Homework 4

STAT 984

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October 3, 2019

Exercise 2.6

Prove Theorem 2.17(a): For a constant c, $X_n \overset{qm}{\to} c$ if and only if $E[X_n] \to c$ and $Var(X_n) \to 0$.

Proof. Assume $E[X_n] \to c$ and $\text{Var}(X_n) \to 0$. Then

$$\begin{aligned} E\left[(X_n - c)^2\right] &= E[X_n^2] - 2cE[X_n] + c^2 \\ &= Var(X_n) + (E[X])^2 - 2cE[X_n] + c^2 \\ &\to 0 + c^2 - 2c^2 + c^2 \\ &= 0. \end{aligned}$$

Therefore, for a constant c, if $E[X_n] \to c$ and $Var(X_n) \to 0.$, then $X_n \overset{qm}{\to} c$.

Now assume $E\left[(X_n - c)^2\right] \to 0$. Then

$$E\left[(X_n - c)^2\right] = Var(X_n) + (E[X] - c)^2.$$

Since $Var(X_n) \geq 0$ and $(E[X] - c)^2 \geq 0, Var(X_n) \to 0$ and $(E[X] - c)^2 \to 0$. Then since $f(x) = \sqrt{x}$ is a continuous function, $E[X_n] - c \to 0$ implies $E[X] \to c$. Therefore, for a constant c, if $X_n \overset{qm}{\to} c$ then $E[X_n] \to c$ and $Var(X_n) \to 0$.

Thus, for a constant c, $X_n \overset{qm}{\to} c$ if and only if $E[X_n] \to c$ and $Var(X_n) \to 0$. □

Exercise 2.9

(a) Prove that if $0 < a < b$, then convergence in b^{th} mean is stronger than convergence in a^{th} mean; i.e. $X_n \overset{b}{\to} X$ implies $X_n \overset{a}{\to} X$.

Hint: Use Exercise 1.40 with $\alpha = b/a$.

Proof. Using the result from Exercise 1.40, we have

$$\begin{aligned} & (E|X_n - X|)^{b/a} \leq E|X_n - X|^{b/a} \\ \Longrightarrow \quad & (E|X_n - X|^a)^{b/a} \leq E|X_n - X|^b \to 0 \\ \Longrightarrow \quad & (E|X_n - X|^a)^{b/a} \to 0 \\ \Longrightarrow \quad & E|X_n - X|^a \to 0. \end{aligned}$$

Thus, $X_n \overset{a}{\to} X$. □

(b) Prove by counterexample that the conclusion of part (a) is not true in general if $0 < b < a$.

Let

$$X_n = \begin{cases} 0 & \text{with probability } 1 - \frac{1}{n^2} \\ n & \text{with probability } \frac{1}{n^2}. \end{cases}$$

Then $X_n \xrightarrow{1} 0$ since

$$E|X_n| = 0 \cdot \left(1 - \frac{1}{n^2}\right) + n\left(\frac{1}{n^2}\right) = \frac{1}{n} \to 0.$$

However,

$$E|X_n^2| = 0^2 \cdot \left(1 - \frac{1}{n^2}\right) + n^2\left(\frac{1}{n^2}\right) = 1 \to 1.$$

Therefore, $X_n \xrightarrow{2} 1$. Thus, $X_n \xrightarrow{1} 0$ does not imply $X_n \xrightarrow{2} 0$.

Exercise 2.10

The goal of this Exercise is to construct an example of an independent sequence $X_1, X_2, ...$ with $E[X_i] = \mu$ such that $\bar{X}_n \overset{P}{\mu}$ but $Var(\bar{X}_n)$ does not converge to 0. There are numerous ways we could proceed, but let us suppose that for some positive constants c_i and p_i, $X_i = c_i Y_i(2Z_i - 1)$, where Y_i and Z_i are independent Bernoulli random variables with $E[Y_i] = p_i$ and $E[Z_i] = 1/2$.

(a) Verify that $E[X_i] = 0$ and find $Var(\bar{X}_n)$.

Suppose there exist a c_i and p_i such that $X_i = c_i Y_i(2Z_i - 1)$ where $Y_i \sim Bern(p_i)$ and $Z_i \sim Bern(1/2)$ with Y_i and Z_i independent. Then

$$X_n = \begin{cases} c_i & \text{with probability } \frac{p_i}{2} \\ -c_i & \text{with probability } \frac{p_i}{2} \\ 0 & \text{with probability } 1 - p_i. \end{cases}$$

Then

$$E[X_i] = c_i\left(\frac{p_i}{2}\right) - c_i\left(\frac{p_i}{2}\right) + 0(1 - p_i) = 0$$

and

$$E[X_i] = c_i^2\left(\frac{p_i}{2}\right) + (-c_i)^2\left(\frac{p_i}{2}\right) + 0^2(1 - p_i) = c_i^2 p_i$$

. Therefore, $\text{Var}(X_i) = c_i^2 p_i$. Thus, $\text{Var}(\bar{X}_i) = \frac{1}{n^2}\sum_{i=1}^n c_i^2 p_i$.

(b) Show that $\bar{X}_n \overset{P}{0}$ if

$$\frac{1}{n}\sum_{i=1}^n c_i p_i \to 0.$$

Hint: Use the triangle inequality to show that if Condition (2.21) is true, then $\bar{X}_n$ converges in mean to 0 (see Definition 2.15).

Consider $E|X_i| = |c_i|\frac{p_i}{2} + |-c_i|\frac{p_i}{2} + |0|(1-p_i) = c_i p_i$. Then by the triangle inequality,

$$E|\bar{X}_n| \leq \frac{1}{n}\sum_{i=1}^{n} E|X_i| = \frac{1}{n}\sum_{i=1}^{n} c_i p_i \to 0.$$

Therefore, by Theroem 2.17 (2), $\bar{X}_n \xrightarrow{1} 0$ implies $\bar{X}_n \xrightarrow{P} 0$.

(c) Now specify c_i and p_i so that $Var(\bar{X}_n)$ does not converge to 0 but Contdition (2.21) holds. Remember that p_i must be less than or equal to 1 because it is the mean of a Bernoulli random variable.

Let $c_i = i^3$ and $p_i = \frac{1}{i^4}$. Then $c_i p_i = \frac{1}{i}$ and $\frac{1}{n}\sum_{i=1}^{n} c_i p_i = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{i} = \frac{\log(n)}{n}\frac{\sum_{i=1}^{n}\frac{1}{i}}{\log(n)} \to 0$ since $\sum_{i=1}^{n}\frac{1}{i} \sim \log(n)$. However, $\mathrm{Var}(\bar{X}_n) = c_i^2 p_i = i^2 \to \infty$.

Exercise 2.13

Let $Y_1, Y_2, \ldots$ be independent and identically distributed with mean μ and variance $\sigma^2 < \infty$. Let

$$X_1 = Y_1, X_2 = \frac{Y_2 + Y_3}{2}, X_3 = \frac{Y_4 + Y_5 + Y_6}{3}, etc.$$

Define δ_n as in Equation (2.14).

(a) Show that δ_n and $\bar{X}_n$ are both consistent estimators of μ.

Consider $E[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma_i^2 = \frac{\sigma^2}{i}$. Then

$$\delta_n = \frac{\sum_{i=1}^{n}\frac{X_i}{\sigma_i^2}}{\sum_{j=1}^{n}\frac{1}{\sigma_j^2}} = \frac{\frac{1}{\sigma^2}\sum_{i=1}^{n} iX_i}{\frac{1}{\sigma^2}\sum_{j=1}^{n} j} = \frac{\sum_{i=1}^{n} iX_i}{\sum_{j=1}^{n} j}.$$

Then,

$$\begin{aligned}
E[\delta_n] &= E\left[\frac{\sum_{i=1}^{n} iX_i}{\sum_{j=1}^{n} j}\right] \\
&= \frac{\sum_{i=1}^{n} iE[X_i]}{\sum_{j=1}^{n} j} \\
&= \frac{\mu\sum_{i=1}^{n} i}{\sum_{j=1}^{n} j} \\
&= \mu
\end{aligned}$$

and

$$
\begin{aligned}
Var[\delta_n] &= Var\left[\frac{\sum_{i=1}^n iX_i}{\sum_{j=1}^n j}\right] \\
&= \frac{\sum_{i=1}^n i^2 Var[X_i]}{\left(\sum_{j=1}^n j\right)^2} \\
&= \frac{\sum_{i=1}^n i^2 \frac{\sigma^2}{i}}{\left(\sum_{j=1}^n j\right)^2} \\
&= \frac{\sigma^2 \sum_{i=1}^n i}{\left(\sum_{j=1}^n j\right)^2} \\
&= \frac{\sigma^2}{\sum_{j=1}^n j}.
\end{aligned}
$$

Then, using Chebyshev's inequlaity,

$$P\left((\delta_n - \mu)^2 \geq \epsilon^2\right) \leq \frac{E\left[(\delta_n - \mu)\right]}{\epsilon^2} \to 0$$

and $\delta_n \xrightarrow{P} \mu$. Thus, δ_n is a consistent esitmator of μ.

Similarly, $E[\bar{X}_n] = \mu$ and $Var(\bar{X}_n) = \frac{1}{n^2}\sum_{i=1}^n \frac{\sigma^2}{i} = \frac{\sigma^2}{n^2}\sum_{i=1}^n \frac{1}{i}$. Therefore,

$$P\left((\bar{X}_n - \mu)^2 \geq \epsilon^2\right) \leq \frac{E\left[(\bar{X}_n - \mu)\right]}{\epsilon^2} \to 0$$

and $\bar{X}_n \xrightarrow{P} \mu$. Thus, $\bar{X}_n$ is a consistent esitmator of μ.

(b) Calculate the relative efficiency $e_{\bar{X}_n,\delta_n}$ of $\bar{X}_n$ to δ_n, defined as $Var(\delta_n)/Var(\bar{X}_n)$, for $n = 5, 10, 20, 50, 100$, and ∞ and report the results in a table. For $n = \infty$, give the limit (with proof) of the efficiency.

$$e_{\bar{X}_n,\delta_n} = \frac{\frac{\sigma^2}{\sum_{j=1}^n j}}{\frac{\sigma^2}{n^2}\sum_{i=1}^n \frac{1}{i}} = \frac{n^2}{\sum_{j=1}^n j \sum_{i=1}^n \frac{1}{i}}.$$

The results are reported in the table below.

n	Eff	Asy	Ratio
5	0.73	1.243	0.587
10	0.621	0.869	0.715
20	0.529	0.668	0.793
50	0.436	0.511	0.852
100	0.382	0.434	0.879
infinity	0	0	NA

Consider $n = \infty$. Then from Example 1.23,

$$\frac{n^2}{\sum_{j=1}^n j \sum_{i=1}^n \frac{1}{i}} = \frac{n^2}{\frac{1}{2}n^2 \log(n)} \frac{\frac{1}{2}n^2}{\sum_{j=1}^n j} \frac{\log(n)}{\sum_{i=1}^n \frac{1}{i}} = \frac{2}{\log(n)} \frac{\frac{1}{2}n^2}{\sum_{j=1}^n j} \frac{\log(n)}{\sum_{i=1}^n \frac{1}{i}} \to 0 \cdot 1 \cdot 1 = 0$$

(c) Using Example 1.23, give a simple expression asymptotically equivalent to $e_{\bar{X}_n, \delta_n}$. Report its values in your table for comparison. How good is the approximation for small n?

Similar to the proof in part (b), consider,

$$\frac{n^2}{\sum_{j=1}^n j \sum_{i=1}^n \frac{1}{i}} \sim \frac{n^2}{\frac{1}{2}n^2 \log(n)} = \frac{2}{\log(n)}.$$

The ratios in the table above indicate that the approximation improves as n increases.

Exercise 2.19

Suppose that (X, Y) is a bivariate normal vector such that both X and Y are marginally standard normal and $\mathrm{Corr}(X, Y) = \rho$. Construct a computer program that simulates the distribution function $F_\rho(x, y)$ of the joint distribution of X and Y. For a given (x, y), the program should generate at least 50,000 random realizations from the distribution of (X, Y), then report the proportion for which $(X, Y) \le (x, y)$. (If you wish, you can also report a confidence interval for the true value.) Use your function to approximate $F_{.5}(1, 1), F_{.25}(-1, -1)$, and $F_{.75}(0, 0)$. As a check of your program, you can try it on $F_0(x, y)$, whose true values are not hard to calculate directly for an arbitrary x and y assuming your software has the ability to evaluate the standard normal distribution function.

Hint: To generate a bivariate normal random vector (X, Y) with covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, start with independent standard normal U and V, then take $X = U$ and $Y = \rho U + \sqrt{1 - \rho^2} V$.

```
BVN_func <- function(rho, x, y, n = 50000) {
    U <- rnorm(n, 0, 1)
    V <- rnorm(n, 0, 1)
    X <- U
    Y <- rho * U + sqrt(1 - rho^2) * V
    results <- cbind(rho, x, y, quantile = mean(X <=
        x & Y <= y))
}
a <- BVN_func(0.5, x = 1, y = 1, n = 50000)
b <- BVN_func(0.25, x = -1, y = -1, n = 50000)
c <- BVN_func(0.75, x = 0, y = 0, n = 50000)
kable(rbind(a, b, c))
```

rho	x	y	quantile
0.50	1	1	0.74382
0.25	-1	-1	0.04246
0.75	0	0	0.38352

Exercise 2.21

Construct a counterexample to show that Slutsky's Theorem 2.39 may not be strengthened by changing $Y_n \xrightarrow{P} c$ to $Y_n \xrightarrow{P} Y$.

Let $Y_n = Z \xrightarrow{P} Z$ and $X_n = -Y_n = -Z \xrightarrow{d} Z$. However,

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \xrightarrow{d} \begin{pmatrix} Z \\ Z \end{pmatrix} \neq \begin{pmatrix} -Z \\ Z \end{pmatrix}.$$