UNIT 3 EULERIAN AND HAMILTONIAN GRAPHS

Structure **Page Nos.**

3.0 INTRODUCTION

Suppose you go to a new city as a salesperson. You would naturally like to familiarise yourself with all the important routes. One way to do this is to buy a map of the city and go around the city. If you do this without proper planning, you may pass through some of the streets more than once. To avoid this, you would need to sit down and plan your route. The most efficient route would involve traversing every street in it only once. But is it possible to find such a route?

This question is so natural that you may not be surprised to know that a similar question was raised more than 250 years ago. Königsberg was a city in what was known as Prussia those days. The Pregel river flowed through this city forming two islands (see B and C in Fig.1).

Fig. 1: A schematic diagram of Königsberg

The two islands and the rest of the city were connected to each other by seven bridges. Some of the citizens used to amuse themselves with the following question: Is it possible to go around the city using each bridge exactly once?

Fig. 2: Leonhard Euler (1707-1783)

In 1736, the great Swiss mathematician Leonhard Euler (pronounced as 'oiler') answered this question by converting this into a problem in graph theory. We will see this problem in Section 3.2 (Sec.3.2 in brief), while discussing graphs named after Euler.

There is one more question similar to the Königsberg problem in recreational mathematics — which figures can be drawn without lifting the pen from the paper and without going over any of the lines twice? This question is also answered in Sec.3.2. A mathematical puzzle invented by Hamilton involves finding a cycle containing all the vertices of a certain graph. Motivated by this, we will discuss conditions for a graph to contain a cycle containing all the vertices of the graph. Such as graph is called a Hamiltonian graph, in honour of Hamilton. In Sec. 3.3 we will give some necessary and sufficient conditions for a graph to be Hamiltonian.

Finally, in Sec.3.4 we discuss a related question, the travelling salesperson problem.

3.1 OBJECTIVES

After studying this unit, you should be able to

- check whether a given graph is Eulerian or not;
- check whether a given graph satisfies certain necessary conditions for a Hamiltonian graph;
- check whether a given graph satisfies certain sufficient conditions for a Hamiltonian graph;
- apply the 1-exchange algorithm to reduce the weight of a Hamiltonian cycle.

3.2 EULERIAN GRAPHS

As we mentioned in the introduction, Euler solved the Königsberg problem by converting it into a problem in graph theory. He represented each land area by a vertex and each bridge by an edge (see Fig.3(a)).

(a) **(b)**

Fig. 3

You might have noticed that the graph in Fig.3(a) is a **multigraph**. Here A and C are connected by two edges; So are C and D. Let us break up one of the edges connecting C and D by adding a new vertex E. Similarly, break up one of the edges joining A and C by adding a new vertex F. Then, we get the simple graph in Fig.3 (b). If we can find a way of going around the graph in Fig.3(b) using each edge only once, then we can do so in the graph in Fig.3 (a) also, and vice-versa. This process of subdividing the edges can be carried out for any multigraph.

To understand the problem, let us introduce some terms first.

Definitions :

i) A **trail** is a walk in which no edge is repeated.
ii) A **circuit** is a trail whose starting vertex and end vertex are the same.
iii) A trail which is not a circuit is sometimes called an **open trail**.
iv) A circuit (resp. trail) in a graph G containing all the edges of G, is called an **Eulerian circuit** (resp. **Eulerian trail**).
v) A **graph is Eulerian** if it contains an Eulerian circuit.

So, we can rephrase the Königsberg bridge problem in the following way:
Is the graph in Fig. 3(b) Eulerian?

Before going further, we give a clarification of our definition of Eulerian graphs in the form of a remark.

Remark: You may have noticed that we made connectedness a part of the definition of Eulerian graphs. This is to avoid examples like the one given in Fig. 4. Here the graph has a circuit which contains all the edges of the graph. However, there is no edge through which we can reach the isolated vertex. Unless there is a very special reason, we will not bother about a place to which there is no access! So, such isolated vertices are of no interest to us. By making connectedness a part of the definition, such situations can be avoided.

Fig.4

Now, let us consider some examples of Eulerian graphs. The simplest class of example is a cycle, for example, C_6 in Fig. 5(a). We can get another example by adding a cycle of length 3 to the graph in Fig. 5(a) at v_1 (see Fig. 5(b)).

(a) **(b)** **(c)**

Fig. 5

This is also Eulerian because we can start at the vertex v_1, traverse the inner triangle, come back to v_1 and traverse the outer cycle. We get yet another Eulerian graph by incorporating a cycle of length 6 at v_1 to Fig.5 (a) (see Fig. 5(c)).

Now you may like to verify whether you have understood the definition of an Eulerian circuit by attempting the following exercise.

E1) Prove that the graph given in Fig.5(c) is Eulerian by producing an Eulerian circuit in it.

E2) What is the difference between an Eulerian graph and an Eulerian circuit?

You probably found E1 easy. In a simple example like this, you can easily prove that a graph is Eulerian by producing an Eulerian circuit by trial and error. This may not be possible in more complicated cases. It is **impossible** to prove that a graph is **not** Eulerian by trial and error — we may miss some clever way of tracing an Eulerian circuit. So, we need a necessary and sufficient condition for a graph to be Eulerian. The condition should also be easy to apply. The next theorem gives such a condition. Euler's proof of the necessary part of the theorem appeared in *Solutio problematis geometriam situs pertinentis* (The solution of a Problem relating to the Geometry of Position). Hierholzer proved the sufficiency part.

Theorem 1: A connected graph G is Eulerian **if and only if** the degree of each of its vertices is even.

Proof: We shall first assume that the graph G is Eulerian and prove that all its vertices have even degree. So, let T be an Eulerian circuit in G. Every time the circuit passes through a vertex, it uses two edges, one to reach the vertex v and one to leave it. What about the vertex v from which we start tracing the circuit? The edge with which we start the circuit is paired with the edge with which we end the circuit. Apart from this, every time we pass through v in the intermediate stages, we will use two edges incident at the vertex as before. Also, we traverse each edge only once. So, all the vertices of the graph have even degree.

To **prove the converse**, consider a connected graph in which each vertex has even degree. We will now prove that G contains an Eulerian circuit, by induction on the number of edges in G.

Suppose that the number of edges is 0. Since we have assumed that the graph is connected, it consists of a single isolated point. Since the edge set is empty the statement that there is an Eulerian circuit containing all the edges is vacuously true.

Next, assume that all the graphs with fewer edges than G contain an Eulerian circuit. All the vertices of G have even degree and G has no vertex of degree 0 (isolated vertex) since it is connected. So, all the vertices have degree at least 2. We can start from an arbitrary point $u = u_0$ and trace a circuit C as follows:

We choose any edge u_0u_1 incident at u_0. Since u_1 has degree at least two, there is another edge incident at u_1, say u_1u_2. We go on tracing a circuit like this, always making sure that we enter and leave any vertex by different edges. During the course of tracing C, we may pass through u_0 several times. The process ends when we reach u_0 and find that there is no unused edge to leave u_0. If the circuit we have obtained contains all the edges, we are done. Otherwise, we remove this circuit from G and call the resulting (possibly disconnected) graph H. All the vertices in each of the components of H have even degree and all the components have fewer edges than G.

So all the components are Eulerian. We now get an Eulerian circuit in G as follows: We start from any vertex v on the circuit C and traverse the edges of C till we come to a vertex that lies on one of the components of H. We then traverse the Eulerian circuit in that component, eventually returning to the circuit C. We continue along C in this fashion, taking Eulerian circuits of components of H as we come to them, finally returning to the vertex v we started with. We would have used each of the edges only once, that is, we have obtained an Eulerian circuit.

Note that, by connectedness of G, each component of H must contain a point of C.

Hence, by induction, the result is true for all graphs satisfying the condition of the converse.

Let us now see if we can solve the Königsberg bridge problem using Theorem 1.

Example 1: Check whether the Königsbergians can go round the city using each bridge only once.

Solution: You may recall that we have reduced the Königsberg bridge problem to finding an Eulerian circuit in Fig.3(b). According to the necessary part of the theorem, if a graph has an Eulerian circuit, it has no edges of odd degree. But, as you can see, all the vertices, except E and F, have odd degree. So, this graph does not have an Eulerian circuit. So, the Königsbergians cannot go around the city using each vertex only once.

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Now, here are some exercises to test your understanding.

E3) After Euler proved his theorem, much water has flown under the bridges in Königsberg. In 1875, an extra bridge was built in Königsberg, joining the land areas A and D (see Fig.6). Is it possible now for the Königsbergians to go round the city, using each bridge only once?

Fig. 6

E4) By writing the degree sequences of the following graphs, check whether they are Eulerian. For the graphs that are Eulerian, write down an Eulerian circuit.

G_1 **G_2**

Fig.7

E5) a) For which values of n is K_n Eulerian?

b) For which values of n and m is $K_{n,m}$ Eulerian?

E6) Check whether Q_3 and Q_4 are Eulerian.

E7) Show that, in a connected Eulerian graph, an Eulerian circuit can be traced starting from any vertex.

Suppose now that the people of Königsberg will be happy if they can go around the city, still using all the bridges only once, but they do not mind ending their tour at a point different from their starting point. Is this possible? Let us now examine this question. We will convert this to a problem in graph theory. But, before that, we need a definition that will be helpful in formulating our problem.

Definition: A graph G is **edge traceable** if G contains an open trail that contains all the edges of G.

For instance, the graph in Fig.8. is edge traceable because it contains the open trail $\{v_5, v_1, v_2, v_5, v_4, v_3, v_2, v_4\}$. This contains all the seven edges of the graph and the end vertices are distinct.

In view of the definition of an edge traceable graph, citizens of Königsberg will have to check whether the graph in Fig.3(b) is edge traceable. As an immediate consequence of Theorem 1 , we get the following characterisation of edge traceable graphs.

Fig.8

Theorem 2: A connected graph G with two or more vertices is edge traceable **if and only if** it has exactly two vertices of odd degree.

Proof: Suppose G is an edge traceable graph. Then, there is an open trail T containing all the edges of G. Suppose x and y are the first and the last vertices of T. We now add a new vertex a and join this to x and y. Let us call the new graph we obtain G'. This is illustrated for a particular case in Fig.9 below:

G **G′**

Fig.9

In the graph G' we get an Eulerian circuit as follows:
We start at a, trace the edge ax, trace the open trail T, and trace the edge ya, So, by Theorem 1 all the edges of G' have even degree. Except for x and y, the degrees of all the vertices are unaffected by the addition of the edges ax and ay. So, all of them must have even degree, considered as vertices in G. In the case of vertices x and y, their degrees have become even after the edges ax and ay are added, i.e., after their degrees are increased by one. So, before the addition of the edges, their degrees must have been odd.

Conversely, suppose that exactly two vertices x and y have odd degree. Then, by adding a new vertex a and two new edges ax and ay, the degrees of all the vertices become even. So, we can find an Eulerian circuit starting at a. Let this Eulerian circuit be $\{v_0 = a, v_1, \ldots.., v_n = a\}$. Since x and y are the only vertices to which a is adjacent, either $v_1 = x$ or $v_{n-1} = x$. If $v_1 = x$, we must have $v_{n-1} = y$ and $\{v_1 = x, v_2, \ldots., v_{n-1} = y\}$ is an open Eulerian trail. Similarly, if $v_1 = y$, we must have $v_{n-1} = x$, and $\{v_1 = y, v_2, \ldots\ldots, v_{n-1} = x\}$ is an open Eulerian trail. Hence, the theorem is proved.

Let us now look at the question that motivated us to prove the theorem above.
Example 2: Check whether it is possible for the Königsbergians to go around the city, still using each bridge only once, but ending the trip at a point different from the starting point (see Fig.3 (b)).

Solution: Referring to Fig.3(b), as we observed before, all the vertices except E and F have odd degree, i.e., there are four vertices of odd degree. So, it is not possible for Königsbergians to tour the city using each bridge only once, even if they are allowed to start and end the tour at two different points.

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Here are some related exercises for you to try.

E8) Consider the situation after the addition of a new bridge in 1875 (see Fig.6). Is it possible to tour the city using each bridge only once, if starting and ending the tour at two different points is permitted?

E9) By writing down the degree sequence, find out which of the following graphs are edge traceable.

(a) **(b)**

Fig. 10

We considered one more problem that we mentioned in the introduction to this unit. This asks for a method for determining whether a given figure can be drawn without lifting the pencil from the paper and without going over any of the lines twice. There is such a method, which we shall now illustrate.

Example 3: Check whether the graph in Fig.11(a) can be drawn without lifting the pencil from the paper and without going over any of the lines twice.

Fig.11(a)

Solution: The method involves 4 steps.

Step 1: **(Add vertices at the junctions where two or more lines meet, and at the ends of line segments.)** In Fig.11 (a) there are three such junctions A, B and C. So, add vertices at A, B and C to get the multigraph with a loop in Fig.11 (b). Note that the curve joining A and B in Fig.11 (a) is replaced by a straight edge in Fig.11 (b). Similarly, the curve joining A and C is represented by the edge AC.

Fig.11(b)

Step 2: **(If there are no loops, go to Step 3. If there are loops, eliminate the loops by adding two vertices of degree two.)** If we add two vertices D and E of degree 2 to the earlier loop at A, we get the figure in Fig.11(c).

Step 3: **(If there are no multiple edges go to Step 4. Otherwise, eliminate the multiple edges by adding vertices of degree 2.)** In Fig.11(c), B and C are connected by two edges. We eliminate one of the multiple edges by adding a vertex F to it.

Fig.11(c)

Step 4: **(Count the number of edges of odd degree in the resulting graph. If there are two vertices of odd degree, the graph is edge traceable. If there is no vertex of odd degree, the graph is Eulerian . So, the graph can be drawn without lifting pen from paper. Therefore, the figure we started with can be traced without lifting pen from paper.)** As you can see from Fig.11(d) there are exactly two edges, B and C, of odd degree. So, the figure can be traced without lifting the pencil from the paper.

Fig.11(d)

* * *

If you go through the example above carefully, you may realize that there is a much easier method for deciding whether a figure can be drawn without lifting the pencil form the paper and without going over any of the lines twice. In analogy with graphs, let us call the number of lines that meet in a junction, the degree of the junction for convenience. Note that, only those junctions where more than two lines meet can give rise to vertices of odd degree. All the other vertices that we added are of even degree. In view of this observation, we have the following result.

Theorem 3: A figure can be drawn without lifting the pencil from the paper and without going over any of the lines twice **if and only if** the number of junctions whose degree is odd, and at least 3, is either 2 or 0.

Here is an opportunity for you to apply the method described above.

E10) Which of the following figures can be drawn without lifting pen from paper and without covering any line segment more than once? (Note that the end points of the vertical line in G_1 are vertices of degree 1.)

G_1 **G_2** **G_3**

Fig.12

E11) Construct, if possible, Eulerian graphs with the following number of vertices and edges. When it is not possible, explain why you think so.

	a	b	c
Number of vertices	5	6	7
Number of edges	10	10	6

So far, we have seen that if all the vertices of a graph have even degree, it is Eulerian. However, there are situations where we know that a graph is Eulerian, but we still may not be able to find an Eulerian circuit in it. There is an algorithm due to Fleury that gives a method of finding an Eulerian circuit in an Eulerian graph. You will study the algorithm in MCS-031.

In this section we were interested in finding circuits in which all the edges of the graph occur exactly once. In the next section we are interested in finding cycles in which all the vertices occur exactly once.

3.3 HAMILTONIAN GRAPHS

Suppose a transport company operates bus services between 10 different places. There are places with no direct bus service between them, but there is always a route between any two places that go through the other places. In this situation, the company wants to offer a round trip that passes through each of the cities exactly once. Is this possible?

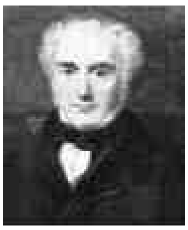

Fig. 13: Hamilton
(1805-1865)

Let us formulate this question as a problem in graph theory. Let us represent the places by vertices. Two vertices are adjacent if a direct bus connects the corresponding places. Since it is possible to go from each place to another, the graph we get is a connected graph. So, the transport company's problem is :
Is there a cycle in the graph in which each vertex occurs precisely once?

A similar question was a basis of the mathematical game described by William Rowan Hamilton. He called this game 'Traveller's Dodecahedron', and also 'A Voyage Round the World'. In this two-player game, we take a regular dodecahedron, each of its 20 vertices representing a city of the world. One player inserts a pin in a vertex.

The other player is supposed to find a 'world tour' starting from this vertex, touching the remaining 19 cities once and returning to the starting vertex. This amounts to finding a cycle covering all the vertices of the regular dodecahedron. Fig.14 gives such a cycle.

Fig.14

Such a cycle is, aptly, named after Hamilton.

Definition : A cycle C in a graph G is called a **Hamiltonian cycle** if it contains all the vertices of G. A graph is called **Hamiltonian** if it contains a Hamiltonian cycle. A graph is called **non-Hamiltonian** if it is not Hamiltonian.

Can you think of examples of Hamiltonian graphs other than the one given in Fig.14? For instance, is any cycle a Hamiltonian graph? Is any graph obtained by adding edges to a Hamiltonian graph also Hamiltonian? The answer to both these questions is 'yes'. For example, the graphs in Fig.15 are Hamiltonian.

Fig.15

Are there any non-Hamiltonian graphs? Trees are obvious examples of non-Hamiltonian graphs. Since they don't have any cycles, they cannot have a cycle containing all the vertices!

Note that, by definition, a Hamiltonian graph contains a cycle containing all the vertices. So, a Hamiltonian graph cannot have cut vertices or pendant vertices. (Recall that **a pendant vertex** is a vertex of degree 1.) This gives a simple method for constructing examples of non-Hamiltonian graphs. For example, the graph in Fig.16 is non-Hamiltonian because it has a cut vertex, namely, the vertex common to both the triangles.

Here are some exercises to help you test your understanding of the discussion above.

Fig.16

E12) Construct a non-Hamiltonian graph on 5 vertices.

E13) i) Is a Hamiltonian graph Eulerian ?

ii) Is an Eulerian graph Hamiltonian ?
Give reasons for your answers.

E14) Check whether the hypercube Q_3 is Hamiltonian.

We have used the existence of a cut vertex to prove that the graph in Fig.16 is not Hamiltonian. However, this does not give us a foolproof method of identifying non-Hamiltonian graphs. For example, $K_{m,n}$, m, n $\geq$ 2, has no cut vertices or pendant vertices, and it is not Hamiltonian when m + n is odd, as we shall now show.

Example 4: Show that $K_{m,n}$ is not Hamiltonian when m + n is odd.

Solution: Since $K_{m,n}$ is bipartite, it does not have cycles of odd length. On the other hand, it has an odd number of vertices. So, a Hamiltonian cycle in this graph, if it exists, must be of odd length. Therefore, $K_{m,n}$ is not Hamiltonian when m + n is odd.

* * *

From the previous example it is clear that to identify non-Hamiltonian graphs we need some conditions which do not depend on the existence of a cut vertex or pendant vertex. The following theorem gives necessary condition for a graph to be Hamiltonian. We will omit the proof of this theorem in this course.

Recall that c(G) denotes the number of components of G.

Theorem 4: If G is a Hamiltonian graph, then for every proper subset S of V(G), we must have $c(G - S) \leq |S|$.

Let us now look at an example to illustrate the use of Theorem 4.

Example 5: Show that $K_{m,n}$ is not Hamiltonian if m < n.

Solution: Recall that the vertex set of $K_{m,n}$ can be partitioned into two disjoint subsets X and Y of cardinality m and n, respectively, in such a way that no two edges in the same subset are adjacent and every vertex in X is adjacent to every vertex in Y. Let us take X to be the set S in Theorem 4. So, $|S| = m$ in this case. If we delete all the vertices in X, the graph becomes totally disconnected. So, there are n components in G – S, one corresponding to each vertex of Y. So, $c(G–S) = n > m = |S|$ in this case. Therefore, by Theorem 4, $K_{m,n}$ is non-Hamiltonian.

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Remark : If the condition given in Theorem 4 is not satisfied, the graph is non-Hamiltonian. However, if the condition is satisfied, it does not mean that the graph is Hamiltonian. For example, consider the Petersen graph in Fig. 17. You can check that for each subset S of V(G), $c(G - S) \leq |S|$. But G is non-Hamiltonian (which is not easy to check!).

Fig.17

Now for some exercises to check your understanding of Theorem 4.

E15) Show that the following graph is non-Hamiltonian.
(**Hint:** Find a set $S \subset V(G)$ such that $c(G - S) > |S|$.)

Fig.18

E16) Check whether the following graphs are Hamiltonian.

Fig.19

So far, we have seen some necessary conditions for a graph to be Hamiltonian. They are helpful if we want to show that a given graph is non-Hamiltonian. They are of no use if we want to show that a given graph is Hamiltonian. We need some sufficient conditions for this purpose. Since we are looking for a cycle covering all the vertices, it is reasonable to expect success whenever, at every vertex, there are enough choices of edges. This is confirmed by the following theorems. Theorem 5 was proved by Gabriel Dirac in 1952. This was generalized to Theorem 6 by Oystein Ore in 1960.

$\delta(G) = \min\{\deg_G(x) \mid x \in V(G)\}$

Theorem 5 (Dirac's criterion) : If G is a simple graph on p vertices, $p \geq 3$, and if $\delta(G) \geq \frac{p}{2}$, then G is Hamiltonian.

Theorem 6 (Ore's criterion) : Let G be a simple graph on p vertices, $p \geq 3$, satisfying the condition that $d(u) + d(v) \geq p$ for any two non-adjacent vertices u and v in G. Then G is Hamiltonian.

Can you see that Dirac's theorem follows from Ore's theorem? This is because if $\delta(G) \geq \frac{p}{2}$, then for **any** two vertices u and v, we have
$d(u) + d(v) \geq 2\delta(G) \geq p$.

So, the conditions of Ore's theorem are satisfied whenever the conditions of Dirac's Theorem are satisfied. So, if we prove Ore's criterion, we will have also proved Dirac's criterion.

Proof of Theorem 6 : We shall prove this result **by contradiction**. Suppose the theorem is false. Then, there are non-Hamiltonian graphs with p vertices satisfying Ore's criterion. So, the following set is non-empty:

$F = \{G \mid |V(G)| = p$, G is non-Hamiltonian and satisfies Ore's condition$\}$
Choose a graph in F with the maximum number of edges among all such graphs. (Such a graph must exist, because there are only finitely many graphs in F.) Let us denote this graph by G_M.

As G_M is non-Hamiltonian, it cannot be complete. So, there are two vertices, call them u and v, which are not adjacent. So, adding the edge e = uv to G_M, we get a new graph G'_M. The number of vertices in G'_M is still p because we haven't removed any vertex. Since we haven't removed any edge, the degrees of each of the vertices has not decreased. So, Ore's condition holds for any two vertices in G'_M also. But then, G'_M must be Hamiltonian. If it is not, it will be in F. This is not possible because $|E(G'_M)| = |E(G_M)| + 1$, and G_M was chosen to be a graph in F with the maximum possible edges.

Now, since G'_M is Hamiltonian, we can choose a Hamiltonian cycle C in G'_M. Since G is non-Hamiltonian, the edge uv must lie on C. (Why?) Removing this edge, we get a path in G containing all the vertices. Let
$P = \{u = u_1, u_2, \ldots, u_p = v\}$ be this path. Define
$S = \{u_j : uu_{j+1} \in E(G_M)\}$, $T = \{u_j : u_jv \in E(G_M)\}$.

Clearly, $u_p = v \notin S \cup T$. (Why?) Hence, $|S \cup T| < p$. Now, if possible, suppose $S \cap T \neq \phi$. Then, let $u_r \in S \cap T$. $\{u_1, \ldots, u_r, u_p, u_{p-1}, \ldots, u_{r+1}, u_1\}$ is a Hamiltonian cycle in the graph G (see Fig.20). This contradicts the assumption that G is non-Hamiltonian.

Fig. 20

Hence, $S \cap T = \phi$, that is, $|S \cap T| = 0$.
But then, $p \leq d_{G_M}(u) + d_{G_M}(v) = |S| + |T| = |S \cup T| < p$, i.e., $p < p$.
This is a contradiction. Thus, our assumption that the theorem is false, is wrong. In other words, every graph G on $p \geq 3$ vertices, satisfying Ore's condition, is Hamiltonian.

Remark: Note that Theorem 5 and Theorem 6 are sufficient conditions. They are not at all necessary. For example, C_n, $n > 4$, is always Hamiltonian, but C_n is a 2-regular graph, and therefore, $d(u) + d(v) = 4 < n$ always.

Here is an example to illustrate the use of the theorems.

Example 6 : To which of the graphs in Fig.21 does Dirac's criterion apply? To which does Ore's criterion apply?

Fig.21

Solution: For the graph in Fig.21(a), p = 6 and d(v) = 3 for each vertex v. So, $\delta(G) = 3$. Thus, Dirac's criterion is satisfied for this graph.

For the graph in Fig.21(b), p = 5, but d(x) = 2. So, Dirac's criterion is not satisfied by this graph. However, $d(u) + d(v) \geq 5$ for all pairs of non-adjacent vertices u and v (in fact, for all pairs u and v). So, Ore's criterion applies in this case.

* * *

Try the following exercise now to test your understanding of the example above.

E17) To which of the following graphs does Ore's criterion apply? To which of these does Dirac's criterion apply?

(a) **(b)** **(c)** **(d)**

Fig.22

So far, we have seen a few necessary conditions and some sufficient conditions for a graph to be Hamiltonian. Are there any conditions that are both necessary and sufficient for a graph to be Hamiltonian? So far no such conditions have been found.

Now, we have come to the end of our discussion on the problem stated in the beginning of the section. In the next section we consider a related, but slightly different problem where we assume that any two places are directly connected by a bus route. We are interested in finding a way of going around all the places, visiting each place only once, and doing so in the shortest possible time.

3.4 TRAVELLING SALESPERSON PROBLEM

A travelling salesperson wants to visit a number of towns and return to her base. The travelling time between any two towns is known. How should she plan her journey so that she spends as short a time as possible but visits each town precisely once? This is known as the **travelling salesperson problem.** Here, one assumes that a direct route connects any two towns without passing through any of the other towns on the list. If we try to represent the towns by vertices and the direct route by edges, then we simply get a complete graph. How should we represent the time required to go from one town to the other? This question leads to the concept of a weighted graph.

Definition: A **weighted graph** is a pair (G, f), where G is a graph and f is a real-valued function on the set E(G).

In simple language, we associate some real number f(e) with each edge e of the graph G. In the case of the travelling salesperson problem, f(e) is simply the time required to travel from one end vertex of e to the other end vertex.

Related to this we have another definition.

Definition: Let W be a walk in a weighted graph G. By the **weight of the walk W,** we mean the sum of the weights of all the edges in W.

So, our traveller's problem reduces to finding a Hamiltonian cycle of minimum weight in a weighted complete graph. One possible approach is to find a Hamiltonian cycle first and then search for edges having smaller weight and modify the cycle using them. The modifications can be made as below:
Let C = $\{v_1, \ldots, v_p, v_1\}$ be a Hamiltonian cycle in a weighted **complete graph**. For a fixed i, first check whether there is a j such that

$$f(v_iv_j) + f(v_{i+1}v_{j+1}) < f(v_iv_{i+1}) + f(v_jv_{j+1}).$$

If this inequality holds, then replace the cycle C by

$$C_{i,j} = \{v_1, \ldots, v_i, v_j, v_{j-1}, \ldots, v_{i+1}, v_{j+1}, v_{j+2}, \ldots, v_p, v_1\}.$$

In Fig.23, we have shown this diagrammatically.

(a) **(b)**

Fig.23

The algorithm for reducing the weight of the cycle is called the **1-exchange heuristic**, and was proposed by Lin and Kernighan**.**

Clearly, the weight of the cycle $C_{i,j}$ is strictly less than that of the cycle C. After performing a sequence of such modifications, one is left with a cycle whose weight cannot be reduced further by this process. Of course, **there is no guarantee that the resulting cycle will have the least possible weight**. There may be other cycles with lower weight. But it will often be fairly good. In fact, finding the minimum weight cycle is an NP-hard problem (ref. the course MCS-031.)

Let us consider an example of how the 1-exchange heuristic is applied.

Example 7: Consider the copy of a weighted K_6 given in Fig.24. Starting with the cycle {L, M, N, O, P, T, L}, modify it to a cycle of lesser weight. The number on the edge e indicates the weight f(e) assigned to it.

Solution: You can check that f(LO) + f(MP) = 80 < f(LM) + f(OP) = 107.
So, we modify the cycle to {L, O, N, M, P, T, L}(see Fig.25(a)).
Now, f(MT) + f(PL) = 121 < f(MP) + f(TL) = 138 (see Fig.25(b)).
So, again we modify the cycle to {L, O, N, M, T, P, L}.
Again, f(OP) + f(NL) = 86 < f(ON) + f(PL) = 87 (see Fig.25(c)).
Hence, modify the cycle to {L, O, P, T, M, N, L} (see Fig.25 (d)).
You can check that we can't decrease the weight of the cycle in the graph further.

Fig.24

(a) **(b)** **(c)** **(d)**

Fig.25

Hence, by this method we have reduced a cycle of weight 237 to a cycle of weight 192.

* * *

Here is a related exercise for you to try !

E18) Start with the cycle $\{v_1, v_2, v_3, v_4, v_5, v_1\}$ in the following weighted copy of K_5. Carry out the reduction step once to get a cycle of lesser weight.

Fig.26

We have now reached the end of our unit. Let us briefly summarise what we have studied in this unit.

3.5 SUMMARY

In this unit we defined the following terms:

i) Eulerian circuit: A circuit in a graph is called Eulerian if each edge of the graph occurs exactly once in the circuit.

ii) Eulerian graph: A connected graph is Eulerian if it contains an Eulerian circuit.

iii) Open trail: A trail is open if the initial and end vertices of the trail are distinct.

iv) Edge traceable graphs: A connected graph is edge traceable if it has an open trail.

v) Hamiltonian cycle: A cycle is Hamiltonian if each vertex of the graph occurs exactly once in the cycle.

vi) Hamiltonian graphs: A graph is called Hamiltonian if it contains a Hamiltonian cycle.

We also discussed the following points in the unit.

1) The proof and application of the statement : A connected graph is Eulerian **iff** the degree of each of its vertices is even.

2) The proof and use of the statement : A connected graph with two or more vertices is edge traceable **iff** it has exactly two vertices of odd degree.

3) The application of an algorithm for checking if a figure can be drawn without lifting pen from paper and without going over any of the lines twice.

4) The application of the fact that if G is Hamiltonian, then $c(G-S) \leq |S| \; \forall \; S \subseteq V(G)$.
We also gave an example to show that this condition is not sufficient.

5) The application of the Dirac and Ore criteria for a graph to be Hamiltonian. According to these criteria, for a simple graph G on p vertices, $p \geq 3$,
   i) if $\delta(G) \geq \frac{p}{2}$, G is Hamiltonian (Dirac)
   ii) if $d(u)+d(v) \geq p$ for any two non-adjacent vertices u and v of G, then G is Hamiltonian (Ore).

6) The travelling salesperson problem, namely, applying the 1-exchange heuristic to obtain a Hamiltonian cycle of smaller weight than that of a given cycle in a complete weighted graph.

3.6 SOLUTIONS/ANSWERS

E1) Here is an Eulerian circuit:
$\{v_1, v_2, v_3, v_4, v_5, v_6, v_1, v_7, v_8, v_4, v_{10}, v_9, v_1\}$
Of course, there are many different Eulerian circuits in the graph, and you may have come up with a different one.

E2) The graph G is the pair (V(G), E(G)), and the circuit is a finite sequence consisting of elements of V and E alternately such that every element of E exists once in this sequence.

E3) The situation will be as in Fig.27. After the addition of the new edge, both the vertices A and D have become even degree vertices. However, B and C still have odd degree. So, it is still not possible for the Königsbergians to go around the city using each bridge exactly once.

Fig.27

E4) The degree sequence of G_1 is {8, 4, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2}. All the vertices are even, and hence the graph is Eulerian. You can check that the following gives an Eulerian circuit in it.
$\{x_1, x_2, x_3, x_4, x_1, x_5, x_6, x_3, x_7, x_8, x_1, x_9, x_{10}, x_{11}, x_1, x_{12}, x_{13}, x_{14}, x_{15}, x_1\}$.
The degree sequence of G_2 is {8, 4, 4, 4, 4, 4, 4, 2, 2, 2, 2}. Since all the degrees are even, it is Eulerian. An Eulerian circuit in G_2 is
$\{x_1, x_2, x_3, x_4, x_5, x_1, x_3, x_5, x_2, x_4, x_1, x_6, x_7, x_8, x_1, x_9, x_7, x_{10}, x_1\}$.

E5) a) K_n is an $(n - 1)$-regular graph. So, it is Eulerian when $n - 1$ is even, i.e., n is odd.

b) $K_{n,m}$ has n vertices of degree $m - 1$ and m vertices of degree $n - 1$. So, it is Eulerian when n, m are odd.

E6) In Q_3, every vertex has degree 3, and hence it is a non-Eulerian graph. On the other hand, all the vertices of Q_4, have degree 4. Hence, Q_4 is Eulerian.

E7) Suppose G is an Eulerian graph and $\{v_0, v_1, \ldots, v_n = v_0\}$ is an Eulerian circuit in it. Let $x = v_i$ be any vertex in G. Then, the following is an Eulerian trail starting and ending at x:
$\{x = v_i, v_{i+1}, \ldots.., v_n = v_0, v_1, \ldots.., v_{i-1}\}$

E8) Refer to Fig.27. After the construction of the new bridge all the vertices except B and C are even, i.e., there are two vertices of odd degree. So, it is possible to go round the city using each bridge only once, starting and ending the trip at two different points.

E9) a) Let us write down the degree sequence of the graph. It is {4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3}. It has eight vertices of odd degree. So, the graph in Fig.10(a) is not edge traceable.

b) The degree sequence of the graph in Fig.10(b) is {4, 3, 3, 2, 2, 2}. So, it has exactly two vertices of odd degree. So, the graph is edge traceable.

E10) Since G_1 has exactly two vertices of odd degree, it can be drawn without lifting pen from paper and without going over any of the vertices twice.
Since G_2 has 6 vertices of odd degree (degree 3), it cannot be traced without lifting pen from paper.
Since G_3 has precisely two vertices of odd degree, this can also be traced without lifting pen from paper.

E11) The solutions for (a) and (b) are given below.

(a) **(b)**

Fig.28

c) Recall that any Eulerian graph is connected. Here, the number of vertices is one more than the number of edges. So, such a graph is a tree, and therefore, does not contain any cycle. Thus, there is no Eulerian graph with the given number of vertices and edges.

E12) For example, consider the graph in Fig.29. This is non-Hamiltonian because the vertex x is a cut vertex.

Fig.29

E13) i) See Fig.30. This has a Hamiltonian cycle $\{v_1, v_2, v_3, v_4, v_5, v_6, v_1\}$. But, it is not Eulerian because the vertices v_2 and v_5 have odd degrees.

ii) The graph given in Fig.29 is Eulerian because all its vertices have even degree. As, we have seen already, it is not Hamiltonian.

E14) A Hamiltonian cycle in Q_3 is
{000, 100, 110, 010, 011, 111, 101, 001, 000}.

E15) If you remove the vertices marked x, y and z in Fig.31, you will get four connected components, namely, one inner triangle and three isolated outer vertices.

Fig.30

Fig.31

Hence, by Theorem 4, the given graph is non-Hamiltonian.

E16) A Hamiltonian cycle in the graph G_1 is $\{x_7, x_3, x_4, x_2, x_6, x_5, x_1, x_7\}$.
The following cycle in the graph G_2 is Hamiltonian :
$\{x_{12}, x_{14}, x_8, x_9, x_{10}, x_{11}, x_{13}, x_7, x_6, x_5, x_4, x_3, x_2, x_1, x_{12}\}$.

E17) a) This is a 4-regular graph. So, $\delta(G) = 4$. Here $p = 6$, and therefore, the condition $\delta(G) \geq \frac{p}{2}$ is satisfied. So, Dirac's criterion (and therefore, Ore's criterion) applies here.

b) Here $p = 7$. The vertices v_6 and v_7 have degree $3 < \frac{7}{2}$. Therefore, Dirac's criterion does not apply. However, the only pairs of non-adjacent vertices in this graph are (v_6, v_4), (v_6, v_5), (v_6, v_3), (v_7, v_4), (v_7, v_5), (v_7, v_1).
Ore's condition is satisfied for these pairs of vertices. So, this graph is Hamiltonian.

c) Here $p = 8$ and the graph is 4-regular. So, Dirac's criterion is satisfied.

d) Here $p = 8$, but the vertices v_8 and v_4 have degree 3 which is less than $\frac{p}{2} = 4$. So, Dirac's criterion is not satisfied. The only pairs of non adjacent vertices are (v_7, v_3), (v_7, v_4), (v_7, v_5), (v_7, v_6), (v_8, v_2), (v_8, v_3), (v_8, v_4), (v_8, v_5). You can check that Ore's criterion is satisfied for these pairs of vertices.

E18) Notice that
$f(v_1 v_2) + f(v_4 v_5) = 51 + 78 = 129$
$f(v_1 v_4) + f(v_2 v_5) = 5 + 36 = 41$

We can modify the given cycle to get the following cycle of smaller weight: $\{v_1, v_4, v_3, v_2, v_5, v_1\}$.

Fig.32