Positivity-Preserving Limiters for the Piecewise P_N approximation

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Radiative Transfer Equation (RTE)

- optics, astrophysics, atmospheric science, remote sensing,$\cdots$
- physical phenomenon of energy transfer
- propagation : absorption, emission and scattering
- $\frac{1}{c}\psi_{,t} + \underline{\Omega} \cdot \nabla_r \psi = -\sigma_t \psi + \frac{1}{4\pi}\left(\sigma_s \int_{\mathbb{S}^2} \psi \, d\underline{\Omega} + \sigma_a B(T) + s\right)$
- ψ : radiation intensity (flux of energy through a surface)
- $B(T) := \dfrac{2h\nu^3}{c^2\left(exp(\frac{h\nu}{kT}) - 1\right)}$: Blackbody source
- k : Boltzmann's constant, h : Planck's constant
- s : external source

Numerical Difficulty in RTE

- rich phase space : $\psi(\underline{r}, \underline{\Omega}, \nu, t)$
- integro-differential equation : $\displaystyle\int_{\mathbb{S}^2} \psi \, d\underline{\Omega}$
- moment closures

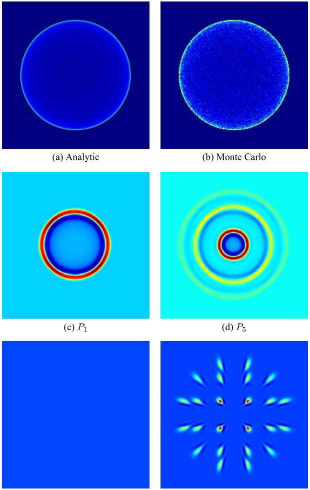
Three Main Approaches

- Implicit Monte Carlo Methods (*IMC*)
- Discrete Ordinates Discretization (S_N)
- Spherical Harmonics Approximation (P_N)
- (Flux Limited Diffusion)

Benefits and Drawbacks of P_N approximation

- **Pros** :
  - rotationally invariant
  - P_N equations converge in L^2 sense as $N \to \infty$
- **Cons** :
  - reduced propagation speed
  - steady-state equations are ill-posed
  - no general theory to BCs for P_N
  - negative particle concentration **[Hauck, 2010& Laiu, 2016]**

Comparison of Solutions-Line Source [Brunner, 2012]



(a) Analytic

(b) Monte Carlo

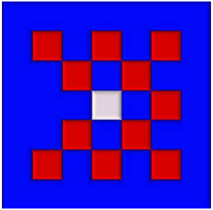
(c) P_1

(d) P_5

(e) Flux Limited Diffusion

(f) S_6

Lattice Problem [Brunner, 2012]

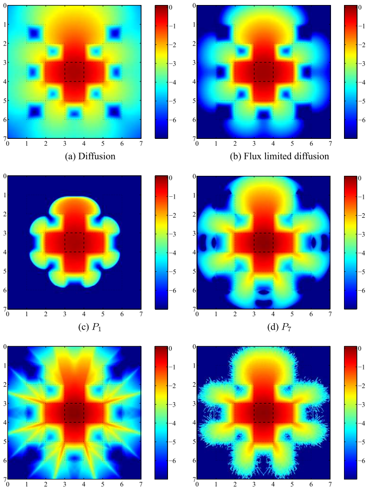


blue and white regions : pure scattering

red regions : pure absorbers

white region : source of particles

Comparison of Solutions [Brunner, 2012]



(a) Diffusion

(b) Flux limited diffusion

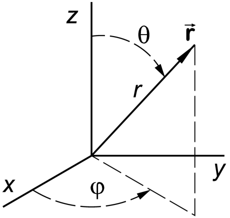
(c) P_1

(d) P_7

(e) S_6

(f) Implicit Monte Carlo

Spherical Harmonics in Transport Equations



- $Y_\ell^m(\mu,\varphi) = \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}} P_\ell^m(\mu) e^{im\varphi}, \quad \mu = \cos\theta$
- Intensity can be expanded in terms of spherical harmonics :
- $I(\underline{r},\underline{\Omega},t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} I_\ell^m(\underline{r},t) Y_\ell^m(\mu,\varphi)$

Linear Transport Equation

- Linear Transport Equation(F : kinetic distribution):

$$F_{,t} + \underline{\Omega} \cdot \nabla_r F + \sigma F = \frac{\sigma}{4\pi} \int_{\mathbb{S}^2} F(\underline{r}, \underline{\Omega}, t)\, d\underline{\Omega}$$

- $\underline{r} = (x, y, z) \in \Gamma \subset \mathbb{R}^d$: position , $\underline{\Omega} \subset \mathbb{S}^2$: angle
- $\int_{\mathcal{D}} F(\underline{r}, \underline{\Omega}, t)\, d\underline{\Omega}\, d\underline{r}$: number of particles at time t

P_N Approximation

- Moments : $\underline{u}(\underline{r}, t) = \int_{\mathbb{S}^2} \underline{p} F(\underline{r}, \underline{\Omega}, t)\, d\underline{\Omega}$, where $\underline{p} : \mathbb{S}^2 \to \mathbb{R}^d$
- Exact Moments Equations :

$$\underline{u}_{,t} + \nabla_r \cdot \int_{\mathbb{S}^2} \underline{\Omega}\underline{p} F\, d\underline{\Omega} + \sigma \int_{\mathbb{S}^2} \underline{p} F\, d\underline{\Omega} = \frac{\sigma}{4\pi} \rho \int_{\mathbb{S}^2} \underline{p}\, d\underline{\Omega}$$

- Density : $\rho = \int_{\mathbb{S}^2} F\, d\underline{\Omega}$
- Moment Closure : $F(\underline{r}, \underline{\Omega}, t) \approx \mathcal{F}(\underline{u}, \underline{p})$ s.t. $\int_{\mathbb{S}^2} \underline{p} \mathcal{F}\, d\underline{\Omega} = \underline{u}$

P_N in 1D

- **Reduced equations:**

$$\bar{F}(z,\mu,t) := \int_0^{2\pi} F(\underline{r},\underline{\Omega},t)\,d\phi$$

$$\bar{F}_{,t} + \mu\bar{F}_{,z} + \sigma\bar{F} = \frac{\sigma}{2}\rho, \quad \rho(z,t) = \int_{-1}^{1} \bar{F}(z,\mu,t)\,d\mu$$

$$\bar{F} := \sum_{k=1}^{N+1} u^k(z,t)p^k(\mu)$$

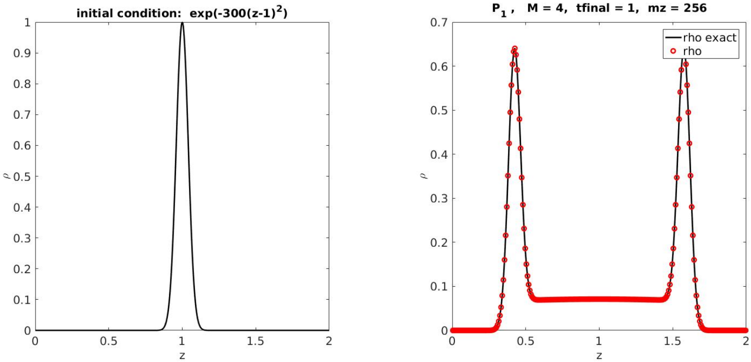
P_N in 1D

- After some algebraic treatment,

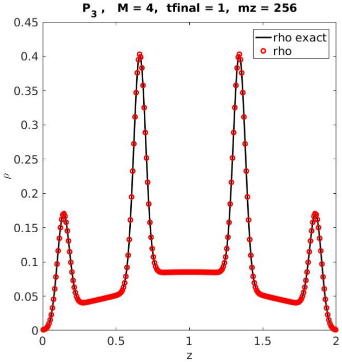
$$\underline{v}_{,t} + \underline{\underline{\Lambda}}\underline{v}_{,z} + \sigma\underline{v} = \sigma\left(\frac{\underline{\rho}}{2}\right)$$

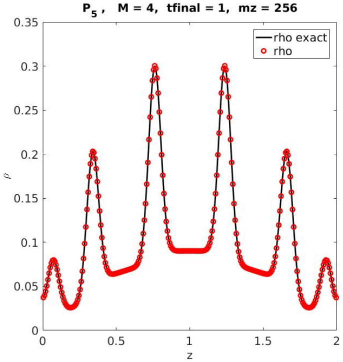
- $\underline{\rho} = (\rho, \cdots, \rho)^T$, where $\rho = \underline{w}^T \underline{v}$
- $\underline{\underline{\Lambda}} = diag(\lambda_0, \cdots, \lambda_N)$, $\quad \underline{w} = (w_0, \cdots, w_N)^T$
- λ_i : zeros of Legendre polynomial P_{N+1}
- w_i : N+1 Gauss-Legendre quadrature weights

P_1 Solution

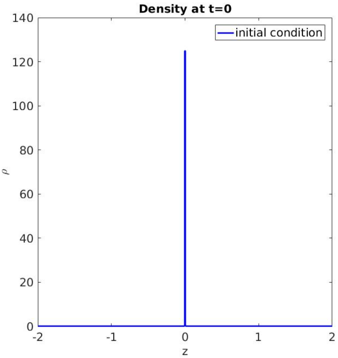

initial condition: exp(-300(z-1)^2)
ρ
z
P1 , M = 4, tfinal = 1, mz = 256
rho exact
rho
ρ
z

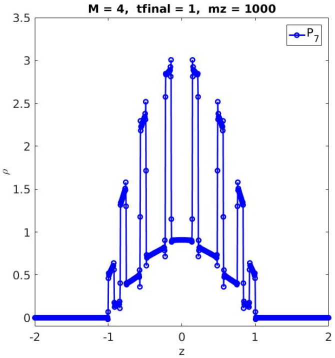
P_3 and P_5 Solutions





P_7 Solution

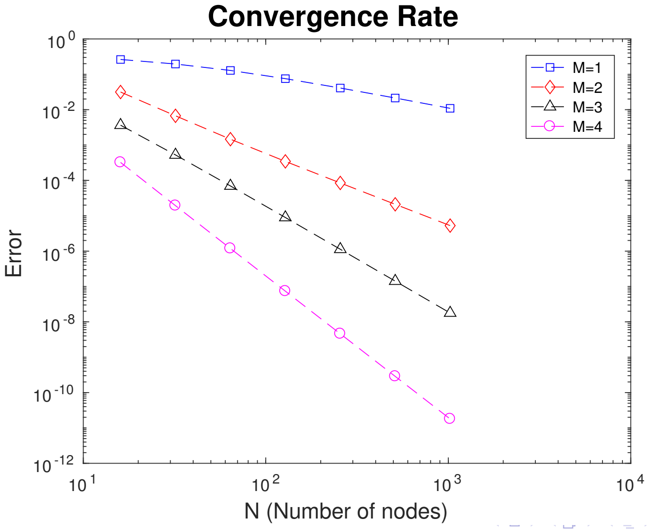

Density at t=0
initial condition
140
120
100
80
60
40
20
0
-2
-1
0
1
2
z
ρ


M = 4, tfinal = 1, mz = 1000
P7
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1
0
1
2
z
ρ

P_1 Convergence Rate in 1D (to Exact P_1 Soln.)

N	error	error ratio
16	0.00032249	0
32	1.9306e-05	16.704
64	1.1865e-06	16.2717
128	7.3742e-08	16.0896
256	4.5994e-09	16.0328
512	2.8722e-10	16.0135
1024	1.7945e-11	16.0061

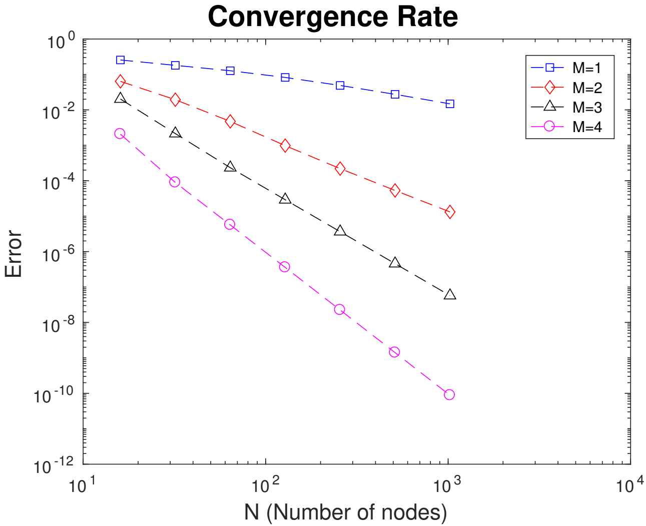
P_1 Convergence Rate in 1D (to Exact P_1 Soln.)



P_3 Convergence Rate in 1D (to Exact P_3 Soln.)

N	error	error ratio
16	0.0020694	0
32	8.9046e-05	23.2401
64	5.658e-06	15.7379
128	3.5727e-07	15.837
256	2.2454e-08	15.9107
512	1.4075e-09	15.953
1024	8.8104e-11	15.9759

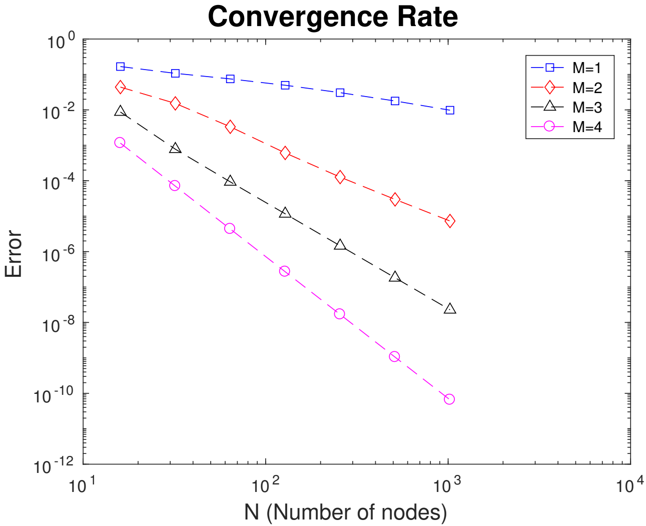
P_3 Convergence Rate in 1D (to Exact P_3 Soln.)



P_5 Convergence Rate in 1D (to Exact P_5 Soln.)

N	error	error ratio
16	0.0011532	0
32	7.0376e-05	16.386
64	4.3402e-06	16.2149
128	2.6986e-07	16.0833
256	1.6822e-08	16.0414
512	1.05e-09	16.0208
1024	6.5585e-11	16.0104

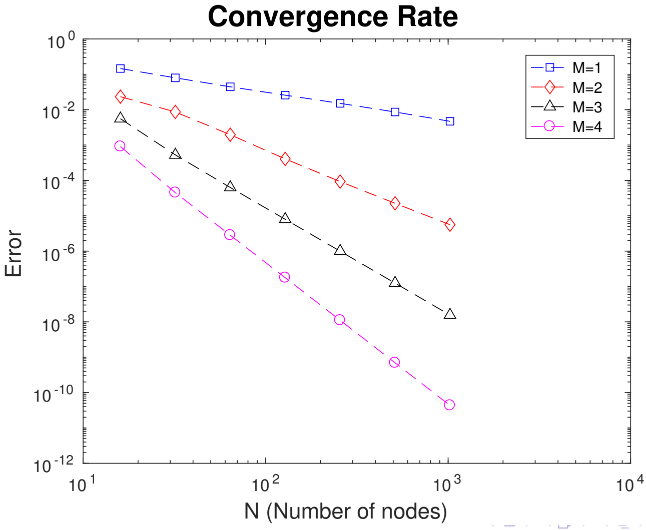
P_5 Convergence Rate in 1D (to Exact P_5 Soln.)



P_7 Convergence Rate in 1D (to Exact P_7 Soln.)

N	error	error ratio
16	0.00090598	0
32	4.4973e-05	20.145
64	2.8197e-06	15.9496
128	1.7704e-07	15.9269
256	1.1086e-08	15.9694
512	6.9362e-10	15.9832
1024	4.3375e-11	15.9911

P_7 Convergence Rate in 1D (to Exact P_7 Soln.)



Piecewise P_N in 1D

- P_N:

$$\bar{F}_{,t} + \mu \bar{F}_{,z} + \sigma \bar{F} = \frac{\sigma}{2}\rho, \quad \rho(z,t) = \int_{-1}^{1} \bar{F}(z,\mu,t)\, d\mu$$

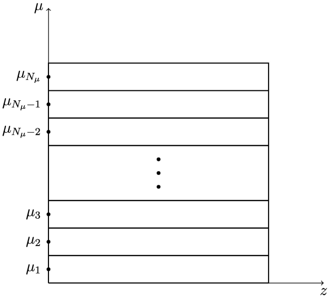
$$\bar{F}(z,\mu,t) := \sum_{k=1}^{N+1} u^k(z,t) p^k(\mu)$$

- **Piecewise** P_N:

$$\bar{F}_j(z,\alpha,t) := \sum_{k=1}^{N+1} u_j^k(z,t) p^k(\alpha), \quad \mu = \mu_j + \alpha\frac{\Delta\mu}{2}$$

$$\rho_j(z,t) := \int_{\mu_j - \frac{\Delta\mu}{2}}^{\mu_j + \frac{\Delta\mu}{2}} \bar{F}(z,\mu,t)\, d\mu = \frac{\Delta\mu}{2} \int_{-1}^{1} \bar{F}_j(z,\alpha,t)\, d\alpha$$

Idea of Piecewise P_N



- P_N : $\mu \in [-1, 1]$
- Piecewise P_N : $\mu = \mu_j + \alpha \dfrac{\Delta\mu}{2}, \;\; \alpha \in [-1, 1]$

Piecewise P_N in 1D

- **Equation:**

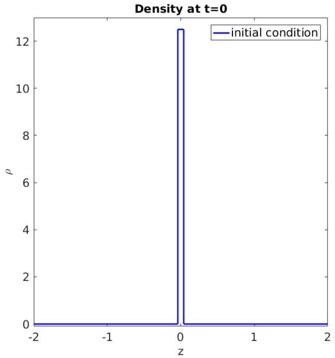
$$\bar{F}_{j,t}(t,z,\alpha) + \left(\mu_j + \alpha\frac{\Delta\mu}{2}\right)\bar{F}_{j,z}(t,z,\alpha) + \sigma\bar{F}_j(t,z,\alpha) = \frac{\sigma}{2}\sum_{j=1}^{N_\mu}\rho_j(t,z)$$

$$\underline{u}_{j,t} + \underline{\underline{A}}_j\underline{u}_{j,z} + \sigma\,\underline{u}_j = \underline{\underline{B}}\sum_{j=1}^{N_\mu}\underline{u}_j\ , \qquad \underline{u}_j = \begin{bmatrix} u_j^1 & u_j^2 & \dots & u_j^{N+1}\end{bmatrix}^T$$

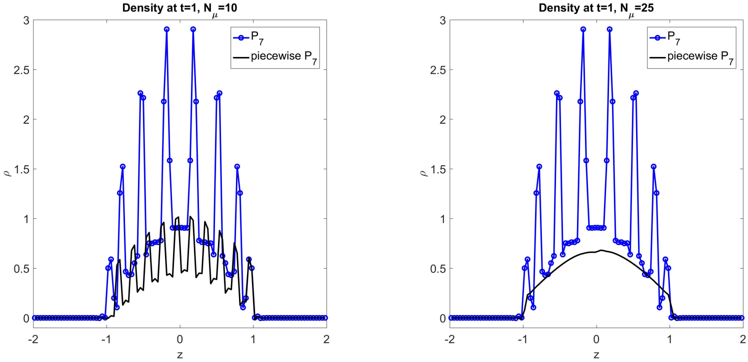
$$A_j^{k\ell} = \mu_j\delta_{k\ell} + \frac{\Delta\mu}{2}\int_{-1}^{1}\alpha\,p^k(\alpha)\,p^\ell(\alpha)\,d\alpha,$$

$$B^{k\ell} = \begin{cases} \frac{\sigma\Delta\mu}{4}, & \text{if } k=\ell=1 \\ 0, & \text{otherwise}\end{cases}$$

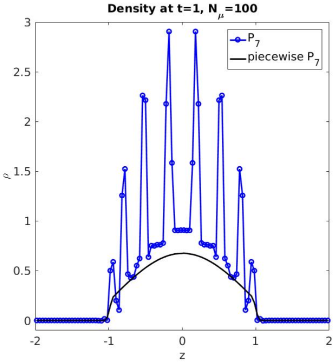
Initial Condition $\delta(z, t = 0)$ $(m_z = 100)$

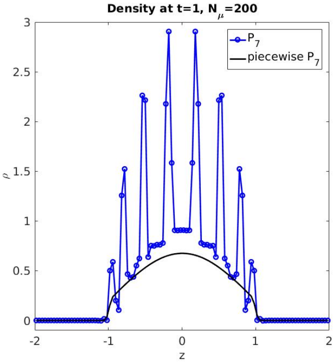


P_N vs Piecewise-P_N $(m_z = 100)$



P_N vs Piecewise-P_N $(m_z = 100)$





SSP-RK-DG Method

$$u_j^k(z,t)\Big|_{z\in[z_i-\frac{\Delta z}{2},z_i+\frac{\Delta z}{2}]} = \sum_{\ell=1}^{M} Q_{ij}^{k\ell}(t)\psi^{\ell}(\xi), \quad \underline{Q}_i^{k\ell}(t) = \begin{bmatrix} Q_{i1}^{k\ell} & Q_{i2}^{k\ell} & \cdots & Q_{iN_\mu}^{k\ell}\end{bmatrix}^T$$

$$\begin{aligned}
\dot{\underline{Q}}_i^{k\ell}(t) = -\Bigg[& \left(\underline{\underline{\Lambda}}^{+}\sum_{\hat{\ell}=1}^{M}\underline{Q}_i^{k\hat{\ell}}(t)\psi^{\hat{\ell}}(1) + \underline{\underline{\Lambda}}^{-}\sum_{\hat{\ell}=1}^{M}\underline{Q}_{i+1}^{k\hat{\ell}}(t)\psi^{\hat{\ell}}(-1)\right)\psi^{\ell}(1) \\
& - \left(\underline{\underline{\Lambda}}^{+}\sum_{\hat{\ell}=1}^{M}\underline{Q}_{i-1}^{k\hat{\ell}}(t)\psi^{\hat{\ell}}(1) + \underline{\underline{\Lambda}}^{-}\sum_{\hat{\ell}=1}^{M}\underline{Q}_i^{k\hat{\ell}}(t)\psi^{\hat{\ell}}(-1)\right)\psi^{\ell}(-1)\Bigg] \\
& + \underline{\underline{\Lambda}}\sum_{\hat{\ell}=1}^{M}\underline{Q}_i^{k\hat{\ell}}(t)\int_{-1}^{1}\psi^{\hat{\ell}}(\xi)\psi^{\ell}_{,\xi}(\xi)\,d\xi \\
& - \underline{\underline{B}}\left[\sum_{\hat{\ell}=1}^{M}\underline{Q}_i^{k\hat{\ell}}(t)\psi^{\hat{\ell}}(1)\psi^{\ell}(1) - \sum_{\hat{\ell}=1}^{M}\underline{Q}_{i-1}^{k\hat{\ell}}(t)\psi^{\hat{\ell}}(1)\psi^{\ell}(-1))\right] \\
& + \underline{\underline{B}}\sum_{\hat{\ell}=1}^{M}\underline{Q}_i^{k\hat{\ell}}(t)\int_{-1}^{1}\psi^{\hat{\ell}}(\xi)\psi^{\ell}_{,\xi}(\xi)\,d\xi - \sigma\underline{Q}_i^{k\ell}(t) \\
& + \frac{\sigma\Delta\mu}{4}\begin{bmatrix}1 & 1 & 1 \cdots & 1\end{bmatrix}^T\int_{-1}^{1}p^{k}(\alpha)\,d\alpha\sum_{j=1}^{N_\mu}\sum_{\hat{k}=1}^{N+1}Q_{ij}^{\hat{k}\ell}(t)\int_{-1}^{1}p^{\hat{k}}(\alpha)\,d\alpha
\end{aligned}$$

SSP-RK-DG Method

$$\underline{\underline{\Lambda}} = \frac{2}{\Delta z} \begin{bmatrix} \mu_1 & 0 & 0 & \dots & 0 \\ 0 & \mu_2 & 0 & \dots & 0 \\ 0 & 0 & \mu_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu_{N_\mu} \end{bmatrix}$$

SSP-RK-DG Method

$$\begin{aligned}
L(Q) = -\Bigg[& \left(\underline{\underline{\Lambda}}^{+} \sum_{\hat{\ell}=1}^{M} \underline{Q}_{i}^{k\hat{\ell}} \psi^{\hat{\ell}}(1) + \underline{\underline{\Lambda}}^{-} \sum_{\hat{\ell}=1}^{M} \underline{Q}_{i+1}^{k\hat{\ell}} \psi^{\hat{\ell}}(-1) \right) \psi^{\ell}(1) \\
& - \left(\underline{\underline{\Lambda}}^{+} \sum_{\hat{\ell}=1}^{M} \underline{Q}_{i-1}^{k\hat{\ell}} \psi^{\hat{\ell}}(1) + \underline{\underline{\Lambda}}^{-} \sum_{\hat{\ell}=1}^{M} \underline{Q}_{i}^{k\hat{\ell}} \psi^{\hat{\ell}}(-1) \right) \psi^{\ell}(-1) \Bigg] \\
& + \underline{\underline{\Lambda}} \sum_{\hat{\ell}=1}^{M} \underline{Q}_{i}^{k\hat{\ell}} \int_{-1}^{1} \psi^{\hat{\ell}}(\xi) \psi^{\ell}_{,\xi}(\xi)\, d\xi \\
& - \underline{\underline{B}} \left[\sum_{\hat{\ell}=1}^{M} \underline{Q}_{i}^{k\hat{\ell}} \psi^{\hat{\ell}}(1) \psi^{\ell}(1) - \sum_{\hat{\ell}=1}^{M} \underline{Q}_{i-1}^{k\hat{\ell}} \psi^{\hat{\ell}}(1) \psi^{\ell}(-1)) \right] \\
& + \underline{\underline{B}} \sum_{\hat{\ell}=1}^{M} \underline{Q}_{i}^{k\hat{\ell}} \int_{-1}^{1} \psi^{\hat{\ell}}(\xi) \psi^{\ell}_{,\xi}(\xi)\, d\xi - \sigma \underline{Q}_{i}^{k\ell} \\
& + \frac{\sigma \Delta\mu}{4} \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^{T} \int_{-1}^{1} p^{k}(\alpha)\, d\alpha \sum_{j=1}^{N_\mu} \sum_{\hat{k}=1}^{N+1} Q_{ij}^{\hat{k}\ell} \int_{-1}^{1} p^{\hat{k}}(\alpha)\, d\alpha
\end{aligned}$$

Low Storage SSP-RK4 [Ketcheson, 2008]

$Q^{(1)} = Q^n;$

$Q^{(2)} = Q^n;$

for $i = 1 : 5$ do

$\quad Q^{(1)} = Q^{(1)} + \frac{\Delta t}{6} L(Q^{(1)});$

end

$Q^{(2)} = \frac{1}{25} Q^{(2)} + \frac{9}{25} Q^{(1)};$

$Q^{(1)} = 15 Q^{(2)} - 5 Q^{(1)};$

for $i = 6 : 9$ do

$\quad Q^{(1)} = Q^{(1)} + \frac{\Delta t}{6} L(Q^{(1)});$

end

$Q^{(n+1)} = Q^{(2)} + \frac{3}{5} Q^{(1)} + \frac{\Delta t}{10} L(Q^{(1)});$

Positive-Preserving Limiters **[Zhang & Shu, 2010]**

- Modified Zhang-Shu Limiter:

$$\bar{F}_{ij}(t,\xi,\alpha) := \sum_{\ell=1}^{M} \sum_{k=1}^{N+1} Q_{ij}^{k\ell}(t)\phi^{\ell}(\xi)p^{k}(\alpha)$$

$$\bar{F}_{ij}(t,\xi,\alpha) = \frac{1}{2}Q_{ij}^{11} + \left(\frac{1}{\sqrt{2}} \sum_{\ell=2}^{M} Q_{ij}^{1\ell}(t)\phi^{\ell}(\xi) + \frac{1}{\sqrt{2}} \sum_{k=2}^{N+1} Q_{ij}^{k1}(t)p^{k}(\alpha) + \sum_{\ell=2}^{M} \sum_{k=2}^{N+1} Q_{ij}^{k\ell}(t)\phi^{\ell}(\xi)p^{k}(\alpha) \right)$$

$$\bar{F}_{ij}(t,\xi,\alpha) \leftarrow \frac{1}{2}Q_{ij}^{11} + \theta_{ij}\left(\bar{F}_{ij} - \frac{1}{2}Q_{ij}^{11} \right)$$

Positive-Preserving Limiters **[Zhang & Shu, 2010]**

- Modified Zhang-Shu Limiter:

$$F_{min} := \min_{(\xi,\alpha)\in S} \bar{F}_{ij}(t,\xi,\alpha), \quad \xi,\alpha \in [-1,1], \quad S = S_1 \otimes S_2$$

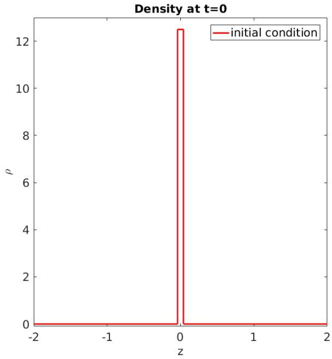
$$S_1 := \text{set of M+1 Gauss-Lobatto quadrature points}$$

$$S_2 := \text{set of N+1 Gauss-Legendre quadrature points}$$

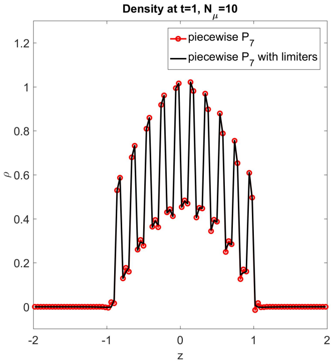
$$\theta_{ij} = \min\left(\frac{\varepsilon - Q_{ij}^{11}}{F_{min} - Q_{ij}^{11}}, 1\right), \quad \varepsilon = \min_{ij}(10^{-13}, Q_{ij}^{11}),$$

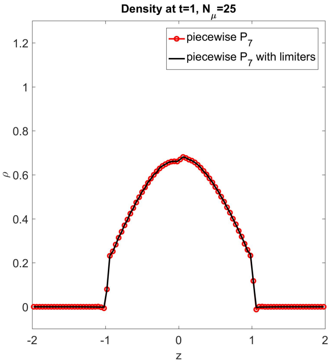
$$Q_{ij}^{k\ell} \leftarrow \theta_{ij} Q_{ij}^{k\ell}\ , \quad \theta_{ij} = 1 \ \text{ if } k = \ell = 1$$

Initial Condition $\delta(z, t = 0)$ $(m_z = 100)$



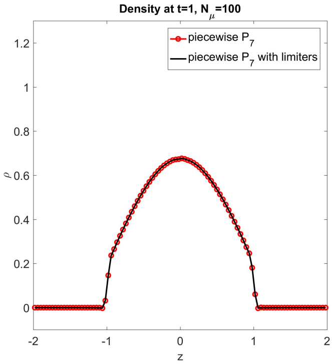
Piecewise-P_N vs Piecewise-P_N w/ limiters (Positivity)

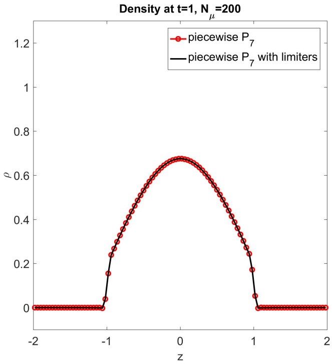




$$m_z = 100$$

Piecewise-P_N vs Piecewise-P_N w/ limiters (Positivity)





m_z=100

P_N Approximation in 2D

- Linear Transport Equation(F : kinetic distribution):

$$F_{,t} + \underline{\Omega} \cdot \nabla_r F + \sigma F = \frac{\sigma}{4\pi} \int_{\mathbb{S}^2} F(\underline{r}, \underline{\Omega}, t)\, d\underline{\Omega}$$

- $Y_\ell^m(\mu, \varphi) = \sqrt{\dfrac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}} P_\ell^m(\mu) e^{im\varphi}, \quad \mu = \cos\theta$
- $P_\ell^m(\mu) = (-1)^m (1-\mu^2)^{\frac{m}{2}} \dfrac{\partial^m}{\partial \mu^m} P_\ell(\mu), \quad m \in [0, \ell]$
- $P_\ell^m(\mu) = (-1)^m \dfrac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^{|m|}, \quad m \in [-\ell, 0)$
- $F(x, z, \underline{\Omega}, t) := \displaystyle\sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} F_\ell^m(x, z, t) Y_\ell^m(\underline{\Omega})$
- $F_\ell^m(x, z, t) = \displaystyle\int_{\mathbb{S}^2} \overline{Y}_\ell^m F(x, z, \underline{\Omega})\, d\underline{\Omega}$

P_N Approximation in 2D

- Multiply it by $\overline{Y}_\ell^m$ and integrate over $\mathbb{S}^2$:
- scattering source term: (Using $Y_0^0 = \overline{Y}_0^0 = 1/\sqrt{4\pi}$)

$$\frac{\sigma}{4\pi}\int_{\mathbb{S}^2}\overline{Y}_\ell^m(\underline{\Omega})\int_{\mathbb{S}^2}F(x,z,\underline{\Omega}',t)\,d\underline{\Omega}'\,d\underline{\Omega}$$

$$=\frac{\sigma}{\sqrt{4\pi}}\int_{\mathbb{S}^2}\overline{Y}_\ell^m(\underline{\Omega})\int_{\mathbb{S}^2}\overline{Y}_0^0(\underline{\Omega}')F(x,z,\underline{\Omega}',t)\,d\underline{\Omega}'\,d\underline{\Omega}$$

$$=\frac{\sigma}{\sqrt{4\pi}}\int_{\mathbb{S}^2}\overline{Y}_\ell^m(\underline{\Omega})F_0^0(x,z,t)\,d\underline{\Omega}$$

$$=\sigma F_0^0(x,z,t)\int_{\mathbb{S}^2}\overline{Y}_\ell^m(\underline{\Omega})Y_0^0\,d\underline{\Omega}$$

$$=\sigma F_0^0(x,z,t)\delta_{\ell 0}\delta_{m0}$$

Properties of Spherical Harmonics

$$\cos\theta Y_\ell^m = A_\ell^m Y_{\ell+1}^m + B_\ell^m Y_{\ell-1}^m$$

$$\sin\theta e^{i\varphi} Y_\ell^m = -C_\ell^m Y_{\ell+1}^{m+1} + D_\ell^m Y_{\ell-1}^{m+1}$$

$$\sin\theta e^{-i\varphi} Y_\ell^m = E_\ell^m Y_{\ell+1}^{m-1} - G_\ell^m Y_{\ell-1}^{m-1}$$

Properties of Spherical Harmonics

$$A_\ell^m = \sqrt{\frac{(\ell - m + 1)(\ell + m + 1)}{(2\ell + 3)(2\ell + 1)}} \qquad B_\ell^m = \sqrt{\frac{(\ell - m)(\ell + m)}{(2\ell + 1)(2\ell - 1)}}$$

$$C_\ell^m = \sqrt{\frac{(\ell + m + 1)(\ell + m + 2)}{(2\ell + 3)(2\ell + 1)}} \qquad D_\ell^m = \sqrt{\frac{(\ell - m)(\ell - m - 1)}{(2\ell + 1)(2\ell - 1)}}$$

$$E_\ell^m = \sqrt{\frac{(\ell - m + 1)(\ell - m + 2)}{(2\ell + 3)(2\ell + 1)}} \qquad G_\ell^m = \sqrt{\frac{(\ell + m)(\ell + m - 1)}{(2\ell + 1)(2\ell - 1)}}$$

One trick in Spherical Harmonics Approximation

$$\text{Let} \quad \underline{\Omega}' = \begin{bmatrix} \sin\theta(\cos\varphi + i\sin\varphi) \\ \sin\theta(\cos\varphi - i\sin\varphi) \\ \cos\theta \end{bmatrix} = \begin{bmatrix} \sin\theta e^{i\varphi} \\ \sin\theta e^{-i\varphi} \\ \cos\theta \end{bmatrix}$$

$$\text{and} \quad \nabla' = \begin{bmatrix} \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \partial_- \\ \partial_+ \\ \partial_z \end{bmatrix}$$

Then, $\underline{\Omega} \cdot \nabla = \underline{\Omega}' \cdot \nabla'$

Streaming Term

$$
\begin{aligned}
&\int_{\mathbb{S}^2} \overline{Y}_\ell^m \underline{\Omega} \cdot \nabla F \, d\underline{\Omega} \\
=&\frac{1}{2}(-C_{\ell-1}^{m-1} F_{\ell-1}^{m-1} + D_{\ell+1}^{m-1} F_{\ell+1}^{m-1} + E_{\ell-1}^{m+1} F_{\ell-1}^{m+1} - G_{\ell+1}^{m+1} F_{\ell+1}^{m+1})_{,x} \\
+&\frac{1}{2} i(C_{\ell-1}^{m-1} F_{\ell-1}^{m-1} - D_{\ell+1}^{m-1} F_{\ell+1}^{m-1} + E_{\ell-1}^{m+1} F_{\ell-1}^{m+1} - G_{\ell+1}^{m+1} F_{\ell+1}^{m+1})_{,y} \\
+&(A_{\ell-1}^m F_{\ell-1}^m + B_{\ell+1}^m F_{\ell+1}^m)_{,z}
\end{aligned}
$$

Simplified Moments Equations

$$
\begin{aligned}
F_\ell^m{}_{,t} + \frac{1}{2}\big(-C_{\ell-1}^{m-1}F_{\ell-1}^{m-1} + D_{\ell+1}^{m-1}F_{\ell+1}^{m-1} + E_{\ell-1}^{m+1}F_{\ell-1}^{m+1} - G_{\ell+1}^{m+1}F_{\ell+1}^{m+1}\big)_{,x} \\
+ \frac{1}{2}i\big(C_{\ell-1}^{m-1}F_{\ell-1}^{m-1} - D_{\ell+1}^{m-1}F_{\ell+1}^{m-1} + E_{\ell-1}^{m+1}F_{\ell-1}^{m+1} - G_{\ell+1}^{m+1}F_{\ell+1}^{m+1}\big)_{,y} \\
+ \big(A_{\ell-1}^{m}F_{\ell-1}^{m} + B_{\ell+1}^{m}F_{\ell+1}^{m}\big)_{,z} + \sigma F_\ell^m = \sigma F_0^0 \delta_{\ell 0}\delta_{m0}
\end{aligned}
$$

for $\;0 \le \ell \le N\;$ and $\;-\ell \le m \le \ell$

Number of Unknowns

$$F_0^0$$

$$F_1^{-1}\, F_1^0 \;\; F_1^1$$

$$F_2^{-2}\, F_2^{-1}\, F_2^0 \;\; F_2^1 \;\; F_2^2$$

$$\vdots$$

$$F_{N-1}^{-N+1}\, F_{N-1}^{-N+2} \cdots F_{N-1}^{0} \cdots \;\; F_{N-1}^{N-2} \;\; F_{N-1}^{N-1}$$

$$F_N^{-N} \;\; F_N^{-N+1}\, F_N^{-N+2} \cdots F_N^0 \quad \cdots \;\; F_N^{N-2} \;\; F_N^{N-1} \;\; F_N^N$$

number of unknowns : $\displaystyle\sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} 1 = N^2 + 2N + 1$

Reduced Number of Unknowns

$$\overline{Y}_\ell^m = (-1)^m Y_\ell^{-m}$$

$$F_\ell^m(\underline{x}, t) = \int_{\mathbb{S}^2} \overline{Y}_\ell^m F(\underline{x}, \underline{\Omega}, t)\, d\underline{\Omega} = (-1)^m \int_{\mathbb{S}^2} Y_\ell^{-m} F(\underline{x}, \underline{\Omega}, t)\, d\underline{\Omega}$$

$$\overline{F}_\ell^m(\underline{x}, t) = (-1)^m \int_{\mathbb{S}^2} \overline{Y}_\ell^{-m} F(\underline{x}, \underline{\Omega}, t)\, d\underline{\Omega} = (-1)^m F_\ell^{-m}$$

$$\underline{q} = [F_0^0, F_1^0, F_2^0, \cdots, F_N^0, F_1^1, F_2^1, \cdots, F_N^1 \cdots F_N^N]^T$$

number of unknowns : $\displaystyle\sum_{\ell=0}^{N} \sum_{m=0}^{\ell} 1 = \frac{1}{2}(N^2 + 3N) + 1$

Reduced Moments Equations

$$\underline{q}_{,t} + \underline{\underline{A}}\underline{q}_{,x} + \underline{\underline{B}}\underline{q}_{,z} = \sigma \underline{\underline{C}}\underline{q}$$

$\underline{\underline{A}}$, $\underline{\underline{B}}$: P_N Jacobians diagonalizable with same eigenvalues.

$$\underline{\underline{C}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}$$

P_1 Jacobians

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 0 & -\sqrt{\frac{2}{3}} \\ 0 & 0 & 0 \\ -\sqrt{\frac{1}{6}} & 0 & 0 \end{bmatrix} \qquad \underline{\underline{B}} = \begin{bmatrix} 0 & \sqrt{\frac{1}{3}} & 0 \\ \sqrt{\frac{1}{3}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

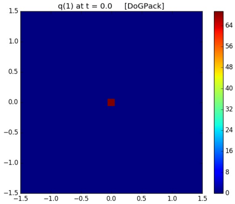
P_3 Jacobians

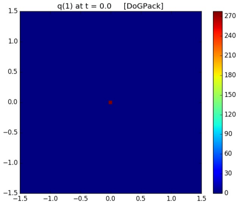
$$\underline{\underline{A}} = \begin{pmatrix}
0 & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{5}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\frac{2}{15}} & 0 & -\sqrt{\frac{12}{35}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{\frac{6}{35}} & 0 & 0 & 0 & 0 \\
-\sqrt{\frac{1}{6}} & 0 & \sqrt{\frac{1}{30}} & 0 & 0 & 0 & 0 & -\sqrt{\frac{1}{5}} & 0 & 0 \\
0 & -\sqrt{\frac{1}{10}} & 0 & \sqrt{\frac{3}{70}} & 0 & 0 & 0 & 0 & -\sqrt{\frac{1}{7}} & 0 \\
0 & 0 & -\sqrt{\frac{3}{35}} & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{70}} & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{\frac{1}{5}} & 0 & \sqrt{\frac{1}{70}} & 0 & 0 & -\sqrt{\frac{3}{14}} \\
0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{1}{7}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{3}{14}} & 0 & 0
\end{pmatrix}$$

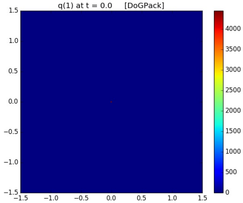
P_3 Jacobians

$$\underline{\underline{B}} = \begin{pmatrix} 0 & \sqrt{\frac{1}{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{4}{15}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{4}{15}} & 0 & \sqrt{\frac{9}{35}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{9}{35}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{5}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{1}{5}} & 0 & \sqrt{\frac{8}{35}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{8}{35}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{7}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{7}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Initial Conditions $\delta(x, z, t = 0)$ in 2D

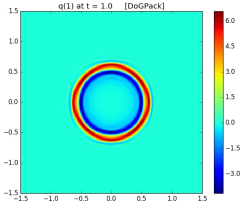


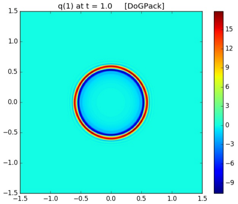


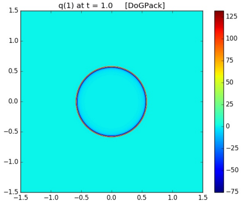


mx=mz=50 mx=mz=100 mx=mz=400

P_1 Approximation in 2D ($t = 1$)

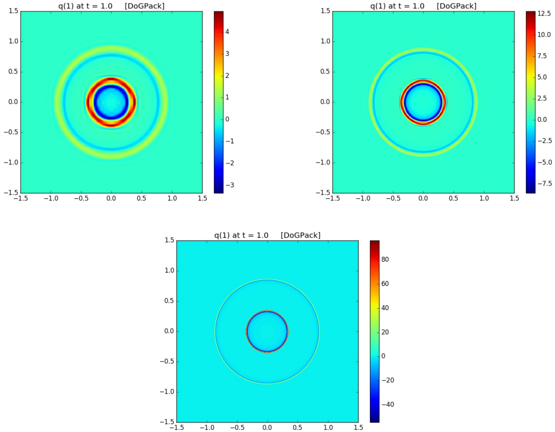






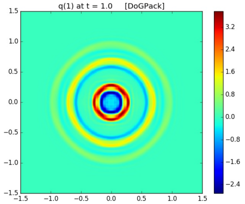
mx=mz=50 mx=mz=100 mx=mz=400

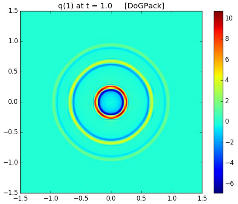
P_3 Approximation in 2D ($t = 1$)

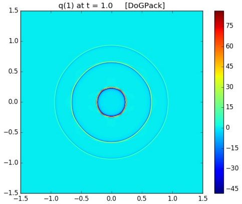


mx=mz=50 mx=mz=100 mx=mz=400

P_5 Approximation in 2D ($t = 1$)

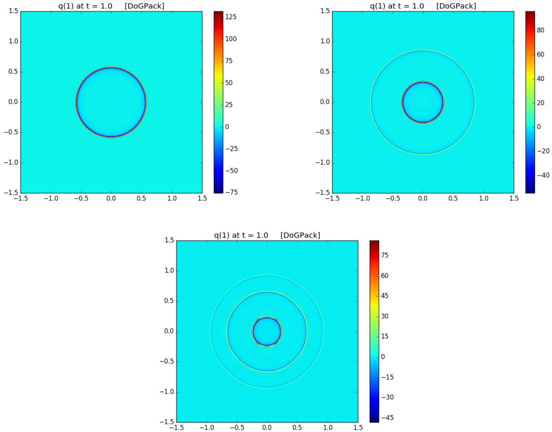






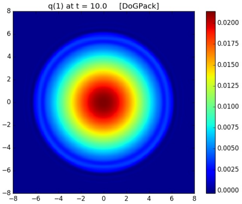
mx=mz=50 mx=mz=100 mx=mz=400

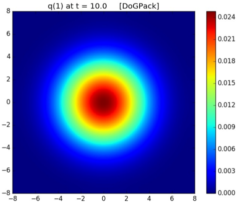
P_1 vs P_3 vs P_5 Approximation in 2D $(t = 1)$

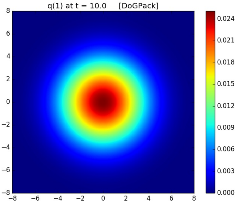


mx=mz=400

P_N Approximation in 2D (Long Term Behavior, $t = 10$)







mx=mz=50

P_1 Approximation in 2D ($m_x = m_z = 50,\ t \in [0, 10]$)

Future Work

- piecewise-P_N approximation in 2D
- positivity-preserving limiters for piecewise-P_N in 2D
- convergence rate to the exact solution

References 1

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The End

(Thank you!)