Homework 6

STAT 984

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Exercise 4.7

Use the Cramer-Wold Theorem along with the univariate Central Limit Theorem (from Example 2.12) to prove Theorem 4.9.

Proof. Let $\boldsymbol{X} \sim N_k(\boldsymbol{0}, \Sigma)$ and take any vector $\boldsymbol{a} \in \mathbb{R}^k$. We wish to show that

$$\boldsymbol{a}^T[\sqrt{n}(\bar{\boldsymbol{X}}_n - \boldsymbol{\mu})] \xrightarrow{d} \boldsymbol{a}^T\boldsymbol{X}.$$

But this follows immediately from the univariate Central Limit Theorem, since $\boldsymbol{a}^T(\boldsymbol{X}_1 - \boldsymbol{\mu}), \boldsymbol{a}^T(\boldsymbol{X}_2 - \boldsymbol{\mu}), ...$ are independent and identically distributed with mean 0 and variance $\boldsymbol{a}^T\Sigma\boldsymbol{a}$. $\square$

Exercise 4.9

In this probelm, we prove the converse of Exercise 4.8, which is the part of the Lindeberg-Feller Theorem due to Feller: Under the assumptions of the Exercise 4.8, the variance condition (4.14) and the asymptotic normality (4.12) together imply the Lindeberg condition (4.13).

(a) Define

$$a_{ni} = \phi_{Y_{ni}}(t/s_n) - 1.$$

Prove that

$$\max_{1\le i\le n} |a_{ni}| \le 2 \max_{1\le i\le n} P(|Y_{ni}| \ge \epsilon s_n) + 2\epsilon|t|$$

and thus

$$\max_{1\le i\le n} |A_{ni}| \to 0 \text{ as } n \to \infty.$$

Hint: Use the result of Exercise 1.43(a) to show that $|\exp\{it\} - 1| \le 2\min\{1, |t|\}$ for $t \in \mathbb{R}$. Then use Chebyshev's inequality along with condition (4.14).

(b) Use part (a) to prove that

$$\sum_{i=1}^{n} |\alpha_{ni}|^2 \to 0$$

as $n \to \infty$.

Hint: Use the result of Exercise 1.43(b) to show that $|\alpha_{ni}| \le t^2\sigma_{ni}^2/s_n^2$. Then write $|\alpha_{ni}|^2 \le |\alpha_{ni} \max_i |\alpha_{ni}|$.

(c) Prove that for n large enough so that $\max_i |\alpha_{ni}| \le 1/2$,

$$\left| \prod_{i=1}^{n} \exp(\alpha_{ni}) - \prod_{i=1}^{n} (1 + \alpha_{ni}) \right| \le \sum_{i=1}^{n} |\alpha_{ni}|^2.$$

Hint: Use the fact that $|\exp(z-1)| = \exp(\text{Re } z - 1) \le 1$ for $|z| \le 1$ to argue that Inequality (4.19) applies. Also use the fact that $|\exp(z) - 1 - z| \le |z|^2$ for $|z| \le 1/2$.

(d) From part (c) and condition (4.12), conclude that

$$\sum_{i=1}^{n} \text{Re } (\alpha_{ni}) \to -\frac{1}{2} t^2.$$

(e) Show that

$$0 \le \sum_{i=1}^{n} \text{E} \left(\cos \frac{tY_{ni}}{s_n} - 1 + \frac{t^2 Y_{ni}^2}{2s_n^2} \right) \to 0.$$

(f) For arbitrary $\epsilon > 0$, choose t large enough so that $t^2/2 > 2/\epsilon^2$. Show that

$$\left(\frac{t^2}{2} - \frac{2}{\epsilon^2} \right) \frac{1}{s_n^2} \sum_{i=1}^{n} \text{E}(Y_{ni}^2 I\{|Y_{ni}| \ge \epsilon s_n\}) \le \sum_{i=1}^{n} \text{E} \left(\cos \frac{tY_{ni}}{s_n} - 1 + \frac{t^2 Y_{ni}^2}{2s_n^2} \right),$$

which completes the proof.

Hint: Bound the expression in part (e) below by using the fact that -1 is a lower bound for $\cos x$. Also note that $|Y_{ni}| \ge \epsilon s_n$ implies $-2 \ge -2Y_{ni}^2/(\epsilon^2 s_n^2)$.

Exercise 4.12

(a) Suppose that $X_1, X_2, ...$ are independent and identically distributed with $\text{E}X_i = \mu$ and $0 < \text{Var}X_i = \sigma^2 < \infty$. Let $a_{n1}, ..., a_{nn}$ be constants satisfying

$$\frac{\max_{i \le n} a_{ni}^2}{\sum_{j=1}^{n} a_{nj}^2} \to 0 \text{ as } n \to \infty.$$

Let $T_n = \sum_{i=1}^{n} a_{ni} X_i$, and prove that $(T_n - \text{E}T_n)/\sqrt{\text{Var}T_n} \xrightarrow{d} N(0,1)$.

Let $\epsilon > 0$ and $m_n = \max_{1 \le i \le n} a_{ni}^2$. Then by Lindeberg condition, we know

$$I\{|a_{ni}(X_i - \mu)| \ge \epsilon s_n\} \le I\{m_n (X_i - \mu)^2 \ge \epsilon^2 s_n^2\}$$

where $s_n^2 = \sum_{i=1}^{n} \text{Var}(a_{ni} X_i)$. Then let

$$Y_i = (X_i - \mu)^2 I\{m_n (X_i - \mu)^2 \ge \epsilon^2 s_n^2\}, \text{ for } 1 \le i \le n$$

where Y_i are iid. Then by the DCT, $\text{E}Y_2 \to 0$. Thus,

$$\frac{1}{s_n^2} \sum_{i=1}^{n} \text{E}[a_{ni}(X_i - \mu)^2 I\{|a_{ni}(X_i - \mu)| \ge \epsilon s_n\}] \le \frac{1}{s_n^2} \sum_{1}^{n} a_{ni}^2 \text{E}Y_i = \frac{\text{E}Y_1}{\sigma^2} \to 0.$$

(b) Reconsider Example 2.22, the simple linear regression case in which

$$\hat{\beta}_{0n} = \sum_{i=1}^{n} v_i^{(n)} Y_i \text{ and } \hat{\beta}_{1n} = \sum_{i=1}^{n} w_i^{(n)} Y_i,$$

where

$$w_i^{(n)} = \frac{z_i - \bar{z}_n}{\sum_{j=1}^{n}(z_j - \bar{z}_n)^2} \text{ and } v_i^{(n)} = \frac{1}{n} - \bar{z}_n w_i^{(n)}$$

for constants $z_1, z_2, \ldots.$ Using part (a), state and prove sufficient conditions on the constants z_i that ensure the asymptotic normality of $\sqrt{n}(\hat{\beta}_{0n} - \beta_0)$ and $\sqrt{n}(\hat{]}beta_{1n} - \beta_1)$. You may assume the results of Example 2.22, where it was shown that $\mathrm{E}\hat{\beta}_{0n} = \beta_0$ and $\mathrm{E}\hat{\beta}_{1n} = \beta_1$

\begin{proof} We will proceed by showing (1) $\frac{(\hat{\beta}_{0n} - \beta_0)}{\sqrt{\mathrm{Var}(\hat{\beta}_{0n})}}$ and $\frac{(\hat{\beta}_{1n} - \beta_1)}{\sqrt{\mathrm{Var}(\hat{\beta}_{1n})}}$ are asymptotically normal and (2) $\sqrt{n\mathrm{Var}(\hat{\beta}_0 n)} \to c_1 > 0$ and $\sqrt{n\mathrm{Var}(\hat{\beta}_0 n)} \to c_2 > 0$.

Assume $\bar{z}_n^2 \to \mu_z$ and $\frac{1}{n}\sum_{j=1}^{n}(z_i - \bar{z}_n)^2 \to \sigma_z^2$. Then

$$n\mathrm{Var}(\hat{\beta}_{0n}) = \sigma^2 + \frac{\sigma^2 \bar{z}_n^2}{\frac{1}{n}\sum_{j=1}^{n}(z_i - \bar{z}_n)^2} \to \sigma^2 + \frac{\sigma^2 \mu_z^2}{\sigma_z^2} = c_1^2$$

and

$$n\mathrm{Var}(\hat{\beta}_{1n}) = \frac{\sigma^2}{\frac{1}{n}\sum_{j=1}^{n}(z_i - \bar{z}_n)^2} \to \frac{\sigma^2}{\sigma_z^2} = c_2^2.$$

Then using part a, we know

$$\frac{\max_{i \le n}(z_i - \bar{z}_n)^2}{\frac{1}{n}\sum_{j=1}^{n}(z_i - \bar{z}_n)^2} \to 0.$$

Thus, (a) is satisfied for $v_i^{(n)}$ and $w_i^{(n)}$ since each Y_i is shifted by ϵ_i iid.Therefore,

$$\frac{1}{\frac{1}{n}\sum_{j=1}^{n}(z_i - \bar{z}_n)^2} \to 0$$

and

$$\frac{\bar{z}_n^2}{\frac{1}{n}\sum_{j=1}^{n}(z_i - \bar{z}_n)^2} \to 0$$

. Thus, $\hat{\beta}_{0n}$ and $\hat{\beta}_{1n}$ are sufficient. \end{proof}

Exercise 4.13

Give an example (with proof) of a sequence of independent random variables $Z_1, Z_2, \ldots$ with $\mathrm{E}(Z_i) = 0, \mathrm{Var}(Z_i) = 1$ such that $\sqrt{n}(\bar{Z}_n)$ does not converge in distribution to $N(0, 1)$.

Let $Z_1, Z_2, \ldots$ be independent with

$$Z_n = \begin{cases} 2^n & \text{with probability } \frac{1}{2^{2n+1}} \\ 0 & \text{with probability } 1 - \frac{1}{n^{2n}} \\ -2^n & \text{with probability } \frac{1}{2^{2n+1}}. \end{cases}$$

Then

$$\mathrm{E}Z_i = 2^n \left(\frac{1}{2^{2n+1}} \right) + 0 \left(\frac{1}{2^{2n}} \right) - 2^n \left(\frac{1}{2^{2n+1}} \right) = 0$$

and

$$\mathrm{Var}(Z_i) = 2^{2n} \left(\frac{1}{2^{2n+1}} \right) + 0^2 \left(\frac{1}{2^{2n}} \right) + 2^{2n} \left(\frac{1}{2^{2n+1}} \right) = 1.$$

Consider $S_n \sum_{i=1}^n (Z_i - \bar{Z}_n)$ and $s_n^2 \sum_{i=1}^n \sigma_i^2$. Then $\max_{i \leq n} \frac{\sigma_i}{s_n} \to 0$ as $n \to \infty$. However,

$$P(S_n = 0) > P(X_i = 0 \forall i) > 1 - \sum_{i=1}^n \frac{1}{2^{2i}} = \frac{2}{3} \neq 0.$$

Thus, $\sqrt{n}\bar{Z}_n \overset{d}{\nrightarrow} N(0,1)$.

Exercise 4.15

Suppose that X_1, X_2, X_3 is a sample of size 3 from a beta $(2,1)$ distribution.

(a) Find $P(X_1 + X_2 + X_3 \leq 1)$ exactly.

Then $f(x_i) = 2x_i^{2-1}(1-x_i)^{1-1} = 2x_i$ for $x \in (0,1)$. Therefore, $f(x_1, x_2, x_3) = 8x_1x_2x_3$. Thus,

$$\begin{aligned} P(X_1 + X_2 + X_3 \leq 1) &= \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} 8x_1x_2x_3 dx_3 dx_2 dx_1 \\ &= \int_0^1 \int_0^{1-x_1} 4x_1x_2(1-x_1-x_2)^2 dx_2 dx_1 \\ &= \frac{1}{3} \int_0^1 (x_1 - 4x_1 + 6x_1^3 - 4x_1^4 + x_1^5) dx_1 \\ &= \frac{1}{90} \\ &= 0.0111. \end{aligned}$$

(b) Find $P(X_1 + X_2 + X_3 \leq 1)$ using a normal approximation derived from the central limit theorem. Consider $\mathrm{E}X_i = \frac{2}{3}$ and $\mathrm{Var}X_i = \frac{1}{18}$. Therefore, using the central limit theorem, $X_1 + X_2 + X_3 \approx N(2, \frac{1}{6})$. Thus,

$$P(X_1 + X_2 + X_3 \leq 1) \approx P\left(Z \leq \frac{1-2}{\sqrt{1/6}} \right) = 0.00715.$$

(c) Let $Z = I\{X_1 + X_2 + X_3 \leq 1\}$. Approximate $\mathrm{E}Z = P(X_1 + X_2 + X_3 \leq 1)$ by $\bar{Z} = \sum_{i=1}^{1000} Z_i/1000$, where $Z_i = I\{X_{i1} + X_{i2} + X_{i3} \leq 1\}$ and the X_{ij} are independent beta $(2,1)$ random variables. In addition to $\bar{Z}$, report $\mathrm{Var}Z$ for your sample. (To think about: What is the theoretical value of $\mathrm{Var}Z$?)

Then $Z \sim \text{Bern}(1/90)$. Thus, the theroretical variance is $\frac{1}{90}\left(1-\frac{1}{90}\right)=0.011$. Simulating 1000 random variables, I got $\hat{p}=0.011$, therefore $V\hat{ar}(\hat{Z})=0.010879$.

```
set.seed(56156)
p <- sum(apply(matrix(rbeta(3000, 2, 1), nrow = 1000),
    1, sum) < 1)/1000
p <- as.numeric(p)
p * (1 - p)
```

```
## [1] 0.010879
```

(d) Approximate $P\left(X_1+X_2+X_3 \leq \frac{3}{2}\right)$ using the normal approximation and the simulation approach. (Don't compute the exact value, which is more difficult to do than in part (a); do you see why?) With the normal approximation, we see that

$$P\left(X_1+X_2+X_3 \leq \frac{3}{2}\right) = P(Z \leq \frac{3/2-2}{\sqrt{1/6}} = 0.1103.$$

Simulating 1000 random variables, I got 0.111616. The triple integral is much more complex in this case.

```
set.seed(56156)
p <- sum(apply(matrix(rbeta(3000, 2, 1), nrow = 1000),
    1, sum) < 3/2)/1000
p <- as.numeric(p)
p * (1 - p)
```

```
## [1] 0.111616
```

Exercise 4.16

Lindeberg and Lyapunov impose sufficient conditions on moments so that asymptotic normality occurs. However, these conditions are not necessary; it is possible to have asymptotic nomrality even if there are no moments at all. Let X_n assume the values $+1$ and -1 with probability $(1-2^{-n})/2$ each and the value 2^k with probability 2^{-k} with probability 2^{-k} for $k > n$.

(a) Show that $\text{E}(X_n^j)=\infty$ for all positive integers j and n.

Then

$$X_a = \begin{cases} 1 & \text{with probability } \frac{1-2^{-a}}{2} \\ -1 & \text{with probability } \frac{1-2^{-a}}{2} \\ 2^k & \text{with probability } \frac{1}{2^k} \text{ for } k=a+1, a+2, ... \end{cases}$$

Consider j odd. Then Then

$$\text{E}(X_n^j) = \sum_{k=a+1}^{\infty} 2^{kj}\left(\frac{1}{2^k}\right) = \sum_{k=a+1}^{\infty} 2^{k(j-1)} = \infty.$$

Consider j even. Then

$$\mathrm{E}(X_n^j) = 1 - 2^{-a} + \sum_{k=a+1}^{\infty} 2^{k(j-1)} = \infty.$$

Therefore, $\mathrm{E}(X_n^j) = \rightarrow infty$.

(b) Show that $\sqrt{(n)}(\bar{X}_n) \xrightarrow{d} N(0,1)$.

Consider the characteristic function of X_a,

$$\begin{aligned}
\mathrm{E}\left[e^{itX_a}\right] &= \cos(t)\left(\frac{1-2^{-a}}{2}\right) + i\sin(t)\left(\frac{1-2^{-a}}{2}\right) \\
&+ \cos(t)\left(\frac{1-2^{-a}}{2}\right) - i\sin(t)\left(\frac{1-2^{-a}}{2}\right) \\
&+ \sum_{k=a+1}^{\infty} \cos(t2^k)\left(\frac{1}{2^k}\right) + i\sum_{k=a+1}^{\infty} \sin(t2^k)\left(\frac{1}{2^k}\right) \\
&= \cos(t)(1-2^{-a}) + \sum_{k=a+1}^{\infty}\left[\left(\cos(t2^k) + isin(t2^k)\right)\left(\frac{1}{2^k}\right)\right].
\end{aligned}$$

Then consider $\sqrt{n}\bar{X}_n = \frac{1}{\sqrt{n}}\sum_{a=1}^{n} X_a$. Then,

$$\begin{aligned}
\mathrm{E}\left[e^{i\frac{t}{\sqrt{n}}\sum_{a=1}^{n} X_a}\right] &= \prod_{a=1}^{n} \mathrm{E}\left[e^{i\frac{t}{\sqrt{n}}X_a}\right] \\
&= \prod a = 1^n \left[\cos(\frac{t}{\sqrt{n}})(1-2^{-a}) + \sum_{k=a+1}^{\infty}\left[\left(\cos(\frac{t}{\sqrt{n}}2^k) + isin(\frac{t}{\sqrt{n}}2^k\right)\left(\frac{1}{2^k}\right)\right]\right] \\
&= \prod a = 1^n \left[\cos(\frac{t}{\sqrt{n}})(1-2^{-a})\right] \text{ since cos and sin bounded by 1} \\
&= \prod a = 1^n (1-2^{-a})(1 - \frac{t^2}{2!n} + \frac{t^4}{4!n} - \frac{t^6}{6!n} + ...) \\
&\rightarrow e^{-t^2/2} \text{ for all } t \in \mathbb{R}.
\end{aligned}$$

Therefore, $\sqrt{n}\bar{X}_n \xrightarrow{d} N(0,1)$.

Exercise 4.18

Suppose that $X_1, X_2, ...$ are independent binomial$(2,p)$ random variables. Define $Y_i = I\{X_i = 0\}$.

(a) Find $\boldsymbol{a}$ such that the joint asymptotic distribution of

$$\sqrt{n}\left[\begin{pmatrix}\bar{X}_n \\ \bar{Y}_n\end{pmatrix} - \boldsymbol{a}\right]$$

is nontirivial, and find this joint asymptotic distribution.

Consider $X_1, X_2, ...$ are independent binomial$(2,p)$ and $Y_i = I\{X_i = 0\} \sim$ Bern $\left((1-p)^2\right)$ .Then from the Central Limit Theorem, we know

$$\sqrt{n}(\bar{X}_n - 2p) \sim N(0, 2p(1-p))$$

and

$$\sqrt{n}(\bar{Y}_n - (1-p)^2) \sim N\left(0, (1-p)^2[1-(1-p)^2]\right).$$

Therefore,

$$\boldsymbol{a} = \begin{pmatrix} 2p \\ (1-p)^2 \end{pmatrix}$$

and the joint asymptotic distribution is $N\left(\mathbf{0}, \Sigma\right)$ with

$$\Sigma = \begin{pmatrix} 2p(1-p) & -2p(1-p)^2 \\ -2p(1-p)^2 & (1-p)^2(1-(1-p)^2) \end{pmatrix}.$$

(b) Using the Cramer-Wold Theorem, Theorem 4.12, find the asymptotic distribution of $sqrtn(\bar{X}_n + \bar{Y}_n - 1 - p^2)$.

Then by Cramer-Wold Theorem, let $A = (11)$. Then

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \left[\begin{pmatrix} \bar{X}_n \\ \bar{Y}_n \end{pmatrix} - \begin{pmatrix} 2p \\ (1-p)^2 \end{pmatrix}\right] = \bar{X}_n + \bar{Y}_n - 1 - p^2 \sim N\left(\mathbf{0}, \Sigma\right)$$

where

$$\Sigma = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 2p(1-p) & -2p(1-p)^2 \\ -2p(1-p)^2 & (1-p)^2(1-(1-p)^2) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$