Solving the Radiative Transfer Equation via the Radon Transform

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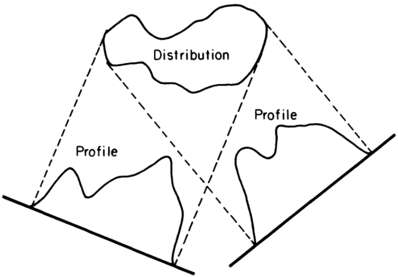
Outline

Imaging Problem

- The imaging problem is to reconstruct a distribution from several profiles taken at different angles
- The Radon transform is a simple way to produce profiles



Deans, S. (1983) *The Radon Transform and Some of Its Applications*. Wiley-Interscience Publication. 1983. Pg 3.

- Certain advection problems are easier to solve profile-by-profile

Our Goal

- To efficiently solve multi-dimensional systems of hyperbolic partial differential equations using the Radon transform
- Three main steps:
  1. Forward Radon transform multi-dimensional problem
  2. Solve family of 1D advection problems
  3. Use inverse Radon transform to original space

Notation

- We will use the following notation in the presentation:

Radon transform	$\mathcal{R}$
Radon transform of a function	$\widehat{f}$
Vector	$\underline{v}$
Matrix	$\underline{\underline{A}}$
Partial derivative	$f_{,t}$

Outline

Coordinate Transform

- To get a single profile, rotate the xy coordinate system by a positive angle ω to obtain the sz coordinate system using a rotation matrix:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\omega) & -\sin(\omega) \\ \sin(\omega) & \cos(\omega) \end{bmatrix} \begin{bmatrix} s \\ z \end{bmatrix}$$

Coordinate Transform

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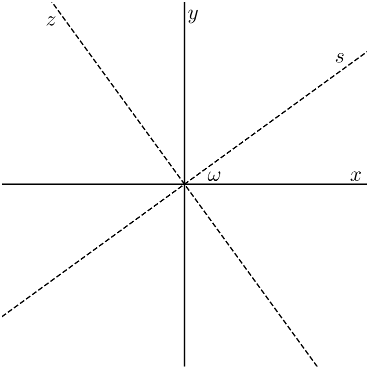
- Use the inverse of the matrix to obtain the reverse transform:

$$\begin{bmatrix} s \\ z \end{bmatrix} = \begin{bmatrix} \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Coordinate Rotation



Coordinate Transform

- Rewrite the rotation matrices:

$$
\begin{aligned}
x(s, z; \omega) &= s\ \cos(\omega) - z\ \sin(\omega) \\
y(s, z; \omega) &= s\ \sin(\omega) + z\ \cos(\omega) \\
s(x, y; \omega) &= x \cos(\omega) + y \sin(\omega) \\
z(x, y; \omega) &= -x \sin(\omega) + y \cos(\omega)
\end{aligned}
$$

- To construct a single profile, ω is constant

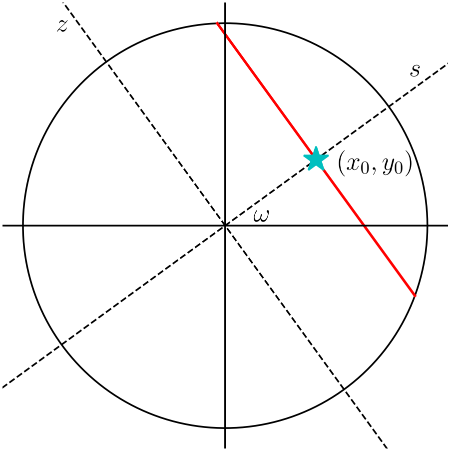
Radon Transform

- Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function with compact support
- After rotating by ω, select a point on the new s-axis and integrate parallel to the new z-axis
- The Radon transform of f is formally defined as follows

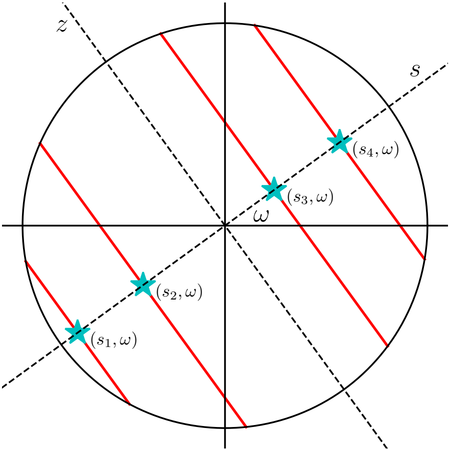
$$\begin{aligned}\mathcal{R}(f) = \widehat{f}(s,\omega) &:= \int_{-\infty}^{\infty} f(\, x(s,z;\omega), y(s,z;\omega)\,)\, dz \\ &= \int_{-\infty}^{\infty} f(s\cos(\omega) - z\sin(\omega), s\sin(\omega) + z\cos(\omega))\, dz\end{aligned}$$

- $\mathcal{R}$ is a linear operator

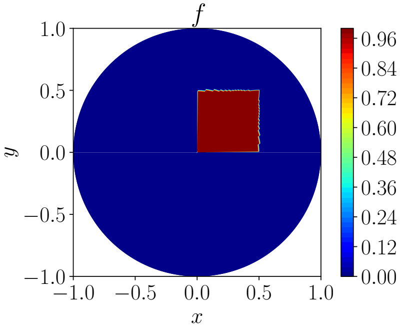
Radon Transform at (x_0, y_0)

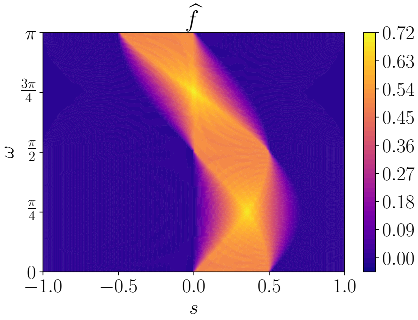


Radon Transform

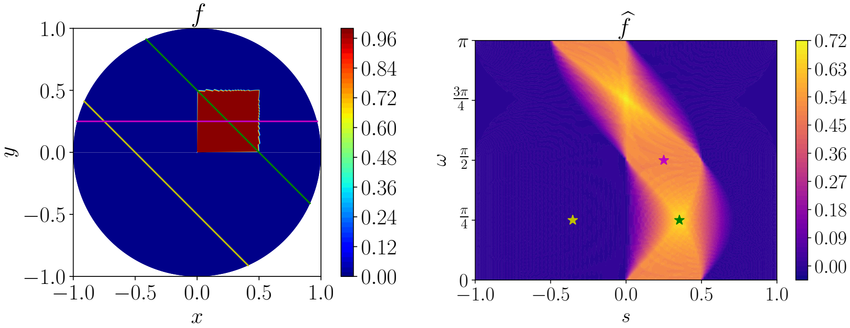


Radon Transform Example





Radon Transform Example



Outline

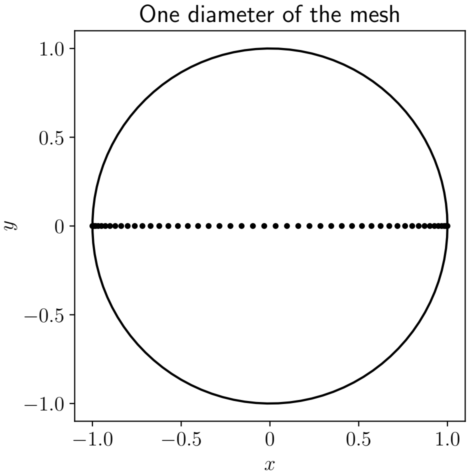
Grid Structure

- Need discretized domain for inverse computation
- Profile-by-profile construction suggests N_ω evenly-spaced diameters
- Along each diameter, create N_s Chebyshev points of the second kind:

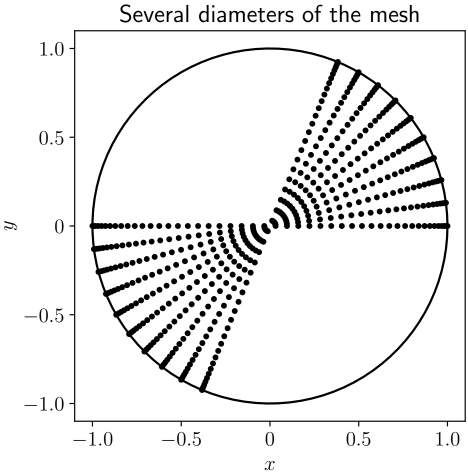
$$s_j = \cos\left(\frac{j\pi}{N_s}\right) \quad \text{for } j = 0, 1, \ldots, N_s$$

  - Allows for spectrally accurate interpolation, differentiation

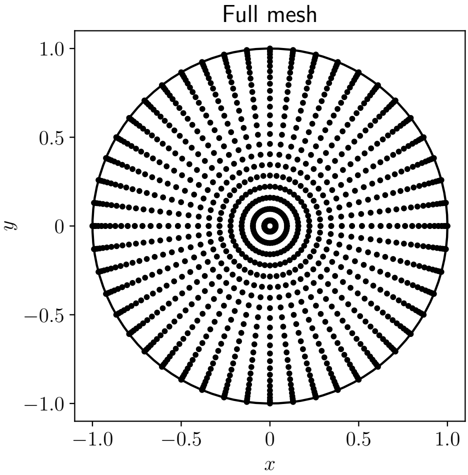
Grid Structure Example



Grid Structure Example



Grid Structure Example

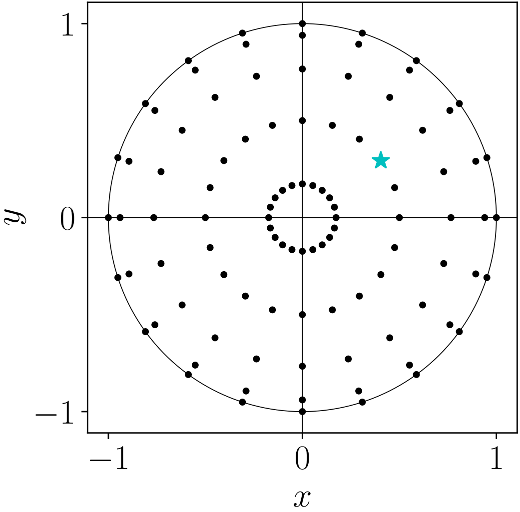


Computing Integrals

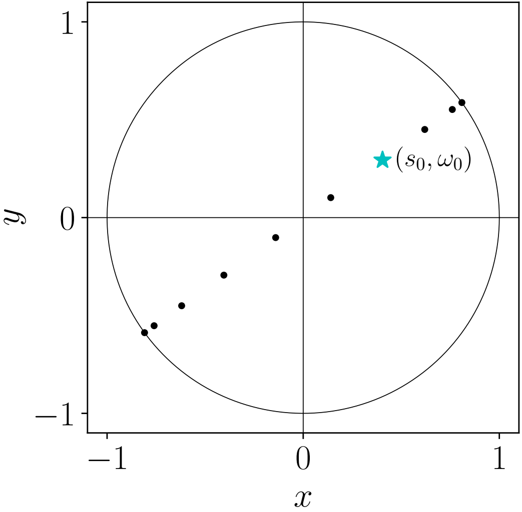
- Compact support of f means we compute line integrals across chords of the domain
- Compute line integrals with Clenshaw-Curtis quadrature

$$\begin{aligned}\mathcal{R}(f) &= \int_{-\infty}^{\infty} f(x,y)\,dz \\ &= \int_{-\tau}^{\tau} f(x,y)\,dz \\ &\approx \sum_{i=1}^{N_q} w_i f(z_i)\end{aligned}$$

Quadrature Example



Quadrature Example



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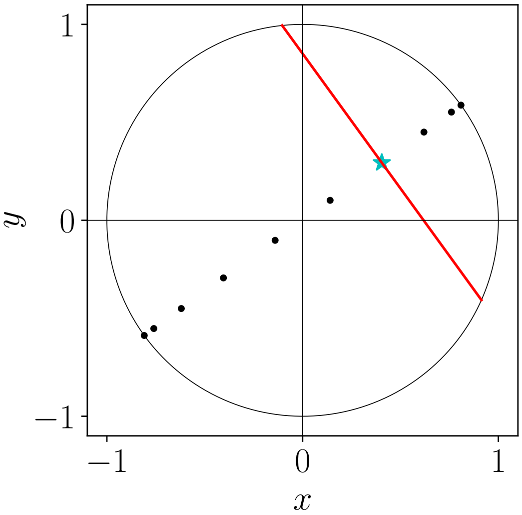

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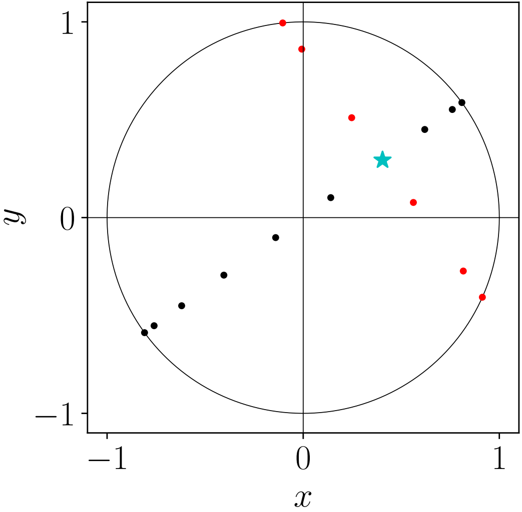

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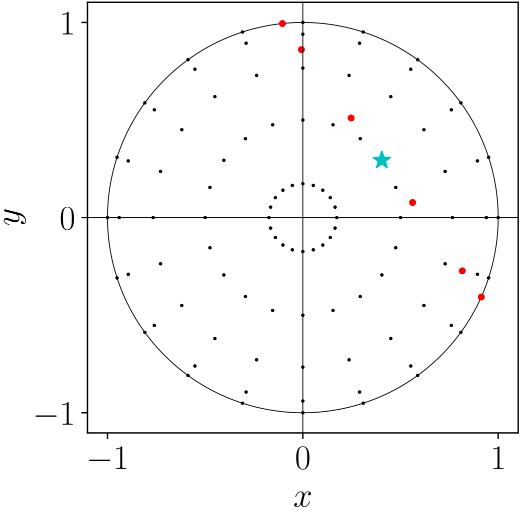
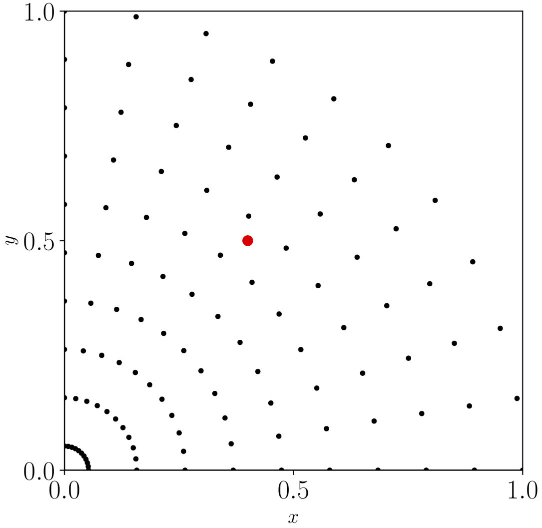


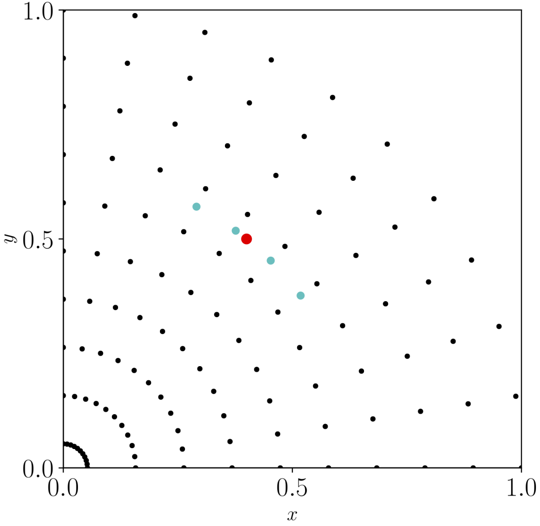
Interpolation Scheme

- Need to sample f at arbitrary quadrature nodes, use interpolation
- Use a series of one-dimensional interpolation schemes:
  - Spectrally accurate on each diameter
  - Fourth order accurate across angles

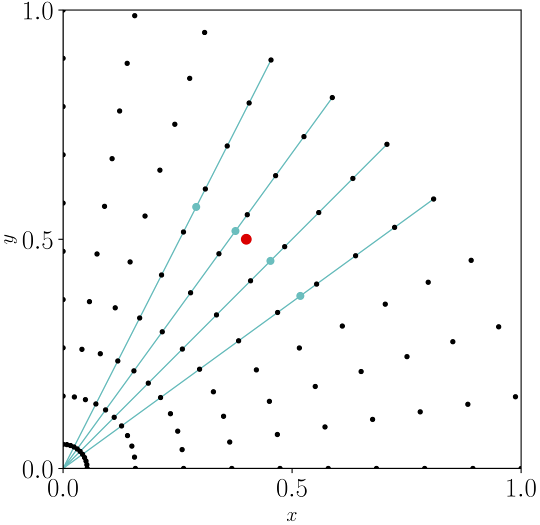
Interpolation Scheme Example



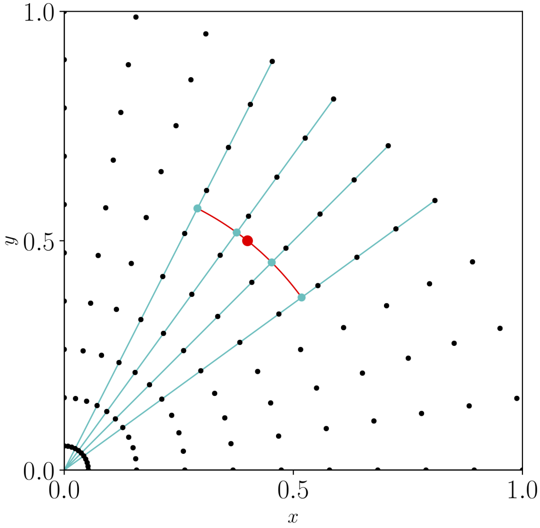
Interpolation Scheme Example (cont.)



Interpolation Scheme Example (cont.)



Interpolation Scheme Example (cont.)



Outline

Ill-posedness of the IRT

- Suppose $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ is a function with compact support, and

$$\mathcal{R}(f) = \widehat{f}$$

- Is there a way of recovering f given only $\widehat{f}$?

Ill-posedness of the IRT (cont.)

- In discretized case, we only know $f, \widehat{f}$ at mesh points
- Recall that our mesh is parameterized by two integers
  - N_ω, the number of angles
  - N_s, the number of Chebyshev points along one angle
  - Total of $N_\omega \times N_s$ mesh points
- Define $\underline{f}, \underline{\widehat{f}} \in \mathbb{R}^{N_\omega \times N_s}$ such that

$$\underline{f} := f \text{ evaluated at mesh points}$$
$$\underline{\widehat{f}} := \widehat{f} \text{ evaluated at mesh points}$$

Ill-posedness of the IRT (cont.)

- Recall we can approximate $\mathcal{R}(f)$ with a matrix-vector multiplication
- We create a matrix $\underline{\underline{R}} \in \mathbb{R}^{(N_s \times N_\omega)\times(N_s \times N_\omega)}$ such that

$$\underline{\underline{R}}\,\underline{f} = \underline{\widehat{f}}$$

where the j^{th} column of $\underline{\underline{R}}$ is the discretized Radon transform evaluated at $\underline{e_j}$

Ill-posedness of the IRT (cont.)

- The system

$$\underline{\underline{R}}\,\underline{f} = \underline{\widehat{f}}$$

is *very hard* to solve with typical linear algebra techniques

Approaches to Computing the IRT

- Note that

$$\underline{\underline{R}}\,\underline{f} = \underline{\widehat{f}} \quad \Longleftrightarrow \quad \underline{\underline{R}}^T \underline{\underline{R}}\,\underline{f} = \underline{\underline{R}}^T \underline{\widehat{f}}$$

- The system

$$\underline{\underline{R}}^T \underline{\underline{R}}\,\underline{f} = \underline{\underline{R}}^T \underline{\widehat{f}}$$

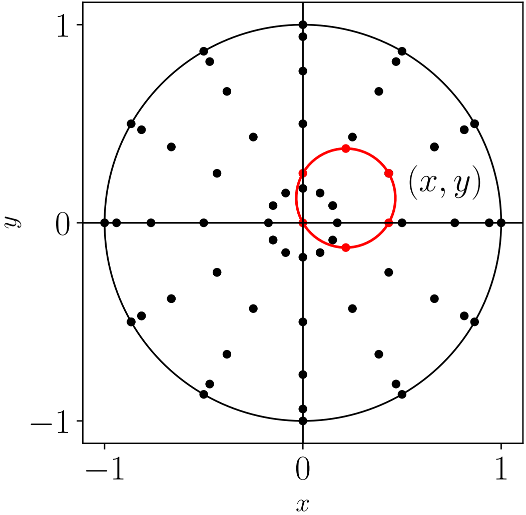
  are the **normal equations**
- What does pre-multiplication by $\underline{\underline{R}}^T$ mean?

Backprojection

- The adjoint of the Radon transform, denoted $\mathcal{R}^*$, is known as **backprojection**
- If $\widehat{f}\,(s,\omega) : \mathbb{R} \times [0,\pi] \to \mathbb{R}$, the backprojection of $\widehat{f}$ is

$$\mathcal{R}^* \left(\widehat{f}\right)(x,y) := \int_0^\pi f(x\cos(\omega) + y\sin(\omega), \omega)\, d\omega$$

Graphical Representation of Backprojection



Backprojection (cont.)

- We can *approximate* the action of $\mathcal{R}^*$ using $\underline{\underline{R}}^T$
- Recall we want to solve

$$\underline{\underline{R}}^T \underline{\underline{R}} \underline{f} = \underline{\underline{R}}^T \underline{\widehat{f}}$$

- However, $\underline{\underline{R}}$ is *very* large in practice
  - N_s and N_ω dictate how well we can recover $\underline{f}$

Using BiCGSTAB to Solve the Normal Equations

- We do not explicitly need $\underline{\underline{R}}^T\,\underline{\underline{R}}$, just the matrix-vector products $\underline{\underline{R}}\,\underline{f}$, $\underline{\underline{R}}^T\left(\underline{\underline{R}}\,\underline{f}\right)$
- We use an iterative method (BiCGSTAB) to solve

$$\underline{\underline{R}}^T\,\underline{\underline{R}}\,\underline{f} = \underline{\underline{R}}^T\,\underline{\widehat{f}}$$

by minimizing successive residuals

$$\left\|\underline{\underline{R}}^T\,\underline{\widehat{f}} - \underline{\underline{R}}^T\,\underline{\underline{R}}\,\underline{f}^{(k)}\right\|_2$$

over span$\{\underline{r_0}, \left(\underline{\underline{R}}^T\,\underline{\underline{R}}\right)\,\underline{r_0}, \ldots,\}$ where $\underline{r_0} = \underline{\underline{R}}^T\,\underline{\underline{R}}\underline{f}^{(0)} - \underline{\underline{R}}^T\,\underline{\widehat{f}}$

Outline

Application: Radiative Transfer

- We want to solve the **Radiative Transfer Equation**

$$F_{,t} + \underline{\Omega} \cdot \underline{\nabla} F + \sigma_t F = \frac{\sigma_s}{4\pi} \int_{\mathbb{S}^2} F \, d\underline{\Omega}$$

- A kinetic model for subatomic particles propagating through a homogeneous medium
  - $\underline{\Omega} \cdot \underline{\nabla} F$ is a transport term
  - $\frac{\sigma_s}{4\pi} \int_{\mathbb{S}^2} F \, d\underline{\Omega} - \sigma_t F$ is a collision term
- $F(t, \underline{x}, \underline{\Omega}) : \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{S}^2 \to \mathbb{R}$ is the distribution function for particles at a specific location and with a specific velocity
- A linear transport equation in $1 + 5$ dimensions

Spherical Harmonics

- Use spherical harmonics to create P_N equations

$$F(t, \underline{x}, \underline{\Omega}) \approx \sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} F_\ell^m(t, \underline{x}) Y_\ell^m(\mu, \phi)$$

- Already know spherical harmonics, $Y_\ell^m(\mu, \phi)$; removes angular dependence from F
- Need to solve for (most) of the $F_\ell^m(t, \underline{x})$

P_N Equations

- For a given positive, odd N, the P_N approximation yields a system of $\frac{1}{2}(N+1)(N+2)$ linear differential equations
- For P_1 in 2D, the equations are

$$\begin{aligned} F^0_{0,t} - \sqrt{\frac{2}{3}}\, F^1_{1,x} + \sqrt{\frac{1}{3}}\, F^0_{1,y} &= -\sigma_a\, F^0_0 \\ F^0_{1,t} + \sqrt{\frac{1}{3}}\, F^0_{0,y} &= -(\sigma_a + \sigma_s)\, F^0_1 \\ F^1_{1,t} - \sqrt{\frac{1}{6}}\, F^0_{0,y} &= -(\sigma_a + \sigma_s)\, F^1_1 \end{aligned}$$

for unknown functions $F^0_0(x,y)$, $F^0_1(x,y)$, and $F^1_1(x,y)$

P_N Equation Matrices

- Can be written as $\underline{F}_{,t} + \underline{\underline{A}}\,\underline{F}_{,x} + \underline{\underline{B}}\,\underline{F}_{,y} = \underline{\underline{C}}\,\underline{F}$ with the following matrices:

$$\underline{F} = \begin{bmatrix} F_0^0 \\ F_1^0 \\ F_1^1 \end{bmatrix} \quad \underline{\underline{A}} = \begin{bmatrix} 0 & 0 & -\sqrt{\frac{2}{3}} \\ 0 & 0 & 0 \\ -\sqrt{\frac{1}{6}} & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{B}} = \begin{bmatrix} 0 & \sqrt{\frac{1}{3}} & 0 \\ \sqrt{\frac{1}{3}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{\underline{C}} = \begin{bmatrix} -\sigma_a & 0 & 0 \\ 0 & -(\sigma_a + \sigma_s) & 0 \\ 0 & 0 & -(\sigma_a + \sigma_s) \end{bmatrix}$$

Outline

Solving the P_N Equations

- We now consider the following system of M differential equations

$$\underline{q}_{,t} + \underline{\underline{A}}\,\underline{q}_{,x} + \underline{\underline{B}}\,\underline{q}_{,y} = \underline{\underline{C}}\,\underline{q}$$
$$\underline{q}(t=0,x,y) = \underline{q}_0(x,y)$$

  where
  - $q(t,x,y) : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \to \mathbb{R}^M$ is a vector valued function
  - $\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}} \in \mathbb{R}^{M \times M}$

- **Claim:** the Radon transform simplifies spatial derivatives

Radon Transform of the PDE

- Apply $\mathcal{R}$ to the PDE at an angle ω

$$\mathcal{R}\left(\underline{q}_{,t} + \underline{\underline{A}}\,\underline{q}_{,x} + \underline{\underline{B}}\,\underline{q}_{,y} = \underline{\underline{C}}\,\underline{q}\right)$$

Radon Transform of the PDE

- Apply $\mathcal{R}$ to the PDE at an angle ω

$$\mathcal{R}\left(\underline{q}_{,t} + \underline{\underline{A}}\,\underline{q}_{,x} + \underline{\underline{B}}\,\underline{q}_{,y} = \underline{\underline{C}}\,\underline{q}\right)$$
$$\implies \mathcal{R}\left(\underline{q}_{,t}\right) + \underline{\underline{A}}\,\mathcal{R}\left(\underline{q}_{,x}\right) + \underline{\underline{B}}\,\mathcal{R}\left(\underline{q}_{,y}\right) = \underline{\underline{C}}\,\mathcal{R}(\underline{q})$$

- **Note:** $\mathcal{R}\left(\underline{f}\right)$ means applying $\mathcal{R}$ to each element of $\underline{f}$

Radon Transforms of Partial Derivatives

- By the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial s}{\partial x}\frac{\partial}{\partial s} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z}$$
$$\frac{\partial}{\partial y} = \frac{\partial s}{\partial y}\frac{\partial}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial}{\partial z}$$

- From our earlier definitions of s and z, we obtain

$$\frac{\partial}{\partial x} = \cos(\omega)\frac{\partial}{\partial s} - \sin(\omega)\frac{\partial}{\partial z}$$
$$\frac{\partial}{\partial y} = \sin(\omega)\frac{\partial}{\partial s} + \cos(\omega)\frac{\partial}{\partial z}$$

Radon Transform of Partial Derivatives

- Recall the definition of the Radon transform, where z is the axis perpendicular to s after a rotation by ω:

$$\mathcal{R}(f(x,y)) = \widehat{f}(s,\omega) = \int_{-\infty}^{\infty} f(\,x,y\,)\,dz$$

Radon Transforms of Partial Derivatives

- We can now compute $\mathcal{R}\left(f_{,x}\right)$

$$\mathcal{R}\left(f_{,x}\right) = \int_{-\infty}^{\infty} f_{,x}\, dz$$

Radon Transforms of Partial Derivatives

- We can now compute $\mathcal{R}\left(f_{,x}\right)$

$$
\begin{aligned}
\mathcal{R}\left(f_{,x}\right) &= \int_{-\infty}^{\infty} f_{,x}\, dz \\
&= \int_{-\infty}^{\infty} \left(\cos(\omega) f_{,s} - \sin(\omega) f_{,z}\right)\, dz
\end{aligned}
$$

Radon Transforms of Partial Derivatives

- We can now compute $\mathcal{R}\left(f_{,x}\right)$

$$\begin{aligned}
\mathcal{R}\left(f_{,x}\right) &= \int_{-\infty}^{\infty} f_{,x}\, dz \\
&= \int_{-\infty}^{\infty} \left(\cos(\omega) f_{,s} - \sin(\omega) f_{,z}\right)\, dz \\
&= \cos(\omega) \int_{-\infty}^{\infty} f_{,s}\, dz - \sin(\omega) \int_{-\infty}^{\infty} f_{,z}\, dz
\end{aligned}$$

Radon Transforms of Partial Derivatives

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$$
\begin{aligned}
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&= \int_{-\infty}^{\infty} \left(\cos(\omega) f_{,s} - \sin(\omega) f_{,z}\right)\, dz \\
&= \cos(\omega) \int_{-\infty}^{\infty} f_{,s}\, dz - \sin(\omega) \int_{-\infty}^{\infty} f_{,z}\, dz \\
&= \cos(\omega) \frac{\partial}{\partial s} \int_{-\infty}^{\infty} f\, dz - \sin(\omega)\left[f(z=+\infty) - f(z=-\infty)\right]
\end{aligned}
$$

Radon Transforms of Partial Derivatives

- We can now compute $\mathcal{R}\left(f_{,x}\right)$

$$\begin{aligned}
\mathcal{R}\left(f_{,x}\right) &= \int_{-\infty}^{\infty} f_{,x}\, dz \\
&= \int_{-\infty}^{\infty} \left(\cos(\omega) f_{,s} - \sin(\omega) f_{,z}\right)\, dz \\
&= \cos(\omega) \int_{-\infty}^{\infty} f_{,s}\, dz - \sin(\omega) \int_{-\infty}^{\infty} f_{,z}\, dz \\
&= \cos(\omega) \frac{\partial}{\partial s} \int_{-\infty}^{\infty} f\, dz - \sin(\omega) \left[f(z = +\infty) - f(z = -\infty) \right] \\
&= \cos(\omega)\, \frac{\partial}{\partial s} \mathcal{R}(f)
\end{aligned}$$

Radon Transforms of Partial Derivatives

- We can now compute $\mathcal{R}\left(f_{,x}\right)$

$$
\begin{aligned}
\mathcal{R}\left(f_{,x}\right) &= \int_{-\infty}^{\infty} f_{,x}\, dz \\
&= \int_{-\infty}^{\infty} \left(\cos(\omega) f_{,s} - \sin(\omega) f_{,z}\right)\, dz \\
&= \cos(\omega) \int_{-\infty}^{\infty} f_{,s}\, dz - \sin(\omega) \int_{-\infty}^{\infty} f_{,z}\, dz \\
&= \cos(\omega) \frac{\partial}{\partial s} \int_{-\infty}^{\infty} f\, dz - \sin(\omega) \left[f(z=+\infty) - f(z=-\infty)\right] \\
&= \cos(\omega)\, \frac{\partial}{\partial s} \mathcal{R}(f) \\
&= \cos(\omega) \widehat{f}_{,s}
\end{aligned}
$$

Radon Transforms of Partial Derivatives

- $\mathcal{R}(f_{,y})$ can be computed similarly

$$\begin{aligned}
\mathcal{R}(f_{,y}) &= \int_{-\infty}^{\infty} f_{,y}\, dz \\
&= \int_{-\infty}^{\infty} \left(\sin(\omega) f_{,s} - \cos(\omega) f_{,z}\right)\, dz \\
&= \sin(\omega) \int_{-\infty}^{\infty} f_{,s}\, dz - \cos(\omega) \int_{-\infty}^{\infty} f_{,z}\, dz \\
&= \sin(\omega) \frac{\partial}{\partial s} \int_{-\infty}^{\infty} f\, dz - \cos(\omega) \left[f(z=+\infty) - f(z=-\infty) \right] \\
&= \sin(\omega)\, \frac{\partial}{\partial s} \mathcal{R}(f) \\
&= \sin(\omega) \widehat{f}_{,s}
\end{aligned}$$

Radon Transforms of Partial Derivatives

- The Radon transform is a spatial transformation, thus:

$$\mathcal{R}\left(f_{,t}\right) = \frac{\partial}{\partial t}\mathcal{R}(f) = \widehat{f}_{,t}$$

- In summary,

$$\begin{aligned} \mathcal{R}(f_{,t}) &= \widehat{f}_{,t} \\ \mathcal{R}(f_{,x}) &= \cos(\omega)\widehat{f}_{,s} \\ \mathcal{R}(f_{,y}) &= \sin(\omega)\widehat{f}_{,s} \end{aligned}$$

- All spatial derivatives in xy space become derivatives of s

Radon Transform of the PDE

- Can now complete the transformation and add like terms:

$$\mathcal{R}\left(\underline{q}_{,t} + \underline{\underline{A}}\,\underline{q}_{,x} + \underline{\underline{B}}\,\underline{q}_{,y} = \underline{\underline{C}}\,\underline{q}\right)$$

Radon Transform of the PDE

- Can now complete the transformation and add like terms:

$$\mathcal{R}\left(\underline{q}_{,t} + \underline{\underline{A}}\,\underline{q}_{,x} + \underline{\underline{B}}\,\underline{q}_{,y} = \underline{\underline{C}}\,\underline{q}\right)$$

$$\Longrightarrow \mathcal{R}\left(\underline{q}_{,t}\right) + \underline{\underline{A}}\,\mathcal{R}\left(\underline{q}_{,x}\right) + \underline{\underline{B}}\,\mathcal{R}\left(\underline{q}_{,y}\right) = \underline{\underline{C}}\,\mathcal{R}(\underline{q})$$

Radon Transform of the PDE

- Can now complete the transformation and add like terms:

$$\mathcal{R}\left(\underline{q}_{,t} + \underline{\underline{A}}\,\underline{q}_{,x} + \underline{\underline{B}}\,\underline{q}_{,y} = \underline{\underline{C}}\,\underline{q}\right)$$

$$\Longrightarrow \mathcal{R}\left(\underline{q}_{,t}\right) + \underline{\underline{A}}\,\mathcal{R}\left(\underline{q}_{,x}\right) + \underline{\underline{B}}\,\mathcal{R}\left(\underline{q}_{,y}\right) = \underline{\underline{C}}\,\mathcal{R}(\underline{q})$$

$$\Longrightarrow \underline{\widehat{q}}_{,t} + \cos(\omega)\,\underline{\underline{A}}\,\underline{\widehat{q}}_{,s} + \sin(\omega)\,\underline{\underline{B}}\,\underline{\widehat{q}}_{,s} = \underline{\underline{C}}\,\underline{\widehat{q}}$$

Radon Transform of the PDE

- Can now complete the transformation and add like terms:

$$\mathcal{R}\left(\underline{q}_{,t} + \underline{\underline{A}}\,\underline{q}_{,x} + \underline{\underline{B}}\,\underline{q}_{,y} = \underline{\underline{C}}\,\underline{q}\right)$$

$$\Longrightarrow \mathcal{R}\left(\underline{q}_{,t}\right) + \underline{\underline{A}}\,\mathcal{R}\left(\underline{q}_{,x}\right) + \underline{\underline{B}}\,\mathcal{R}\left(\underline{q}_{,y}\right) = \underline{\underline{C}}\,\mathcal{R}(\underline{q})$$

$$\Longrightarrow \widehat{\underline{q}}_{,t} + \cos(\omega)\,\underline{\underline{A}}\,\widehat{\underline{q}}_{,s} + \sin(\omega)\,\underline{\underline{B}}\,\widehat{\underline{q}}_{,s} = \underline{\underline{C}}\,\widehat{\underline{q}}$$

$$\Longrightarrow \widehat{\underline{q}}_{,t} + \left(\cos(\omega)\,\underline{\underline{A}} + \sin(\omega)\,\underline{\underline{B}}\right)\widehat{\underline{q}}_{,s} = \underline{\underline{C}}\,\widehat{\underline{q}}$$

Radon Transform of the PDE

- Given

$$\underline{\widehat{q}}_{,t} + \left(\cos(\omega)\,\underline{\underline{A}} + \sin(\omega)\,\underline{\underline{B}}\right)\underline{\widehat{q}}_{,s} = \underline{\underline{C}}\,\underline{\widehat{q}}$$

let $\widetilde{\underline{\underline{A}}}(\omega) := \cos(\omega)\,\underline{\underline{A}} + \sin(\omega)\,\underline{\underline{B}}$ so that

$$\underline{\widehat{q}}_{,t} + \widetilde{\underline{\underline{A}}}(\omega)\underline{\widehat{q}}_{,s} = \underline{\underline{C}}\,\underline{\widehat{q}}$$
$$\underline{\widehat{q}}(t = 0, s, \omega) = \underline{\widehat{q}}_0(s, \omega)$$

- For each ω, we now have a system of one dimensional PDEs which can be solved with traditional methods

Radon Transform of the PDE

- To recap: we transformed the 2D system of PDEs

$$\underline{q}_{,t} + \underline{\underline{A}}\,\underline{q}_{,x} + \underline{\underline{B}}\,\underline{q}_{,y} = \underline{\underline{C}}\,\underline{q}$$
$$\underline{q}(t=0,x,y) = \underline{q_0}(x,y)$$

into a collection of 1D systems of PDEs

$$\underline{\widehat{q}}_{,t} + \underline{\underline{\widetilde{A}}}(\omega)\,\underline{\widehat{q}}_{,s} = \underline{\underline{C}}\,\underline{\widehat{q}}$$
$$\underline{\widehat{q}}(t=0,s,\omega) = \underline{\widehat{q_0}}(s,\omega)$$

that we can now solve, angle by angle

Defining Hyperbolicity

- We say the PDE $\underline{q}_{,t} + \underline{\underline{A}}\,\underline{q}_{,x} + \underline{\underline{B}}\,\underline{q}_{,y} = \underline{\underline{C}}\,\underline{q}$ is **hyperbolic** if

$$\widetilde{\underline{\underline{A}}}(\alpha) := \cos(\alpha)\,\underline{\underline{A}} + \sin(\alpha)\,\underline{\underline{B}}$$

  is diagonalizable with only real eigenvalues for all $\alpha \in \mathbb{R}$
- Physically, this means that information in the system travels at a finite speed
- The wave equation and P_N equations are hyperbolic

Outline

Decoupling the System

- After transforming, our P_N formulation is of the form

$$\underline{\widehat{q}}_{,t} + \left(\cos(\omega)\,\underline{\underline{A}} + \sin(\omega)\,\underline{\underline{B}}\right)\underline{\widehat{q}}_{,s} = \underline{\underline{C}}\,\underline{\widehat{q}}$$
$$\underline{\widehat{q}}_{,t} + \underline{\underline{\widetilde{A}}}(\omega)\underline{\widehat{q}}_{,s} = \underline{\underline{C}}\,\underline{\widehat{q}}$$

- Recall that $\underline{\underline{\widetilde{A}}}(\omega)$ is diagonalizable with only real eigenvalues for all $\omega \in \mathbb{R}$
- We can then write

$$\underline{\underline{\widetilde{A}}}(\omega) = \underline{\underline{P}}\,\underline{\underline{\Lambda}}\,\underline{\underline{P}}^{-1}$$

where $\underline{\underline{\Lambda}}$ is a diagonal matrix of the eigenvalues of $\underline{\underline{\widetilde{A}}}(\omega)$

Decoupling the System

- Rewrite system using eigendecomposition of $\underline{\underline{\widetilde{A}}}(\omega)$

$$\underline{\widehat{q}}_{,t} + \underline{\underline{\widetilde{A}}}(\omega)\,\underline{\widehat{q}}_{,s} = \underline{\underline{C}}\,\underline{\widehat{q}}$$

Decoupling the System

- Rewrite system using eigendecomposition of $\underline{\underline{\widetilde{A}}}(\omega)$

$$
\begin{aligned}
&\underline{\widehat{q}}_{,t} + \underline{\underline{\widetilde{A}}}(\omega)\,\underline{\widehat{q}}_{,s} = \underline{\underline{C}}\,\underline{\widehat{q}} \\
\Rightarrow\ &\underline{\widehat{q}}_{,t} + \underline{\underline{P}}\,\underline{\underline{\Lambda}}\,\underline{\underline{P}}^{-1}\,\underline{\widehat{q}}_{,s} = \underline{\underline{C}}\,\underline{\widehat{q}}
\end{aligned}
$$

Decoupling the System

- Rewrite system using eigendecomposition of $\underline{\underline{\widetilde{A}}}(\omega)$

$$
\begin{aligned}
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\Rightarrow\ &\underline{\widehat{q}}_{,t} + \underline{\underline{P}}\,\underline{\underline{\Lambda}}\,\underline{\underline{P}}^{-1}\,\underline{\widehat{q}}_{,s} = \underline{\underline{C}}\,\underline{\widehat{q}} \\
\Rightarrow\ &\underline{\underline{P}}^{-1}\,\underline{\widehat{q}}_{,t} + \underline{\underline{\Lambda}}\,\underline{\underline{P}}^{-1}\,\underline{\widehat{q}}_{,s} = \underline{\underline{P}}^{-1}\,\underline{\underline{C}}\,\underline{\widehat{q}}
\end{aligned}
$$

Decoupling the System

- Rewrite system using eigendecomposition of $\underline{\underline{\widetilde{A}}}(\omega)$

$$
\begin{aligned}
& \underline{\widehat{q}}_{,t} + \underline{\underline{\widetilde{A}}}(\omega)\,\underline{\widehat{q}}_{,s} = \underline{\underline{C}}\,\underline{\widehat{q}} \\
\Rightarrow\ & \underline{\widehat{q}}_{,t} + \underline{\underline{P}}\,\underline{\underline{\Lambda}}\,\underline{\underline{P}}^{-1}\,\underline{\widehat{q}}_{,s} = \underline{\underline{C}}\,\underline{\widehat{q}} \\
\Rightarrow\ & \underline{\underline{P}}^{-1}\,\underline{\widehat{q}}_{,t} + \underline{\underline{\Lambda}}\,\underline{\underline{P}}^{-1}\,\underline{\widehat{q}}_{,s} = \underline{\underline{P}}^{-1}\,\underline{\underline{C}}\,\underline{\widehat{q}} \\
\Rightarrow\ & \frac{\partial}{\partial t}\left(\underline{\underline{P}}^{-1}\,\underline{\widehat{q}}\right) + \underline{\underline{\Lambda}}\,\frac{\partial}{\partial s}\left(\underline{\underline{P}}^{-1}\,\underline{\widehat{q}}\right) = \underline{\underline{P}}^{-1}\,\underline{\underline{C}}\,\underline{\widehat{q}}
\end{aligned}
$$

Decoupling the System

- Rewrite system using eigendecomposition of $\underline{\underline{\widetilde{A}}}(\omega)$

$$
\begin{aligned}
\underline{\widehat{q}}_{,t} + \underline{\underline{\widetilde{A}}}(\omega)\,\underline{\widehat{q}}_{,s} &= \underline{\underline{C}}\,\underline{\widehat{q}} \\
\Rightarrow \underline{\widehat{q}}_{,t} + \underline{\underline{P}}\,\underline{\underline{\Lambda}}\,\underline{\underline{P}}^{-1}\underline{\widehat{q}}_{,s} &= \underline{\underline{C}}\,\underline{\widehat{q}} \\
\Rightarrow \underline{\underline{P}}^{-1}\underline{\widehat{q}}_{,t} + \underline{\underline{\Lambda}}\,\underline{\underline{P}}^{-1}\underline{\widehat{q}}_{,s} &= \underline{\underline{P}}^{-1}\underline{\underline{C}}\,\underline{\widehat{q}} \\
\Rightarrow \frac{\partial}{\partial t}\left(\underline{\underline{P}}^{-1}\underline{\widehat{q}}\right) + \underline{\underline{\Lambda}}\,\frac{\partial}{\partial s}\left(\underline{\underline{P}}^{-1}\underline{\widehat{q}}\right) &= \underline{\underline{P}}^{-1}\underline{\underline{C}}\,\underline{\widehat{q}}
\end{aligned}
$$

- Define the vector of **characteristic variables**

$$
\underline{w} := \underline{\underline{P}}^{-1}\underline{\widehat{q}}, \qquad \underline{\widehat{q}} = \underline{\underline{P}}\,\underline{w}
$$

to obtain

$$
\underline{w}_{,t} + \underline{\underline{\Lambda}}\,\underline{w}_{,s} = \underline{\underline{P}}^{-1}\underline{\underline{C}}\,\underline{\underline{P}}\,\underline{w}
$$

Decoupling the System

- Taking $\underline{\underline{F}} := \underline{\underline{P}}^{-1}\,\underline{\underline{C}}\,\underline{\underline{P}}$, we now have in characteristic variables

$$\underline{w}_{,t} + \underline{\underline{\Lambda}}\,\underline{w}_{,s} = \underline{\underline{F}}\,\underline{w}$$
$$\underline{w}(t = 0, s, \omega) = \underline{w_0}(s, \omega)$$

- The above system is fully decoupled if $\underline{\underline{C}} \equiv \underline{\underline{0}}$
  - Otherwise, the system is only partially decoupled

Space Discretization

- Recall ω is fixed for each 1D family of equations
- Discretize w_p in space at the Chebyshev points along ω

$$w_{p,t} + \lambda_p w_{p,s} = \sum_{q=1}^{M} F_{pq} w_q \quad \rightarrow \quad \underline{W_p}_{,t} + \lambda_p \underline{W_p}_{,s} = \sum_{q=1}^{M} F_{pq}\, \underline{W_q}$$

Space Discretization

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- Discretize w_p in space at the Chebyshev points along ω

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- Discretize spatial derivatives and enforce boundary conditions by sign of λ_p

$$\underline{W_p}_{,t} + \lambda_p \underline{W_p}_{,s} = \sum_{q=1}^{M} F_{pq} \underline{W_q} \quad \rightarrow \quad \underline{W_p}_{,t} + \lambda_p \underline{\underline{D_p}} \, \underline{W_p} = \sum_{q=1}^{M} F_{pq} \underline{W_q}$$

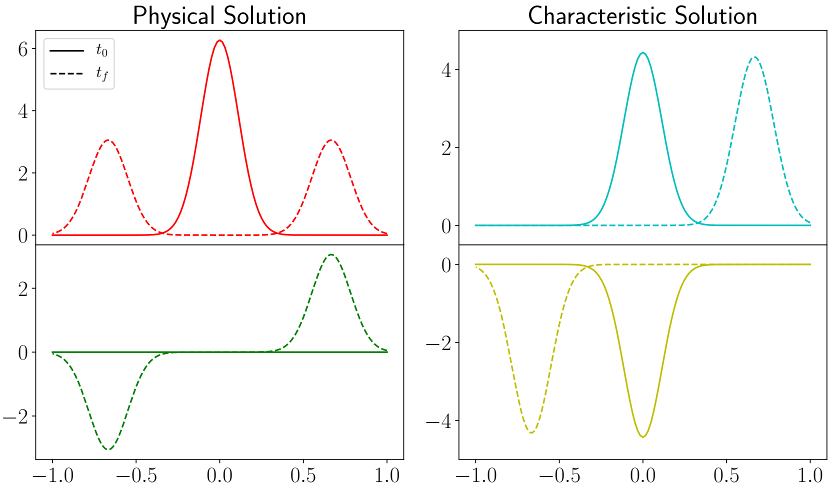
Time-stepping Methods

- We now have M semi-discrete equations

$$\underline{W_p}_{,t} + \lambda_p \underline{\underline{D_p}}\, \underline{W_p} = \sum_{q=1}^{M} F_{pq}\, \underline{W_q}$$

- More freedom in selecting a time-stepping method
- We use a **third-order** method

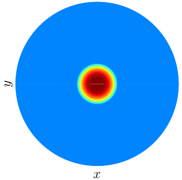
Advection Example



Outline

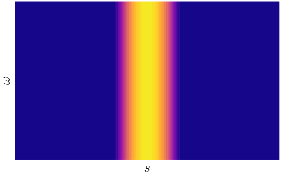
Full Example: Wave Equation

f_0



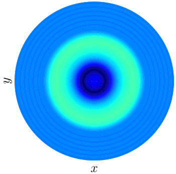
$\mathcal{R} \rightarrow$

$\widehat{f_0}$



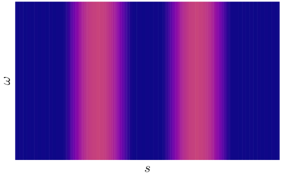
$\downarrow$ solve 1D equations

f_T



$\leftarrow \mathcal{R}^{-1}$

$\widehat{f_T}$



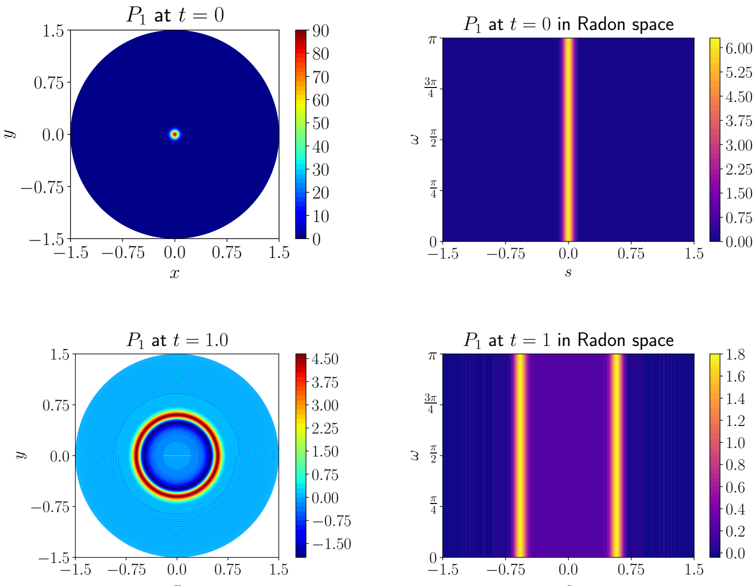
Standard P_N Reference Solution

- In following the standard for testing the P_N equations, we use the following approximate delta function as initial condition with $\alpha = 0.003$

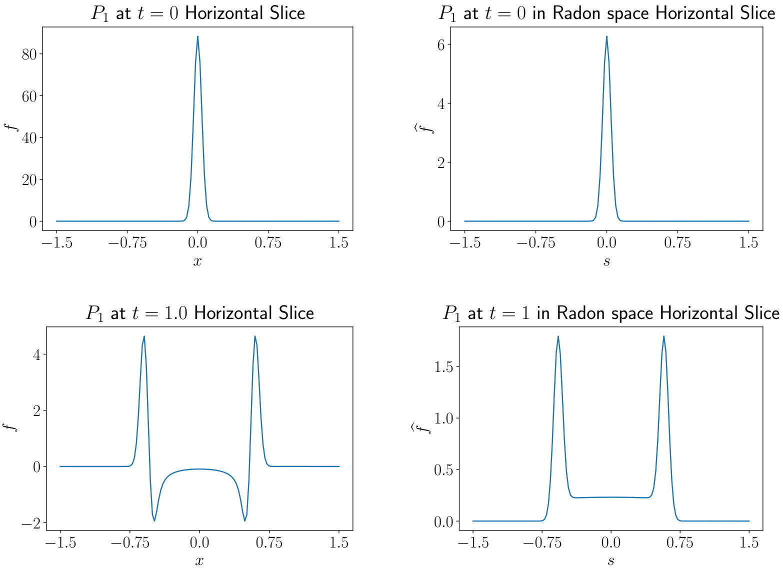
$$q_0(x,y) = \frac{1}{4\pi\alpha^2} e^{\frac{-(x^2+y^2)}{4\alpha^2}}$$

- An exact solution to the radially symmetric case can be computed numerically with high accuracy
- Can translate the initial condition and reference solution to test non-symmetric problems

Full Example: P_1 Approximation



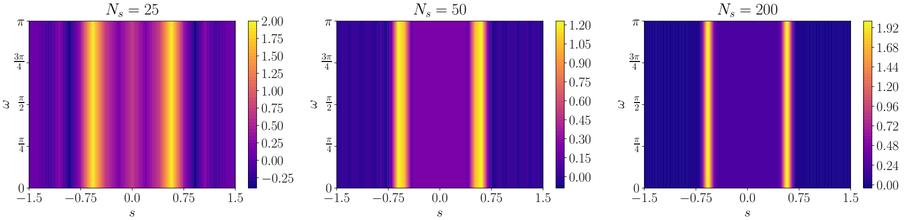
Full Example: P_1 Approximation



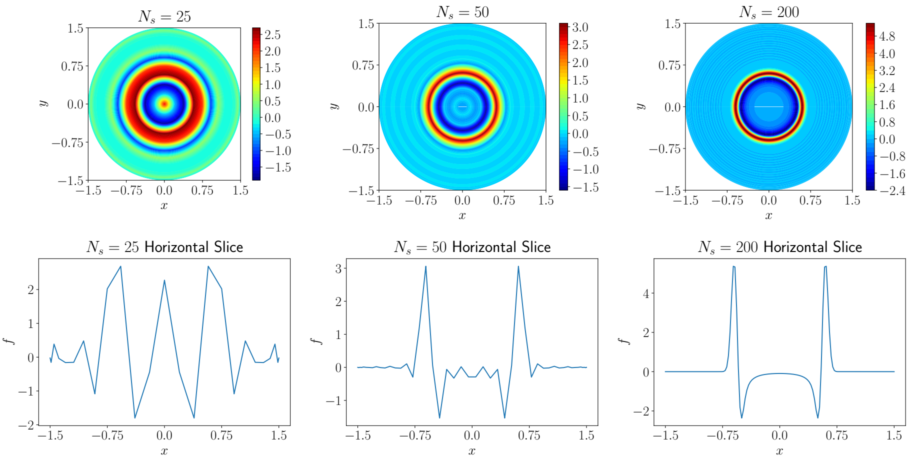
Outline

Radial Symmetry: Convergence Study on N_s

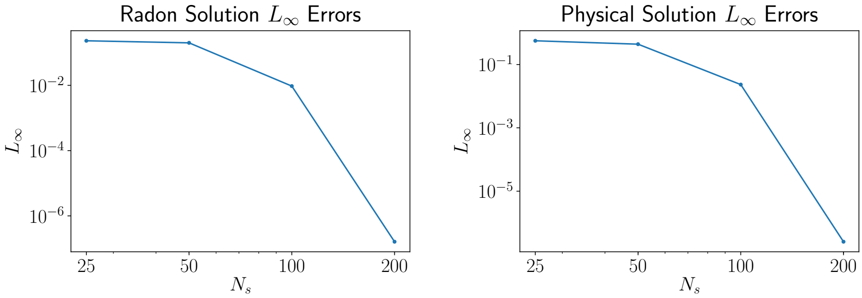
- Radial symmetry means number of angles N_ω irrelevant
- Primary error is in resolving initial condition



Radial Symmetry: Convergence Study on N_s



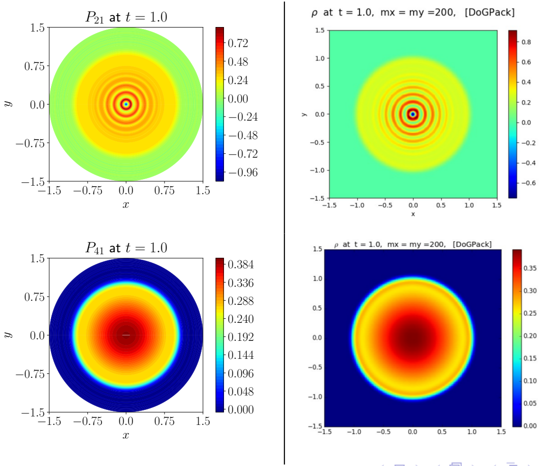
Radial Symmetry: Convergence Study on N_s



- Component methods have varying convergence rates
- Solution in Radon space is spectrally accurate in N_s
- If radially symmetric, spectrally accurate in physical space

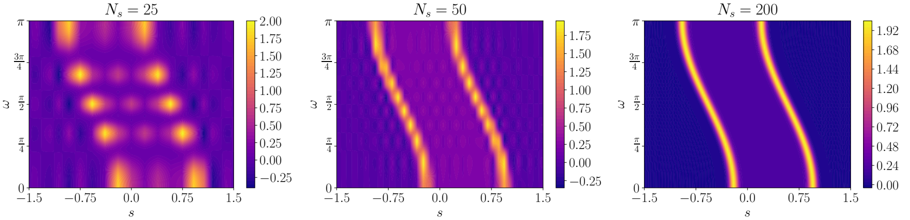
Radial Symmetry: Other P_N Solutions

- Justified in comparison to existing literature [Shin 2019] due to uniformity of methods across P_N

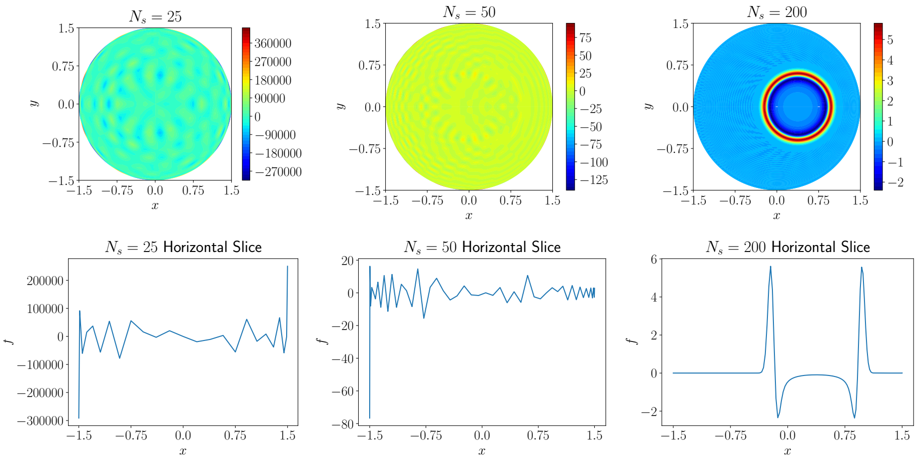


Convergence Study on N_s $(N_\omega = 200)$

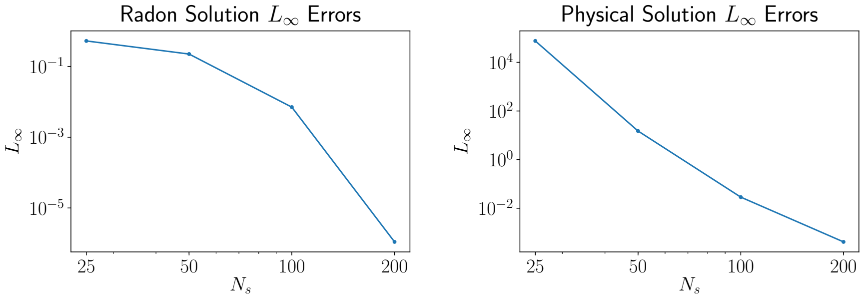
- Translating the initial condition disrupts symmetry
- Transport step in method independent of symmetry



Convergence Study on N_s $\quad (N_\omega = 200)$



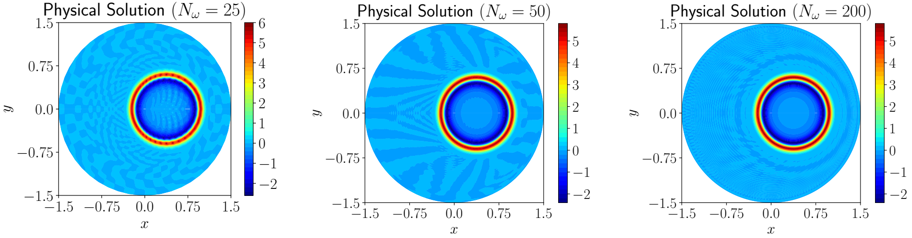
Convergence Study on N_s $\quad (N_\omega = 200)$



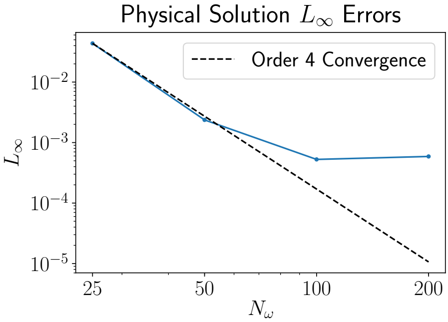
- N_s indirectly affects convergence of physical solution

Convergence Study on N_ω $\quad (N_s = 200)$

- Need to consider effect of angular discretization seperately
- Solving each 1D system is independent of choice of N_ω



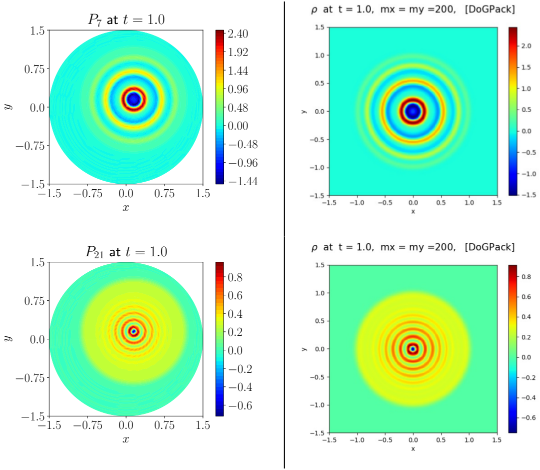
Convergence Study on N_ω ($N_s = 200$)



- Transport step identical across tests
- Interpolation across anges is 4th order

Nonsymmetric: Other P_N examples

- Primary source of instability is singularity at the origin



Summary

- Use the forward Radon transform on a linear system of hyperbolic PDEs
- Solve each system in the collection of 1D transport equations
- Use the inverse Radon transform to bring solution back to physical space

Acknowledgments

- NSF Grant DMS-1457443
- Iowa State University
- Dr. Rossmanith
- Christine Wiersma

Thank You!

- Future Work
  - Adding spatially-dependent collision terms to the P_N equations
  - Implementing more sophisticated/higher order timestepping schemes
  - Improving efficiency through a parallelization of transport computations

Thank You!

- Future Work
  - Adding spatially-dependent collision terms to the P_N equations
  - Implementing more sophisticated/higher order timestepping schemes
  - Improving efficiency through a parallelization of transport computations

- Questions?

References

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(7) Trefethen, L. (2000). *Spectral Methods in MATLAB*, Society for Industrial and Applied Mathematics, 2000.

Clenshaw-Curtis Quadrature

- Let τ be half the length of the chord perpendicular to (s, ω)

$$\int_{-\tau}^{\tau} f(s_i \cos(\omega_j) - z \sin(\omega_j),\ s_i \sin(\omega_j) + z \cos(\omega_j))\, dz$$
$$= \tau \int_{-1}^{1} f(s_i \cos(\omega_j) - \tau t \sin(\omega_j),\ s_i \sin(\omega_j) + \tau t \cos(\omega_j))\, dt$$
$$\approx \tau \sum_{k=0}^{N_q} w_k f(s_i \cos(\omega_j) - \tau t_k \sin(\omega_j),\ s_i \sin(\omega_j) + \tau t_k \cos(\omega_j))$$

where t_k are Chebyshev points of the second kind in $[-1, 1]$

Barycentric Interpolation

- Stabilized barycentric interpolation formula:

$$H_n(x) = \frac{\sum_{j=0}^{n} \alpha_j \frac{f_j}{x-x_j}}{\sum_{j=0}^{n} \alpha_j \frac{1}{x-x_j}}$$

$$\alpha_0 = \frac{1}{2}, \quad \alpha_{1:n-1} = (-1)^{1:n-1}, \quad \alpha_n = \frac{1}{2}(-1)^n$$

4 Point Polynomial Interpolation

- Let θ_0 be the midpoint of the arc containing h_i, $i = 1, \ldots, 4$

$$
\begin{aligned}
p(\theta) = & -\frac{1}{16}(h_1 - 9h_2 - 9h_3 + h_4) \\
& + \frac{1}{24d\omega}(h_1 - 27h_2 + 27h_3 - h_4)(\theta - \theta_0) \\
& + \frac{1}{4d\omega^2}(h_1 - h_2 - h_3 + h_4)(\theta - \theta_0)^2 \\
& - \frac{1}{6d\omega^3}(h_1 - 3h_2 + 3h_3 - h_4)(\theta - \theta_0)^3
\end{aligned}
$$

Spectral Differentiation Matrix

$$
\underline{\underline{D_{N+1}}} =
\left[\begin{array}{c|c|c}
\dfrac{2N^2+1}{6} & 2\dfrac{(-1)^j}{1-x_j} & \dfrac{1}{2}(-1)^N \\
\hline
-\dfrac{1}{2}\dfrac{(-1)^i}{1-x_i} &
\begin{array}{ccc}
 & & \dfrac{(-1)^{i+j}}{x_i - x_j} \\
 & \dfrac{-x_j}{2(1-x_j^2)} & \\
\dfrac{(-1)^{i+j}}{x_i - x_j} & &
\end{array}
& \dfrac{1}{2}\dfrac{(-1)^{N+i}}{1+x_i} \\
\hline
\dfrac{1}{2}(-1)^N & -2\dfrac{(-1)^{N+j}}{1+x_j} & -\dfrac{2N^2+1}{6}
\end{array}\right]
$$

Boundary Conditions

- Sign of λ_p determines direction of inflow.
- Negative $\implies$ left

$$\underline{\underline{D}}_{\text{left}} = \left[\begin{array}{c|cc} 0 & \ldots & 0 \\ \hline \vdots & \ddots & \\ 0 & & D \end{array}\right] \qquad \underline{\underline{D}}_{\text{right}} = \left[\begin{array}{cc|c} D & & 0 \\ & \ddots & \vdots \\ \hline 0 & \ldots & 0 \end{array}\right]$$

IMEX Method Butcher Tableu

$$
\begin{array}{c|c}
\tilde{c} & \tilde{A} \\
\hline
 & \tilde{w}
\end{array}
=
\begin{array}{c|cccc}
0 & 0 & & & \\
0 & 0 & 0 & & \\
1 & 0 & 1 & 0 & \\
\frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\
\hline
 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3}
\end{array}
\qquad
\begin{array}{c|c}
c & A \\
\hline
 & w
\end{array}
=
\begin{array}{c|cccc}
\alpha & \alpha & & & \\
0 & -\alpha & \alpha & & \\
1 & 0 & 1-\alpha & \alpha & \\
\frac{1}{2} & \beta & \eta & \frac{1}{2}-\beta-\eta & \alpha \\
\hline
 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3}
\end{array}
$$

3rd-Order IMEX

- Each stage must be computed for $p = 1, \ldots, M$

$$w_p^{(1)} = w_p^k - \Delta t \, a_{11} \, \lambda_p \, w_{p,s}^{(1)},$$

for $i = 2, \ldots, \nu$

$$w_p^{(i)} = w_p^k + \Delta t \sum_{\ell=1}^{i-1} \tilde{a}_{i\ell} \sum_{q=1}^{M} F_{pq} \, w_q^{(\ell)} - \Delta t \sum_{\ell=1}^{i} a_{i\ell} \, \lambda_p \, w_{p,s}^{(\ell)}$$

$$w_p^{k+1} = w_p^k + \Delta t \sum_{i=1}^{\nu} \tilde{w}_i \sum_{q=1}^{M} F_{pq} \, w_q^{(i)} - \Delta t \sum_{i=1}^{\nu} w_i \, \lambda_p \, w_{p,s}^{(i)}$$

Reference Solution PDE

- The following coupled PDE in physical space governs the P_1 approximation with radially symmetric initial conditions

$$\begin{bmatrix} p \\ u_r \end{bmatrix}_{,t} + \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} p \\ u_r \end{bmatrix}_{,r} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \frac{1}{r} u_r \\ -\sigma u_r \end{bmatrix}$$

- Can rewrite in characteristic variables and solve exactly
- Run with high resolution ($N_s = N_\omega = 400$) to use in convergence studies