ON 3D U(1) LATTICE GAUGE THEORY AND RANDOM HEIGHT FUNCTIONS

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ABSTRACT. We discuss the duality relationship between three-dimensional $U(1)$ lattice gauge theory and random height functions. Past results, possible directions towards a new proof of the area law, and a new qualitative result regarding level-set percolations of random height functions are also mentioned. [Under construction as of December 2024]

1. INTRODUCTION

In [GM82], it was shown that the three-dimensional $U(1)$ lattice gauge theory with Villain action follows the area law and exhibits exponential decay of correlations at all temperatures. With small modifications, this result holds for the Wilson action as well. Their idea was to start by making a duality transformation, so that the question becomes one about the integer-valued Gaussian free field, and then to transform it to a question about the Gaussian free field by inserting the integrality condition with δ functions and applying the Poisson summation formula. A similar representation was used by [FS82] to show that there is a phase transition in four dimensions.

Recently, [Lam23b] proved that the exponential decay-polynomial decay phase transition for the XY model coincides with the localization-delocalization transition in a dual (two-dimensional) height functions model. A simple relationship between the exponential decay rates in the two models was provided. Duality methods were also used directly in [vEL23b] to give an elementary proof that a transition occurs in the XY model, although not that the dual transition coincides.

We explain how to begin to extend these new ideas in the direction of $U(1)$ lattice gauge theory. To do this, we will consider integer-valued height functions models of the form

$$\mu_\Lambda(h) = \frac{1}{Z_\Lambda} \prod_{xy \in E_\Lambda} e^{-V(h_y - h_x)},$$

where here Λ is some finite subset of $\mathbb{Z}^3$, E_Λ are edges, and V is some super-Gaussian potential function. In order for this measure to have finite mass, we also specify 0 boundary conditions. It would be useful to have

$$\lim_{\Lambda \uparrow \mathbb{Z}^3} \mu_\Lambda(h_0 h_x) \leqslant Ce^{-c|x|}.$$

This is known for the integer-valued Gaussian free field due to [GM82]. In notation we will define later, [Lam23a] proves that it is sufficient that

$$\mu(0 \overset{\mathcal{L}_0^\diamond}{\longleftrightarrow} x) \leqslant Ce^{c|x|},$$

which is statement about the probability that the origin is connected to x by some percolation. We will spend some time discussing possible approaches to this.

1.1. **Organization.** In section 2, we describe the lattice gauge theory results we are after and derive the duality relationship with height functions. In section 3, we set up a framework to study these height functions in a similar way to [Lam23a], and we prove a result about the relevant percolations. We then discuss possible approaches to the question posed in the introduction and a possible extension in section 4.

2. Lattice Gauge Theory

2.1. Exterior Difference Calculus. We follow the expositions of [FS82] and [CCA23], although we use different notation. The technicalities related to orientations of k-cells will not be very important in the present article, but we include them for completeness.

Let $e_1, \dots, e_d$ be the standard basis of $\mathbb{R}^d$. We will usually drop the word 'oriented' in the following definition.

Definition 2.1. *(k-cell). An oriented k-cell in $\mathbb{Z}^d$ is specified (not uniquely) by a signed ordered tuple $s(x, e_{i_1}, \dots e_{i_k})$ where $s = \pm 1$, $x \in \mathbb{Z}^d$, and $i_1, \dots, i_k$ are distinct. It consists of the convex hull of the set of vertices $\{x + a_1 e_{i_1} + \dots + a_k e_{i_k} : a_1 \dots a_k \in \{0,1\}\}$ and has orientation equal to $s \cdot \operatorname{sgn}(\sigma)$, where σ is the permutation such that $i_{\sigma(1)}, \dots, i_{\sigma(k)}$ is increasing.*

Definition 2.2. *(Containment of cells). A k-cell $c_k = s(x, e_{i_1}, \dots e_{i_k})$ is said to contain the $(k-1)$-cell c_{k-1}, denoted $c_{k-1} \subset c_k$, if $c_{k-1} = (-1)^j s(x, e_{i_1}, \dots, e_{i_{j-1}}, e_{i_{j+1}}, \dots e_{i_k})$ for some $j = 1, \dots, k$.*

Definition 2.3. *(Dual cells). The dual of the k-cell c_k is the unique $(d-k)$-cell c_k^* on the dual lattice $\mathbb{Z}^d + (\frac{1}{2}, \dots, \frac{1}{2})$ which intersects c_k and has the same orientation.*

Let $R = \mathbb{R}$ or $\mathbb{Z}$. We say that a k-form is a function

$$\alpha : k\text{-cells} \to R$$

with finite support such that flipping the orientation changes the sign: $\alpha(-c_k) = -\alpha(c_k)$. We define a function from k-forms to $(k+1)$-forms called the coboundary operator as follows:

$$\delta\alpha(c_{k+1}) = \sum_{c_k \subset c_{k+1}} \alpha(c_k).$$

There is a natural inner product on the space of k-forms for any k

$$(\alpha, \alpha') = \sum_{c_k} \alpha(c_k)\alpha'(c_k),$$

and we let ∂ be the adjoint of δ from $(k+1)$-forms to k-forms:

$$(\delta\alpha, \beta) = (\alpha, \partial\beta).$$

This is called the boundary operator.

Lemma 2.4. *(Poincaré). If α is a k-form such that $\partial\alpha = 0$, then there exists some $(k+1)$-form β such that*

$$\alpha = \partial\beta$$

and $\operatorname{supp}\beta$ is contained in the smallest hypercube containing $\operatorname{supp}\alpha$.

We also define the Hodge star operator which takes a k-form on $\mathbb{Z}^d$ to a $(d-k)$-form on the dual lattice:

$$\star\alpha(c_k^*) = \alpha(c_k),$$

where c_k^* is the dual $(d-k)$-form corresponding to c_k. This satisfies the important property

$$\star\partial\alpha = \delta \star \alpha. \tag{1}$$

2.2. **U(1) Lattice Gauge Theory.** Let $\Lambda \Subset \mathbb{Z}^d$ be a finite subset built from plaquettes (2-cells). For concreteness, one may think of $\Lambda_N = [-N, N]^d \cap \mathbb{Z}^d$. Let E_Λ° and P_Λ be the set of positively oriented edges and plaquettes, respectively, with all vertices in Λ. With inverse coupling strength or inverse temperature $\beta > 0$ and the typical Wilson action $\varphi_\beta(\cdot) = e^{\beta \cos \cdot}$, $U(1)$ lattice gauge theory is defined as the measure on configurations $\theta \in [-\pi, \pi)^{E_\Lambda^\circ}$ given by

$$d\nu_{\Lambda,\beta}(\theta) = \frac{1}{Z_{\Lambda,\beta}} \prod_{p \in P_\Lambda} \varphi_\beta(\delta\theta_p) \prod_{xy \in E_\Lambda^\circ} d\theta_{xy},$$

where $d\theta_{xy}$ are normalized Lebesgue measures on $[-\pi, \pi)$. For technical reasons related to duality, it can be easier to study the same model with Villain action

$$\varphi_\beta(\cdot) = \sum_{n \in \mathbb{Z}} \exp\left(-\frac{\beta}{2}(\cdot + 2\pi n)^2\right).$$

The observables of interest in this theory are Wilson loop observables.

Definition 2.5. *(Wilson loop observable). Take a closed path of edges $\gamma = xy_1, \ldots, xy_m$. The associated Wilson loop observable is*

$$W_\gamma = \prod_{j=1}^m \exp\left(i\theta_{xy_j}\right).$$

That is, it is the product of the associated elements of $U(1)$ along the loop.

We are interested in two problems in three dimensions, which we take from [Cha18].

Theorem 2.6. *(Area Law in $\mathbb{Z}^3$) Restrict to $d = 3$. Consider ν_β some associated infinite-volume Gibbs measure, and let $\gamma_{R,T}$ be some rectangular R by T loop. Regardless of our choice of β, there exist positive constants c and C depending only on β such that*

$$\left|\nu_\beta(W_{\gamma_{R,T}})\right| \leqslant Ce^{-cRT}.$$

Theorem 2.7. *(Mass gap in $\mathbb{Z}^3$). Restrict to $d = 3$. Consider ν_β some associated infinite-volume Gibbs measure, with respect to which we define a covariance cov_{ν_β}. Regardless of our choice of β, there exists a positive constant m depending only on β such that*

$$-\lim_{|x| \to \infty} \frac{\log \mathrm{cov}_{\nu_\beta}(W_{p_0}, W_{p_x})}{|x|} = m,$$

where p_x denotes some plaquette closest to $x \in \mathbb{R}^d$.

These were proven by [GM82]. This behaviour is indeed special for $d = 3$. In dimension $d = 4$, we instead have a phase transition, where they both hold for small β only, proven in [FS82]. Interestingly, it is not yet proven that this phase transition is sharp in the sense that past this critical β, the quantity in theorem 2.6 decays no faster than exponentially in the perimeter.

2.3. **Getting to height functions.** We now establish the duality relationship which was a starting point for both [GM82] and [FS82]. Letting $\hat{\varphi}$ denote the Fourier transform of φ, we have that for any $\mathbb{R}$-valued 2-form ϵ,

$$\nu_{\Lambda,\beta}(e^{-i(\delta\theta,\epsilon)}) = \frac{1}{Z_{\Lambda,\beta}} \int \prod_{p \in \mathcal{P}_\Lambda} \varphi_\beta(\delta\theta_p) e^{-i\epsilon_p \delta\theta_p} \prod_{xy \in E_\Lambda^\circ} d\theta_{xy}$$

$$= \frac{1}{Z_{\Lambda,\beta}} \int \prod_{p \in \mathcal{P}_\Lambda} \Big(\sum_{n_p \in \mathbb{Z}} \hat{\varphi}_\beta(n_p + \epsilon_p) e^{i n_p \delta\theta_p} \Big) \prod_{xy \in E_\Lambda^\circ} d\theta_{xy}$$

$$= \frac{1}{Z_{\Lambda,\beta}} \sum_{n:\mathcal{P}_\Lambda \to \mathbb{Z}} \prod_{p \in \mathcal{P}_\Lambda} \hat{\varphi}_\beta(n_p + \epsilon_p) \int e^{i\langle \delta\theta, n\rangle} \prod_{xy \in E_\Lambda^\circ} d\theta_{xy}$$

$$= \frac{1}{Z_{\Lambda,\beta}} \sum_{n:\mathcal{P}_\Lambda \to \mathbb{Z}} \prod_{p \in \mathcal{P}_\Lambda} \hat{\varphi}_\beta(n_p + \epsilon_p) \prod_{xy \in E_\Lambda^\circ} \int e^{i\theta_{xy} \partial n_{xy}} d\theta_{xy}$$

$$= \frac{1}{Z_{\Lambda,\beta}} \sum_{n:\partial n = 0} \prod_{p \in \mathcal{P}_\Lambda} \hat{\varphi}_\beta(n_p + \epsilon_p) = \mu_{\Lambda,\beta}\Big(\prod_{p \in \mathcal{P}_\Lambda} \frac{\hat{\varphi}_\beta(n_p + \epsilon_p)}{\hat{\varphi}_\beta(n_p)} \Big).$$

Here $\mu_{\Lambda,\beta}$ is the probability measure (the total mass is 1 by substituting $\epsilon = 0$) on integer-valued 2-forms on $\mathcal{P}_\Lambda$ in $\ker \partial$ such that

$$\mu_{\Lambda,\beta}(n) = \frac{1}{Z_{\Lambda,\beta}} \prod_{p \in \mathcal{P}_\Lambda} \hat{\varphi}_\beta(n_p).$$

The same reasoning going the other way yields

$$\mu_{\Lambda,\beta}(e^{i(n,\epsilon)}) = \nu_{\Lambda,\beta}\Big(\prod_{p \in \mathcal{P}_\Lambda} \frac{\varphi_\beta(d\theta_p + \epsilon_p)}{\varphi_\beta(d\theta_p)} \Big). \tag{2}$$

Now suppose $\Lambda = \Lambda_N$. The case of three dimensions is special because each plaquette belongs to exactly 2 cubes. By lemma 2.4, it follows that such 2-forms on $\mathcal{P}_{\Lambda_N}$ in $\ker \partial$ are in bijection with with 3-forms with support contained in the 3-cells with all vertices in Λ_N through the relationship $n = \partial m$. These in turn are in bijection with scalar functions h on the dual lattice, which by equation (1) satisfy $n = \star\delta h$. This leads us to view $\mu_{\Lambda_N,\beta}$ as a height functions model with 0 boundary condition:

$$\mu_{\Lambda,\beta}(h) = \frac{1}{Z_{\Lambda,\beta}} \prod_{xy \in \mathcal{P}_\Lambda^*} \hat{\varphi}_\beta(\delta h_{xy}).$$

Remark 2.8. *For the Villain action, we have that*

$$\hat{\varphi}_\beta(k) = \frac{1}{\sqrt{2\pi\beta}} e^{-\frac{1}{2\beta}k^2},$$

so our height functions model is the integer-valued Gaussian free field.

Remark 2.9. *For the Wilson action, we have that*

$$\hat{\varphi}_\beta(k) = -\log(I_n(\beta)),$$

where $I_n(\beta)$ is the modified Bessel function of the first kind. The resulting height functions model can also be reached by starting with a Taylor expansion instead of a Fourier transform (as in [Lam23b]). This fits into the framework of the following section when $\beta \leqslant 1$, but other values can be dealt with by modifying the underlying graph (as in [vEL23a]).

Theorem 2.10. *(Covariance duality). Let $U = -\log\varphi_\beta$ be the spin potential and ϵ^1, ϵ^2 be any pair of real-valued 2-forms. Then the following holds for both the Villain and Wilson action:*

$$\mu_{\Lambda,\beta}((n,\epsilon^1)(n,\epsilon^2)) + \nu_{\Lambda,\beta}((U'(\theta),\epsilon^1)(U'(\theta),\epsilon^2)) = \sum_{xy \in E_\Lambda^\circ} \mu(U''(\theta_{xy}))\epsilon^1_{xy}\epsilon^2_{xy}.$$

Proof. The result follows from setting $\epsilon = t_1\epsilon^1 + t_2\epsilon_2$ in equation (2) and taking the mixed partial derivative in t_1, t_2 at 0 in the same way as lemma 2.2 of [vEL23b]. □

Remark 2.11. *As in [vEL23b], the theorem above holds for a larger class of actions. In our case, the actions are infinitely differentiable, so we can derive the more general formula*

$$\mu_{\Lambda,\beta}(i^m(n,\epsilon^1)\dots(n,\epsilon^m)) = \nu_{\Lambda,\beta}\Big(\sum_{f:\mathbb{N}_{\leqslant m}\to\mathcal{P}_\Lambda} \frac{\prod_{p\in\mathcal{P}_\Lambda}\varphi_\beta^{(|f^{-1}(p)|)}(d\theta_p)}{\prod_{p\in\mathcal{P}_\Lambda}\varphi_\beta(d\theta_p)}\epsilon^1_{f(1)}\dots\epsilon^m_{f(m)}\Big).$$

3. Height functions

3.1. **The model.** In order to study the height functions that arise in section 2.3, we study a more general object. The following is an extension to three dimensions of the setup in [Lam23a]. Let $V : \mathbb{Z} \to \mathbb{R}$ be an unbounded convex symmetric potential function with nonincreasing second derivative. Such a function is called super-Gaussian. Let $\Lambda \Subset \mathbb{Z}^3$ be some finite subset, and let E_Λ be the unoriented edges with at least one vertex in Λ. For notational convenience, we will say that $xy \in E_\Lambda$ if $y \in \Lambda$, so that edges are written as though oriented inward. For a function on the edge boundary $\xi : \partial_e\Lambda \to \mathbb{Z}$, we say

$$\mu_{\Lambda,\xi} = \frac{1}{Z_{\Lambda,\xi}}\Big(\prod_{xy\in E_\Lambda} e^{-V(\delta h_{xy})}\Big)\lambda^\Lambda \times \prod_{xy\in E_\Lambda} 1_{\{\rho_{xy}\geqslant 0\}}e^{-\rho_{xy}}d\rho_{xy}, \tag{3}$$

where λ is the counting measure on the integers, $d\rho_{xy}$ is the Lebesgue measure, and δh_{xy} is taken to be $h_y - \xi_{xy}$ whenever $x \notin \Lambda$. In general, we will write h_y and h_x to mean h_y and ξ_{xy} whenever $x \notin \Lambda$.

This is no longer just a height functions model on vertices since we assigned independent standard exponential random variables to the edges. More precisely, we now have a measure on $\mathbb{Z}^\Lambda \times \mathbb{R}^{E_\Lambda}$. This gives us a rigorous way of defining a coupling with several percolation models, which will be described in the next section.

First, we describe some important results.

Lemma 3.1. *(Localization). For any super-Gaussian potential,*

$$\lim_{\Lambda\uparrow\mathbb{Z}^3} \operatorname{var}_{\mu_{\Lambda,0}}(h_0^2) < \infty.$$

Proof. This is the same as [vEL23b], but we include a proof restricted to our setting for completeness. Denote $\Delta = \partial\delta$ the graph Laplacian on $\mathbb{Z}^3$. Let f be the 0-form which takes value 1 at 0 and value 0 everywhere else. Then by covariance duality theorem 2.10 with $\epsilon^1 = \epsilon^2 = \star\delta\Delta^{-1}f$ restricted to an appropriate domain,

$$\begin{aligned}\mu_{\Lambda,0}(h_0^2) = \mu_{\Lambda,0}((h,f)^2) = \mu_{\Lambda,0}((n,\star\delta\Delta^{-1}f)^2) &\leqslant C(\delta\Delta^{-1}f, \delta\Delta^{-1}f)\\ &= C(\Delta^{-1}f, f) = C(\Delta^{-1}f)_0\end{aligned}$$

for some constant $C > 0$ depending only on the spin potential. Since the simple random walk on $\mathbb{Z}^3$ is transient, it follows that this upper bound is finite (see for example [LL10]). □

Remark 3.2. *The existence of the limit of the variances is standard but a priori not trivial. It follows from the fact that h_0^2 is increasing and measureable in $|h|$, along with the monotonicity result in lemma 3.2 of [Lam23a].*

Theorem 3.3. *(Existence of infinite-volume limit). As $\Lambda \uparrow \mathbb{Z}^3$, we have that $\mu_{\Lambda,0}$ converges to some ergodic extremal Gibbs measure μ in the topology of local convergence.*

Proof. The result follows immediately from from lemma 3.1, which is an equivalent notion in the setting of super-Gaussian potentials. See the proof of lemma 9.5 in [Lam23a] for more details. ☐

3.2. **Percolation.**

Definition 3.4. *(Level lines). For $a \in \mathbb{Z}$ we define*

$$\mathcal{L}_a = \{xy^* \in E_\Lambda^* : V\big(|h_x - a| + |h_y - a|\big) \leqslant V(\delta h_{xy}) + \rho_{xy}\}.$$

This is a percolation on dual plaquettes, where each plaquette is included if 'the total energy on the edge is more than the energy it would take to go from h_x to a to h_y.' We also let

$$\mathcal{L}_{\geqslant a} = \bigcup_{b \geqslant a} \mathcal{L}_b.$$

Its dual complement

$$\mathcal{L}_a^\diamond = \{xy \in E_\Lambda : V\big(|h_x - a| + |h_y - a|\big) > V(\delta h_{xy}) + \rho_{xy}\}$$

also plays an important role and will give us information about the two-point correlation function. The following result is suggestive of this fact.

Lemma 3.5. *($\mathcal{L}_a^\diamond$ is a random FK-Ising model). Conditional on $|h - a|$, the law of $(\operatorname{sgn}(h-a), \mathcal{L}_a^\diamond)$ is that of the Edwards-Sokal coupling of a percolation and an Ising model with coupling constants*

$$K_{xy} = \frac{1}{2}\Big(V\big(|h_x - a| + |h_y - a|\big) - V\big(|h_x - a| - |h_y - a|\big)\Big)$$

and boundary conditions $\operatorname{sgn}(\xi_{xy})$.

The following is a qualitative result which seems to be in the right direction. A quantitative version of this might give exponential decay of the two-point covariance function. It is essentialy a combination of arguments from [Lam23a] and [DGR+20].

Theorem 3.6. *(Percolation occurs). For any super-Gaussian potential, $\mathcal{L}_1^\diamond$ and $\mathcal{L}_0$ percolate almost surely with respect to μ.*

Proof. Let $0 \in T \subset \Lambda \Subset \mathbb{Z}^3$. By simple manipulations and lemma 3.5,

$$\mu_{\Lambda,0}(0 \xleftrightarrow{\mathcal{L}_{\geqslant 1}^\diamond} T^c) \geqslant \mu_{\Lambda,0}(0 \xleftrightarrow{\mathcal{L}_{\geqslant 1}^\diamond} \Lambda^c) = \mu_{\Lambda,0}(0 \xleftrightarrow{\mathcal{L}_1^\diamond} \Lambda^c)$$
$$\mu_{\Lambda,0}\big(\mu_{\Lambda,0}(0 \xleftrightarrow{\mathcal{L}_1^\diamond} \Lambda^c | |h-1|)\big) = \mu_{\Lambda,0}(-\operatorname{sgn}(h_0 - 1)).$$

Both sides of the inequality are written in terms of probabilities of local events. Taking $\Lambda \uparrow \mathbb{Z}^3$ and keeping T fixed,

$$\mu(0 \xleftrightarrow{\mathcal{L}_{\geqslant 1}^\diamond} T^c) \geqslant \mu(-\operatorname{sgn}(h_0 - 1)).$$

Since the law of h_0 in μ is symmetric about 0 and log concave (see [She06]), the right-hand side is strictly positive. Letting $T \uparrow \mathbb{Z}^3$, we find that $\mathcal{L}_{\geqslant 1}^\diamond$ percolates with probability. Since μ is extremal (or ergodic), this occurs almost surely. By flip symmetry, $\mathcal{L}_{\leqslant -1}^\diamond$ also percolates almost surely. When both percolations occur, there will be some infinite dual-plaquette interface between nonpositive and nonnegative vertices. These plaquettes are all open in $\mathcal{L}_0$. Explicitly,

$$\{\exists N : \Lambda_N \xleftrightarrow{\mathcal{L}_{\geqslant 1}^\diamond} \infty \wedge \Lambda_N \xleftrightarrow{\mathcal{L}_{\leqslant -1}^\diamond} \infty\}$$
$$\supset \{\exists N : \Lambda_N \xleftrightarrow{\mathcal{L}_0} \infty\},$$

so $\mathcal{L}_0$ percolates almost surely.

□

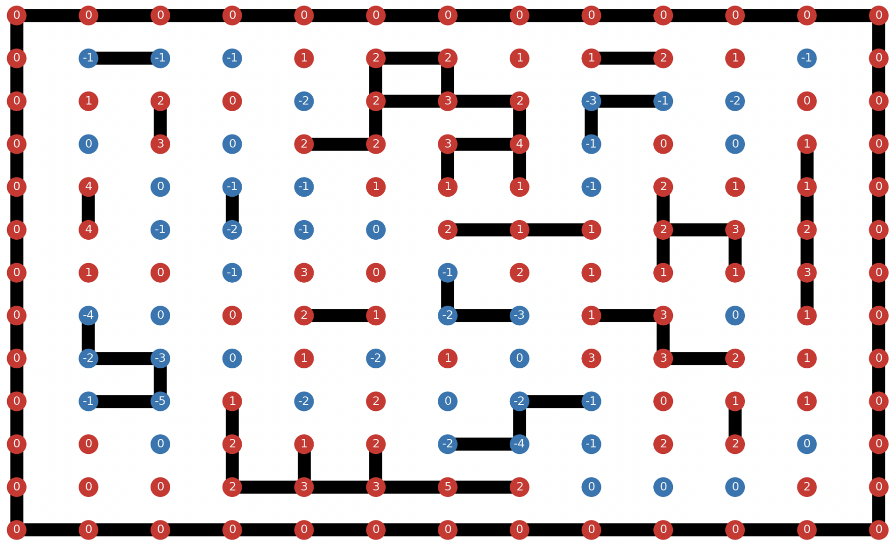

FIGURE 1. Realization of 2D version of integer-valued GFF with $\beta = .05$ on Λ_5 ($\mu_{\Lambda_5,0}$). The $\mathcal{L}_0^\diamond$ percolation is drawn in black.

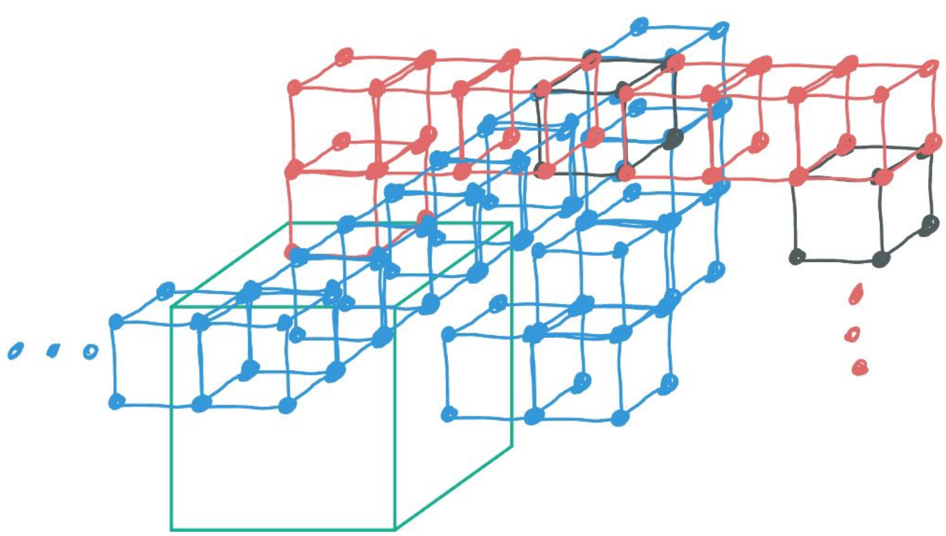

FIGURE 2. Diagram for argument in theorem 3.6. The green cube represents Λ_N, the red cubes represent positive vertices, the blue cubes represent negative vertices, and the gray cubes represent 0. The interface (not drawn) belongs to $\mathcal{L}_0.x$

4. Further directions and techniques

4.1. Pisztora renormalization. An important tool in the study of two dimensional height functions was the second coarse-graining inequality from [Lam23a] which was inspired by [DST17]. In the original, it shows that a certain observable in the critical planar FK-Ising model is either bounded below or decays exponentially fast. Similar arguments were used to show that a similar observable involving $\mathcal{L}_1$ (now a percolation on dual edges) decays exponentially fast whenever the height function localizes. This was possible due to a general RSW theory for planar percolation, which is not available in higher dimensions.

One way to try to adapt this is to make use of a different renormalization scheme. In [DGR20], Pisztora renormalization is used, where the starting point was that for an FK-percolation on Λ_{2k} with coupling constants $\beta > 0$ and boundary condition ξ, denoted $\phi^{\xi}_{\Lambda_{2k}}$ (β is implicit), we have

Theorem 4.1. *For subcritical temperature and any boundary conditions,*

$$\phi^{\xi}_{\Lambda_{2k}}\Big(\text{The percolation restricted to } \Lambda_k \text{ touches all } 6 \text{ faces of } \Lambda_k \text{ and contains all length } k \text{ paths.}\Big)$$
$$> 1 - e^{-ck}$$

for some constant c depending only on temperature.

It may be possible to prove a similar result which replaces $\phi^{\xi}_{\Lambda_{2k}}$ with $\mu_{\Lambda_{2k},\xi}(\cdot||h+1|)$ and $\mu_{\Lambda_{2k},\xi}(\cdot)$ with some restriction on ξ, such as $\xi \geqslant 0$. The relevant percolation here could be $\mathcal{L}^{\diamond}_{-1}$. By theorem theorem 3.6, we can view this as a random low-temperature FK-Ising percolation. Of course, more work is needed since the random coupling constants in lemma 3.5 are not translation-invariant, and the rate of exponential decay would vary with the choice of $|h+1|$. Maybe one could show that a single low-temperature FK-percolation stochastically dominates the law of $\mathcal{L}^{\diamond}_{-1}$, but this seems unlikely to be the case since we expect several random coupling constants to be 0 (so for example the probability of 'many singleton components' may always be lower for a fixed FK-percolation model). Another approach could be to show that certain important events are more likely for $\mathcal{L}^{\diamond}_{-1}$ than for some fixed FK-percolation model, in a way similar to the approach in [DGR+20].

It may instead be useful to prove results similar to [Sev24] (see also [Bod03]) in our setting. This is a recent simple proof of the fact that the so-called slab percolation threshold of the FK-percolation model coincides with the percolation threshold, which is what makes Pisztora renormalization possible.

4.2. Differential inequalities. In [DRT18], exponential decay of two-point connectivity function throughout the entire high-temperature phase of FK-percolation is proven through the use of differential inequalities. In particular, the following general result holds

Theorem 4.2. *If $f_n : [a,b] \to [0,M]$ is a convergent sequence of increasing differentiable functions such that*

$$f_n' \geqslant \frac{n}{\sum_{k=0}^{n-1} f_k} f_n,$$

then there exists $\beta_c \in [a,b]$ such that

(1) For $\beta < \beta_c$, there exist C and c positive depending only on β such that $f_n(\beta) \leqslant Ce^{-cn}$
(2) For $\beta > \beta_c$, the limit $f = \lim_{n\to\infty} f_n$ satisfies $f(\beta) \geqslant \beta - \beta_c$.

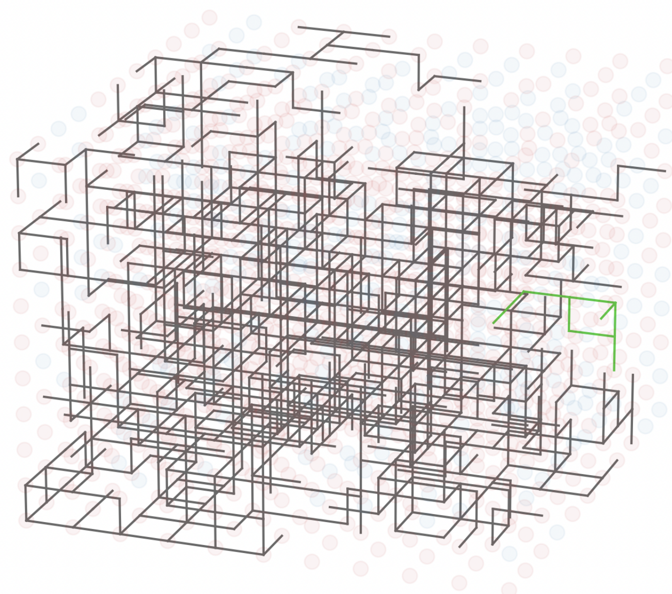

FIGURE 3. Realization of integer-valued GFF with $\beta = .05$ on Λ_{10} ($\mu_{\Lambda_{10},0}$). The figure shows Λ_5 and the largest cluster of $\mathcal{L}^{\diamond}_{-1}$ when restricted to this set. The event described in the paragraph following theorem 4.1 does not occur because there is a separate cluster of diameter 5, colored green.

In their setting, $f_n(\beta) = \phi^1_{\Lambda_{2n}}(0 \leftrightarrow \Lambda_n^c)$ satisfied the desired properties. An important step was the use of the OSSS inequality, which is also available in our setting.

To make something like this work, we should specify how our model will depend on temperature. As described in section 3, there is no temperature. A natural way to do this would be to replace V with βV. Another would be to multiply the total energy by β, which amounts to multiplying V and ρ each by β in equation (3). These turn out to be equivalent in terms of the resulting percolations, so we choose the first for concreteness.

We would like to use some reversed version of this for $f_n(\beta) = \mu_{\Lambda_{2n},0}(0 \xleftrightarrow{\mathcal{L}_0^{\diamond}} \Lambda_n^c)$. If it is the case that $\mathcal{L}_0^{\diamond}$ does not percolate (which is already known for the integer-valued Gaussian free field), then lemma 3.2 of [Lam23a] implies that this converges to 0. Working in slightly greater generality, we have the following.

Theorem 4.3. *For any $\mathcal{L}_0^{\diamond}$-measureable function X, we have that*

$$\frac{\partial}{\partial\beta}\mu_{\Lambda,0}(X) = \sum_{xy\in E_\Lambda}\Bigg[\mu_{\Lambda,0}\Big(\frac{2K_{xy}}{1-e^{-2\beta K_{xy}}}\operatorname{cov}_{\mu_{\Lambda,0}}(1_{\{xy\in\mathcal{L}_0^{\diamond}\}}, X||h|)\Big)$$

$$-\operatorname{cov}_{\mu_{\Lambda,0}}\big(V(\delta h_{xy}), \mu_{\Lambda,0}(X||h|)\big)\Bigg],$$

where each K_{xy} is a function of $|h|$ not depending on β, as in lemma 3.5.

Proof. One may write $\mu_{\Lambda,0}(\cdot) = \mu_{\Lambda,0}\big(\mu_{\Lambda,0}(\cdot||h|)\big)$. Writing out these measures explicitly using equation (3) and lemma 3.5 and interchanging the summation and differentiation yields the result. □

Unfortunately, it is not immediately clear whether this is negative for our choice of X and Λ. Looking at the left hand side, we have two opposing effects: as we increase β, the heights will be closer to 0, but at any fixed height, the event $\{\beta(V(h_y + h_x) - V(h_y - h_x)) \leqslant \rho_{xy}\}$ is less likely to occur. Looking at the right hand side, the first term is positive by the FKG inequality, but the second term is unclear since $\mu_{\Lambda,0}(V(\delta h_{xy})||h|)$ is not monotonic in $|h|$. However, if we are able to prove that something of the form theorem 4.2 hold wherever the function is monotonic, then we would only have problems at local maxima, which would be good progress regardless. It seems that the only difficulty is at small β (high temperature), so an alternative would be to deal with this case separately and then prove that the desired result holds for all larger β.

4.3. **Four dimensional lattice gauge theory.** A natural question is whether a similar approach could work in four dimensions. In this case, we would be dealing with equivalence classes of height 1-forms. Could there be some connection between these models and the plaquette percolation of [DS24]? Does the fact that plaquettes are dual to plaquettes simplify the analysis? Can something of this form be used to prove sharpness of the phase transition?

5. Acknowledgements

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