Homework 8

STAT 984

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Exercise 6.1

For a given n, let $X_1, ..., X_n$ be independent and identically distributed with distribution function

$$P(X_i \leq t) = \frac{t^3 + \theta^3}{2\theta^3} \text{ for } t \in [-\theta, \theta].$$

Let $X_{(1)}$ denote the first order statistic from the sample of size n; that is, $X_{(1)}$ is the smallest of the X_i.

(a) Prove that $-X_{(1)}$ is consistent for θ.

(b) Prove that

$$n(\theta + X_{(1)}) \xrightarrow{d} Y,$$

where Y is a random variable with an exponential distribution. Find $\mathrm{E}(Y)$ in terms of θ.

(c) For a fixed α, define

$$\delta_{\alpha,n} = -\left(1 + \frac{\alpha}{n} X_{(1)}\right).$$

Find, with proof, α^* such that

$$n(\theta - \delta_{\alpha^*,n}) \xrightarrow{d} Y - \mathrm{E}(Y),$$

where Y is the same random variable as in part (b).

(d) Compare the two consistent θ-estimators $\delta_{\alpha^*,n}$ and $-X_{(1)}$ empirically as follows. For $n \in \{10^2, 10^3, 10^4\}$, take $\theta = 1$ and simulate 1000 samples of size n from the distribution of X_i. From these 1000 samples, estimate the bias and mean squared error of each estimator. Which of the two appears better? Do your empiriccal results agree with the theoretical results in parts (c) and (d)?

Exercise 6.2

Let $X_1, X_2, ...$ be independent uniform $(0, \theta)$ random variables. Let $X_{(n)} = \max\{X_1, ..., X_n\}$ and consider the three estimators

$$\delta_n^0 = X_{(n)} \delta_n^1 = \frac{n}{n-1} X_{(n)} \qquad \delta_n^2 = \left(\frac{n}{n-1}\right)^2 X_{(n)}$$

(a) Prove that each estimator is consistent for θ.

(b) Perform an empirical comparison of these three estimators for $n = 10^2, 10^3, 10^4$. Use $\theta = 1$ and simulate 1000 samples of size n from uniform $(0, 1)$. From these 1000 samples, estimate the bias and mean squared error of each estimator. Which one of the three appears to be best?

(c) Find the asymptotic distribution of $n(\theta - \delta_n^i)$ for $i = 0, 1, 2$. Based on your results, which of the three appears to be the best estimator and why? (For the latter question, don't attempt to make a rigorous mathematical argument; simply give an educated guess.)

Exercise 6.5

Let $X_1, ..., X_n$ be a simple random sample from the distribution function $F(x) = [1 - (1/x)]I\{x > 1\}$.

(a) Find the joint asymptotic distribution of $(X_{(n-1)}/n, X_{(n)/n})$.

Hint: Proceed as in Example 6.5.

(b) FInd the asymptotic distribution of $X_{(n-1)}/n, X_{(n)/n}$.

Exercise 6.8

Let $X_1, ..., X_n$ be independent uniform$(0, 2\theta)$ random variables.

(a) Let $M = (X_{(1)} + X_{(n)})/2$. Find the asymptotic distribution of $n(M - \theta)$.

(b) Compare the asymptotic performance of the three estimators $M, \bar{X}_n$, and the sample median $\tilde{X}_n$ by considering their relative efficiencies.

(c) For $n \in \{101, 1001, 10001\}$, generate 500 samples of size n, taking $\theta = 1$. Keep track of $M, \bar{X}_n$, and $\tilde{X}_n$ for each sample. Construct a 3×3 table in which you report the sample variance of each estimator for each value of n. Do your simulation results agree with your theoretical results in part (b)?

Exercise 6.12

Let $X_1, ..., X_n$ be a random sample from Uniform$(0, 2\theta)$. Find the asymptotic distributions of the median, the midquartile range, and $\frac{2}{3}Q_3$, where Q_3 denotes the third quartile and the midquartile range is the mean of the 1st and 3rd quartiles. Compare these three estimates of θ based on their asymptotic variances.