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Graphic user interface design for flow curves and yield loci in sheet metal forming

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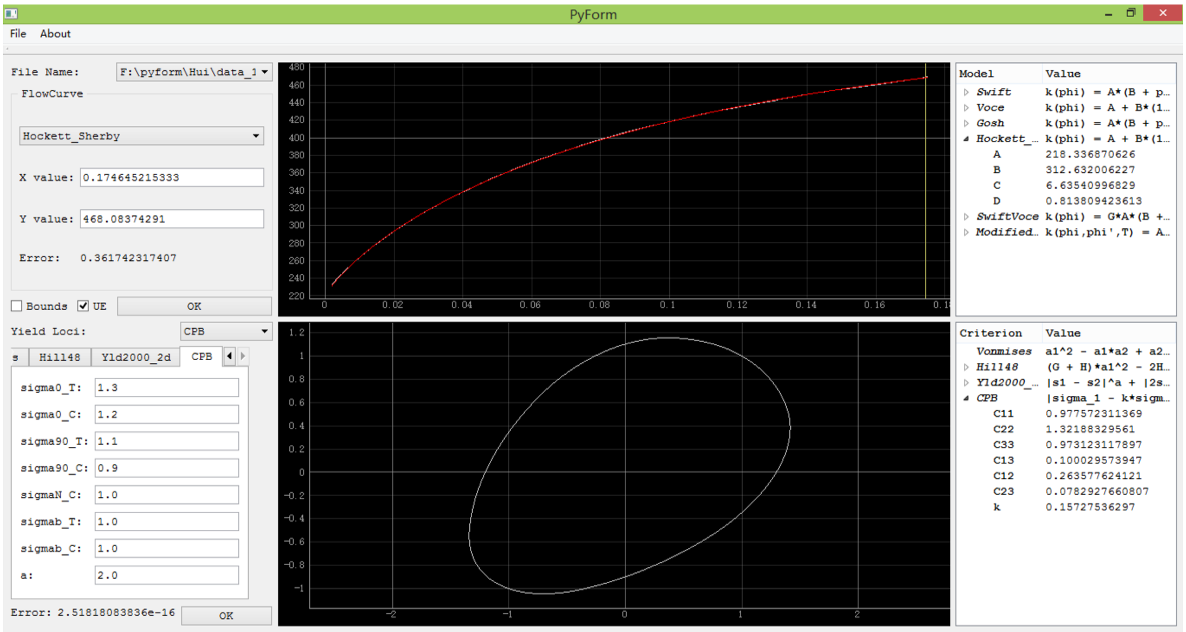

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Abstract

In this paper, flow curves and yield loci in the field of sheet metal forming will be implemented into a graphic user interface(GUI). They are used to describe the mechanical properties of materials such as steel, aluminum and titanium. The GUI has following functions:

- Users can select single or multiple data files directly from test-machines to apply flow curve fitting or input self-defined values to generate yield loci. The GUI will sample and catalogue data for further treatment.
- Multiple choices of flow curves and yield loci are available where the flow curves include Swift, Voce, Gosh, Hockett Sherby, mixed Swift and Voce, modified Zener Hollomon and the yield loci include von Mises, Hill48, Yld2000-2d, Cazacu-Plunkett-Barlat 2006 (CPB06).
- The GUI will show plots of the flow curves and yield loci. Original data and the corresponding fitting will appear in the same picture so as to compare the difference. Meanwhile the error will also be shown so that the user can choose the best fitting model. In addition, parameters will be generated for further treatment such as finite element analysis.

Contents

1 Introduction

In the field of sheet metal forming, finite element analysis (FEA) has become common and mature nowadays, especially with the popularity of modern FEA commercial softwares and powerful hardware. In order to increase the accuracy of the simulations, people try to find more accurate models to describe and predict the material behaviour. In this paper, we focus on the flow curves and yield loci which are essential for nonlinear FEA. Most common flow curves and yield loci will be collected in a GUI form. It helps the user to find quickly the most suitable model to describe the material behavior and bridge the gap between the experiment and simulation. Thus, the objective is to develop a software which analyzes the results of material test and generates the FEA software input parameters.

For the convenience of reading, some basic concepts are introduced here. Let stress and strain tensors be

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \tag{1.1}$$

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix} \tag{1.2}$$

where stress tensor $\boldsymbol{\sigma}$ and strain tensor $\boldsymbol{\epsilon}$ are both symmetric. Under the coordinate rotation $\boldsymbol{Q}$, the tensors become

$$\boldsymbol{\sigma}^{\boldsymbol{\theta}} = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T \tag{1.3}$$

$$\boldsymbol{\epsilon}^{\boldsymbol{\theta}} = \boldsymbol{Q}\boldsymbol{\epsilon}\boldsymbol{Q}^T \tag{1.4}$$

where θ is the angle of rotation around a certain axis.

The generalized Hooke's law can be written by Einstein summation,

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{1.5}$$

where C_{ijkl} is the tensor of the elastic constants of the material and composed of 21 independent constants due to symmetry conditions ($C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$). Hooke's law is a constitutive law applicable for elastic materials. Besides, for general materials, the strain can be decomposed into elastic part and plastic part,

$$\epsilon_{ij} = \epsilon^e_{ij} + \epsilon^p_{ij} \tag{1.6}$$

where the elastic strain is governed by Hooke's law. The plastic strain depends on the history of plastic deformation,

$$\epsilon^p_{ij} = \epsilon^p_{ij}(0) + \int_0^t \dot{\epsilon}^p_{ij}\mathrm{d}t \tag{1.7}$$

where $\epsilon^p_{ij}(0)$ is the initial value of the plastic strain at $t = 0$ and $\dot{\epsilon}^p_{ij}$ is the plastic strain rate. Therefore, how to compute the increment or rate of plastic strain is important for the whole diagram of the stress and the strain. In other words, a flow rule is needed.

For strain hardening material like metals, assume that there exits a yield function $f(\sigma_{ij})$

which obeys the following assumptions,

- $f = 0$ is a closed surface in stress space, which is called the yield surface or yield criterion. It tells whether the yielding occurs or not.
- No change in plastic strain occurs when $f < 0$. In other words, only the change of elastic strain is allowed inside the yield surface.
- Change in plastic strain occurs when $f = 0$. Specifically, the flow rule tells how much the plastic strain changes at that time, i.e. the increment or rate of plastic strain.
- $f > 0$ has no meaning and will not occur.

and the flow rule should satisfy the following conditions[7]

convexity The yield surface and all subsequent loading surfaces must be convex

normality The plastic strain increment vector must be normal to the loading surface at a regular point, and it must lie between adjacent normals to the loading surface at a corner of the surface

linearity The rate of change of plastic strain must be a linear function of the rate of change of the stress

Mathematically, the normality condition can be expressed as

$$\mathrm{d}\epsilon_{ij}^{p} = \mathrm{d}\lambda \frac{\partial f}{\partial \sigma_{ij}} \tag{1.8}$$

where $\mathrm{d}\lambda$ is a nonnegative constant called the plastic multiplier.

Due to the limitations of experimental devices, the stress and strain tensors cannot be observed from each direction. Instead, some simple tests (e.g. uniaxial tensile test and uniform compression test) will unveil the stress-strain relationship (flow curve) at certain directions. Normally, the flow curve can be express as a function $\sigma(\epsilon, \kappa)$ where κ depends on the history of plastic deformation. For example, the function becomes $\sigma(\epsilon, \dot{\epsilon}, T)$ when considering the time and temperature effect. The experimental results from these simple tests facilitate the validation of the yield function $f(\sigma_{ij})$ in view of the fact that the rule applicable in the 3D case should also be applicable in 1D cases.

In Chapter 2, flow laws for the simple tests will be presented. Especially, a modified Zener Hollomon model will be introduced for metals deformed under high temperature and strain rate. In Chapter 3, an isotropic yield criterion and orthotropic criterions are included. In Chapter 4, theory about flow curves and yield criterions will be implemented into the GUI. An overview of the program will be presented.

2 Flow curves

In the general uniaxial test, engineering stress σ_{eng} and engineering strain ϵ_{eng} are used,

$$\sigma_{eng} = \frac{F}{A_0} \tag{2.1}$$

$$\epsilon_{eng} = \frac{\Delta L}{L_0} = \frac{L - L_0}{L_0} \tag{2.2}$$

where F is the load normal to the cross section of the specimen, A_0 is the initial area of the cross section, L is the current length of the specimen, L_0 is the initial length. In comparison, true stress σ and true strain ϵ are defined as

$$\sigma = \frac{F}{A} \tag{2.3}$$

$$\epsilon = \int_{L_0}^{L} \frac{\mathrm{d}L}{L} = \ln(\frac{L}{L_0}) \tag{2.4}$$

where A is the current area of the cross section, $\mathrm{d}L$ is the infinitesimal length increment. The incompressibility condition requires,

$$A_0 L_0 = AL \tag{2.5}$$

which yields the relationship between true stress and engineering stress,

$$\sigma = \sigma_{eng}(1 + \epsilon_{eng}) \tag{2.6}$$

$$\epsilon = \ln(1 + \epsilon_{eng}) \tag{2.7}$$

A typical stress-strain curve of the metal is depicted in Fig.(1) where Eq.(1.6) still holds, but $\epsilon^e \ll \epsilon^p$ so it is reasonable to assume $\epsilon \approx \epsilon^p$[6]. The uniform elongation ϵ^u_{eng} is the strain corresponding to the maximum stress, which implies

$$\frac{\mathrm{d}\sigma_{eng}}{\mathrm{d}\epsilon_{eng}}|_{\epsilon^u_{eng}} = 0 \tag{2.8}$$

Substitute Eq.(2.6) and Eq.(2.7) in Eq.(2.8),

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\epsilon}|_{\epsilon^u} = \sigma(\epsilon^u) \tag{2.9}$$

where ϵ^u is the corresponding uniform elongation in true stress-strain curve, i.e. $\epsilon^u = \ln(1 + \epsilon^u_{eng})$. The uniform elongation is the separation point of uniform deformation and nonuniform deformation, which also indicates the start of necking. Because nearly all the subsequent elongation occurs in the neck, the area of the cross section can no longer be calculated from Eq.2.5. Therefore, Eq.(2.6) and Eq.(2.7) are only valid in the range of uniform deformation. The true stress-strain curve is monotonically increasing in this range.

The experimental flow curve is discrete and constrained in small strain range. A mathematical model is required to tell the stress or strain at any point of the curve and predict the material behavior beyond the point of localization of the uniaxial specimen, which is quite common in nonlinear FEA. Thus, many constitutive laws are proposed to fit and extrapolate

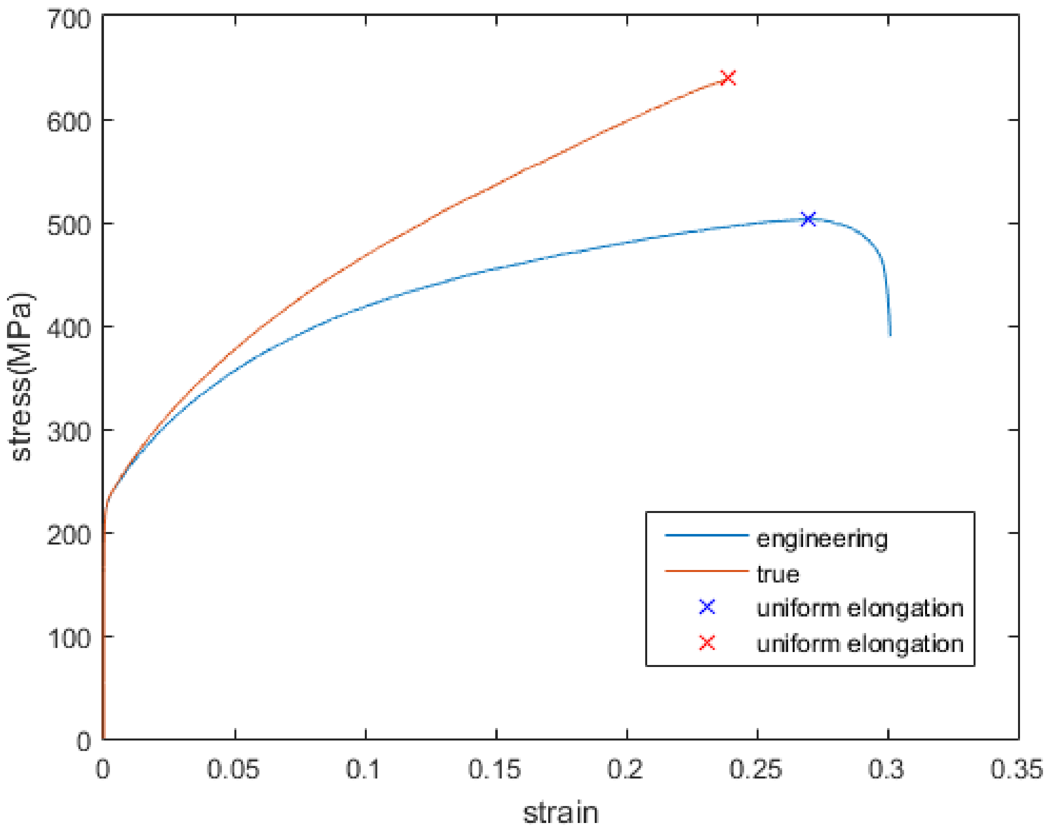

Figure 1: Comparison of engineering and true stress-strain

the experimental flow curve [8]. For example, Hollomon law is in the form

$$\sigma = K\epsilon^n \tag{2.10}$$

where K is the strength coefficient and n is the strain hardening index. Moreover, laws which will be implemented in the GUI are listed as follows,

$$Swift : \sigma(\epsilon) = C_1(C_2 + \epsilon)^{C_3} \tag{2.11}$$

$$Voce : \sigma(\epsilon) = C_1 + (C_2 - C_1)\mathrm{e}^{-C_3\epsilon} \tag{2.12}$$

$$Gosh : \sigma(\epsilon) = C_1 + C_2(C_3 + \epsilon)^{C_4} \tag{2.13}$$

$$HockettSherby : \sigma(\epsilon) = C_1 + (C_2 - C_1)\mathrm{e}^{-C_2\epsilon^{C_4}} \tag{2.14}$$

$$MixedSwiftVoce : \sigma(\epsilon) = C_1\sigma_{Swift}(\epsilon) + (1 - C_1)\sigma_{Voce}(\epsilon) \tag{2.15}$$

The Swift law can be regarded as a shifted Hollomon law for prestrained material[1]. The Voce law is used for flow curve at high temperature where C_1 is the saturation stress and C_2 is the back-extrapolated stress to $\epsilon = 0$ and $\frac{1}{C_3}$ is the strain parameter that defines the velocity with which the saturation stress is achieved [2]. The Gosh law adds a vertical shift C_1 on the Swift law. The Hockett Sherby law adds a strain hardening index C_4 on the Voce law. The mixed Swift Voce law is the combination of Swift law and Voce law. The fitting and extrapolation of these laws are depicted in Fig.(2). Most aforementioned laws are suitable for normal situations (i.e. room temperature, low strain rate). For materials deformed under high

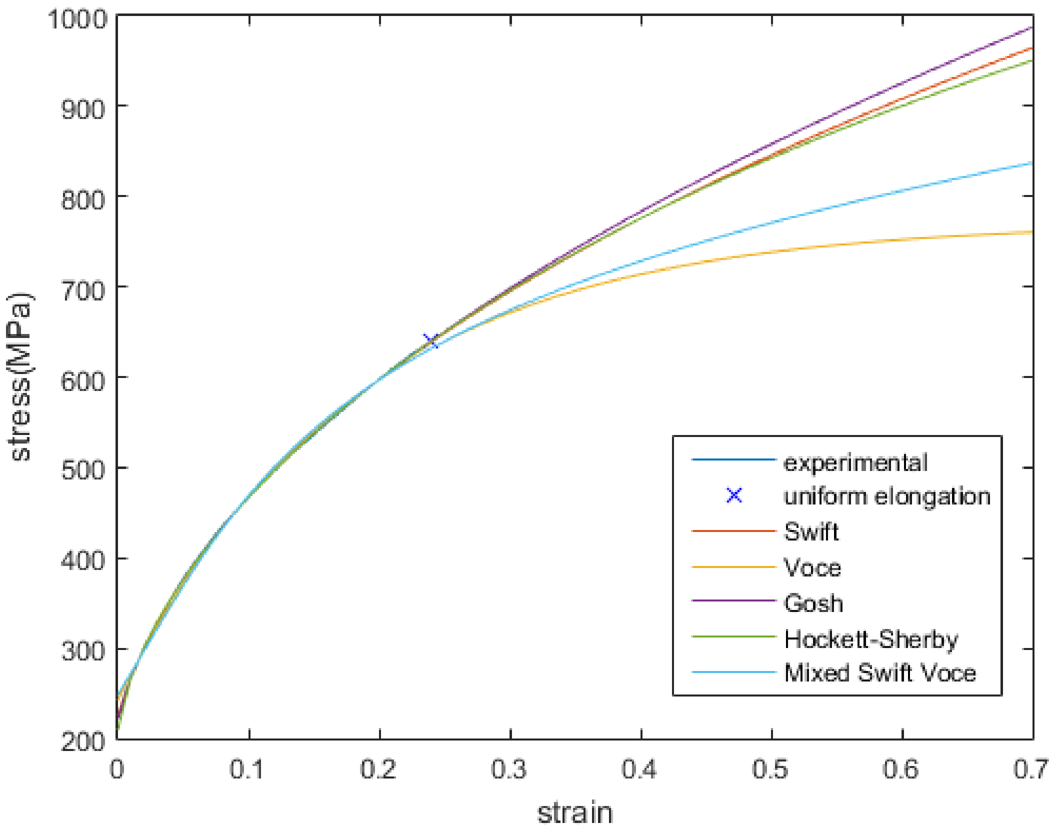

Figure 2: Flow curve fitting

temperature and high strain rate, Tong [11] proposed a modified Zener Hollomon model,

$$\sigma(\epsilon, \dot{\epsilon}, T) = A\mathrm{e}^{\frac{Q}{RT}}\dot{\epsilon}^m(1+\alpha\mathrm{e}^{-c(\epsilon-\epsilon_0)^2})(1-\beta\mathrm{e}^{-N\phi^n}) \tag{2.16}$$

where R is the gas constant ($R \approx 8.314$), $A\mathrm{e}^{\frac{Q}{RT}}\dot{\epsilon}^m$ is taken from the Zener Hollomon model,$(1+\alpha\mathrm{e}^{-c(\epsilon-\epsilon_0)^2})$ describes the softening effect and $(1-\beta\mathrm{e}^{-N\phi^n})$ describes the strain hardening effect. The modified Zener Hollomon model has 9 parameters: $A, Q, m, \alpha, c, \epsilon_0, \beta, N, n$ altogether. The application of the modified Zener Hollomon model on different temperatures are shown in Fig.(3). For further discussion about constitutive equations applied in metal forming, readers are referred to [9].

To implement the fitting, the basic idea is to solve the optimization problem,

$$\begin{aligned} &\min_x \|\sigma(x,\epsilon)-\sigma\|_2^2 = \min_x \sum_{i=1}^{n}(\sigma(x,\epsilon_i)-\sigma_i)^2 \\ s.t.&\frac{\mathrm{d}\sigma}{\mathrm{d}\epsilon}|_{\epsilon^u} = \sigma(\epsilon^u) \end{aligned} \tag{2.17}$$

where (ϵ_i, σ_i) is the data for fitting, n is the number of data and x is a vector containing the parameters of the model. Many gradient-based methods (e.g. the Gauss-Newton method) are suitable for the data fitting problem. However, in this case, the solution of problem (2.17) highly depends on the initial value if using a gradient-based method. Namely, the solution will easily drop into a local minimum for an unreasonable initial guess. Therefore, the differential evolution method [10] is adopted to ensure the global minimum under a reasonable domain

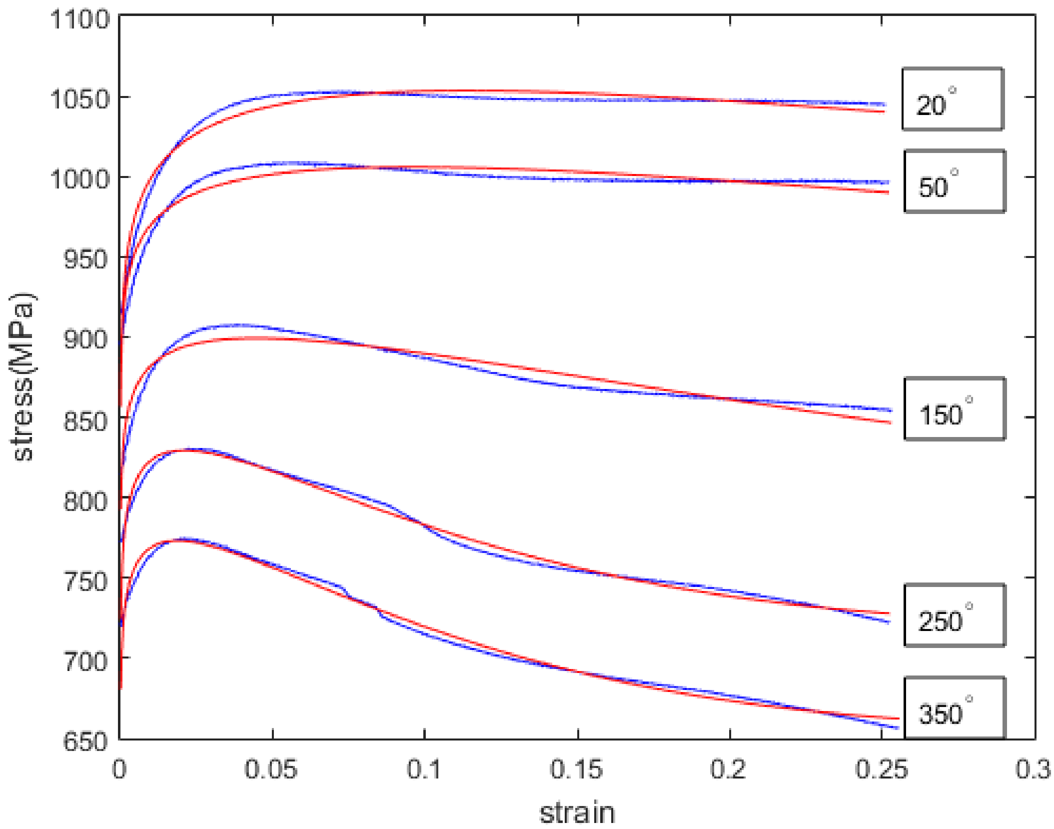

Figure 3: modified Zener Hollomon model (red) of stainless steel AISI 316L under $\dot{\epsilon} = 0.5$

of the initial value. In detail, the penalty method is used to build the objective function of problem (2.17), that is

$$f(x) = \sum_{i=1}^{n}(\sigma(x, \epsilon_i) - \sigma_i)^2 + w(\frac{\mathrm{d}\sigma}{\mathrm{d}\epsilon}|_{\epsilon^u} - \sigma(\epsilon^u))^2 \tag{2.18}$$

where w is the penalty factor which is usually a large number e.g. $w = 1000$ in the GUI.

3 Yield loci

A yield locus is the projection of the yield surface on the plane of principal stresses $\sigma_1 - \sigma_2$. For an isotropic material, based on the assumption that the material will yield when it reaches the critical distortion energy, the von Mises yield criterion is in the form,

$$\sigma_v^2 = \frac{1}{2}[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\sigma_{yz}^2 + \sigma_{xz}^2 + \sigma_{xy}^2)] \tag{3.1}$$

where σ_v is the equivalent stress. Convert Eq.(3.1) in the principal stress space $(\sigma_1, \sigma_2, \sigma_3)$,

$$\sigma_v^2 = \frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \tag{3.2}$$

and project the von Mises yield surface into the plane $\sigma_3 = 0$,

$$\sigma_v^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 \tag{3.3}$$

Take $\sigma_v = 1.0$, the yield locus is depicted in Fig.(4). Though the von Mises yield criterion is

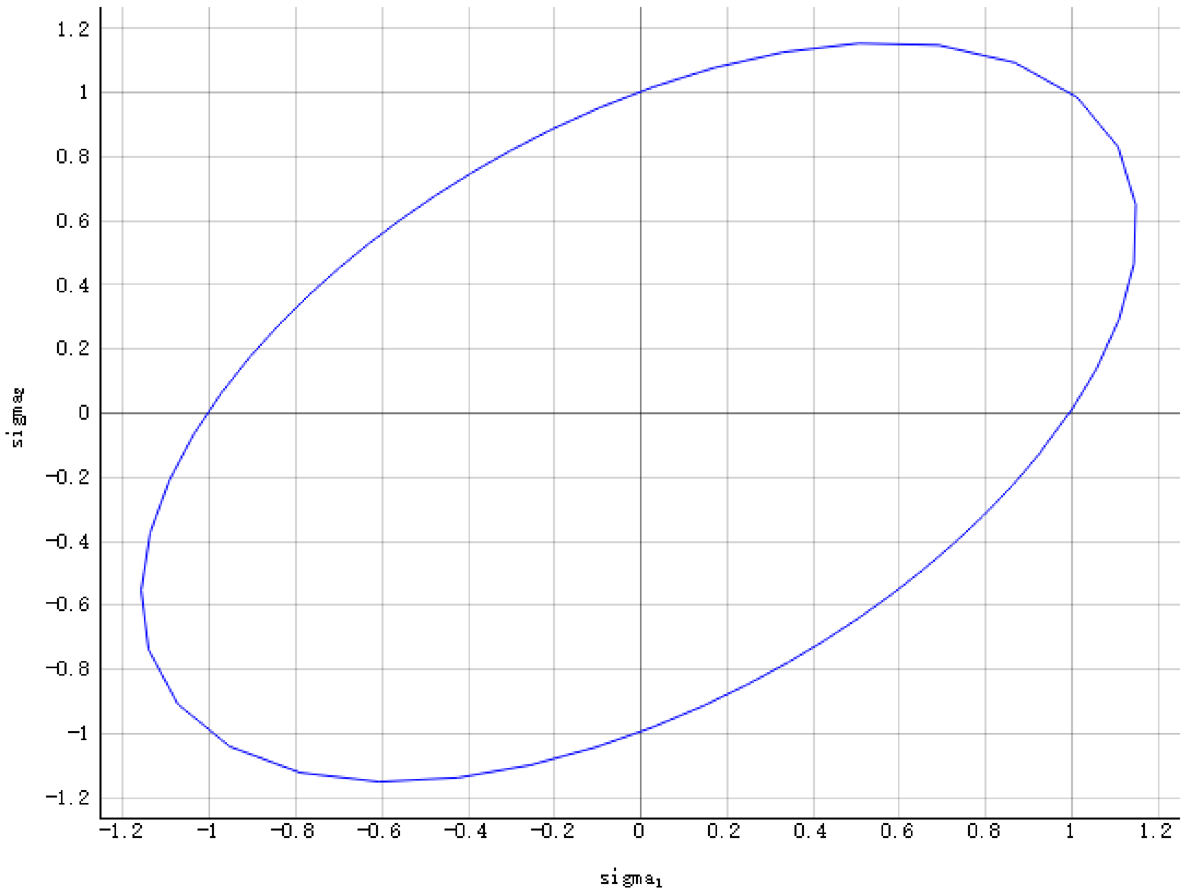

Figure 4: von Mises yield locus

basic and theoretical complete, it's not applicable for the anisotropic material. Thus, some specific yield loci in sheet metal forming will be introduced here. Nevertheless, all the yield criterions fulfill the convexity, normality and linearity conditions. For sheet metals, plane stress state is assumed since the thickness is small compared with the other dimensions. Let x,y,z directions be rolling direction, transverse direction and normal direction, in the uniaxial tensile test on sheet specimens on the θ direction (Figure 5), the coordinate rotation can be

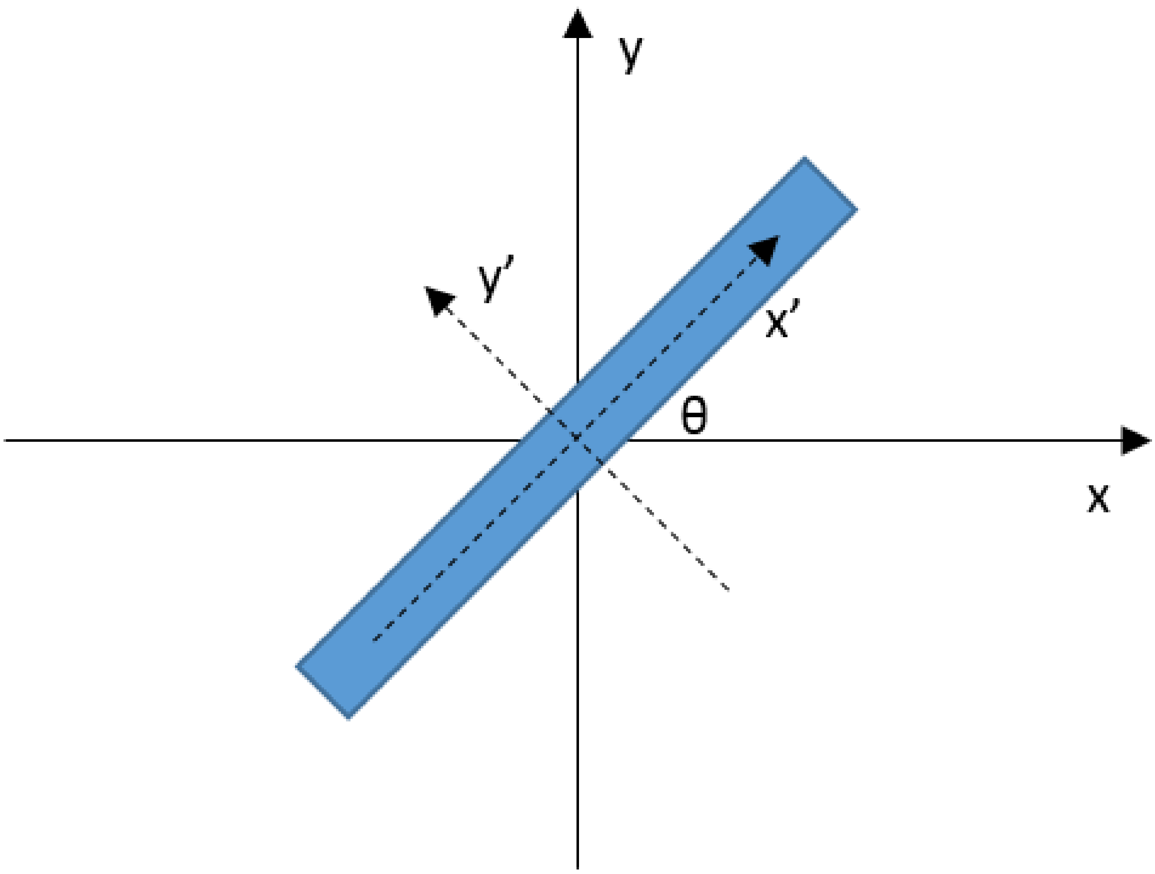

Figure 5: θ: the rotation angle around the z axis

express as

$$\boldsymbol{Q} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.4}$$

Substitute Eq.(3.4) and plane stress state ($\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$) in Eq.(1.3),

$$\begin{pmatrix} \sigma_{xx}^{\theta} \\ \sigma_{yy}^{\theta} \\ \sigma_{xy}^{\theta} \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} \tag{3.5}$$

Similarly, the strain tensor $\boldsymbol{\epsilon}$

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix} \tag{3.6}$$

under the rotated coordinates can be expressed as

$$\begin{pmatrix} \epsilon_{xx}^{\theta} \\ \epsilon_{yy}^{\theta} \\ \epsilon_{xy}^{\theta} \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{pmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{pmatrix} \tag{3.7}$$

If a general uniaxial stress in θ direction is

$$\begin{pmatrix} \sigma_{xx}^{\theta} \\ \sigma_{yy}^{\theta} \\ \sigma_{xy}^{\theta} \end{pmatrix} = \begin{pmatrix} \sigma_{\theta} \\ 0 \\ 0 \end{pmatrix} \tag{3.8}$$

then, from equation (3.5), the corresponding stress in the global coordinate system is

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} \sigma_{\theta} \cos^2\theta \\ \sigma_{\theta} \sin^2\theta \\ \sigma_{\theta} \cos\theta \sin\theta \end{pmatrix} \tag{3.9}$$

As for the biaxial test, the stress in the global coordinate system is

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} \sigma_{b} \\ \sigma_{b} \\ 0 \end{pmatrix} \tag{3.10}$$

where σ_b represents the yield stress in biaxial test.

Moreover, the Lankford parameter or r value is introduced to describe the anisotropy of the sheet metal and it is defined as

$$r_{\theta} = \frac{\mathrm{d}\epsilon_{yy}^{\theta}}{\mathrm{d}\epsilon_{zz}} \tag{3.11}$$

Due to incompressible condition: $\mathrm{d}\epsilon_{xx}^{\theta} + \mathrm{d}_{yy}^{\theta} + \mathrm{d}\epsilon_{zz} = 0$,

$$r_{\theta} = -\frac{\mathrm{d}\epsilon_{yy}^{\theta}}{\mathrm{d}\epsilon_{yy}^{\theta} + \mathrm{d}\epsilon_{xx}^{\theta}} \tag{3.12}$$

In addition, the biaxial Lankford parameter r_b is defined as

$$r_{b} = \frac{\mathrm{d}\epsilon_{yy}}{\mathrm{d}\epsilon_{xx}} \tag{3.13}$$

Substitute Eq.(3.7) in Eq.(3.12),

$$r_{\theta} = \frac{\mathrm{d}\epsilon_{yy}^{\theta}}{\mathrm{d}\epsilon_{zz}} = -\frac{\sin^2\theta \mathrm{d}\epsilon_{xx} + \cos^2\theta \mathrm{d}\epsilon_{yy} - 2\cos\theta\sin\theta \mathrm{d}\epsilon_{xy}}{\mathrm{d}\epsilon_{xx} + \mathrm{d}\epsilon_{yy}} \tag{3.14}$$

Taking into account Eq.(1.8),

$$r_{\theta} = -\frac{\sin^2\theta \frac{\partial f}{\partial \sigma_{xx}} + \cos^2\theta \frac{\partial f}{\partial \sigma_{yy}} - 2\cos\theta\sin\theta \frac{\partial f}{\partial \sigma_{xy}}}{\frac{\partial f}{\partial \sigma_{xx}} + \frac{\partial f}{\partial \sigma_{yy}}} \tag{3.15}$$

and the biaxial Lankford parameter becomes

$$r_{b} = \frac{\frac{\partial f}{\partial \sigma_{yy}}}{\frac{\partial f}{\partial \sigma_{xx}}} \tag{3.16}$$

In practice, uniaxial tensile test is usually taken on sheet specimen on $\theta = 0°, 45°, 90°$ direction. Since the uniaxial tensile test is not sufficient for describing the mechanical properties, the biaxial test is also taken. Thus, the corresponding yield stresses and r values are denoted as $\sigma_0, \sigma_{45}, \sigma_{90}, \sigma_b$ and r_0, r_{45}, r_{90}, r_b, which are the input values for the numerical optimization to generate the parameters of some yield criterions. In the following, the yield criterions for anisotropic material include Hill48[3], Yld2000-2d[4], CPB[5].

3.1 Hill48

Let the energy density (energy per volume) be ψ,

$$\psi = \boldsymbol{\sigma} : \boldsymbol{\epsilon} = \sigma_{ij}\epsilon_{ij} \tag{3.17}$$

where symbol ':' is contraction and subscripts i, j follow Einstein summation. Convert Hooke's law Eq.(1.5) in another form,

$$\boldsymbol{\epsilon} = \boldsymbol{D} : \boldsymbol{\sigma} = D_{ijkl}\sigma_{kl} \tag{3.18}$$

where $\boldsymbol{D}$ follows the symmetric condition ($D_{ijkl} = D_{jikl} = D_{ijlk} = D_{klij}$) and has 21 independent terms as a result. Assume the material to be orthotropic, $\boldsymbol{D}$ is reduced to 9 independent terms. Moreover, considering the incompressible condition ($tr(\boldsymbol{\epsilon}) = 0$) for metals, $\boldsymbol{D}$ decreases to 6 independent terms. Substitute Eq.(3.18) to Eq.(3.17),

$$\psi = \sigma_{ij}D_{ijkl}\sigma_{kl} \tag{3.19}$$

The Hill48 yield criterion is in the form,

$$2f(\sigma_{ij}) = F(\sigma_{yy}-\sigma_{zz})^2+G(\sigma_{zz}-\sigma_{xx})^2+H(\sigma_{xx}-\sigma_{yy})^2+2L\sigma_{yz}^2+2M\sigma_{xz}^2+2N\sigma_{xy}^2 = 1 \tag{3.20}$$

which is a quadratic function with 6 coefficients of anisotropy. Convert the Hill48 criterion in the plane stress state ($\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$),

$$2f(\sigma_{ij}) = F(\sigma_{yy})^2 + G(\sigma_{xx})^2 + H(\sigma_{xx} - \sigma_{yy})^2 + 2N\sigma_{xy}^2 = 1 \tag{3.21}$$

Furthermore, taking into account the test results σ_0, σ_{90},

$$(G + H)\sigma_0^2 = 1 \tag{3.22}$$

$$(F + H)\sigma_{90}^2 = 1 \tag{3.23}$$

and combined with Eq.(3.15) and Eq.(3.16),

$$r_0 = \frac{H}{G} \tag{3.24}$$

$$r_{90} = \frac{H}{F} \tag{3.25}$$

An interesting relationship can be obtained from the Eqs.(3.22), (3.23), (3.24), (3.25),

$$\frac{\sigma_0^2}{\sigma_{90}^2} = \frac{F + H}{G + H} = \frac{\frac{1}{r_{90}} + 1}{\frac{1}{r_0} + 1} \tag{3.26}$$

which indicates that $\sigma_{90} > \sigma_0$ if $r_{90} > r_0$ and vice versa. Most materials like aluminum alloy don't follow Eq.(3.26), so the Hill48 yield criterion only suits for some materials like steel.

By substituting F, G, H from Eqs.(3.22), (3.24), (3.25), the Hill48 criterion is expressed in the principal stress plane σ_1-σ_2 as follows,

$$\sigma_1^2 - \frac{2r_0}{1+r_0}\sigma_1\sigma_2 + \frac{r_0(1+r_{90})}{r_{90}(1+r_0)}\sigma_2^2 = \sigma_0^2 \tag{3.27}$$

If $\sigma_0 = \sigma_v$ and $r_0 = r_{90} = 1$, the Hill criterion is compatible with the von Mises criterion.

3.2 Yld2000-2d

The Yld2000 yield criterion is suitable for the materials with body-centered-cubic (bcc) or face-centered-cubic (fcc) structure, which is,

$$f(\sigma_{ij}) = |s_1' - s_2'|^a + |s_2' - s_3'|^a + |s_3' - s_1'|^a = 2\sigma_e^a \tag{3.28}$$

with

$$\boldsymbol{s'} = \boldsymbol{L\sigma} \tag{3.29}$$

where $\boldsymbol{L}$ is a linear transformation contributing the anisotropy and s_1', s_2', s_3' are the principal values of $\boldsymbol{s'}$ which fulfills $tr(\boldsymbol{s'}) = 0$. In particular, $\boldsymbol{s'}$ is the stress deviator $\boldsymbol{s}$ for an isotropic material. Generally, $\boldsymbol{s'}$ is the linear transformation of $\boldsymbol{s}$

$$\boldsymbol{s'} = \boldsymbol{Cs} \tag{3.30}$$

As a linear transformation of stress does not influence the convexity of yield function, different transformations are allowed. Especially for the plane stress, the Yld2000-2d yield criterion is

$$f(\sigma_{ij}) = f_1 + f_2 = |s_1' - s_2'|^a + |2s_2'' + s_1''|^a + |2s_1'' + s_2''|^a = 2\sigma_e^a \tag{3.31}$$

where $f_1 = |s_1' - s_2'|^a$ and $f_2 = |2s_2'' + s_1''|^a + |2s_1'' + s_2''|^a$. Note that different transformations $\boldsymbol{C'}$,$\boldsymbol{L'}$ and $\boldsymbol{C''}$,$\boldsymbol{L''}$ are used. Assuming the material is orthotropically symmetric, we have

$$\begin{pmatrix} s'_{xx} \\ s'_{yy} \\ x'_{xy} \end{pmatrix} = \begin{pmatrix} C'_{11} & C'_{12} & 0 \\ C'_{21} & C'_{22} & 0 \\ 0 & 0 & C'_{66} \end{pmatrix} \begin{pmatrix} s_{xx} \\ s_{yy} \\ s_{xy} \end{pmatrix} \tag{3.32}$$

$$\begin{pmatrix} s''_{xx} \\ s''_{yy} \\ x''_{xy} \end{pmatrix} = \begin{pmatrix} C''_{11} & C''_{12} & 0 \\ C''_{21} & C''_{22} & 0 \\ 0 & 0 & C''_{66} \end{pmatrix} \begin{pmatrix} s_{xx} \\ s_{yy} \\ s_{xy} \end{pmatrix} \tag{3.33}$$

Since,

$$\begin{pmatrix} s_{xx} \\ s_{yy} \\ s_{xy} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} \tag{3.34}$$

$$\boldsymbol{L'} = \begin{pmatrix} L'_{11} & L'_{12} & 0 \\ L'_{21} & L'_{22} & 0 \\ 0 & 0 & L'_{66} \end{pmatrix} = \begin{pmatrix} C'_{11} & C'_{12} & 0 \\ C'_{21} & C'_{22} & 0 \\ 0 & 0 & C'_{66} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.35}$$

Therefore,

$$\boldsymbol{L'} = \frac{1}{3} \begin{pmatrix} 2C'_{11} - C'_{12} & -C'_{11} + 2C'_{12} & 0 \\ 2C'_{21} - C'_{22} & -C'_{21} + 2C'_{22} & 0 \\ 0 & 0 & 3C'_{66} \end{pmatrix} \tag{3.36}$$

Similarly,

$$\boldsymbol{L''} = \frac{1}{3} \begin{pmatrix} 2C''_{11} - C''_{12} & -C''_{11} + 2C''_{12} & 0 \\ 2C''_{21} - C''_{22} & -C''_{21} + 2C''_{22} & 0 \\ 0 & 0 & 3C''_{66} \end{pmatrix} \tag{3.37}$$

Especially, Let $C'_{12} = C'_{21} = 0$, $\boldsymbol{C'}$ together with $\boldsymbol{C''}$ are reduced to 8 terms which are $C'_{11}, C'_{22}, C'_{33}, C''_{11}, C''_{22}, C''_{33}, C''_{12}, C''_{21}$ and denote

$$\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8] \tag{3.38}$$

where

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{pmatrix} = \begin{pmatrix} C'_{11} \\ C'_{22} \\ 2C''_{21} + C''_{11} \\ C''_{22} + \frac{C''_{12}}{2} \\ C''_{11} + \frac{C''_{21}}{2} \\ 2C''_{12} + C''_{22} \\ C'_{66} \\ C''_{66} \end{pmatrix} \tag{3.39}$$

$\boldsymbol{\alpha}$ is a linear transformation of $\boldsymbol{C'}$ and $\boldsymbol{C''}$. $\boldsymbol{\alpha}$ is used instead of $\boldsymbol{C'}$ and $\boldsymbol{C''}$ because the yield criterion is an isotropic criterion when all the terms of $\boldsymbol{\alpha}$ equal 1.0. Furthermore, replacing $\boldsymbol{C'}$ and $\boldsymbol{C''}$ in L' and L'' with a transformation of $\boldsymbol{\alpha}$,

$$\begin{pmatrix} L'_{11} \\ L'_{12} \\ L'_{21} \\ L'_{22} \\ L'_{66} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_7 \end{pmatrix} \tag{3.40}$$

$$\begin{pmatrix} L''_{11} \\ L''_{12} \\ L''_{21} \\ L''_{22} \\ L''_{66} \end{pmatrix} = \frac{1}{9} \begin{pmatrix} -2 & 2 & 8 & -2 & 0 \\ 1 & -4 & -4 & 4 & 0 \\ 4 & -4 & -4 & 1 & 0 \\ -2 & 8 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_8 \end{pmatrix} \tag{3.41}$$

To get the values of $\boldsymbol{\alpha}$, 8 independent equations are necessary. Uniaxial tensile specimen test at $0°, 45°, 90°$ with respect to rolling direction (x axis) and a biaxial test are usually carried out. They yield 8 experimental results: $\sigma_0, \sigma_{45}, \sigma_{90}, \sigma_b$ and r_0, r_{45}, r_{90}, r_b. All of the experimental results should fulfill the yield function $f(\boldsymbol{\sigma}, \boldsymbol{\alpha})$ and the associated flow rule. Thus, put $\sigma_0, \sigma_{45}, \sigma_{90}, \sigma_b$ into the yield function Eq.(3.31) to get the expressions of equivalent stress σ_e,

$$\sigma(\sigma_0, \boldsymbol{\alpha}) = \sigma_e \tag{3.42a}$$

$$\sigma(\sigma_{45}, \boldsymbol{\alpha}) = \sigma_e \tag{3.42b}$$

$$\sigma(\sigma_{90}, \boldsymbol{\alpha}) = \sigma_e \tag{3.42c}$$

$$\sigma(\sigma_b, \boldsymbol{\alpha}) = \sigma_e \tag{3.42d}$$

It is noted that σ_e is a constant in equations (3.42), so if we take $\sigma_e = \sigma(\sigma_0, \boldsymbol{\alpha})$ as a reference and substitute it in other 3 equations, Eqs.(3.42) are actually composed by 3 equations. By the same way, put $\sigma_0, \sigma_{45}, \sigma_{90}, \sigma_b$ into the instantaneous Lankford expression (3.15) and (3.16), the results of these equations should equal to the experimental results. Namely,

$$r(\sigma_0, \boldsymbol{\alpha}) = r_0 \tag{3.43a}$$

$$r(\sigma_{45}, \boldsymbol{\alpha}) = r_{45} \tag{3.43b}$$

$$r(\sigma_{90}, \boldsymbol{\alpha}) = r_{90} \tag{3.43c}$$

$$r(\sigma_b, \boldsymbol{\alpha}) = r_b \tag{3.43d}$$

So far, 8(actually 7) equations have been obtained for solving unknown parameters $\boldsymbol{\alpha}$. Unfortunately, they cannot be solved analytically. In other words, optimization methods such as the least square method are needed to get the parameters numerically. Thus, the objective function for the yield criterion is set as the sum of error between analytical estimation and experimental result. That is,

$$\min_{\boldsymbol{\alpha}} \sum_{\theta=0,45,90,b} (\frac{\sigma(\sigma_\theta, \boldsymbol{\alpha}) - \sigma_e}{\sigma_e})^2 + (\frac{r(\sigma_\theta, \boldsymbol{\alpha}) - r_\theta}{r_\theta})^2 \tag{3.44}$$

The utilization of such type of error: $(\frac{\sigma(\sigma_\theta, \boldsymbol{\alpha}) - \sigma_e}{\sigma_e})^2$ but not just $(\sigma(\sigma_\theta, \boldsymbol{\alpha}) - \sigma_e)^2$ is to prevent some term to be dominant while others are negligible. Unlike the optimization of flow curves, the gradient based optimization method will always find the global minimum because of the convexity of yield function.

To carry out the optimization, $\sigma(\sigma_\theta, \boldsymbol{\alpha})$ and $r(\sigma_\theta, \boldsymbol{\alpha})$ have to be found. For the function $\sigma(\sigma_\theta, \boldsymbol{\alpha})$, the procedures are

1. convert σ_θ to plane stress $\boldsymbol{\sigma}$ by equations (3.9) and (3.10)
2. obtain $\boldsymbol{s'} = \boldsymbol{L'}\boldsymbol{\sigma}$ and $\boldsymbol{s''} = \boldsymbol{L''}\boldsymbol{\sigma}$ by using equations (3.40) and (3.41) respectively.
3. find the eigenvalues $[s'_1, s'_2]$ and $[s''_1, s''_2]$ from $\boldsymbol{s'}$ and $\boldsymbol{s''}$
4. substitute $[s'_1, s'_2]$ and $[s''_1, s''_2]$ into yield function $f(\boldsymbol{\sigma})$ by (3.31)
5. return equivalent stress $\sigma_e = (\frac{f(\boldsymbol{\sigma})}{2.0})^{\frac{1}{a}}$

Especially, to get the eigenvalues $[s'_1, s'_2]$ or $[s''_1, s''_2]$, the eigenvalue problem of a 2D symmetric matrix $\boldsymbol{s}$ needs to be solved. Fortunately, a 2D matrix has analytical expressions of their eigenvalues,

$$s_1 = \frac{1}{2}(s_{xx} + s_{yy} + \sqrt{(s_{xx} - s_{yy})^2 + 4s_{xy}^2}) \tag{3.45a}$$

$$s_2 = \frac{1}{2}(s_{xx} + s_{yy} - \sqrt{(s_{xx} - s_{yy})^2 + 4s_{xy}^2}) \tag{3.45b}$$

Likewise, the procedure for the functions $r(\sigma_\theta, \boldsymbol{\alpha})$ are

1. convert σ_θ to plane stress $\boldsymbol{\sigma}$ by equations (3.9) and (3.10)
2. obtain derivatives $\frac{\partial f}{\partial \boldsymbol{\sigma}}$ using the chain rule
3. substitute $\frac{\partial f}{\partial \boldsymbol{\sigma}}$ into equation (3.15) and (3.16) to get r_θ
4. return the Lankford parameter r_θ

The most complicated step is in the application of the chain rule. So $\frac{\partial f}{\partial \boldsymbol{\sigma}}$ is expressed in detail here. The chain rule is

$$\begin{aligned} \frac{\partial f}{\partial \boldsymbol{\sigma}} &= \frac{\partial f}{\partial \boldsymbol{s}} : \frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\sigma}} \\ &= (\frac{\partial f}{\partial s_1}\frac{\partial s_1}{\partial \boldsymbol{s}} + \frac{\partial f}{\partial s_2}\frac{\partial s_2}{\partial \boldsymbol{s}}) : L \end{aligned} \tag{3.46}$$

The derivation of equations (3.45) with respect to $\boldsymbol{s}$ yields $\frac{\partial s_1}{\partial \boldsymbol{s}}$ and $\frac{\partial s_2}{\partial \boldsymbol{s}}$. For convenience, let $\Delta = (s_{xx} - s_{yy})^2 + 4s_{xy}^2$, if $\Delta \neq 0$, then

$$\frac{\partial s_1}{\partial s_{xx}} = \frac{1}{2}(1 + \frac{s_{xx} - s_{yy}}{\sqrt{\Delta}}) \tag{3.47a}$$

$$\frac{\partial s_1}{\partial s_{yy}} = \frac{1}{2}(1 - \frac{s_{xx} - s_{yy}}{\sqrt{\Delta}}) \tag{3.47b}$$

$$\frac{\partial s_1}{\partial s_{xy}} = \frac{2s_{xy}}{\sqrt{\Delta}} \tag{3.47c}$$

$$\frac{\partial s_2}{\partial s_{xx}} = \frac{1}{2}(1 - \frac{s_{xx} - s_{yy}}{\sqrt{\Delta}}) \tag{3.48a}$$

$$\frac{\partial s_2}{\partial s_{yy}} = \frac{1}{2}(1 + \frac{s_{xx} - s_{yy}}{\sqrt{\Delta}}) \tag{3.48b}$$

$$\frac{\partial s_2}{\partial s_{xy}} = -\frac{2s_{xy}}{\sqrt{\Delta}} \tag{3.48c}$$

For the case of f_1,

$$\frac{\partial f_1}{\partial s_1'} = a(s_1' - s_2')^{a-1}\mathrm{sign}(s_1' - s_2') \tag{3.49a}$$

$$\frac{\partial f_1}{\partial s_2'} = -a(s_1' - s_2')^{a-1}\mathrm{sign}(s_1' - s_2') \tag{3.49b}$$

where the function $\mathrm{sign}(x) = 1$ if $x > 0$;-1 if $x < 0$;0 if $x = 0$. If $\Delta' \neq 0$, $\frac{\partial s_1'}{\partial \boldsymbol{s'}}$ and $\frac{\partial s_2'}{\partial \boldsymbol{s'}}$ follow from the equations (3.47) and (3.48) respectively. Otherwise, if $\Delta' = 0$,

$$\frac{\partial f_1}{\partial s'_{xx}} = 0 \tag{3.50a}$$

$$\frac{\partial f_1}{\partial s'_{yy}} = 0 \tag{3.50b}$$

$$\frac{\partial f_1}{\partial s'_{xy}} = 0 \tag{3.50c}$$

For the case of f_2, the derivatives of f_2 are,

$$\frac{\partial f_2}{\partial s''_1} = a|2s''_2 + s''_1|^{a-1}\mathrm{sign}(2s''_2 + s''_1) + 2a|2s''_1 + s''_2|^{a-1}\mathrm{sign}(2s''_1 + s''_2) \tag{3.51a}$$

$$\frac{\partial f_2}{\partial s''_2} = 2a|2s''_2 + s''_1|^{a-1}\mathrm{sign}(2s''_2 + s''_1) + a|2s''_1 + s''_2|^{a-1}\mathrm{sign}(2s''_1 + s''_2) \tag{3.51b}$$

if $\Delta'' \neq 0$, $\frac{\partial s''_1}{\partial \boldsymbol{s}''}$ and $\frac{\partial s''_2}{\partial \boldsymbol{s}''}$ follow from the equations (3.47) and (3.48) respectively. Otherwise, if $\Delta'' = 0$,

$$\frac{\partial f_2}{\partial s''_{xx}} = \frac{\partial f_2}{\partial s''_1} \tag{3.52a}$$

$$\frac{\partial f_2}{\partial s''_{yy}} = \frac{\partial f_2}{\partial s''_2} \tag{3.52b}$$

$$\frac{\partial f_2}{\partial s''_{xy}} = 0 \tag{3.52c}$$

In summary, functions $\sigma(\sigma_\theta, \boldsymbol{\alpha})$ and $r(\sigma_\theta, \boldsymbol{\alpha})$ are listed for the help of programming.

Algorithm 1 Yld2000-2d yield criterion

function YLD2D($\sigma_\theta, \theta, a, \boldsymbol{\alpha}$)
  if $\theta = b$ **then** ▷ Biaxial Stress
    $\sigma_{xx} = \sigma_b$
    $\sigma_{yy} = \sigma_b$
    $\sigma_{xy} = 0$
  else ▷ Uniaxial Stress
    $\sigma_{xx} = \sigma_\theta \cos^2\theta$
    $\sigma_{yy} = \sigma_\theta \sin^2\theta$
    $\sigma_{xy} = \sigma_\theta \cos\theta \sin\theta$
  end if ▷ $\boldsymbol{L'}$,$\boldsymbol{L''}$
  $L'_{11} = \frac{2}{3}\alpha_1$
  $L'_{12} = -\frac{1}{3}\alpha_1$
  $L'_{21} = -\frac{1}{3}\alpha_2$
  $L'_{22} = \frac{2}{3}\alpha_2$
  $L'_{66} = \alpha_7$
  $L''_{11} = \frac{1}{9}(-2\alpha_3 + 2\alpha_4 + 8\alpha_5 - 2\alpha_6)$
  $L''_{12} = \frac{1}{9}(\alpha_3 - 4\alpha_4 - 4\alpha_5 + 4\alpha_6)$
  $L''_{21} = \frac{1}{9}(4\alpha_3 - 4\alpha_4 - 4\alpha_5 + \alpha_6)$
  $L''_{22} = \frac{1}{9}(-2\alpha_3 + 8\alpha_4 + 2\alpha_5 - 2\alpha_6)$ ▷ $\boldsymbol{s'}$,$\boldsymbol{s''}$
  $s'_{xx} = L'_{11}\sigma_{xx} + L'_{12}\sigma_{yy}$
  $s'_{yy} = L'_{21}\sigma_{xx} + L'_{22}\sigma_{yy}$
  $s'_{xy} = L'_{66}\sigma_{xy}$
  $s''_{xx} = L''_{11}\sigma_{xx} + L''_{12}\sigma_{yy}$
  $s''_{yy} = L''_{21}\sigma_{xx} + L''_{22}\sigma_{yy}$
  $s''_{xy} = L''_{66}\sigma_{xy}$ ▷ Δ',Δ''
  $\Delta' = (s'_{xx} - s'_{yy})^2 + 4{s'_{xy}}^2$
  $\Delta'' = (s''_{xx} - s''_{yy})^2 + 4{s''_{xy}}^2$ ▷ s'_1, s'_2, s''_1, s''_2
  $s'_1 = \frac{1}{2}(s'_{xx} + s'_{yy} + \sqrt{\Delta'})$
  $s'_2 = \frac{1}{2}(s'_{xx} + s'_{yy} - \sqrt{\Delta'})$
  $s''_1 = \frac{1}{2}(s''_{xx} + s''_{yy} + \sqrt{\Delta''})$
  $s''_2 = \frac{1}{2}(s''_{xx} + s''_{yy} - \sqrt{\Delta''})$ ▷ f_1, f_2, f, σ_e
  $f_1 = |s'_1 - s'_2|^a$
  $f_2 = |2s''_2 + s''_1|^a + |2s''_1 + s''_2|^a$
  $f = f_1 + f_2$
  $\sigma_e = (\frac{f}{2})^{\frac{1}{a}}$ ▷ $\frac{\partial f_1}{\partial s'_1}, \frac{\partial f_1}{\partial s'_2}, \frac{\partial f_2}{\partial s''_1}, \frac{\partial f_2}{\partial s''_2}$

Algorithm 1 Yld2000-2d yield criterion(continued)

$\frac{\partial f_1}{\partial s'_1} = a(s'_1 - s'_2)^{a-1}\mathrm{sign}(s'_1 - s'_2)$
$\frac{\partial f_1}{\partial s'_2} = -a(s'_1 - s'_2)^{a-1}\mathrm{sign}(s'_1 - s'_2)$
$\frac{\partial f_2}{\partial s''_1} = a|2s''_2 + s''_1|^{a-1}\mathrm{sign}(2s''_2 + s''_1) + 2a|2s''_1 + s''_2|^{a-1}\mathrm{sign}(2s''_1 + s''_2)$
$\frac{\partial f_2}{\partial s''_2} = 2a|2s''_2 + s''_1|^{a-1}\mathrm{sign}(2s''_2 + s''_1) + a|2s''_1 + s''_2|^{a-1}\mathrm{sign}(2s''_1 + s''_2)$
if $\Delta' \neq 0$ **then**
$\frac{\partial s'_1}{\partial s'_{xx}} = \frac{1}{2}(1 + \frac{s'_{xx} - s'_{yy}}{\sqrt{\Delta'}})$
$\frac{\partial s'_1}{\partial s'_{yy}} = \frac{1}{2}(1 - \frac{s'_{xx} - s'_{yy}}{\sqrt{\Delta'}})$
$\frac{\partial s'_1}{\partial s'_{xy}} = \frac{2s'_{xy}}{\sqrt{\Delta'}}$
$\frac{\partial s'_2}{\partial s'_{xx}} = \frac{1}{2}(1 - \frac{s'_{xx} - s'_{yy}}{\sqrt{\Delta'}})$
$\frac{\partial s'_2}{\partial s'_{yy}} = \frac{1}{2}(1 + \frac{s'_{xx} - s'_{yy}}{\sqrt{\Delta'}})$
$\frac{\partial s'_2}{\partial s'_{xy}} = -\frac{2s'_{xy}}{\sqrt{\Delta'}}$
$\frac{\partial f_1}{\partial s'_{xx}} = \frac{\partial f_1}{\partial s'_1}\frac{\partial s'_1}{\partial s'_{xx}} + \frac{\partial f_1}{\partial s'_2}\frac{\partial s'_2}{\partial s'_{xx}}$
$\frac{\partial f_1}{\partial s'_{yy}} = \frac{\partial f_1}{\partial s'_1}\frac{\partial s'_1}{\partial s'_{yy}} + \frac{\partial f_1}{\partial s'_2}\frac{\partial s'_2}{\partial s'_{yy}}$
$\frac{\partial f_1}{\partial s'_{xy}} = \frac{\partial f_1}{\partial s'_1}\frac{\partial s'_1}{\partial s'_{xy}} + \frac{\partial f_1}{\partial s'_2}\frac{\partial s'_2}{\partial s'_{xy}}$
else
$\frac{\partial f_1}{\partial s'_{xx}} = 0$
$\frac{\partial f_1}{\partial s'_{yy}} = 0$
$\frac{\partial f_1}{\partial s'_{xy}} = 0$
end if
if $\Delta'' \neq 0$ **then**
$\frac{\partial s''_1}{\partial s''_{xx}} = \frac{1}{2}(1 + \frac{s''_{xx} - s''_{yy}}{\sqrt{\Delta''}})$
$\frac{\partial s''_1}{\partial s''_{yy}} = \frac{1}{2}(1 - \frac{s''_{xx} - s''_{yy}}{\sqrt{\Delta''}})$
$\frac{\partial s''_1}{\partial s''_{xy}} = \frac{2s''_{xy}}{\sqrt{\Delta''}}$
$\frac{\partial s''_2}{\partial s''_{xx}} = \frac{1}{2}(1 - \frac{s''_{xx} - s''_{yy}}{\sqrt{\Delta''}})$
$\frac{\partial s''_2}{\partial s''_{yy}} = \frac{1}{2}(1 + \frac{s''_{xx} - s''_{yy}}{\sqrt{\Delta''}})$
$\frac{\partial s''_2}{\partial s''_{xy}} = -\frac{2s''_{xy}}{\sqrt{\Delta''}}$
$\frac{\partial f_2}{\partial s''_{xx}} = \frac{\partial f_2}{\partial s''_1}\frac{\partial s''_1}{\partial s''_{xx}} + \frac{\partial f_2}{\partial s''_2}\frac{\partial s''_2}{\partial s''_{xx}}$
$\frac{\partial f_2}{\partial s''_{yy}} = \frac{\partial f_2}{\partial s''_1}\frac{\partial s''_1}{\partial s''_{yy}} + \frac{\partial f_2}{\partial s''_2}\frac{\partial s''_2}{\partial s''_{yy}}$
$\frac{\partial f_2}{\partial s''_{xy}} = \frac{\partial f_2}{\partial s''_1}\frac{\partial s''_1}{\partial s''_{xy}} + \frac{\partial f_2}{\partial s''_2}\frac{\partial s''_2}{\partial s''_{xy}}$
else
$\frac{\partial f_2}{\partial s''_{xx}} = \frac{\partial f_2}{\partial s''_1}$
$\frac{\partial f_2}{\partial s''_{yy}} = \frac{\partial f_2}{\partial s''_2}$
$\frac{\partial f_2}{\partial s''_{xy}} = 0$
end if $\triangleright$ $\frac{\partial f}{\boldsymbol{\sigma}}$

Algorithm 1 Yld2000-2d yield criterion(continued)

$\frac{\partial f}{\sigma_{xx}} = \frac{\partial f_1}{s'_{xx}} L'_{11} + \frac{\partial f_1}{s'_{yy}} L'_{21} + \frac{\partial f_2}{s''_{xx}} L''_{11} + \frac{\partial f_2}{s''_{yy}} L''_{21}$

$\frac{\partial f}{\sigma_{yy}} = \frac{\partial f_1}{s'_{xx}} L'_{12} + \frac{\partial f_1}{s'_{yy}} L'_{22} + \frac{\partial f_2}{s''_{xx}} L''_{12} + \frac{\partial f_2}{s''_{yy}} L''_{22}$

$\frac{\partial f}{\sigma_{xy}} = \frac{\partial f_1}{s'_{xy}} L'_{66} + \frac{\partial f_2}{s''_{xy}} L''_{66}$ ▷ r_θ

if $\theta = b$ **then** ▷ Biaxial Stress

$r_\theta = \frac{\frac{\partial f}{\partial \sigma_{yy}}}{\frac{\partial f}{\partial \sigma_{xx}}}$

else ▷ Uniaxial Stress

$r_\theta = -\frac{\sin^2\theta \frac{\partial f}{\partial \sigma_{xx}} + \cos^2\theta \frac{\partial f}{\partial \sigma_{yy}} - 2\cos\theta\sin\theta \frac{\partial f}{\partial \sigma_{xy}}}{\frac{\partial f}{\partial \sigma_{xx}} + \frac{\partial f}{\partial \sigma_{yy}}}$

end if

return σ_e, r_θ

end function

Algorithm (1) can be split into two functions $\sigma(\sigma_\theta, \theta, a, \boldsymbol{\alpha})$ with return value σ_e and $r(\sigma_\theta, \theta, a, \boldsymbol{\alpha})$ with return value r_θ, then optimization (3.44) with $\sigma_0, \sigma_{45}, \sigma_{90}, \sigma_b, r_0, r_{45}, r_{90}, r_b, a$ as input will output the parameters of the yield function, which are $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$. It is noted that the input constant a can be selected as 8 for fcc structure and 6 for bcc structure. For example, set all the input $\sigma_0 = \sigma_{45} = \sigma_{90} = \sigma_b = r_0 = r_{45} = r_{90} = r_b = 1.0$ and $a = 8$, the Yld2000-2d yield locus is depicted in Fig.(6),

3.3 CPB06

Observing the yield loci mentioned before, the flow stresses in tension σ_0^T and σ_{90}^T are always equal to the absolute values of the flow stresses in compression σ_0^C and σ_{90}^C. In fact, tension and compression are not distinguished in the von Mises, Hill48, Yld2000-2d yield criterions. However, for materials with hexagonal closed packed (hcp) structure, tensile test and compression test yield different values of flow stress. The previous yield criterions are inapplicable for such materials. Hence, in 2006, O.Cazacu, B.Plunkett and F.Barlat proposed an orthotropic yield criterion (CPB06)[5] and applied it to the plane stress case. Introducing

$$\boldsymbol{s'} = \boldsymbol{C}\boldsymbol{s} \tag{3.53}$$

which is $s'_{ij} = C_{ijkl}s_{kl} = C_{ijkl}(\sigma_{kl} - \frac{1}{3}\delta_{kl}\sigma_{mm})$ in the form of Einstein summation. Then, the CPB06 yield criterion is,

$$(|s'_1| - ks'_1)^a + (|s'_2| - ks'_2)^a + (|s'_3| - ks'_3)^a = F \tag{3.54}$$

where s'_1, s'_2, s'_3 are the eigenvalues of $\boldsymbol{s'}$, integer $a \geq 1$ and $k \in [-1, 1]$. These restrictions ensure convexity. In Voigt notation, $\boldsymbol{C}$ has following symmetric form because of orthotropy,

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix} \tag{3.55}$$

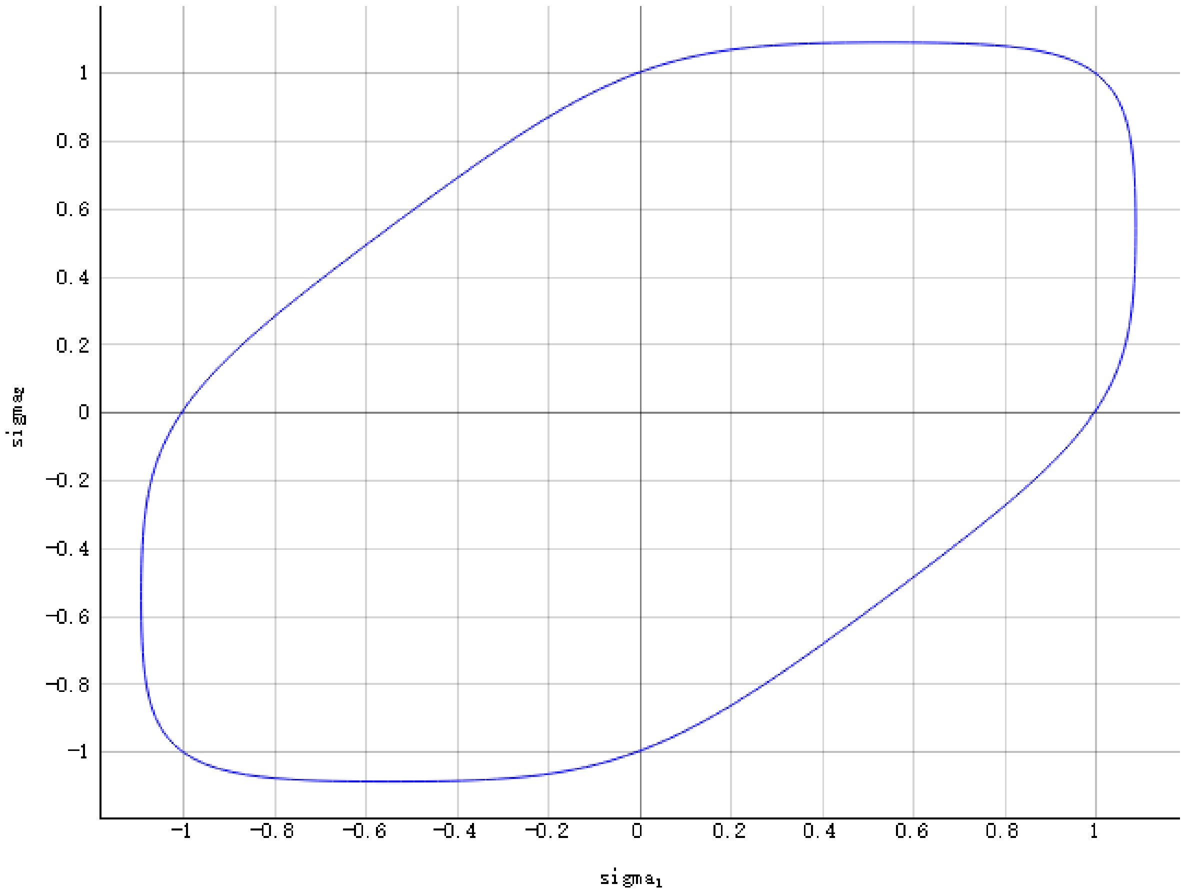

Figure 6: Yld2000-2d yield locus

Substituting equation (3.55) into equation (3.53) and expressing it in the form of $\boldsymbol{\sigma}$,

$$
\begin{aligned}
s'_{xx} &= (\frac{2}{3}C_{11} - \frac{1}{3}C_{12} - \frac{1}{3}C_{13})\sigma_{xx} + (-\frac{1}{3}C_{11} + \frac{2}{3}C_{12} - \frac{1}{3}C_{13})\sigma_{yy} \\
&\quad + (-\frac{1}{3}C_{11} - \frac{1}{3}C_{12} + \frac{2}{3}C_{13})\sigma_{zz} \\
s'_{yy} &= (\frac{2}{3}C_{21} - \frac{1}{3}C_{22} - \frac{1}{3}C_{23})\sigma_{xx} + (-\frac{1}{3}C_{21} + \frac{2}{3}C_{22} - \frac{1}{3}C_{23})\sigma_{yy} \\
&\quad + (-\frac{1}{3}C_{21} - \frac{1}{3}C_{22} + \frac{2}{3}C_{23})\sigma_{zz} \\
s'_{zz} &= (\frac{2}{3}C_{31} - \frac{1}{3}C_{32} - \frac{1}{3}C_{33})\sigma_{xx} + (-\frac{1}{3}C_{31} + \frac{2}{3}C_{32} - \frac{1}{3}C_{33})\sigma_{yy} \\
&\quad + (-\frac{1}{3}C_{31} - \frac{1}{3}C_{32} + \frac{2}{3}C_{33})\sigma_{zz} \\
s'_{yz} &= C_{44}\sigma_{yz} \\
s'_{xz} &= C_{55}\sigma_{xz} \\
s'_{xy} &= C_{66}\sigma_{xy}
\end{aligned}
\tag{3.56}
$$

Thus, we have a function $F(\boldsymbol{\sigma}, a, \boldsymbol{C}, k)$ which returns F as value of the yield criterion (3.54). For simplicity afterwards, following the notation in the paper [5], let

$$\begin{aligned}\Phi_1 &= \frac{2}{3}C_{11} - \frac{1}{3}C_{12} - \frac{1}{3}C_{13} \\ \Phi_2 &= \frac{2}{3}C_{21} - \frac{1}{3}C_{22} - \frac{1}{3}C_{23} \\ \Phi_3 &= \frac{2}{3}C_{31} - \frac{1}{3}C_{32} - \frac{1}{3}C_{33}\end{aligned} \tag{3.57}$$

$$\begin{aligned}\Psi_1 &= -\frac{1}{3}C_{11} + \frac{2}{3}C_{12} - \frac{1}{3}C_{13} \\ \Psi_2 &= -\frac{1}{3}C_{21} + \frac{2}{3}C_{22} - \frac{1}{3}C_{23} \\ \Psi_3 &= -\frac{1}{3}C_{31} + \frac{2}{3}C_{32} - \frac{1}{3}C_{33}\end{aligned} \tag{3.58}$$

$$\begin{aligned}\Omega_1 &= -\frac{1}{3}C_{11} - \frac{1}{3}C_{12} + \frac{2}{3}C_{13} \\ \Omega_2 &= -\frac{1}{3}C_{21} - \frac{1}{3}C_{22} + \frac{2}{3}C_{23} \\ \Omega_3 &= -\frac{1}{3}C_{31} - \frac{1}{3}C_{32} + \frac{2}{3}C_{33}\end{aligned} \tag{3.59}$$

Considering a 3D tension and compression case without shear stress, $\boldsymbol{\sigma}$ turns to be diagonal: $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ and $\boldsymbol{C}$ can be reduced to 6 independent terms which are C_{11},C_{22},C_{33},C_{23},C_{13},C_{12}. Including k, the CPB06 yield criterion has 7 unknown parameters(because a is set to 2 in general), which means 7 equations, in other words, 7 experimental values are necessary to get the solution. Therefore, we adopt tension and compression stress: $\sigma_0^T, \sigma_0^C, \sigma_{90}^T, \sigma_{90}^C$ and biaxial stress: σ_b^T, σ_b^C where 'T','C' denote tension and compression respectively. It is noted that we take absolute values of stress here so σ_θ^T and σ_θ^C are both positive values. Compared with the Hill48 criterion σ_{45} is not used because its stress tensor is not in the diagonal form. Moreover, the Lankford parameter r_θ is not used either here for simplicity. Instead, a normal stress σ_n^C which results from stack compression tests is added into the input. It is the compression stress in z axis. For future improvement of GUI, the expressions of the Lankford parameters and pure shear from the paper [5] will be appended here because including the Lankford parameter estimation in the objective function will raise the accuracy of optimization and shear stress $\sigma_{xy}, \sigma_{xz}, \sigma_{yz}$ will contribute the determination of the other terms: C_{66}, C_{55}, C_{44}. In conclusion, stress inputs of function $F(\boldsymbol{\sigma}, a, \boldsymbol{C}, k)$ are $\sigma_0^T, \sigma_0^C, \sigma_{90}^T, \sigma_{90}^C, \sigma_b^T, \sigma_b^C, \sigma_n^C$, which generate the parameters $C_{11}, C_{22}, C_{33}, C_{23}, C_{13}, C_{12}, k$. Based on paper [5], stresses can be expressed as follows,

$$\begin{aligned}\sigma_0^T &= \left[\frac{F}{(|\Phi_1| - k\Phi_1)^a + (|\Phi_2| - k\Phi_2)^a + (|\Phi_3| - k\Phi_3)^a}\right]^{\frac{1}{a}} \\ \sigma_0^C &= \left[\frac{F}{(|\Phi_1| + k\Phi_1)^a + (|\Phi_2| + k\Phi_2)^a + (|\Phi_3| + k\Phi_3)^a}\right]^{\frac{1}{a}}\end{aligned} \tag{3.60}$$

$$\begin{aligned}\sigma_{90}^T &= \left[\frac{F}{(|\Psi_1| - k\Psi_1)^a + (|\Psi_2| - k\Psi_2)^a + (|\Psi_3| - k\Psi_3)^a}\right]^{\frac{1}{a}} \\ \sigma_{90}^C &= \left[\frac{F}{(|\Psi_1| + k\Psi_1)^a + (|\Psi_2| + k\Psi_2)^a + (|\Psi_3| + k\Psi_3)^a}\right]^{\frac{1}{a}}\end{aligned} \tag{3.61}$$

$$\begin{aligned}\sigma_b^T &= \left[\frac{F}{(|\Omega_1|+k\Omega_1)^a+(|\Omega_2|+k\Omega_2)^a+(|\Omega_3|+k\Omega_3)^a}\right]^{\frac{1}{a}}\\ \sigma_b^C &= \left[\frac{F}{(|\Omega_1|-k\Omega_1)^a+(|\Omega_2|-k\Omega_2)^a+(|\Omega_3|-k\Omega_3)^a}\right]^{\frac{1}{a}}\end{aligned} \tag{3.62}$$

$$\begin{aligned}\sigma_n^T &= \left[\frac{F}{(|\Omega_1|-k\Omega_1)^a+(|\Omega_2|-k\Omega_2)^a+(|\Omega_3|-k\Omega_3)^a}\right]^{\frac{1}{a}}\\ \sigma_n^C &= \left[\frac{F}{(|\Omega_1|+k\Omega_1)^a+(|\Omega_2|+k\Omega_2)^a+(|\Omega_3|+k\Omega_3)^a}\right]^{\frac{1}{a}}\end{aligned} \tag{3.63}$$

Note that σ_b^T and σ_n^C (or σ_b^C and σ_n^T) have the same expression but different physical meaning. In fact, it is the hydraulic pressure $-\sigma_n^C$ that contributes biaxial stress σ_b^T in a real test. The expression of σ_n^T is for theoretical completeness but only σ_n^C is for the test. That's because it's not easy to do tensile test on thickness direction of sheet metal. In addition, pure shear τ_0 and the Lankford parameters r_0, r_{90} of plane stress state are appended here:

$$\tau_0 = \left[\frac{F}{(|C_{66}|+kC_{66})^a+(|C_{66}|-kC_{66})^a}\right]^{\frac{1}{a}} \tag{3.64}$$

$$\begin{aligned}r_0^T &= -\frac{(1-k)^a\Phi_1^{a-1}\Psi_1+(-1-k)^a(\Phi_2^{a-1}\Psi_2+\Phi_3^{a-1}\Psi_3)}{(1-k)^a\Phi_1^{a-1}(\Psi_1+\Phi_1)+(-1-k)^a(\Phi_2^{a-1}\Psi_2+\Phi_3^{a-1}\Psi_3+\Phi_2^a+\Phi_3^a)}\\ r_0^C &= -\frac{(-1-k)^a\Phi_1^{a-1}\Psi_1+(1-k)^a(\Phi_2^{a-1}\Psi_2+\Phi_3^{a-1}\Psi_3)}{(-1-k)^a\Phi_1^{a-1}(\Psi_1+\Phi_1)+(1-k)^a(\Phi_2^{a-1}\Psi_2+\Phi_3^{a-1}\Psi_3+\Phi_2^a+\Phi_3^a)}\end{aligned} \tag{3.65}$$

$$\begin{aligned}r_{90}^T &= -\frac{(1-k)^a\Psi_2^{a-1}\Phi_2+(-1-k)^a(\Psi_1^{a-1}\Phi_1+\Psi_3^{a-1}\Phi_3)}{(1-k)^a\Psi_2^{a-1}(\Phi_2+\Psi_2)+(-1-k)^a(\Psi_1^{a-1}\Phi_1+\Psi_3^{a-1}\Phi_3+\Psi_1^a+\Psi_3^a)}\\ r_{90}^C &= -\frac{(-1-k)^a\Psi_2^{a-1}\Phi_2+(1-k)^a(\Psi_1^{a-1}\Phi_1+\Psi_3^{a-1}\Phi_3)}{(-1-k)^a\Psi_2^{a-1}(\Phi_2+\Psi_2)+(1-k)^a(\Psi_1^{a-1}\Phi_1+\Psi_3^{a-1}\Phi_3+\Psi_1^a+\Psi_3^a)}\end{aligned} \tag{3.66}$$

A function $\sigma_\theta(F, a, \boldsymbol{C}, k)$ returns the vector $\sigma_\theta = [\sigma_\theta^T, \sigma_\theta^C]$. Since the test results σ_0^T, σ_0^C, σ_{90}^T, σ_{90}^C, σ_b^T, σ_b^C, σ_n^C should locate on the yield locus, the error $\|\frac{\sigma_\theta(F,a,\boldsymbol{C},k)-\sigma_\theta}{\sigma_\theta}\|^2$ should be minimized. Therefore, the objective function is

$$\min_{\boldsymbol{C}} \sum_{\theta=0,90,b,n} \|\frac{\sigma_\theta(F,a,\boldsymbol{C},k)-\sigma_\theta}{\sigma_\theta}\|^2 \tag{3.67}$$

where '$\|\cdot\|$' is vector norm. By a nonlinear least square method, the optimization will yield the parameters $C_{11}, C_{22}, C_{33}, C_{23}, C_{13}, C_{12}, k$. The convexity of the yield function ensures the solution to be the global minimum. To prevent that parameter k exceeds the interval $[-1, 1]$, constraints $k \in [-1, 1]$ could be added on objective function. However, it is generally unnecessary if we set an appropriate initial guess of k such as 0.

To help programming, the procedures of building the function $\sigma_\theta(F, a, \boldsymbol{C}, k)$ are listed here. The constant F needs to be specified first. Without loss of generality, let $F = F(\sigma_0^T, a, \boldsymbol{C}, k)$. Obviously, $\sigma_0(F, a, \boldsymbol{C}, k) = \sigma_0$. Then, substituting the value of F into Eqs.(3.60), (3.61), (3.62), (3.63) and put them into the objective function (3.67). Here is the key procedure in summary:

Algorithm 2 CPB06 yield criterion

function $F(\boldsymbol{\sigma}, a, \boldsymbol{C}, k)$ $\triangleright\ \sigma_{xx}, \sigma_{yy}, \sigma_{zz} \to s'_{xx}, s'_{yy}, s'_{zz}$

$s'_{xx} = (\frac{2}{3}C_{11} - \frac{1}{3}C_{12} - \frac{1}{3}C_{13})\sigma_{xx} + (-\frac{1}{3}C_{11} + \frac{2}{3}C_{12} - \frac{1}{3}C_{13})\sigma_{yy} + (-\frac{1}{3}C_{11} - \frac{1}{3}C_{12} + \frac{2}{3}C_{13})\sigma_{zz}$

$s'_{yy} = (\frac{2}{3}C_{21} - \frac{1}{3}C_{22} - \frac{1}{3}C_{23})\sigma_{xx} + (-\frac{1}{3}C_{21} + \frac{2}{3}C_{22} - \frac{1}{3}C_{23})\sigma_{yy} + (-\frac{1}{3}C_{21} - \frac{1}{3}C_{22} + \frac{2}{3}C_{23})\sigma_{zz}$

$s'_{zz} = (\frac{2}{3}C_{31} - \frac{1}{3}C_{32} - \frac{1}{3}C_{33})\sigma_{xx} + (-\frac{1}{3}C_{31} + \frac{2}{3}C_{32} - \frac{1}{3}C_{33})\sigma_{yy} + (-\frac{1}{3}C_{31} - \frac{1}{3}C_{32} + \frac{2}{3}C_{33})\sigma_{zz}$

$\triangleright\ s'_1, s'_2, s'_3$

$s'_1 = s'_{xx}$

$s'_2 = s'_{yy}$

$s'_3 = s'_{zz}$

$F = (|s'_1| - ks'_1)^a + (|s'_2| - ks'_2)^a + (|s'_3| - ks'_3)^a$ **return** F

end function

function $\sigma_0(F, a, \boldsymbol{C}, k)$

$\Phi_1 = \frac{2}{3}C_{11} - \frac{1}{3}C_{12} - \frac{1}{3}C_{13}$

$\Phi_2 = \frac{2}{3}C_{21} - \frac{1}{3}C_{22} - \frac{1}{3}C_{23}$

$\Phi_3 = \frac{2}{3}C_{31} - \frac{1}{3}C_{32} - \frac{1}{3}C_{33}$

$\sigma_0^T = [\frac{F}{(|\Phi_1|-k\Phi_1)^a+(|\Phi_2|-k\Phi_2)^a+(|\Phi_3|-k\Phi_3)^a}]^{\frac{1}{a}}$

$\sigma_0^C = [\frac{F}{(|\Phi_1|+k\Phi_1)^a+(|\Phi_2|+k\Phi_2)^a+(|\Phi_3|+k\Phi_3)^a}]^{\frac{1}{a}}$ **return** σ_0^T, σ_0^C

end function

function $\sigma_{90}(F, a, \boldsymbol{C}, k)$

$\Psi_1 = -\frac{1}{3}C_{11} + \frac{2}{3}C_{12} - \frac{1}{3}C_{13}$

$\Psi_2 = -\frac{1}{3}C_{21} + \frac{2}{3}C_{22} - \frac{1}{3}C_{23}$

$\Psi_3 = -\frac{1}{3}C_{31} + \frac{2}{3}C_{32} - \frac{1}{3}C_{33}$

$\sigma_{90}^T = [\frac{F}{(|\Psi_1|-k\Psi_1)^a+(|\Psi_2|-k\Psi_2)^a+(|\Psi_3|-k\Psi_3)^a}]^{\frac{1}{a}}$

$\sigma_{90}^C = [\frac{F}{(|\Psi_1|+k\Psi_1)^a+(|\Psi_2|+k\Psi_2)^a+(|\Psi_3|+k\Psi_3)^a}]^{\frac{1}{a}}$ **return** $\sigma_{90}^T, \sigma_{90}^C$

end function

function $\sigma_b(F, a, \boldsymbol{C}, k)$

$\Omega_1 = -\frac{1}{3}C_{11} - \frac{1}{3}C_{12} + \frac{2}{3}C_{13}$

$\Omega_2 = -\frac{1}{3}C_{21} - \frac{1}{3}C_{22} + \frac{2}{3}C_{23}$

$\Omega_3 = -\frac{1}{3}C_{31} - \frac{1}{3}C_{32} + \frac{2}{3}C_{33}$

$\sigma_b^T = [\frac{F}{(|\Omega_1|+k\Omega_1)^a+(|\Omega_2|+k\Omega_2)^a+(|\Omega_3|+k\Omega_3)^a}]^{\frac{1}{a}}$

$\sigma_b^C = [\frac{F}{(|\Omega_1|-k\Omega_1)^a+(|\Omega_2|-k\Omega_2)^a+(|\Omega_3|-k\Omega_3)^a}]^{\frac{1}{a}}$ **return** σ_b^T, σ_b^C

end function

function $\sigma_n(F, a, \boldsymbol{C}, k)$

$\Omega_1 = -\frac{1}{3}C_{11} - \frac{1}{3}C_{12} + \frac{2}{3}C_{13}$

$\Omega_2 = -\frac{1}{3}C_{21} - \frac{1}{3}C_{22} + \frac{2}{3}C_{23}$

$\Omega_3 = -\frac{1}{3}C_{31} - \frac{1}{3}C_{32} + \frac{2}{3}C_{33}$

$\sigma_n^T = [\frac{F}{(|\Omega_1|-k\Omega_1)^a+(|\Omega_2|-k\Omega_2)^a+(|\Omega_3|-k\Omega_3)^a}]^{\frac{1}{a}}$

$\sigma_n^C = [\frac{F}{(|\Omega_1|+k\Omega_1)^a+(|\Omega_2|+k\Omega_2)^a+(|\Omega_3|+k\Omega_3)^a}]^{\frac{1}{a}}$ **return** σ_n^T, σ_n^C

end function

The CPB06 yield criterion specialized in describing hcp material like Titanium alloys. It is compatible with the von Mises yield criterion if setting $\sigma_0^T = \sigma_0^C = \sigma_{90}^T = \sigma_{90}^C = \sigma_b^T = \sigma_b^C = \sigma_n^C = 1.0$ and $a = 2.0$. For an anisotropic example, set $\sigma_0^T, \sigma_0^C, \sigma_{90}^T, \sigma_{90}^C, \sigma_b^T, \sigma_b^C, \sigma_n^C$ to be $[1.3, 1.0, 1.2, 1.0, 1.1, 0.9, 1.0]$ and $a = 2.0$, the yield locus is shown in Figure (7).

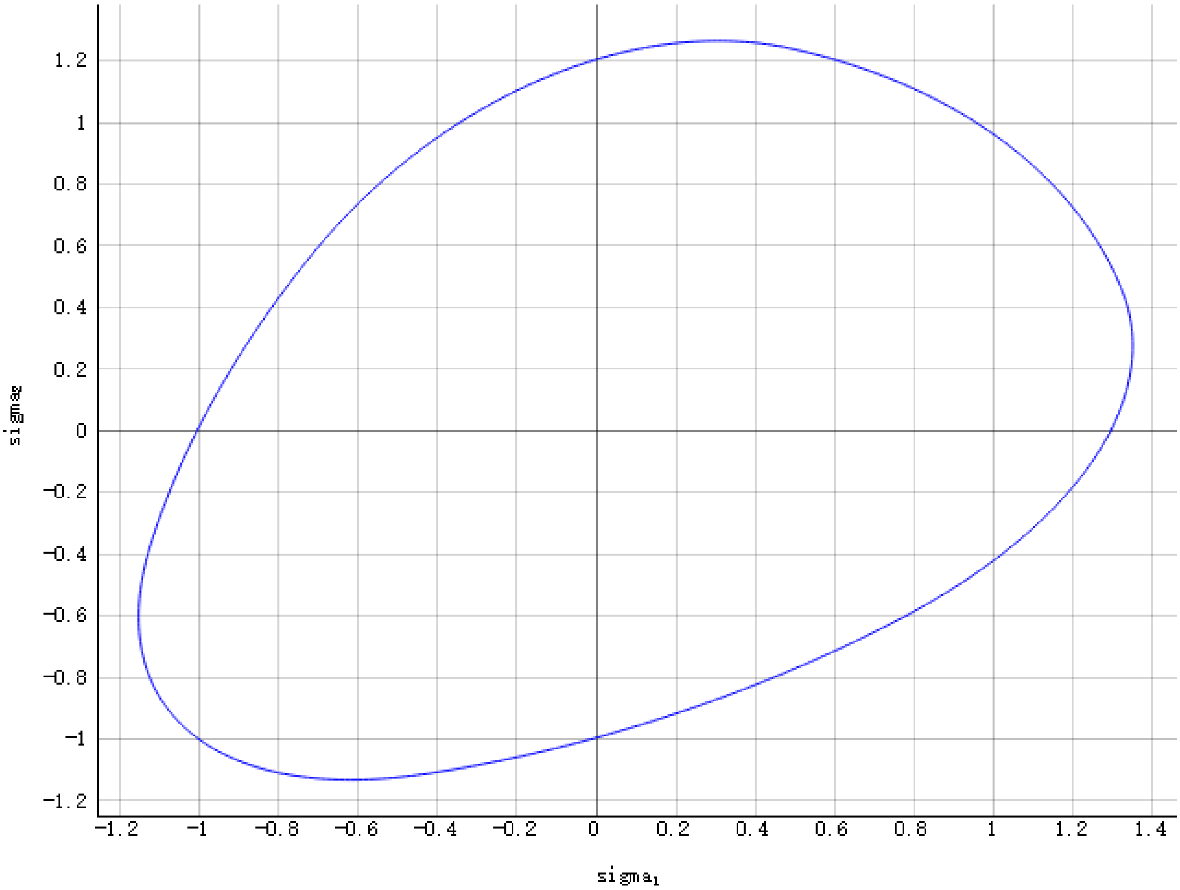

Figure 7: CPB06 yield criterion

4 Implementation

In this chapter, the aforementioned theory will be implemented into the GUI. The basic structure of the program is composed of main file, reader class, yield loci class, flow curves class and user interface(ui) files. The main file combines all the classes and ui files together as Fig.(8) shows. As a result, the main file generates the GUI as shown in Fig.(9)

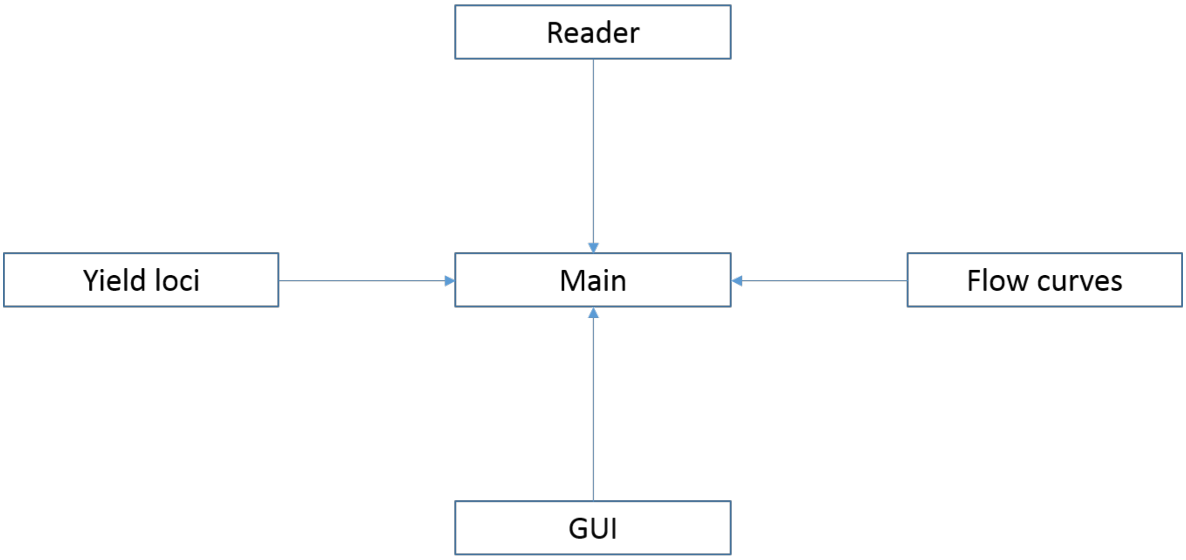

Figure 8: Program structure

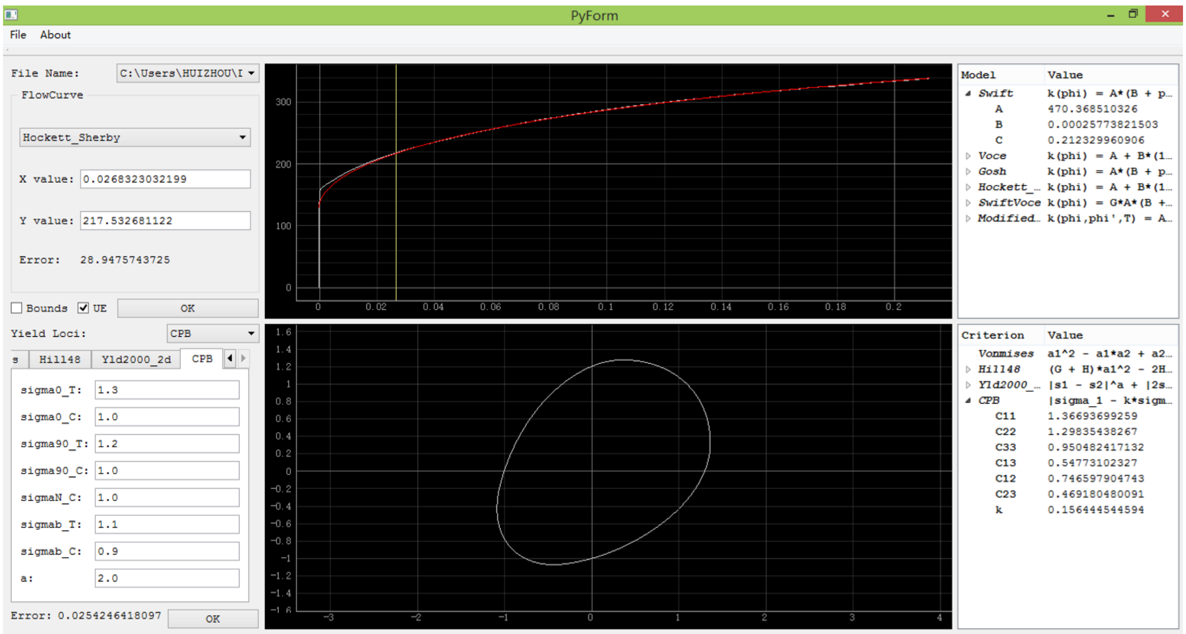

Figure 9: GUI

4.1 Flow curves

Fig.(10) shows the basic process of flow curve fitting in the main file. Boxes connected to the GUI with black lines represent the available widgets for the user in the GUI. Boxes connected

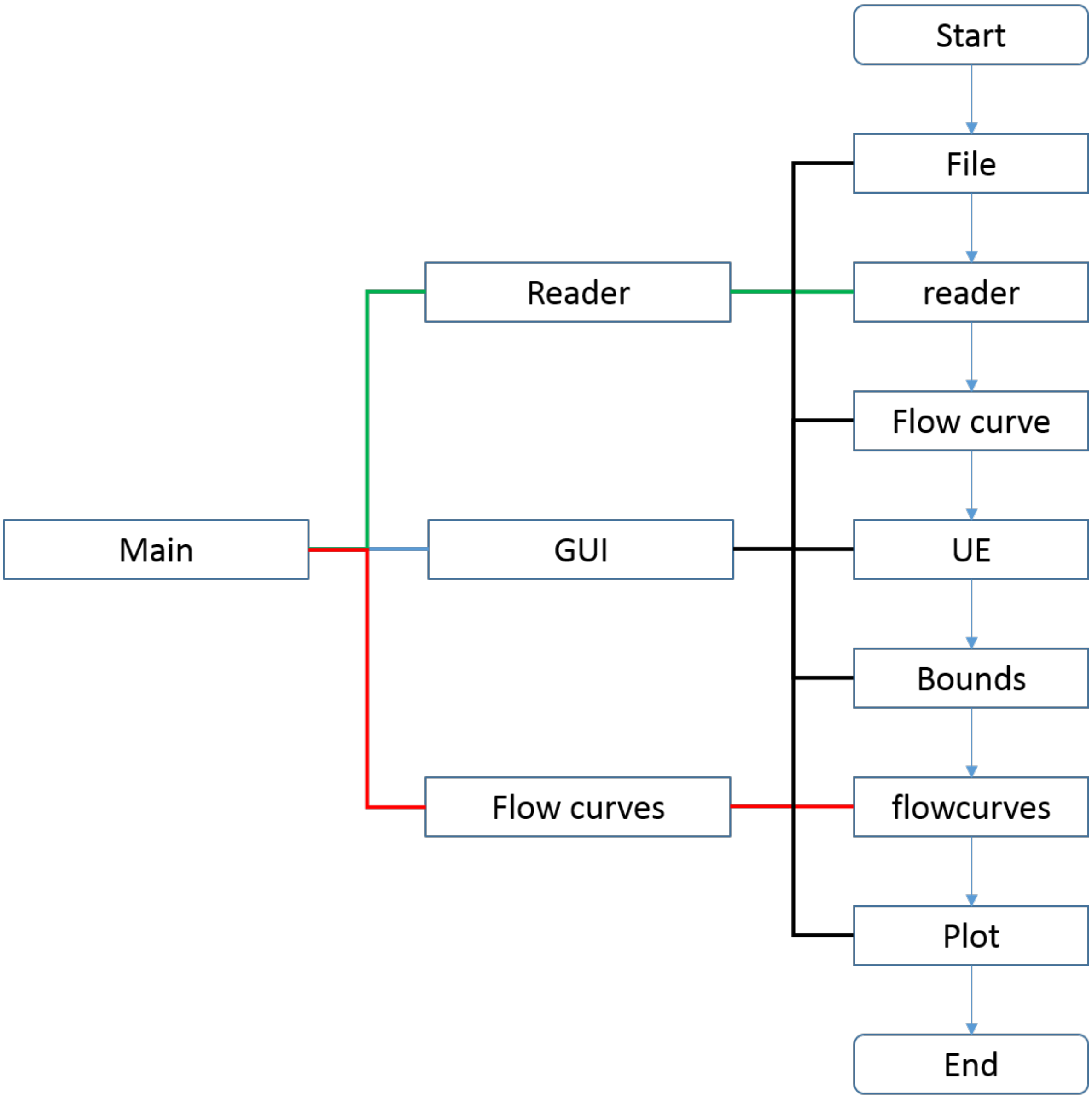

Figure 10: The process of flow curve fitting

to the Reader and Flow curves with red and green lines represent the calculation process done by the reader class and flow curves class. Note that ”UE” means uniform elongation.

First, experimental data need to be imported from the file. This function is realized by the process shown in Fig.(11). The experimental data (temperature, strain rate, strain, stress) will be input by the reader class. Otherwise, the user should check the file format. The failure of data input is usually caused by not specifying the header of each column of the data. In

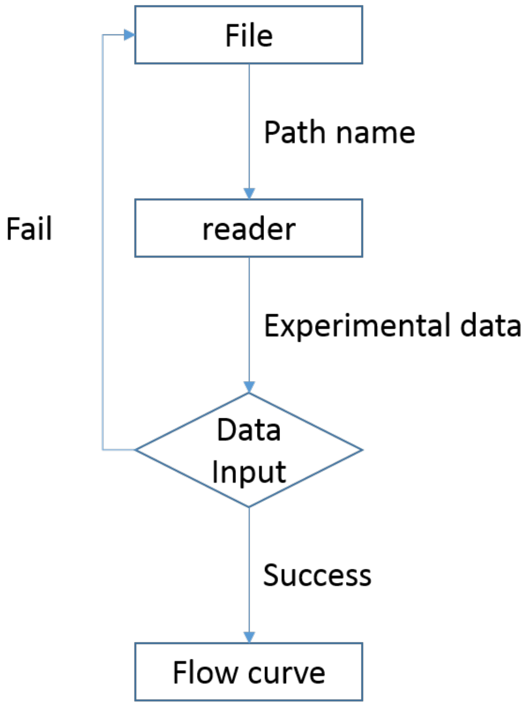

Figure 11: File import

fact, the reader class catalogues the data by the column header. It follows the process shown in Fig.(12). The data is split into four parts (temperature, strain rate, strain, stress). Besides, the uniform elongation will be found and then each part will be shrunk inside the uniform deformation range and become (temp, rate, strain, stress).

Second, the processed data will be passed to the flow curves class. The flow curves class is composed of the flow function and the objective function, and returns the parameters. In others words, the flow curve fitting is done in the flow curve class. For example, the Hollomon flow function is defined to be $\sigma(\epsilon) = K\epsilon^n$ and uses the differential evolution method. The code can be written as

```
# penalty factor
    w = 1000
# uniform elongation
    epsilon_u = 0.22
# sigma = K*epsilon**n
def Hollomon(params, epsilon):
    # params = [K,n]
    sigma = params[0]*epsilon**params[1]
    return sigma
# sigma' = K*n*epsilon**(n-1)
def HollomonPrime(params, epsilon):
```

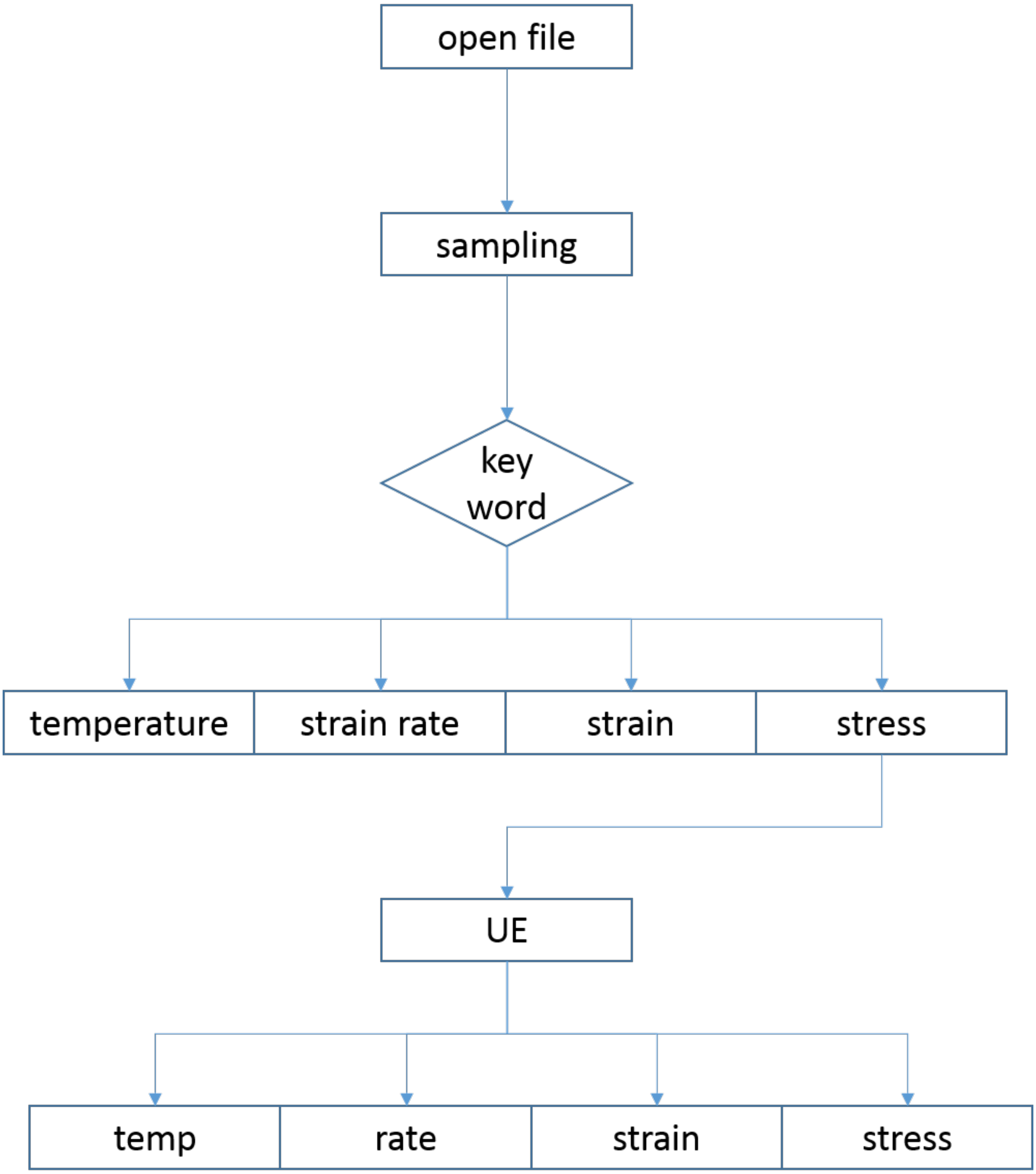

Figure 12: reader class

```
    dsigdep = params[0]*params[1]*epsilon**(params[2]-1)
    return dsigdep
# Objective function
def HollomonObj(params, epsilon, sigma):
    # experimental data (epsilon, sigma)
    cons = w*square(HollomonPrime(params, epsilon_u) - Hollomon(params, epsilon_u))
    return sum(square(Hollomon(params, epsilon)-sigma)) + cons
# Fitting
def HollomonFit(bounds, epsilon, sigma):
    params = differential_evolution(HollomonObj, bounds, args=(epsilon, sigma))
    return params
```

The example codes used here is the implementation of Eq.(2.18). Each flow curve model in Eq.(2.11)-Eq.(2.15) is included the flow curve class. However, for the modified Zener Hollomon model (Eq.2.16), temperature and strain rate should be considered,

```
# modified Zener Hollomon
def ModifiedZenerHollomon(params, temp, rate, epsilon):
    R = 8.314
    First = params[0]*exp(params[1]/(R*temp))*rate**(params[2])
    Second = 1+params[3]*np.exp(-params[4]*(epsilon-params[5])**2)
    Third = 1-params[6]*np.exp(-params[7]*epsilon**(params[8]))
    return First*Second*Third
# Objective function
def ModifiedZenerHollomonObj(params, temp, rate, epsilon, sigma):
    return ModifiedZenerHollomon(params, temp, rate, epsilon)-sigma
# Fitting
def ModifiedZenerHollomonFit(temp, rate, epsilon, sigma):
    # initial guess
    init = [1,1,1,1,1,1,1,1,1]
    params = leastsq(ModifiedZenerHollomonObj, init, args=(temp, rate, epsilon, sigma))
    return params
```

Note that the nonlinear least square method is used for the fitting of the modified Zener Hollomon model instead of the differential evolution method because the gradient based method saves computing time for the model with many parameters. Besides, uniform elongation is not required by the modified Zener Hollomon model.

4.2 Yield loci

The basic process of the yield locus fitting in the main file is as Fig.(13) shows. The yield loci class will be presented in detail since it not only generates the parameters of the yield locus but also the points of the yield locus for the plot. Specifically, the polar coordinates are used to help plotting. From Eq.(3.3), the code for the von Mises yield criterion is written as

```
def vonMises(sigma_v):
    # using polar coordinate
    # (R, theta)<->(sigma_1, sigma_2)
    point = {'X':[], 'Y':[]}
    for theta in linspace(0,2*pi):
        R = absolute(sigma_v)/sqrt(1-cos(theta)*sin(theta))
```

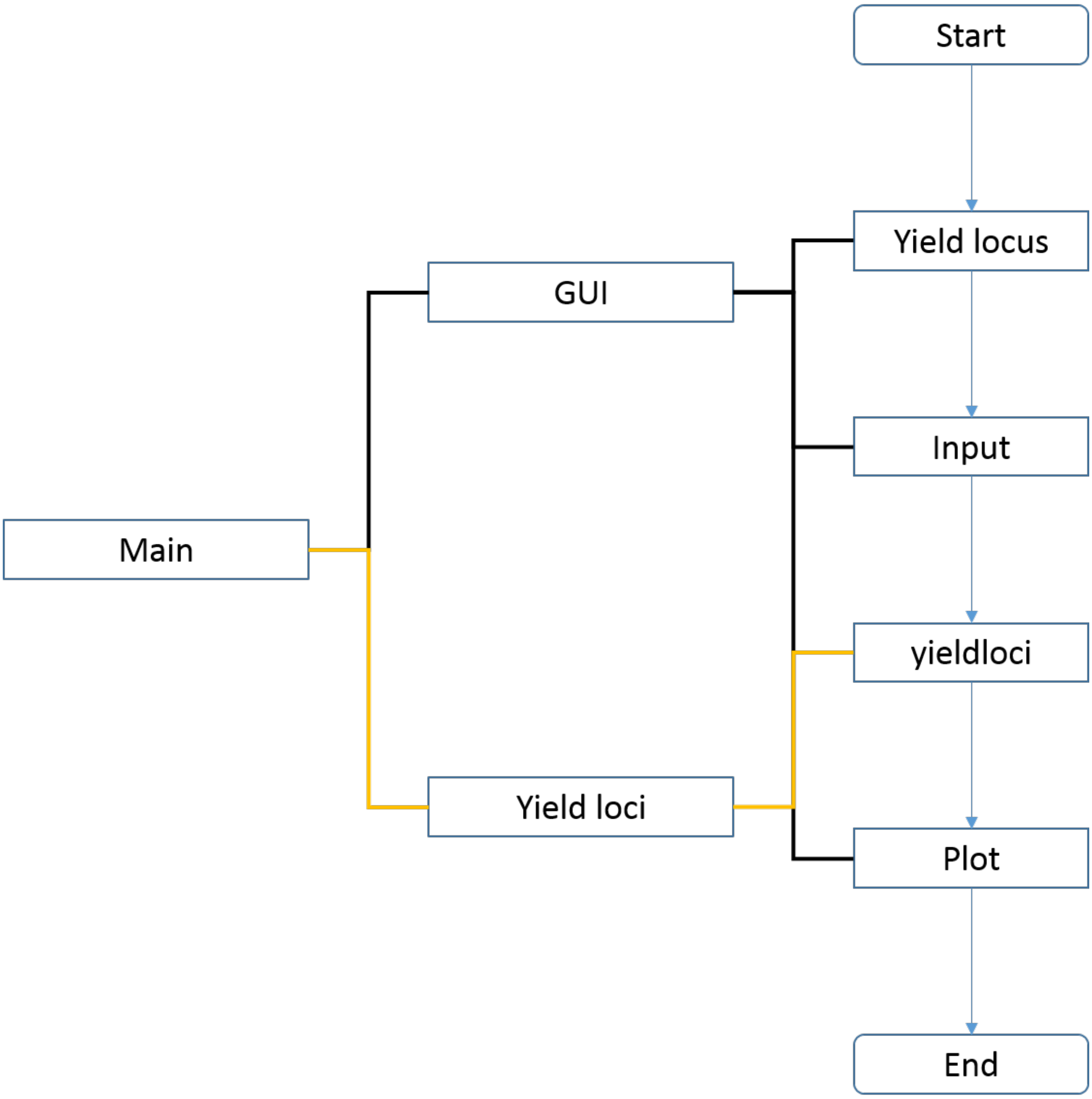

Figure 13: The process of yield locus fitting

```
        x = R*cos(theta) # sigma_1
        y = R*sin(theta) # sigma_2
        point['X'].append(x)
        point['Y'].append(y)
    return point
```

From Eq.(3.27), the code for the Hill48 yield criterion is written as

```
def Hill48(sig0,r0,r90):
    # sigma_0: sig0; r_0: r0; r_90: r90
    # (G+H)*sigma_1**2 - 2H*sigma_1*sigma_2 + (H+F)*sigma_2**2 = 1
    GH = 1/sig0**2 # G+H
    H = r0/(1+r0)/sig0**2
    HF=r0*(1+r90)/r90/(1+r0)/sig0**2 # H+F
    G = GH-H
    F = HF-H
    return [G,H,F]
def Hill48_plot(G,H,F):
    # plotting using polar coordinate
    point = {'X':[], 'Y':[]}
    GH = G+H
    HF = H+F
    for theta in linspace(0,2*pi):
        R = 1/sqrt(GH*cos(theta)**2-2*H*cos(theta)*sin(theta)+HF*sin(theta)**2)
        x = R*cos(theta) #sigma_1
        y = R*sin(theta) #sigma_2
        point['X'].append(x)
        point['Y'].append(y)
    return point
```

From Algorithm (1) and Eq.(3.44), the Yld2000-2d yield criterion can be written as

```
# algorithm
def Yld2000_sig(sig_theta,theta,a,alpha):
    ...
    return sig_v # equivalent stress
def Yld2000_r(sig_theta,theta,a,alpha):
    ...
    return r # r value
# objective function
def Yld2000_Obj(alpha,sig0,sig45,sig90,sigb,r0,r45,r90,rb,a):
    sig_v = Yld2000_sig(sig0,0,a,alpha)
    # error
    err_sig45 = (Yld2000_sig(sig45,45,a,alpha)-sig_v)/sig_v
    err_sig90 = (Yld2000_sig(sig90,90,a,alpha)-sig_v)/sig_v
    err_sigb = (self.Yld2000_a(sigb,'b',a,alpha)-sig_v)/sig_v
    err_r0 = (Yld2000_r(sig0,0,a,alpha)-r0)/r0
    err_r45 = (Yld2000_r(sig45,45,a,alpha)-r45)/r45
    err_r90 = (Yld2000_r(sig90,90,a,alpha) - r90)/r90
    err_rb = (Yld2000_r(sigb,'b',a,alpha) - rb)/rb
    return [0,err_sig45,err_sig90,err_sigb,err_r0,err_r45,err_r90,err_rb]
# fitting
```

```
def Yld2000_Fit(sig0,sig45,sig90,sigb,r0,r45,r90,rb,a):
    # initial guess
    init = [1,1,1,1,1,1,1,1]
    alpha = leastsq(Yld2000_Obj,init,args=sig0,sig45,sig90,sigb,r0,r45,r90,rb,a)
    return alpha
# plotting
def Yld2000_Plot(sig0,sig45,sig90,sigb,r0,r45,r90,rb,a):
    alpha = Yld2000_Fit(sig0,sig45,sig90,sigb,r0,r45,r90,rb,a)
    sig_v = Yld2000_sig(sig0,0,a,alpha)
    point={'X':[],'Y':[]}
    for theta in linspace(0,2*pi):
        # calculate R
        g = lambda Y: self.Yld2000_sig(Y,theta,a,alpha)-sig_v
        init = 1.0
        R = fsolve(g,init)
        x = R*cos(theta)
        y = R*sin(theta)
        point['X'].append(x)
        point['Y'].append(y)
    return point
```

From Algorithmn (2) and Eq.(3.67), the CPB06 yield criterion can be written as

```
# algorithmn
def CPB_F(sigxx,sigyy,sigzz,sigyz,sigxz,sigxy,a,Ck):
    # sigma_xx: sigxx; sigma_yy: sigyy; ... sigma_xy: sigxy
    # Ck: [C11,C22,C33,C13,C12,C23,k]
    C66 = 1.0
    C44 = 1.0
    C55 = 1.0
    ...
    return F
# T: tension
# C: compression
def CPB_sig0(F,a,Ck):
    ...
    return [sig0_T,sig0_C]
def CPB_sig90(F,a,Ck):
    ...
    return [sig90_T,sig90_C]
def CPB_sigN(F,a,Ck):
    ...
    return [sigN_T,sigN_C]
def CPB_sigb(F,a,Ck):
    ...
    return [sigb_T,sigb_C]
# objective function
def CPB_Obj(Ck,sig0T,sig0C,sig90T,sig90C,sigNC,sigbT,sigbC,a):
    F = CPB_F(sig0T,0,0,0,0,0,a,Ck)
    [sig0_T,sig0_C] = CPB_sig0(F,a,Ck)
    [sig90_T,sig90_C] = CPB_sig90(F,a,Ck)
```

```
    [sigN_T, sigN_C] = CPB_sigN(F,a,Ck)
    [sigb_T, sigb_C] = CPB_sigb(F,a,Ck)
    # error
    err_sig0T = (sig0_T - sig0T)/sig0T
    err_sig0C = (sig0_C - sig0C)/sig0C
    err_sig90T = (sig90_T - sig90T)/sig90T
    err_sig90C = (sig90_C - sig90C)/sig90C
    err_sigNC = (sigN_C - sigNC)/sigNC
    err_sigbT = (sigb_T - sigbT)/sigbT
    err_sigbC = (sigb_C - sigbC)/sigbC
    return [err_sig0T, err_sig0C, err_sig90T, err_sig90C, err_sigNC, err_sigbT, err_sigbC]
# fitting
def CPB_Fit(sig0T, sig0C, sig90T, sig90C, sigNC, sigbT, sigbC, a):
    # initial guess
    init = [1,1,1,0,0,0,0]
    Ck = leastsq(CPB_Obj, init, args=(sig0T, sig0C, sig90T, sig90C, sigNC, sigbT, sigbC, a))
    return Ck
# plotting
def CPB_Plot(sig0T, sig0C, sig90T, sig90C, sigNC, sigbT, sigbC, a):
    Ck = CPB_Fit(sig0T, sig0C, sig90T, sig90C, sigNC, sigbT, sigbC, a)
    F = CPB_F(sig0T,0,0,0,0,0,a,Ck)
    point={'X':[], 'Y':[]}
    for theta in np.linspace(0,2*np.pi,600):
        # calculate R
        g = lambda Y: self.CPB_F(Y*cos(theta),Y*sin(theta),0,0,0,0,a,Ck) - F
        init = 1
        R = fsolve(g, Init)
        x = R*cos(theta)
        y = R*sin(theta)
        point['X'].append(x)
        point['Y'].append(y)
    return point
```

5 Conclusion

Basic flow curves and yield loci in the field of sheet metal forming are covered in the GUI. Generally, it follows that (1) build the objective function by taking the model from the theory, (2) fit the model by choosing the appropriate optimization method. Particularly for the flow curves, the uniform elongation constraint is considered by using the penalty method. The GUI provides an option for users to choose whether to adopt this constraint. Usually, the fitting without the constraint has less fitting error than the one with the constraint. So the fulfillment of the uniform elongation in the expense of fitting error need to be improved for the future development. As for the future development of the yield loci, a hardening rule such as kinematic hardening or mixed hardening need to be included for the whole diagram of the strain-stress relation. Namely, the yield function $f(\sigma_{ij})$ could turn into the loading function $f(\sigma_{ij}, \epsilon_{ij}^p, \kappa)$ where κ depends on the history of plastic deformation.

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