

5 Poisson Processes

5.1 Exponential Distribution

Definition. A random variable T is said to have an exponential distribution with rate $\lambda > 0$, if

$$P(T \leq t) = 1 - e^{-\lambda t} \quad \text{for all } t \geq 0.$$

Let us denote it as $T \sim \text{Exp}(\lambda)$. Its density function will be

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{if } t \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

It is easy to see that $E(T) = 1/\lambda$ and $\text{Var}(T) = 1/\lambda^2$.

Example. Let $S \sim \text{Exp}(1)$ and λ be a positive number. Then $S/\lambda \sim \text{Exp}(\lambda)$.

Example. Let T_1 and T_2 be independent exponential random variables with rate λ_1 and λ_2 respectively. Find $E(T_1 | T_1 < T_2)$.

Lack of Memory. Let X be $\text{Exp}(\lambda)$. Then

$$P(T > s + t | T > s) = P(X > t).$$

Example 1. The life time of a TV is exponentially distributed with mean 8 years. If you buy a 5 year-old TV which is in good condition. What is the probability that the TV will work for another 3 years?

Example 2. Consider a bank with two tellers. Three people, Joe, Steve, and Stacy enter the bank at the same time and in that order. Joe and Steve go directly into service while Stacy waits for the first available teller. Suppose that the service times for each customer are exponentially distributed with mean 4 minutes. What is the probability that Stacy is the last person of the three people to leave?

Exponential Races. Let $S \sim \text{Exp}(\lambda)$ and $T \sim \text{Exp}(\mu)$ be independent. Then,

$$P(\min(S, T) > t) = e^{-(\lambda + \mu)t}$$

and

$$P(S < T) = \frac{\lambda}{\lambda + \mu}.$$

The above can be easily generalized: Let T_1, T_2, \dots, T_n be independent exponential random variables with rate $\lambda_1, \lambda_2, \dots, \lambda_n$ respectively. Then,

$$P(\min(T_1, T_2, \dots, T_n) > t) = e^{-(\lambda_1 + \lambda_2 + \dots + \lambda_n)t}$$

and

$$P(T_j = \min(T_1, T_2, \dots, T_n)) = \frac{\lambda_j}{\lambda_1 + \lambda_2 + \dots + \lambda_n}$$

Example. Jen and Betty enter a beauty parlor at the same time, Jen to get a manicure and Betty to get a haircut. Suppose the time for a manicure and a haircut exponentially distributed with mean 20 and 30 minutes respectively.

- (a) What is the probability that Betty gets done first?
- (c) What is the expected time for Jen gets done if it is known that Betty gets done first?

$$E(T_1 | T_1 < T_2)$$

- (d) What is the expected amount of time until both get done?

$$E(T) = E(T_1 | T_1 < T_2)P(T_1 < T_2) + E(T_1 | T_1 > T_2)P(T_1 > T_2)$$

Theorem. Let t_1, t_2, \dots, t_n be independent exponential random variables with rate λ . Then the sum $T_n = t_1 + t_2 + \dots + t_n$ has a gamma distribution, $\Gamma(n, \lambda)$. Its density function given by

$$f(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \quad \text{for } t \geq 0.$$

5.2 Poisson Processes

One of the most important types of counting processes is the Poisson process,

Let t_1, t_2, \dots be independent $\text{Exp}(\lambda)$ random variables and let $T_n = t_1 + t_2 + \dots + t_n$ for $n \geq 1$. Define $N(s) = \max\{n : T_n \leq s\}$. Then the density function of $N(s)$ is given by

$$P(N(s) = n) = e^{-\lambda s} \frac{(\lambda s)^n}{n!} \quad \text{for } s \geq 0.$$

Proof. It is clear that $N(s) = n$ if and only if $T_n \leq s < T_{n+1}$.

$$\begin{aligned} P(N(s) = n) &= \int_0^s P(T_n \leq s < T_{n+1} | T_n = t) P(T_n = t) dt = \\ &= \int_0^s P(T_{n+1} > s | T_n = t) P(T_n = t) dt = \int_0^s P(t_{n+1} > s - t) P(T_n = t) dt \end{aligned}$$

We may think of the t_n as times between arrivals of customers at a bank. Then $N(s)$ is total number customers arrive at the bank during s time period.

Example. Suppose that people immigrate into a territory at a Poisson rate λ per day.

- (a) What is the expected time until the tenth immigrant arrives?
- (b) What is the probability that the elapsed time between the tenth and the eleventh arrival exceeds two day?

Lemma. Let r , s , and t be positive numbers. Then, $N(s+t) - N(s)$ is $\text{Poisson}(\lambda)$ and independent of $N(r)$ whenever $r \leq s$

Lemma. $N(t)$ has independent increments, that is, $N(t_1) - N(t_0)$, $N(t_2) - N(t_1)$, \dots , $N(t_n) - N(t_{n-1})$ are independent if $t_0 < t_2 < \dots < t_n$.

Proof. It follows from the lack of memory property of exponential distribution.

Definition 1. A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ , $\lambda > 0$, if

- (i) $N(0) = 0$.
- (ii) The process has independent increments.
- (iii) The number of events in any interval of length t is Poisson distributed with mean λt . That is, for all $s, t \geq 0$,

$$P(N(t+s) - N(s) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n = 1, 2, \dots$$

Properties

- (i) Let $N(t)$ be a Poisson process with rate λ . Then $\text{Var}(N(t)) = \lambda$ and $E(N(t)) = \lambda$
- (ii) Let $N_i(t)$, $i = 1, 2, \dots, n$ be independent Poisson processes with rates λ_i $i = 1, 2, \dots, n$ respectively. Then $N_1(t) + N_2(t) + \dots + N_n(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2 + \dots + \lambda_n$.
- (iii) For $s < t$, $P(X_1 < s | N(t) = 1) = s/t$

Example 1. Suppose $N(t)$ is a Poisson process with rate 2. Compute the conditional probabilities

$$(a) P(N(3) = 4 | N(1) = 1) \quad (b) P(N(1) = 1 | N(3) = 4)$$

Solution. $P(N(3) = 4 | N(1) = 1) = P(N(3) - N(1) = 4 - N(1) | N(1) = 1) = P(N(3) - N(1) = 3, N(1) = 1) / P(N(1) = 1)$

Example 2. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ . Calculate $E(N(t) \cdot N(t+s))$.

Example 3. An insurance company pays out claims at times of a Poisson process with rate 4 per week. Writing K as shorthand for "thousands of dollars", suppose that the mean payment is $10K$ and the standard deviation is $6K$. Find the mean and standard deviation of the total payments for 4 weeks.

Poisson Approximation. Let $X \sim B(n, p)$. If n is very large and $np \rightarrow \lambda$ then

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \approx \frac{e^{-\lambda} (\lambda)^x}{x!}.$$

In other words, $B(n, p) \sim \text{Poisson}(\lambda)$, where $np \rightarrow \lambda$.

Theorem. For each n let $X_{n,m}$ be independent random variables with $P(X_{n,m} = 1) = p_{n,m}$ and $P(X_{n,m} = 0) = 1 - p_{n,m}$. Let

$$S_{n,n} = X_{n,1} + X_{n,2} + \cdots + X_{n,n}, \quad \lambda_n = E(S_n) = p_{n,1} + p_{n,2} + \cdots + p_{n,n}$$

and $Z_n = \text{Poisson}(\lambda_n)$. Then for any x

$$|P(S_n \leq x) - P(Z_n \leq x)| \leq \sum_{m=1}^n p_{n,m}^2.$$

In some applications it is important to have an arrival rate that depends on time. It leads to **nonhomogeneous Poisson processes**.

We say that $\{N(s), s \geq 0\}$ is said to be a Poisson process having rate $\lambda(r)$, $\lambda > 0$, if

- (i) $N(0) = 0$,
- (ii) The process has independent increments, and
- (iii) $N(t + s) - N(s)$ is Poisson with mean $\int_s^{s+t} \lambda(r) dr$.

Example. Suppose that the customer arrival rate at a store starts at 0 at 10:00AM, increases to 4 at 12:00PM, to 6 by 2:00PM, drops to 2 by 4:00PM and decreases to 0 by the time the store closes at 6:00. Assume that the arrival rates are linear between these time points.

- a. What is the distribution of the number of arrivals in a day?
- b. What is the probability no one arrives before noon?

If the store is closed at 5:30PM instead 6:00PM,

- c. What is the expected number of customers that are lost?
- d. What is the probability that at least one customer arrives to find the store closed?

5.3 Decomposition of Poisson Processes

Example. Suppose that the number of people who visit a bank each day is Poisson random variable with mean λ . Suppose further that each person who visits is, independently, male with probability p or female with probability $(1 - p)$. Find the joint probability that exactly n women, and m men visit the bank.

Solution. Let N denote the total number of cars, let N_0 and N_1 be the number women and men respectively.

$$\begin{aligned} P(N_2 = n, N_4 = m) &= \sum_{j=0}^{\infty} P(N_2 = n, N_2 = m | N = j) P(N = j) = \\ &= P(N_2 = n, N_2 = m | N = n + m) P(N = n + m) = \\ &= \binom{n+m}{n} p^n (1-p)^m e^{-\lambda} \frac{\lambda^{n+m}}{(n+m)!}. \end{aligned}$$

Suppose that each event of a Poisson process with rate λ is classified as being either a type-I or type-II event, and suppose that the probability of an event being classified as type-I depends on the time at which it occurs. Suppose that if an event occurs at time s , then, independently of all else, it is classified as being type a type-I with probability $P(s)$ and a type-II event with probability $1 - P(s)$.

Proposition. If $N_1(t)$ and $N_2(t)$ represent the number of type-I and type-II events that occur by time t , then $N_1(t)$ and $N_2(t)$ are independent Poisson random variables having respective means λtp and $\lambda t(1 - p)$, where

$$p = \frac{1}{t} \int_0^t P(s) ds.$$

Example 4. Ellen catches fish at times of a Poisson process with rate 2 per hour. 40% of fish are salmon, while 60% of the fish are trout. What is the probability she will catch exactly 1 salmon and 2 trout if she fishes for 2 hours 30 minutes?

5.4 Interarrival and Waiting Time Distribution

Sequence Interarrival Times

Consider a Poisson process, and let X_1 denote the time of the first event. Further, for $n \geq 1$, let X_n denote the time between the $(n - 1)$ st and the n th event. The sequence $\{X_n, n \geq 1\}$ is called the *sequence of interarrival times*.

We shall now determine the distribution of the X_n . Let us note that the event $\{X_1 > t\}$ takes place if and only if no events of the Poisson process occur in the interval $[0, t]$. That is,

$$P(X_1 > t) = P(N(t) = 0) = e^{-\lambda t}.$$

Since X_1 and X_2 are independent,

$$\begin{aligned} P(X_2 > t) &= P(X_2 > t | X_1 = s) = P(\text{no event in } (s, s + t] | X_1 = s) = \\ &= P(\text{no event in } (0, t] |) = e^{-\lambda t} \end{aligned}$$

EXERCISES.

- A submarine has three navigational devices but can remain at sea if at least two are working. Suppose that the failure times are exponential with means 1 year, 1.5 years, and 3 years. What is the average length of time the boat can remain at sea?

$$\begin{aligned}
 E(T) &= \sum_{1 \leq i, j; k \leq 3} E(T | T_i < T_j < T_k) P(T_i < T_j < T_k) = \\
 &= \sum_{1 \leq i, j; k \leq 3} E(T_j | T_i < T_j < T_k) P(T_i < T_j < T_k) = \\
 &= \sum_{1 \leq i, j; k \leq 3} \int_0^\infty P(T_j = t, T_i < t < T_k) dt
 \end{aligned}$$

- Three people are fishing and each catches fish at rate 2 per hour. How long do will it take until everyone has caught at least one fish?

$$\begin{aligned}
 E(T) &= 6E(T | T_1 < T_2 < T_3) P(T_1 < T_2 < T_3) = \\
 &= 6E(T_3 | T_1 < T_2 < T_3) P(T_1 < T_2 < T_3) = \\
 &= 6 \int_0^\infty P(T_3 = t, T_1 < T_2 < t) dt
 \end{aligned}$$

- Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ . Calculate $E[N(t)N(t+s)]$.
- Consider a bank with two tellers. Three people, Joe, Steve, and Stacy enter the bank at the same time and in that order. Joe and Steve go directly into service while Stacy waits for the first available teller. Suppose that the service times for each customer are exponentially distributed with mean 4 minutes. What is the probability that Stacy is the last person of the three people to leave?
- For a Poisson process show, for $s < t$, that

$$P(N(s) = k | N(t) = n) = \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}, \quad k = 0, 1, \dots, n$$

- Signals are transmitted according to a Poisson process with rate λ . Each signal is successfully transmitted with probability p and lost with probability $1 - p$. The fates of different signals are independent. For $t \geq 0$ let $N_1(t)$ be the number of signals successfully transmitted and let $N_2(t)$ be the number that are lost up to time t . Find the distribution of $(N_1(t), N_2(t))$.