Ramanujan's Summation

Eashan Gandotra

1 Analytic Continuation and the Gamma Function

Perhaps you have heard of Ramanujan's now internet-famous result that $1+2+3+... = -\frac{1}{12}$. As bizarre and unintuitive this result may seem, it can be explained by analytic continuation (and is revisited in section 4). Recall a function $f(z)$ for $z \in \mathbb{C}$ is analytic on some set S if the derivative of f exists everywhere in S. That is, for every point $z_0 \in S$, $f'(z_0)$ exists in some neighborhood of z_0. Analytic continuation is a technique used to expand the domain on which a function is analytic. More formally, suppose we have a function $f(z)$ that is on analytic on some region R. Suppose there is some region of the plane R_1 where $f_1(z) = f(z)$ and another region R_2 where $f_2(z) = f(z)$. Then f_1 is the analytic continuation of f_2 on the region $R_1 \backslash R_2$ and f_2 is the analytic continuation of f_1 on the region $R_2 \backslash R_1$. While this definition may seem confusing, it is more easily understood via the examples in the following section.

One classic example of analytic continuation, which we will use later in section 4, is the gamma function. It extends the domain of the factorial function (which only takes non-negative integers) to $\mathbb{C}\backslash\{n \in \mathbb{Z}, n \leq 0\}$. The gamma function is defined as $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$ for such z. We can see this is the factorial function for positive integers using integration by parts. Let n be a positive integer and $u = x^n$ and $dv = e^{-x}dx$. Then

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}dx = uv - \int vdu = -x^n e^{-x}|_0^\infty + n\int_0^\infty x^{n-1}e^{-x}dx = n\Gamma(n)$$

This is the recursive definition of the factorial function (for which the base case is defined as $0! = 1$). This function is explored further in sections 2 and 3.

2 Problems and Examples

Most of the following problems come from *Fundamentals of Complex Analysis with Applications to Engineering and Science* by Saff and Snider. All mentions of theorems and corollaries in the following section refer to this book.

1) Find a function that extends the domain of $f(x) = x^2+1, x \in \mathbb{R}$ to the complex numbers.

Note that for $g(z) = z^2+1, z \in \mathbb{C}$ we have $g(z) = f(z), z \in \mathbb{R}$ and $g(z)$ is well-defined (and analytic) on the complex plane, so it serves as a valid extension. This is a trivial example of extending a domain, but goes to show how subtle the process can be. It also provides a simple/intuitive way to think about extending domains and the uniqueness of analytic continuations.

2) Given that $f(z)$ is analytic at $z = 0$ and that $f(\frac{1}{n}) = \frac{1}{n^2}, n \in 1, 2, ...,$ find $f(z)$.

Let D be a neighborhood of 0 containing $\{1, \frac{1}{2}, \frac{1}{3}, ...\}$. Then suppose we have a function $g : D \to \mathbb{C}$ with $g(z) = f(z) - z^2$. Since g is analytic and $g(z) = 0$ for all points in D, $f(z) = z^2$. This follows form the more general theorem that if F is analytic on in a domain D and and vanishes on some open disk contained in D, then it vanishes throughout D (see Theorem 19). In this case $F = f - g$.

3) Prove that if $f(z)$ is analytic and agrees with polynomial $\sum_{j=0}^{n} a_j x^j$ for $z = x$ on a segment of the real axis, then $f(z) = \sum_{j=0}^{n} a_j z^j$ everywhere.

Note that the segment of the real axis on which f is analytic and agrees with the given polynomial is a closed set with infinitely many points. It follows form corollary 5 of Theorem 19 that $f(z) = \sum_{j=0}^{n} a_j z^j$ everywhere. The proof of this corollary comes from applying theorem 19 (stated in the previous problem) to the difference of f and the polynomial.

4) Does there exist a function $f(z)$, not identically zero, which is analytic in the open disk $D : |z| < 1$ and vanishes at infinitely many points in D.

Let $f(z) = \cos(\frac{1}{1-z})$. Then since cosine is differentiable everywhere in the complex plane and $\frac{1}{1-z}$, the geometric series, is analytic on $|z| < 1$, f is also analytic on D. Clearly f is not identically zero as at $f(\frac{1}{2}) = \cos(2) \neq 0$. We also have infinitely many points of the form $z = 1 - \frac{1}{2n\pi}, n \in \mathbb{Z}, n \neq 0$. Since $|1 - \frac{1}{2n\pi}| < 1$, such points are in D. Also $f(1 - \frac{1}{2\pi n}) = \cos(2\pi n) = 0$. Hence $f(z) = \cos(\frac{1}{1-z})$ serves as an example of a function with the desired qualities.

5) Prove that if f is analytic in a punctured neighborhood of $z = 0$ and if $f(\frac{1}{n}) = 0$ for all $n = \pm 1, \pm 2, ...,$ then either f is identically zero or f has an essential singularity at $z = 0$.

Since f is analytic in a neighborhood of 0, f has a singularity at 0. suppose this neighborhood is $D = \{z | 0 < |z| < m\}$. If the singularity is removable, we can use a similar approach to in the second problem. Since the singularity is removable there must be some function g that is equal to f on D. Then $f(z) = g(z) = 0, z \in \{\pm 1, \pm \frac{1}{2} ...\}$. This is a sequence converging to 0, so, by theorem 22, $g(z) = 0$ and thus $f(z) = 0$ on D. Applying theorem 19 again and viewing D as the disk on which f vanishes, f is identically zero. Clearly f isn't a pole because $f(\frac{1}{n}) = 0$ for all non-zero integers n, which means that as $z \to \infty$, $|f(z)|$ doesn't approach 0. Hence $f(z)$ is identically zero if 0 is a removable singularity and 0 is an essential singularity otherwise.

6) Let $f(z) = \sum_{j-0}^{\infty} z^j$ for $|z| < 1$. For what values of α ($|\alpha| < 1$) does the Taylor expansion of $f(z)$ about $z = \alpha$ yield a direct analytic continuation of $f(z)$ to a disk extending outside the circle $|z| < 1$.

Note that this is simply a geometric series, so $f(z) = \frac{1}{1-z}, |z| < 1$. Then the Taylor series for f about z_0 is $f(z) = \sum_{j=0}^{\infty} \frac{(z-z_0)^j}{j!} \cdot \frac{j!}{(1-z_0)^{j+1}} = \sum_{j=0}^{\infty} \frac{(z-z_0)^j}{(1-z_0)^{j+1}}$. This works when the distance from z_0 to the new center and that from the origin are together less than 1 (in other words

$|z-z_0|+|z_0|<1)$. Clearly, since this is a geometric series, the radius of convergence is $|1-z_0|$ (you can use ratio test for more rigor). Now we want the region on which this series converges, so $|1-z_0|>1-|z_0| \Rightarrow |1-re^{i\theta}|>1-r \Rightarrow 1>\cos(\theta)$, which always holds unless $\theta=0+2\pi n$. Hence we have analytic continuation extending outside $|z|<1$ if z does not lie on the positive real axis and $|z|<1$.

7) The gamma function is defined as $\Gamma(z)=\int_0^\infty x^{z-1}e^{-x}dx$ for $z\in\mathbb{C}\backslash\{n\in\mathbb{Z}, z\leq 0\}$. Show $\Gamma(z+1)=z\Gamma(z)$.

This follows the same process I used in section 1 to show the gamma function is the factorial function (shifted over by one) for positive integers. Let $u=x^z$ and $dv=e^{-x}dx$. Then we have $\Gamma(z+1)=\int_0^\infty x^z e^{-x}dz = uv-\int vdu = -x^z e^{-x}|_0^\infty + z\int_0^\infty x^{z-1}e^{-x}dx = z\Gamma(z)$.

8) Assuming the Gamma function is analytic in the right half of the plane, use the functional equation proven in section 1 to analytically continue $\Gamma(z)$ to the entire plane.

From the last problem, we have $\Gamma(z+1)=z\Gamma(z)$. We can shift the index by 1 to see $\Gamma(z+2)=(z+1)\Gamma(z+1)=(z+1)(z)\Gamma(z)$. In general, $\Gamma(z+1+n)=(z+n)(z+n-1)...(z)\Gamma(z) \Rightarrow \Gamma(z)=\frac{\Gamma(z+1+n)}{(z+n)(z+n-1)...(z)}$. Hence we have an analytic continuation for $\Gamma(z)$ to the entire plane. The function is analytic everywhere except at the singularities. Setting the denominator to 0 we see the singularities are simply the non-positive integers, as desired. Note this works because we can choose n such that $z+n+1$ is in the right half of the plane, which we assumed to be analytic.

9) Evaluate $((-\frac{1}{2})!)^2$.

Using the Gamma function as the to evaluate the factorial at $-\frac{1}{2}$, we have $(-\frac{1}{2})!=\Gamma(-\frac{1}{2}+1)=\Gamma(\frac{1}{2})=\int_0^\infty x^{\frac{1}{2}-1}e^{-x}dx=\int_0^\infty x^{-\frac{1}{2}}e^{-x}dx$. Letting $u^2=x\Rightarrow 2udu=dx$, the integral becomes $\int_0^\infty u^{-1}e^{-u^2}\cdot 2udu = 2\int_0^\infty e^{-u^2}du=\int_{-\infty}^\infty e^{-u^2}du$. Using this result, we have:

$$((-\tfrac{1}{2})!)^2=\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-x^2}e^{-y^2}dxdy=\int_0^\infty r\int_0^{2\pi}e^{-r^2}d\theta dr=2\pi\int_0^\infty re^{-r^2}dr=2\pi\cdot\tfrac{1}{2}=\pi$$

Here I use polar coordinates and perform a simple $u-$substitution with $v=-r^2$ in the last step (left out for clarity). I find this a neat result!

10) What is the general formula for $\Gamma(\frac{n}{2})$, $n\in\mathbb{Z}, n>0$.

We have two cases. First assume n is even. Then, letting $n=2k, k\in\mathbb{Z}$, we have $\Gamma(\frac{n}{2})=\Gamma(k)=(k-1)!$. Now suppose n is odd. Then, letting $n=2k+1$, we have $\Gamma(\frac{n}{2})=\Gamma(\frac{1}{2}+k)$. Using the functional equation from 8 and the result from 9, this simplifies to $\Gamma(\frac{1}{2}+k)=(\frac{1}{2}+k-1)(\frac{1}{2}+k-2)...(\frac{1}{2})\Gamma(\frac{1}{2})=(\frac{1}{2})(\frac{3}{2})...(\frac{2k-1}{2})(\sqrt{\pi})=\sqrt{\pi}(\frac{(2k)!}{2^{2k}k!})$.

In addition to having a relationship to geometry, suggested by the integrals involving π above, the Gamma function appears in various other fields of mathematics. One notable example is the Gamma distribution, which can be used to model elapsed times and various financial variables.

3 Using the Riemann Zeta Function

I'll start this section with a proof Euler came up with to show there are infinitely many primes. This relies on the clever insight that $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+... = \prod_p(\frac{1}{1-\frac{1}{p}})$, where p take each prime value. To see this, first note that by the geometric series formula $\prod_p \frac{1}{1-\frac{1}{p}} = (1+\frac{1}{2}+\frac{1}{2^2}+...)(1+\frac{1}{3}+\frac{1}{3^2}+...)....$ Now consider some term $\frac{1}{n}$ on the left hand side. The Fundamental Theorem of Arithmetic tells us that n can be expressed as $n = p_1^{a_1}p_2^{a_2}...p_m^{a_m}$ for primes p_i. It follows that $\frac{1}{n} = \frac{1}{p_1^{a_1}}\cdots\frac{1}{p_m^{a_m}}$. Now each term on the left hand side of the original equation is the product of one term in each $(1+\frac{1}{p}+\frac{1}{p^2}...)$ for all primes p. Then to get $\frac{1}{n}$ we can take the term $\frac{1}{p_i^{a_i}}$ from $(1+\frac{1}{p_i}+\frac{1}{p_i^2}...)$ for the primes that appear in n, and 1 from the other terms. Hence $\frac{1}{n}$ appears on the left hand side. Furthermore, since n has a unique prime factorization, $\frac{1}{n}$ appears exactly once. Since n is arbitrary, the claim follows. It is well known that the harmonic series $1+\frac{1}{2}+\frac{1}{3}...$ diverges. This implies our product $\prod_p(\frac{1}{1-\frac{1}{p}})$ also diverges. If there were finitely many primes, the product would also be finite. Hence there must be infinitely many primes.

You can only wonder what went on in Euler's mind that made him come up with such an elegant proof. You may be thinking "Wow, this is cool... but isn't there that much simpler way to show there are infinite primes?" You would be correct. Euclid had argued hundreds of years before Euler that if you assume finitely many primes $p_1, p_2, ..., p_n$, then $p_1p_2...p_n+1$ isn't divisible by any p_i, and so must be prime itself, contradicting the assumption. Euler's proof, however, offers far more insight. Euler could go the other way and argue that since there are infinitely many primes, $\prod_p(\frac{1}{1-\frac{1}{p}})$ diverges, and then so must $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+....$ A natural follow up to this is to define a function $\zeta(z) = \frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+... = \sum_{n=1}^{\infty} n^{-z}$ and ask for what values of s does this series diverge/converge. For $s \in \mathbb{C}$, this is known as the Riemann zeta function.

Now we will relate the Riemann zeta function to a more easy to evaluate function by the following proof. We have:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \Rightarrow \frac{1}{2^{z-1}}\zeta(z) = \frac{1}{2^{z-1}}\sum_{n=1}^{\infty}\frac{1}{n^z} = \sum_{n=1}^{\infty}\frac{2}{(2n)^{z-1}}$$

Now we subtract the two equations to get:

$$(1-\frac{1}{2^{z-1}})\zeta(z) = \frac{1}{1^z}-\frac{1}{2^z}+\frac{1}{3^z}-\frac{1}{4^z}+... = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^z} = \eta(z)$$

Simplifying the left-hand side, we get:

$$(2^{1-z}-1)\zeta(z) = \eta(z)$$

Note that I ignored the case where $2^{z-1}-1=0$, which would prevent the function from being analytic, because it can be shown to be a removable singularity. The function we called $\eta(z)$ is known as the Dirichlet eta function. The technique used here by subtracting a multiple of a series from itself can be used to solve general arithmetico-geometric series, many of which are interesting (see 2018 GT HSMC Free Response #12 for a clever one with Fibonacci numbers). There is one more relation we need before we can evaluate the zeta function, and it brings back our friend from section 1, the gamma function: By $u-$substitution with $u = xn$ and $du = ndx$:

$$\int_0^\infty x^{z-1}e^{-nx}dx = \int_0^\infty \frac{1}{u}(\frac{u}{n})^z e^{-u}du = \frac{1}{n^z}\int_0^\infty u^{z-1}e^{-u}du = \frac{1}{n^z}\Gamma(z)$$

Then by the geometric series formula on $\frac{1}{1-(-e^x)}$ and applying the integral above (the more general form of which is known as a Mellin transform), we have:

$$\int_0^\infty \frac{x^{z-1}}{e^x+1}dx = \int_0^\infty x^{z-1}\sum_{n=1}^\infty (-1)^{n-1}e^{-nx}dx = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^z}\int_0^\infty x^{z-1}e^{-x}dx = \eta(z)\Gamma(z)$$

This is yet another beautiful proof that, along with our previous result, allows us to evaluate the zeta function as follows:

$$(1-2^{-z})\zeta(z)\Gamma(z) = \eta(z)\Gamma(z) = \int_0^\infty \frac{x^{z-1}}{e^x+1}dx$$

We perform integration by parts twice, which increases the input of the gamma function by 2. This will be useful later. First we use $u = \frac{1}{e^x+1}, dv = x^{z-1}$.

$$\int_0^\infty \frac{x^{z-1}}{e^x+1}dx = \frac{x^z}{z(e^x+1)}|_0^\infty - \int_0^\infty (-\frac{e^x}{(e^x+1)^2})(\frac{1}{z}x^2)dx = \frac{1}{z}\int_0^\infty \frac{x^z e^x}{(e^x+1)^2}dx$$

Next we use $u = \frac{1}{(e^x+1)^2} \Rightarrow du = \frac{e^x(e^{2x}+2e^x+1)-2e^{2x}(e^x+1)}{(e^x+1)^4} = \frac{e^x-e^{3x}}{(e^x+1)^3}$ and $dv = x^z \Rightarrow v = \frac{x^{z+1}}{z+1}$.

$$\frac{1}{z}\int_0^\infty \frac{x^z e^x}{(e^x+1)^2}dx = \frac{1}{z}(\frac{e^x}{(e^x+1)^2}\cdot\frac{x^{z+1}}{z+1})|_0^\infty - \int_0^\infty \frac{x^{z+1}}{z+1}\cdot\frac{e^x-e^{2x}}{(e^x+1)^3}dx = \frac{1}{z(z+1)}\int_0^\infty \frac{x^{z+1}(e^{2x}-e^x)}{(e^x+1)^3}$$

Hence We have

$$(1-2^{1-z})\zeta(z)\Gamma(z)z(z+1) = (1-2^{1-z})\zeta(z)\Gamma(z+2) = \int_0^\infty \frac{x^{z+1}(e^{2x}-e^x)}{(e^x+1)^3}$$

Suppose we want to evaluate the expression at -1. Then the integral on the right-hand side is

$$\int_0^\infty \frac{x^{z+1}(e^{2x}-e^x)}{(e^x+1)^3} = \int_0^\infty \frac{e^{2x}-e^x}{(e^x+1)^3}dx = \int_1^\infty \frac{u-1}{(u+1)^3}du = \int_2^\infty \frac{v-2}{v^3}dv = v^{-2}-v^{-1}|_2^\infty = \frac{1}{4}$$

Finally, we can evaluate ζ at -1:

$$(1-2^{1-z})\zeta(z)\Gamma(z+2) = (1-2^2)\zeta(-1)\Gamma(1) = \frac{1}{4} \Rightarrow \zeta(-1) = \frac{1}{4}\cdot -\frac{1}{3}\cdot 1 = -\frac{1}{12}$$

Recall that our definition for the ζ function was $\zeta(z) = \sum_{n=1}^\infty n^{-z}$. It follows that

$$\zeta(-1) = 1+2+3+... = -\frac{1}{12}$$

4 Remarks

When looking at proofs for this famous summation, I found many other interesting results along the way, including a neat functional equation for the Dirichlet eta function due to G.H. Hardy. While it may seem purely theoretical, many of these results, including this sum, are extremely useful in modern physics. It's amazing what people do with series, and I look forward to learning more about the field. What I find most incredible, however, is that Ramanujan, who was poor and virtually self-educated, somehow managed to come up with such counter-intuitive and involved results on his own.

5 References

"Analytic Continuation." Complex Analysis with Applications in Science and Engineering, by Harold Cohen, 2nd ed., Springer, 2007, p. 399.

"Analytic Functions." Complex Variables and Applications, by James Ward Brown and Ruel Vance Churchill, McGraw-Hill Education, 2014, pp. 37–86.

Andreescu, Titu, and Gabriel Dospinescu. Problems from the Book. XYZ Press, 2010.

Saff, E. B., and Arthur David Snider. Fundamentals of Complex Analysis with Applications to Engineering and Science. Pearson India Education Services, 2019.