APPM 5610 - Homework 7

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1. Determine the order and the region of absolute stability of the s-step Adams-Bashforth methods for $s = 2, 3$.

$$s = 2: \quad y_{n+2} = y_{n+1} + h\left[\frac{3}{2}f\left(t_{n+1}, y_{n+1}\right) - \frac{1}{2}f\left(t_n, y_n\right)\right]$$
$$s = 3: \quad y_{n+3} = y_{n+2} + h\left[\frac{23}{12}f\left(t_{n+2}, y_{n+2}\right) - \frac{4}{3}f\left(t_{n+1}, y_{n+1}\right) + \frac{5}{12}f\left(t_n, y_n\right)\right]$$

and the order and the region of absolute stability of the 2-step Adams-Moulton method,

$$s = 2: \quad y_{n+2} = y_{n+1} + h\left[\frac{5}{12}f\left(t_{n+2}, y_{n+2}\right) + \frac{2}{3}f\left(t_{n+1}, y_{n+1}\right) - \frac{1}{12}f\left(t_n, y_n\right)\right].$$

(a) 2-step Adams-Bashforth

To determine the order of the method, we use condition 2.11 in Iserles.

The coefficients of the above linear multistep method are given by $a_0 = 0$, $a_1 = -1$, $a_2 = 1$ and $b_0 = -\frac{1}{2}$, $b_1 = \frac{3}{2}$, and $b_2 = 0$.

We now check the conditions outlined in 2.11:

$$\sum_{m=0}^{2} a_m = 0 - 1 + 1 = 0$$
$$\sum_{m=0}^{2} m^1 a_m = 0^1(0) + 1(-1) + 2(1) = 1$$
$$1\left(\sum_{m=0}^{2} m^0 b_m\right) = 1\left(-\frac{1}{2}\right) + 1\left(\frac{3}{2}\right) = 1$$
$$\sum_{m=0}^{2} m^2 a_m = 0^2(0) + 1^2(-1) + 2^2(1) = 3$$
$$2\left(\sum_{m=0}^{2} m^1 b_m\right) = 2\left(\frac{3}{2}\right) = 3$$
$$\sum_{m=0}^{2} m^3 a_m = 0^3(0) + 1^3(-1) + 2^3(1) = 7$$
$$3\left(\sum_{m=0}^{2} m^2 b_m\right) = 3\left(\frac{3}{2}\right) = \frac{9}{2}.$$

Thus, we see that the 2-step Adams-Bashforth method is second-order.

To determine the absolute stability region of this method, we apply the method to the model problem $y'(x) = \lambda y$, $y(x_0) = y_0$. Upon application of the method to the model problem, we obtain the associated stability polynomial

$$\zeta^2 + \left(-1 - \frac{3}{2}z\right)\zeta^2 + \frac{1}{2}z = 0,$$

where we have defined $z = \lambda h$. We use Mathematica to compute and plot the absolute stability region from the above stability polynomial, i.e., we compute the roots of the stability polynomial, determine which values of z result in all the roots having magnitude less than one, and plot the corresponding z in the complex-plane. This process is repeated for the following methods.

(b) 3-step Adams-Bashforth

To determine the order of the method, we use condition 2.11 in Iserles.

The coefficients of the above linear multistep method are given by $a_0 = 0$, $a_1 = 0$, $a_2 = -1$, $a_3 = 1$ and $b_0 = \frac{5}{12}$, $b_1 = -\frac{4}{3}$, $b_2 = \frac{23}{12}$, $b_3 = 0$.

We now check the conditions outlined in 2.11:

$$\begin{aligned}
\sum_{m=0}^{3} a_m &= 0 + 0 - 1 + 1 = 0 \\
\sum_{m=0}^{3} m^1 a_m &= 0^1\,(0) + 1^1\,(0) + 2^1\,(-1) + 3^1\,(1) = 1 \\
1\left(\sum_{m=0}^{3} m^0 b_m\right) &= 1\left(\frac{5}{12}\right) + 1\left(-\frac{4}{3}\right) + 1\left(\frac{23}{12}\right) = 1 \\
\sum_{m=0}^{3} m^2 a_m &= 0^2\,(0) + 1^2\,(0) + 2^2\,(-1) + 3^2\,(1) = 5 \\
2\left(\sum_{m=0}^{3} m^1 b_m\right) &= 2\left(-\frac{4}{3} + 2\left(\frac{23}{12}\right)\right) = 5 \\
\sum_{m=0}^{3} m^3 a_m &= 0^3\,(0) + 1^3\,(0) + 2^3\,(-1) + 3^3\,(1) = 19 \\
3\left(\sum_{m=0}^{3} m^2 b_m\right) &= 3\left(-\frac{4}{3} + 2^2\left(\frac{23}{12}\right)\right) = 19 \\
\sum_{m=0}^{3} m^4 a_m &= 0^4\,(0) + 1^4\,(0) + 2^4\,(-1) + 3^4\,(1) = 65 \\
4\left(\sum_{m=0}^{3} m^3 b_m\right) &= 4\left(-\frac{4}{3} + 2^3\left(\frac{23}{12}\right)\right) = \frac{260}{3}.
\end{aligned}$$

Thus, we see that the 3-step Adams-Bashforth method is third-order.

To determine the absolute stability region of this method, we apply the method to the model problem $y'(x) = \lambda y$, $y(x_0) = y_0$. Upon application of the method to the model problem, we obtain the associated stability polynomial

$$z^3 + \left(-1 - \frac{23}{12}z\right)\zeta^2 + \frac{4}{3}z\zeta - \frac{5}{12}z = 0,$$

where we have defined $z = \lambda h$. We use Mathematica to compute and plot the absolute stability region from the above stability polynomial.

(c) 2-step Adams-Moulton

To determine the order of the method, we use condition 2.11 in Iserles.

The coefficients of the above linear multistep method are given by $a_0 = 0$, $a_1 = -1$, $a_2 = 1$ and $b_0 = -\frac{1}{12}$, $b_1 = \frac{2}{3}$, $b_2 = \frac{5}{12}$.

We now check the conditions outlined in 2.11:

$$\sum_{m=0}^{2} a_m = 0 - 1 + 1 = 0$$

$$\sum_{m=0}^{2} m^1 a_m = 0^1 (0) + 1^1 (-1) + 2^1 (-1) = 1$$

$$1 \left(\sum_{m=0}^{2} m^0 b_m \right) = 1 \left(-\frac{1}{12} \right) + 1 \left(\frac{2}{3} \right) + 1 \left(\frac{5}{12} \right) = 1$$

$$\sum_{m=0}^{2} m^2 a_m = 0^2 (0) + 1^2 (-1) + 2^2 (1) = 3$$

$$2 \left(\sum_{m=0}^{2} m^1 b_m \right) = 2 \left(\frac{2}{3} + 2 \left(\frac{5}{12} \right) \right) = 3$$

$$\sum_{m=0}^{2} m^3 a_m = 0^3 (0) + 1^3 (-1) + 2^3 (1) = 7$$

$$3 \left(\sum_{m=0}^{2} m^2 b_m \right) = 3 \left(\frac{2}{3} + 2^2 \left(\frac{5}{12} \right) \right) = 7$$

$$\sum_{m=0}^{2} m^4 a_m = 0^4 (0) + 1^4 (-1) + 2^4 (1) = 15$$

$$4 \left(\sum_{m=0}^{3} m^3 b_m \right) = 4 \left(\frac{2}{3} + 2^3 \left(\frac{5}{12} \right) \right) = 16.$$

Thus, we see that the 2-step Adams-Moulton method is third-order.

To determine the absolute stability region of this method, we apply the method to the model problem $y'(x) = \lambda y$, $y(x_0) = y_0$. Upon application of the method to the model problem, we obtain the associated stability polynomial

$$\left(1 - \frac{5}{12} z \right) \zeta^2 + \left(-1 - \frac{2}{3} z \right) \zeta^2 + \frac{1}{12} z = 0,$$

where we have defined $z = \lambda h$. We use Mathematica to compute and plot the absolute stability region from the above stability polynomial.

2. Determine the order and the region of absolute stability of the backward differentiation formulas (BDF) methods of order $s = 2, 3$:

$$\begin{aligned} s = 2: &\quad y_{n+2} - \frac{4}{3} y_{n+1} + \frac{1}{3} y_n = \frac{2}{3} h f(x_{n+2}, y_{n+2}) \\ s = 3: &\quad y_{n+3} - \frac{18}{11} y_{n+2} + \frac{9}{11} y_{n+1} - \frac{2}{11} y_n = \frac{6}{11} h f(x_{n+3}, y_{n+3}). \end{aligned}$$

In all problems, state the order and plot the corresponding regions of absolute stability (using the approach described in class).

(a) 2-step BDF

To determine the order of the method, we use condition 2.11 in Iserles.

The coefficients of the above linear multistep method are given by $a_0 = \frac{1}{3}$, $a_1 = -\frac{4}{3}$, $a_2 = 1$ and $b_0 = 0$, $b_1 = 0$, $b_2 = \frac{2}{3}$.

We now check the conditions outlined in 2.11:

$$\sum_{m=0}^{2} a_m = \frac{1}{3} - \frac{4}{3} + 1 = 0$$

$$\sum_{m=0}^{2} m^1 a_m = 0^1\left(\frac{1}{3}\right) + 1^1\left(-\frac{4}{3}\right) + 2^1(1) = \frac{2}{3}$$

$$1\left(\sum_{m=0}^{2} m^0 b_m\right) = 1(0) + 1(0) + 1\left(\frac{2}{3}\right) = \frac{2}{3}$$

$$\sum_{m=0}^{2} m^2 a_m = 0^2\left(\frac{1}{3}\right) + 1^2\left(-\frac{4}{3}\right) + 2^2(1) = \frac{8}{3}$$

$$2\left(\sum_{m=0}^{2} m^1 b_m\right) = 2\left(0 + 2\left(\frac{2}{3}\right)\right) = \frac{8}{3}$$

$$\sum_{m=0}^{2} m^3 a_m = 0^3\left(\frac{1}{3}\right) + 1^3\left(-\frac{4}{3}\right) + 2^3\left(\frac{2}{3}\right) = 4$$

$$3\left(\sum_{m=0}^{2} m^2 b_m\right) = 3\left(2^2\left(\frac{2}{3}\right)\right) = 8$$

Thus, we see that the 2-step BDF is second-order.

To determine the absolute stability region of this method, we apply the method to the model problem $y'(x) = \lambda y$, $y(x_0) = y_0$. Upon application of the method to the model problem, we obtain the associated stability polynomial

$$\left(1 - \frac{2}{3}z\right)\zeta^2 - \frac{4}{3}\zeta + \frac{1}{3} = 0,$$

where we have defined $z = \lambda h$. We use Mathematica to compute and plot the absolute stability region from the above stability polynomial.

(b) 3-step BDF

To determine the order of the method, we use condition 2.11 in Iserles.

The coefficients of the above linear multistep method are given by $a_0 = -\frac{2}{11}$, $a_1 = \frac{9}{11}$, $a_2 = -\frac{18}{11}$, $a_3 = 1$ and $b_0 = 0$, $b_1 = 0$, $b_2 = 0$, $b_3 = \frac{6}{11}$.

We now check the conditions outlined in 2.11:

$$\sum_{m=0}^{3} a_m = -\frac{2}{11} + \frac{9}{11} - \frac{18}{11} + 1 = 0$$

$$\sum_{m=0}^{3} m^1 a_m = 0^1 \left(-\frac{2}{11}\right) + 1^1 \left(\frac{9}{11}\right) + 2^1 \left(-\frac{18}{11}\right) + 3^1 (1) = \frac{6}{11}$$

$$1 \left(\sum_{m=0}^{3} m^0 b_m\right) = 1 \left(\frac{6}{11}\right) = \frac{6}{11}$$

$$\sum_{m=0}^{3} m^2 a_m = 0^2 \left(-\frac{2}{11}\right) + 1^2 \left(\frac{9}{11}\right) + 2^2 \left(-\frac{18}{11}\right) + 3^2 (1) = \frac{36}{11}$$

$$2 \left(\sum_{m=0}^{3} m^1 b_m\right) = 2 \left(3 \left(\frac{6}{11}\right)\right) = \frac{36}{11}$$

$$\sum_{m=0}^{3} m^3 a_m = 0^3 \left(-\frac{2}{11}\right) + 1^3 \left(\frac{9}{11}\right) + 2^3 \left(-\frac{18}{11}\right) + 3^3 (1) = \frac{162}{11}$$

$$3 \left(\sum_{m=0}^{3} m^2 b_m\right) = 3 \left(3^2 \left(\frac{6}{11}\right)\right) = \frac{162}{11}$$

$$\sum_{m=0}^{3} m^4 a_m = 0^4 \left(-\frac{2}{11}\right) + 1^4 \left(\frac{9}{11}\right) + 2^4 \left(-\frac{18}{11}\right) + 3^4 (1) \approx 55.63$$

$$4 \left(\sum_{m=0}^{3} m^3 b_m\right) = 4 \left(3^3 \left(\frac{6}{11}\right)\right) \approx 14.72.$$

Thus, we see that the 3-step BDF method is third-order.

To determine the absolute stability region of this method, we apply the method to the model problem $y'(x) = \lambda y, y(x_0) = y_0$. Upon application of the method to the model problem, we obtain the associated stability polynomial

$$\left(1 - \frac{6}{11} z\right) \zeta^3 - \frac{18}{11} \zeta^2 + \frac{9}{11} \zeta - \frac{2}{11} = 0,$$

where we have defined $z = \lambda h$. We use Mathematica to compute and plot the absolute stability region from the above stability polynomial.

In[21]:=
```
AB2Roots = z /. Solve[z^(2) - (1 + (3 / 2) * r) * z + (1 / 2) * r == 0, z, Complexes]
```

Out[21]= $\left\{\frac{1}{4}\left(2+3\,r-\sqrt{4+4\,r+9\,r^{2}}\right), \frac{1}{4}\left(2+3\,r+\sqrt{4+4\,r+9\,r^{2}}\right)\right\}$

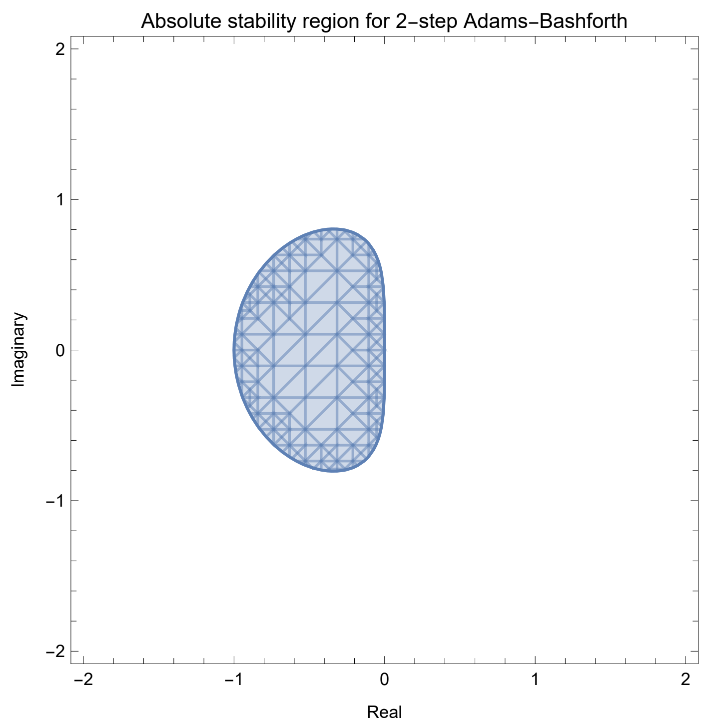
In[12]:=
```
AB2Roots1 = AB2Roots[[1]]
AB2Roots2 = AB2Roots[[2]]
```

Out[12]= $\frac{1}{4}\left(2+3\,r-\sqrt{4+4\,r+9\,r^{2}}\right)$

Out[13]= $\frac{1}{4}\left(2+3\,r+\sqrt{4+4\,r+9\,r^{2}}\right)$

In[48]:=
```
ComplexRegionPlot[{Abs[AB2Roots1] ≤ 1 && Abs[AB2Roots2] ≤ 1},
 {r, 2}, FrameLabel → {"Real", "Imaginary"},
 PlotLabel → "Absolute stability region for 2-step Adams-Bashforth"]
```

Out[48]=



In[18]:=
```
AB3Roots =
  z /. Solve[z^(3) - (1 + (23 / 12) * r) * z^(2) + (4 / 3) * r * z - (5 / 12) * r == 0, z, Complexes]
```

Out[18]=

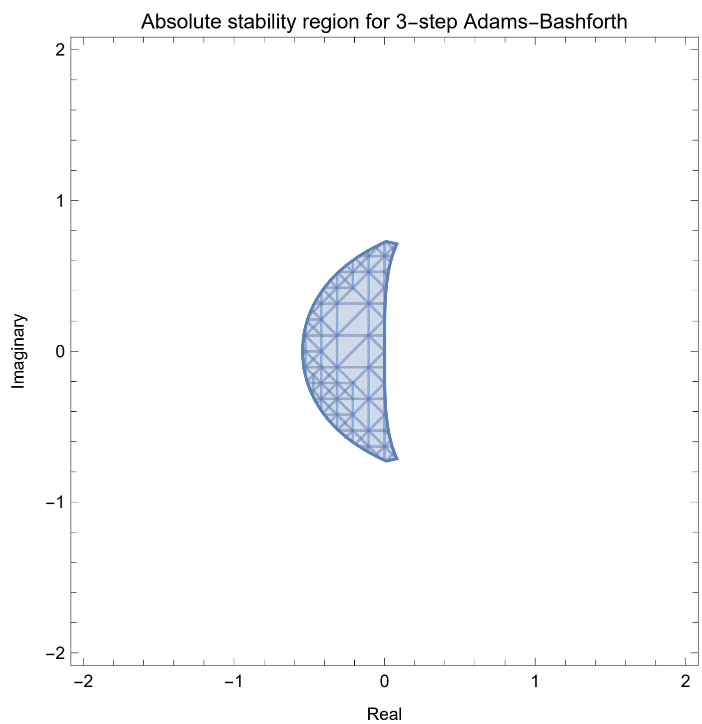
$$\left\{\frac{1}{36}(12+23r) - \frac{4\left(-1+\frac{r}{6}-\frac{529r^2}{144}\right)}{\left(1728+9288r-828r^2+12167r^3+108\sqrt{r}\sqrt{2880+4308r+3228r^2+8993r^3}\right)^{1/3}} + \frac{1}{36}\left(1728+9288r-828r^2+12167r^3+108\sqrt{r}\sqrt{2880+4308r+3228r^2+8993r^3}\right)^{1/3},\right.$$

$$\frac{1}{36}(12+23r) + \frac{2\left(1+i\sqrt{3}\right)\left(-1+\frac{r}{6}-\frac{529r^2}{144}\right)}{\left(1728+9288r-828r^2+12167r^3+108\sqrt{r}\sqrt{2880+4308r+3228r^2+8993r^3}\right)^{1/3}} - \frac{1}{72}\left(1-i\sqrt{3}\right)\left(1728+9288r-828r^2+12167r^3+108\sqrt{r}\sqrt{2880+4308r+3228r^2+8993r^3}\right)^{1/3},$$

$$\left.\frac{1}{36}(12+23r) + \frac{2\left(1-i\sqrt{3}\right)\left(-1+\frac{r}{6}-\frac{529r^2}{144}\right)}{\left(1728+9288r-828r^2+12167r^3+108\sqrt{r}\sqrt{2880+4308r+3228r^2+8993r^3}\right)^{1/3}} - \frac{1}{72}\left(1+i\sqrt{3}\right)\left(1728+9288r-828r^2+12167r^3+108\sqrt{r}\sqrt{2880+4308r+3228r^2+8993r^3}\right)^{1/3}\right\}$$

In[50]:=
```
ComplexRegionPlot[
  {Abs[AB3Roots[[1]]] ≤ 1 && Abs[AB3Roots[[2]]] ≤ 1 && Abs[AB3Roots[[3]]] ≤ 1},
  {r, 2}, FrameLabel → {"Real", "Imaginary"},
  PlotLabel → "Absolute stability region for 3-step Adams-Bashforth"]
```

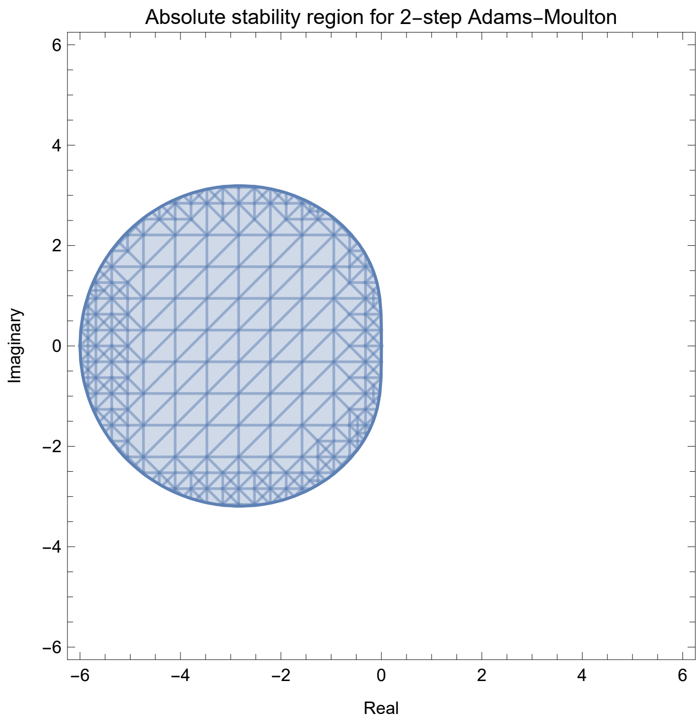
Out[50]=



In[27]:=
```
AM2Roots =
 z /. Solve[(1 - (5 / 12) * r) * z^(2) - (1 + (2 / 3) * r) * z + (1 / 12) * r == 0, z, Complexes]
```

Out[27]=
$$\left\{\frac{-6-4r-\sqrt{3}\ \sqrt{12+12r+7r^2}}{-12+5r}, \frac{-6-4r+\sqrt{3}\ \sqrt{12+12r+7r^2}}{-12+5r}\right\}$$

In[36]:=
```
ComplexRegionPlot[{Abs[AM2Roots[[1]]] ≤ 1 && Abs[AM2Roots[[2]]] ≤ 1},
 {r, 6}, FrameLabel → {"Real", "Imaginary"},
 PlotLabel → "Absolute stability region for 2-step Adams-Moulton"]
```

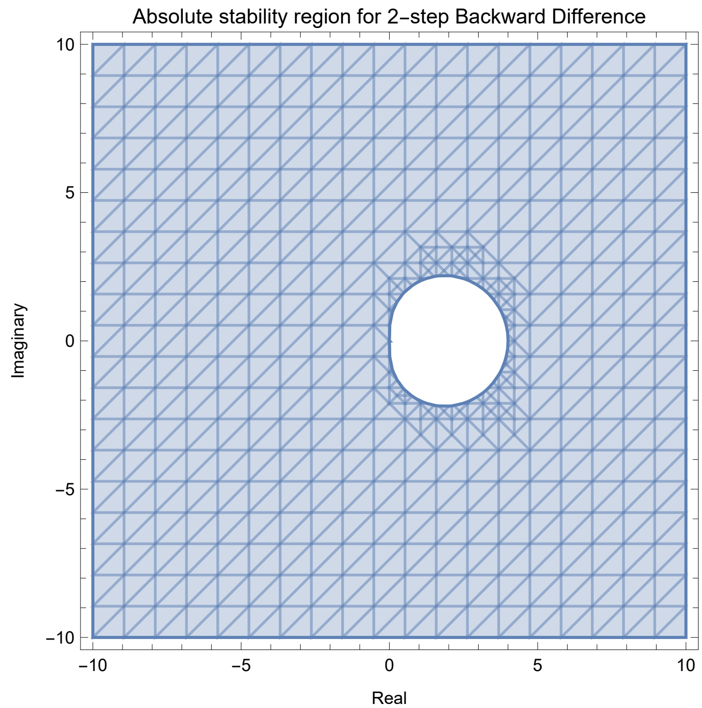
Out[36]=


In[37]:=
```
BDF2Roots = z /. Solve[(1 - (2 / 3) * r) * z^(2) - (4 / 3) * z + (1 / 3) == 0, z, Complexes]
```

Out[37]=
$$\left\{\frac{-2-\sqrt{1+2r}}{-3+2r}, \frac{-2+\sqrt{1+2r}}{-3+2r}\right\}$$

In[42]:=
```
ComplexRegionPlot[{Abs[BDF2Roots[[1]]] ≤ 1 && Abs[BDF2Roots[[2]]] ≤ 1},
 {r, 10}, FrameLabel → {"Real", "Imaginary"},
 PlotLabel → "Absolute stability region for 2-step Backward Difference"]
```

Out[42]=



In[39]:=
```
BDF3Roots = z /.
  Solve[(1 - (6 / 11) * r) * z^(3) - (18 / 11) * z^(2) + (9 / 11) * z - (2 / 11) == 0, z, Complexes]
```

Out[39]=

$$\left\{\frac{6}{11-6r}-\frac{-27-162r}{9(11-6r)\left(40+30r+36r^2+\sqrt{1573+1914r+864r^2-3672r^3+1296r^4}\right)^{1/3}}+\frac{\left(40+30r+36r^2+\sqrt{1573+1914r+864r^2-3672r^3+1296r^4}\right)^{1/3}}{11-6r},\right.$$

$$\frac{6}{11-6r}+\frac{\left(1+i\sqrt{3}\right)(-27-162r)}{18(11-6r)\left(40+30r+36r^2+\sqrt{1573+1914r+864r^2-3672r^3+1296r^4}\right)^{1/3}}-\frac{\left(1-i\sqrt{3}\right)\left(40+30r+36r^2+\sqrt{1573+1914r+864r^2-3672r^3+1296r^4}\right)^{1/3}}{2(11-6r)},$$

$$\left.\frac{6}{11-6r}+\frac{\left(1-i\sqrt{3}\right)(-27-162r)}{18(11-6r)\left(40+30r+36r^2+\sqrt{1573+1914r+864r^2-3672r^3+1296r^4}\right)^{1/3}}-\frac{\left(1+i\sqrt{3}\right)\left(40+30r+36r^2+\sqrt{1573+1914r+864r^2-3672r^3+1296r^4}\right)^{1/3}}{2(11-6r)}\right\}$$

In[41]:=
```
ComplexRegionPlot[
 {Abs[BDF3Roots[[1]]] ≤ 1 && Abs[BDF3Roots[[2]]] ≤ 1 && Abs[BDF3Roots[[3]]] ≤ 1},
 {r, 10}, FrameLabel → {"Real", "Imaginary"},
 PlotLabel → "Absolute stability region for 3-step Backward Difference"]
```

LessEqual: Internal precision limit $MaxExtraPrecision = 50.` reached while evaluating

$$\frac{5}{11} - \frac{3}{11\left(40 + 11\sqrt{13}\right)^{1/3}} - \frac{1}{11}\left(40 + 11\sqrt{13}\right)^{1/3}.$$

Out[41]=