APPM 5600 - Homework 4

Eappen Nelluvelil

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1. Prove the following for $\mathbf{x} \in \mathbb{C}^n$:

   (a) $||\mathbf{x}||_\infty \leq ||\mathbf{x}||_1 \leq n\,||\mathbf{x}||_\infty$
   Let $\mathbf{x} \in \mathbb{C}^n$ be given. We see that

$$\begin{aligned}||\mathbf{x}||_\infty &= \max_{1\leq i\leq n} |\mathbf{x}_i| \\ &\leq |\mathbf{x}_1| + |\mathbf{x}_2| + \ldots + |\mathbf{x}_{n-1}| + |\mathbf{x}_n| \\ &= ||\mathbf{x}||_1 .\end{aligned}$$

   We also see that

$$\begin{aligned}||\mathbf{x}||_1 &= |\mathbf{x}_1| + |\mathbf{x}_2| + \ldots + |\mathbf{x}_{n-1}| + |\mathbf{x}_n| \\ &\leq n\left(\max_{1\leq i\leq n} |\mathbf{x}_i|\right) \\ &= n\,||\mathbf{x}||_\infty .\end{aligned}$$

   Thus, $||\mathbf{x}||_\infty \leq ||\mathbf{x}||_1 \leq n\,||\mathbf{x}||_\infty$, as desired.

   (b) $||\mathbf{x}||_\infty \leq ||\mathbf{x}||_2 \leq \sqrt{n}\,||\mathbf{x}||_\infty$
   We see that

$$\begin{aligned}(||\mathbf{x}||_\infty)^2 &= \left(\max_{i\leq i\leq n} |\mathbf{x}|\right)^2 \\ &\leq \left(|\mathbf{x}_1|^2 + |\mathbf{x}_2|^2 + \ldots + |\mathbf{x}_{n-1}|^2 + |\mathbf{x}_n|^2\right) \\ &= ||\mathbf{x}||_2^2 ,\end{aligned}$$

   which implies that $||\mathbf{x}||_\infty \leq ||\mathbf{x}||_2$. We also see that

$$\begin{aligned}||\mathbf{x}||_2^2 &= \left(|\mathbf{x}_1|^2 + |\mathbf{x}_2|^2 + \ldots + |\mathbf{x}_{n-1}|^2 + |\mathbf{x}_n|^2\right) \\ &\leq n\left(\max_{1\leq i\leq n} |\mathbf{x}_i|\right)^2 \\ &= n\,||\mathbf{x}||_\infty^2 ,\end{aligned}$$

   which implies that $||\mathbf{x}||_2 \leq \sqrt{n}\,||\mathbf{x}||_\infty$

   (c) $||\mathbf{x}||_2 \leq ||\mathbf{x}||_1 \leq \sqrt{n}\,||\mathbf{x}||_2$

We see that

$$
\begin{aligned}
||\mathbf{x}||_2^2 &= \sum_{i=1}^{n} |\mathbf{x}_i|^2 \\
&\leq \sum_{i=1}^{n} |\mathbf{x}_i| \left(\sum_{j=1}^{n} |\mathbf{x}_j| \right) \\
&= (|\mathbf{x}_1| + |\mathbf{x}_2| + \ldots + |\mathbf{x}_{n-1}| + |\mathbf{x}_n|)\,(|\mathbf{x}_1| + |\mathbf{x}_2| + \ldots + |\mathbf{x}_{n-1}| + |\mathbf{x}_n|) \\
&= ||\mathbf{x}||_1^2,
\end{aligned}
$$

which implies that $||\mathbf{x}||_2 \leq ||\mathbf{x}||_1$. Now, consider the vector $\widetilde{\mathbf{x}} = \left[|\mathbf{x_1}|, \ldots, |\mathbf{x_n}|\right]^T$. We see that

$$
\begin{aligned}
\langle \widetilde{\mathbf{x}}, \mathbf{1} \rangle &= \sum_{j=1}^{n} |\mathbf{x_j}| \\
&= ||\mathbf{x}||_1,
\end{aligned}
$$

where $\mathbf{1}$ is the vector of 1's in $\mathbb{R}^n$. Using the Cauchy-Schwarz inequality, we have that

$$
\begin{aligned}
|\langle \widetilde{\mathbf{x}}, \mathbf{1} \rangle|^2 &= ||\mathbf{x}||_1^2 \\
&\leq ||\mathbf{x}||_2^2\, ||\mathbf{1}||_2^2 \\
&= n\, ||\mathbf{x}||_2^2
\end{aligned}
$$

because $||\mathbf{1}||_2^2 = \sum_{j=1}^{n} 1^2 = n$. Taking the square root of both sides, we get that $||\mathbf{x}||_1 \leq \sqrt{n}\, ||\mathbf{x}||_2$, as desired.

2. (a) Let $\mathbf{A} \in \mathbb{R}^{n \times m}$ be a non-zero matrix with rank r. Write down the singular-value decomposition of $\mathbf{A}$. List the properties of the matrices you use in your decomposition.

   Without loss of generality, we assume that $n \geq m$. The singular value decomposition of $\mathbf{A}$ is given by

   $$\mathbf{A} = \mathbf{U\Sigma V^T},$$

   where $\mathbf{U} \in \mathbb{R}^{n \times n}$ and $\mathbf{V} \in \mathbb{R}^{m \times m}$ are unitary matrices, and $\mathbf{\Sigma} \in \mathbb{R}^{n \times m}$ is a matrix with diagonal entries $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ in the $m \times m$ sub-matrix in its first m rows and m columns, and zeros everywhere else.

   (b) Show that $\mathbb{R}^m$ has an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, $\mathbb{R}^n$ has an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, and there exist $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ such that

   $$
   \mathbf{Av_i} = \begin{cases} \sigma_i \mathbf{u}_i & i = 1, \ldots, r \\ \mathbf{0} & i = r+1, \ldots, m \end{cases}
   $$

   $$
   \mathbf{A^T u_i} = \begin{cases} \sigma_i \mathbf{v_i} & i = 1, \ldots, r \\ \mathbf{0} & i = r+1, \ldots, n. \end{cases}
   $$

   From (a), we have that the SVD of $\mathbf{A} = \mathbf{U\Sigma V^T}$. Since $\mathbf{V} \in \mathbb{R}^{m \times m}$ is a unitary matrix, by construction, its columns $\{\mathbf{v_1}, \ldots, \mathbf{v_m}\}$ form an orthonormal basis of $\mathbb{R}^m$. Furthermore, since $\mathbf{U} \in \mathbb{R}^{n \times n}$ is a unitary matrix, by construction, its columns $\{\mathbf{u_1}, \ldots, \mathbf{u_n}\}$ form an orthonormal basis of $\mathbb{R}^n$.

   We see that

   $$
   \begin{aligned}
   \mathbf{Av_i} &= \mathbf{U\Sigma V^T v_i} \\
   &= \mathbf{U\Sigma e_{i,m}} \\
   &= \mathbf{U}\sigma_i \mathbf{e_{i,n}} \\
   &= \sigma_i \mathbf{u_i},
   \end{aligned}
   $$

where $\mathbf{e_{i,m}}$ and $\mathbf{e_{i,n}}$ are the i^{th} canonical basis vectors of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. If $i \in \{1, 2, \ldots, r\}$, $\sigma_i > 0$, so it follows that $\mathbf{Av_i} = \sigma_i \mathbf{u_i}$. If $i \in \{r+1, \ldots, m\}$, $\sigma_i = 0$, so it follows that $\mathbf{Av_i} = \mathbf{0}$, as desired.

We also see that

$$\begin{aligned}
\mathbf{A^T u_i} &= \mathbf{V \Sigma^T U^T u_i} \\
&= \mathbf{V \Sigma^T e_{i,n}} \\
&= \mathbf{V} \sigma_i \mathbf{e_{i,m}} \\
&= \sigma_i \mathbf{v_i}.
\end{aligned}$$

If $i \in \{1, 2, \ldots, r\}$, $\sigma_i > 0$, so it follows that $\mathbf{Au_i} = \sigma_i \mathbf{v_i}$. If $i \in \{r+1, \ldots, m\}$, $\sigma_i = 0$, so it follows that $\mathbf{A^T u_i} = \mathbf{0}$, as desired.

(c) Argue that

$$\begin{aligned}
\text{Range}\,(\mathbf{A}) &= \text{span}\,\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}\,, \\
\text{Null}\,(\mathbf{A}) &= \text{span}\,\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_m\}\,, \\
\text{Range}\left(\mathbf{A^T}\right) &= \text{span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}\,, \\
\text{Null}\left(\mathbf{A^T}\right) &= \text{span}\,\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n\}\,.
\end{aligned}$$

i. To show that Range $(\mathbf{A}) =$ span $\{\mathbf{u_1}, \ldots, \mathbf{u_r}\}$, we have to show that Range $(\mathbf{A}) \subseteq$ span $\{\mathbf{u_1}, \ldots, \mathbf{u_r}\}$ and span $\{\mathbf{u_1}, \ldots, \mathbf{u_r}\} \subseteq$ Range $(\mathbf{A})$.
We will show the first set inclusion. Let $\mathbf{b} \in$ Range $(\mathbf{A})$ be given, which means that there exists $\mathbf{x} \in \mathbb{R}^m$ such that $\mathbf{Ax} = \mathbf{b}$. Since $\mathbf{Ax}$ is a linear combination of $\mathbf{u_1}, \ldots, \mathbf{u_r}$ (the remaining columns $\mathbf{u_{r+1}}, \ldots, \mathbf{u_n}$ are multiplied by zeros after right-multiplication by $\mathbf{\Sigma}$), $\mathbf{b} \in$ span $(\mathbf{u_1}, \ldots, \mathbf{u_r}\}$. Thus, it must be the case that Range $(\mathbf{A}) \subseteq$ span $\{\mathbf{u_1}, \ldots, \mathbf{u_r}\}$.
To show the set inclusion in the other direction, it suffices to show that $\mathbf{u_1}, \ldots, \mathbf{u_r} \in$ Range $(\mathbf{A})$. From earlier, we showed that $\mathbf{Av_i} = \sigma_i \mathbf{u_i}$ for $i = 1, \ldots, r$ (which implies that $\mathbf{A}\left(\frac{1}{\sigma_i}\mathbf{v_i}\right) = \mathbf{u_i}$). Thus it must be the case that $\mathbf{u_1}, \ldots, \mathbf{u_r} \in$ Range $(\mathbf{A})$, which implies that span $\{\mathbf{u_1}, \ldots, \mathbf{u_r}\} \subseteq$ Range $(\mathbf{A})$.
Thus, Range $\{\mathbf{A}) =$ span $\{\mathbf{u_1}, \ldots, \mathbf{u_r}\}$

ii. To show that Null $(\mathbf{A}) =$ span $\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_m\}$, we show set inclusion both ways.
We will show the first set inclusion. Let $\mathbf{x} \in$ Null $(\mathbf{A})$ be given, i.e., $\mathbf{Ax} = \mathbf{0}$. From the SVD of $\mathbf{A}$, we have that

$$\begin{aligned}
\mathbf{Ax} &= \sum_{i=1}^{r} \left(\sigma_i \mathbf{v_i^T x}\right) \mathbf{u_i} \\
&= \mathbf{0}.
\end{aligned}$$

Taking the inner product of both sides with $\mathbf{u_j}$, $j = 1, \ldots, r$, we get that

$$\begin{aligned}
\mathbf{u_j^T A x} &= \sigma_j \mathbf{v_j^T x} \\
&= 0.
\end{aligned}$$

Since $\sigma_i > 0$ for $i = 1, \ldots, r$, it must be the case that $\mathbf{v_j^T x} = 0$. This is only possible if $\mathbf{x}$ is a linear combination of $\mathbf{v_{r+1}}, \ldots, \mathbf{v_m}$ by orthogonality of the columns of $\mathbf{V}$. Thus, it must be the case that $\mathbf{x} \in$ span $\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_m\}$, which implies that Null $(\mathbf{A}) \subseteq$ span $\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_m\}$.
To show the set inclusion in the other direction, it suffices to show that $\mathbf{v_{r+1}}, \ldots, \mathbf{v_m} \in$ Null $(\mathbf{A})$. From earlier, we showed that $\mathbf{Av_i} = \mathbf{0}$ for $i = r+1, \ldots, m$. Thus, it must be the case that $\mathbf{v_{r+1}}, \ldots, \mathbf{v_m} \in$ Null $(\mathbf{A})$, which implies that span $\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_m\} \subseteq$ Null $(\mathbf{A})$.
Thus, Null $(\mathbf{A}) =$ span $\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_m\}$.

iii. To show that Range $\left(\mathbf{A^T}\right) = \text{span}\left\{\mathbf{v_1}, \ldots, \mathbf{v_r}\right\}$, we have to show that Range $\left(\mathbf{A^T}\right) \subseteq \text{span}\left\{\mathbf{v_1}, \ldots, \mathbf{v_r}\right\}$ and span $\left\{\mathbf{v_1}, \ldots, \mathbf{v_r}\right\} \subseteq$ Range $\left(\mathbf{A^T}\right)$.
We will show the first set inclusion. Let $\mathbf{b} \in$ Range $\left(\mathbf{A^T}\right)$ be given, which means that there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{A^T x} = \mathbf{b}$. Since $\mathbf{A^T x}$ is a linear combination of $\mathbf{v_1}, \ldots, \mathbf{v_r}$ (the remaining columns $\mathbf{v_{r+1}}, \ldots \mathbf{v_m}$ are multiplied by zeros after right-multiplication by $\mathbf{\Sigma^T}$), $\mathbf{b} \in \text{span}\left\{\mathbf{v_1}, \ldots, \mathbf{v_r}\right\}$. Thus, it must be the case that Range $\left(\mathbf{A^T}\right) \subseteq \text{span}\left(\mathbf{v_1}, \ldots, \mathbf{v_r}\right\}$.
To show the set inclusion in the other direction, it suffices to show that $\mathbf{v_1}, \ldots, \mathbf{v_r} \in$ Range $\left(\mathbf{A^T}\right)$. From earlier, we showed that $\mathbf{A^T u_i} = \sigma_i \mathbf{v_i}$ for $i = 1, \ldots, r$ (which implies that $\mathbf{A^T}\left(\frac{1}{\sigma_i}\mathbf{u_i}\right) = \mathbf{v_i}$). Thus it must be the case that $\mathbf{v_1}, \ldots, \mathbf{v_r} \in$ Range $\left(\mathbf{A^T}\right)$, which implies that span $\left\{\mathbf{v_1}, \ldots, \mathbf{v_r}\right\} \subseteq$ Range $\left(\mathbf{A^T}\right)$.
Thus, Range $\left\{\mathbf{A^T}\right) = \text{span}\left\{\mathbf{v_1}, \ldots, \mathbf{v_r}\right\}$.

iv. To show that Null $\left(\mathbf{A^T}\right) = \text{span}\left\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n\right\}$, we show set inclusion both ways.
We will show the first set inclusion. Let $\mathbf{x} \in$ Null $\left(\mathbf{A^T}\right)$ be given, i.e., $\mathbf{A^T x} = \mathbf{0}$. From the SVD of $\mathbf{A^T}$, we have that

$$\begin{aligned}\mathbf{A^T x} &= \sum_{i=1}^{r}\left(\sigma_i \mathbf{u_i^T x}\right)\mathbf{v_i} \\ &= \mathbf{0}.\end{aligned}$$

Taking the inner product of both sides with $\mathbf{v_j}$, $j = 1, \ldots, r$, we get that

$$\begin{aligned}\mathbf{v_j^T A x} &= \sigma_j \mathbf{u_j^T x} \\ &= 0.\end{aligned}$$

Since $\sigma_i > 0$ for $i = 1, \ldots, r$, it must be the case that $\mathbf{u_j^T x} = 0$. This is only possible if $\mathbf{x}$ is a linear combination of $\mathbf{u_{r+1}}, \ldots, \mathbf{u_n}$ by orthogonality of the columns of $\mathbf{U}$. Thus, it must be the case that $\mathbf{x} \in \text{span}\left\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n\right\}$, which implies that Null $\left(\mathbf{A^T}\right) \subseteq \text{span}\left\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n\right\}$.
To show the set inclusion in the other direction, it suffices to show that $\mathbf{u_{r+1}}, \ldots, \mathbf{u_n} \in$ Null $\left(\mathbf{A^T}\right)$. From earlier, we showed that $\mathbf{A^T u_i} = \mathbf{0}$ for $i = r+1, \ldots, n$. Thus, it must be the case that $\mathbf{u_{r+1}}, \ldots, \mathbf{u_n} \in$ Null $\left(\mathbf{A^T}\right)$, which implies that span $\left\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n\right\} \subseteq$ Null $\left(\mathbf{A^T}\right)$.
Thus, Null $\left(\mathbf{A^T}\right) = \text{span}\left\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n\right\}$.

(d) Now show that Range $\left(\mathbf{A^T}\right)$ is orthogonal to Null $\left(\mathbf{A}\right)$.
Earlier, we showed that Range $\left(\mathbf{A^T}\right) = \text{span}\left\{\mathbf{v_1}, \ldots, \mathbf{v_r}\right\}$ and Null $\left(\mathbf{A}\right) = \text{span}\left\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_m\right\}$. Since $\mathbf{V}$ is unitary, its columns are orthonormal, and it must be the case that $\mathbf{v_1}, \ldots, \mathbf{v_r}$ are orthogonal to $\mathbf{v_{r+1}}, \ldots, \mathbf{v_m}$. Thus, Range $\left(\mathbf{A^T}\right)$ is orthogonal to Null $\left(\mathbf{A}\right)$, as desired.

3. (a) Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be non-singular and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Prove the following matrix identity (Sherman-Morrison):

$$\left(\mathbf{A} + \mathbf{u v^T}\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u v^T A^{-1}}}{1 + \mathbf{v^T A^{-1} u}}.$$

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a given non-singular matrix, and let $\mathbf{u}$ and $\mathbf{v}$ be given vectors in $\mathbb{R}^n$. To show that the above identity is the inverse of $\left(\mathbf{A} + \mathbf{u v^T}\right)$, we multiply it through on the left and right of $\left(\mathbf{A} + \mathbf{u v^T}\right)$. We see that $\left(\mathbf{A} + \mathbf{u v^T}\right)\left(\mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u v^T A^{-1}}}{1 + \mathbf{v^T A^{-1} u}}\right)$ is given by

$$\mathbf{A A^{-1}} - \mathbf{A}\frac{\mathbf{A}^{-1}\mathbf{u v^T A^{-1}}}{1 + \mathbf{v^T A^{-1} u}} + \mathbf{u v^T A^{-1}} - \mathbf{u v^T}\frac{\mathbf{A}^{-1}\mathbf{u v^T A^{-1}}}{1 + \mathbf{v^T A^{-1} u}},$$

which, when simplified, yields

$$\mathbf{I} - \frac{\mathbf{u v^T A^{-1}}}{1 + \mathbf{v^T A^{-1} u}} + \frac{\mathbf{u v^T A^{-1}}}{1 + \mathbf{v^T A^{-1} u}} + \left(\mathbf{v^T A^{-1} u}\right)\frac{\mathbf{u v^T A^{-1}}}{1 + \mathbf{v^T A^{-1} u}} - \left(\mathbf{v^T A^{-1} u}\right)\frac{\mathbf{u v^T A^{-1}}}{1 + \mathbf{v^T A^{-1} u}} = \mathbf{I}.$$

Similarly, we see that $\left(\mathbf{A}^{-1} - \frac{\mathbf{A^{-1}uv^TA^{-1}}}{1+\mathbf{v^TA^{-1}u}}\right)\left(\mathbf{A} + \mathbf{uv^T}\right)$ is given by

$$\mathbf{A^{-1}A} + \mathbf{A^{-1}uv^T} - \frac{\mathbf{A}^{-1}\mathbf{uv^TA^{-1}}}{1+\mathbf{v^TA^{-1}u}}\mathbf{A} - \frac{\mathbf{A}^{-1}\mathbf{uv^TA^{-1}}}{1+\mathbf{v^TA^{-1}u}}\mathbf{uv^T},$$

which, when simplified, yields,

$$\mathbf{I} + \frac{\mathbf{A^{-1}uv^T}}{1+\mathbf{v^TA^{-1}u}} + \left(\mathbf{v^TA^{-1}u}\right)\frac{\mathbf{A^{-1}uv^T}}{1+\mathbf{v^TA^{-1}u}} - \frac{\mathbf{A}^{-1}\mathbf{uv^T}}{1+\mathbf{v^TA^{-1}u}} - \left(\mathbf{v^TA^{-1}u}\right)\frac{\mathbf{A^{-1}uv^T}}{1+\mathbf{v^TA^{-1}u}} = \mathbf{I}.$$

Thus, the Sherman-Morrison identity is the inverse of $\left(\mathbf{A} + \mathbf{uv^T}\right)$.

(b) Suppose that the LU factorization of $\mathbf{A}$ is available, e.g., because you computed it. Explain how the Sherman-Morrison identity can be used to solve the system $\left(\mathbf{A} + \mathbf{uv^T}\right)\mathbf{x} = \mathbf{b}$.

Suppose that we have the LU factorization of $\mathbf{A}$, i.e., $\mathbf{A} = \mathbf{LU}$, and $\mathbf{A}$ is non-singular. We know that $\mathbf{A^{-1}} = \mathbf{U^{-1}L^{-1}}$. To solve the system $\left(\mathbf{A} + \mathbf{uv^T}\right)\mathbf{x} = \mathbf{b}$, we left multiply both sides of the system by the Sherman-Morrison identity to obtain the solution $\mathbf{x}$ as follows:

$$\begin{aligned}\mathbf{x} &= \left(\mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{uv^TA^{-1}}}{1+\mathbf{v^TA^{-1}u}}\right)\mathbf{b} \\ &= \left(\mathbf{U^{-1}L^{-1}b} - \frac{\mathbf{U^{-1}L^{-1}uv^TU^{-1}L^{-1}b}}{1+\mathbf{v^TU^{-1}L^{-1}u}}\right).\end{aligned}$$

(c) What is the operation count of the solution approach you made in part (b)? **NOTE**: The power of the Sherman-Morrison identity lies in this part. In some applications, one has to solve the linear system with many different $\mathbf{u}$ and $\mathbf{v}$. This identity saves from having to compute the dense inverse again.

We first note that if an upper or lower triangular matrix is invertible, then their inverses are also upper or lower triangular, respectively. We then note that the computation of $\mathbf{U^{-1}y}$ or $\mathbf{L^{-1}y}$, where $\mathbf{y}$ is a vector in $\mathbb{R}^n$, involves $\sum_{j=1}^{n} j = \frac{n(n-1)}{2}$ multiplications and $\sum_{j=1}^{n-1} j = \frac{(n-1)(n-2)}{2}$ additions, which is a total of $\left(n^2 - 2n + 1\right)$ FLOPs. Furthermore, an inner product of two vectors in $\mathbb{R}^n$ involves n multiplications and $n-1$ additions, which is a total of $2n-1$ FLOPs.

Summing over operation counts of the matrix-vector products and inner products in the solution, we see the operation count of the solution approach in part (b) is $4\left(n^2 - 2n + 1\right) + 2\left(2n-1\right) + 4$ FLOPs, i.e., $\mathcal{O}\left(n^2\right)$ FLOPs.