Module 2: Completely Randomized Designs

Inference for Treatment Means

Inference for Treatment Means

ANOVA tells us whether *any* means differ.

Now we want:

- Estimates of each treatment mean (μ_i)
- Uncertainty of those treatment means (SEs + CIs)
- Interpretation in context

Example 2.1: Running Shoes

Response: Lap time (seconds)

Treatment structure:

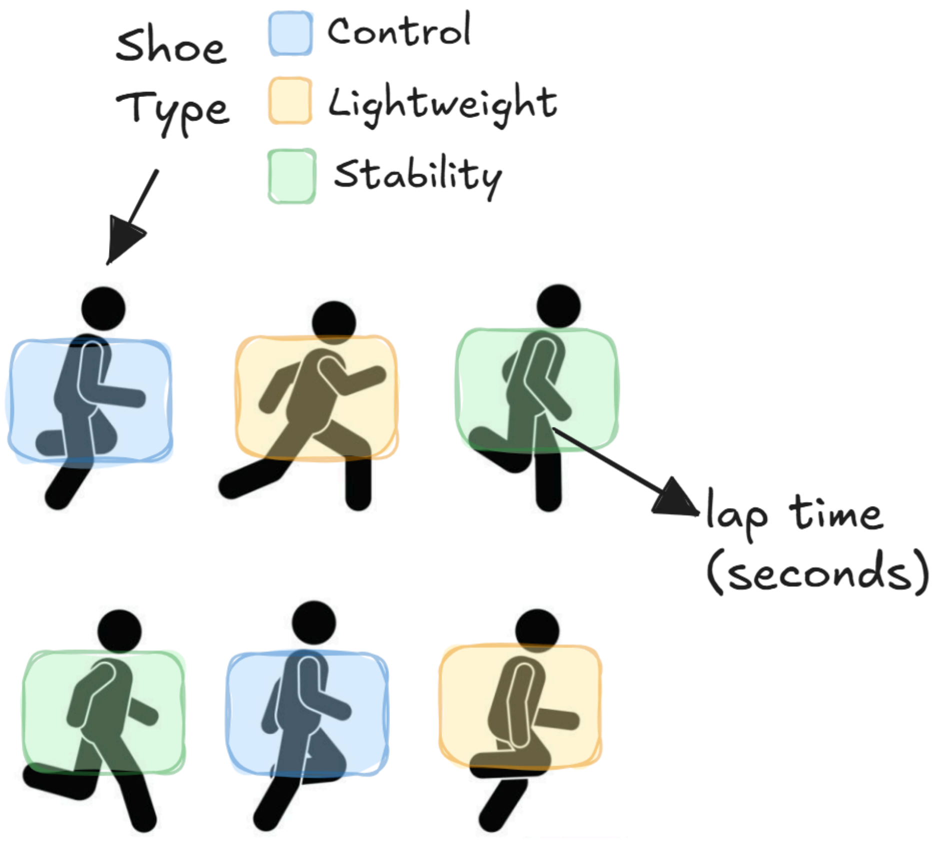
- One-way
- Factor: Shoe type
- 3 Levels: control, lightweight, and stability
- t = 3

Experimental structure:

- CRD
- Experimental Unit: Individual (r = 2)
- Measurement Unit: Individual (N = 6)

Goal: Determine whether shoe type affects mean lap time.

Example 2.1: Running Shoes (Blueprint)



Example 2.1: Running Shoes (Effects Model)

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij} \text{ with } \varepsilon_{ij} \text{ iid } \sim N(0, \sigma^2)$$

for $i = 1, 2, 3$ and $j = 1, 2$

Where:

- y_{ij} - the *observed* lap time for the j^{th} runner wearing the i^{th} shoe type.
- μ - the overall mean lap time.
- τ_i - the effect of the i^{th} shoe type.
- ϵ_{ij} - the experimental error associated with the j^{th} runner wearing the i^{th} shoe type.

What we Estimated

Model Reminder: $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$

```
Analysis of Variance Table

Response: Lap Time (seconds)
          Df Sum Sq Mean Sq F value  Pr(>F)
Shoe       2    172      86    21.5 0.01666 *
Residuals  3     12       4
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
Call:
lm(formula = `Lap Time (seconds)` ~ Shoe, data = shoe_data)

Residuals:
 1  2  3  4  5  6
-1  1 -1  1  2 -2

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  62.0000     0.8165  75.934 5.03e-06 ***
Shoe1        -1.0000     1.1547  -0.866  0.45018
Shoe2         7.0000     1.1547   6.062  0.00901 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2 on 3 degrees of freedom
Multiple R-squared:  0.9348,    Adjusted R-squared:  0.8913
F-statistic:  21.5 on 2 and 3 DF,  p-value: 0.01666
```

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Ratio
Model	2	172.00000	86.0000	21.5000
Error	3	12.00000	4.0000	**Prob > F**
C. Total	5	184.00000		0.0167*

Expanded Estimates

Nominal factors expanded to all levels

Term	Estimate	Std Error	t Ratio	Prob>|t|
Intercept	62	0.816497	75.93	<.0001*
Shoe[Control]	-1	1.154701	-0.87	0.4502
Shoe[Lightweight]	7	1.154701	6.06	0.0090*
Shoe[Stability]	-6	1.154701	-5.20	0.0138*

Estimating Treatment Means $\hat{\mu}_i$ (Balanced CRD)

From the data, each treatment has a sample mean:

$$\bar{y}_{i\cdot}$$

From the model, each treatment has an estimated *least squares mean*:

$$\hat{\mu}_i = \hat{\mu} + \hat{\tau}_i$$

In a **balanced** CRD (i.e., each treatment has r replications), these are the same quantity:

$$\hat{\mu}_i = \bar{y}_{i\cdot}$$

Sampling Variability of $\bar{y}_{i\cdot}$

Under the model:

$$Var(\bar{y}_{i\cdot}) = \frac{\sigma^2}{r}$$

We don't know σ^2, so we estimate it with:

$$\widehat{\sigma}^2 = MSE$$

Standard Error of a Treatment Mean $SE(\hat{\mu}_i)$

Balanced CRD (each treatment has r replications):

$$SE(\hat{\mu}_i) = \sqrt{\frac{MSE}{r}}$$

Degrees of Freedom

All inference uses the **error degrees of freedom**:

$$df_E = N - t = (r - 1)t$$

This is the same df used for the F-test in ANOVA.

Confidence Interval for a Treatment Mean

A CI for μ_i:

$$\hat{\mu}_i \pm t^*_{df_E,\alpha/2}\ SE(\hat{\mu}_i)$$

where t^* comes from the t distribution with $df_E = N - t = (r-1)t$ and the $\alpha/2$ quantile associated with the confidence level.

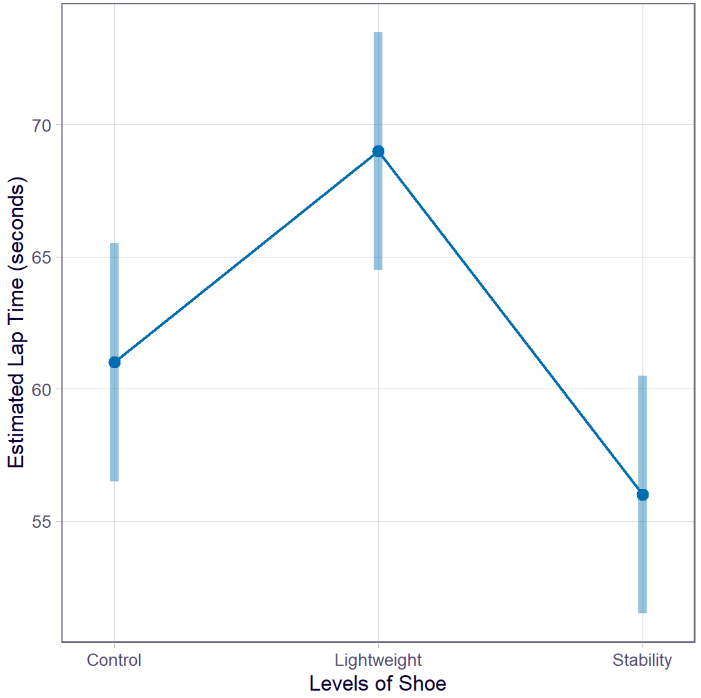
R: Estimated Means + SE + CI

```r
library(emmeans)
shoe_lsmeans <- emmeans(shoe_mod, specs = ~ Shoe)
shoe_lsmeans
```

```
 Shoe        emmean   SE df lower.CL upper.CL
 Control         61 1.41  3     56.5     65.5
 Lightweight     69 1.41  3     64.5     73.5
 Stability       56 1.41  3     51.5     60.5

Confidence level used: 0.95
```

```r
emmip(shoe_mod, ~ Shoe, CIs = T) +
  labs(y = "Estimated Lap Time (seconds)")
```



JMP: Estimated Means + SE + CI

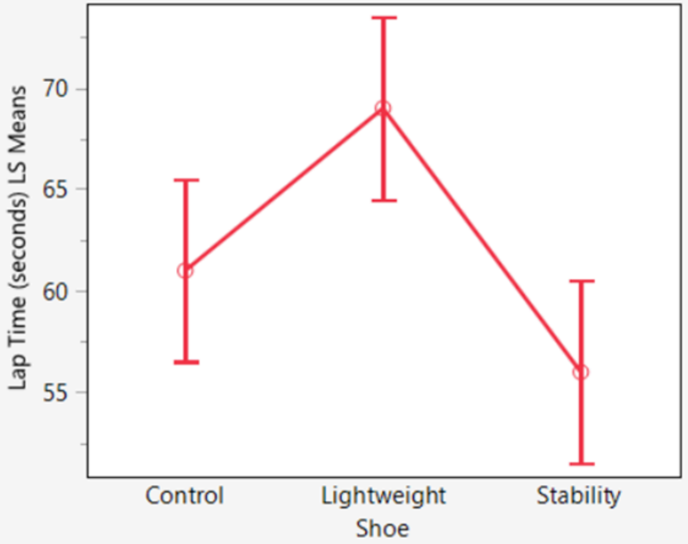
Response > Multiple Comparisons > Click “Show Least Squares Means” Plot > OK

Multiple Comparisons for Shoe

Least Squares Means Estimates

Shoe	Estimate	Std Error	DF	Lower 95%	Upper 95%	Arithmetic Mean Estimate	N
Control	61.000000	1.4142136	3	56.499341	65.500659	61.000000	2
Lightweight	69.000000	1.4142136	3	64.499341	73.500659	69.000000	2
Stability	56.000000	1.4142136	3	51.499341	60.500659	56.000000	2

Least Squares Means Plot



Interpretation of Results

The estimated mean 400m lap time for runners wearing the control shoe is 62 seconds (s.e. = 1.41).

We are 95% confident the population mean 400m lap time for all runners wearing the control shoe is between 51.5 and 60.5 seconds.

These results apply to all runner similar to those in our study under similar conditions.