APPM 5600 - Homework 9

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1. We saw a discrete orthogonal Fourier basis in class, as the columns of the Vandermonde matrix for the equispaced trigonometric interpolation problem. Denote these orthogonal basis vectors by $\mathbf{v}^k$, with $k = 0, \ldots, 2n$. The j^{th} entry of the k^{th} vector is

$$\mathbf{v}_j^k = e^{\frac{2\pi ijk}{2n+1}},$$

where $i = \sqrt{-1}$. A circulant matrix of size $(2n+1) \times (2n+1)$ has the form

$$\mathbf{C} = \begin{bmatrix} a_0 & a_1 & \ldots & \ldots & a_{2n} \\ a_{2n} & a_0 & a_1 & \ldots & a_{2n-1} \\ a_{2n-1} & a_{2n} & a_0 & \ldots & a_{2n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_1 & a_2 & \ldots & a_{2n} & a_0 \end{bmatrix}.$$

Let $\mathbf{S}$ denote the matrix that shifts the index of a vector by 1, i.e., for any vector $\mathbf{w}$, the j^{th} entry of $\mathbf{Sw}$ is

$$(\mathbf{Sw})_j = \mathbf{w}_{j+1}, \quad j = 0, \ldots, 2n-1,$$

and

$$(\mathbf{Sw})_{2n} = \mathbf{w}_0.$$

(a) Show that any circular matrix can be written as a polynomial of the $\mathbf{S}$ matrix.
The matrix S, which is of size $(2n+1) \times (2n+1)$ and where $n \in \mathbb{N}$, is given by the following:

$$\begin{aligned} S(i, i+1) &= 1, \quad 0 \leq i \leq 2n-1, \\ S(2n, 0) &= 1, \end{aligned}$$

and S is zero everywhere else. That is, S has ones along above the main diagonal and one in the last row and first column. The matrix S has the property that taking integer powers of S leads to the shifting of the super-diagonal entries by the given power. Thus, we can write the circulant matrix $\mathbf{C}$ as

$$\mathbf{C} = \sum_{j=0}^{2n} a_j \mathbf{S}^j.$$

(b) Prove that the vectors $\mathbf{v}^k$ are all the eigenvectors of the circular matrix. What are the eigenvalues?
Let $\mathbf{v}^k$ be the k^{th} orthogonal basis vector as described above. We know that

$$\mathbf{v}^k = \begin{bmatrix} 1 \\ \omega_k \\ \omega_k^2 \\ \vdots \\ \omega_k^{2n} \end{bmatrix},$$

where $\omega_k = \exp\left(\frac{2\pi i k}{2n+1}\right)$ is the k^{th} root of unity. We see that

$$\mathbf{C}\mathbf{v}^k = \begin{bmatrix} \sum_{m=0}^{2n} a_m \omega_k^m \\ \vdots \\ \dots \end{bmatrix}.$$

The candidate eigenvalue corresponding to $\mathbf{v}^k$, which we denote λ_k, is given by

$$\lambda_k = \sum_{m=0}^{2n} a_m \omega_k^m.$$

Using the result from part 1a, we know that

$$\begin{aligned}
\left(\sum_{m=0}^{2n} a_m \mathbf{S}^m\right)\mathbf{v}^k &= a_0 \begin{bmatrix} 1 \\ \omega_k \\ \omega_k^2 \\ \vdots \\ \omega_k^{2n-2} \\ \omega_k^{2n-1} \\ \omega_k^{2n} \end{bmatrix} + a_1 \begin{bmatrix} \omega_k \\ \omega_k^2 \\ \omega_k^3 \\ \vdots \\ \omega_k^{2n-1} \\ \omega_k^{2n} \\ 1 \end{bmatrix} + \dots + a_{2n-1} \begin{bmatrix} \omega_k^{2n-1} \\ \omega_k^{2n} \\ 1 \\ \vdots \\ \omega_k^{2n-4} \\ \omega_k^{2n-3} \\ \omega_k^{2n-2} \end{bmatrix} + a_{2n} \begin{bmatrix} \omega_k^{2n} \\ 1 \\ \omega_k \\ \vdots \\ \omega_k^{2n-3} \\ \omega_k^{2n-2} \\ \omega_k^{2n-1} \end{bmatrix} \\
&= \begin{bmatrix} a_0 + a_1\omega_k + \dots + a_{2n-1}\omega_k^{2n-1} + a_{2n}\omega_k^{2n} \\ a_0\omega_k + a_1\omega_k^2 + \dots + a_{2n-1}\omega_k^{2n} + a_{2n} \\ a_0\omega_k^2 + a_1\omega_k^3 + \dots + a_{2n-1} + a_{2n}\omega_k \\ \vdots \\ a_0\omega_k^{2n-2} + a_1\omega_k^{2n-1} + \dots + a_{2n-1}\omega_k^{2n-4} + a_{2n}\omega_k^{2n-3} \\ a_0\omega_k^{2n-1} + a_1\omega_k^{2n} + \dots + a_{2n-1}\omega_k^{2n-3} + a_{2n}\omega_k^{2n-2} \\ a_0\omega_k^{2n} + a_1 + \dots + a_{2n-1}\omega_k^{2n-2} + a_{2n}\omega k^{2n-1} \end{bmatrix} \\
&= \begin{bmatrix} a_0 + a_1\omega_k + \dots + a_{2n-1}\omega_k^{2n-1} + a_{2n}\omega_k^{2n} \\ \omega_k\left(a_0 + a_1\omega_k + \dots + a_{2n-1}\omega_k^{2n-1} + a_{2n}\omega_k^{2n}\right) \\ \omega_k^2\left(a_0 + a_1\omega_k + \dots + a_{2n-1}\omega_k^{2n-1} + a_{2n}\omega_k^{2n}\right) \\ \vdots \\ \omega_k^{2n-2}\left(a_0 + a_1\omega_k + \dots + a_{2n-1}\omega_k^{2n-1} + a_{2n}\omega_k^{2n}\right) \\ \omega_k^{2n-1}\left(a_0 + a_1\omega_k + \dots + a_{2n-1}\omega_k^{2n-1} + a_{2n}\omega_k^{2n}\right) \\ \omega_k^{2n}\left(a_0 + a_1\omega_k + \dots + a_{2n-1}\omega_k^{2n-1} + a_{2n}\omega_k^{2n}\right) \end{bmatrix} \\
&= \left(\sum_{m=0}^{2n} a_0 \omega_k^m\right) \begin{bmatrix} 1 \\ \omega_k \\ \omega_k^2 \\ \vdots \\ \omega_k^{2n-2} \\ \omega_k^{2n-1} \\ \omega_k^{2n} \end{bmatrix},
\end{aligned}$$

as desired. Thus, the $\mathbf{v}^k$'s are eigenvectors of the circulant matrix, with corresponding eigenvalues $\lambda_k = \sum_{m=0}^{2n} a_0 \omega_k^m$ and $\omega_k = \exp\left(\frac{2i\pi k}{2n+1}\right)$.

2. Let $0 \le t_0 < t_1 < \dots < t_{2n} < 2\pi$, and consider the trigonometric polynomial interpolation problem. Define

$$\ell_j(t) = \prod_{k=0, k\neq j}^{2n} \frac{\sin\left(\frac{1}{2}(t - t_k)\right)}{\sin\left(\frac{1}{2}(t_j - t_k)\right)}$$

for $j = 0, 1, \ldots, 2n$. It is easy to show that $\ell_j(t_i) = \delta_{ij}$, $0 \le i, j \le 2n$.

Show that $\ell_j(t)$ is a trigonometric polynomial of degree less than or equal to n. Then the solution of the trigonometric interpolation problem is given by

$$p_n(t) = \sum_{j=0}^{2n} f(t_j)\,\ell_j(t).$$

Hint: Use induction on n and standard trigonometric identities.

We will prove that ℓ_j is a trigonometric polynomial of degree less than or equal to n via induction.

(a) **Base case:** $n = 1$.

Without loss of generality, we consider the case where $j = 0$ in the product. If $n = 1$, we have that

$$\begin{aligned}
\prod_{k=0,k\neq 0}^{2} \frac{\sin\left(\frac{1}{2}(t-t_k)\right)}{\sin\left(\frac{1}{2}(t_j-t_k)\right)} &= \frac{\sin\left(\frac{1}{2}(t-t_1)\right)\sin\left(\frac{1}{2}(t-t_2)\right)}{\sin\left(\frac{1}{2}(t_0-t_1)\right)\sin\left(\frac{1}{2}(t_0-t_2)\right)} \\
&= \frac{1}{2}\left(\frac{\cos\left(\frac{1}{2}(t_2-t_1)\right) - \cos\left(t - \frac{1}{2}(t_1+t_2)\right)}{\sin\left(\frac{1}{2}(t_0-t_1)\right)\sin\left(\frac{1}{2}(t_0-t_2)\right)}\right) \\
&= \frac{1}{2}\left(\frac{\cos\left(\frac{1}{2}(t_2-t_1)\right)}{\sin\left(\frac{1}{2}(t_0-t_1)\right)\sin\left(\frac{1}{2}(t_0-t_2)\right)} - \frac{\left(\cos(t)\cos\left(\frac{1}{2}(t_1+t_2)\right) - \sin(t)\sin\left(\frac{1}{2}(t_1+t_2)\right)\right)}{\sin\left(\frac{1}{2}(t_0-t_1)\right)\sin\left(\frac{1}{2}(t_0-t_2)\right)}\right) \\
&= a_0 + a_1\cos(t) + b_1\sin(t),
\end{aligned}$$

where

$$\begin{aligned}
a_0 &= \frac{1}{2}\left(\frac{\cos\left(\frac{1}{2}(t_2-t_1)\right)}{\sin\left(\frac{1}{2}(t_0-t_1)\right)\sin\left(\frac{1}{2}(t_0-t_2)\right)}\right), \\
a_1 &= -\frac{1}{2}\left(\frac{\cos\left(\frac{1}{2}(t_1+t_2)\right)}{\sin\left(\frac{1}{2}(t_0-t_1)\right)\sin\left(\frac{1}{2}(t_0-t_2)\right)}\right), \\
b_1 &= \frac{1}{2}\left(\frac{\cos\left(\frac{1}{2}(t_1+t_2)\right)}{\sin\left(\frac{1}{2}(t_0-t_1)\right)\sin\left(\frac{1}{2}(t_0-t_2)\right)}\right).
\end{aligned}$$

Thus, we have that $\ell_0(x)$ is a trigonometric polynomial of degree one. We can perform similar calculations to obtain that ℓ_1 and ℓ_2 are also trigonometric polynomials of degree one. Thus, the base case holds.

(b) **Inductive hypothesis:** We assume that for $n \ge 1$,

$$\prod_{k=0,k\neq j}^{2n} \frac{\sin\left(\frac{1}{2}(t-t_k)\right)}{\sin\left(\frac{1}{2}(t_j-t_k)\right)}$$

is a trigonometric polynomial of degree n or less, where $0 \le t_0 < t_1 < \ldots < t_{2n} < 2\pi$. That is, we assume we can write

$$\prod_{k=0,k\neq j}^{2n} \frac{\sin\left(\frac{1}{2}(t-t_k)\right)}{\sin\left(\frac{1}{2}(t_j-t_k)\right)} = a_0 + \sum_{k=1}^{n} a_k\cos(kt) + \sum_{k=1}^{n} b_k\sin(kt).$$

(c) **Inductive step:** We want to show that for $n+1$, the product

$$\prod_{k=0,k\neq j}^{2n+2} \frac{\sin\left(\frac{1}{2}(t-t_k)\right)}{\sin\left(\frac{1}{2}(t_j-t_k)\right)}$$

is a trigonometric polynomial of degree $n+1$ or less. We know that we can write the above product as

$$\begin{aligned}\prod_{k=0,k\neq j}^{2n+2}\frac{\sin\left(\frac{1}{2}\left(t-t_k\right)\right)}{\sin\left(\frac{1}{2}\left(t_j-t_k\right)\right)} &= \left(\widetilde{a}_0+\sum_{k=1}^{n}\widetilde{a}_k\cos\left(kt\right)+\sum_{k=1}^{n}\widetilde{b}_k\sin\left(kt\right)\right)\left(\frac{\sin\left(\frac{1}{2}\left(t-t_{2n+1}\right)\right)\sin\left(\frac{1}{2}\left(t-t_{2n+2}\right)\right)}{\sin\left(\frac{1}{2}\left(t_j-t_{2n+1}\right)\right)\sin\left(\frac{1}{2}\left(t_j-t_{2n+2}\right)\right)}\right)\\ &= \left(\widetilde{a}_0+\sum_{k=1}^{n}\widetilde{a}_k\cos\left(kt\right)+\sum_{k=1}^{n}\widetilde{b}_k\sin\left(kt\right)\right)\left(c_0+c_1\cos\left(t\right)+c_2\sin\left(t\right)\right),\end{aligned}$$

by the inductive hypothesis and base case, where

$$\begin{aligned}c_0&=\frac{1}{2}\left(\frac{\cos\left(\frac{1}{2}\left(t_{2n+2}-t_{2n+1}\right)\right)}{\sin\left(\frac{1}{2}\left(t_j-t_{2n+1}\right)\right)\sin\left(\frac{1}{2}\left(t_j-t_{2n+2}\right)\right)}\right),\\ c_1&=-\frac{1}{2}\left(\frac{\cos\left(\frac{1}{2}\left(t_{2n+1}+t_{2n+2}\right)\right)}{\sin\left(\frac{1}{2}\left(t_j-t_{2n+1}\right)\right)\sin\left(\frac{1}{2}\left(t_j-t_{2n+2}\right)\right)}\right),\\ c_2&=\frac{1}{2}\left(\frac{\cos\left(\frac{1}{2}\left(t_{2n+1}+t_{2n+2}\right)\right)}{\sin\left(\frac{1}{2}\left(t_j-t_{2n+1}\right)\right)\sin\left(\frac{1}{2}\left(t_j-t_{2n+2}\right)\right)}\right).\end{aligned}$$

Multiplying through, we get the following terms:

i. $\widetilde{a}_0c_0+\sum_{k=1}^{n}c_0\widetilde{a}_k\cos\left(kt\right)+\sum_{k=1}^{n}c_0\widetilde{b}_k\sin\left(kt\right)$
The above term is a trigonometric polynomial of degree n or less.

ii. $c_1\widetilde{a}_0\cos\left(t\right)+\sum_{k=1}^{n}c_1\widetilde{a}_k\cos\left(kt\right)\cos\left(t\right)+\sum_{k=1}^{n}c_1\widetilde{b}_k\sin\left(kt\right)\cos\left(t\right)$
We see that that the above term becomes

$$c_1\widetilde{a}_0\cos\left(t\right)+\frac{1}{2}\sum_{k=1}^{n}c_1\widetilde{a}_k\left(\cos\left(\left(k-1\right)t\right)+\cos\left(\left(k+1\right)t\right)\right)+\frac{1}{2}\sum_{k=1}^{n}c_1\widetilde{b}_k\left(\sin\left(\left(k+1\right)t\right)+\sin\left(\left(k-1\right)t\right)\right),$$

which is a trigonometric polynomial of degree $n+1$ or less.

iii. $c_2\widetilde{a}_0\sin\left(t\right)+\sum_{k=1}^{n}c_2\widetilde{a}_k\cos\left(kt\right)\sin\left(t\right)+\sum_{k=1}^{n}c_2\widetilde{b}_k\sin\left(kt\right)\sin\left(t\right)$
We see that that the above term becomes

$$c_2\widetilde{a}_0\cos\left(t\right)+\frac{1}{2}\sum_{k=1}^{n}c_2\widetilde{a}_k\left(\sin\left(\left(k+1\right)t\right)-\sin\left(\left(k-1\right)t\right)\right)+\frac{1}{2}\sum_{k=1}^{n}c_2\widetilde{b}_k\left(\cos\left(\left(k-1\right)t\right)-\cos\left(\left(k+1\right)t\right)\right),$$

which is a trigonometric polynomial of degree $n+1$ or less.

Thus, the product is a trigonometric polynomial of degree $n+1$ or less, as desired. This concludes the inductive step.

Thus, we have that the ℓ_j's defined above are trigonometric polynomials of degree n or less.