SCIENCE PROJECT

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<u>BASICS</u>

Before we begin, we will have a look at some basic definations that will help us in understanding the following concepts.

1.) **Polynomial Sequence** - A sequence of polynomials indexed by the nonnegative integers 0, 1, 2, 3, ..., in which each index is equal to the degree of the corresponding polynomial.

Alternative Defination - A sequence of polynomials $p_i(x)$, for i=0, 1, 2, ..., where $p_i(x)$ is exactly of degree i for all i.

2.) **Second-order linear differential equation** - A second order differential equation is an equation involving the unknown function y, its derivatives y' and y'', and the variable x.

3.) **Moment of Inertia** - Moment of inertia is the rotational analogue to mass. The mass moment of inertia about a fixed axis is the property of a body that measures the body's resistance to rotational acceleration.

1.) LAGUERRE POLYNOMIAL

- Laguerre's equation : **xy'' + (1-x)y' + ny = 0** which is a second-order linear differential equation).

- **Laguerre polynomials are solution to Laguerre's equation.**

This equation has nonsingular solutions only if n is a non-negative integer.

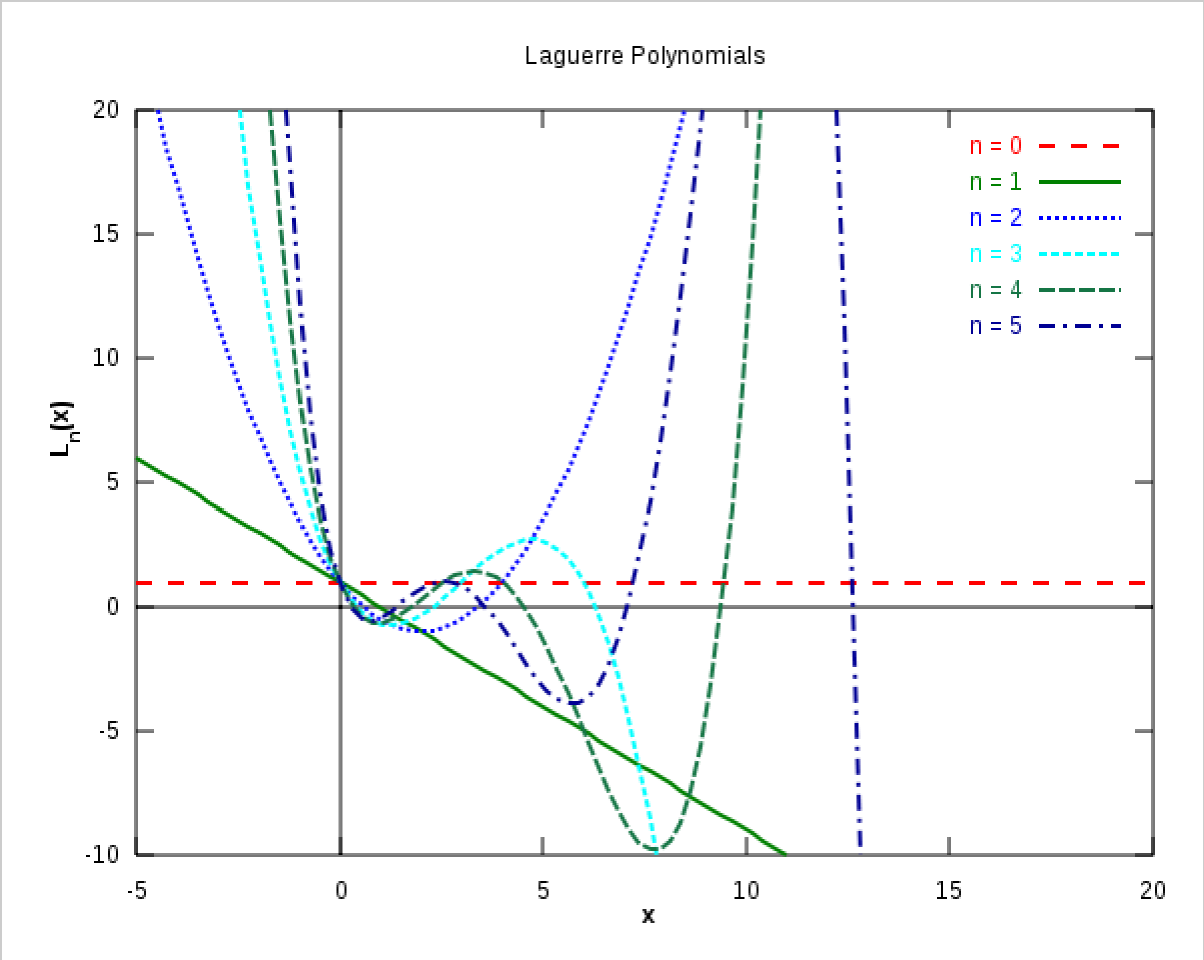
- **Defination** :

$$L_n(t) = \sum_{k=0}^{n} \frac{(-1)^k n! t^k}{(k!)^2 (n-k)!} \qquad n = 0, 1, 2, \cdots, \ 0 \le t < \infty$$

- First few Laguerre polynomials are as given in the table below:

n	$L_n(x)$
0	1
1	$-x+1$
2	$\frac{1}{2}(x^2-4x+2)$
3	$\frac{1}{6}(-x^3+9x^2-18x+6)$
4	$\frac{1}{24}(x^4-16x^3+72x^2-96x+24)$
5	$\frac{1}{120}(-x^5+25x^4-200x^3+600x^2-600x+120)$
6	$\frac{1}{720}(x^6-36x^5+450x^4-2400x^3+5400x^2-4320x+720)$

- The Laguerre polynomials arise in quantum mechanics, in the radial part of the solution of the Schrodinger equation for a one-electron atom.
- They further enter in the quantum mechanics of the Morse potential and of the 3D isotropic harmonic oscillator.
- Their graphs are as shown below:



The first six Laguerre polynomials.

- Also,

  One can also the first two polynomials of Laguerre polynomials are:
  L(0)(x) = 1
  L(1)(x) = 1-x

- **Generating Function**:

$$w(t,x) = (1-x)^{-1} \exp\left[-\frac{tx}{1-x}\right] = \sum_{n=0}^{\infty} L_n(t) x^n \qquad |x| < 1,\ 0 \le t < \infty$$

- Using **Recurrance Relation** :

  The recurrance relation for any k>=1 can be given as:

$$L_{k+1}(x) = \frac{(2k+1-x)L_k(x) - kL_{k-1}(x)}{k+1}.$$

The **closed form** is:

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} x^k.$$

Extending it to **Generality**:

More generally, the name Laguerre polynomials is used for solutions of :
xy'' + (α+1-x)y' + ny = 0 ,
for any arbitary real α.

2.) HERMITE PLOYNOMIAL

- Hermite polynomials were defined by Laplace.

- There are two different ways of standardizing the Hermite polynomials:

  - The **"probabilists' Hermite polynomials"** are given by

$$He_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} = \left(x - \frac{d}{dx}\right)^n \cdot 1,$$

  - while the **"physicists' Hermite polynomials"** are given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = \left(2x - \frac{d}{dx}\right)^n \cdot 1.$$

These two definitions are not exactly identical; each one is a rescaling of the other,

$$H_n(x) = 2^{\frac{n}{2}} He_n(\sqrt{2}\,x), \qquad He_n(x) = 2^{-\frac{n}{2}} H_n\left(\frac{x}{\sqrt{2}}\right).$$

- The first Eleven Hermite polynomials are shown below:

The first eleven probabilists' Hermite polynomials are:

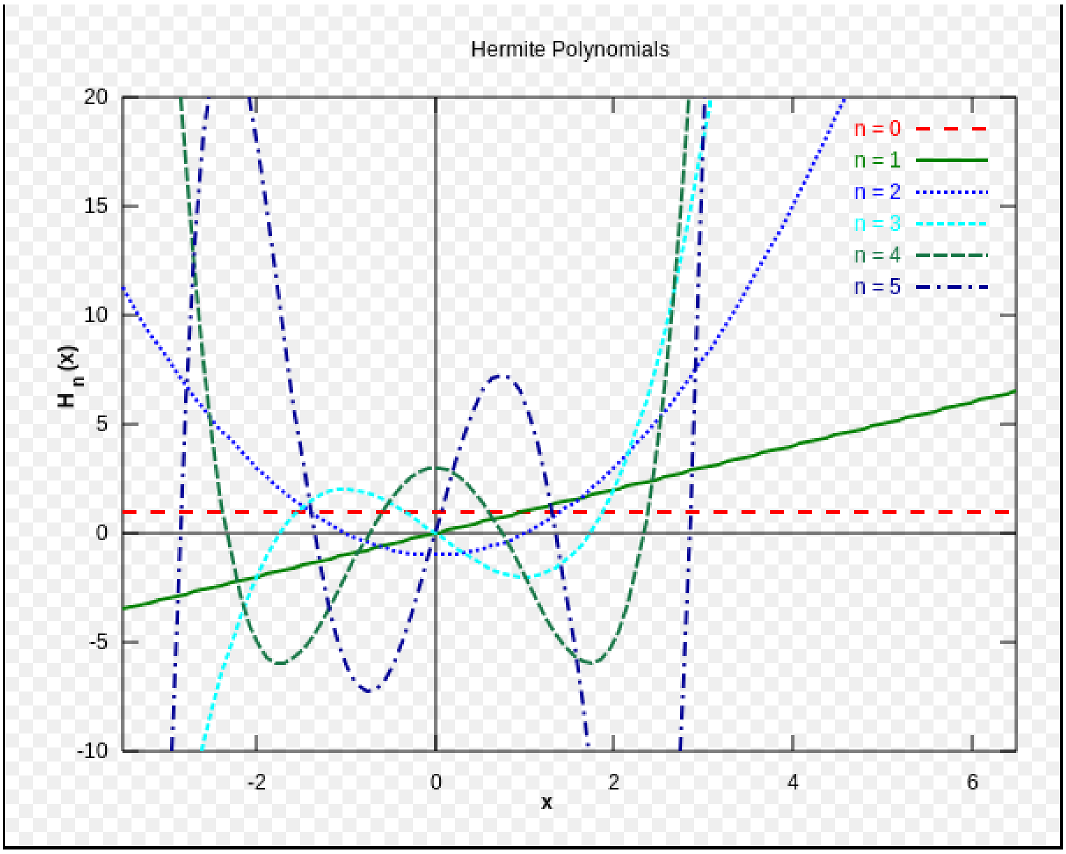
$$
\begin{aligned}
He_0(x) &= 1 \\
He_1(x) &= x \\
He_2(x) &= x^2 - 1 \\
He_3(x) &= x^3 - 3x \\
He_4(x) &= x^4 - 6x^2 + 3 \\
He_5(x) &= x^5 - 10x^3 + 15x \\
He_6(x) &= x^6 - 15x^4 + 45x^2 - 15 \\
He_7(x) &= x^7 - 21x^5 + 105x^3 - 105x \\
He_8(x) &= x^8 - 28x^6 + 210x^4 - 420x^2 + 105 \\
He_9(x) &= x^9 - 36x^7 + 378x^5 - 1260x^3 + 945x \\
He_{10}(x) &= x^{10} - 45x^8 + 630x^6 - 3150x^4 + 4725x^2 - 945
\end{aligned}
$$

and the first eleven physicists' Hermite polynomials are:

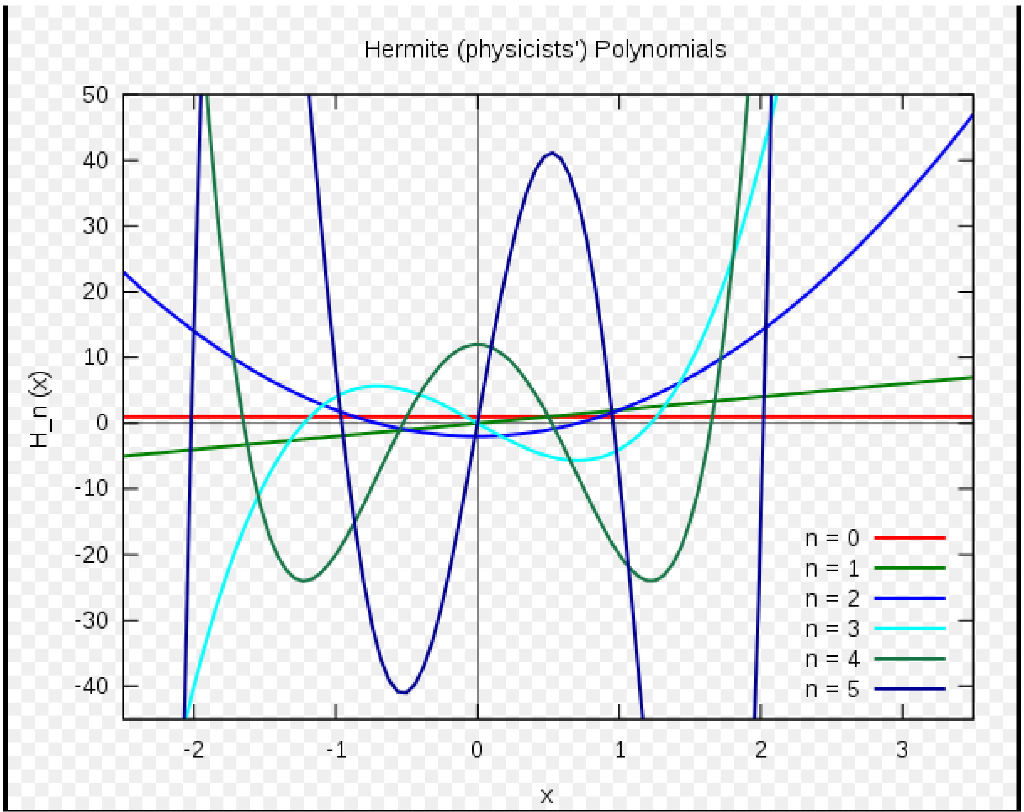
$$
\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 \\
H_3(x) &= 8x^3 - 12x \\
H_4(x) &= 16x^4 - 48x^2 + 12 \\
H_5(x) &= 32x^5 - 160x^3 + 120x \\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120 \\
H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x \\
H_8(x) &= 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680 \\
H_9(x) &= 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x \\
H_{10}(x) &= 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240
\end{aligned}
$$

GRAPHS

- **The first six probabilists' Hermite polynomials *Hen*(x).**



- **The first six (physicists') Hermite polynomials *Hn*(x).**



3.) MOMENT OF INERTIA MATRIX

Defination: We defined Moment of Intertia above. Also,

- **Moment of momentum** - Moment of momentum measures an objects tendency to continue to spin, it describes the rotary inertia of a system in motion about an axis.

  The moment of momentum, $\underline{h}_p$ about a fixed point o is

  defined as : r x mv

  Where:
  r is the position expressed as a displacement vector from the origin
  x represents the vector cross product
  mv is the linear momentum (mass x velocity)

- Now, moment of momentum , h_o can be expressed as:
  $h_o = [I]w$

  where I is the Inertia Matrix.

- I can be expressed as :

  Where $[I_p]$ is the Inertia Matrix

$$I_p = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & I_{-yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

- Problems where the moment of momentum vector, h is parallel to $\underline{\omega}$ are easier to solve, so the moment of momentum can be expressed as

$$\underline{h}_p = [I_p]\underline{\omega}$$