Homework 3

STAT 984

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Exercise 1.38

Let $f(x)$ be a convex function on some interval, and let x_0 be any point on the interior of that interval.

(a) Prove that

$$\lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0} \tag{1.38}$$

exists and is finite, that is, a one-sided derivative exists at x_0.

Hint: Using Definition 1.30, show that the fraction in expression (1.38) is non-increasing and bounded below as x decreases to x_0.

(b) Prove that there exists a linear function $g(x) = ax + b$ such that $g(x_0) = f(x_0)$ and $g(x) \leq f(x)$ for all x in the interval. This fact is the supporting hyperplane property in the case of a convex function taking a real argument.

Hint: Let $f'(x_0+)$ denote the one-sided derivative of part (a). Consider the line $f(x_0) + f'(x_0+)(x - x_0)$.

Exercise 1.39

Prove Holder's inequality: For random variables X and Y and positive p and a such that $p + q = 1$,

$$E|XY| \leq (E|X|^{1/p})^p (E|Y|^{1/q})^q. \tag{1.39}$$

(If $p = q = 1/2$, inequality 1.39 is also called the Cauchy-Schwartz inequality.)

Hint: Use the convexity of $\exp(x)$ to prove that $|abXY| \leq p|aX|^{1/p} + q|bY|^{1/q}$ whenever $aX \neq 0$ and $bY \neq 0$ (the same inequality is also true if $aX = 0$ or $bY = 0$). Take expectations, then find values for the scalars a and b that give the desired result when the right side of inequality (1.39) is nonzero.

Exercise 1.40

Use Holder's Inequality (1.39) to prove that if $\alpha > 1$, then

$$(E|X|)^\alpha \leq E|X|^\alpha.$$

Hint: Take Y to be a constant in Inequality (1.39).

Exercise 1.45

For any nonnegative random variable Y with finite expectation, prove that

$$\sum_{i=1}^{\infty} P(Y \geq i) \leq EY. \tag{1.43}$$

Hint: First, prove that equality holds if Y is supported on the nonnegative integers. Then note for a general Y that $E\lfloor Y \rfloor \leq EY$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Though we will not do so here, it is possible to prove a statement stronger than inequality (1.43) for nonnegative random variables, namely,

$$\int_0^{\infty} P(Y \geq t)dt = EY.$$

(This equation remains true if $EY = \infty$.) To sketch a proof, note that if we can prove $\int Ef(Y,t)dt = E\int f(Y,t)dt$, the result follows immediately by taking $f(Y,t) = I\{Y \geq t\}$.

Exercise 2.1

For each of the three cases below, prove that $X_n \xrightarrow{P} 1$:

(a) $X_n = 1 + nY_n$, where Y_n is a Bernoulli random variable with mean $1/n$.

(b) $X_n = Y_n/\log n$, where Y_n is a Poisson random variable with mean $\sum_{i=1}^n (1/i)$.

(c) $X_n = \frac{1}{n}\sum_{i=1}^n Y_i^2$, where the Y_i are independent standard normal random variables.

Exercise 2.2

This exercise deals with bounded in probability sequences; see Definition 2.6.

(a) Prove that if $X_n \xrightarrow{d} X$ for some random variable X, then X_n is bounded in probability.

Hint: You may use the fact that any interval of real numbers must contain a point of continuity of $F(x)$. Also, recall that $F(x) \to 1$ as $x \to \infty$.

(b) Prove that if X_n is bounded in probability and $Y_n \xrightarrow{P} 0$, then $X_nY_n \xrightarrow{P} 0$.

Hint: For fixed $\epsilon > 0$, argue that there must be M and N such that $P(|X_n| < M) > 1 - \epsilon/2$ and $P(|Y_n| < \epsilon/M) > 1 - \epsilon/2$ for all $n > N$. What is then the smallest possible value of $P(|X_n| < M \text{ and } |Y_n| < \epsilon/M)$? Use this result to prove $X_nY_n \xrightarrow{P} 0$.

Exercise 2.4

Suppose that $X_1, ...X_n$ are independent and identically distributed Uniform(0,1) random variables. For a real number t, let

$$G_n(t) = \sum_{i=1}^n I\{X_i \leq t\}.$$

(a) What is the distribution of $G_n(t)$ if $0 < t < 1$?

(b) Suppose $c > 0$. Find the distribution of a random variable X such that $G_n(c/n) \xrightarrow{d} X$. Justify your answer.

(c) How does your answer to part (b) change if $X_1, ..., X_n$ are from a standard exponential distribution instead of a uniform distribution? The standard exponential distribution function is $F(t) = 1 - e^{-t}$.