Computational Physics
Homework 4

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1 Problem 1

Newtonian mechanic describes planetary orbit in second order differential equation.

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{l^2} \tag{1}$$

Solving the equation analytically and numerically. Euler method, two stages Runge-Kutta and four stages Runge-Kutta method are applied. Compare behaviors such as error, convergence rate between each method.

1.1 Theory

(a)prove exact solution of equation (1) is

$$u = \frac{GM}{l^2}(1 + \epsilon cos\phi) \tag{2}$$

Explicitly, we have homogenious and inhomogenious solution

$$\begin{aligned} u_0 &= \frac{GM}{l^2} && (3)\\ u_H &= Acos(\phi + \phi_0) && (4) \end{aligned}$$

where A and ϕ_0 are arbitrary parameter depends on initial condition. We choose $\phi_0 = 0$, and $A = \epsilon\frac{GM}{l^2}$where

$$\epsilon = \sqrt{1 + 2EL^2/(GMm)^2} \tag{5}$$

hence, the exact solution is u_H plus u_0

$$\begin{aligned} u(\phi) &= u_0 + u_H \\ &= \frac{GM}{l^2}(1 + \epsilon cos(\phi)) \end{aligned} \tag{6}$$

An ellipse with eccentricity ϵ are obtained.

1.2 Algorithms

1.2.1 Euler Method

Suppose we we want to solve a first order ODE with initial value

$$y'(t) = f(t, y(t)), y(t_0) = y_0 \tag{7}$$

Set step size$h$, one step of Euler method from t_n to t_{n+1} is

$$y_{n+1} = y_n + hf(t_n, y_n) \tag{8}$$

Now derive local truncation error, using Taylor expansion in equation (8),

$$y(t_{n+1} + h) = y(t_n) + hy'(t_n) + \frac{1}{2}y''(t_n) + O(h^3) \tag{9}$$

Therefore,

$$\delta = y(t_0 + h) - y_1 = \frac{1}{2}h^2 y''(t_0) + O(h^3) \tag{10}$$

which shows that truncation error is proportional to h^2.

In this homework, we can reduce second order ODE into two first order ODE problem.

$$\begin{aligned} \frac{du}{d\phi} = v = f_1(u, n, t) &\qquad , u(0) = 1 + \epsilon, u'(0) = 0 \\ \frac{dv}{d\phi} = -u + 1 = f_2(v, n, t) &\qquad , v(0) = 0, v'(0) = 0 \end{aligned} \tag{11}$$

where we let $\frac{GM}{l^2} = 1$ with proper unit. Therefore, Euler method become

$$\begin{aligned} u_{n+1} &= u_n + hf_1(u_n, v_n, t_n) \\ v_{n+1} &= v_n + hf_2(u_n, v_n, t_n) \end{aligned} \tag{12}$$

1.2.2 2-order Runge-Kutta Method

2-order Runge-Kutta method are given by formula

$$y_{n+1} = y_n + hf(t_n + \frac{h}{2}), y_n + \frac{h}{2}f(t_n, y_n) \tag{13}$$

truncation error is proportional to h^2.

In our practice, the formula become

$$\begin{aligned}
l_1 &= f_1(u_n + \frac{h}{2}f_1(u_n, v_n, t_n), v_n, t_n + \frac{h}{2}) \\
k_1 &= f_2(u_n + \frac{h}{2}f_1(u_n, v_n, t_n), v_n + \frac{h}{2}f_2(u_n, v_n, t_n), t_n + \frac{h}{2}) \\
u_{n+1} &= u_n + hl_1 \\
v_{n+1} &= v_n + hk_1
\end{aligned} \tag{14}$$

1.2.3 4-order Runge-Kutta Method

4-order Runge-Kutta method are given by formula

$$\begin{aligned}
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k4) \\
k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1) \\
k_3 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2) \\
k_4 &= f(t_n + h, y_n + hk_3)
\end{aligned} \tag{15}$$

truncation error is proportional to h^4.

Again, in our problem the formula become

$$\begin{aligned}
l_1 &= f_1(t_n, u_n, v_n) \\
k_1 &= f(t_n, u_n + \frac{h}{2}l_1, v_n) \\
l_2 &= f(t_n + \frac{h}{2}, u_n + \frac{h}{2}l_1, v_n + \frac{h}{2}k_1) \\
k_2 &= f(t_n + \frac{h}{2}, u_n + \frac{h}{2}l_2, v_n + \frac{h}{2}k_1) \\
l_3 &= f(t_n + \frac{h}{2}, u_n + \frac{h}{2}l_2, v_n + \frac{h}{2}k_2)
\end{aligned}$$

$$
\begin{aligned}
k_3 &= f(t_n + \frac{h}{2}, u_n + \frac{h}{2} l_3, v_n + \frac{h}{2} k_2) \\
l_4 &= f(t_n + h, u_n + h l_3, v_n + 2k_3) \\
k_4 &= f(t_n + h, u_n + h l_4, v_n + 2k_3) \\
u_{n+1} &= u_n + \frac{h}{6}(l_1 + 2l_2 + 2l_3 + l4) \\
v_{n+1} &= v_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k4)
\end{aligned} \tag{16}
$$

1.3 Sample output

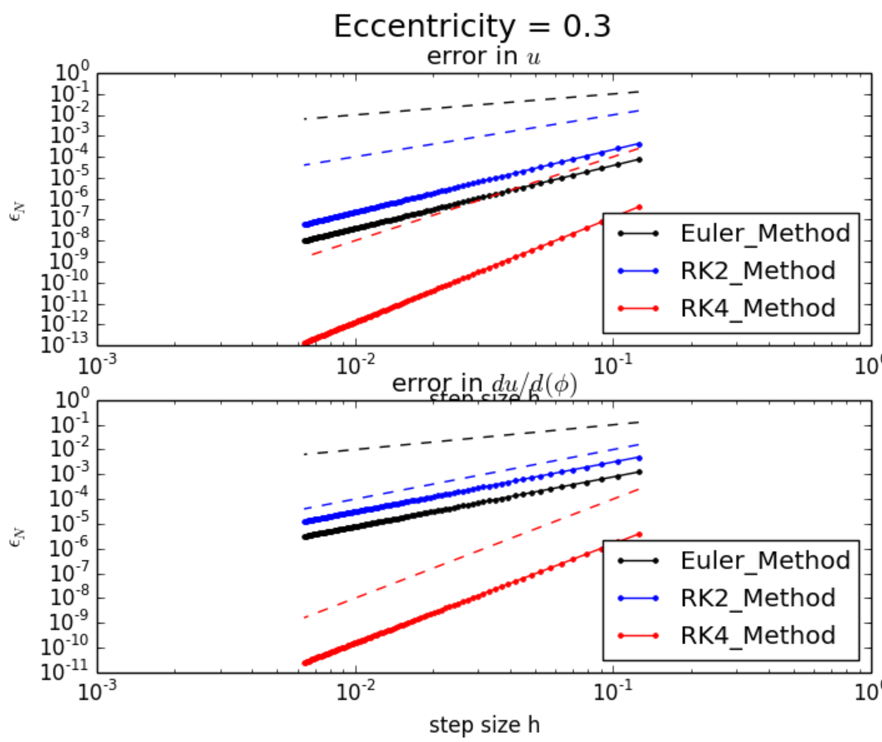

Figure 1: Sample output eccentricity=0.3

From sample output, RK2 and RK4 converge roughly follow the theoritical predict. But, Euler method go faster then it should be.

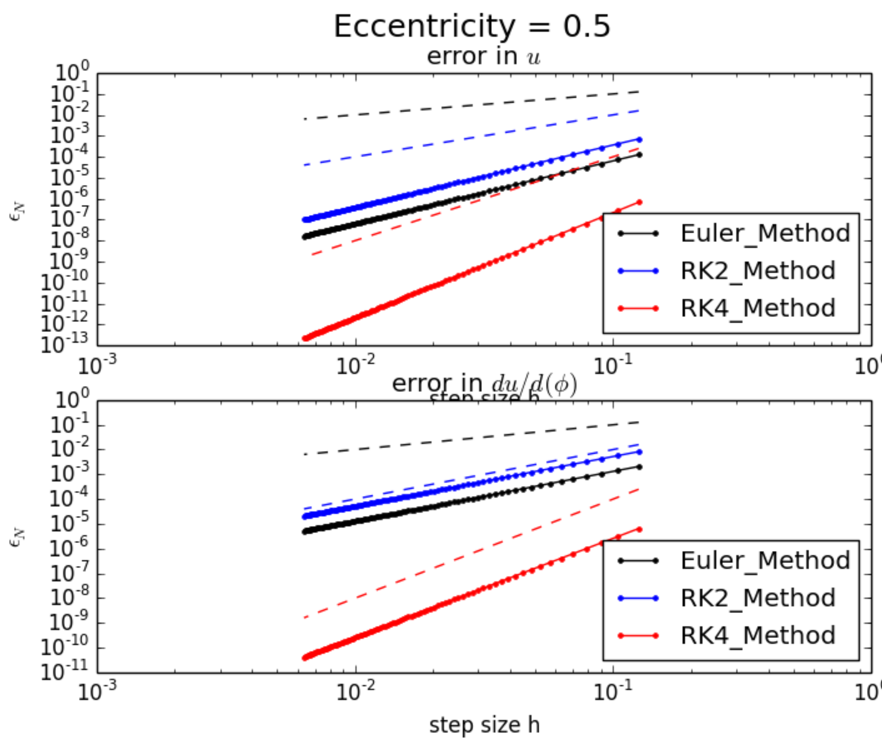

Figure 2: Sample output eccentricity=0.5

2 Problem 2

2.1 Theory

General relativity corrects equation (1) to

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{l^2} + \frac{3GM}{c^2}u^2 \tag{17}$$

or rewrite into

$$u'' + u = 1 + 3\lambda u^2 \tag{18}$$

from perturbation theory, we assume the solution is homogenious exact solution plus perturbation term

$$u = u_0 + \lambda * f_1(\phi) \tag{19}$$

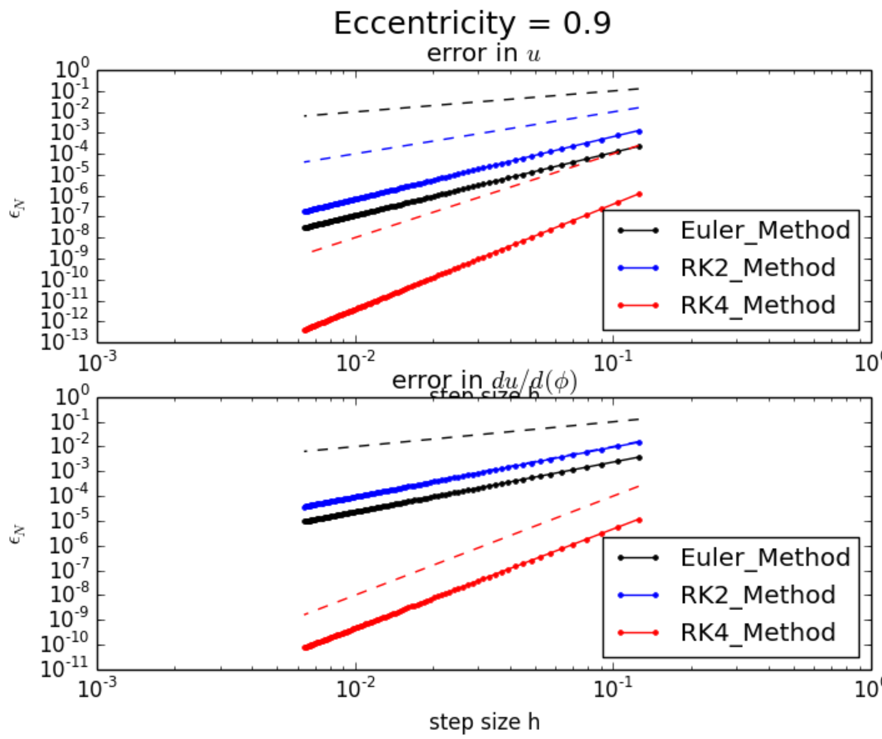

Figure 3: Sample output eccentricity=0.9

where $u_0 = 1+\epsilon cos(\phi)$ as we shown before, and λ is a small number. Substitute equation(19) into equation (18), we have

$$\begin{aligned} \lambda f'' + \lambda f &= 3\lambda(1+\epsilon cos(\phi))^2 \\ f'' + f &= 3(1+2\epsilon cos(\phi)) + O(\epsilon^2) \end{aligned} \tag{20}$$

Solving $f(\phi)$, we have

$$f(\phi) = 3(1+\epsilon\phi sin(\phi)) \tag{21}$$

Therefore, equation (19) becomes

$$u = 1+\epsilon cos(\phi) + 3\lambda(1+\epsilon\phi sin(\phi)) \tag{22}$$

for small λ, one can write

$$\begin{aligned} cos(\phi(1-3\alpha)) &= cos\phi cos3\lambda\phi + sin\phi sin3\lambda\phi \\ &\approx cos\phi + 3\lambda\phi sin\phi \end{aligned} \tag{23}$$

$$u \approx 1+\epsilon cos[\phi(1-3\lambda)] \tag{24}$$

finally,

$$\Delta\phi^{shift} = \frac{2\pi}{1-3\lambda} - 2\pi = 6\pi\lambda + O(\lambda^2) \tag{25}$$

shows that $\Delta\phi^{shift}$ is proportional to small number of λ, the slope is 6π.

2.2 Algorithm

4-order Runge-Kutta Method were described before. Here we just modified $f_2(u, v, t)$ into

$$\frac{dv}{d\phi} = f_2(u, v, t) = -u + 1 + 3\lambda u^2 \tag{26}$$

2.3 Sample output

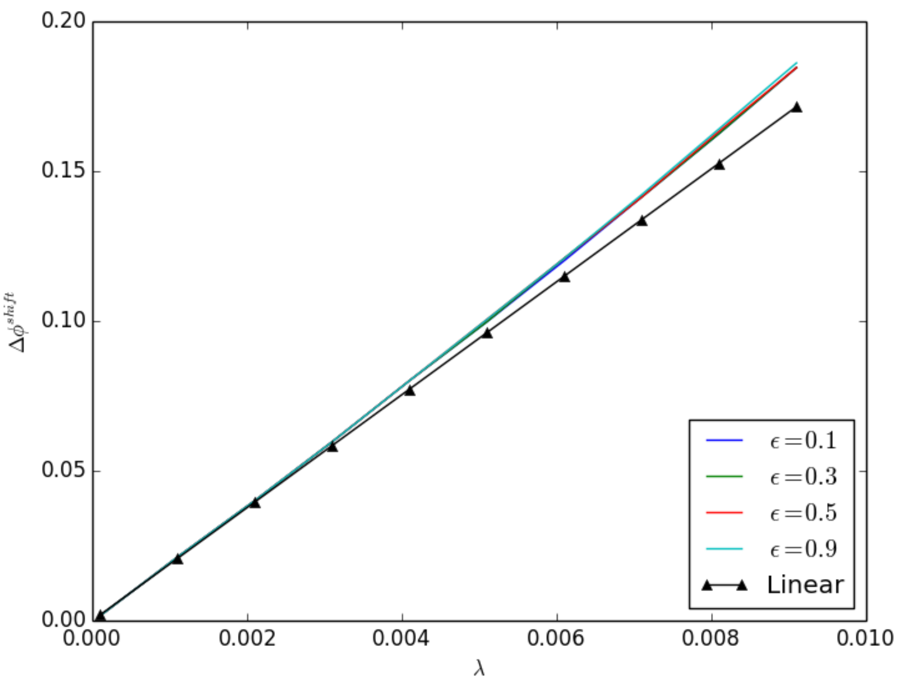

Figure 4: Relationship between λ and $\Delta\phi^{shift}$ follow linear approximation when λ is small. When λ goes larger, the numerical approach behave like quadratic due to higher order term

2.4 problem 2 (c)

From NASA database, the key fact of Mercury

$$\begin{aligned}
G &= 6.67 \times 10^{11} m^3 kg^{-1} s^{-2} \\
M_{sun} &= 1.988 \times 10^{30} kg \\
c &= 3.0 \times 10^8 ms^{-1} \\
R_{orbit} &= 46 \times 10^9 m \\
v &= 58.98 \times 10^3 ms^{-1} \\
T_{period} &= 87.968 days
\end{aligned} \tag{27}$$

Hence

$$\lambda = \frac{GM^2}{lc} = 2.6541 \times 10^{-8} \tag{28}$$

Therefore, $\Delta\phi^{shift} perperiod$

$$\Delta\phi^{shift} = 6\pi\lambda = 0.103191" \tag{29}$$

and shift per century are given by

$$\Delta\phi^{shift}_{century} = \Delta\phi^{shift} \frac{100}{T_{period}} = 42.8458" \tag{30}$$