APPM 5610 - Homework 10

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1. Solve Poisson's equation

$$\nabla \cdot \nabla u = f$$

on the square $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ with homogeneous Dirichlet boundary conditions. Assume that the function f is well approximated by

$$f(x, y) = \sum_{m=1}^{N} \sum_{n=1}^{N} a_{m,n} \sin(m\pi x) \sin(n\pi y).$$

Choose an appropriate discretization for $f(x, y)$ and organize computation to use the fast Fourier transform. Verify your results on several examples.

In the code, we use $N = 2^7 = 128$ points in the x- and y-directions to discretize the unit square. For a given right-hand side f, we evaluate f at the gridpoints of the discretization. Afterward, we apply the two-dimensional discrete sine transform (DST) to obtain f's Fourier sine coefficients, denoted $a_{m,n}$ for $1 \leq m \leq N$, $1 \leq n \leq N$. We then assume that the solution is well approximated by

$$u(x, y) = \sum_{m=1}^{N} \sum_{n=1}^{N} b_{m,n} \sin(m\pi x) \sin(n\pi y),$$

where $b_{m,n}$ are u's Fourier sine coefficients for $1 \leq m \leq N$, $1 \leq n \leq N$. Then, it follows that the Laplacian of u is given by

$$\nabla \cdot \nabla u(x, y) = -\pi^2 \sum_{m=1}^{N} \sum_{n=1}^{N} \left(m^2 + n^2\right) b_{m,n} \sin(m\pi x) \sin(n\pi y).$$

We can then obtain $b_{m,n}$ by equating the coefficients of $\sin(m\pi x)\sin(n\pi y)$ in the above sums, i.e.,

$$b_{m,n} = -\frac{a_{m,n}}{\pi^2 (m^2 + n^2)}, \quad 1 \leq m \leq N, 1 \leq n \leq N.$$

To then recover u at the discretization gridpoints, we apply the two-dimensional inverse discrete sine transform (IDST) to the $b_{m,n}$'s.

For verification of results, we consider the following right-hand sides for f:

(a) $f(x, y) = \sin(\pi x) \sin(\pi y)$.

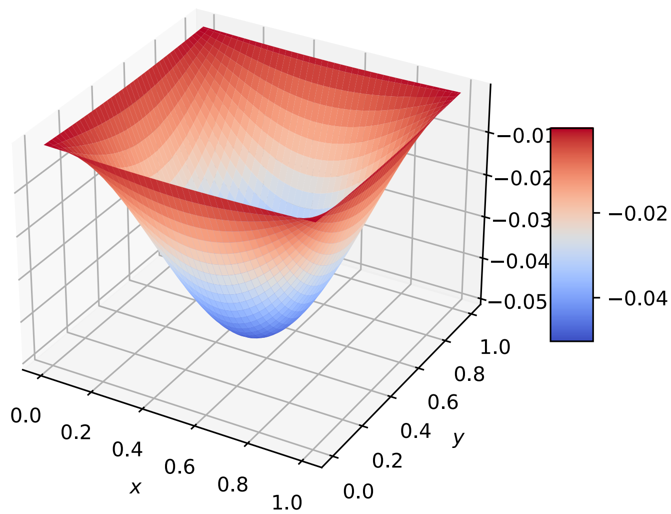

Figure 1: Numerical solution computed via DST and IDST for $f\left(x,y\right)=\sin\left(\pi x\right)\sin\left(\pi y\right)$

(b) $f\left(x,y\right)=\sin\left(4\pi x\right)\sin\left(\pi y\right)$.

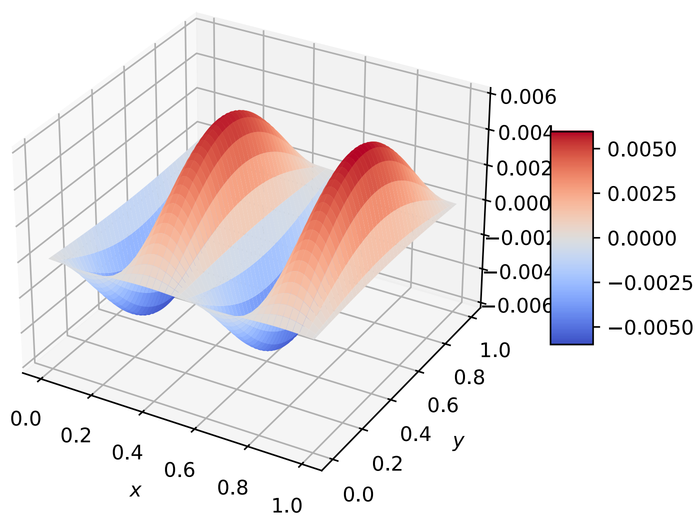

Figure 2: Numerical solution computed via DST and IDST for $f\left(x,y\right)=\sin\left(4\pi x\right)\sin\left(\pi y\right)$

(c) $f(x, y) = -\sin(\pi x) \sin(5 * \pi y)$.

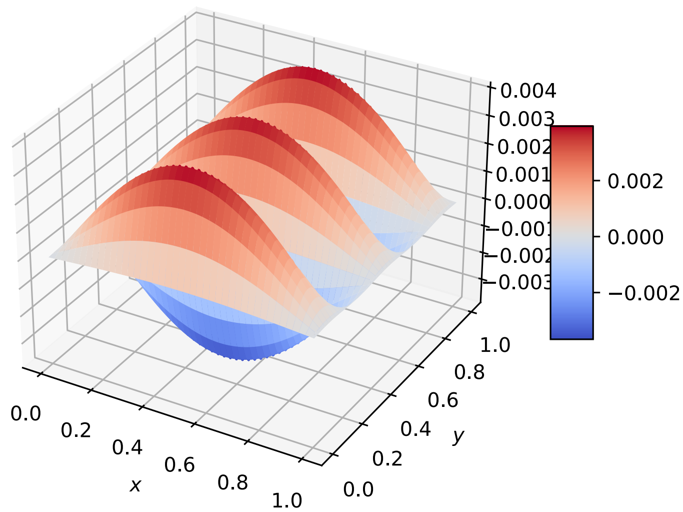

Figure 3: Numerical solution computed via DST and IDST for $f(x, y) = -\sin(\pi x) \sin(5\pi y)$

The code is given below:

```
from re import M
from scipy.fft import dstn, idstn
import numpy as np
import matplotlib.pyplot as plt
from matplotlib import cm

def main():
    # f = lambda x, y: np.sin(np.pi*x)*np.sin(np.pi*y)
    # f = lambda x, y: np.sin(4*np.pi*x)*np.sin(np.pi*y)
    f = lambda x, y: -np.sin(np.pi*x)*np.sin(5*np.pi*y)

    # Discretize grid
    a = 0
    b = 1
    p = 7
    # To efficiently use the Cooley-Tukey algorithm, we use 2**p points in
    # either direction
    q = 2**p

    x = np.linspace(a, b, q)
    y = x
    X, Y = np.meshgrid(x, y)
```

```
    # Compute sine wavenumbers
    m = np.arange(1, q + 1)
    n = m
    M, N = np.meshgrid(m, n)

    # Compute u at gridpoints via the DST and IDST
    f_hat = dstn(f(X, Y), norm="ortho")
    u_hat = np.divide(-f_hat, (np.pi)**2*(M**2 + N**2))
    u = idstn(u_hat, norm="ortho")

    # Plot solution
    fig, ax = plt.subplots(subplot_kw={"projection": "3d"})
    surf = ax.plot_surface(X, Y, u, cmap="coolwarm",
                           linewidth=0, antialiased=True)
    fig.colorbar(surf, shrink=0.4, aspect=5)
    ax.set_xlabel("$x$")
    ax.set_ylabel("$y$")
    ax.set_zlabel("$u ( x, y )$")
    plt.show()

main()
```

2. Let ∂D be the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Consider the boundary value problem

$$\begin{aligned} \nabla \cdot \nabla u &= 1, \quad (x, y) \in D, \\ u &= x^4 + y^4, \quad (x, y) \in \partial D. \end{aligned}$$

(a) Reduce the problem to that with homogeneous boundary conditions.

We first define $v(x, y) = u(x, y) - a(x, y)$, and we seek a function a such that $v(x, y) = 0$ for $(x, y) \in \partial D$. Since $u(x, y) = x^4 + y^4$ for $(x, y) \in \partial D$, we require that $a(x, y) = x^4 + y^4$ for $(x, y) \in \partial D$.

One such choice for a is $a(x, y) = x^4 + y^4$. Making the substitution v into the equation, we have that

$$\begin{aligned} \nabla \cdot \nabla v &= 1 - 12x^2 - 12y^2, \quad (x, y) \in D \\ v &= 0, \quad (x, y) \in \partial D. \end{aligned}$$

(b) Reduce the problem to the Dirichlet problem for the Laplace equation.

We first define $v(x, y) = u(x, y) - a(x, y)$, and we seek a function a such that $\nabla \cdot \nabla v = 0$. Since $\nabla \cdot \nabla u = 1$, we require that $\nabla \cdot \nabla a = 1$. One such choice for a is $a(x, y) = \frac{1}{4}x^2 + \frac{1}{4}y^2$. Making the substitution v into the equation, we have that

$$\begin{aligned} \nabla \cdot \nabla v &= 0, \quad (x, y) \in D \\ v &= x^4 + y^4 - \frac{1}{4}x^2 - \frac{1}{4}y^2, \quad (x, y) \in \partial D. \end{aligned}$$