Homework 3

STAT 984

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Exercise 1.38

Let $f(x)$ be a convex function on some interval, and let x_0 be any point on the interior of that interval.

(a) Prove that

$$\lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0} \tag{1.38}$$

exists and is finite, that is, a one-sided derivative exists at x_0.

Hint: Using Definition 1.30, show that the fraction in expression (1.38) is non-increasing and bounded below as x decreases to x_0.

Proof. Consider $f(x)$ on convex on $[a, b]$ where $a < x_0 < b$. Let there be $x_2 \in (x_0, b)$. Then $x_2 = \alpha x_0 + (1-\alpha)b$ and $f(x_2) = f(\alpha x_0 + (1-\alpha)b) \leq \alpha f(x_0) + (1-\alpha)f(b)$. Therefore,

$$\frac{f(x_2) - f(x_0)}{x_2 - x_0} \leq \frac{\alpha f(x_0) + (1-\alpha)f(b) - f(x_0)}{\alpha x_0 + (1-\alpha)b - x_0} = \frac{f(b) - f(x_0)}{b - x_0}.$$

Now let there be $x_1 \in (x_0, x_2)$. Then $x_1 = \alpha x_0 + (1-\alpha)x_2$ and $f(x_1) = f(\alpha x_0 + (1-\alpha)x_2) \leq \alpha f(x_0) + (1-\alpha)f(x_2)$. Therefore,

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \leq \frac{\alpha f(x_0) + (1-\alpha)f(x_2) - f(x_0)}{\alpha x_0 + (1-\alpha)x_2 - x_0} = \frac{f(x_2) - f(x_0)}{x_2 - x_0}.$$

Thus, for $x_0 < x_1 < x_2 < b$,

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \leq \frac{f(x_2) - f(x_0)}{x_2 - x_0} \leq \frac{f(b) - f(x_0)}{b - x_0}.$$

Therefore, $\frac{f(x) - f(x_0)}{x - x_0}$ is non-increasing as $x \to x_0+$ and f has a right handed derivative $f'(x_0+)$.

Now consider $x_0 \in (a, x_1)$. Then $x_0 = \alpha a + (1-\alpha)x_1$ and $f(x_0) = f(\alpha a + (1-\alpha)x_1) \leq \alpha f(a) + (1-\alpha)f(x_1)$. Therefore,

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \geq \frac{f(x_1) - f(a)}{x_1 - a}.$$

Therefore, $\frac{f(x)-f(x_0)}{x-x_0}$ is bounded below.

Thus, since

$$\frac{f(x_1)-f(a)}{x_1-a} \leq \frac{f(x_1)-f(x_0)}{x_1-x_0} \leq \frac{f(b)-f(x_0)}{b-x_0},$$

$\frac{f(x)-f(x_0)}{x-x_0}$ is finite and exists. □

(b) Prove that there exists a linear function $g(x) = ax + b$ such that $g(x_0) = f(x_0)$ and $g(x) \leq f(x)$ for all x in the interval. This fact is the supporting hyperplane property in the case of a convex function taking a real argument.

Hint: Let $f'(x_0+)$ denote the one-sided derivative of part (a). Consider the line $f(x_0) + f'(x_0+)(x - x_0)$.

Proof. Using part a, consider $g(x) = f(x_0) + f'(x_0+)(x - x_0)$. Then

$$g(x) = f(x_0) + f'(x_0+)(x - x_0) \leq f(x_0) + \frac{f(x)-f(x_0)}{x-x_0}(x - x_0) = f(x).$$

Therefore, $g(x) \leq f(x)$. Then

$$g(x_0) = f(x_0) + f'(x_0+)(x_0 - x_0) = f(x_0).$$

Therefore, $g(x_0) = f(x_0)$. □

Exercise 1.39

Prove Holder's inequality: For random variables X and Y and positive p and a such that $p + q = 1$,

$$E|XY| \leq (E|X|^{1/p})^p (E|Y|^{1/q})^q. \tag{1.39}$$

(If $p = q = 1/2$, inequality 1.39 is also called the Cauchy-Schwartz inequality.)

Hint: Use the convexity of $\exp(x)$ to prove that $|abXY| \leq p|aX|^{1/p} + q|bY|^{1/q}$ whenever $aX \neq 0$ and $bY \neq 0$ (the same inequality is also true if $aX = 0$ or $bY = 0$). Take expectations, then find values for the scalars a and b that give the desired result when the right side of inequality (1.39) is nonzero.

Proof. Consider $f(x) = e^x$ convex. Let $\alpha = p$, thus $1 - \alpha = q$. Then by the definition of convex,

$$f(\alpha A + (1-\alpha)B) = f(pA + qB) \leq pf(A) + qf(B).$$

Now let $A = \frac{\log|aX|}{p}$ and $B = \frac{\log|bY|}{q}$. Then

$$\begin{aligned}
& f\left(p\frac{\log|aX|}{p} + q\frac{\log|bY|}{q}\right) \leq pf\left(\frac{\log|aX|}{p}\right) + qf\left(\frac{\log|bY|}{q}\right) \\
\Longrightarrow \quad & f(\log|abXY|) \leq pe^{\log(|aX|^{1/p})} + qe^{\log|bY|^{1/q}} \\
\Longrightarrow \quad & e^{\log|abXY|} \leq p|aX|^{1/p} + q|bY|^{1/q} \\
\Longrightarrow \quad & |abXY| \leq p|aX|^{1/p} + q|bY|^{1/q}.
\end{aligned}$$

Now let $a = (E|X|^{1/p})^{-p}$ and $b = (E|Y|^{1/q})^{-q}$. Then

$$
\begin{aligned}
& & E|abXY| &\le E\left[p|aX|^{1/p} + q|bY|^{1/q}\right] \\
&\implies & |ab|E|XY| &\le p|a|^{1/p}E|X|^{1/p} + q|b|^{1/q}E|Y|^{1/q} \\
&\implies & (E|X|^{1/p})^{-p}(E|Y|^{1/q})^{-q}E|XY| &\le p\left[(E|X|^{1/p})^{-p}\right]^{1/p} E|X|^{1/p} + q\left[(E|Y|^{1/q})^{-q}\right]^{1/q} E|Y|^{1/q} \\
&\implies & \frac{E|XY|}{(E|X|^{1/p})^p(E|Y|^{1/q})^q} &\le p\frac{E|X|^{1/p}}{E|X|^{1/p}} + q\frac{E|Y|^{1/q}}{E|Y|^{1/q}} \\
&\implies & \frac{E|XY|}{(E|X|^{1/p})^p(E|Y|^{1/q})^q} &\le p + q \\
&\implies & \frac{E|XY|}{(E|X|^{1/p})^p(E|Y|^{1/q})^q} &\le 1 \\
&\implies & E|XY| &\le (E|X|^{1/p})^p(E|Y|^{1/q})^q.
\end{aligned}
$$

□

Exercise 1.40

Use Holder's Inequality (1.39) to prove that if $\alpha > 1$, then

$$(E|X|)^\alpha \le E|X|^\alpha.$$

Hint: Take Y to be a constant in Inequality (1.39).

Proof. Let $Y = c$, constant. Then by Holder's Inequality,

$$
\begin{aligned}
& & E|XY| &\le (E|X|^{1/p})^p(E|Y|^{1/q})^q \\
&\implies & |c|E|X| &\le (E|X|^{1/p})^p(|c|^{1/q})^q \\
&\implies & E|X| &\le (E|X|^{1/p})^p \\
&\implies & (E|X|)^{1/p} &\le E|X|^{1/p}.
\end{aligned}
$$

Then let $\alpha = 1/p$. Since $p + q = 1$. Then $p < 1$ implies $1/p > 1$. Thus, $\alpha > 1$. Therefore, $(E|X|)^\alpha \le E|X|^\alpha$. □

Exercise 1.45

For any nonnegative random variable Y with finite expectation, prove that

$$\sum_{i=1}^{\infty} P(Y \ge i) \le EY. \qquad (1.43)$$

Hint: First, prove that equality holds if Y is supported on the nonnegative integers. Then note for a general Y that $E\lfloor Y \rfloor \le EY$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Though we will not do so here, it is possible to prove a statement stronger than inequality (1.43) for nonnegative random variables, namely,

$$\int_0^\infty P(Y \geq t)dt = EY.$$

(This equation remains true if $EY = \infty$.) To sketch a proof, note that if we can prove $\int Ef(Y,t)dt = E\int f(Y,t)dt$, the result follows immediately by taking $f(Y,t) = I\{Y \geq t\}$.

Proof. Assume Y is supported on the non-negative integers. Then

$$\begin{aligned}
\sum_{i=1}^{\infty} P(Y \geq i) &= [P(Y=1) + P(Y=2) + \cdots] \\
&\quad + [P(Y=2) + P(Y=3) + \cdots] \\
&\quad + [P(Y=3) + P(Y=4) + \cdots] + \cdots \\
&= 1 \cdot P(Y=1) + 2 \cdot P(Y=2) + 3 \cdot P(Y=3) + \cdots \\
&= \sum_{i=1}^{\infty} iP(Y=i) \\
&= \sum_{i=1}^{\infty} yP(Y=y) \\
&= E[Y].
\end{aligned}$$

Then note, for a general non-negative Y, $E\lfloor Y \rfloor \leq EY$. Therefore, $\sum_{i=1}^{\infty} P(Y \geq i) \leq EY$.

□

Exercise 2.1

For each of the three cases below, prove that $X_n \xrightarrow{P} 1$:

(a) $X_n = 1 + nY_n$, where Y_n is a Bernoulli random variable with mean $1/n$.

Proof. Let $\epsilon > 0$. Then

$$\begin{aligned}
P(|X_n - 1| < \epsilon) &= P(|1 + nY_n - 1| < \epsilon) \\
&= P(|nY_n| < \epsilon) \\
&= P(Y_n < \epsilon/n) \\
&= 1 - 1/n \\
&\to 1.
\end{aligned}$$

Thus, $X_n \xrightarrow{P} 1$. □

(b) $X_n = Y_n/\log n$, where Y_n is a Poisson random variable with mean $\sum_{i=1}^{n}(1/i)$.

Proof. Consider $E(X_n) = \frac{\sum_{i=1}^n (1/i)}{\log n} \to 1$ and $Var(X_n) = \frac{\sum_{i=1}^n (1/i)}{(\log n)^2} \to 0$. Thus, X_n is asymptotically unbiased. Then by Chebyshev's Inequality,

$$P(|X_n - 1| \geq \epsilon) = P(|X_n - E[X_n]| \geq \epsilon) \leq \frac{Var(X_n)}{\epsilon^2} \to 0.$$

Therefore, $P(|X_n - 1| < \epsilon) \to 1$ and thus, $X_n \xrightarrow{P} 1$. □

(c) $X_n = \frac{1}{n}\sum_{i=1}^n Y_i^2$, where the Y_i are independent standard normal random variables.

Proof. Let $X_n = \frac{1}{n}\sum_{i=1}^n Y_i^2$. Then $X_n \sim Gamma(n, 1/n)$. Then $E(X_n) = 1$ and $Var(X_n) \to 0$. Thus, X_n is unbiased. Then by Chebyshev's Inequality,

$$P(|X_n - 1| \geq \epsilon) = P(|X_n - E[X_n]| \geq \epsilon) \leq \frac{Var(X_n)}{\epsilon^2} \to 0.$$

Therefore, $P(|X_n - 1| < \epsilon) \to 1$ and thus, $X_n \xrightarrow{P} 1$. □

Exercise 2.2

This exercise deals with bounded in probability sequences; see Definition 2.6.

(a) Prove that if $X_n \xrightarrow{d} X$ for some random variable X, then X_n is bounded in probability.

Hint: You may use the fact that any interval of real numbers must contain a point of continuity of $F(x)$. Also, recall that $F(x) \to 1$ as $x \to \infty$.

Proof. Let $\epsilon > 0$. Then since any interval of real numbers must contain a point of continuity of $F(x)$, there exists a such that $F(x) < \epsilon/4$ and b such that $F(b) > 1 - \epsilon/4$. Then by Definition 2.32, since $X_n \to X$, we know $F_n(a) \to F(a)$ and $F_n(b) \to F(b)$. Then by definition of convergence, there exists an N_1 such that for all $n > N_1$, $|F_n(a) - F(a)| < \epsilon/4$. Therefore, since $Fn(a) > F(a)$, we know $F_n(a) < \epsilon/2$. Then similarly, there exists an N_2 such that for all $n > N_2$, $|F_n(b) - F(b)| < \epsilon/4$. Therefore, since $F_n(b) < F(b)$, we know $F_n(b) > 1 - \epsilon/2$. Let $M = \max\{|a|, |b|\}$ and $N = \max\{N_1, N_2\}$. Then $P(|X_n| \leq M) \geq P(|a| \leq X_n \leq b) = F_n(b) - F_n(a) > 1 - \epsilon$. Therefore, X_n is bounded in probability. □

(b) Prove that if X_n is bounded in probability and $Y_n \xrightarrow{P} 0$, then $X_n Y_n \xrightarrow{P} 0$.

Hint: For fixed $\epsilon > 0$, argue that there must be M and N such that $P(|X_n| < M) > 1 - \epsilon/2$ and $P(|Y_n| < \epsilon/M) > 1 - \epsilon/2$ for all $n > N$. What is then the smallest possible value of $P(|X_n| < M$ and $|Y_n| < \epsilon/M)$? Use this result to prove $X_n Y_n \xrightarrow{P} 0$.

Proof. Let $\epsilon > 0$. Then there exists M and N such that $P(|X_n| < M) > 1 - \epsilon/2$ and $P(|Y_n| < \epsilon/M) > 1 - \epsilon/2$ for all $n > N$. Then

$$\begin{aligned} P(|X_n Y_n| < \epsilon) &\geq P(|X_n| < M \cap |Y_n| < \epsilon/M) \\ &= P(|X_n| < M) + P(|Y_n| < \epsilon/M) - P(|X_n| < M \cup |Y_n| < \epsilon/M) \\ &\geq P(|X_n| < M) + P(|Y_n| < \epsilon/M) - 1 \\ &> (1 - \epsilon/2) + (1 - \epsilon/2) - 1 \\ &= 1 - \epsilon. \end{aligned}$$

Therefore, $X_n Y_n \xrightarrow{P} 0$. $\square$

Exercise 2.4

Suppose that $X_1, ...X_n$ are independent and identically distributed Uniform(0,1) random variables. For a real number t, let

$$G_n(t) = \sum_{i=1}^{n} I\{X_i \leq t\}.$$

(a) What is the distribution of $G_n(t)$ if $0 < t < 1$?

Consider $I(X_i \leq t) \sim Bern(t)$ for $0 < t < 1$. Then, \$$G_n(t) = \sum_{i=1}^{n} I\{X_i \leq t\} \sim Bin(n,t)$.

(b) Suppose $c > 0$. Find the distribution of a random variable X such that $G_n(c/n) \xrightarrow{d} X$. Justify your answer.

Consider $Y_n \sim Bin(n, p_n)$ where $p_n \to c$. Then for $y = 0, 1, 2, ...,$

$$\begin{aligned}
\lim_{n\to\infty} f_n(y) &= \lim_{n\to\infty} \binom{n}{y} p_n^y (1-p_n)^{n-y} \\
&= \lim_{n\to\infty} \frac{n(n-1(n-2)\cdots(n-y+1)}{y!} \left(\frac{c}{n}\right)^y \left(1-\frac{c}{n}\right)^{n-y} \\
&= \frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-y+1}{n}\cdot\frac{c^y}{y!}\left(1-\frac{c}{n}\right)^n\left(1-\frac{c}{n}\right)^{-y} \\
&= \frac{c^y e^{-c}}{y!}.
\end{aligned}$$

Therefore, $Y_n \xrightarrow{d} Poisson(c)$.

Let $Y_n = G_n(c/n) \sim Bin(n, c/n)$. Then $n \cdot \frac{c}{n} \to c$. Thus, using the result from above, $Y_n \xrightarrow{d} Poisson(c)$.

(c) How does your answer to part (b) change if $X_1, ..., X_n$ are from a standard exponential distribution instead of a uniform distribution? The standard exponential distribution function is $F(t) = 1 - e^{-t}$.

We now have $I(X_i \leq t) \sim Bern(1-e^{-t})$ for $0 < t < 1$. Then, $G_n(t) = \sum_{i=1}^{n} I\{X_i \leq t\} \sim Bin(n, 1-e^{-t})$. Thus, let $Y_n = G_n(c/n) \sim Bin(n, 1-e^{-c/n})$. Then using Taylors series expansion

$$n(1-e^{-c/n}) = n\left(1-\left(1-\frac{c}{n}+\frac{c^2}{2n^2}\right)\right) = c + o\left(\frac{1}{n}\right).$$

Thus, using the result from above, $Y_n \xrightarrow{d} Poisson(c)$.