Homework 5

STAT 984

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Exercise 2.24

Prove Slutsky's Theorem, Theorem 2.39, using the following approach:

(a) Prove the following lemma:

Lemma 2.42 Let $\boldsymbol{V}_n$ and $\boldsymbol{W}_n$ be k-dimensional random vectors on the same sample space.

$$\text{If } \boldsymbol{V}_n \xrightarrow{d} \boldsymbol{V} \text{ and } \boldsymbol{W}_n \xrightarrow{P} \boldsymbol{0}, \text{ then } \boldsymbol{V}_n + \boldsymbol{W}_n \xrightarrow{d} \boldsymbol{V}.$$

Hint: For $\epsilon > 0$, let $\boldsymbol{\epsilon}$ denote the k-dimentional vector all of whose entries are ϵ. Take $\boldsymbol{a} \in \mathbb{R}^k$ to be a continuity point of $\boldsymbol{F_v(v)}$. Now argue that $\boldsymbol{a}$, since it is a point of continuity, must be contained in a neighborhood consisting only of points of continuity; therefore, ϵ may be taken small enough so that $\boldsymbol{a} - \boldsymbol{\epsilon}$ and $\boldsymbol{a} + \boldsymbol{\epsilon}$ are also points of continuity. Prove that

$$\begin{aligned} P(\boldsymbol{V}_n \leq \boldsymbol{a} - \boldsymbol{\epsilon}) - P(||\boldsymbol{W}_n|| \geq \epsilon) &\leq P(\boldsymbol{V}_n + \boldsymbol{W}_n \leq \boldsymbol{a}) \\ &\leq P(\boldsymbol{V}_n \leq \boldsymbol{a} + \boldsymbol{\epsilon}) + P(||\boldsymbol{W}_n|| \geq \epsilon). \end{aligned}$$

Next, take $\limsup_n$ and $\liminf_n$. Finally, let $\epsilon \to 0$.

Proof. □

(b) Show how to prove Theorem 2.39 using Lemma 2.42.

Hint: Consider the random vectors

$$\boldsymbol{V}_n = \begin{pmatrix} \boldsymbol{X}_n \\ \boldsymbol{c} \end{pmatrix} \text{ and } \boldsymbol{W}_n = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{Y}_n - \boldsymbol{c} \end{pmatrix}.$$

Proof. □

Exercise 3.2

The diagram at the end of this section suggests that neither $X_n \xrightarrow{a.s.} X$ nor $X_n \xrightarrow{qm} X$ implies the other. Construct two counterexamples, one to show that $X_n \xrightarrow{a.s.} X$ does not imply $X_n \xrightarrow{qm} X$ and the other to show that $X_n \xrightarrow{qm} X$ does not imply $X_n \xrightarrow{a.s.} X$.

(1)

(2)

Exercise 3.3

Let $B_1, B_2, ...$ denote a sequence of events. Let B_n i.o., which stands for B_n infinitely often, denote the set

$$B_n \text{ i.o.} \stackrel{\text{def}}{=} \{\omega \in \Omega : \text{ for every } n, \text{ there exists } k \geq n \text{ such that } \omega \in B_k\}.$$

Prove the *First Borel-Cantelli Lemma*, which states that if $\sum_{n=1}^{\infty} P(B_n) < \infty$, then $P(B_n \text{ i.o.}) = 0$.

Hint: Argue that

$$B_n \text{ i.o.} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k,$$

then adapt the proof of Lemma 3.9.

Exercise 3.4

Use the steps below to prove a version of the Strong Law of Large Numbers for the special case in which the random variables $X_1, X_2, ...$ have a finite fourth moment, $\mathrm{E}[X_1^4] < \infty$.

(a) Assume without loss of generality that $\mathrm{E}[X_1] = 0$. Expand $\mathrm{E}[(X_1 + ... + X_n)^4]$ and then count the nonzero terms. **Hint:** The only nonzero terms are of the form $\mathrm{E}[X_i^4]$ or $(\mathrm{E}[X_i^2])^2$.

(b) Use Markov's inequality (1.35) with $r = 4$. to put an upper bound on

$$P(|\bar{X}_n| > \epsilon)$$

involving $\mathrm{E}[(X_1 + ... + X_n)^4]$.

(c) Combind parts (a) and (b) with Lemma 3.9 to show that $\bar{X}_n \stackrel{a.s.}{\to} 0$. **Hint:** Use the fact that $\sum_{n=1}^{\infty} n^{-2} < \infty$.

Exercise 3.13

Prove that if there exists $\epsilon > 0$ such that $\sup_n \mathrm{E}[Y_n]^{1+\epsilon} < \infty$, then $Y_1, Y_2, ...$ is uniformly integrable sequence.

Hint: First prove that

$$|Y_n| I\{|Y_n| \geq \alpha\} \leq \frac{1}{\alpha^{\epsilon}} |Y_n|^{1+\epsilon}.$$

Exercise 3.14

Prove that if there exists a random variable Z such that $\mathrm{E}|Z| = \mu < \infty$ and $P(|Y_n| \geq t) \leq P(|Z| \geq t)$ for all n and for all $t > 0$, then $Y_1, Y_2, ...$ is a uniformly integrable sequence. You may use the fact (without proof) that for a nonngative X,

$$\mathrm{E}[X] = \int_0^{\infty} P(X \geq t) dt.$$

Hint: Consider the random variables $|Y_n|I\{|Y_n| \geq t\}$ and $|Z|I\{|Z| \geq t\}$. In addition, use the fact that

$$\mathrm{E}|Z| = \sum_{i=1}^{\infty} \mathrm{E}\left[|Z|I\{i = 1 \leq |Z| < i\}\right]$$

to argue that $\mathrm{E}[|Z|I\{|Z| < \alpha\}] \to \mathrm{E}|Z|$ as $\alpha \to \infty$.