APPM 5600 — In class assignment # 7
Interpolation

1. Consider the tasks of interpolating $f(x) = e^x$ in the interval $[0, 1]$ with equispaced nodes $x_j = jh$ where $h = 1/n$. Create a sequence of polynomial interpolants in increasing order to say 20^{th} order. Plot $f(x)$ and the collection of interpolants on the interval $[0, 1]$. What happens as the order increases?

2. Now consider a similar task with $f(x) = \frac{1}{1+x^2}$ on the interval $[-5, 5]$. Create the polynomial interpolants with equispaced nodes with increasing order. Plot $f(x)$ and the collection of interpolants on $[-5, 5]$. What happens as the order increases? Is this similar or different from the behavior you observed in the first problem? What you should observe is something called the *Runge phenomena*.

3. Next, do the same exercise with a different set of interpolation nodes. Let your interpolation nodes be $x_j = \cos\left(\left(\frac{(2j+1)\pi}{2(n+1)}\right)\right.$ for $j = 0, \ldots n$. (NOTE: you have to scale the nodes so that they are in the appropriate interval.) Are the result similar to the results from the previous problem?

4. The polynomials that are associated with using the nodes in the last problem are called the Chebychev polynomials. They can be written via a three term recursion

$$T_0(x) = 1,\ T_1(x) = x,\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

   The polynomials can also be written as a trig function $T_{n+1}(x) = \cos((n+1)\cos^{-1}(x))$. Prove that the x_j in the previous problem are the roots of the Chebychev polynomials.

5. Do you have an intuition of why one set of interpolation nodes is better than another set? Hint: Think about the basis being used in creating the interpolating polynomial. (You can look at the Newton basis. We will go into this in more detail next week.)

6. Consider the tasks of interpolating the following function:

$$f(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0 \end{cases}$$

   on $[-1, 1]$ using both equispaced nodes and Chebychev nodes. Is one better than the other? If you are observing oscillations near 0, this is called the *Gibbs phenomena*. Do you have any ideas for how to reduce them? Try them to see if anything is better.

7. Find the second degree polynomial $p_2(x)$ that interpolates $[5, 7, 12]$ at the points $[1, 2, 3]$ by hand using Newton-Divided Differences.