# APPM 5610 - Homework 3

Eappen Nelluvelil

February 2022

1. Prove that

   (a) If all singular values of matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ are equal, then $\mathbf{A} = \gamma \widetilde{\mathbf{U}}$, where $\widetilde{\mathbf{U}}$ is a unitary matrix and $\gamma$ is a constant.

   Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be given, and suppose that $\mathbf{A}$'s singular values are the same, i.e., $\sigma_1 = \ldots = \sigma_n = \gamma$, where $\gamma$ is some constant.

   We can write the SVD of $\mathbf{A}$ as $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$, where $\mathbf{U}, \mathbf{V} \in \mathbb{C}^{n \times n}$ are unitary and $\mathbf{\Sigma} = \text{diag}\left(\sigma_1, \ldots, \sigma_n\right) \in \mathbb{R}^{n \times n}$.

   Since $\sigma_1 = \ldots = \sigma_n = \gamma$, we can write $\mathbf{\Sigma} = \gamma \mathbf{I}$.

   This means that

$$
\begin{aligned}
\mathbf{A} &= \mathbf{U}\mathbf{\Sigma}\mathbf{V}^* \\
&= \mathbf{U}\left(\gamma \mathbf{I}\right)\mathbf{V}^* \\
&= \gamma \mathbf{U}\mathbf{V}^* \\
&= a\widetilde{\mathbf{U}},
\end{aligned}
$$

   where $\widetilde{\mathbf{U}} = \mathbf{U}\mathbf{V}^*$. By construction, $\widetilde{\mathbf{U}}$ is unitary. Thus, $\mathbf{A} = \gamma \widetilde{\mathbf{U}}$, as desired.

   (b) If $\mathbf{A} \in \mathbb{C}^{n \times n}$ is non-singular and $\lambda$ is an eigenvalue of $\mathbf{A}$, then $\left|\left|\mathbf{A}\right|\right|_2^{-1} \leq |\lambda| \leq \left|\left|\mathbf{A}\right|\right|_2$.

   Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be given such $\mathbf{A}$ is non-singular, and let $(\lambda, \mathbf{x})$ and $\left(\frac{1}{\lambda}, \mathbf{y}\right)$ be eigenpairs of $\mathbf{A}$ and $\mathbf{A}^{-1}$, respectively.

   We know that $\left|\left|\mathbf{A}\mathbf{x}\right|\right|_2 = \left|\left|\lambda \mathbf{x}\right|\right|_2 = |\lambda| \left|\left|\mathbf{x}\right|\right|_2$. But we also know that $\left|\left|\mathbf{A}\mathbf{x}\right|\right|_2 \leq \left|\left|\mathbf{A}\right|\right|_2 \left|\left|\mathbf{x}\right|\right|_2$, which implies that

$$
|\lambda| \left|\left|\mathbf{x}\right|\right|_2 \leq \left|\left|\mathbf{A}\right|\right|_2 \left|\left|\mathbf{x}\right|\right|_2 \implies |\lambda| \leq \left|\left|\mathbf{A}\right|\right|_2.
$$

   We know that $\left|\left|\mathbf{A}^{-1}\mathbf{y}\right|\right|_2 = \left|\left|\frac{1}{\lambda}\mathbf{y}\right|\right|_2 = \left|\frac{1}{\lambda}\right| \left|\left|\mathbf{y}\right|\right|_2$. But we also know that $\left|\left|\mathbf{A}^{-1}\mathbf{y}\right|\right|_2 \leq \left|\left|\mathbf{A}^{-1}\right|\right|_2 \left|\left|\mathbf{y}\right|\right|_2$, which implies that

$$
\begin{aligned}
\left|\frac{1}{\lambda}\right| \left|\left|\mathbf{y}\right|\right|_2 \leq \left|\left|\mathbf{A}^{-1}\right|\right|_2 \left|\left|\mathbf{y}\right|\right|_2 &\implies \left|\frac{1}{\lambda}\right| \leq \left|\left|\mathbf{A}^{-1}\right|\right|_2 \\
&\implies \left|\left|\mathbf{A}^{-1}\right|\right|_2^{-1} \leq |\lambda|.
\end{aligned}
$$

   Thus, $\left|\left|\mathbf{A}^{-1}\right|\right|_2^{-1} \leq |\lambda| \leq \left|\left|\mathbf{A}\right|\right|_2$, as desired.

2. Show that any square matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ may be represented in the form $\mathbf{A} = \mathbf{S}\mathbf{U}$, where $\mathbf{S}$ is a Hermitian non-negative definite matrix and $\mathbf{U}$ is a unitary matrix. Show that if $\mathbf{A}$ is invertible, such representation is unique.

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be given. The SVD of $\mathbf{A}$ is given by $\mathbf{A} = \mathbf{U\Sigma V}^*$, where $\mathbf{U}, \mathbf{V} \in \mathbb{C}^{n\times r}$ have orthonormal columns, $\mathbf{\Sigma} = \text{diag}\left(\sigma_1, \ldots, \sigma_r\right)$, and $r$ is the rank of $\mathbf{A}$. We see that

$$\begin{aligned}\mathbf{A} &= \mathbf{U\Sigma V}^* \\ &= \mathbf{U\Sigma U}^*\mathbf{UV}^* \\ &= \mathbf{S}\widetilde{\mathbf{U}},\end{aligned}$$

where $\mathbf{S} = \mathbf{U}\Sigma\mathbf{U}^*$ and $\widetilde{\mathbf{U}} = \mathbf{UV}^*$. By construction, $\mathbf{S}$ is Hermitian, and for any $\mathbf{x} \in \mathbb{C}^n$, we have that

$$\begin{aligned}\mathbf{x}^*\mathbf{U}\Sigma\mathbf{U}^*\mathbf{x} &= \mathbf{y}^*\mathbf{\Sigma y} \\ &\geq 0\end{aligned}$$

since $\mathbf{\Sigma}$ is a diagonal matrix whose entries are positive, and $\mathbf{y} = \mathbf{U}^*\mathbf{x}$. Thus, $\mathbf{S}$ is also positive semi-definite. Furthermore, $\widetilde{\mathbf{U}}$ is unitary by construction:

$$\begin{aligned}\widetilde{\mathbf{U}}^*\widetilde{\mathbf{U}} &= \mathbf{VU}^*\mathbf{UV}^* \\ &= \mathbf{VV}^* \\ &= \mathbf{I}, \\ \widetilde{\mathbf{U}}\widetilde{\mathbf{U}}^* &= \mathbf{UV}^*\mathbf{VU}^* \\ &= \mathbf{UU}^* \\ &= \mathbf{I}.\end{aligned}$$

Thus, $\mathbf{A} = \mathbf{S}\widetilde{\mathbf{U}}$, where $\mathbf{S}$ is Hermitian, positive semi-definite and $\widetilde{\mathbf{U}}$ is unitary, as desired.

We also see that $\mathbf{AA}^* = \mathbf{SUU}^*\mathbf{S} = \mathbf{S}^2$, from which we get that $\mathbf{S} = \sqrt{(\mathbf{AA}^*)}$. Since $\mathbf{AA}^*$ is Hermitian, positive semi-definite, the matrix $\mathbf{S}$ is uniquely determined.

Now, suppose that $\mathbf{A}$ is also non-singular, i.e., $\mathbf{A}^{-1}$ exists. This implies that $\mathbf{S}$ is invertible, and we get that

$$\mathbf{A} = \mathbf{SU} \implies \mathbf{S}^{-1}\mathbf{A} = \mathbf{U}.$$

Since $\mathbf{S}^{-1}$ is unique, we get that $\mathbf{U}$ is also uniquely determined if $\mathbf{A}$ is non-singular.

3. Consider the Discrete Fourier Transform (DFT) matrix $\mathbf{F} \in \mathbb{C}^{n\times n}$,

$$\mathbf{F} = \begin{bmatrix}\mathbf{F}_0 & \mathbf{F}_1 & \ldots & \mathbf{F}_{n-1}\end{bmatrix},$$

where $\omega = e^{\frac{2\pi i}{n}}$ is the $n^{th}$ root of unity. Show that $\mathbf{F}^*\mathbf{F} = n\mathbf{I}$, where $\mathbf{I}$ is the $n\times n$ identity matrix.

Note that we can write $\mathbf{F}$ as

$$\mathbf{F} = \begin{bmatrix}\mathbf{F}_0 & \mathbf{F}_1 & \ldots & \mathbf{F}_{n-1},\end{bmatrix}$$

where

$$\mathbf{F}_\ell = \begin{bmatrix}\left(\omega^0\right)^\ell \\ \left(\omega^1\right)^\ell \\ \vdots \\ \left(\omega^{n-1}\right)^\ell\end{bmatrix}$$

Then, we have that $\left(\mathbf{F}^*\mathbf{F}\right)_{k\ell}$, where $0 \leq k, \ell \leq n-1$, is given by the following two cases:

(a) $k \neq \ell$

In this case,

$$\begin{aligned}(\mathbf{F}^*\mathbf{F})_{k\ell} &= \sum_{j=0}^{n-1} \left(\exp\left(-\frac{2\pi i j}{n}\right)\right)^k \left(\exp\left(\frac{2\pi i j}{n}\right)\right)^\ell \\ &= \sum_{j=0}^{n-1} \exp\left(\frac{2\pi i j\left(\ell - k\right)}{n}\right)\end{aligned}$$

This is a geometric series of the form $\sum_{k=0}^{n} r^k$, whose sum is given by $\frac{1-r^{n+1}}{1-r}$. Thus,

$$\begin{aligned}\sum_{j=0}^{n-1} \exp\left(\frac{2\pi i j\left(\ell - k\right)}{n}\right) &= \frac{1 - \exp\left(\frac{2\pi i(\ell-k)}{n}\right)^n}{1 - \exp\left(\frac{2\pi i(\ell-k)}{n}\right)} \\ &= \frac{1 - \exp\left(2\pi i\left(\ell - k\right)\right)}{1 - \exp\left(\frac{2\pi i(\ell-k)}{n}\right)} \\ &= 0\end{aligned}$$

since $\ell - k$ is an integer. Thus, the off-diagonal entries of $\mathbf{F}^*\mathbf{F}$ are zero.

(b) $k = \ell$

In this case,

$$\begin{aligned}(\mathbf{F}^*\mathbf{F})_{kk} &= \sum_{j=0}^{n-1} \left(\exp\left(-\frac{2\pi i j}{n}\right)\right)^k \left(\exp\left(\frac{2\pi i j}{n}\right)\right)^k \\ &= \sum_{j=0}^{n-1} \exp\left(0\right) \\ &= n.\end{aligned}$$

Thus, the diagonal-entries of $\mathbf{F}^*\mathbf{F}$ are $n$.

Thus, $\mathbf{F}^*\mathbf{F} = n\mathbf{I}$.