APPM 5600 - Homework 11

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1. Define $S_n(x) = \frac{1}{n+1} T'_{n+1}(x)$, $n \geq 0$, with $T_{n+1}(x)$ being the Chebyshev polynomial of degree $n+1$. The polynomials $S_n(x)$ are called the *Chebyshev polynomials of the second kind.*

   (a) Show that $\{S_n(x) \mid n \geq 0\}$ is an orthogonal family on $[-1, 1]$ with respect to the weight function $w(x) = \sqrt{1-x^2}$.

   We wish to show that for integers n and m,

$$\begin{aligned}\int_{-1}^{1} S_n(x) S_m(x) \sqrt{1-x^2}\,\mathrm{d}x &= \int_{-1}^{1} \frac{1}{n+1} T'_{n+1}(x) \frac{1}{m+1} T'_{m+1}(x) \sqrt{1-x^2}\,\mathrm{d}x \\ &= \begin{cases} c_n & \text{if } n = m, \text{ where } c_n \neq 0, \\ 0 & \text{if } n \neq m. \end{cases}\end{aligned}$$

   We know that $T_n(x) = \cos(n\theta)$, where $\theta = \arccos(x)$ for $n = 0, 1, \ldots$, and $T'_n(x) = \frac{n \sin(n\theta)}{\sqrt{1-x^2}}$ Thus,

$$\begin{aligned} S_n(x) &= \frac{1}{n+1} \frac{(n+1)\sin((n+1)\theta)}{\sqrt{1-x^2}} \\ &= \frac{\sin((n+1)\theta)}{\sqrt{1-x^2}} \\ &= \frac{1}{\sqrt{1-x^2}} \left(\sin(n\theta)\cos(\theta) + \cos(n\theta)\sin(\theta)\right)\end{aligned}$$

   Then, we have that for non-negative integers m and n,

$$\begin{aligned}\int_{-1}^{1} S_n(x) S_m(x) \sqrt{1-x^2}\,\mathrm{d}x &= \int_{0}^{\pi} \left(\sin(n\theta)\cos(\theta) + \cos(n\theta)\sin(\theta)\right)\left(\sin(m\theta)\cos(\theta) + \cos(m\theta)\sin(\theta)\right)\,\mathrm{d}\theta \\ &= \begin{cases} \frac{1}{2}\left(\frac{\sin(\pi(m-n))}{m-n} - \frac{\sin(\pi(m+n+2))}{m+n+2}\right) & \text{if } m \neq n \\ \frac{\pi}{2} - \frac{\sin(2\pi n)}{4(n+1)} & \text{if } m = n \end{cases} \\ &= \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n, \end{cases}\end{aligned}$$

   where we made use of the u-substitution $\theta = \arccos(x)$. Thus, we have that $S_n(x) = \frac{1}{n+1} T'_{n+1}(x)$, $n \geq 0$ is an orthogonal family on $[-1, 1]$ with respect to the weight function $w(x) = \sqrt{1-x^2}$, as desired.

   (b) Show that the family $\{S_n(x)\}$ satisfies the same triple recursion relation (equation 4.4.13 in Atkinson) as the family $\{T_n(x)\}$.

   To show that the family satisfies the same triple recursion relation, which is given by

$$S_{n+1}(x) = 2x S_n(x) - S_{n-1}(x)$$

   for $n \geq 1$, we consider two cases.

i. $n = 1$
We want to show that

$$S_2(x) = 2xS_1(x) - S_0(x).$$

From Atkinson, we know that

$$\begin{aligned} T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x. \end{aligned}$$

Thus,

$$\begin{aligned} S_0(x) &= \frac{1}{\sqrt{1-x^2}}, \\ S_1(x) &= \frac{2x}{\sqrt{1-x^2}}, \\ S_2(x) &= \frac{4x^2-1}{\sqrt{1-x^2}} \end{aligned}$$

We see that

$$\begin{aligned} 2xS_1(x) - S_0(x) &= \frac{4x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \\ &= \frac{4x^2-1}{\sqrt{1-x^2}} \\ &= S_2(x), \end{aligned}$$

as desired.

ii. $n > 1$
Using our substitution $x = \cos(\theta)$, we want to show that

$$S_{n+1}(x) = 2xS_n(x) - S_{n-1}(x).$$

We know that $\sin(\theta) = \sqrt{1-x^2}$, as well as that

$$\begin{aligned} S_n(x) &= \frac{\sin((n+1)\theta)}{\sin(\theta)} \\ S_{n+1}(x) &= \frac{\sin((n+2)\theta)}{\sin(\theta)} \\ S_{n+2}(x) &= \frac{\sin((n+3)\theta)}{\sin(\theta)}. \end{aligned}$$

By direct calculation, we have that

$$\begin{aligned} 2xS_{n+1}(x) - S_n(x) &= \frac{1}{\sin(\theta)} \left(2\cos(\theta)\sin((n+2)\theta) - \sin((n+1)\theta)\right) \\ &= \frac{1}{\sin(\theta)} \left(\sin((n+3)\theta) + \sin((n+1)\theta) - \sin((n+1)\theta)\right) \\ &= \frac{\sin((n+3)\theta)}{\sin(\theta)} \\ &= S_{n+2}(x), \end{aligned}$$

as desired.

Thus, the S_n's satisfy the triple recursion relation 4.4.13 in Atkinson as the T_n's.

(c) Given $f \in \mathcal{C}([-1,1])$ solve the problem

$$\min \int_{-1}^{1} |f(x) - p_n(x)|^2 \sqrt{1-x^2}\, \mathrm{d}x,$$

where $p_n(x)$ is allowed to range over all polynomials of degree $\leq n$.
From earlier, we showed that $\{S_n(x) \mid n \geq 0\}$ is an orthogonal family on $[-1,1]$, so we can write any polynomial of degree n, p_n, on $[-1,1]$ as

$$p_n(x) = \sum_{j=0}^{n} b_j S_j(x),$$

where b_j are scalars that are to be determined. We seek to minimize $||f - p_n||_w^2$, the weighted inner-product of $f - p_n$ on $[-1,1]$, where $w(x) = \sqrt{1-x^2}$. We know that

$$\begin{aligned} ||f - p_n||_w^2 &= ||f||_w^2 - 2\langle f, p_n\rangle_w + ||p_n||_w^2 \\ &= ||f||_w^2 - 2\sum_{j=0}^{n} b_j \langle f, S_j\rangle_w + \sum_{j=0}^{n} b_j^2 ||S_j||_w^2 \\ &= ||f||_w^2 - 2\sum_{j=0}^{n} b_j \langle f, S_j\rangle_w + \frac{\pi}{2}\sum_{j=0}^{n} b_j^2. \end{aligned}$$

To find the minimizer, we take the derivative of the above with respect to the b_j's and set it equal to 0, from which we obtain that for each $j = 0, \ldots, n$,

$$-2\langle f, S_j\rangle_w + \pi b_j = 0 \implies b_j = \frac{2}{\pi}\langle f, S_j\rangle_w.$$

Thus, the minimizer of the original problem is given by

$$p_n(x) = \frac{2}{\pi}\sum_{j=0}^{n} \langle f, S_j\rangle_w S_j(x).$$

2. Let $\{\phi_i(x)\}_{i=0}^{n}$ be a family of *orthonormal* polynomials with respect to the weighted L^2 inner product on $[a,b]$, where $w(x) \geq 0$ is the weight function. Define the kernel

$$K(x,y) = \sum_{i=0}^{n} \phi_i(x)\,\phi_i(y).$$

Recall that this means $K(x,y)$ is a separable kernel. Prove that the following formula defines the optimal weighted L^2 polynomial

$$p(x) = \int_a^b K(x,y)\, f(y)\, w(y)\ \mathrm{d}y.$$

Taking p as above, we have that

$$
\begin{aligned}
p(x) &= \int_a^b \left(\sum_{i=0}^{n} \phi_i(x)\, \phi_i(y) \right) f(y)\, w(y)\ \mathrm{d}y \\
&= \int_a^b \left(\sum_{i=0}^{n} \phi_i(x)\, \phi_i(y)\, f(y)\, w(y) \right) \mathrm{d}y \\
&= \sum_{i=0}^{n} \left(\int_a^b \phi_i(x)\, \phi_i(y)\, f(y)\, w(y)\ \mathrm{d}y \right) \\
&= \sum_{i=0}^{n} \left(\int_a^b \phi_i(y)\, f(y)\, w(y)\ \mathrm{d}y \right) \phi_i(x) \\
&= \sum_{i=0}^{n} \langle f, \phi_i \rangle_w\, \phi_i(x)\,.
\end{aligned}
$$

We know that the optimal weighted L^2 polynomial of degree n or less that approximates f is given by

$$
\begin{aligned}
p_n(x) &= \sum_{j=0}^{n} \frac{\langle f, \phi_i \rangle_w}{||\phi_i||_w^2} \phi_i(x) \\
&= \sum_{j=0}^{n} \langle f, \phi_i \rangle_w\, \phi_i(x) \\
&= \int_a^b K(x, y)\, f(y)\, w(y)\ \mathrm{d}y,
\end{aligned}
$$

since the ϕ_i's are orthonormal on $[a, b]$ with respect to the weighted L^2 inner product on $[a, b]$, as desired.

3. Let $p(x) = \sum_{i=0}^{n} c_i \phi_i(x)$ be the optimal degree-n polynomial approximation of the function $f \in \mathcal{C}([a, b])$, where $\phi(x)$ are *orthonormal* polynomials. Prove the following (for any weighted L^2 norm):

   (a) Bessel's inequality: $||p|| \leq ||f||$.
   We know that $c_n = \langle f, \phi_i \rangle_w$ by orthonormality of the ϕ_i's with respect to the weighted L^2 inner product on $[a, b]$ and by p being the optimal degree n polynomial approximation of f with respect to the weighted L^2 norm on $[a, b]$. Furthermore, we also know that

$$
\begin{aligned}
0 &\leq ||f - p||_w^2 \\
&= ||f||_w^2 - 2 \sum_{i=0}^{n} c_i \langle f, \phi_i \rangle_w + \sum_{i=0}^{n} c_i^2\, ||\phi_i||_w^2 \\
&= ||f||_w^2 - 2 \sum_{i=0}^{n} c_i^2 + \sum_{i=0}^{n} c_i^2 \\
&= ||f||_w^2 - \sum_{i=0}^{n} c_i^2.
\end{aligned}
$$

   Rearranging the above, we get

$$
\sum_{i=0}^{n} c_i^2 \leq ||f||_w^2\,,
$$

but we note that

$$\begin{aligned}
||p||_w^2 &= \sum_{i=0}^{n} \sum_{j=0}^{n} c_i c_j \left\langle \phi_i, \phi_j \right\rangle_w \\
&= \sum_{i=0}^{n} c_i^2.
\end{aligned}$$

Thus, we have that

$$||p||_w^2 \leq ||f||_w^2 \implies ||p||_w \leq ||f||_w \, ,$$

as desired.

(b) Parseval's relation: $\sum_{i=0}^{\infty} c_i^2 = ||f||^2$.

Suppose that $\{\phi_i\}_{i=0}^{\infty}$ is a family of orthonormal polynomials with respect to the weighted L^2 inner product on $[a, b]$. Further suppose that $||f||_w^2 < \infty$.

We know that the optimal degree n polynomial approximation to f with respect to the weighted L^2 inner product on $[a, b]$ is given by $p_n(x) = \sum_{i=0}^{n} c_i \phi_i(x)$, where $c_i = \left\langle f, \phi_i \right\rangle_w$ for each i. In part (a), we also showed that

$$\begin{aligned}
||f||_w^2 &= ||p_n||_w^2 + ||f - p_n||_w^2 \\
&= \sum_{i=0}^{n} c_i^2 + ||f - p_n||_w^2 \, .
\end{aligned}$$

Since $[a, b]$ is assumed to be finite and the ϕ_i's are orthonormal, by theorem 4.5.7 in Atkinson, we have that

$$\lim_{n \to \infty} ||f - p_n||_w = 0.$$

Thus, we have that

$$\begin{aligned}
\lim_{n \to \infty} ||f||_w^2 = \lim_{n \to \infty} \left(\sum_{i=0}^{n} c_i^2 + ||f - p_n||_w^2 \right) &\implies ||f||_w^2 = \lim_{n \to \infty} \sum_{i=0}^{n} c_i^2 \\
&\implies ||f||_w^2 = \sum_{i=0}^{\infty} c_i^2,
\end{aligned}$$

as desired.