# Homework 1

STAT 984

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**Exercise 1.1**

Assume that $a_n \to a$ and $b_n \to b$, where $a$ and $b$ are real numbers.

a. Prove that $a_n b_n \to ab$.

*Proof.* Let $a_n$ and $b_n$ be sequences and suppose $a_n \to a$ and $b_n \to b$ for real numbers $a$ and $b$. Then $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are bounded.Thus, there exists an $M > 0$ such that $|a_n| \leq M$ for all $n \geq 1$. Consider two cases: $b = 0$ and $b \neq 0$.

Case 1: Let $b = 0$. By definition of convergence (Definition 1.1), there exists some $N_1 > 0$ such that for all $n \geq N_1$, $|b_n - b| < \frac{\epsilon}{M}$. Then

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - a_n b + a_b - ab| \\
&= |a_n(b_n - b) + b(a_n - a)| \\
&\leq |a_n(b_n - b)| + |b(a_n - a)| \\
&= |a_n||b_n - b| + |b||a_n - a| \\
&< M\left(\frac{\epsilon}{M}\right) + 0 \\
&= \epsilon.
\end{aligned}$$

Case 2: Let $b \neq 0$. Then there exists $N_2 > 0$ such that for all $n \geq N_2$, $|b_n - b| < \frac{\epsilon}{2M}$ and there exists $N_3 > 0$ such that for all $n \geq N_3$, $|a_n - a| < \frac{\epsilon}{2|b|}$. Then

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - a_n b + a_b - ab| \\
&= |a_n(b_n - b) + b(a_n - a)| \\
&\leq |a_n(b_n - b)| + |b(a_n - a)| \\
&= |a_n||b_n - b| + |b||a_n - a| \\
&< M\left(\frac{\epsilon}{2M}\right) + |b|\left(\frac{\epsilon}{2|b|}\right) \\
&= \epsilon.
\end{aligned}$$

Therefore, for all $\epsilon > 0$, there exists $N = max\{N_1, N_2, N_3\}$, such that for all $n \geq N$, $|a_n b_n - ab| < \epsilon$. Thus, $a_n b_n \to ab$. □

b. Prove that if $b \neq 0$, $a_n/b_n \to a/b$.

*Proof.* Let $a_n$ and $b_n$ be sequences and suppose $a_n \to a$ and $b_n \to b$ for $b \neq 0$. Let $\epsilon > 0$. Consider $\epsilon_1 = \frac{\epsilon}{2}$. There exists an $N_1$ such that for all $n \geq N_1$, $|b_n - b| \leq \frac{|b|}{2}$. Therefore, $|b_n| > \frac{|b|}{2}$ which implies $\frac{1}{|b_n|} < \frac{1}{|b|/2}$. Now consider $\epsilon_2 = \frac{|b|^2\epsilon}{2}$. There exists $N_2 > 0$ such that for all $n > N_2$, $|b_n - b| < \frac{|b|^2\epsilon}{2}$. Let $N = max\{N_1, N_2\}$. Then for all $n \geq N$,

$$\begin{aligned}
\left|\frac{1}{b_n} - \frac{1}{b}\right| &= \left|\frac{b - b_n}{b_n b}\right| \\
&= \left|(b - b_n)\left(\frac{1}{b_n b}\right)\right| \\
&= |b - b_n|\left|\frac{1}{b_n b}\right| \\
&= |b_n - b|\frac{1}{|b_n||b|} \\
&< \frac{|b|^2\epsilon}{2}\frac{1}{|b|^2/2} \\
&= \epsilon.
\end{aligned}$$

Thus, $\frac{1}{b_n} \to \frac{1}{b}$. Then by part (a),

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|a_n\left(\frac{1}{b_n}\right) - a\left(\frac{1}{b}\right)\right| \to a\frac{1}{b} = \frac{a}{b}$$

Thus, if $b \neq 0$, $a_n/b_n \to a/b$. ☐

### Exercise 1.2

For a fixed real number c, define $a_n(c) = (1 + c/n)^n$. Then Equation (1.9) states that $a_n(c) \to \exp(c)$. A different sequence with the same limit is obtained from the power series expansion of exp(c):

$$b_n(c) = \sum_{i=0}^{n-1} \frac{c^i}{i!}$$

For each of the values $c \in -10, -1, 0.2, 1, 5$, find the smallest value of $n$ such that

$$|a_n(c) - \exp(c)|/\exp(c) < .01.$$

Now replace $a_n(c)$ by $b_n(c)$ and repeat. Comment on any general differences you observe between the two sequences.

The sequence $a_n(c)$, meets the convergence criteria at $n = \{4982, 51, 2, 50, 1241\}$ for respective values of $c$.

```r
ex_1.2_an <- function(c, epsilon = 0.01, maxIter) {
    for (n in 1:maxIter) {
        value <- (1 + c/n)^n
        Conv <- abs(value - exp(c))/exp(c)
        if (Conv < epsilon)
            break
    }
    return(list(c = c, value = value, Convergence = Conv <
        epsilon, n = n))
}

resultsFunc <- function(f, c_seq) {
    results <- matrix(NA, length(c_seq), 4)
    colnames(results) <- c("c", "value", "Convergence",
        "n")

    for (k in 1:length(c_seq)) {
        c_results <- f(c = c_seq[k], epsilon = 0.01,
            maxIter = 10000)
        results[k, "c"] <- c_results$c
        results[k, "value"] <- round(c_results$value,
            5)
        results[k, "Convergence"] <- as.character(c_results$Convergence)
        results[k, "n"] <- c_results$n
    }

    as.data.frame(results)
    return(results)
}

c_seq <- c(-10, -1, 0.2, 1, 5)
an_results <- resultsFunc(ex_1.2_an, c_seq)
kable(an_results)
```

| c | value | Convergence | n |
|---|---|---|---|
| -10 | 4e-05 | TRUE | 4982 |
| -1 | 0.36424 | TRUE | 51 |
| 0.2 | 1.21 | TRUE | 2 |
| 1 | 2.69159 | TRUE | 50 |
| 5 | 146.92973 | TRUE | 1241 |

The sequence $b_n(c)$, meets the convergence criteria at $n = \{38, 6, 3, 5, 12\}$ for respective values of $c$.

```
ex_1.2_bn <- function(c, epsilon = 0.01, maxIter) {
    for (n in 1:maxIter) {
        i <- seq(0, n - 1, 1)
        value <- sum(c^i/factorial(i))
        Conv <- abs(value - exp(c))/exp(c)
        if (Conv < epsilon)
            break
    }
    return(list(c = c, value = value, Convergence = Conv <
        epsilon, n = n))
}

bn_results <- resultsFunc(ex_1.2_bn, c_seq)
kable(bn_results)
```

| c | value | Convergence | n |
|---|---|---|---|
| -10 | 5e-05 | TRUE | 38 |
| -1 | 0.36667 | TRUE | 6 |
| 0.2 | 1.22 | TRUE | 3 |
| 1 | 2.70833 | TRUE | 5 |
| 5 | 147.60385 | TRUE | 12 |

**Exercise 1.3**

a. Suppose that $a_k \to c$ as $k \to \infty$ for a sequence of real numbers $a_1, a_2, \ldots$. Prove that this implies convergence in the sense of Cesaro, which means that

$$\frac{1}{n}\sum_{k=1}^{n} a_k \to c \text{ as } n \to \infty. \tag{1.3}$$

In this case, $c$ may be real or it may be $\pm\infty$.

*Proof.* Suppose that $a_k \to c$ as $k \to \infty$ for a sequence of real numbers $a_1, a_2, \ldots$. Let $\epsilon > 0$. Consider three cases: $a_k \to c$, $a_k \to \infty$, and $a_k \to -\infty$.

Case 1: Consider $a_k \to c$ where $c \in \mathbb{R}$. Then there exists an $N > 0$ such that for all $k > N$,

$|a_k - c| < \epsilon$. Then

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=1}^{n}(a_n) - c\right| = \left|\frac{1}{n}\left(\sum_{k=1}^{n}(a_k) - nc\right)\right| & \leq \frac{1}{n}\sum_{k=1}^{n}|a_k - c| \\
&= \frac{1}{n}\sum_{k=1}^{N}|a_k - c| + \frac{1}{n}\sum_{k=N+1}^{n}|a_k - c| \qquad \text{(first term} \to 0 \text{ since finite sum)} \\
&< \frac{1}{n}\sum_{k=N+1}^{n}\epsilon \\
&= \frac{n-N}{n}\epsilon \\
&< \epsilon \qquad \left(\text{since} \frac{n-N}{n} < 1\right).
\end{aligned}$$

Thus, $\frac{1}{n}\sum_{k=1}^{n} a_k \to c$ for $c \in \mathbb{R}$.

Case 2: Consider $a_k \to \infty$. Then for all $M > 0$, there exists an $N > 0$ such that $a_n > 2M$ if $n \geq N$. Then

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} a_k &= \frac{1}{n}\sum_{k=1}^{N} a_k + \frac{1}{n}\sum_{k=N+1}^{n} a_k \qquad \text{(first term} \to 0 \text{ since finite sum)} \\
&= \frac{1}{n}\sum_{k=N+1}^{n} a_k \\
&= \frac{1}{n}(a_{N+1} + a_{N+2} + ...) \\
&> \frac{1}{n}(2M + 2M + ...) \\
&= \frac{2(n-N)}{n} \cdot M \\
&> M \text{ if } \frac{n-N}{n} > 0.5 \qquad \text{(i.e. for large enough } n; n > 2N).
\end{aligned}$$

Thus, $\frac{1}{n}\sum_{k=1}^{\infty} a_n \to \infty$.

Case 3: Consider $a_k \to -\infty$. A similar argument follows as to Case 2 with $a_n < -2M$.

Thus, in all three cases, we have shown that

$$\frac{1}{n}\sum_{k=1}^{n} a_k \to c \text{ as } n \to \infty.$$

□

b. Is the converse true? In other words, does (1.3) imply $a_k \to c$?

No, the converse is not true. Consider $a_k = (-1)^{k-1}$. Then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1} = \lim_{n\to\infty}\{1/1, 1/2, 2/3, 2/4, 3/5, 3/6, ...\} = 1/2.$$

However, $a_k$ oscillates between 1 and -1, thus $a_k$ is divergent even though the Cesaro converges to 1/2.

**Exercise 1.5**

Let $a_n = \sin n$ for $n = 1, 2, ....$

a. What is $\sup_n a_n$? Does $\max_n a_n$ exist?

The $\sup_n(a_n) = 1$ and $\max_n a_n$ does not exit.

b. What is the set of limit points of $\{a_1, a_2, ...\}$? What are $\limsup_n a_n$ and $\liminf_n a_n$? (Recall that a limit point is any point that is the limit of a subsequence $a_{k_1}, a_{k_2}, ...$, where $k_1 < k_2 < \cdots$,)

The set of limit points is $\{\sin(1), \sin(2), ...\}$ while the $\limsup_n a_n = 1$ and the $\liminf_n a_n = -1$.

c. As usual in mathematics, we assume above that angles are measured in radians. How do the answers to (a) and (b) chnge if we use degrees instead (i.e., $a_n = \sin n°$)?

The limit points will change to $\{\sin(1°), \sin(2°), ..., \sin(90°), \sin(270°), ..., \sin(360°)\}$ and the $max_n a_n$ will exist at $n = 90$ and $n = 270$. The $\sup(a_n)$, $\limsup_n a_n$, and $\liminf_n a_n$ do not change.

**Exercise 1.8**

Define $F(t)$ as in Example 1.15 (and as pictured in Figure 1.1). This function is not continuous so Theorem 1.16 does not apply. That is, $a_n \to a$ does not imply that $F(a_n) \to F(a)$.

a. Give an example of a sequence $\{a_n\}$ and a real number $a$ such that $a_n \to a$ but $\limsup_n F(a_n) \neq F(a)$.

Let $a_n = -\frac{1}{n}$. Then $a_n$ is an increasing sequence and $a_n \to 0$. The limit points of $F(a_n)$ are $\{0\}$. Thus, $\limsup_n F(a_n) = 0 \neq \frac{1}{2} = F(0)$.

b. Change your answer to part (a) so that $a_n \to a$ and $\limsup_n F(a_n) = F(a)$, but $\lim_n F(a_n)$ does not exist.

Let $a_n = 1+(-1)^n \frac{1}{n}$. Then $a_n$ jumps below and above 1 until $a_n \to 1$. The limit points of $F(a_n)$ are $\{\frac{1}{2}, 1\}$. Thus, $\limsup_n F(a_n) = 1 = F(1)$ and $\liminf_n F(a_n) = \frac{1}{2}$. Then $lim inf$_n F(a_n) \neq \limsup_n F(a_n)$. Thus, $\lim_n F(a_n)$ does not exist.

c. Explain why it is not possible to change your answer so that $a_n \to a$ and $\liminf_n F(a_n) = F(a)$, but $\lim_n F(a_n)$ does not exist.

It is not possible to select a sequence $a_n$ such that $a_n \to a$ and $\liminf_n F(a_n) = F(a)$, but $\lim_n F(a_n)$ does not exist since $F(x)$ is only right continuous at $x = 0$ and $x = 1$. Therefore, if $a_n$ is a decreasing sequence which converges to either 0 or 1, $\limsup_n F(a_n) = \liminf_n F(a_n)$, and thus, $\lim_n a_n$ exists. However, if we select $a_n$ similar to part (b) where $a_n$ oscillates above and below 0 or 1 so that $\lim_n F(a_n)$ does not exist, when $a_n$ converges to 0 or 1, $\liminf_n F(a_n) < F(a)$.

**Exercise 1.14**

The gamma function $\Gamma(x)$ is defined for positive real $x$ as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

[in fact, equation (1.14) is also valid for complex $x$ with positive real part]. The gamma function may be viewed as a continuous version of the factorial function in the sense that $\Gamma(n) = (n-1)!$ for all positive integers n. The gamma function satisfies the identity

$$\Gamma(x+1) = x\Gamma(x)$$

even for noninteger positive values of $x$. Since $\Gamma(x)$ grows very quickly as $x$ increases, it is often convenient in numerical calculations to deal with the logarithm of the gamma function, which we term the log-gamma function. The *digamma function* $\Psi(x)$ is defined to be the derivative of the log-gamma function; this function often arises in statistical calculations involving certain distributions that use the gamma function.

a. Apply the result of Exercise 1.13(b) using $h = 1$ to demonstrate how to obtain the approximation

$$\Psi(x) \approx \frac{1}{2} log[x(x-1)]$$

for $x > 2$.

Hint: Use Identity (1.15).

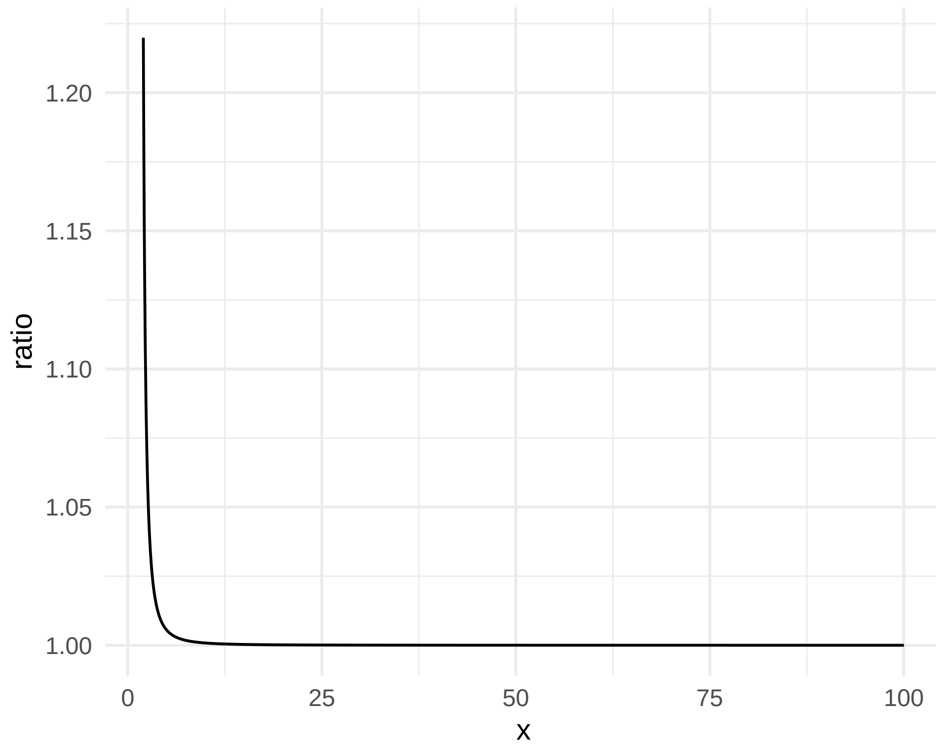
The result from Exercise 1.13 states $f'(a) \approx \frac{f(a+x)-f(a-h)}{2h}$. Then

$$\begin{aligned}
\Psi(x) &= \frac{d}{dx} log\Gamma(x) \\
&\approx \frac{1}{2}\left(\log(\Gamma(x+1) - \log\Gamma(x-1)\right) && \text{(Exercise 1.13(b))} \\
&= \frac{1}{2}\left(\log(x\Gamma(x) - \log\Gamma(x)/(x-1)\right) && \text{(from 1.15)} \\
&= \frac{1}{2}\left(\log(x) + \log\Gamma(x) - \log\Gamma(x) + \log(x-1)\right) \\
&= \frac{1}{2}\left(\log(x(x-1))\right). && (1.16)
\end{aligned}$$

b. Test Approximation (1.16) numerically for all $x$ in the interval (2, 100) by plotting the ratio of the approximation to the true $\Psi(x)$. What do you notice about the quality of the approximation? If you are using R or Splus, then `digamma(x)` gives the value of $\Psi(x)$.

The approximation does not appear to perform well for $x$ close to 2. However, the approximation performs better for larger values of $x$.

```r
x <- seq(from = 2, to = 100, by = 0.01)
ratio <- 2 * digamma(x)/log(x * (x - 1))
data_1.14b <- as.data.frame(cbind(x, ratio))
library(ggplot2)
ggplot(data = data_1.14b, aes(x = x, y = ratio)) +
    geom_line() + theme_minimal()
```




**Exercise 1.15**

The second derivative of the log-gamma function is called the trigamma function:

$$\Psi'(x) = \frac{d^2}{dx^2} \log \Gamma(x).$$

Like the digamma function, it often arises in statistical calculations; for example, see Exercise 1.35.

a. Using the method of Exercise 1.13(c) with $h = 1$ [that is, expanding $f(x + 2h), f(x + h), f(x - h)$, and $f(x - 2h)$ and then finding a linear combination that makes all but the *second* derivative of the log-gamma function disappear], show how to derive the following approximation to $\Psi'(x)$ for $x > 2$ :

$$\Psi'(x) \approx \frac{1}{12} \log \left[ \left( \frac{x}{x-1} \right)^{15} \left( \frac{x-2}{x+1} \right) \right]$$

Let $f(x) = \log \Gamma(x)$. Then $f'(x) = \Psi(x) = \frac{d}{dx} \log \Gamma(x)$ and $f''(x) = \Psi'(x) = \frac{d^2}{dx^2} \log \Gamma(x)$. Using Taylor Series to expand:

$$
\begin{aligned}
f(x+1) &\approx f(x) + f'(x) + \frac{1}{2}f''(x) + \frac{1}{6}f'''(x) + \frac{1}{24}f''''(x) \\
f(x) &\approx f(x) \\
f(x-1) &\approx f(x) - f'(x) + \frac{1}{2}f''(x) - \frac{1}{6}f'''(x) + \frac{1}{24}f''''(x)
\end{aligned}
$$

$$
\begin{aligned}
f(x+2) &\approx f(x) + f'(x) + 2f''(x) + \frac{8}{6}f'''(x) + \frac{16}{24}f''''(x) \\
f(x) &\approx f(x) \\
f(x-2) &\approx f(x) - f'(x) + 2f''(x) - \frac{8}{6}f'''(x) + \frac{16}{24}f''''(x)
\end{aligned}
$$

Then, $f(x+1) - 2f(x) + f(x-1) = f''(x) + \frac{1}{12}f''''(x)$ and $f(x+2) - 2f(x) + f(x-2) = 4f''(x) + \frac{16}{12}f''''(x)$. Therefore,

$$
f''(x) \approx \frac{16[f(x+1) - 2f(x) + f(x-1)] - [f(x+2) - 2f(x) + f(x-2)]}{12}.
$$

Then using equation 1.15,

$$
\begin{aligned}
\Gamma(x+1) &= x\Gamma(x) \\
\Gamma(x+2) &= (x+1)x\Gamma(x) \\
\Gamma(x-1) &= \frac{\Gamma(x)}{x-1} \\
\Gamma(x-2) &= \frac{\Gamma(x)}{(x-1)(x-2)}.
\end{aligned}
$$

Thus, plugging in $f(x) = \log\Gamma(x)$ and using the above identities, we obtain,

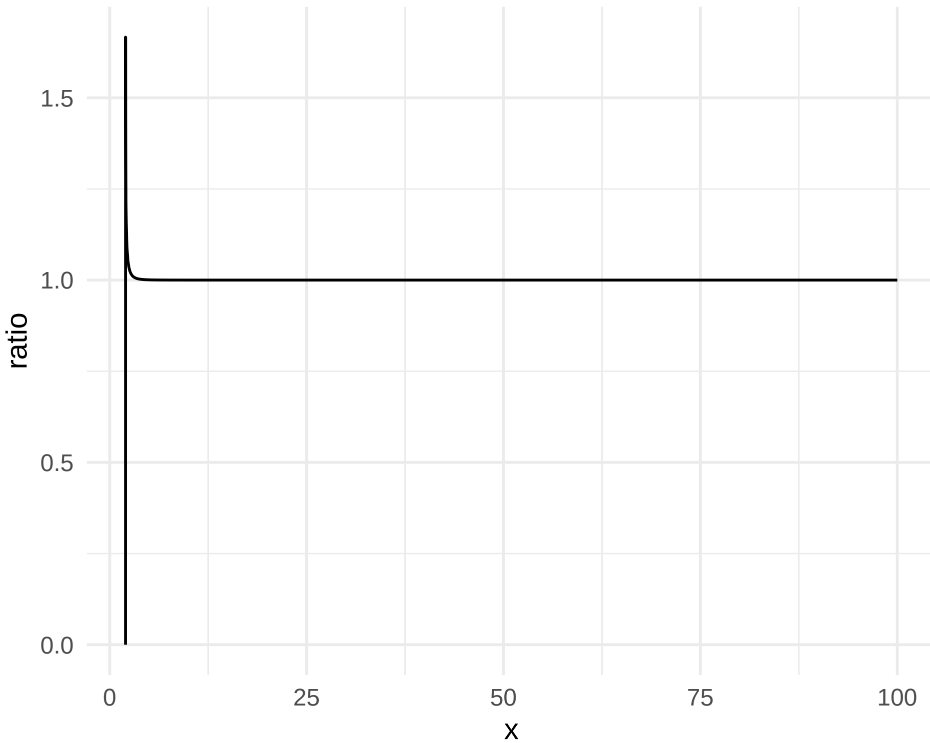
$$
\Psi'(x) \approx \frac{1}{12}\log\left(\left(\frac{x}{x-1}\right)^{15}\left(\frac{x-2}{x+1}\right)\right).
$$

b. Test Approximation 1.18 numerically as in Exercise 1.14(b). In R or Splus, `trigamma(x)` gives the value of $\Psi'(x)$

Similar to 1.14(b), the approximation does not appear to perform well for $x$ close to 2. However, the approximation performs better for larger values of $x$.

```
x <- seq(from = 2, to = 100, by = 0.01)
ratio <- 12 * trigamma(x)/log((x/(x - 1))^{
    15
} * ((x - 2)/(x + 1)))
data_1.15b <- as.data.frame(cbind(x, ratio))
```

```r
library(ggplot2)
ggplot(data = data_1.15b, aes(x = x, y = ratio)) +
    geom_line() + theme_minimal()
```




**Exercise 1.18**

Suppose that $a_n \sim b_n$ and $c_n \sim d_n$.

a. Prove that $a_n c_n \sim b_n d_n$.

*Proof.* Since $a_n \sim b_n$ and $c_n \sim d_n$, then $a_n/b_n \to 1$ and $c_n/d_n \to 1$. Then

$$\frac{a_n c_n}{b_n d_n} = \frac{a_n/b_n}{c_n/d_n} \to \frac{1}{1} = 1.$$

Therefore, $a_n c_n \sim b_n d_n$. $\square$

b. Show by counterexample that it is not generally true that $a_n + c_n \sim b_n + d_n$.

Let $a_n = n, b_n = n+1, c_n = -n$, and $d_n = -n$. Then $\frac{a_n}{b_n} = \frac{n}{n+1} \to 1$ and $\frac{c_n}{d_n} = \frac{-n}{-n} \to 1$. Therefore, $a_n \sim b_n$ and $c_n \sim d_n$. However,

$$\frac{a_n + c_n}{b_n + d_n} = \frac{n - n}{n + 1 - n} = 0/1 \to 0.$$

Thus, $a_n + c_n \sim b_n + d_n$ does not hold.

c. Prove that $|a_n| + |c_n| \sim |b_n| + |d_n|$.

*Proof.* Since $a_n \sim b_n$ and $c_n \sim d_n$, $\left|\frac{a_n - b_n}{a_n}\right| \to 0$ and $\left|\frac{c_n - d_n}{c_n}\right| \to 0$. Then

$$\begin{aligned}
\left|\frac{(|a_n| + |c_n|) - (|b_n| + |d_n|)}{|a_n| + |c_n|}\right| &= \left|\frac{(|a_n| - |b_n|) + (|c_n| - |d_n|)}{|a_n| + |c_n|}\right| \\
&= \left|\left(\frac{|a_n|}{|a_n| + |c_n|}\right)\left(\frac{|a_n| - |b_n|}{|a_n|}\right) + \left(\frac{|c_n|}{|a_n| + |c_n|}\right)\left(\frac{|c_n| - |d_n|}{|c_n|}\right)\right| \\
&\to C_1 \cdot 0 + C_2 \cdot 0 \\
&= 0.
\end{aligned}$$

Thus, since

$$\left|\frac{(|a_n| + |c_n|) - (|b_n| + |d_n|)}{|a_n| + |c_n|}\right| \to 0,$$

$|a_n| + |c_n| \sim |b_n| + |d_n|$. □

d. Show by counterexample that it is not generally true that $f(a_n) \sim f(b_n)$ for a continuous function $f(x)$.

Let $a_n = n^2 + n$ and $b_n = n^2$. Consider $f(x) = e^x$. Then $\frac{a_n}{b_n} = \frac{n^2+n}{n^2} \to 1$, but $\frac{e^{n^2+n}}{e^{n^2}} = e^x \to \infty$. Therefore, $f(a_n) \sim f(b_n)$ does not hold.