Solving the Radiative Transfer Equation via the Radon Transform

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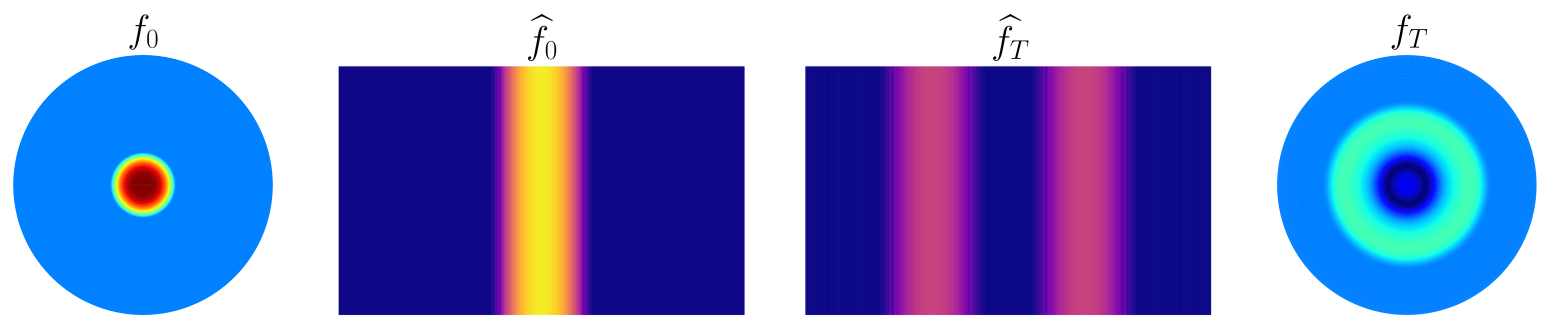
Problem Statement

- **Goal:** Use Radon transform to reduce dimensionality of hyperbolic PDEs
  - Apply Radon transform to high dimensional PDE ($f_0 \to \widehat{f}_0$)
  - Solve each system in a family of 1-dimensional PDEs ($\widehat{f}_0 \to \widehat{f}_T$)
  - Use inverse Radon transform to get solution in physical space ($\widehat{f}_T \to f_T$)



Hyperbolic PDE: The P_N Approximation

- We want to solve the **Radiative Transfer Equation (RTE)**

$$F_{,t} + \Omega \cdot \nabla F + \sigma_t F = \frac{\sigma_s}{4\pi}\int_{\mathbb{S}^2} F \, d\Omega$$

- Model for subatomic particles propagating through a homogeneous medium
- $F(t, \underline{x}, \underline{\Omega}) : \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ is the distribution function for particles
- **RTE** is a linear transport equation in as many as $1 + 5$ dimensions
- F can be approximated with spherical harmonics in $\underline{\Omega}$
- This yields the P_N equations, a system of $\frac{1}{2}(N+1)(N+2)$ PDEs in 2D:

$$\underline{q}_{,t} + \underline{\underline{A}}\,\underline{q}_{,x} + \underline{\underline{B}}\,\underline{q}_{,y} = \underline{\underline{C}}\,\underline{q}$$

- **Hyperbolicity:** $\cos(\alpha)\,\underline{\underline{A}} + \sin(\alpha)\,\underline{\underline{B}}$ is diagonalizable with $\lambda_i \in \mathbb{R}\ \forall i$

Radon Transform

- The **Radon transform (RT)** rotates the xy plane by an angle ω to form an sz plane, then integrates f parallel to the z axis for several values of s
- Given a function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ with compact support:

$$\mathcal{R}(f)(s, \omega) := \int_{-\infty}^{\infty} f(s\cos(\omega) - z\sin(\omega), s\sin(\omega) + z\cos(\omega))\, dz$$

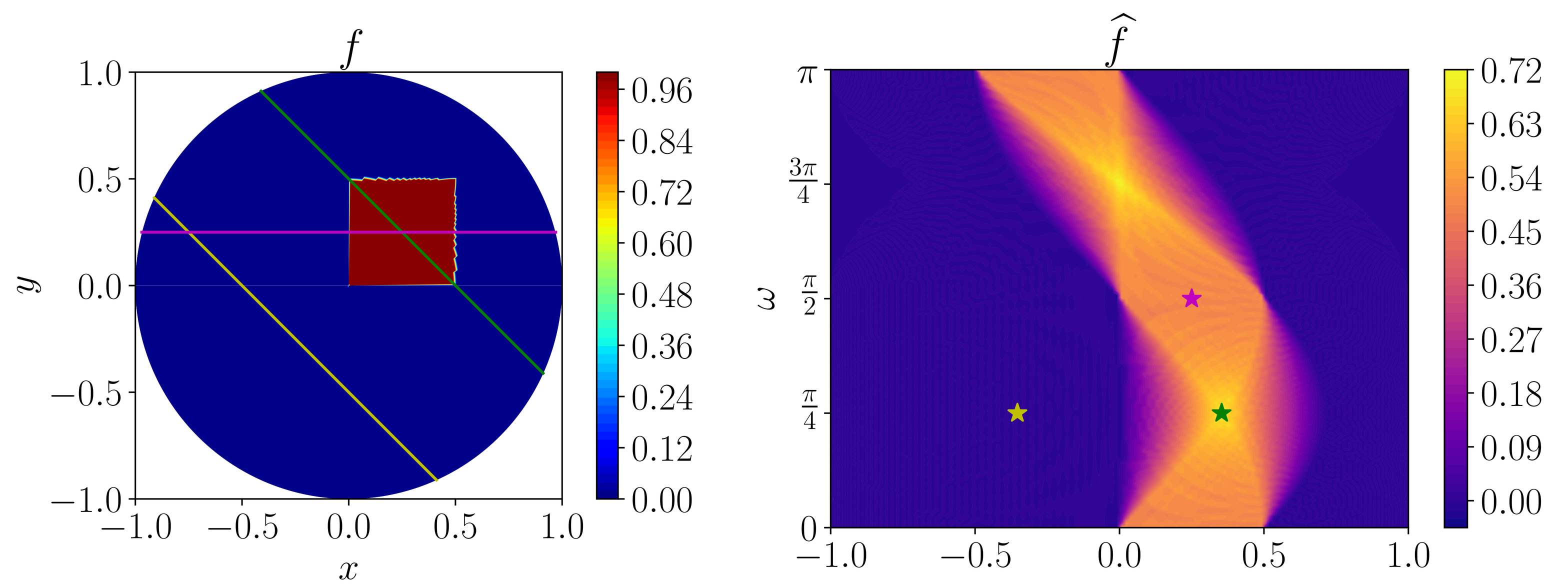
- For each angle ω, the **RT** simplifies the hyperbolic PDE:

$$\mathcal{R}\left(\underline{q}_{,t} + \underline{\underline{A}}\,\underline{q}_{,x} + \underline{\underline{B}}\,\underline{q}_{,y} = \underline{\underline{C}}\,\underline{q}\right) \implies \underline{\widehat{q}}_{,t} + \underline{\underline{\widetilde{A}}}(\omega)\underline{\widehat{q}}_{,s} = \underline{\underline{C}}\,\underline{\widehat{q}}$$

where $\underline{\underline{\widetilde{A}}}(\omega) := \cos(\omega)\,\underline{\underline{A}} + \sin(\omega)\,\underline{\underline{B}}$ and is diagonalizable

- Because ω is constant for each system, $\underline{\widehat{q}}(t, s, \omega)$ can be computed at the final time T for many angles through standard 1D timestepping methods
- The **inverse Radon transform (IRT)** is used to bring these solutions, $\underline{\widehat{q}}$, back to physical space, forming the vector of solutions $\underline{q}(t = T, x, y)$

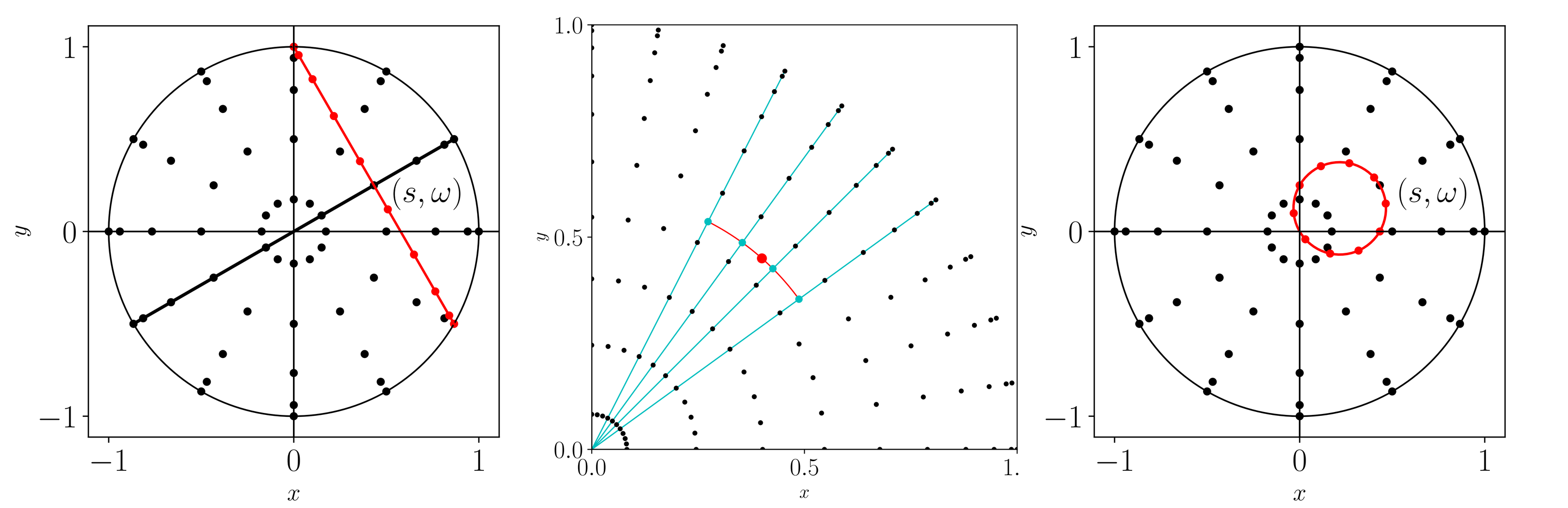
Discretizing the Radon Transform



Square of unit height

RT of square

- Quadrature on irregular mesh requires modified interpolation scheme



Integration for **RT**

Interpolation Scheme

Backprojection

Inverse Radon Transform

- Given $\widehat{f}(s, \omega) : \mathbb{R} \times [0, \pi] \to \mathbb{R}$, the **backprojection** is the adjoint of $\mathcal{R}$:

$$\mathcal{R}^*\left(\widehat{f}(s, \omega)\right)(x, y) := \int_0^{\pi} \widehat{f}(x\cos(\omega) + y\sin(\omega), \omega)\, d\omega$$

- Given $\widehat{f}$, recover f by solving the normal equations

$$\mathcal{R}^*(\mathcal{R}(f)) = \mathcal{R}^*\left(\widehat{f}\right)$$

- For discretized $\underline{f}$, $\underline{\widehat{f}}$, solve the normal equations via BiCGSTAB

Solving Systems of One-dimensional PDEs

- Diagonalizability of $\underline{\underline{\widetilde{A}}}(\omega)$ implies $\underline{\underline{\widetilde{A}}}(\omega) = \underline{\underline{P}}\,\underline{\underline{\Lambda}}\,\underline{\underline{P}}^{-1}$; partially decouples PDE

$$\underline{\widehat{q}}_{,t} + \underline{\underline{\widetilde{A}}}(\omega)\underline{\widehat{q}}_{,s} = \underline{\underline{C}}\,\underline{\widehat{q}} \implies \underline{w}_{,t} + \underline{\underline{\Lambda}}\,\underline{w}_{,s} = \underline{\underline{F}}\,\underline{w}$$

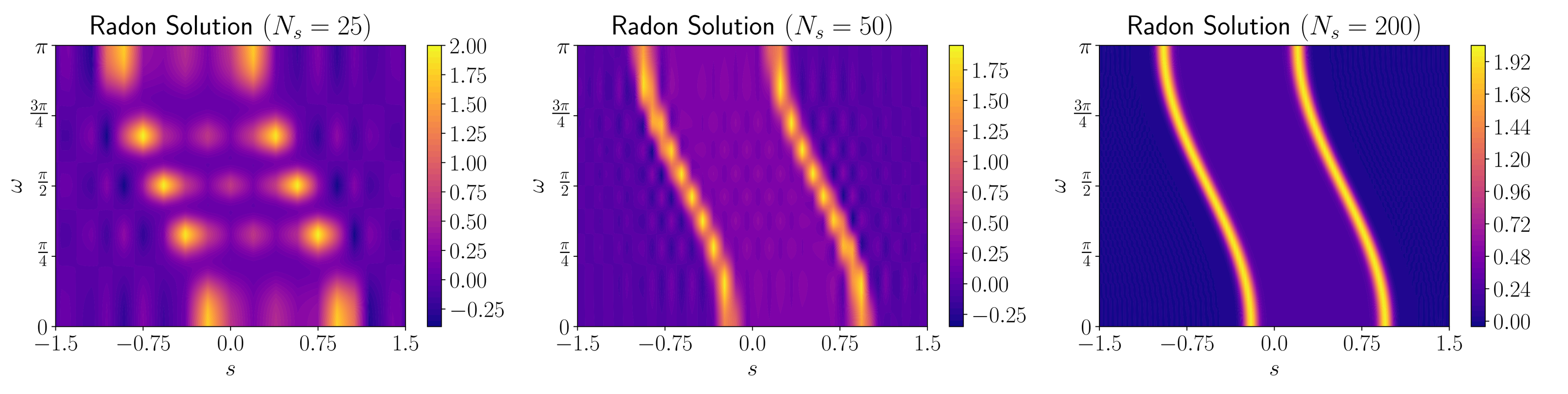
- Discretize each variable in space and approximate spatial derivative

$$\underline{w}_{,t} + \underline{\underline{\Lambda}}\,\underline{w}_{,s} = \underline{\underline{F}}\,\underline{w} \implies \underline{W}_{p,t} + \lambda_p \underline{\underline{D}}\,\underline{W}_p = \sum_{q=1}^{M} F_{pq}\,\underline{W}_q$$

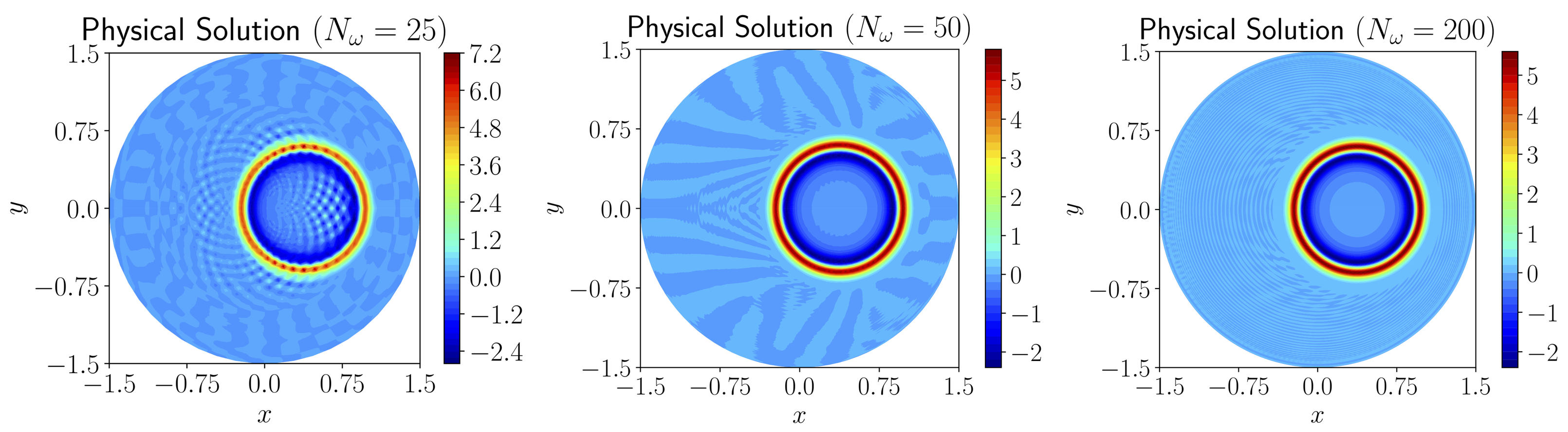
- Modify $\underline{\underline{D}}$ to enforce inflow Dirichlet boundary conditions using sign of λ_p
- Use an L-stable, third-order IMEX scheme (IMEX-SSP3(4,3,3))

Results

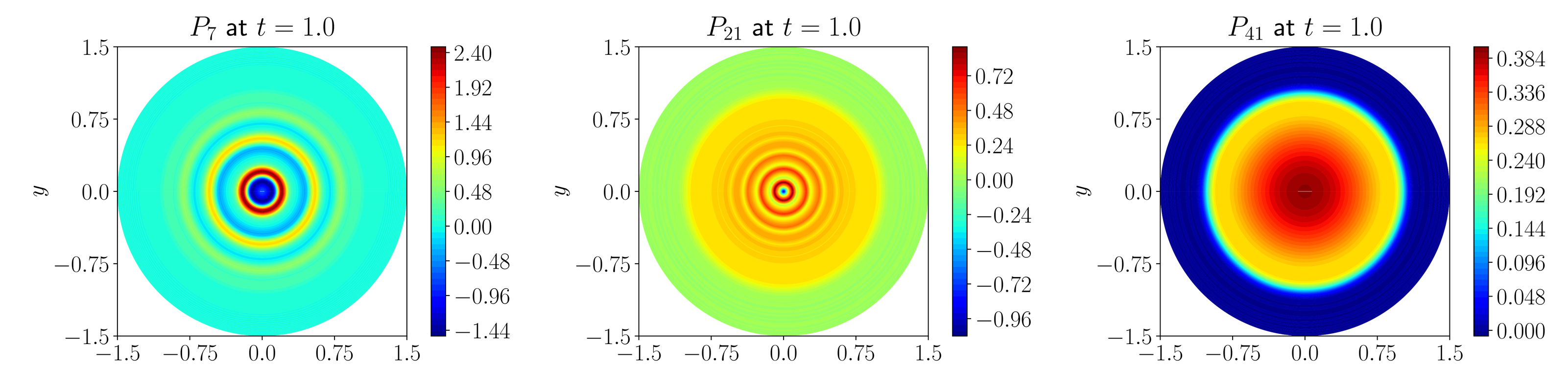
- Overall method is minimally 4^{th} order accurate
- Transport quality independent of angular discretization ($N_\omega = 200$)

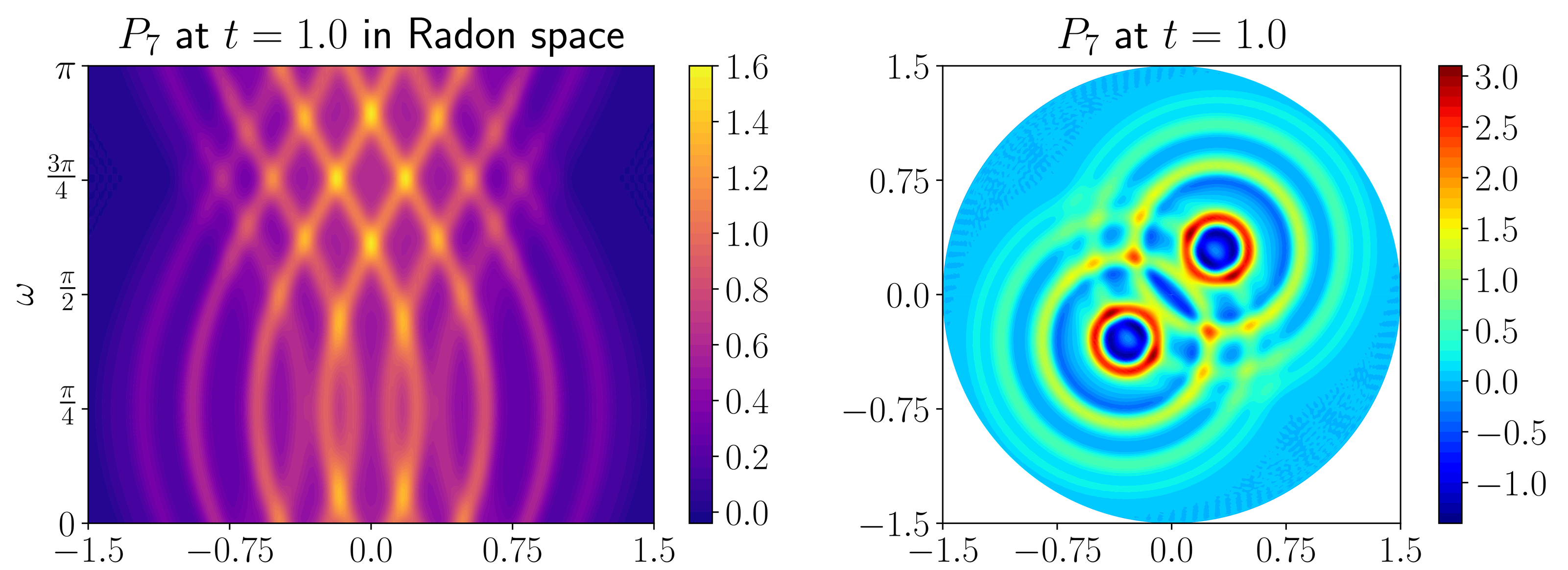


- **IRT** quality independent of spatial discretization ($N_s = 200$)



- Accuracy of P_N approximation increases with N





Future Work

- Adding spatially-dependent collision terms to the P_N equations
- Implementing more sophisticated/higher order timestepping schemes
- Improving efficiency through a parallelization of transport computations

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