APPM 5600 - Homework 3

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1. Let x_0 and x_1 be two successive points from a secant method applied to solving $f(x) = 0$, with $f_0 = f(x_0)$ and $f_1 = f(x_1)$. Show that regardless of which point x_0 or x_1 is regarded as the most recent point, the new point derived from the secant step will be the same.

   The secant iteration is given by

$$x_{n+1} = x_n - f_n \frac{x_n - x_{n-1}}{f_n - f_{n-1}},$$

   where $n \geq 1$. If we take x_1 as our most recent iterate, we see that

$$\begin{aligned} x_1 - f_1 \frac{x_1 - x_0}{f_1 - f_0} &= \frac{(f_1 - f_0) x_1}{f_1 - f_0} - \frac{f_1 (x_1 - x_0)}{f_1 - f_0} \\ &= \frac{f_1 x_1 - f_0 x_1 - f_1 x_1 + f_1 x_0}{f_1 - f_0} \\ &= \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}. \end{aligned}$$

   If we take x_0 as our most recent iterate, we see that

$$\begin{aligned} x_0 - f_0 \frac{x_0 - x_1}{f_0 - f_1} &= \frac{(f_0 - f_1) x_0}{f_0 - f_1} - \frac{f_0 (x_0 - x_1)}{f_0 - f_1} \\ &= \frac{f_0 x_0 - f_1 x_0 - f_0 x_0 + f_0 x_1}{f_0 - f_1} \\ &= \frac{-f_1 x_0 + f_0 x_1}{f_0 - f_1} \\ &= (-1) \left(\frac{f_1 x_0 - f_0 x_1}{f_0 - f_1} \right) \\ &= \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}, \end{aligned}$$

   which we see is the same point derived from the secant iteration by taking x_1 as the most recent iterate. Thus, we have shown that regardless of which point x_0 or x_1 we take as our most recent iterate, the new point derived from the secant iteration will be the same

2. Determine whether the following sets of vectors are linearly dependent or linearly independent.

   (a) $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ -5 \\ 11 \end{bmatrix}$

The above vectors are linearly dependent. When we form a matrix whose columns are the above vectors and row-reduce the matrix, we obtain two rows of zero vectors:

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 9 \\ -1 & 1 & -5 \\ 3 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & -7 & 7 \\ 0 & 4 & -4 \\ 0 & -8 & 8 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since we are not able to obtain a pivot in every column of the above matrix, this indicates that the matrix columns are linearly dependent, i.e., the original vectors are linearly dependent.

(b) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

The above vectors are linearly independent. When we form a matrix whose columns are the above vectors and row-reduce the matrix, we obtain a pivot in every diagonal entry:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since we able to obtain a pivot in every column of the above matrix, this indicates that the matrix columns are linearly independent, i.e., the original vectors are linearly independent.

3. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be linearly dependent vectors in $\mathbb{R}^n$, and let $\mathbf{A}$ be a non-singular $n \times n$ matrix. Define $\mathbf{y}_i = \mathbf{A}\mathbf{x}_i$ for $i = 1, 2, \ldots, k$. Show that $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ are linearly independent.

   For the sake of contradiction, suppose that $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ are linearly dependent, i.e., there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$, not all zero, such that $\sum_{i=1}^k \alpha_i \mathbf{y}_i = \mathbf{0}$. We see that

$$\sum_{i=1}^k \alpha_i \mathbf{y}_i = \mathbf{0} \iff \sum_{i=1}^k \alpha_i \mathbf{A}\mathbf{x}_i = \mathbf{0}$$
$$\iff \mathbf{A}\left(\sum_{i=1}^k \alpha_i \mathbf{x}_i\right) = \mathbf{0}.$$

   Since $\mathbf{A}$ is non-singular, it must be the case that $\sum_{i=1}^k \alpha_i \mathbf{x}_i = \mathbf{0}$. However, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are linearly independent by assumption, which implies that $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$. This is a contradiction as $\alpha_1, \alpha_2, \ldots, \alpha_k$ were assumed to not all be identically zero. Thus, it must be the case that $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ are linearly independent, as desired.

4. Given the orthogonal vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix},$$

produce a third vector $\mathbf{u}_3$ such that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. Normalize the vectors to create an orthonormal basis.

To obtain a third vector that is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$, we can take their cross-product:

$$\begin{aligned}
\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3 \\
&= \left| \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 1 & 1 & 3 \end{bmatrix} \right| \\
&= \begin{bmatrix} 7 \\ -4 \\ -1 \end{bmatrix}.
\end{aligned}$$

By construction, $\mathbf{u}_3$ is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$. We then normalize $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ to obtain $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$:

$$\begin{aligned}
\mathbf{v}_1 &= \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \\
\mathbf{v}_2 &= \frac{1}{\sqrt{11}} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \\
\mathbf{v}_3 &= \frac{1}{\sqrt{66}} \begin{bmatrix} 7 \\ -4 \\ -1 \end{bmatrix}.
\end{aligned}$$

Since $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are three orthonormal vectors in $\mathbb{R}^3$, they form an orthonormal basis for $\mathbb{R}^3$.

5. Prove that similar matrices have the same eigenvalues and that there is a one-to-one correspondence of the eigenvectors.

   Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices that are similar, i.e., there exists a non-singular $n \times n$ matrix $\mathbf{P}$ such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$. Furthermore, let $(\lambda, \mathbf{v})$ be an eigenpair of $\mathbf{A}$. Since $\mathbf{P}$ is non-singular, there exists a vector $\mathbf{u} \in \mathbb{R}^n$ such that $\mathbf{P}\mathbf{u} = \mathbf{v}$. It follows that

$$\begin{aligned}
\mathbf{B}\mathbf{u} &= \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{u} \\
&= \mathbf{P}^{-1}\mathbf{A}\mathbf{v} \\
&= \mathbf{P}^{-1}\lambda\mathbf{v} \\
&= \lambda\mathbf{P}^{-1}\mathbf{v} \\
&= \lambda\mathbf{u}.
\end{aligned}$$

   It follows that $(\lambda, \mathbf{u})$ is an eigenpair of $\mathbf{B}$, and we can obtain $\mathbf{u}$ by solving $\mathbf{P}\mathbf{u} = \mathbf{v}$, i.e., $\mathbf{u} = \mathbf{P}^{-1}\mathbf{v}$. This is a one-to-one correspondence between the eigenvectors of $\mathbf{A}$ and $\mathbf{B}$ since $\mathbf{P}$ is non-singular.

6. A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is **positive-definite** if and only if $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle > 0$ for all $\mathbf{x} \in \mathbb{R}^n$, where $\mathbf{x} \neq 0$. Prove that if $\mathbf{A}$ is positive-definite, then $\mathbf{A}$ is non-singular.

   Let $\mathbf{A}$ be a positive-definite matrix. For the sake of contradiction, suppose that $\mathbf{A}$ is singular, i.e., there exists a vector $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} \neq \mathbf{0}$ and

$$\mathbf{A}\mathbf{x} = \mathbf{0}.$$

   We take the inner product of both sides of the equation with $\mathbf{x}$ and see that

$$\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} = 0.$$

   However, since $\mathbf{x}$ was not identically zero and we know that $\mathbf{y}^{\mathbf{T}}\mathbf{A}\mathbf{y} > 0$ for any $\mathbf{y} \in \mathbb{R}^n$ that is not identically zero, it cannot be the case that $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} = 0$. Thus, it follows that $\mathbf{A}$ is non-singular, as desired.

7. Let $\mathbf{M}$ be any real $n \times n$ non-singular matrix, and let $\mathbf{A} = \mathbf{M^T M}$. Prove that $\mathbf{A}$ is positive-definite.

   Let $\mathbf{x} \in \mathbb{R}^n$ be given, where $\mathbf{x} \neq \mathbf{0}$. Taking the inner-product of $\mathbf{x}$ with $\mathbf{Ax}$, we see that

$$\begin{aligned} \mathbf{x^T A x} &= \mathbf{x^T M^T M x} \\ &= ||\mathbf{Mx}||_2^2 . \end{aligned}$$

   Since $\mathbf{M}$ is non-singular by assumption and $\mathbf{x} \neq \mathbf{0}$, it cannot be the case that $\mathbf{Mx} = \mathbf{0}$. Thus, it follows that $||\mathbf{Mx}||_2^2$ is positive, which implies that $\mathbf{A}$ is positive-definite, as desired.