Introduction to Olympiad Inequalities

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This is a compilation of common inequalities used in Olympiad level contest math and their applications. It is primarily based on Dr. Baker's 4802 (Georgia Tech's Mathematical Problem Solving for the Putnam) lecture on inequalities, "111 Problems in Algebra and Number Theory" by Adrian Andreescu and Vinjai Vale, and problems/solutions from various posts on AoPS fora and wiki. I'm just getting started on exploring Oly inequalities myself, so think of this handout as by a beginner for beginners.

1 Rearrangement Inequality

Given two multisets of real numbers, $x_1, x_2, \cdots x_n$ and $y_1, y_2, \cdots y_n$, and k, some permutation of $1, 2, \cdots, n$, then

$$x_n y_1 + x_{n-1} y_2 + \cdots + x_1 y_n \le \sum_{i=1}^{n} x_{k_i} y_i \le x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

A more intuitive way to think about is that you have two unordered sets of numbers of the same size and you take the sum of the pairwise products of them. This quantity is maximized when both sets are sorted and minimized when one set is sorted and the other is sorted in reverse order. This is easier to understand with the proof and a little playing around. This concept is really pretty straightforward and quite useful in a variety of proofs.

1.1 Proof

Consider the case where $n = 2$. We have our two sets x_1, x_2 and y_1, y_2. WLOG $x_2 \ge x_1$ and $y_2 \ge y_1$. Then we have $(x_2 - x_1)(y_2 - y_1) \ge 0 \Rightarrow x_1 y_1 \ge x_2 y_2$. This same logic can be applied to sets for $n > 2$ as well. If we assume for the sake of contradiction that there is some maximizing sum that is not $x_1 y_1 + \cdots + x_n y_n$, then there is at least one pair $x_i y_j$ and $x_k y_l$ in the expansion where $i \ne, j, k \ne l$. Then the argument for size two can be applied to these to terms to produce a contradiction. The case for minimization follows from a similar argument.

1.2 Example 1

1.2.1 Problem

Prove Chebyshev's Sum Inequality: $n \sum_{i=1}^{n} x_i y_i \ge (\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)$

1.2.2 Solution

Note that the expansion of the RHS gives the sum of all pairwise products $x_i y_j$, with x possibly equal to y. By the rearrangement inequality we have

$\sum_{i=1}^{n} x_i y_i \ge x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$
$\sum_{i=1}^{n} x_i y_i \ge x_1 y_2 + x_2 y_3 + \cdots + x_n y_1$
$\cdots$
$\sum_{i=1}^{n} x_i y_i \ge x_1 y_n + x_2 y_1 + \cdots + x_n y_{n-1}$

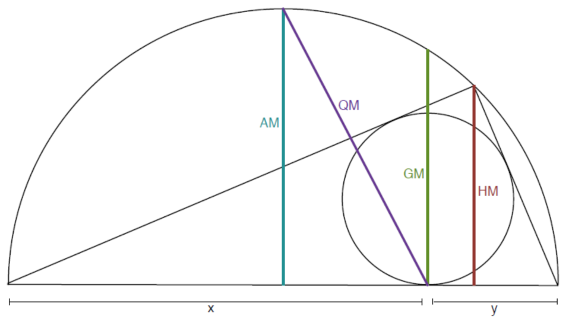
Adding these produces the desired result since the RHS will include all pairwise products and there are exactly n inequalities.

2 RMS-AM-GM-HM

$$\sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}} \ge \frac{x_1 + x_2 + \cdots + x_n}{n} \ge \sqrt[n]{x_1 \cdot x_2 \cdot \cdots \cdot x_n} \ge \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

As the name implies, this inequality states that the root-mean-square is greater than or equal to the arithmetic mean, which is greater than or equal to the geometric mean, which is greater than or equal to the harmonic mean. The equality case is when all x_i are equal. This is undoubtedly the most commonly used inequality in this handout (so much so that I included it in my earlier handout, "Formulas for Pre-Olympiad Competition Math"). In particular, the middle portion of this inequality, AM-GM, is extremely useful. I'll focus on this part in the following proofs/examples.

2.1 Proof



This image, from the AoPS Proofs Without Words page, is a beautiful way to think about AM-GM for $n = 2$. Since the two values x, y sum to the diameter, their arithmetic mean is just the radius. By power of a point we know $x \cdot y$ is the height of the circle at the point between x and y squared. Then $\sqrt{xy} = \text{GM}$ is that height. A more rigorous algebraic proof is below.

Suppose we have some sequence a_i such that $a_i = \sqrt[n]{x}_i$. Then $\sum_{i=1}^n x_i = \sum_{i=1}^n \prod_{j=1}^n a_i$. By the rearrangement inequality, $\prod_{j=1}^n a_i \geq \prod_{j=1}^n a_{i+j}$. The RHS of this inequality just comes out to be $\sqrt[n]{x_1} \cdot \sqrt[n]{x_2} \cdots \sqrt[n]{x_n} = \sqrt[n]{x_1 \cdots x_n}$, so $\sum_{i=1}^n \prod_{j=1}^n a_{i+j} = n \cdot \sqrt[n]{x_1 \cdots x_n}$. Thus $\sum_{i=1}^n x_i \geq n \sqrt[n]{x_1 \cdots x_n} \Rightarrow \frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$, which is the desired result.

2.2 Example 1

2.2.1 Problem

(AIME 1983) Find the minimum value of $\frac{9x^2 \sin^2 x + 4}{x \sin x}$ for $0 < x < \pi$.

2.2.2 Solution

Note that the given fraction can be written as $9x \sin x + \frac{4}{x \sin x}$. Then by AM-GM $9x \sin x + \frac{4}{x \sin x} \geq 2\sqrt{9x \sin x * \frac{4}{x \sin x}} = 2\sqrt{36} = \boxed{12}$. The fact that this is a minimization problem and that the nasty $x \sin x$ expression appears in the numerator and denominator in the decomposed fraction strongly motivates the use of AM-GM here.

2.3 Example 2

2.3.1 Problem

If a, b, c are positive real numbers, prove that

$$a^4 + b^4 + c^4 \geq abc(a + b + c)$$

.

2.3.2 Solution

By AM-GM we have that $a^4 + b^4 \geq 2\sqrt{a^4 b^4} = 2a^2 b^2$. Similarly, $b^4 + c^4 \geq 2b^2 c^2$ and $a^4 + c^4 \geq 2a^2 c^2$. Adding these inequalities we get $2(a^4 + b^4 + c^4) \geq 2a^2 b^2 + 2b^2 c^2 + 2a^2 c^2 \Rightarrow a^4 + b^4 + c^4 \geq a^2 b^2 + b^2 c^2 + a^2 c^2$. Then we repeat this process. By AM-GM again, we have $a^2 b^2 + b^2 c^2 \geq 2\sqrt{a^2 b^4 c^2} = 2ab^2 c$, $b^2 c^2 + a^2 c^2 \geq 2abc^2$ and $a^2 c^2 + a^2 b^2 \geq 2a^2 bc$. Adding and dividing both sides by two we get:

$$a^4 + b^4 + c^4 \geq a^2 b^2 + b^2 c^2 + a^2 c^2 \geq a^2 bc + ab^2 c + abc^2 = abc(a + b + c)$$

Using AM-GM on smaller portions of an equation and then adding the results together is a strategy used in many inequality problems.

2.4 Weighted Form

Given $w = w_1 + w_2 + \cdots + w_n$, where w_i represents a weight, the weighted AM-GM inequality states that

$$\frac{w_1x_1+w_2x_2+\cdots+w_nx_n}{n} \geq \sqrt[w]{x_1^{w_1} + x_2^{w_2} + \cdots + x_n^{w_n}}$$

This form can be proven by Jensen's inequality (see section 4.2).

3 Cauchy-Schwarz Inequality

Given real numbers $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$:

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$$

The equality case is when, for all $0 \leq i \leq n$, $\frac{a_i}{b_i} = \lambda$, where λ is a non-zero constant.

3.1 Proof

I found this elegant proof in AoPS Volume 2. Consider the vectors $a = \langle a_1, a_2, ..., a_n\rangle$ and $b = \langle b_1, b_2, ..., b_n\rangle$ with angle θ between them. Then $(a \cdot b)^2 = (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$, which is the RHS of the inequality. But $(a \cdot b)^2 = (||a|| \cdot ||b|| \cos\theta)^2 = ||a||^2 \cdot ||b||^2 \cos^2\theta$. Since $0 \leq \cos^2\theta \leq 1$, $||a||^2 \cdot ||b||^2 \cos^2\theta \leq ||a||^2 \cdot ||b||^2 = (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$. This is the LHS of the inequality, so we are done.

3.2 Example 1

3.2.1 Problem

Prove Titu's Lemma:

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \cdots + \frac{x_n^2}{y_n} \geq \frac{(x_1+x_2+\cdots+x_n)^2}{y_1+y_2+\cdots+y_n}$$

3.2.2 Solution

Multiplying both sides by $y_1 + y_2 + \cdots + y_n$, we get

$$(y_1 + y_2 + \cdots + y_n)\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \cdots + \frac{x_n^2}{y_n} \geq (x_1 + x_2 + \cdots + x_n)^2$$

This is true by Cauchy-Schwarz (where $a_i = \sqrt{y_i}$ and $b_i = \frac{x_i}{\sqrt{y_i}}$). This form of Cauchy-Schwarz appears frequently. It's commonly called Titu's lemma after AwesomeMath founder and former IMO coach Titu Andreescu.

3.3 Example 2

3.3.1 Problem

Prove Nesbitt's Inequality:

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

3.3.2 Solution

We can rewrite the LHS in a form similar to that of Titu's lemma and apply it:

$$\text{LHS} = \frac{a^2}{ab+ca} + \frac{b^2}{ab+bc} + \frac{c^2}{bc+ca} \geq \frac{(a+b+c)^2}{2ab+2bc+2ca}$$

Note that by the Rearrangement inequality $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) \geq 3(ab+bc+ca)$. Thus $\frac{(a+b+c)^2}{2ab+2bc+2ca} \geq \frac{3ab+3bc+3ca}{2ab+2bc+2ca} = \frac{3}{2}$. For me, a key take away from this problem was to not be afraid to let x_i and/or y_i be relatively complicated expressions (especially is there is some nice symmetry involved).

4 Jensen's Inequality

Let f be a convex function, $p_1, p_2, ..., p_n$ be positive values such that $p_1 + p_2 + \cdots p_n = 1$, and $x_1, x_2, ..., x_n$ be real numbers. Then

$$p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \geq f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)$$

Since each p_i can be thought of a probability, Jensen's Inequality states that the expected value of some convex function is greater than or equal to that function of the expected value, or $E(f(x)) \geq f(E(X))$, where f is convex. If f is concave, the inequality is flipped.

4.1 Proof

Jensen's inequality is easier to think about graphically. Consider the tangent line, $l = ax + b$, to f at $E(x) = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n$. Since f is convex, it lies above l. Thus, since $f(x) \geq l(x)$, the expected value of $f(x)$, which is the LHS of Jensen's inequality, is greater than or equal to the expected value of $l(x)$. Then we have the following:

$$\begin{gathered} p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) = E(f(x)) \geq E(l(x)) = E(ax+b) = \\ aE(x) + b = l(E(x)) = f(E(x)) = p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) = E(f(x)) \end{gathered}$$

Note that $l(E(x)) = f(E(x))$ since we defined l as tangent to f at $E(x)$. Also, $E(ax+b) = aE(x)+b$ since the expected value of a constant is itself.

4.2 Example 1

4.2.1 Problem

Prove that for any $\triangle ABC$, we have

$$\sin A + \sin B + \sin C \leq \tfrac{3\sqrt{3}}{2}$$

4.2.2 Solution

The sine function is concave for $0 \leq x \leq \pi$. Let our probabilities all be $\frac{1}{3}$ and f be the sine function. Then by Jensen's inequality (on a concave function), we have

$$\begin{gathered} \tfrac{1}{3}\sin A + \tfrac{1}{3}\sin B + \tfrac{1}{3}\sin c \leq \sin \tfrac{A+B+C}{3} \\ \Rightarrow \sin A + \sin B + \sin C \leq 3 \sin 60^\circ = \tfrac{3\sqrt{3}}{2} \end{gathered}$$

4.3 Example 2

4.3.1 Problem

Prove the weighted form of AM-GM using Jensen's inequality.

4.3.2 Solution

Let $f = \ln x$ be our concave function and $p_i = \frac{1}{n}$. Then by Jensen's inequality:

$$\begin{gathered} \ln \tfrac{x_1 + x_2 + \cdots + x_n}{n} \geq \tfrac{1}{n}\ln x_1 + \tfrac{1}{n}\ln x_2 + \cdots + \tfrac{1}{n}\ln x_n \\ = \ln x_1^{\frac{1}{n}} + \ln x_2^{\frac{1}{n}} + \cdots + \ln x_n^{\frac{1}{n}} = \ln \sqrt[n]{x_1 \cdot x_2 \cdots x_n} \end{gathered}$$

Undoing the natural logarithm on both sides yields the desired result. Because of their many properties, logarithms are often useful with Jensen's inequality.

5 Schur's Inequality

For non-negative real numbers a, b, c and positive r,

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0$$

most important cases of this inequality are when $r = 1$ and $r = 2$:

$$r = 1\text{: } a^3 + b^3 + c^3 + 3abc \geq a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$$
$$r = 2\text{: } a^4 + b^4 + c^4 + abc(a+b+c) \geq a^3b + a^3c + b^3a + b^3c + c^3a + c^3b$$

5.1 Proof

Without loss of generality, let $a \geq b \geq c$. Then

$$\begin{aligned} a^r(a-b)(a-c) + b^r(b-c)(b-a) &= a^r(a-b)(a-c) - b^r(b-c)(a-b) \\ &= (a-b)(a^r(a-c) - b^r(b-c)) \end{aligned}$$

Since $a^r \geq b^r$ and $a - c \geq b - c$, $a^r(a-c) - b^r(b-c)$ is non-negative. Thus, since we have a product of two non-negative terms, the $RHS \geq 0$. This takes care of the first two terms in the statement of Schur's inequality. Note that $c^r(c-a)(c-b)$ is the product of one non-negative term and two terms strictly less than or equal to zero. Therefore the product is always at least zero. Adding this to the above inequality completes the proof.

5.2 Example

5.2.1 Problem

Prove that for positive real numbers a, b, c,

$$abc \geq (a+b-c)(b+c-a)(c+a-b)$$

5.2.2 Solution 1

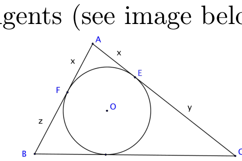
We can expand the RHS and to see that this is simply Schur's inequality for $r = 1$.

$$\begin{aligned} LHS - RHS = abc - (-a^3 - b^3 - c^3 + a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) \geq 0 \\ \Rightarrow a^3 + b^3 + c^3 + 3abc \geq a^2b + a^2c + b^2a + b^2c + c^2a + c^2b \end{aligned}$$

5.2.3 Solution 2 (Ravi Substitution)

Whenever you're given strictly positive variables in an inequality, it's a good idea to consider a geometric interpretation. The following proof illustrates a nice geometric connection to Schur's inequality.

Ravi substitution is a slick and useful tool (as anyone who has taken AMSP Geo II can attest to) for solving inequalities. Given sides of a triangle a, b, c, we can let $a = x + y$, $b = y + z$, and $c = z + x$ because we can express each side as the sum of the distances from the its endpoints to the tangency point of the incircle on that side. We can then repeat the variables because the incircle forms three pairs of equal tangents (see image below).



Applying this principle, we can write the problem statement as

$$(x+y)(y+z)(z+x) \geq (x+y+y+z-x-z)(y+z+z+x-x-y)(z+x+x+y-y-z)$$

This simplifies to $(x+y)(y+z)(z+x) \geq 8xyz$, which an introductory exercise in AM-GM. This inequality also has a connection to Euler's inequality ($R \geq 2r$, where R is the circumradius and r is the inradius). This is left as an exercise, but it can be done by using triangle area formulas to express R and r in terms of x, y, z.

6 Statements of Other Useful Inequalities

I'm not that familiar with these inequalities/techniques, and, in my opinion, they go beyond the scope of an introduction. They are, however, still useful.

6.1 Hölder's Inequality

Let a_ij be sequences of non-negative real numbers such that $1 \leq i \leq n$ and $1 \leq j \leq m$. Also, let $p_1, p_2, ..., p_n$ be a sequence of non-negtive reals such that $p_1 + p_2 + \cdots + p_n = 1$. Then

$$\sum_j \prod_i a_{ij}^{p_i} \leq \prod_i \left(\sum_j a_{ij} \right)^{p_i}$$

6.2 Muirhead's Inequality

Suppose we have non-increasing sequences $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ such that for all $1 \leq k \leq n$, $\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i$ (if this condition holds, sequence A "majorizes" sequence B). Then given positive reals $x_1, x_2, ..., x_n$,

$$\sum_{\text{sym}} x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n} \geq \sum_{\text{sym}} x_1^{b_1} + x_2^{b_2} + \cdots + x_n^{b_n}$$

The AoPS wiki gives the following example. The sequence $(5, 1)$ majorizes $(4, 2)$, so for any positive x, y, by Muirhead's inequality $x^5y^1 + y^5x^1 \geq x^4y^2 + y^4x^2$.

6.3 Erdős-Mordell Inequality

Given a point P inside $\triangle ABC$, let X, Y, Z be the feet of the perpendiculars from P to sides a, b, c. Then:

$$PA + PB + PC \geq 2(PX + PY + PZ)$$

This geometric inequality (another key topic from AMSP Geo II) has many interesting proofs that can be found online.

6.4 Homogenization

This is a simple yet powerful technique in solving inequalities with multiple variables. If all the terms in an inequality have the same degree, you can impose arbitrary constraints on it (since the terms "scale" by the same amount). This is easier to see with an example:

In $x^3 + y^3 + z^3 - 3xyz \geq 0$, the degree of each term is three. We can let $x = ka$,$y = kb$, and $z = kc$ such that $a + b + c = 3$. Then our inequality becomes $k^3a^3 + k^3b^3 + k^3c^3 - 3(ka)(kb)(kc) \geq 0 \Rightarrow a^3 + b^3 + c^3 - 3abc \geq 0$. By this process the inequality is preserved and we now have a constraint. In this case the inequality was in the correct form, but note that you can often force an inequality that can't be homogenized into a form that can by performing substitutions.

7 Remarks

In this handout, I only included what I thought to be the most commonly used/seen (non-trivial) inequalities in contest math. There are, however, many more useful and very powerful inequalities. There are also numerous intricate and elegant applications for each of the formulae I've talked about that I didn't go over. In my opinion, the key to improving problem solving abilities is not to memorize formulas, but rather to build a strong intuition of common techniques and patterns that show up when solving such problems. Therefore, please keep in mind that this handout really is just the tip of the iceberg.