② Fast Fourier Transform.

- DFT takes N^2 operations to calculate values
- FFT takes $N\log(N)$ steps.

$x[n]$ – n^{th} sample of signal x.

X_k – DFT coefficient $(k = 0 : N-1)$

$$X_k = \sum_{n=0}^{N-1} x[n] \exp\left(\frac{-2\pi j k n}{N}\right).$$

$$\boxed{\begin{aligned} &\text{Let } k = \frac{N}{2}k_1 + k_2 \\ &\quad k_1 = 0/1 \\ &\quad k_2 \Rightarrow 0 \le k_2 \le \frac{N}{2} - 1 \end{aligned}}$$

$$X_k = \sum_{n=0}^{N-1} x[n] \exp\left(\frac{-2\pi j n}{N}\left(\frac{N}{2}k_1 + k_2\right)\right)$$

$$= \sum_{n=0}^{N-1} x[n] \exp\left(\frac{-2\pi j n}{\cancel{N}}\right)\left(\frac{\cancel{N}}{\cancel{2}}k_1\right) \cdot \exp\left(\frac{-2\pi j n}{N}k_2\right)$$

$$= \sum_{n=0}^{N-1} x[n] \exp(-\pi j n k_1) \exp\left(\frac{-2\pi j n k_2}{N}\right)$$

$$= \sum_{n=0}^{N-1} x[n] (-1)^{nk_1} \exp\left(\frac{-2\pi j k_2 n}{N}\right)$$

Now, split 'n' into even and odd indices (2n) and (2n+1)

$$X_k = \sum_{n=0}^{N/2-1} x[2n]\,(-1)^{k_1 2n} \cdot \exp\left(\frac{-2\pi j k_2 (2n)}{N}\right)$$

$$+ \sum_{n=0}^{N/2-1} x[2n+1]\,(-1)^{k_1(2n+1)} \exp\left(\frac{-2\pi j k_2 (2n+1)}{N}\right)$$

$$= \sum_{n=0}^{N/2-1} x[2n]\,(1) \exp\left(\frac{-2\pi j k_2 (2n)}{N}\right)$$

$$+ \sum_{n=0}^{N/2-1} x[2n+1] \exp\left(\frac{-2\pi j k_2 n}{N}\right) \exp\left(\frac{-2\pi j k_2}{N}\right) \times (-1)^{k_1}$$

$$= \sum_{n=0}^{N/2-1} x[2n] \exp\left[\frac{-2\pi j k_2 (2n)}{N}\right]$$

$$+ (-1)^{k_1} \exp\left[\frac{-2\pi j k_2}{N}\right] \cancel{\sum_{n=0}^{N/2-1} x[2n+1] \exp(-2}$$

$$* \sum_{n=0}^{N/2-1} x[2n+1] \exp\left(\frac{-2\pi j k_2 n}{N}\right)$$

$$= F_0 + Q F_1$$

When $\boxed{k_1 = 0}$

① $\boxed{X_{k_2} = F_0 + QF_1}$ → FFT of N/2 elements

FFT of $\frac{N}{2}$ elements

$\boxed{k_1 = 1}$

② $\boxed{X_{\frac{N}{2}+k_2} = F_0 - QF_1}$

F_0 = DFT of $\frac{N}{2}$ even terms.

F_1 = DFT of $\frac{N}{2}$ odd terms

To calculate DFT of an N sample signal, we break it down (recursively) into parts of $\frac{N}{2}$ each time.

$\therefore T_N = 2T_{\frac{N}{2}} + 2N$ → $\frac{N}{2}$ multiplications

$\frac{N}{2}$ additions for

$k_1 = 0/1$.

$\therefore T_N = O(N \log N)$ → Master's theorem