recall the Lagrange basis functions ℓ_j defined for $x_0, \ldots, x_n$ are defined by

$$\ell_j(x) = \prod_{i \neq j} \left(\frac{x - x_i}{x_j - x_i} \right)$$

for $i = 0, \ldots, n$.

(a) Prove that for any $n \geq 1$,

$$\sum_{j=0}^{n} \ell_j(x) = 1$$

for all $x \in \mathbb{R}$.

Define ℓ to be the polynomial $\ell(x) = \sum_{j=0}^{n} \ell_j(x)$. We know that ℓ interpolates the data points $(x_i, 1)$ for $i = 0, \ldots, n$ by definition of the ℓ_j's. However, ℓ is an n^{th} degree polynomial that interpolates 1 at $n+1$ locations. This implies that ℓ must be identically equal to the constant 1 for all $x \in \mathbb{R}$, as desired.

(b) Define $\psi(x) = (x - x_0) \cdots (x - x_n)$. Show that the polynomial interpolant of degree n that interpolates the data $(x_i, f(x_i))$ for $i = 0, \ldots, n$ can be written in the form

$$p_n(x) = \sum_{i=0}^{n} \frac{\psi(x)}{(x - x_i)\,\psi'(x_i)} f(x_i),$$

provided $x \neq x_i$ for all i.

Taking the logarithm of ψ, we have that

$$\log\left(\psi\left(x\right)\right) = \sum_{i=0}^{n} \log\left(x - x_i\right).$$

We know that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\log\left(\psi\left(x\right)\right)\right) = \frac{\psi^{'}(x)}{\psi(x)}$, and we also know that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\log\left(\psi\left(x\right)\right)\right) = \sum_{i=0}^{n} \frac{1}{x - x_i},$$

which is defined for all $x \neq x_0, \ldots, x_n$. Equating the two expressions, we have that

$$\frac{\psi^{'}\left(x\right)}{\psi\left(x\right)} = \sum_{i=0}^{n} \frac{1}{x - x_i} \implies \psi^{'}\left(x\right) = \psi\left(x\right) \sum_{i=0}^{n} \frac{1}{x - x_i},$$

which is defined for all $x \neq x_0, \ldots x_n$. Then, we have that $\psi^{'}\left(x_i\right)$ is given by

$$\begin{aligned}
\psi^{'}\left(x_i\right) &= \psi\left(x_i\right) \sum_{j=0}^{n} \frac{1}{x_i - x_j} \\
&= \left(x_i - x_0\right) \cdots \left(x_i - x_n\right) \sum_{j=0}^{n} \frac{1}{x_i - x_j} \\
&= \sum_{j=0}^{n} \frac{\left(x_i - x_0\right) \cdots \left(x_i - x_n\right)}{x_i - x_j} \\
&= \sum_{j=0}^{n} \left(x_i - x_0\right) \cdots \left(x_i - x_{j-1}\right)\left(x_i - x_{j+1}\right) \cdots \left(x_i - x_n\right) \\
&= \prod_{j=0, j \neq i}^{n} \left(x_i - x_j\right).
\end{aligned}$$

We see that the i^{th} term of p_n is given by

$$\begin{aligned}
\frac{\psi\left(x\right)}{\left(x - x_i\right)\psi'\left(x_i\right)} f\left(x_i\right) &= \frac{\prod_{j=0, j\neq i}^{n} x - x_j}{\prod_{j=0, j\neq i}^{n} x_i - x_j} f\left(x_i\right) \\
&= \ell_i\left(x\right) f\left(x_i\right).
\end{aligned}$$

Thus, we see that i^{th} term of p_n is the i^{th} Lagrange polynomial multiplied by $f\left(x_i\right)$. That is, p_n is given by

$$p_n\left(x\right) = \sum_{i=0}^{n} \ell_i\left(x\right) f\left(x_i\right),$$

which we know is the unique polynomial interpolant of degree n that interpolates the data $\left(x_i, f\left(x_i\right)\right)$ for $i = 0, \ldots, n$.

(c) Define next

$$w_i = \frac{1}{\psi'\left(x_i\right)}.$$

Show that the polynomial interpolant of degree n that interpolates the data $(x_i, f(x_i))$ for $i = 0, \dots, n$ can be written in the form

$$p_n(x) = \frac{\sum_{i=0}^{n} \frac{w_i f(x_i)}{x - x_i}}{\sum_{i=0}^{n} \frac{w_i}{x - x_i}},$$

provided $x \neq x_i$ for all i. This form is called the barycentric representation of p_n, giving it as a weighted sum of the values $f(x_0), \dots, f(x_n)$.
From part (b), we note that the i^{th} Lagrange polynomial is given by

$$\begin{aligned} \ell_i(x) &= \frac{\psi(x)}{(x - x_i)\psi'(x_i)} \\ &= \psi(x) \frac{w_i}{(x - x_i)}. \end{aligned}$$

Then, the unique polynomial interpolant of degree n that interpolates the data $(x_i, f(x_i))$ for $i = 0, \dots, n$ is given by

$$\begin{aligned} p_n(x) &= \sum_{i=0}^{n} \ell_i(x) f(x_i) \\ &= \sum_{i=0}^{n} \psi(x) \frac{w_i f(x_i)}{(x - x_i)} \\ &= \psi(x) \sum_{i=0}^{n} \frac{w_i f(x_i)}{(x - x_i)}. \end{aligned}$$

We also recall from part (a) that

$$\begin{aligned} \sum_{i=0}^{n} \ell_i(x) &= \sum_{i=0}^{n} \psi(x) \frac{w_i}{(x - x_i)} \\ &= \psi(x) \sum_{i=0}^{n} \frac{w_i}{(x - x_i)} \\ &= 1. \end{aligned}$$

Rearranging the above equation to solve for $\psi(x)$, we have that

$$\psi(x) = \frac{1}{\sum_{i=0}^{n} \frac{w_i}{(x - x_i)}}.$$

Thus, we have that

$$\begin{aligned} p_n(x) &= \psi(x) \sum_{i=0}^{n} \frac{w_i f(x_i)}{(x - x_i)} \\ &= \frac{\sum_{i=0}^{n} \frac{w_i f(x_i)}{(x - x_i)}}{\sum_{i=0}^{n} \frac{w_i}{(x - x_i)}}, \end{aligned}$$

as desired.

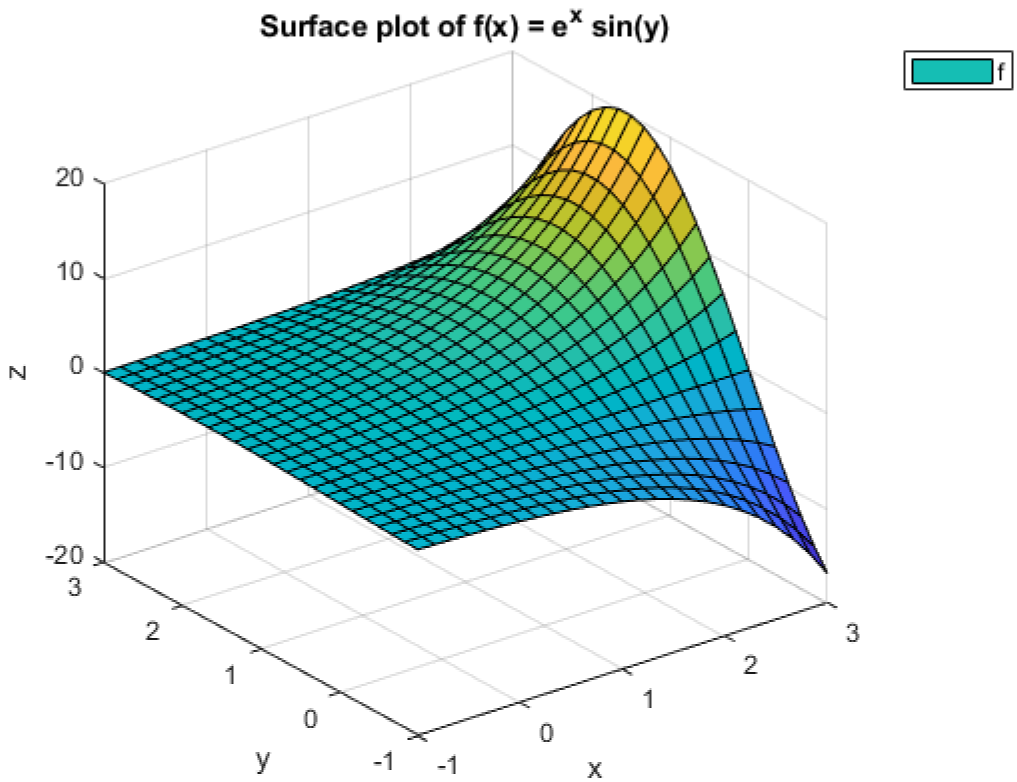
Homework 7

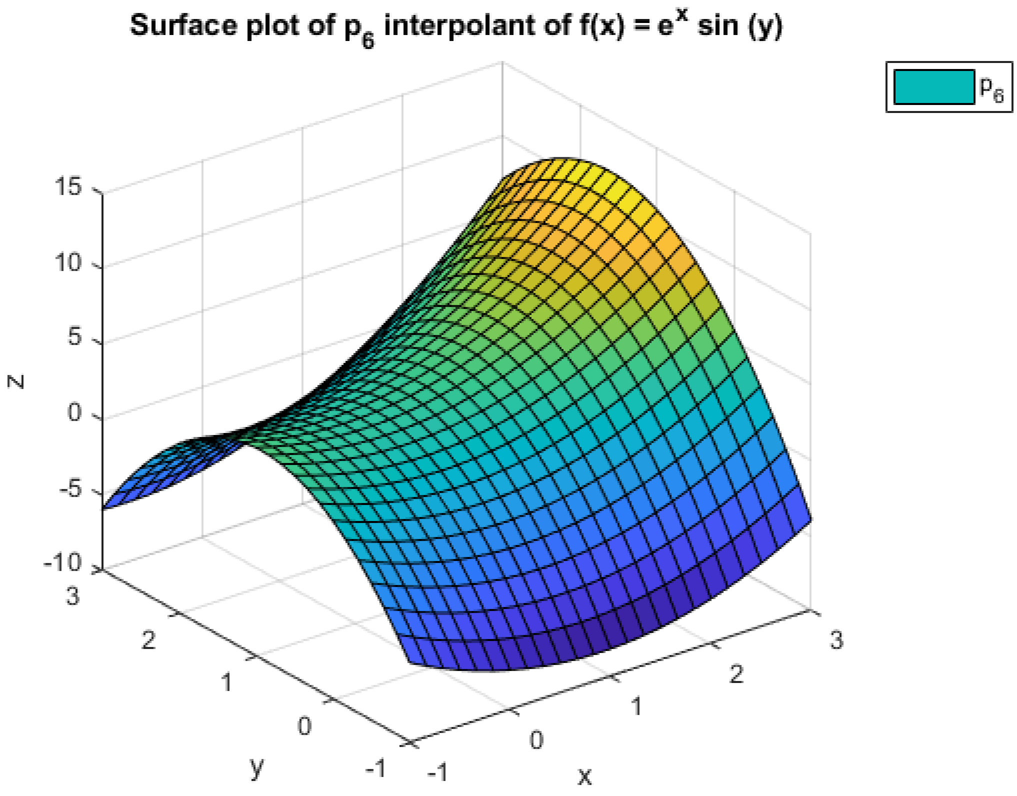
Problem 2

```
clc;
clear;
close all;

f = @(x, y) exp(x).*sin(y);

x_interp = [0; 0; 1; 1; 2; 2];
y_interp = [0; 2; 0; 2; 1; 3];

A = [ones(size(x_interp)), x_interp, y_interp, x_interp.*y_interp, ...
     x_interp.^(2), y_interp.^(2)];

b = f(x_interp, y_interp);

% Solve for the multivariate interpolant coefficients
c = A\b;
fprintf("Coefficients for question 2b: \n");
disp(c);

f_interp = @(c, x, y) c(1)*ones(size(x)) + c(2)*x + c(3)*y + c(4)*x.*y +
 c(5)*x.^(2) + c(6)*y.^(2);

a = -1; b = 3; n = 25;
[XX, YY] = meshgrid(linspace(a, b, n), linspace(a, b, n));
ZZ_f = f(XX, YY);
ZZ_f_interp = f_interp(c, XX, YY);

figure(1);
surf_f = surf(XX, YY, ZZ_f);
xlabel("x");
ylabel("y");
zlabel("z");
title("Surface plot of f(x) = e^{x} sin(y)");
legend(surf_f, {'f'});

figure(2);
surf_f_interp = surf(XX, YY, ZZ_f_interp);
xlabel("x");
ylabel("y");
zlabel("z");
title("Surface plot of p_{6} interpolant of f(x) = e^{x} sin (y)");
legend(surf_f_interp, {'p_6'});
```

```
Coefficients for question 2b:
         0
   -0.9492
    5.0592
    0.7812
```

```
  0.9492
 -2.3023
```





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