# Lesson Plan
# SI Session #13
# September 5, 2017

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Course: Math 18
Academic Quarter: Summer Session 2 2017
Instructor: Professor Drimbe

Topics Covered: Basis & Span, Cramer's Rule, Eigenvalues, Eigenvectors

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## Opener Activity:

**5:05pm - 5:10pm**
Basis and Span

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}.$$

Then $\text{span}(S) = \mathbb{R}^2$. (Exercise). In fact, any two of the elements of $S$ span $\mathbb{R}^2$. (Exercise). So we can throw out any one of them, for example, the second one, obtaining the set

$$\widehat{S} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}.$$

And this smaller set $\widehat{S}$ also spans $\mathbb{R}^2$. (There are two other possibilities for subsets of $S$ that also span $\mathbb{R}^2$.) But we can't discard an element of $\widehat{S}$ and still span $\mathbb{R}^2$ with the remaining one vector.

(Why not? Suppose we discard the second vector of $\widehat{S}$, leaving us with the set

$$\tilde{S} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

Now $\text{span}(\tilde{S})$ consists of all scalar multiples of this single vector (a line through $\mathbf{0}$). But anything not on this line, for instance the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is not in the span. So $\tilde{S}$ does not span $\mathbb{R}^2$.)

## Activity 1

**5:10pm - 5:30pm**
Cramer's Rule

Practice problem 1a:
$-2x + 3y = 7$
$4x - 3y = 6$

Solutions for Practice Problem 1a:

First we hunt down *D*, the coefficient determinant:

$$\begin{vmatrix} -2 & 3 \\ 4 & -3 \end{vmatrix}$$

We multiply down the diagonal from left to right and then subtract the value we get by multiplying up the diagonal from left to right:

$D$ = (-2)(-3) – (4)(3) = 6 – 12 = -6

To find *x*'s determinant, we delete the *x*-values from the matrix and substitute in the constant values:

$$D_x = \begin{vmatrix} 7 & 3 \\ 6 & -3 \end{vmatrix}$$

Then we use Cramer's Rule as usual with the new values:

$D_x$ = (7)(-3) – (6)(3) = -21 – 18 = -39

We do the same thing to find the *y* determinant:

$$D_y = \begin{vmatrix} -2 & 7 \\ 4 & 6 \end{vmatrix}$$

Follow up with Cramer's Rule:

$D_y$ = (-2)(6) – (4)(7) = -12 – 28 = -40

Now we can find *x* and *y*.

$$x = \frac{D_x}{D} = \frac{-39}{-6} = \frac{13}{2}$$

$$y = \frac{D_y}{D} = \frac{-40}{-6} = \frac{20}{3}$$

Practice Problem 1b:

$$2x + 3y + z = 10$$
$$x - y + z = 4$$
$$4x - y - 5z = -8$$

Practice Problem 1b Solutions:

$$x = \frac{\begin{vmatrix} 10 & 3 & 1 \\ 4 & -1 & 1 \\ -8 & -1 & -5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 1 \\ 4 & -1 & -5 \end{vmatrix}} \quad y = \frac{\begin{vmatrix} 2 & 10 & 1 \\ 1 & 4 & 1 \\ 4 & -8 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 1 \\ 4 & -1 & -5 \end{vmatrix}} \quad z = \frac{\begin{vmatrix} 2 & 3 & 10 \\ 1 & -1 & 4 \\ 4 & -1 & -8 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 1 \\ 4 & -1 & -5 \end{vmatrix}}$$

$$\begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 1 \\ 4 & -1 & -5 \end{vmatrix} = (4)\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + (-5)\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}$$

$$4(3 - (-1)) + (2 - 1) - 5(-2 -3)$$
$$= 4(4) + (1) - 5(-5)$$
$$= 16 + 1 + 25$$
$$= 42$$

$$x = \frac{\begin{vmatrix} 10 & 3 & 1 \\ 4 & -1 & 1 \\ -8 & -1 & -5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 1 \\ 4 & -1 & -5 \end{vmatrix}} = \frac{10\begin{vmatrix} -1 & 1 \\ -1 & -5 \end{vmatrix} - 3\begin{vmatrix} 4 & 1 \\ -8 & -5 \end{vmatrix} + 1\begin{vmatrix} 4 & -1 \\ -8 & -1 \end{vmatrix}}{42}$$

$$= \frac{10(5 - (-1)) - 3(-20 - (-8)) + (-4 - 8)}{42}$$
$$= \frac{10(6) - 3(-12) + (-12)}{42}$$
$$= \frac{60 + 36 -12}{42}$$
$$= \frac{84}{42}$$
$$= 2$$

$$y = \frac{\begin{vmatrix} 2 & 10 & 1 \\ 1 & 4 & 1 \\ 4 & -8 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 1 \\ 4 & -1 & -5 \end{vmatrix}} = \frac{2\begin{vmatrix} 4 & 1 \\ -8 & -5 \end{vmatrix} - 10\begin{vmatrix} 1 & 1 \\ 4 & -5 \end{vmatrix} + 1\begin{vmatrix} 1 & 4 \\ 4 & -8 \end{vmatrix}}{42}$$

$$= \frac{2(-20 -(-8)) - 10(-5 - 4) + (-8 - 16)}{42}$$

$$= \frac{2(-12) - 10(-9) + (-24)}{42}$$

$$= \frac{-24 + 90 - 24}{42}$$

$$= \frac{42}{42}$$

$$= 1$$

$$z = \frac{\begin{vmatrix} 2 & 3 & 10 \\ 1 & -1 & 4 \\ 4 & -1 & -8 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 1 \\ 4 & -1 & -5 \end{vmatrix}} = \frac{2\begin{vmatrix} -1 & 4 \\ -1 & -8 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 \\ 4 & -8 \end{vmatrix} + 10\begin{vmatrix} 1 & -1 \\ 4 & -1 \end{vmatrix}}{42}$$

$$= \frac{2(8 -(-4)) - 3(-8 - 16) + 10(-1 - (-4))}{42}$$

$$= \frac{2(12) - 3(-24) + 10(3)}{42}$$

$$= \frac{24 + 72 + 30}{42}$$

$$= \frac{126}{42}$$

$$= 3$$

(x,y,z) = (2,1,3)

**Activity 2**

**5:30pm - 5:45pm**

Eigenvectors and Eigenvalues

Practice Problem 2a:

**2.** Is $\lambda = -2$ an eigenvalue of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$? Why or why not?

Solution to Practice Problem 2a:

**2**. The number $-2$ is an eigenvalue of $A$ if and only if the equation $A\mathbf{x} = -2\mathbf{x}$ has a nontrivial solution. This equation is equivalent to $(A+2I)\mathbf{x} = \mathbf{0}$. Compute

$$A+2I = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

The columns of $A$ are obviously linearly dependent, so $(A+2I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, and so $-2$ is an eigenvalue of $A$.

**3**. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 29 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is *not* an eigenvector of $A$.

**4**. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -1+2\sqrt{2} \\ 3+\sqrt{2} \end{bmatrix}$ The second entries of $\mathbf{x}$ and $A\mathbf{x}$ shows that if $A\mathbf{x}$ is a multiple of $\mathbf{x}$, then that multiple must be $3+\sqrt{2}$. Check $3+\sqrt{2}$ times the first entry of $\mathbf{x}$:

$$(3+\sqrt{2})(-1+\sqrt{2}) = -3 + \left(\sqrt{2}\right)^2 + 2\sqrt{2} = -1 + 2\sqrt{2}$$

This matches the first entry of $A\mathbf{x}$, so $\begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}$ is an eigenvector of $A$, and the corresponding eigenvalue is $3+\sqrt{2}$.

Practice Problem 2b:

Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}$$

Solution to Practice Problem 2b:

The first thing that we need to do is find the eigenvalues. That means we need the following matrix,

$$A - \lambda I = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 7 \\ -1 & -6-\lambda \end{pmatrix}$$

In particular we need to determine where the determinant of this matrix is zero.

$$\det\left(A - \lambda I\right) = \left(2-\lambda\right)\left(-6-\lambda\right) + 7 = \lambda^2 + 4\lambda - 5 = \left(\lambda+5\right)\left(\lambda-1\right)$$

$\lambda_1 = -5$ :

In this case we need to solve the following system.

$$\begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix} \vec{\eta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Recall that officially to solve this system we use the following augmented matrix.

$$\begin{pmatrix} 7 & 7 & 0 \\ -1 & -1 & 0 \end{pmatrix} \begin{matrix} \frac{1}{7}R_1 + R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 7 & 7 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Upon reducing down we see that we get a single equation

$$7\eta_1 + 7\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = -\eta_2$$

$\lambda_2 = 1$ :

We'll do much less work with this part than we did with the previous part.

$$\begin{pmatrix} 1 & 7 \\ -1 & -7 \end{pmatrix} \vec{\eta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Clearly both rows are multiples of each other and so we will get infinitely n
many minus signs floating around. Doing this gives us,

$$\eta_1 + 7\eta_2 = 0 \qquad \eta_1 = -7\eta_2$$

$$\lambda_1 = -5 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} -7 \\ 1 \end{pmatrix}$$

<u>**Closure- Survey/ Feedback**</u>

<u>**5:45pm- 5:50pm**</u>

- Wrap-up:
	- Please share with the group one thing you gained understanding of through the session today.

- Make a note to yourself/ write down anything you need to review/ do more practice problems on.

- Survey/ Feedback:
    1. How fun was the session? (1-10)
    2. How useful was the session? (1-10)
    3. Would you come back? (yes or no)
    4. Optional: Comments (pace of the activity), questions, concerns, suggestions, feedback on the back or wherever

Please recommend SI to your friends/ peers if you found the session useful! Thanks for coming and have a great day :)

# PLANNING THE SI SESSION

**SI Leader:**
**Session Date & Day of Week**:
**Course:**
**Course Instructor**:

| Warm-up/ Opening: **(2-4 min.)** | **Content to cover:** | **Collaborative Learning Technique** | **Strategy to be used:** |
|---|---|---|---|
| | | | |

Please provide a **DETAILED BREAKDOWN** of warm-up activity **OR** attach corresponding document(s)

| Cool-down/ Closing: **(2-4 min.)** | **Content to cover:** | **Collaborative Learning Technique** | **Strategy to be used:** |
|---|---|---|---|
| | | | |

Please provide a **DETAILED BREAKDOWN** of cool-down activity **OR** attach corresponding document(s)

| Workout: **(44-46 min.)** | **Content to cover:** | **Collaborative Learning Technique(s)** | **Strategy(ies) to be used:** |
|---|---|---|---|
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Please provide a **DETAILED BREAKDOWN** of workout activity **OR** attach corresponding document(s)