Homework 9

STAT 984

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Exercise 7.3

Suppose that $X_1, ..., X_n$ are independent and identically distributed with density $f_\theta(x)$, where $\theta \in (0, \infty)$. For each of the following forms of $f_\theta(x)$, prove that the likelihood equation has a unique solution and that this solution maximizes the likelihood function.

(a) *Weibull:* For some constant $a > 0$,

$$f_\theta(x) = a\theta^a x^{a-1} \exp\{-(\theta x)^a\} I\{x > 0\}$$

(b) *Cauchy:*

$$f_\theta(x) = \frac{\theta}{\pi} \frac{1}{x^2 + \theta^2}$$

(c)

$$f_\theta(x) = \frac{3\theta^2\sqrt{3}}{2\pi(x^3 + \theta^3)} I\{x > 0\}$$

Exercise 7.8

Prove Theorem 7.9

Hint: Start with $\sqrt{n}(\delta_n - \theta_0) = \sqrt{n}(\delta_n - \tilde{\theta}_n) + \sqrt{n}(\tilde{\theta}_n - \theta_0)$, then expand $\ell'(\tilde{\theta}_n)$ in a Taylor series about θ_0 and substitute the result into Equation (7.15). After simplifying. use the result of Exercise 2.2 along with arguments similar to those leading up to Theorem 7.8.

Exercise 7.9

Suppose that the following is a random sample from a logistic density with distribution function $F_\theta(x) = (1 + \exp\{\theta - x\})^{-1}$(I'll cheat and tell you that I used $\theta = 2$.)

1.0944	6.4723	3.118	3.8318	4.1262

1.2853	1.0439	1.7472	4.9483	1.7001

1.0422	0.169	3.6111	0.997	2.9438

(a) Evaluate the unique root of the likelihood equation numerically. Then, taking the sample median as our known $\sqrt{n}$-consistent estimator $\tilde{\theta}_n$ of θ, evaluate the estimator δ_n in Equation (7.15) numerically.

(b) Find the asymptotic distributions of $\sqrt{n})(\tilde{\theta}_n - 2)$ and $\sqrt{n}(\delta_n - 2)$. Then, simulate 200 samples of size $n = 15$ from the logistic distribution with $\theta = 2$. Find the sample variances of the resulting sample medians and δ_n-estimators. How well does the asymptotic theory match reality?

Exercise 7.11

If $f_\theta(x)$ forms a location family, so that $f_\theta(x) = f(x - \theta)$ for some density $f(x)$, then the Fisher information $I(\theta)$ is a constant (you may assume this fact without proof).

(a) Verify that for the Cauchy location family,

$$f_\theta(x) = \frac{1}{\pi\{1 + (x - \theta)^2\}},$$

we have $I(\theta) = \frac{1}{2}$.

(b) For 500 samples of size $n = 51$ from a standard Cauchy distribution, calculate the sample median $\tilde{\theta}_n$ and the efficient estimator δ_n^* of Equation (7.19). Compare the variances of $\tilde{\theta}_n$ and δ_n^* with their theoretical asymptotic limits.

Exercise 7.15

Suppose that $\boldsymbol{\theta} \in \mathbb{R}x\mathbb{R}_+$ (that is $\theta_1 \in \mathbb{R}$ and $\theta_2 \in (0, \infty)$) and

$$f_{\boldsymbol{\theta}}(x) = \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right)$$

for some continuous, differentiable density $f(x)$ that is symmetric about the origin. Find $I(\boldsymbol{\theta})$.