Professor Um
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Math 20B Lecture 13 1/26

Winter 2013

Midterm 2 #3

Final Ex #3, 4

Fall 2012 #4, #5

Midterm 2 #4, #5

Final #7, #8

10.4 Absolute and Conditional Convergence

$s_1 < s_2 < s_3 < s_4 \rightarrow \{s_N\}$ inc

$a_1 \quad a_1+a_2 \quad a_1+a_2+a_3$

$s_N < L$ # $\lim_{N\to\infty} s_N$ exists

$a_1+a_2+a_3+\cdots = \sum a_n$: conv

for $a_n > 0$. if $\lim_{N\to\infty} s_N$

Consider $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

As we see $a_n = (-1)^{n-1}|a_n|$ or $(-1)^n|a_n|$

These series are called "Alternating Series"

We already know $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

$S_{2N} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{2N-1} - \frac{1}{2N}\right)$.

$\{S_{2N}\}$ increasing $S_2 < S_4 < S_6 \cdots$

$S_{2N} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \cdots - \left(\frac{1}{2N-2} - \frac{1}{2N-1}\right) - \frac{1}{2N} < L = 1$

$\lim_{N\to\infty} S_{2N} = S$ (exists)

$$\lim_{N\to\infty} S_{2N+1} = \lim_{N\to\infty} S_{2N} + a_{2N+1}$$

$$S_{2N+1} = \underbrace{a_1 + a_2 \cdots + a_{2N}}_{S_{2N}} + a_{2N+1}$$

$$= \lim_{N\to\infty} S_{2N} + \frac{1}{2N+1}$$

$$\nearrow S$$

$$= S$$

So $\lim_{N\to\infty} S_N = S \Rightarrow \sum \frac{(-1)^{n-1}}{n}$ ∴ conv.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}$$

The alternating series test

(*) $\sum_{n=1}^{\infty} \underbrace{(-1)^{n-1} b_n}_{a_n} = b_1 - b_2 + b_3 - b_4 + \cdots \quad b_n > 0$

satisfies (i) $\lim_{n\to\infty} b_n = 0$

(ii) $\{b_n\}$: decreasing

then the series (*) is convergent

Ex $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$: conv or div?

$b_n = \frac{1}{\sqrt{n}} > 0$

(i) $\lim_{n\to\infty} b_n = \frac{1}{\sqrt{n}} = 0$

(ii) $\{b_n\}$ decreasing $\Leftarrow$ $b_n > b_{n+1}$: $\sqrt{n} < \sqrt{n+1}$, $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$

or $f' < 0$ when $b_n = f(n)$

By A.S.T, series converges.

Ex $\sum_{n=1}^{\infty} \frac{\cos n\pi}{\sqrt{n}}$ conv or div?

$$\frac{\cos \pi}{\sqrt{1}} + \frac{\cos 2\pi}{\sqrt{2}} + \frac{\cos 3\pi}{\sqrt{3}} + \dots$$

$b_n = \frac{1}{\sqrt{n}}$

i) $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n^2}\right)$. A.S

$b_n = \cos\left(\frac{1}{n^2}\right)$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \cos\frac{1}{n^2} = 1$

$\lim_{n\to\infty} a_n \neq b_y$ div test $\sum a_n$: div.

(ii) $\sum \frac{\cos(2n\pi)=1}{\sqrt{n}}$ Not A.S

$= \sum_{1}^{\infty} \frac{1}{\sqrt{n}}$: div (p-series test $p = \frac{1}{2} < 1$

Ex1 $\sum_{n=3}^{\infty} \frac{(-1)^n \ln n}{n}$ A.S By A.S.T., series converges.

$b_n = \frac{\ln n}{n}$

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln n}{n}$

$= \lim_{x\to\infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{(\ln x)'}{(x)'} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

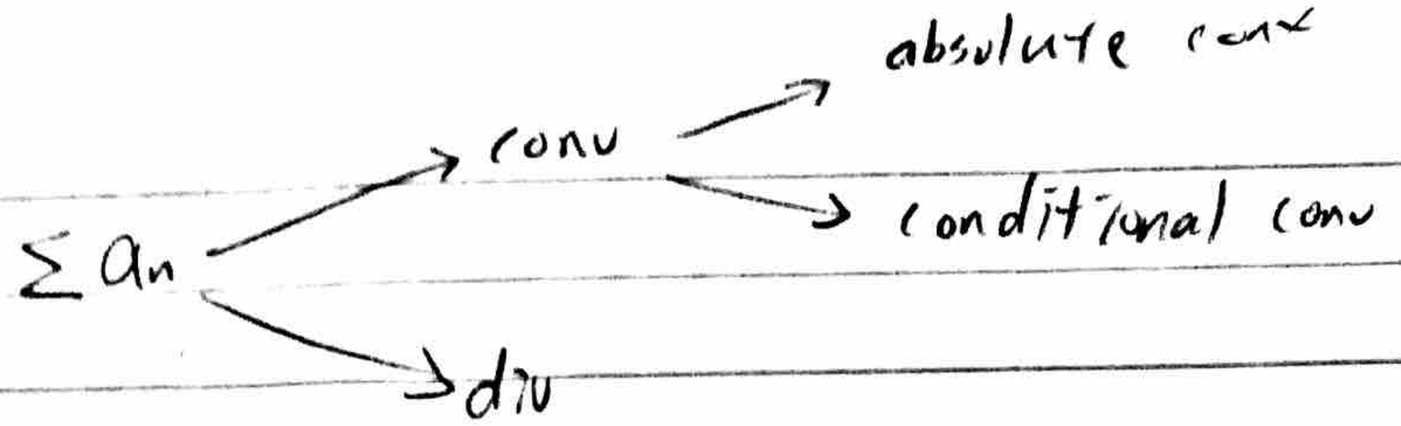
(ii) $\{b_n\}$ decreasing? yes ✓

$f(x) = \frac{\ln x}{x}$ $f' < 0$

$f'(x) \overset{q.r}{=} \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \frac{1-\ln x}{x^2} < 0$

$x^2 \to +$

$x \geq 3 \to \ln x > 1 \Rightarrow 1 - \ln x < 0$



Def: The series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ is convergent.

$\sum |a_n| = |a_1| + |a_2| + |a_3| \cdots$

Ex. $\sum \frac{(-1)^n}{n^2}$ is absolutely convergent

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum \frac{1}{n^2}$: conv (by p-series test $p = 2 > 1$

So $\sum \frac{(-1)^n}{n^2}$: abs. conv.

Thm: If $\sum a_n$ abs: conv, then $\sum a_n$; conv.
($\sum |a_n|$: conv, then $\sum a_n$ conv.)

$a_n \le |a_n|$

$0 \le a_n + |a_n| \le 2|a_n|$

$\sum 0 \le \sum (a_n + |a_n|) \le \sum 2|a_n|$

$\sum (a_n + |a_n|)$ conv by $\sum a_n$ abs conv.
(conv)

$\sum a_n = \sum (a_n + |a_n| - |a_n|)$

$= \sum (a_n + |a_n|) - \sum |a_n|$
conv conv

∴, conv

The series $\sum a_n$ is called "conditionally convergent" if $\sum a_n$: conv and $\sum |a_n|$: div.

Ex. Is $\sum \frac{(-1)^{n-1}}{\sqrt{n}}$ absolutely conv? conditionally conv? ✓ divergent?

By A.S.T series converges

$$\sum \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum \frac{1}{\sqrt{n}} : \text{div} \left(\begin{array}{c} p\text{-series for} \\ p = \frac{1}{2} < 1 \end{array} \right)$$

$\sum a_n$ is conditionally conv.

Motivation for the definitions the term "conditionally" comes from the fact that if we take another arrangement of a conditionally conv series, we get a different answer for the series.

conditionally conv. (*) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} \cdots = \ln 2$

$\times \frac{1}{2}$ $\quad \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} \cdots = \frac{1}{2}\ln 2$

rearrangement of (*) $\quad 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} \cdots = \frac{3}{2}\ln 2$