Homework 5

STAT 984

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Exercise 2.24

Prove Slutsky's Theorem, Theorem 2.39, using the following approach:

(a) Prove the following lemma:

Lemma 2.42 Let $\boldsymbol{V}_n$ and $\boldsymbol{W}_n$ be k-dimensional random vectors on the same sample space.

$$\text{If } \boldsymbol{V}_n \xrightarrow{d} \boldsymbol{V} \text{ and } \boldsymbol{W}_n \xrightarrow{P} \mathbf{0}, \text{ then } \boldsymbol{V}_n + \boldsymbol{W}_n \xrightarrow{d} \boldsymbol{V}.$$

Hint: For $\epsilon > 0$, let $\boldsymbol{\epsilon}$ denote the k-dimentional vector all of whose entries are ϵ. Take $\boldsymbol{a} \in \mathbb{R}^k$ to be a continuity point of $\boldsymbol{F_v(v)}$. Now argue that $\boldsymbol{a}$, since it is a point of continuity, must be contained in a neighborhood consisting only of points of continuity; therefore, ϵ may be taken small enough so that $\boldsymbol{a} - \boldsymbol{\epsilon}$ and $\boldsymbol{a} + \boldsymbol{\epsilon}$ are also points of continuity. Prove that

$$\begin{aligned} P(\boldsymbol{V}_n \le \boldsymbol{a} - \boldsymbol{\epsilon}) - P(||\boldsymbol{W}_n|| \ge \epsilon) \le P(\boldsymbol{V}_n + \boldsymbol{W}_n \le \boldsymbol{a}) \\ \le P(\boldsymbol{V}_n \le \boldsymbol{a} + \boldsymbol{\epsilon}) + P(||\boldsymbol{W}_n|| \ge \epsilon). \end{aligned}$$

Next, take limsup_n and liminf_n. Finally, let $\epsilon \to 0$.

Proof. Let $\epsilon > 0$, let $\boldsymbol{\epsilon}$ denote the k-dimentional vector all of whose entries are ϵ. Then there exists a continuity point of $\boldsymbol{F_v(v)}, \boldsymbol{a} \in \mathbb{R}^k$. Then since it is a point of continuity, $\boldsymbol{a}$, must be contained in a neighborhood consisting only of points of continuity; therefore, ϵ may be taken small enough so that $\boldsymbol{a} - \boldsymbol{\epsilon}$ and $\boldsymbol{a} + \boldsymbol{\epsilon}$ are also points of continuity. Then whenever $\boldsymbol{V}_n + \boldsymbol{W}_n \le \boldsymbol{a}$ it must be true that either $\boldsymbol{V}_n \le \boldsymbol{a} + \boldsymbol{\epsilon}$ or $||\boldsymbol{W}_n|| > \boldsymbol{\epsilon}$. Therefore,

$$\begin{aligned} & P(\boldsymbol{V}_n \le \boldsymbol{a} - \boldsymbol{\epsilon}) - P(||\boldsymbol{W}_n|| \ge \boldsymbol{\epsilon}) \le P(\boldsymbol{V}_n + \boldsymbol{W}_n \le \boldsymbol{a}) \le P(\boldsymbol{V}_n \le \boldsymbol{a} + \boldsymbol{\epsilon}) + P(||\boldsymbol{W}_n|| \ge \epsilon) \\ \implies & F_{\boldsymbol{V}_n}(\boldsymbol{a} - \boldsymbol{\epsilon}) - P(||\boldsymbol{W}_n|| \ge \boldsymbol{\epsilon}) \le F_{\boldsymbol{V}_n + \boldsymbol{W}_n}(\boldsymbol{a}) \le F_{\boldsymbol{V}_n}(\boldsymbol{a} + \boldsymbol{\epsilon}) + P(||\boldsymbol{W}_n|| \ge \epsilon) \\ \to & F_{\boldsymbol{V}}(\boldsymbol{a} - \boldsymbol{\epsilon}) - 0 \le F_{\boldsymbol{V}_n + \boldsymbol{W}_n}(\boldsymbol{a}) \le F_{\boldsymbol{V}}(\boldsymbol{a} + \boldsymbol{\epsilon}) + 0 \\ \implies & F_{\boldsymbol{V}}(\boldsymbol{a} - \boldsymbol{\epsilon}) \le \text{liminf}_n F_{\boldsymbol{V}_n + \boldsymbol{W}_n}(\boldsymbol{a}) \le \text{limsup}_n F_{\boldsymbol{V}_n + \boldsymbol{W}_n}(\boldsymbol{a}) \le F_{\boldsymbol{V}}(\boldsymbol{a} + \boldsymbol{\epsilon}) \\ \implies & F_{\boldsymbol{V}}(\boldsymbol{a}) \le \text{liminf}_n F_{\boldsymbol{V}_n + \boldsymbol{W}_n}(\boldsymbol{a}) \le \text{limsup}_n F_{\boldsymbol{V}_n + \boldsymbol{W}_n}(\boldsymbol{a}) \le F_{\boldsymbol{V}}(\boldsymbol{a}) \text{ as } \epsilon \to 0. \\ \implies & F_{\boldsymbol{V}}(\boldsymbol{a}) = \text{liminf}_n F_{\boldsymbol{V}_n + \boldsymbol{W}_n}(\boldsymbol{a}) = \text{limsup}_n F_{\boldsymbol{V}_n + \boldsymbol{W}_n}(\boldsymbol{a}) \\ \implies & F_{\boldsymbol{V}_n + \boldsymbol{W}_n}(\boldsymbol{a}) \to F_{\boldsymbol{V}}(\boldsymbol{a}). \end{aligned}$$

Therefore, $\boldsymbol{V}_n + \boldsymbol{W}_n \xrightarrow{d} \boldsymbol{V}$. $\square$

(b) Show how to prove Theorem 2.39 using Lemma 2.42.

Hint: Consider the random vectors

$$\boldsymbol{V}_n = \begin{pmatrix} \boldsymbol{X}_n \\ \boldsymbol{c} \end{pmatrix} \text{ and } \boldsymbol{W}_n = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{Y}_n - \boldsymbol{c} \end{pmatrix}.$$

Proof. Let $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{P} c$. Then consider

$$\boldsymbol{V}_n = \begin{pmatrix} \boldsymbol{X}_n \\ \boldsymbol{c} \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{c} \end{pmatrix} \text{ and } \boldsymbol{W}_n = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{Y}_n - \boldsymbol{c} \end{pmatrix} \xrightarrow{P} \boldsymbol{0}.$$

Then by Lemma 2.42,

$$\boldsymbol{V}_n + \boldsymbol{W}_n \xrightarrow{d} \boldsymbol{V} \implies \begin{pmatrix} \boldsymbol{X}_n \\ \boldsymbol{Y}_n \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{c} \end{pmatrix}.$$

Therefore, Slutsky's Theorem holds. □

Exercise 3.2

The diagram at the end of this section suggests that neither $X_n \xrightarrow{a.s.} X$ nor $X_n \xrightarrow{qm} X$ implies the other. Construct two counterexamples, one to show that $X_n \xrightarrow{a.s.} X$ does not imply $X_n \xrightarrow{qm} X$ and the other to show that $X_n \xrightarrow{qm} X$ does not imply $X_n \xrightarrow{a.s.} X$.

(1) Consider $X_n \xrightarrow{a.s.} X$, but $X_n \overset{qm}{\nrightarrow} X$. Let

$$X_n = \begin{cases} n^2 & \text{with probability } \frac{1}{n^2} \\ 0 & \text{with probability } 1 - \frac{1}{n^2}. \end{cases}$$

Then $X_n \xrightarrow{P} 0$ since

$$P(|X_n| < \epsilon) = P(X_n = 0) = 1 - \frac{1}{n^2} \to 1$$

and $X_n \xrightarrow{a.s.} 0$ since

$$\begin{aligned} &\lim_{k\to\infty} \prod_{j=0}^{k} \left(1 - \frac{1}{n^2}\right)\left(1 - \frac{1}{(n+1)^2}\right) \cdots \left(1 - \frac{1}{(n+k)^2}\right) \\ &= \lim_{k\to\infty} \prod_{j=0}^{k} \left(\frac{n^2-1}{n^2}\right)\left(\frac{(n+1)^2-1}{(n+1)^2}\right) \cdots \left(\frac{(n+k)^2-1}{(n+k)^2}\right) \\ &\to 1. \end{aligned}$$

However, $X_n \overset{P}{\nrightarrow} 0$ since

$$\mathrm{E}[X_n] = n^2\left(\frac{1}{n^2}\right) + 0\left(1 - \frac{1}{n^2}\right) = 1.$$

(2) Consider $X_n \overset{qm}{\to} X$, but $X_n \overset{a.s.}{\nrightarrow} X$. Let

$$X_n = \begin{cases} \sqrt[3]{n} & \text{with probability } \frac{1}{2n} \\ -\sqrt[3]{n} & \text{with probability } \frac{1}{2n} \\ 0 & \text{with probability } 1 - \frac{1}{n}. \end{cases}$$

Then $X_n \overset{qm}{\to} 0$ since

$$\mathrm{E}[X_n] = \sqrt[3]{n}\left(\frac{1}{2n}\right) - \sqrt[3]{n}\left(\frac{1}{2n}\right) + 0\left(1 - \frac{1}{n}\right) = 0 \to 0$$

and

$$\mathrm{Var}(X_n) = n^{2/3}\left(\frac{1}{2n}\right) + (-n)^{2/3}\left(\frac{1}{2n}\right) + 0^2\left(1 - \frac{1}{n}\right) = \frac{n^{2/3}}{n} = \frac{1}{n^{1/3}} \to 0.$$

However, $X_n \overset{a.s.}{\nrightarrow} 0$ since

$$\begin{aligned} &\lim_{k\to\infty} \prod_{j=0}^{k} \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n+1}\right)\cdots\left(1 - \frac{1}{n+k}\right) \\ &= \lim_{k\to\infty} \prod_{j=0}^{k} \left(\frac{n-1}{n}\right)\left(\frac{n}{n+1}\right)\cdots\left(\frac{n+k-1}{n+k}\right) \\ &= \lim_{k\to\infty} \prod_{j=0}^{k} \left(\frac{n-1}{n+k}\right) \\ &\to 0 \neq 1. \end{aligned}$$

Exercise 3.3

Let $B_1, B_2, ...$ denote a sequence of events. Let B_n i.o., which stands for B_n infinitely often, denote the set

$$B_n \text{ i.o. } \overset{\text{def}}{=} \{\omega \in \Omega : \text{ for every } n, \text{ there exists } k \geq n \text{ such that } \omega \in B_k\}.$$

Prove the *First Borel-Cantelli Lemma*, which states that if $\sum_{n=1}^{\infty} P(B_n) < \infty$, then $P(B_n \text{ i.o.}) = 0$.

Hint: Argue that

$$B_n \text{ i.o. } = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k,$$

then adapt the proof of Lemma 3.9.

Proof. There exists some $k \geq n$ such that $\omega \in B_k$ is captured by the union and true for all n is captured by the intersection. Therefore,

$$B_n \text{ i.o. } = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k.$$

Let $\epsilon > 0$. Then if $\sum_{n=1}^{\infty} P(B_n) < \infty$, there exists an N such that $\sum_{n=N}^{\infty} P(B_n) < \epsilon$. Then,

$$P\left(B_n \text{ i.o. }\right) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k\right) \leq P\left(\bigcup_{k=N}^{\infty} B_k\right) \leq \sum_{k=N}^{\infty} P(B_k) < \epsilon.$$

Therefore, if $\sum_{n=1}^{\infty} P(B_n) < \infty$, then $P(B_n \text{ i.o.}) = 0$. $\square$

Exercise 3.4

Use the steps below to prove a version of the Strong Law of Large Numbers for the special case in which the random variables $X_1, X_2, ...$ have a finite fourth moment, $\mathrm{E}[X_1^4] < \infty$.

(a) Assume without loss of generality that $\mathrm{E}[X_1] = 0$. Expand $\mathrm{E}[(X_1 + ... + X_n)^4]$ and then count the nonzero terms. **Hint:** The only nonzero terms are of the form $\mathrm{E}[X_i^4]$ or $(\mathrm{E}[X_i^2])^2$.

Consider

$$\begin{aligned}
\mathrm{E}[(X_1 + X_2 + ... + X_n)^4] &= \mathrm{E}[X_1^4] + \mathrm{E}[X_2^4] + ... + \mathrm{E}[X_n^4] && (n \text{ of these}) \\
&\quad + \mathrm{E}[X_1^3 X_2] + ... + \mathrm{E}[X_n^3 X_{n-1}] && (= 0) \\
&\quad + \mathrm{E}[X_1^2 X_2^2] + \mathrm{E}[X_1^2 X_3^2] + ... + \mathrm{E}[X_1^2 X_n^2] && (n-1 \text{ columns of these}) \\
&\quad + \mathrm{E}[X_2^2 X_1^2] + \mathrm{E}[X_2^2 X_3^2] + ... + \mathrm{E}[X_2^2 X_n^2] && (3n \text{ rows of these}) \\
&\quad . \\
&\quad . \\
&\quad . \\
&\quad + \mathrm{E}[X_1^2 X_2^2] + \mathrm{E}[X_1^2 X_3^2] + ... + \mathrm{E}[X_1^2 X_n^2] \\
&= n\mathrm{E}[X_i^4] + 3n(n-1)\left(\mathrm{E}[X_i^2]\right)^2.
\end{aligned}$$

Therefore, there are $n + 3n(n-1) = n(3n-2)$ nonzero terms.

(b) Use Markov's inequality (1.35) with $r = 4$. to put an upper bound on

$$P(|\bar{X}_n| > \epsilon)$$

involving $\mathrm{E}[(X_1 + ... + X_n)^4]$. Consider

$$\begin{aligned}
P\left(|\bar{X}_n| > \epsilon\right) &= P\left(|X_n| > n\epsilon\right) \\
&\leq \frac{\mathrm{E}[|X_n|^4]}{(n\epsilon)^4} && \text{(by Markov's Inequality)} \\
&= \frac{n\mathrm{E}[X_i^4] + 3n(n-1)\left(\mathrm{E}[X_i^2]\right)^2}{(n\epsilon)^4} \\
&= \frac{\mathrm{E}[X_i^4] + 3(n-1)\sigma^4}{\epsilon^4 n^3} \\
&= o\left(\frac{1}{n^2}\right).
\end{aligned}$$

(c) Combind parts (a) and (b) with Lemma 3.9 to show that $\bar{X}_n \overset{a.s.}{\to} 0$. **Hint:** Use the fact that $\sum_{n=1}^{\infty} n^{-2} < \infty$. Consider

$$\sum_{n=1}^{\infty} P\left(|\bar{X}_n| > \epsilon\right) \leq \sum_{n=1}^{\infty} \frac{\mathrm{E}[X_i^4] + 3(n-1)\sigma^4}{\epsilon^4 n^3} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Therefore, $\bar{X}_n \overset{a.s.}{\to} 0$.

Exercise 3.13

Prove that if there exists $\epsilon > 0$ such that $\sup_n \mathrm{E}[Y_n]^{1+\epsilon} < \infty$, then $Y_1, Y_2, ...$ is uniformly integrable sequence.

Hint: First prove that

$$|Y_n| I\{|Y_n| \geq \alpha\} \leq \frac{1}{\alpha^\epsilon} |Y_n|^{1+\epsilon}.$$

Proof. By the Markov Inequality,

$$P(|Y_n| \geq \alpha) \leq \frac{1}{\alpha^\epsilon} \mathrm{E}|Y_n|^\epsilon.$$

Then multipliying by $|Y_n|$ and taking the absolute value of both sides,

$$\mathrm{E}\left[|Y_n| I\{|Y_n| \geq \alpha\}\right] \leq \mathrm{E}\left[\frac{1}{\alpha^\epsilon} \mathrm{E}|Y_n|^\epsilon |Y_n|\right] = \frac{1}{\alpha^\epsilon} \mathrm{E}|Y_n|^\epsilon \mathrm{E}|Y_n| = \frac{1}{\alpha^\epsilon} \mathrm{E}|Y_n|^{\epsilon+1}.$$

Then since $\sup_n \mathrm{E}|Y_n|^{\epsilon+1} < \infty$,

$$\sup_n \mathrm{E}\left[|Y_n| I\{|Y_n| \geq \alpha\}\right] \leq \sup_n \frac{1}{\alpha^\epsilon} \mathrm{E}|Y_n|^{\epsilon+1} = \frac{1}{\alpha^\epsilon} \sup_n \mathrm{E}|Y_n|^{\epsilon+1} \to 0 \text{ (as } \alpha \to \infty).$$

□

Exercise 3.14

Prove that if there exists a random variable Z such that $\mathrm{E}|Z| = \mu < \infty$ and $P(|Y_n| \geq t) \leq P(|Z| \geq t)$ for all n and for all $t > 0$, then $Y_1, Y_2, ...$ is a uniformly integrable sequence. You may use the fact (without proof) that for a nonngative X,

$$\mathrm{E}[X] = \int_0^\infty P(X \geq t) dt.$$

Hint: Consider the random variables $|Y_n| I\{|Y_n| \geq t\}$ and $|Z| I\{|Z| \geq t\}$. In addition, use the fact that

$$\mathrm{E}|Z| = \sum_{i=1}^{\infty} \mathrm{E}\left[|Z| I\{i = 1 \leq |Z| < i\}\right]$$

to argue that $\mathrm{E}[|Z| I\{|Z| < \alpha\}] \to \mathrm{E}|Z|$ as $\alpha \to \infty$.

Proof. Consider the random variable Z such that $\mathrm{E}|Z| = \mu < \infty$ and $P(|Y_n| \geq t) \leq P(|Z| \geq t)$ for all n and for all $t > 0$. Then

$$\mathrm{E}|Y_n| = \int_0^\infty P(|Y_n| \geq t)dt \leq \int_0^\infty P(|Z| \geq t)dt = \mathrm{E}|Z|.$$

Similarly, consider the random variables $|Y_n|I\{|Y_n| \geq t\}$ and $|Z|I\{|Z| \geq t\}$, it follows that

$$|Y_n|I\{|Y_n| \geq t\} \leq |Z|I\{|Z| \geq t\}.$$

Notice that $|z| < \infty$. Therefore, $|Z|I\{|Z| \geq t\} \to 0$ as $t \to \infty$. Then,

$$|Y_n|I\{|Y_n| \geq t\} \leq |Z|I\{|Z| \geq t\} \to 0 \text{ (as } t \to \infty).$$

Therefore, $|Y_n|I\{|Y_n| \geq t\} \to 0$ as $t \to \infty$. Thus, Y_n is a uniformly integrable sequence. $\square$