Module 6: Random Effects & Linear Mixed Models

Linear Mixed Models

Example 6.3: Knee Brace

> Researchers study 4 knee braces: Control, Air Armor 2, Breg, Don Joy
>
> 10 high school running backs with previous ACL injury participate. Each player: Wears all 4 braces, Order randomized
>
> Time to complete 40-yard agility course recorded

Run										
1	Control	Air	Control	Control	Air	Breg	Air	Breg	Air	Control
2	Breg	Don	Don	Air	Breg	Control	Control	Air	Don	Don
3	Air	Control	Air	Breg	Don	Air	Don	Control	Control	Air
4	Don	Breg	Breg	Don	Control	Don	Breg	Don	Breg	Breg

Example 6.3: Skeleton ANOVA

Do we care only about these 10 running backs?

Source of Variation	DF

When one factor is fixed and one is random

Linear Mixed Models (LMM)

Mixed models come about when one factor is fixed and one factor is random. A mixed model should be used to study the effects of one factor on the population mean and the effects of another factor on the population variance.

Statistical Effecs Model

$$y_{ij} = \mu + \tau_i + p_j + \epsilon_{ij} \text{ with } p_j \sim \text{iid } N(0, \sigma^2_{blk}) \text{ and } \epsilon_{ij} \sim \text{iid } N(0, \sigma^2_\epsilon)$$

for $i = 1, 2, 3; j = 1, 2, \ldots, 10$

where:

- y_{ij} is the time to complete a 40-yard agility course for the j^{th} running back wearing the i^{th} knee brace
- μ is the overall mean time to complete the agility course
- τ_i is the fixed effect of the i^{th} knee brace
- p_j is the random effect of the j^{th} running back (block)
- ϵ_{ij} is the random error term associated with the j^{th} running back wearing the i^th knee brace

Scope of Inference

Note

- If blocks were fixed: Inference limited to these 10 running backs
- If blocks are random: Inference extends to *all* similar running backs

The Data

```r
library(tidyverse)
kneebrace_data <- read_csv("data/06-kneebrace-data.csv") |>
  mutate(across(RunningBack:KneeBrace, as.factor))
head(kneebrace_data, n = 12)
```

```
# A tibble: 12 × 3
   RunningBack KneeBrace AgilityTime
   <fct>       <fct>           <dbl>
 1 1           Control          19.1
 2 1           AirArmor         18  
 3 1           Breg             18.0
 4 1           DonJoy           17.9
 5 2           Control          19  
 6 2           AirArmor         18.4
 7 2           Breg             18.2
 8 2           DonJoy           17.7
 9 3           Control          19.9
10 3           AirArmor         18.7
11 3           Breg             18.6
12 3           DonJoy           18.7
```

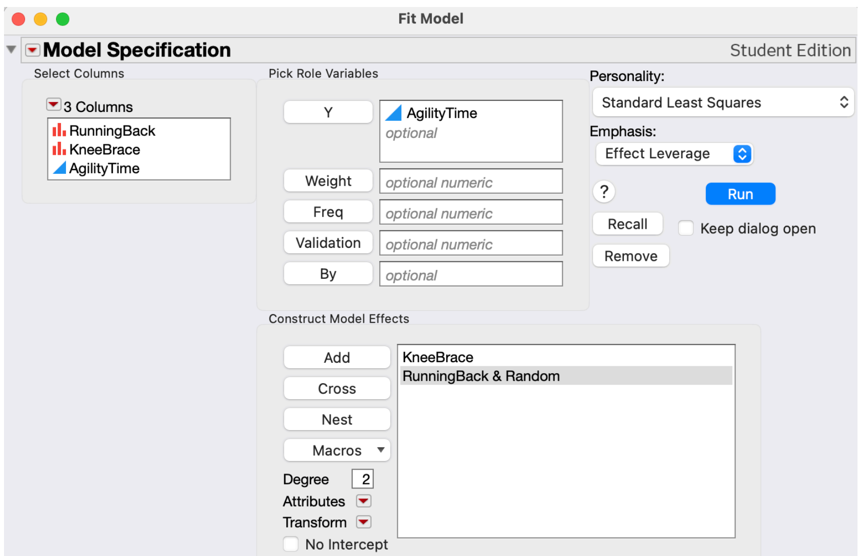
R: Fitting a LMM

```r
library(lme4)
kneebrace_mod <- lmer(AgilityTime ~ KneeBrace + (1 | RunningBack), data = kneebrace_data)
summary(kneebrace_mod)
```

```
Linear mixed model fit by REML ['lmerMod']
Formula: AgilityTime ~ KneeBrace + (1 | RunningBack)
   Data: kneebrace_data

REML criterion at convergence: 1.2

Scaled residuals:
     Min       1Q   Median       3Q      Max
-1.72607 -0.47036 -0.03709  0.53329  1.71255

Random effects:
 Groups      Name        Variance Std.Dev.
 RunningBack (Intercept) 0.15453  0.3931
 Residual                0.01965  0.1402
Number of obs: 40, groups:  RunningBack, 10

Fixed effects:
                 Estimate Std. Error t value
(Intercept)      18.24200    0.13198  138.22
KneeBraceBreg    -0.01000    0.06269   -0.16
KneeBraceControl  0.97500    0.06269   15.55
KneeBraceDonJoy  -0.16800    0.06269   -2.68

Correlation of Fixed Effects:
            (Intr) KnBrcB KnBrcC
KneeBracBrg -0.237
```

JMP: Fitting a LMM

`Analyze > Fit Model`



Parameter Estimates

Term	Estimate	Std Error	DFDen	t Ratio	Prob>|t|
Intercept	18.44125	0.126272	9	146.04	<.0001*
KneeBrace[AirArmor]	-0.19925	0.038388	27	-5.19	<.0001*
KneeBrace[Breg]	-0.20925	0.038388	27	-5.45	<.0001*
KneeBrace[Control]	0.77575	0.038388	27	20.21	<.0001*

REML Variance Component Estimates

Random Effect	Var Ratio	Var Component	Std Error	95% Lower	95% Upper	Wald p-Value	Pct of Total
RunningBack	7.8649595	0.1545341	0.0751755	0.0071927	0.3018754	0.0398*	88.720
Residual		0.0196484	0.0053476	0.0122818	0.0364025		11.280
Total		0.1741825	0.0752706	0.0867447	0.5112768		100.000

Testing Hypotheses About the Treatment Means

$$H_0 : \tau_1 = \tau_2 = \tau_3 = \tau_4 \text{ vs } H_A : \text{ at least one } \tau_i \text{ differs}$$

```r
library(lmerTest)
anova(kneebrace_mod)
```

```
Analysis of Variance Table
          npar Sum Sq Mean Sq F value
KneeBrace    3 8.2015  2.7338  139.14
```

Fixed Effect Tests

Source	Nparm	DF	DFDen	F Ratio	Prob > F
KneeBrace	3	3	27	139.1370	<.0001*

Estimating Treatment Means (balanced design)

Recall $y_{ij} = \mu + \tau_i + p_j + \epsilon_{ij}$

In the mixed model:

- $E[y_{ij}] = \hat{\mu} + \hat{\tau_i}$
- $Var(y_{ij}) = \hat{\sigma_{blk}} + \hat{\sigma_\epsilon^2}$.

Thus, the estimated mean and variance are given by:

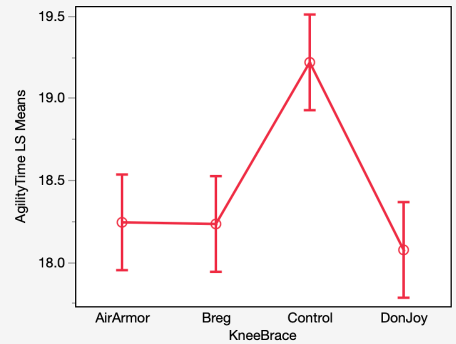
- Mean: $\hat{\mu_i} = \hat{\mu} + \hat{\tau_i}$
- SE of Mean: $SE(\hat{\mu_i}) = \sqrt{\frac{\hat{\sigma_{blk}^2} + \hat{\sigma_\epsilon^2}}{r}}$
- SE of Diff in Means: $SE(\hat{\mu_i} - \hat{\mu_{i'}}) = \sqrt{\frac{2}{r}\sigma_\epsilon^2}$

Estimating Treatment Means

```r
library(emmeans)
library(multcomp)
# emmip(kneebrace_mod, ~ KneeBrace, CI = T)
emmeans(kneebrace_mod, specs = ~KneeBrace) |>
  cld(Letters = LETTERS, decreasing = T, adjust = "tukey")
```

```
 KneeBrace emmean    SE   df lower.CL upper.CL .group
 Control     19.2 0.132 10.7     18.8     19.6  A    
 AirArmor    18.2 0.132 10.7     17.8     18.6   B   
 Breg        18.2 0.132 10.7     17.8     18.6   B   
 DonJoy      18.1 0.132 10.7     17.7     18.5   B   

Degrees-of-freedom method: kenward-roger 
Confidence level used: 0.95 
Conf-level adjustment: sidak method for 4 estimates 
P value adjustment: tukey method for comparing a family of 4 estimates 
significance level used: alpha = 0.05 
NOTE: If two or more means share the same grouping symbol,
      then we cannot show them to be different.
      But we also did not show them to be the same.
```

Least Squares Means Plot



LSMeans Differences Tukey HSD

α= 0.050 Q= 2.73656

Level		Least Sq Mean	Std Error
Control	A	19.217	0.13198
AirArmor	B	18.242	0.13198
Breg	B	18.232	0.13198
DonJoy	B	18.074	0.13198

Levels not connected by same letter are significantly different.

Level	- Level	Difference	Std Err Dif	Lower CL	Upper CL	p-Value
Control	DonJoy	1.143000	0.0626872	0.971452	1.314548	<.0001*
Control	Breg	0.985000	0.0626872	0.813452	1.156548	<.0001*
Control	AirArmor	0.975000	0.0626872	0.803452	1.146548	<.0001*
AirArmor	DonJoy	0.168000	0.0626872	-0.003548	0.339548	0.0565
Breg	DonJoy	0.158000	0.0626872	-0.013548	0.329548	0.0792
AirArmor	Breg	0.010000	0.0626872	-0.161548	0.181548	0.9985

Effect on Standard Errors

Treating blocks as random (instead of fixed) (1) Increases SE of treatment means (2) Does NOT change SE of treatment differences

Running back Fixed (LM)

```r
kneebrace_lm <- lm(AgilityTime ~ KneeBrace + RunningBack,
                   data = kneebrace_data)
emmeans(kneebrace_lm, specs = ~KneeBrace)
```

```
 KneeBrace emmean     SE df lower.CL upper.CL
 AirArmor   18.24 0.0443 27    18.15    18.33
 Breg       18.23 0.0443 27    18.14    18.32
 Control    19.22 0.0443 27    19.13    19.31
 DonJoy     18.07 0.0443 27    17.98    18.16

Results are averaged over the levels of: RunningBack
Confidence level used: 0.95
```

```r
emmeans(kneebrace_lm, specs = ~KneeBrace) |> pairs()
```

```
 contrast           estimate     SE df t.ratio p.value
 AirArmor - Breg       0.010 0.0627 27   0.160  0.9985
 AirArmor - Control   -0.975 0.0627 27 -15.553 <0.0001
 AirArmor - DonJoy     0.168 0.0627 27   2.680  0.0565
 Breg - Control       -0.985 0.0627 27 -15.713 <0.0001
 Breg - DonJoy         0.158 0.0627 27   2.520  0.0792
 Control - DonJoy      1.143 0.0627 27  18.233 <0.0001

Results are averaged over the levels of: RunningBack
P value adjustment: tukey method for comparing a family of 4 estimates
```

Running back Random (LMM)

```r
kneebrace_lmer <- lmer(AgilityTime ~ KneeBrace + (1 | RunningBack),
                       data = kneebrace_data)
emmeans(kneebrace_lmer, specs = ~KneeBrace)
```

```
 KneeBrace emmean    SE   df lower.CL upper.CL
 AirArmor    18.2 0.132 10.7     18.0     18.5
 Breg        18.2 0.132 10.7     17.9     18.5
 Control     19.2 0.132 10.7     18.9     19.5
 DonJoy      18.1 0.132 10.7     17.8     18.4

Degrees-of-freedom method: kenward-roger
Confidence level used: 0.95
```

```r
emmeans(kneebrace_lmer, specs = ~KneeBrace) |> pairs()
```

```
 contrast           estimate     SE df t.ratio p.value
 AirArmor - Breg       0.010 0.0627 27   0.160  0.9985
 AirArmor - Control   -0.975 0.0627 27 -15.553 <0.0001
 AirArmor - DonJoy     0.168 0.0627 27   2.680  0.0565
 Breg - Control       -0.985 0.0627 27 -15.713 <0.0001
 Breg - DonJoy         0.158 0.0627 27   2.520  0.0792
 Control - DonJoy      1.143 0.0627 27  18.233 <0.0001

Degrees-of-freedom method: kenward-roger
P value adjustment: tukey method for comparing a family of 4 estimates
```