Homework 2

STAT 984

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Exercise 1.21

Let $x_1, ..., x_n$ be a simple random sample from an exponential distribution with density $f(x) = \theta e^{-\theta x}$ and consider the estimator $\delta_n(x) = \sum_{i=1}^n \frac{x_i}{n+2}$ of $g(\theta) = \frac{1}{theta}$. Show that for some constants c_1 and c_2 depending on θ,

$$\text{bias of } \delta_n \sim c_1 \text{variance of } \delta_n) \sim \frac{c_2}{n}$$

as $n \to \infty$. The bias of δ_n equals its expectation minus $\frac{1}{\theta}$.

Let $g(\theta) = \frac{1}{\theta}$. Then $f(x) = \frac{1}{g(\theta)} e^{-\frac{x}{g(\theta)}}$. Therefore, $E[x_i] = g(\theta) = \frac{1}{\theta}$ and $Var(x_i) = g(\theta)^2 = \frac{1}{\theta^2}$. Consider

$$E[\hat{\delta}_n] = E\left[\sum_{i=1}^n \frac{x_i}{n+2}\right] = \frac{1}{n+2}\sum_{i=1}^n E[x_i] = \frac{n}{n+2}E[x_i] = \frac{n}{\theta(n+2)}$$

and

$$Var[\hat{\delta}_n] = Var\left[\sum_{i=1}^n \frac{x_i}{n+2}\right] = \frac{1}{(n+2)^2}\sum_{i=1}^n Var[x_i] = \frac{n}{(n+2)^2}Var[x_i] = \frac{n}{\theta^2(n+2)^2}.$$

Therefore,

$$Bias(\hat{\delta}_n) = \frac{n}{\theta(n+2)} - \frac{1}{\theta} = \frac{n-n-2}{\theta(n+2)} = -\frac{2}{\theta(n+2)}.$$

Let $c_1 = -2\theta$. Then

$$\frac{Bias(\hat{\delta}_n)}{c_1 Var(\hat{\delta}_n)} = \frac{-\frac{2}{\theta(n+2)}}{-2\theta\frac{n}{\theta^2(n_2)^2}} = \frac{-2\theta^2(n+2)^2}{-2\theta^2 n(n+2)} = \frac{n^2+4n_4}{n^2+2n} \to 1.$$

Let $c_2 = -\frac{2}{\theta}$. Then

$$\frac{c_1 Var(\hat{\delta}_n)}{\frac{c_2}{n}} = \frac{-2\theta\frac{n}{\theta^2(n+2)^2}}{-\frac{2}{\theta}/n} = \frac{-2\theta^2 n}{-2\theta^2(n+2)^2} = \frac{n^2}{n^2+4n+4} \to 1.$$

Thus, for $c_1 = -2\theta$ and $c_2 = -\frac{2}{\theta}$, bias of $\delta_n \sim c_1$variance of $\delta_n) \sim \frac{c_2}{n}$.

Exercise 1.24

Prove that if $f(x)$ is everwhere twice differentiable and $f''(x) \geq 0$ for x, then $f(x)$ is convex.

Let $a = \alpha x + (1 - \alpha y)$. Since $f''(x) \geq 0$, the Mean Value Theorem implies, $r_1(x, a) = \frac{1}{2}(x-a)^2 f''(x^*) \geq 0.$,then by Taylors' expansion,

$$\alpha f(x) \geq \alpha f(a) + \alpha(x-a)f'(a)$$

and

$$(1-\alpha)f(y) \geq (1-\alpha)f(a) + (1-\alpha)(x-a)f'(a).$$

Therefore,

$$\begin{aligned}
\alpha f(x) + (1-\alpha)f(x) &\geq f(a) + f'(a)\left(\alpha(x-a) + (1-\alpha)(y-a)\right) \\
&= f(a) + f'(a)\left(\alpha x - \alpha a + y - \alpha y - a + \alpha a\right) \\
&= f(a) + f'(a)\left([\alpha x + (1-\alpha)y] - a\right) \\
&= f(\alpha x - (1-\alpha)y).
\end{aligned}$$

Thus, since $\alpha f(x) + (1-\alpha)f(x) \geq f(\alpha x - (1-\alpha)y)$, $f(x)$ is convex.

Exercise 1.27

Recall that $\log n$ always denotes the natural logarithm of n. Assuming that $\log n$ means $\log_{10} n$ will change some of the answers in this exercise!

a. The following 5 sequences have the property that each tends to 1 as $n \to \infty$, and for any pair of sequences, one is little-o of the other. List them in order of rate of increase from slowest to fastest. In other words, give an ordering such that first sequence = o(second sequence), second sequence = o(third sequence), etc.

$$n \qquad \sqrt{\log n!} \qquad \sum_{i=1}^{n} {}^{3}\sqrt{i} \qquad 2^{\log n} \qquad (\log n)^{\log \log n}$$

Prove the 4 order relationships that result from your list.

Hint: Here and in part (b), using a computer to evaluate some of the sequences for large values of n can be helpful in suggesting the correct ordering. However, note that this procedure does not constitute a proof!

Ordering the 5 sequences above from slowest to fastest is:

$$(\log n)^{\log \log n}, \sqrt{\log n!}, 2^{\log n}, n, \sum_{i=1}^{n} {}^{3}\sqrt{i}.$$

Many of the following proofs depend on the equivalence of a function, $f(x)$, existing similar to the sequence on the positive integers and therefore, uses l'Hopital's rule. Then proving the following:

(1) $(\log n)^{\log\log n} = o\left(\sqrt{\log n!}\right)$

Proceeding by induction. Let n = 6. Then $\log n! = \log 6! = 6.57 > 6 = n$. Assume it is true for $n = k$ for $k \geq 6$. Then let $n = k+1$. Then $\log((k+1)!) = \log((k+1)k!) = \log(k!) + \log(k+1) > k + \log(k+1) > k+1$. Thus, for $n \geq 6$, $\log n! > n$ implies $\sqrt{\log n!} > \sqrt{n}$. Then since $-\log$ is a convex function, taking the $-\log$ of both sides, we can show that $-\log(\log(n))^{\log(\log(n)))} = o\left(-\log\sqrt{n}\right)$. Denote $a = \log n$. Consider,

$$\lim_{n\to\infty} \frac{(\log(\log(n)))^2}{\frac{1}{2}\log n} = \lim_{a\to\infty} \frac{\log(a)^2}{\frac{1}{2}a}.$$

Then let $f(x) = \frac{\log(x)^2}{\frac{1}{2}x}$, by l'Hopitals' rule, taking the derivative two times,

$$\lim_{x\to\infty} \frac{\log(x)^2}{\frac{1}{2}x} = \lim_{x\to\infty} \frac{2\log a}{a\frac{1}{2}} = \lim_{x\to\infty} \frac{4}{a^2} = 0.$$

Therefore, $-\log(\log(n))^2 = o\left(-\frac{1}{2}\log n\right)$ implies $\log(n)^{\log(\log(n))} = o\left(\sqrt{\log n!}\right)$.

(2) $\sqrt{\log n!} = o\left(2^{\log n}\right)$

Consider $\log n! = \log(n) + \log(n+1) + ... \leq n\log n$. and $2^{\log n} = e^{\log(2)\log(n)} = n^{\log(2)}$. Then

$$\lim_{n\to\infty} \frac{\sqrt{n\log n}}{n^{\log 2}} = \lim_{n\to\infty} \frac{\sqrt{n}\sqrt{\log n}}{n^{\log 2}} = \lim_{n\to\infty} \frac{\sqrt{\log n}}{n^{\log 2 - 0.5}}.$$

Consider $f(x) = \frac{\sqrt{\log x}}{x^{\log 2 - 0.5}}$. Then by l'Hopital's rule and order of polynomials,

$$\lim_{x\to\infty} = \frac{\sqrt{\log x}}{x^{\log 2 - 0.5}} = \lim_{x\to\infty} = \frac{\frac{1}{2x\sqrt{\log x}}}{\frac{0.19}{x^{0.81}}} = \lim_{x\to\infty} \frac{x^{0.81}}{(0.19)2x\sqrt{\log x}} = 0.$$

Therefore, $\frac{\sqrt{n\log n}}{n^{\log 2}} \to 0$ and $\sqrt{n\log n} = o\left(n^{\log 2}\right)$ implies $\sqrt{\log n!} = o\left(2^{\log n}\right)$.

(3) $2^{\log n} = o(n)$

From (2), we know that $2^{\log n} = n^{\log(2)}$. Then

$$\frac{n^{\log 2}}{n} = \frac{n^{0.69}}{n^1} \to 0.$$

Therefore, $\frac{2^{\log n}}{n} \to 0$ and $2^{\log n} = o(n)$.

(4) $n = o\left(\sum_{i=1}^{n} {}^3\sqrt{i}\right)$

Using the geometric series, we know $\lim_{n\to\infty} \frac{\frac{4}{3}n^{4/3}}{\sum_{i=1}^{n} \sqrt[3]{i}} \leq 1$. Then,

$$\frac{n}{\sum_{i=1}^{n} \sqrt[3]{i}} = \frac{n}{\frac{4}{3}n^{4/3}} \frac{\frac{4}{3}n^{4/3}}{\sum_{i=1}^{n} \sqrt[3]{i}} \to 0.$$

Thus, $n = o\left(\sum_{i=1}^{n} \sqrt[3]{i}\right)$.

b. Follow the directions of part (a) for the following 13 sequences.

$$\begin{array}{cccccc} & \log(n!) & n^2 & n^n & 3^n & \\ \log(\log n) & n & \log n & 2^{3\log n} & n^{n/2} \\ & n! & 2^{2^n} & n^{\log n} & (\log n)^n & \end{array}$$

Proving the 12 order relationships is challenging but not quite as tedious as it sounds; some of the proofs will be very short.

Ordering the 12 sequences above from slowest to fastest is:

$$\log(\log n), \log(n), n, \log(n!), n^2, 2^{3\log n}, n^{\log n}, 3^n, \log(n)^n, n^{n/2}, n!, n^n, 2^{2^n}$$

Then proving the following:

Many of the following proofs depend on the equivalence of a function, $f(x)$, existing similar to the sequence on the positive integers and therefore, uses l'Hopital's rule. Then proving the following:

(1) $\log(\log n) = o\left(\log(n)\right)$

Consider $f(x) = \frac{\log\log n}{\log n}$. Then by l'Hopital's rule,

$$\lim_{x\to\infty} \frac{\log\log x}{\log x} = \lim_{x\to\infty} \frac{\frac{1}{x\log x}}{\frac{1}{x}} = \lim_{x\to\infty} = \frac{1}{\log x} = 0.$$

Therefore, $\frac{\log\log n}{\log n} \to 0$.

(2) $\log(n) = o\left(n\right)$

Consdier $f(x) = \frac{\log x}{x}$. Then by l'Hopital's rule,

$$\lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{x} = \lim_{x\to\infty} \frac{1}{x^2} = 0.$$

Therefore, $\frac{\log n}{n} \to 0$.

(3) $n = o\left(\log(n!)\right)$

In part a (1) above, we used induction to show that $n > \log n!$ for $n \geq 6$. Thus, $n = o\left(\log(n!)\right)$.

(4) $\log(n!) = o\left(n^2\right)$

Recall, $\log n! \leq n \log n$. Then using (2) from above,

$$\lim_{n\to\infty} \frac{n \log n}{n^2} = \lim_{n\to\infty} \frac{\log n}{n} = 0.$$

Therefore, $\frac{\log n!}{n^2} \to 0$.

(5) $n^2 = o\left(2^{3 \log n}\right)$

Recall from part (a), $2^{\log n} = n^{\log 2}$. Then

$$\lim_{n\to\infty} \frac{n^2}{2^{3\log n}} = \lim_{n\to\infty} \frac{n^2}{n^{3\log 2}} = \lim_{n\to\infty} \frac{n^2}{n^{2.08}} = 0.$$

Therefore, $\frac{n^2}{2^{3\log n}} \to 0$.

(6) $2^{3\log n} = o\left(n^{\log n}\right)$

Notice, $\log n > 3 \log 2$ for $n > 8$. Then,

$$\lim_{n\to\infty} \frac{n^{3\log 2}}{n^{\log n}} = 0.$$

Therefore, $\frac{n^{3\log 2}}{n^{\log n}} \to 0$.

(7) $n^{\log n} = o\left(3^n\right)$

Consider the convex function, $f(x) = e^{\log x}$. Then, from equation 1.26, with $\alpha = 2$ and $\beta = 1$,

$$\lim_{n\to\infty} \frac{n^{\log n}}{3^n} = \lim_{n\to\infty} \frac{e^{\log(n)^2}}{e^{n\log(3)}} = 0$$

since logarithm gorws slower than polynomial. Therefore, $n^{\log n} = o\left(3^n\right)$.

(8) $3^n = o\left(\log(n)^n\right)$

Consider the convex function, $f(x) = e^{\log x}$. Then,

$$\lim_{n\to\infty} \frac{3^n}{\log(n)^n} = \lim_{n\to\infty} \frac{e^{n\log 3}}{e^{n\log n}} = \lim_{n\to\infty} \frac{(e^n)^{\log 3}}{(e^n)^{\log n}} = 0.$$

Therefore, $3^n = o\left(\log(n)^n\right)$.

(9) $\log(n)^n = o\left(n^{n/2}\right)$

Consider $f(x) = \frac{\log x}{x^{1/2}}$. Then by l'Hopitals 's rule, taking the derivative twice,

$$\lim_{x\to\infty} \frac{\log x}{x^{1/2}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2x^{1/2}}{x} = 0.$$

Therefore, $\frac{\log(n)^n}{(n^{1/2})^n} \to 0$ and $\log(n)^n = o\left(n^{n/2}\right)$.

(10) $n^{n/2} = o\,(n!)$

Consider

$$\lim_{n\to\infty} \frac{n^{n/2}}{n!} = \lim_{n\to\infty} \frac{n^{n/2}}{n^n + ...} = 0.$$

Therefore, $n^{n/2} = o\,(n!)$.

(11) $n! = o\,(n^n)$

Let $n = 2$. Then $2! = 2 < 4 = 2^2$. Assume it is true for $n = k$. Let $n = k+1$. Then

$$(k+1)! = (k+1)k! < (k+1)k^k < (k+1)(k+1)^k = (k+1)^{k+1}.$$

Therefore, for all $n \geq 2, n! < n^n$. Thus, $n! = o\,(n^n)\,.$

(12) $n^n = o\left(n^{2^n}\right)$

Consider $f(x) = \frac{x \log x}{2^x}$. Then using l'Hopital's rule,

$$\lim_{x\to\infty} \frac{x \log x}{2^x} = \lim_{x\to\infty} \frac{1+\log n}{2^n \log(2)} = \lim_{x\to\infty} \frac{1}{n(2^n \log(2)^2)} = 0.$$

Thus $\frac{n \log n}{2^n} \to 0$. Then taking the convex function, $f(x) = e^{\log(x)}$,

$$\lim_{n\to\infty} \frac{n^n}{2^{2^n}} = \lim_{n\to\infty} \frac{e^{n\log n}}{e^{2^n \log 2}} = \lim_{n\to\infty} e^{n\log n - 2^n \log 2} = \lim_{n\to\infty} e^{2^n\left(\frac{n\log n}{2^n} - \log 2\right)} = \lim_{n\to\infty} e^{-2^n \log 2} = 0.$$

Therefore, $n^n = o\left(n^{2^n}\right)$.

Exercise 1.29

Suppose that $a_{nj} \to c_j$ as $n \to \infty$ for $j = 1, ..., k$. Prove that if $f : \mathbb{R}^k \to \mathbb{R}$ is continuous at the point $\boldsymbol{c}$, then $f(\boldsymbol{a}_n) \to f(\boldsymbol{c})$. This proves every part of Exercise 1.1. (The hard work of an exercise like 1.1(b) is in showing that multiplication is continuous.)

Proof. We need to show that for any $\epsilon > 0$, there exists an N such that for all $n > n$, $||f(\boldsymbol{a_n}) - f(\boldsymbol{c})|| < \epsilon$. Then from the definition of continuity, we know there exists some $\delta > 0$ such that $||f(\boldsymbol{x}) - f(\boldsymbol{c})|| < \epsilon$ for all x such that $||\boldsymbol{x} - \boldsymbol{c}|| < \delta$. Since $a_{nj} \to c_j$ for all $1 \leq j \leq k$, then we know $\boldsymbol{a_n} \to \boldsymbol{c}$ as $n \to \infty$. Then since $\delta > 0$, there must by definition be some N such that $||\boldsymbol{a_n} - \boldsymbol{c}|| < \delta$ for all $n > N$. We conclude that for all $n > N$, $||f(\boldsymbol{a_n}) - f(\boldsymbol{c})|| < \epsilon$. $\square$

Exercise 1.31

Prove that the converse of Theorem 1.38 is not true by finding a function that is not differentiable at some point but whose partial derivatives at that point all exist.

Consider $f(x, y) = I(xy = 0)$. Then along the x and y axis, $f(x, y) = 1$. Therefore, the partial derivatives with respect to both x and y exist at the orgin where the two lines cross. However, since $f(x, y) = 0$ everywhere else, $f(x, y)$ is not continuous at the orgin. Thus, $\nabla f(x, y)$ does not exist.

Exercise 1.34

a. Find the Hessian matrix of the loglikelihood function defined in Exercise 1.32.

Consider

$$
\begin{aligned}
& & f(\boldsymbol{x};\mu,\sigma^2) &= \frac{exp\{-\frac{1}{2\sigma^2}(x_i-\mu)^2\}}{\sqrt{(2\pi\sigma^2)}} \\
\Longrightarrow & & L(\mu,\sigma^2;\boldsymbol{x}) &= \prod_{i=1}^{n} \frac{1}{\sqrt{(2\pi\sigma^2)}} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2} \\
& & &= (s\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu^2)} \\
\Longrightarrow & & \log L(\mu,\sigma^2;\boldsymbol{x}) &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2.
\end{aligned}
$$

Then taking partial derivatives,

$$
\nabla \log L(\mu,\sigma^2;\boldsymbol{x}) = \begin{bmatrix} \frac{d^2}{d\mu^2}\log L(\mu,\sigma^2;\boldsymbol{x}) & \frac{d^2}{d\mu d\sigma^2}\log L(\mu,\sigma^2;\boldsymbol{x}) \\ \frac{d^2}{d\mu d\sigma^2}\log L(\mu,\sigma^2;\boldsymbol{x}) & \frac{d^2}{d\sigma^4}\log L(\mu,\sigma^2;\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} -\frac{n}{\sigma^2} & -\frac{\sum_{i=1}^{n}(x_i-\mu)}{\sigma^4} \\ -\frac{\sum_{i=1}^{n}(x_i-\mu)}{\sigma^4} & \frac{n}{2\sigma^4} - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^6} \end{bmatrix}
$$

b. Suppose that $n = 10$ and that we observe this sample:

```r
x <- c(2.946, 0.975, 1.333, 4.484, 1.711, 2.627, -0.628,
    2.476, 2.599, 2.143)
kable(t(x))
```

2.946	0.975	1.333	4.484	1.711	2.627	-0.628	2.476	2.599	2.143

Evaluate the Hessian matrix at the maximum likelihood estimator $(\hat{\mu}, \hat{\sigma}^2)$. (A formula for the MLE is given in the hint to Exercise 1.32)

```r
n <- 10
muhat <- mean(x)
sigma2hat <- mean((x - muhat)^2)
Hessian <- matrix(c(-n/sigma2hat, -(sum(x - rep(muhat,
    n)))/sigma2hat^2, -(sum(x - rep(muhat, n)))/sigma2hat^2,
    n/(2 * sigma2hat^2) - (sum((x - rep(muhat, n))^2))/sigma2hat^3),
    2, 2, byrow = T)
kable(Hessian)
```

-6.0587	0.000000
0.0000	-1.835392