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Math 20B Lecture 15 8/1

10.6 Power Series

A power series with center c is an infinite series

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 \cdots$$

Determine the values of x for which the following converges.

Ex. (i) $\sum_{n=0}^{\infty} x^n = 1+x+x^2 \cdots$ geometric series $r=x$ $|x|<1 \Rightarrow -1<x<1$

(ii) $\sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow$ By the ratio test, find limit of the ratio.

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} : \frac{n!}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$ absolutely conv.

$= \sum \frac{x^n}{n!}$: converges for any x $(-\infty,\infty)$.

(iii) $\sum_{n=0}^{\infty} n!\, x^n$ By the ratio test $\rho = \lim_{n\to\infty} \left|\frac{(n+1)!}{n!} \frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty} (n+1)|x|$

The only x that makes $\sum n!\, x^n$ conv is $x=0$.

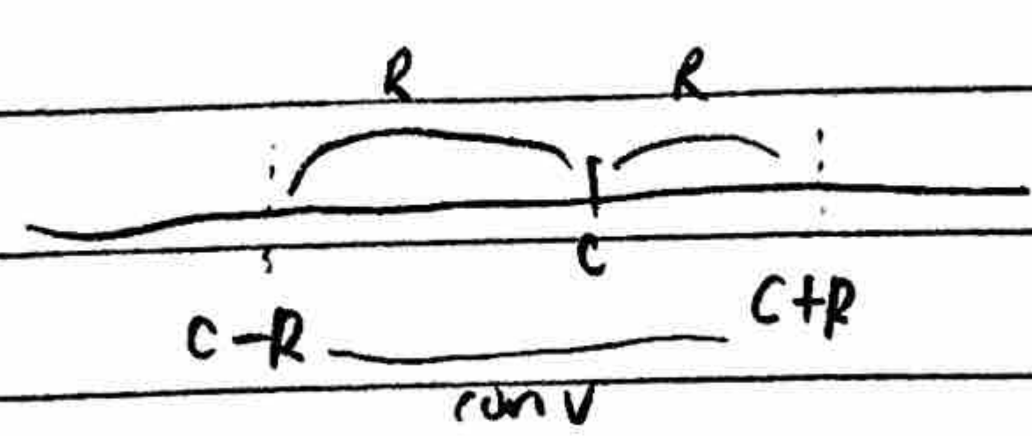
$$= \begin{cases} \infty & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Thm: For any power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, there are three possibilities

(i) The series conv. only when $x=c$

(ii) The series converges for all x

(iii) There exists positive # R such that the series converges if $|x-c|<R$
diverges if $|x-c|>R$



such R is called "the radius of convergence"

The interval of conv is the interval which consist of all values of x for which series converges.

ex.

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x-3)^n}{n \cdot 4^n}$$ Find the radius of conv. & interval of conv.

To use the ratio test

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|(-1)^{n+1} \frac{(x-3)^{n+1}}{(n+1) \cdot 4^{n+1}} \cdot \frac{n \cdot 4^n}{(-1)^n (x-3)^n}\right|.$$

$$= \lim_{n\to\infty} \left|\frac{n}{n+1} \cdot \frac{1}{4}(x-3)\right| = \frac{1}{4}|x-3|$$

If $\rho = \frac{1}{4}|x-3| < 1$, then

(*) $|x-3| < 4 \rightarrow -4 < x-3 < 4 \rightarrow -1 < x < 7$: conv.

If $\rho = \frac{1}{4}|x-3| > 1$, then $|x-3| > 4 \rightarrow$ diverges

If $x = -1$, $\sum_{n=1}^{\infty} (-1)^n \frac{(-1-3)^n}{n \cdot 4^n} = \sum (-1)^n \cdot \frac{(-4)^n}{n \cdot 4^n} = \sum \frac{((-1)(-4))^n}{n \cdot 4^n}$

$= \sum_{1}^{\infty} \frac{1}{n}$: div (p-series

If $x = 7$, $* = \sum_{n=1}^{\infty} \frac{(-1)^n (7-3)^n}{n \cdot 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ ($b_n = \frac{1}{n}$, AST

(1) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

~~Interval of conv of (*)~~

$-1 < x \leq 7$

(2) ~~$\{b_n\}$ decre~~ ✓ $\frac{1}{n} > \frac{1}{n+1}$

AS.T.

Find power series Expansion and the radius convg. R.

(1) $f(x) = \frac{3}{1+x} = 3 \cdot \frac{1}{1+x} = 3 \frac{1}{1-(-x)}$ $|x| < 1$

$3 \sum_{n=0}^{\infty} (-x)^n$

$= 3 \sum_{n=0}^{\infty} (-1)^n x^n$ if $|x| < 1$

$= \sum 3(-1)^n x^n$, $R = 1$

(71) $f(x) = \frac{5}{2+8x^2} = \frac{5}{2} \cdot \frac{1}{1+4x^2} = \frac{5}{2} \frac{1}{1-(-4x^2)}$

$= \frac{5}{2} \sum_{n=0}^{\infty} (-4x^2)^n$

$= \frac{5}{2} \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}$

$= \sum_{n=0}^{\infty} \frac{5}{2} (-1)^n \cdot 4^n \cdot x^{2n}$ if $|4x^2| < 1$

$|x|^2 < \frac{1}{4}$

$-\frac{1}{2} < |x| < \frac{1}{2}$

if $|x| < \frac{1}{2}$ $R = \frac{1}{2}$

Thm: If the power series $\sum a_n (x-c)^n$ has a radius of conv. R.
then $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ is differentiable in $(c-R, c-R)$