Homework 4

STAT 984

Emily Robinson

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Exercise 2.6

Prove Theorem 2.17(a): For a constant c, $X_n \stackrel{qm}{\to} c$ if and only if $E[X_n] \to c$ and $Var(X_n) \to 0$.

Exercise 2.9

(a) Prove that if $0 < a < b$, then convergence in b^{th} mean is stronger than convergence in a^{th} mean; i.e. $X_n \stackrel{b}{\to} X$ implies $X_n \stackrel{a}{\to} X$.

Hint: Use Exercise 1.40 with $\alpha = b/a$.

(b) Prove by counterexample that the conclusion of part (a) is not true in general if $0 < b < a$.

Exercise 2.10

The goal of this Exercise is to construct an example of an independent sequence $X_1, X_2, ...$ with $E[X_i] = \mu$ such that $\bar{X}_n \stackrel{P}{} \mu$ but $Var(\bar{X}_n)$ does not converge to 0. There are numerous ways we could proceed, but let us suppose that for some positive constants c_i and p_i, $X_i = c_i Y_i (2Z_i - 1)$, where Y_i and Z_i are independent Bernoulli random variables with $E[Y_i] = p_i$ and $E[Z_i] = 1/2$.

(a) Verify that $E[X_i] = 0$ and find $Var(\bar{X}_n)$.

(b) Show that $\bar{X}_n \stackrel{P}{} 0$ if

$$\frac{1}{n} \sum_{i=1}^{n} c_i p_i \to 0.$$

Hint: Use the triangle inequality to show that if Condition (2.21) is true, then $\bar{X}_n$ converges in mean to 0 (see Definition 2.15).

(c) Now specify c_i and p_i so that $Var(\bar{X}_n)$ does not converge to 0 but Contdition (2.21) holds. Remember that p_i must be less than or equal to 1 because it is the mean of a Bernoulli random variable.

Exercise 2.13

Let $Y_1, Y_2, ...$ be independent and identically distributed with mean μ and variance $\sigma^2 < \infty$. Let

$$X_1 = Y_1, X_2 = \frac{Y_2 + Y_3}{2}, X_3 = \frac{Y_4 + Y_5 + Y_6}{3}, etc.$$

Define δ_n as in Equation (2.14).

(a) Show that δ_n and $\bar{X}_n$ are both consistent estimators of μ.

(b) Calculate the relative efficiency $e_{\bar{X}_n,\delta_n}$ of $\bar{X}_n$ to δ_n, defined as $Var(\delta_n)/Var(\bar{X}_n)$, for $n = 5, 10, 20, 50, 100$, and ∞ and report the results in a table. For $n = \infty$, give the limit (with proof) of the efficiency.

(c) Using Example 1.23, give a simple expression asymptotically equivalent to $e_{\bar{X}_n,\delta_n}$. Report its values in your table for comparison. How good is the approximation for small n?

Exercise 2.19

Suppose that (X, Y) is a bivariate normal vector such that both X and Y are marginally standard normal and $\mathrm{Corr}(X, Y) = \rho$. Construct a computer program that simulates the distribution function $F_\rho(x, y)$ of the joint distribution of X and Y. For a given (x, y), the program should generate at least 50,000 random realizations from the distribution of (X, Y), then report the proportion for which $(X, Y) \le (x, y)$. (If you wish, you can also report a confidence interval for the true value.) Use your function to approximate $F_{.5}(1, 1), F_{.25}(-1, -1)$, and $F_{.75}(0, 0)$. As a check of your program, you can try it on $F_0(x, y)$, whose true values are not hard to calculate directly for an arbitrary x and y assuming your software has the ability to evaluate the standard normal distribution function.

Hint: To generate a bivariate normal random vector (X, Y) with covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, start with independent standard normal U and V, then take $X = U$ and $Y = \rho U + \sqrt{1-\rho^2}V$.

Exercise 2.21

Construct a counterexample to show that Slutsky's Theorem 2.39 may not be strengthened by changing $Y_n \xrightarrow{P} c$ to $Y_n \xrightarrow{P} Y$.