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4.3 Linear Independent Set; bases

Def: V vector space, $\{v_1, \dots, v_p\} \subseteq V$ is linearly independent if the equation $c_1 v_1 + \dots + c_p v_p = 0$ has only the trivial solution $c_1 = \dots = c_p = 0$.
and linearly dependent if there exist $c_1, c_2, \dots, c_p \in \mathbb{R}$, not all zeros, s.t. $c_1 v_1 + \dots + c_p v_p = 0$.

Ex 1) $V = \mathbb{P}_1 = \{a + bt \mid a, b \in \mathbb{R}\}$

$p_1(t) = 1+t$, $p_2(t) = 1-t$, $p_3(t) = 3+t$

$2p_1 + p_2 - p_3 = 0 \Rightarrow \{p_1, p_2, p_3\}$ linear dep.

2) $V = C[0,1] = \{f: [0,1] \to \mathbb{R} \text{ continuous}\} \Rightarrow \{\sin t, \cos t, t+1\}$ is lin indep.

Soln: Let us take $c_1, c_2, c_3 \in \mathbb{R}$ s.t. $c_1 \cos t + c_2 \sin t + c_3 (t+1) = 0$, for all $t \in [0, 10]$.

The goal is to prove $c_1 = c_2 = c_3 = 0$.

$$\left.\begin{array}{l} t = 0 \Rightarrow c_1 + c_3 = 0 \\ t = \frac{\pi}{2} \Rightarrow c_2 + c_3(\frac{\pi}{2} + 1) = 0 \\ t = \pi \Rightarrow -c_1 + c_3(\pi + 1) = 0 \end{array}\right\} c_1 = c_2 = c_3 = 0.$$

Thus $c_1 = c_2 = c_3 = 0 \Rightarrow \{\sin t, \cos t, t+1\}$ is lin indep.

Def: Let H be a subspace of vector space V. A subset $\mathcal{B} = \{b_1, \dots, b_p\}$ in V is a basis for H if (1) $\mathcal{B}$ is lin. indep.
(2) span $\mathcal{B} = H$. i.e. $\{a_1 b_1 + \dots + a_p b_p \mid a_1, \dots, a_p \in \mathbb{R}\} = H$

Remark: $\mathcal{B} \subset H$.

Ex 1) $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \dots e_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow \{e_1, \dots, e_n\}$ is a basis for $\mathbb{R}^n$.

$\left(x \in \mathbb{R}^n, x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + \dots x_n e_n \in \text{Span}\{e_1 \dots e_n\} \right.$.

2) A $n \times n$, $A = [a_1\ a_2\ \dots\ a_n] \Rightarrow \{a_1, \dots a_n\}$ is a basis for $\mathbb{R}^n \Leftrightarrow A$ is invertible.

Sol The mtx inv. thm.

A is invertible $\Leftrightarrow \{a_1, \dots a_n\}$ is lin. indep $\Leftrightarrow \text{Span}\{a_1, \dots a_n\} = \mathbb{R}$.

$\Rightarrow A$ invertible $\Leftrightarrow \{a_1, \dots, a_n\}$ is a basis.

ii) $A = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow$ A is invertible iff $\{b_1, \dots b_n\}$ is a basis for $\mathbb{R}^n$.

3) $\mathbb{P}_n = \{a_0 + a_1 t + \dots a_n t^n \mid a_0, \dots a_n \mid$

$\mathcal{B} = \{1, t, t^2, \dots, t^n\}$ is a basis for $\mathbb{P}^n$

4. $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$, $H = \text{span}\{v_1, v_2, v_3\}$

Find a basis for H

Sol $\vec{v}_3 = \vec{v}_1 + \vec{v}_2 \Rightarrow v_3 \in \text{Span}\{v_1, v_2\} \Rightarrow H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$

But $\{v_1, v_2\}$ is lin. indep. $\longrightarrow$ $\{v_1, v_2\}$ is a basis for H.

Thm (The spanning set thm).

Let $S = \{v_1, v_2, \dots v_p\}$ subset in V, $H = \text{span}\{v_1, \dots v_p\}$

a) if $v_k \in \text{Span}\{v_1, v_2, \dots v_{k-1}, v_{k+1}, \dots, v_p\}$, then $H = \text{Span}\{v_1, v_2, \dots v_{k-1}, v_{k+1}, \dots, v_p\}$.

b) if $H \neq \{0\}$ then some subset of S is a basis for H.

Proof a) suppose $k = p$.

$v_p \in \text{Span}\{v_1, \dots v_{p-1}\} \Rightarrow \exists\ c_1 \dots c_{p-1} \in \mathbb{R}$ s.t. $v_p = c_1 v_1 \dots c_{p-1} v_{p-1}$.

Take $x \in H. \Rightarrow \exists\ a_1, \dots a_p \in$ s.t.

$x = a_1 v_1 + \dots + a_p v_p + a_p v_p$.

$= (a_1 + c_1 p c_1) v_1 + \dots + a_p + a_p c_p) v_{p-1} \Rightarrow x \in S\ \text{Span}\{v_1 \dots v_{p-1}\}$.

b) If S is not linindependent, then $\exists\ v_k \in \text{Span}\{S \setminus \{v_k\}\}$
$\Rightarrow H = \text{span}(S \setminus \{v_k\})$
If $S \setminus \{v_k\}$ is lin indp $\rightarrow S \setminus \{v_k\}$ is a basis.
If not, repeat.

Bases for NulA and ColA (A $m \times n$).

Case : The matrix is in echelon form

$$B = [b_1 \ b_2 \ b_3 \ b_4 \ b_5] = \begin{bmatrix} 1 & 2 & 0 & 7 & 8 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b_1, b_3, b_5 is lin ind.
$b_2, b_4 \in \text{Span}\{b_1, b_3, b_5\} \Rightarrow \text{Span}\{b_1, b_3, b_5\} = \text{Span}\{b_1, b_2, b_3, b_4, b_5\}$
$\Rightarrow \{b_1, b_3, b_5\}$ is a basis for $\text{Col}(B)$ $= \text{Col}(B)$.

$\text{Col}(B) = \text{Span}\{b_1, b_2, b_3, b_4, b_5\}$

Remark : 1) The pivot colm is lin indp.
2) Each non-pivot column can be generated by the pivot colm.

$\text{Nul}(B) = \{x \in \mathbb{R}^5 \mid Bx = 0\}$.

$$Bx = 0 \iff \begin{cases} x_1 + 2x_2 + 7x_4 + 8x_5 = 0 & x_1 = -2x_2 - 7x_4 \\ x_3 + 3x_4 = 0 & 3x_3 = -3x_4 \\ +x_5 = 0 & \\ x_2 \ x_4 \text{ free} & \end{cases}$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 7x_4 \\ x_2 \\ -3x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

$\text{Nul} B = \text{Span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{4}{3} \\ 0 \end{bmatrix} \right\}$

But $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix} \right\}$ is lin indep $\Rightarrow \{u, v\}$ is a base for Nul B.

Remark: 1. The vectors found with the weight, the free variables, span Nul(B).
2. is lin indep.

$u = \begin{bmatrix} u_1 \\ \vdots \\ u_5 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ \vdots \\ v_5 \end{bmatrix}$ $u_2 = 1, v_2 = 0$
$u_4 = 0, v_4 = 1.$

Case 2: general case

Remark: $A \sim B$, A, B $m \times n$. $\Rightarrow Ax = 0$ iff $Bx = 0$

If $A = [a_1 \ldots a_n]$, $B = [b_1 \ldots b_n]$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.
$\Rightarrow x_1 a_1 + \ldots + x_n a_n = 0 \Longleftrightarrow x_1 b_1 + \ldots x_n b_n = 0.$

Thus, $\{a_{i_1}, a_{i_2}, \ldots a_{i_8}\}$ is lin dep. iff

① $\{b_{i_1}, \ldots b_{i_8}\}$ is lin dep, where $i_1 < i_2 < \ldots < i_8$. j-$\{a_1, a_2, \ldots a_8\}$ is lin dep iff
$\{b_1 \ldots b_2 \ldots b_8\}$ is lin dep.

② $a_s \in \text{Span}\{a_1, a_2\, a_8\}$ iff $b_s \in \text{Span}\{b_1, b_2\, b_8\}$.

Take A general matrix, $A \sim B$, B echelon form.
Remarks $\Rightarrow$ Nul A = Nul B $\Rightarrow$ the basis formed for Nul B is good for Nul A.

For Col A: Thm 6: The pivot col of A form a basis for Col A.
For simplicity, suppose $\{b_1, b_2, b_8\}$ are the pivot columns of B.

By case, $\{b_1, b_2, b_8\}$ is a basis for Col B.
Remark ① + ② get $\{a_1, a_2, a_8\}$ is a basis for Col(A).

Ex. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$Col(A) = span(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$ $\qquad$ $Col(A) \neq Col(B)$

$Col(B) = span(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$

use the pivot cols of A not B.

4.5 The Dimension of Vector Space.

Thm 9 $\left.\begin{array}{l} B = \{b_1, \dots, b_n\} \text{ basis for a vector space } V \\ S = \{u_1, \dots, u_p\} \in V,\ p > n \end{array}\right\} \Rightarrow S$ is lin dep.

(The general case Thm 8, Ch 1) $\mathbb{R}^n$)

Proof: $u_1 \in V = Span\ B \Rightarrow \exists\ a_{11}, a_{21}, a_{n1}$ s.t. $u_1 = a_{11}b_1 + a_{21}b_2 + \dots + a_{n1}b_n$.

$u_2 \in V \dots u_2 = a_{12}b_1, b_1 + a_{22}b_2 + \dots a_{n2}b_n$.

$u_p \in V \dots\ u_p = a_{1p}b_1 + a_{2p}b_2 + \dots a_{np}b_n$.

$A = \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{np} \end{bmatrix}$ $n \times p \Rightarrow Ax = 0$ has a nontrivial solution

$c = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} \Rightarrow c_1u_1 + c_pu_p = 0 \Rightarrow \{u_1, \dots, u_p\}$ is linearly dep.

Thm 10: Let B_1 and B_2 be two bases for a vector space $V \Rightarrow B$ and B_2 have the same number of elements.

pf: $\# B_1 = n$

$\# B_2 = m$

Suppose $n \neq m$. Suppose $n < m$.

B_1 is a basis for V } $\overset{\text{Thm 9}}{\Rightarrow}$ B_2 is lin dep $\Rightarrow$ $\times$.
$B_2 = m > n$

<u>Def.</u> 1) Let V be a vector space. V is <u>finite dimensional</u> if it's spanned by a finite set, otherwise V is <u>infinite dimensional</u>
2) If $B = \{b_1, \dots, b_n\}$ is a basis for a finite dimensional vector space V, then $\dim V = n$, the dim of V.

Remark: Because of Thm 10, $\dim V$ is well defined. $\dim\{0\} = 0$.

Ex 1) $\dim \mathbb{R}^n = n$.
2) $\dim \mathbb{P}_n = n+1$ ($\{1, t, t^2, \dots t^n\}$ is a basis for $\mathbb{P}_n$)
3) $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\Rightarrow$ $\dim(\text{span}\{v_1, v_2\}) = 2$, $\subseteq \mathbb{R}^3$
"4) $\mathbb{P}_\infty$ = { polynomials of degree maxim 1 in the variables $x_1, x_2, x_3, \dots$ }
$x_1 + 1 \in P_\infty$
$x_2 - x_1 + x_{12} \in P_\infty$.

4) $P = \{a_0 + a_1 t + \dots + a_n t^n \mid n \geq 0, a_0, \dots, a_n \in \mathbb{R}\}$.
Exc. Prove P is infinite dimensional.

<u>Thm 11.</u> Let H be a subspace of a fin dim vector space V.

$\Rightarrow$ i) any lin indep. set in H can be extended to be a basis in H
ii) $\dim H \leq \dim V$

<u>Proof</u>: If $H = \{0\}$ ✓.
Suppose $H \neq \{0\}$.
i) Let us consider a lin. ind. set S in H.
If span $S = H \Rightarrow S$ is a basis for H.
If not take $u_1 \in H$.
$u_1 \notin$ span S.
$S_1 = S \cup \{u_1\}$ Exc. S_1 is lin ind.
If span $S_1 = H \Rightarrow$ ✓.

ii) Take B to be a basis for H $\Rightarrow$ B is lin independent in V.
By i), extend B to B_1 to be a basis in V:

$B \subset B_1 \Rightarrow \dim H \leq \dim V$.

Exr) (use thm 9 to give a 2nd proof for ii))

Thm 12 V vector space, $\dim V = p < \infty$ $S \subset V$ is a subspace.

1) if S is linearly indep, then S is a basis.
2) if span S $V = V$, then S is a basis.

Proof: ① S is lin indep in V $\xRightarrow{\text{Thm 11}}$ $\exists$ $S_1 \subseteq V$. s.t. $S_1 \supset S$ & S_1 is a basis. ($H = V$)

$\Rightarrow \# S_1 = p$

$$\left.\begin{array}{l} \# S = p \\ S \subset S_1 \end{array}\right\} \Rightarrow S = S_1 \text{ which is a basis}$$

② Span $S = V$ The span set thm $\Rightarrow$ $\exists$ $S_0 \subset S$ s.t. S_0 is a basis for V.

$$\left.\begin{array}{l} \dim V = p \\ S \text{ is a basis} \end{array}\right\} \Rightarrow \# S_0 = p \text{ for } V.$$

$$\left.\begin{array}{l} \# S_0 = p \\ \# S = p \\ S_0 \subset S \end{array}\right\} \Rightarrow S_0 = S \text{ is a basis for } V.$$

The dim of Nul A and Col A.

Proposition A $m \times n$ $\Rightarrow$ $\dim \operatorname{Nul} A = \#$ free variables
$\dim \operatorname{Col} A = \#$ basic variables (pivot cols)

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 0 \\ 2 & 2 & 0 & 3 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$\dim \operatorname{Nul} A = 1$

$\dim \operatorname{Col} A = 3$,