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Math 20B Lecture 11 7/24

10.2 Summing an Infinite Series.

Suppose we are given $\{a_n\}$

"Infinite Series" $= a_1 + a_2 + a_3 + a_4 \dots a_n + \dots$ ⊛

We denote ⊛ by $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$

To find $\sum_{n=1}^{\infty} a_n$, consider

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + \dots + a_N$

S_n's are called partial sum.

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N \qquad \sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \overbrace{\boxed{\sum_{n=1}^{N} a_n}}^{S_N}$$

If $\lim_{N\to\infty} S_N = \underset{\text{some \#}}{S}$ exists, then $\sum_{n=1}^{\infty} a_n$: convergent

If $\lim_{N\to\infty}$ does not exist, " : divergent.

Thm If $\sum_{n=1}^{\infty} a_n$: convergent, then $\lim_{n\to\infty} a_n = 0$.

$a_n = S_n - S_{n-1}$

$\Downarrow$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1})$

$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$ (Since $\sum a_n$: conv, $\lim_{N\to\infty} S_n = S$ exists)

$= S - S = 0$

divergent test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$: divergent

Ex $\sum_{n=1}^{\infty} \frac{n}{3n+2}$ conv or div?

$= \left(\frac{1}{3\cdot1+2} + \frac{2}{3\cdot2+2} + \frac{3}{3\cdot3+2} \cdots + \underbrace{\frac{100}{302}}_{\approx\frac{1}{3}} + \underbrace{\frac{101}{305}}_{\approx\frac{1}{3}} \cdots = \infty \right)$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n}{3n+2}\right) = \frac{1}{3} \neq 0$, by div test, $\sum a_n$: div.

Note : $\lim_{n\to\infty} a_n = 0$ does not imply $\sum a_n$: conv.

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \cdots\cdots$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum \frac{1}{n}$: div

$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots \frac{1}{16}\right) + \frac{1}{17} \cdots\cdots \geq$

$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{\frac{1}{2}} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{\frac{1}{2}} + \underbrace{\left(\cdots\cdots\right)}_{\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{2} + \cdots = \infty$

When $\lim_{n\to\infty} a_n = 0$, $\sum a_n$ may conv or may div.

Ex. $\sum_{n=1}^{\infty} \underbrace{\frac{1}{(n+1)(n+2)}}_{=a_n}$ conv or div?

$\sum a_n$ conv if $\lim_{N\to\infty} S_N$ exist
div if .. DNE.

$S_N = \sum_{n=1}^{N} \frac{1}{(n+1)(n+2)}$ $\left(\frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \cdots + \frac{1}{(N+1)(N+2)} \right)$

partial fraction

$= \sum_{n=1}^{N} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$

$= \left(\frac{1}{1+1} - \cancel{\frac{1}{1+2}} \right) + \left(\cancel{\frac{1}{2+1}} - \cancel{\frac{1}{2+2}} \right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} \right) + \cdots$ $\left(\cancel{\frac{1}{N+1}} - \frac{1}{N+2} \right)$

$= \frac{1}{2} - \frac{1}{N+2}$

$$\lim_{N\to\infty} S_N = \lim_{n\to\infty}\left(\frac{1}{2} - \cancel{\frac{1}{N+2}}\right) = \frac{1}{2}$$

By definition, $\sum a_n$: conv.

Linearity of Infinite Series

If $\sum a_n$, $\sum b_n$ conv, & c : const

then so are $\sum a_n \pm b_n = \sum a_n \pm \sum b_n$

$$\sum (c \cdot a_n) = c \sum_{n=1}^{\infty} a_n$$

The geometric series

$$\sum_{n=0} ar^n = a + ar^1 + ar^2 + \cdots ar^3 + \cdots$$

$$S_N = a + ar + ar^2 + \cdots ar^N$$

$$- \quad r \cdot S_N = ar + ar^2 + ar^3 + \cdots + ar^{N+1}$$

$$S_N - r \cdot S_N = a - ar^{N+1}$$

$$S_N = \begin{cases} \dfrac{a(1-r^{N+1})}{1-r} & \text{if } r \neq 1 \\ (N+1)a & r = 1 \end{cases}$$

$S_N = a + a + \cdots + a$

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \text{conv} & \left(\text{value} = \dfrac{a}{1-r}\right) = \dfrac{\text{first term}}{1-\text{ratio}} \quad \text{if } -1 < r < 1 \\ \text{div} & \text{otherwise} \end{cases}$$

Find the sum of $6 + 6 \cdot \frac{5}{9} + 6 \cdot \frac{5^2}{9^2} + 6 \cdot \frac{5^3}{9^3} + \cdots$ (*) Geo Ser

$r = \frac{5}{9}$ since $-1 < \frac{5}{9} < 1$, (*) is conv.

The value $= \dfrac{\text{first term}}{1 - \text{ratio}} = \dfrac{6}{1-\frac{5}{9}} = \dfrac{6}{\frac{4}{9}} = 6 \cdot \dfrac{9}{4} = \dfrac{27}{2}$.

(ii) $\sum_{n=0}^{\infty} \frac{3\cdot(-2)^{2n}-5^{n+1}}{8^n}$

$= \sum_{n=0}^{\infty} \frac{3(-2)^{2n}}{8^n} - \sum_{n=0}^{\infty} \frac{5^{n+1}}{8^n}$

$= \sum_{n=0}^{\infty} 3\left(\frac{4}{8}\right)^n - \sum_{n=0}^{\infty} 5\left(\frac{5}{8}\right)^n$

$= \quad -1 < r_1 = \frac{1}{2} < 1$ converges $\qquad -1 < r_2 = \frac{5}{8} < 1$ conv.

$= \frac{3}{1-\frac{1}{2}} - \frac{5}{1-\frac{5}{8}}$

$2.\overline{32} = 2.32323232$

$= 2 + 0.32 + 0.0032$

$a = 0.32$

$r = 0.01$

Ex (i) $S_n = \frac{2n^2+1}{3n^2-2}$ $n \geq 1$ $\qquad a_n = \frac{2n^2+1}{3n^2-2}$ $n \geq 1$

$\{a_n\}$ conv. or div? $=0$

$\{a_n\}$ conv or div?

Σa_n conv or div

Σa_n conv or div

$\lim_{n\to\infty} S_n = \frac{2}{3}$