Kinematics of a Differential Drive Robot

Forward Kinematics

We will begin by solving the forward kinematics problem. Providing we know the pose of the robot at time t, Given the wheel angles at a time $t+1$, the forward kinematics allow us to find the pose of the robot at $t+1$. Here we are assuming the robot follows a body twist for $t=1$ in arbitrary time units, so we are solving the forward kinematics for 1 timestep.

In terms of transformations, with the world frame given as T_W and the body frame of the robot at time t known to be T_{WB}, we will use forward kinematics to find $T_{WB'}$ at time $t+1$

D : half the distance between the wheel centers
r : wheel radius
ϕ : wheel angle
$\dot{V}_{x_l}$: velocity of the left wheel
$\dot{V}_{x_r}$: velocity of the right wheel
$\dot{\theta}$: θ component of the body twist
$\dot{V}_x$: x component of the body twist
$\dot{V}_y$: y component of the body twist

Equations 1 and 2 descibe the relation between the body twist and the wheel velocities for the left and right wheels respectively

$$\begin{bmatrix} \dot{\theta} \\ \dot{V}_{x_l} \\ \dot{V}_{y_l} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -D & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{V}_x \\ \dot{V}_y \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -D\dot{\theta} + V_x \\ 0 \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} \dot{\theta} \\ \dot{V}_{x_r} \\ \dot{V}_{y_r} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ D & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{V}_x \\ \dot{V}_y \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -D\dot{\theta} + V_x \\ 0 \end{bmatrix} \tag{2}$$

Since we are using traditional wheels that can only spin forward or backward (no mecanum wheels or omniwheels) and we are assuming that there is no slipping, $V_y = 0$. Furthermore, the linear distance travelled by a wheel is given as $r\phi$. Since here we consider that the robot follows the body twist for $t=1$, the wheel speed in this case is simply the difference in wheel angles, $\Delta\phi_l = \dot{\phi}_l$ and $\Delta\phi_r = \dot{\phi}_r$. Therefore, $V_{x_l} = r\dot{\phi}_l$, and $V_{x_r} = r\dot{\phi}_r$ measured in radians per unit time. Now solving Equations 1 and 2 for $\dot{\phi}_l$ and $\dot{\phi}_r$ we get:

$$\begin{aligned} \dot{\phi}_l &= \frac{-D\dot{\theta} + V_x}{r} \\ \dot{\phi}_r &= \frac{D\dot{\theta} + V_x}{r} \end{aligned} \tag{3}$$

Now we will want to find the body twist that brings the robot from its starting configuration to the new configuration, or in other words we want the transformation $T_{BB'}$. For this we need the body twist in which we can find from Equations 3. Solving for $\dot{\theta}$, then for V_x, and finally recalling that since we assert there be no slipping, $V_y = 0$, we obtain the following equations for the three components of the body twist.

$$\begin{aligned}
\dot{\theta} &= \frac{r}{2D}\left(\dot{\phi}_r - \dot{\phi}_l\right) \\
\mathcal{V}_x &= \frac{1}{r}\left(\dot{\phi}_r + \dot{\phi}_l\right) \\
\mathcal{V}_y &= 0
\end{aligned} \tag{4}$$

Finally, we can integrate this twist using the *rigid2D* library to obtain $T_{BB'}$, and from there we obtain $T_{WB'} = T_{WB} \cdot T_{BB'}$. The pose of the robot can now be easily obtained from $T_{WB'}$.

Inverse Kinematics

Given some desired body twist, solving the inverse kinematics problem allows us to determine the wheel velocities needed to achieve that twist. Assumptions regarding slipping and only considering a unit time step as in the forward kinematics problem hold here as well.

$$\begin{bmatrix} \dot{\phi}_l \\ \dot{\phi}_r \end{bmatrix} = \frac{1}{r} \begin{bmatrix} -D & 1 & 0 \\ D & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{V}_x \\ \dot{V}_y \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} \dot{\phi}_l \\ \dot{\phi}_r \end{bmatrix} = \frac{1}{r} \begin{bmatrix} -D\dot{\theta} + V_x \\ D\dot{\theta} + V_x + V_y \end{bmatrix} \tag{6}$$

Equation 6 shows the wheel velocities, $\dot{\phi}_l$ and $\dot{\phi}_r$ needed to achieve the desired body twist. Note that as we have stated previously, for no slipping to occur, $V_y = 0$