APPM 5600 - Homework 1

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1. How would you perform the following calculations to avoid cancellation? Justify your answers.

   (a) Evaluate $\sqrt{x+1}-1$ for $x \simeq 0$.
   We can multiply $\sqrt{x+1}-1$ by $\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$, which we see is

$$\begin{aligned}\left(\sqrt{x+1}-1\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) &= \frac{(x+1)-1}{\sqrt{x+1}+1} \\ &= \frac{x}{\sqrt{x+1}+1}.\end{aligned}$$

   This avoids the cancellation issue present in the numerator of the original expression. This is because we are no longer taking the difference of $\sqrt{x+1}$ and 1 for $x \simeq 0$ in the new expression.

   (b) Evaluate $\sin(x) - \sin(y)$ for $x \simeq y$.
   We use the sum-to-product identity for $\sin$ and evaluate this expression via

$$\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right).$$

   The resulting expression is preferable over the original expression as the subtraction occurs within the $\sin$ evaluation.

   (c) Evaluate $\frac{1-\cos(x)}{\sin(x)}$ for $x \simeq 0$.
   We can multiply $\frac{1-\cos(x)}{\sin(x)}$ by $\frac{1+\cos(x)}{1+\cos(x)}$, which we see is

$$\begin{aligned}\frac{1-\cos(x)}{\sin(x)}\left(\frac{1+\cos(x)}{1+\cos(x)}\right) &= \frac{1-\cos^2(x)}{\sin(x)(1+\cos(x))} \\ &= \frac{\sin^2(x)}{\sin(x)(1+\cos(x))} \\ &= \frac{\sin(x)}{1+\cos(x)}.\end{aligned}$$

   This avoids the cancellation issue present in the numerator of the original expression. This is because we are no longer taking the difference of 1 and $\cos(x)$ for $x \simeq 0$ in the new expression.

2. Consider the polynomial

$$\begin{aligned}p(x) &= (x-2)^9 \\ &= x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512.\end{aligned}$$

   (a) Plot $p(x)$ for $x = 1.920, 1.921, 1.922, \ldots, 2.080$, i.e., `x = [1.920:0.001:2.080]`, evaluating p via its coefficients.
   See attached plot.

(b) Produce the same plot again, now evaluating p via the expression $(x-2)^9$.

See attached plot.

(c) What is the difference? What is causing the discrepancy? Which plot is correct?

When p is evaluated directly, the resulting graph is smooth. However, when p is evaluated via its coefficients, we see that the the resulting graph does not look like the first plot.

Since $x \simeq 2$, $(x-2)^9$ will be a small value, but when evaluating p via its coefficients, we form large numbers that have alternating signs and add them up. This discrepancy is caused by the loss of significant digits when adding up these relatively large numbers that have alternating signs.

3. **Cancellation of terms.** Consider computing $y = x_1 - x_2$, with $\widetilde{x}_1 = x_1 + \Delta x_1$ and $\widetilde{x}_2 = x_2 + \Delta x_2$ being approximations to the exact values. If the operation $x_1 - x_2$ is carried out exactly, we have

$$\begin{aligned}\widetilde{y} &= y + (\Delta x_1 - \Delta x_2)\\ &= y + \Delta y,\end{aligned}$$

where $\Delta y = (\Delta x_1 - \Delta x_2)$.

(a) Find upper bounds on the absolute error $|\Delta y|$ and the relative error $\left|\frac{\Delta y}{y}\right|$. When is the relative error large?

We can obtain an upper bound on the absolute error by noting that

$$\begin{aligned}|\Delta y| &= |\Delta x_1 - \Delta x_2|\\ &\leq |\Delta x_1| + |\Delta x_2|,\end{aligned}$$

where we obtained the last inequality via the triangle inequality. We can obtain an upper bound on the relative error by noting that

$$\begin{aligned}\left|\frac{\Delta y}{y}\right| &= \left|\frac{\Delta x_1 - \Delta x_2}{x_1 - x_2}\right|\\ &\leq \frac{1}{|x_1 - x_2|}\left(|\Delta x_1| + |\Delta x_2|\right),\end{aligned}$$

where we obtained the last inequality also via the triangle inequality. The relative error is large when x_1 and x_2 are close to one another, i.e., when $x_1 \simeq x_2$.

(b) First manipulate $\cos(x+\delta) - \cos(x)$ into an expression without subtraction. Then, tabulate or plot the difference between your expression and $\cos(x+\delta) - \cos(x)$ for $\delta = 10^{-16}, 10^{-15}, \ldots, 10^{-2}, 10^{-1}, 10^{0}$.

Using the cos sum-to-product identity

$$\cos(\theta) - \cos(\omega) = -2\sin\left(\frac{\theta+\omega}{2}\right)\sin\left(\frac{\theta-\omega}{2}\right)$$

for angles θ and ω, we see that

$$\begin{aligned}\cos(x+\delta) - \cos(x) &= -2\sin\left(\frac{x+\delta+x}{2}\right)\sin\left(\frac{x+\delta-x}{2}\right)\\ &= -2\sin\left(x+\frac{\delta}{2}\right)\sin\left(\frac{\delta}{2}\right).\end{aligned}$$

(c) Taylor expansion yields $f(x+\delta) - f(x) = \delta f'(x) + \frac{\delta^2}{2!}f''(\xi)$, where $\xi \in [x, x+\delta]$. Use this expression to approximate $\cos(x+\delta) - \cos(x)$ for the same values of δ as in (b). Determine for which values of δ each method is better.

In the attached code and plot, we plot the relative error between $\cos(x+\delta) - \cos(x)$ and the two functions:

i. $-2\sin\left(x+\frac{\delta}{2}\right)\sin\left(\frac{\delta}{2}\right)$, and

ii. the first-order Taylor expansion of $\cos(x+\delta)-\cos(x)$, which is given by $-\delta\cos(x)+-\frac{\delta^2}{2}\sin(\xi)$, where we take $\xi = x+\frac{\delta}{2}$.

Here, we take the relative errors to be $\left|\frac{(\cos(x+\delta)-\cos(x))-y(x)}{(\cos(x+\delta)-\cos(x))}\right|$, where y is either the trigonometric identity or Taylor expansion. We compare the relative error at $x_1=\pi$ and $x_2=10^{10}$.

In the plot of the relative errors, we see that for x_1, the relative errors associated with the trigonometric identity and Taylor expansion agree up until δ becomes larger than 10^{-4}. When $\delta > 10^{-4}$, the relative error associated with the Taylor expansion increases, whereas the relative error associated with the trigonometric identity continues to decrease.

For x_2, the relative errors associated with the trigonometric identity and Taylor expansion agree up until δ becomes larger than 10^{-2}. When $\delta > 10^{-2}$, the relative error associated with the Taylor expansion increases, whereas the relative error associated with the trigonometric identity continues to decrease.

4. Show that $(1+x)^n = 1+nx+o(x)$ as $x\to 0$, where $n\in\mathbb{Z}$.

   The Taylor series of $(x)=(1+x)^n$ centered at 0, where n is a natural number, is given by

$$\begin{aligned}(1+x)^n &= 1+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n+\binom{n}{n+1}\frac{x^{n+1}}{1+\xi}\\ &= 1+nx+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n+\binom{n}{n+1}\frac{x^{n+1}}{1+\xi},\end{aligned}$$

   where ξ is a number between 0 and x. We see that the limit as x approaches 0 of the absolute value of the Taylor remainder (starting from the term $\binom{n}{2}x^2$ and onward) divided by x is

$$\begin{aligned}\lim_{x\to 0}\left|\frac{\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n+\binom{n}{n+1}\frac{x^{n+1}}{1+\xi}}{x}\right| &= \lim_{x\to 0}\left|\binom{n}{2}x+\ldots+\binom{n}{n}x^{n-1}+\binom{n}{n+1}\frac{x^n}{1+\xi}\right|\\ &= 0,\end{aligned}$$

   as desired.

5. Show that $x\sin(\sqrt{x}) = \mathcal{O}\left(x^{3/2}\right)$ as $x\to 0^+$.

   To show that $x\sin(\sqrt{x}) = \mathcal{O}\left(x^{3/2}\right)$, we have to show that there exists a positive constant M and a constant x_0 such that

$$\left|x\sin\left(\sqrt{x}\right)\right| \le M\left|x^{3/2}\right|$$

   for all $x > x_0$. We know that

$$\begin{aligned}\left|x\sin\left(\sqrt{x}\right)\right| &\le |x|\\ &\le \left|x\sqrt{x}\right|\\ &= \left|x^{3/2}\right|,\end{aligned}$$

   for all $x > x_0 = 1$, where we obtain the second inequality by noting that $|\sin(\sqrt{x})| \le 1$ for all x. Thus, by taking our constants $M=1$ and $x_0=1$, we have shown that $x\sin(\sqrt{x}) = \mathcal{O}\left(x^{3/2}\right)$.

6. The function $f(x) = (x-5)^9$ has a root (with multiplicity 9) at $x=5$ and is monotonically increasing (decreasing) for $x>5$ ($x<5$) and should thus be suitable candidate for your function above.

   (a) Set `a = 4.8`, `b = 5.31`, and `tol = 1e-4`, and use `bisection` with

i. $f(x) = (x-5)^9$.

ii. the expanded version of $(x-5)^9$, i.e., $f(x) = x^9 - 45x^8 + \ldots - 1953125$.

(b) Explain what is happening.

Since $f(x) = (x-5)^9$ and `bisect` returns a first approximate root of $x_0 = \frac{4.8+5.31}{2} = 5.055$, when f is evaluated directly at x_0, the resulting value is a number that is close to 0, but not equal to 0 in machine precision. Because of this, the `bisect` method carries out additional iterations. When f is evaluated via the expanded form at x_0, the resulting value is computed by adding large numbers of alternating signs that are roughly of the same magnitude, which results in a loss of significant digits as the additions are carried out. Thus, the resulting value is close enough to 0 to machine precision the `bisect` method stops after one iteration, which yields an approximate root that is not close to 5.

The attached plot of the absolute error between f evaluated directly and evaluated via its coefficients on the interval `[4.8:0.001:5.31]` shows how the evaluation of f via its coefficients is unstable for $x \simeq 5$.