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Math 18 Lecture 4 Aug 17

Thm 12: $T: \mathbb{R}^n \to \mathbb{R}^m$ linear transf, $T(\vec{x}) = A\vec{x}$.
$\Rightarrow$ a) T is onto iff the colⁿ of A span $\mathbb{R}^m$.
b) T is one-to-one iff colⁿ of A lin independent.

Proof: a) T is onto $\overset{\text{def}}{\Longleftrightarrow}$ for all $b \in \mathbb{R}^m$, there exist at least one $x \in \mathbb{R}^n$ s.t. $T(x) = b$.
$\Leftrightarrow$ for all $b \in \mathbb{R}$, $A\vec{x} = \vec{b}$ is consistent.
$\underset{\text{Thm 4 Sec 1.4}}{\Leftrightarrow}$ the colⁿ of A span $\mathbb{R}^m$.

b) T is one to one $\overset{\text{def}}{\Longleftrightarrow}$ for all $b \in \mathbb{R}^m$ $\exists$ at most one $x \in \mathbb{R}^n$ s.t. $T(x) = b$
$\Leftrightarrow$ $T(x) = 0$ has unique sol
$\Leftrightarrow$ $Ax = 0$ has unique sol
$\Leftrightarrow$ col of A are lin. indep.

Ex. 1) $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2, x_1 + 3x_2)$
T onto? T 1-1?

Solⁿ: $A = [\,T([\begin{smallmatrix}1\\0\end{smallmatrix}])\ T([\begin{smallmatrix}0\\1\end{smallmatrix}])\,] = \underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 3 \end{bmatrix}}$ $\Rightarrow T(x) = Ax$ $3 > 2$.

A has max of 2 pivots $\underset{\text{Thm 4, sec 1.4}}{\Longrightarrow}$ colⁿ of A don't span $\mathbb{R}^3$ $\underset{\text{Thm 12}}{\Longrightarrow}$ T is NOT onto.

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \neq \alpha \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ $\Rightarrow$ colⁿ of A are lin. indep. $\overset{\text{Thm 12}}{\Longrightarrow}$ T is 1-to-1.

colⁿ of A not multiple of each other.

2) $T: \mathbb{R}^7 \to \mathbb{R}^5$, $T(x) = Ax$, A 5×7 has 5 pivots.
$\Rightarrow$ T is onto & is not 1-1.

Matrix Operations:

A $m\times n$, $A = [a_1 \dots a_n]$, $a_1, \dots, a_n \in \mathbb{R}^m$

$A = [a_{ij}]$, where a_{ij} is the ith entry on the jth col'n a_j.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$ a_{ij} is the (i, j)-entry of A.

$\underbrace{\text{second column}}_{= a_2}$

If $m = n$ $a_{11}, a_{22}, \dots a_{nn}$ are the <u>diagonal entries</u> of A.

A is called a <u>diagonal matrix</u> if all its non-diagonal entries are zero.

ex. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}$ $0_n = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$.

Def: i) $A = [a_1 \dots a_n]$ $m\times n$
$B = [b_1 \dots b_t]$ $s\times t$.

$A \overset{def}{=} B$ if $m = s, n = t$
$a_1 = b_1, \dots, a_n = b_n$.

ii) A, B $m\times n$, $A = [a_1, \dots, a_n]$
$B = [b_1, \dots, b_n]$

$A + B \overset{def}{=} [a_1 + b_1 \ \ a_n + b_n]$
$m\times n$.

iii) A $m \times n$ $r \in \mathbb{R}$, $A = [a_1 \ldots a_n]$

$rA \overset{def}{=} [r a_1, \ldots r a_n]$

Ex. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 2 \\ 4 & 6 & 7 \end{bmatrix}$

Ex. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ $\neq$ is not well defined.

Ex. $(-1) \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & -3 \end{bmatrix}$

Thm 1 A, B, C $m \times n$ $r, s \in \mathbb{R}$.

a) $A+B = B+A$

b) $(A+B)+C = A+(B+C)$

c) $A+0 = A$

d) $r(A+B) = rA + rB$.

e) $(r+s)A = rA + sA$.

f) $r(sA) = (rs)A$.

Matrix Multiplication

A, B two matrices, $\vec{x}$ vector $A\,(\underbrace{\overrightarrow{Bx}}_{\text{vector}}) = (A \underset{\text{define}}{\cdot} B)\,\vec{x}$

$\underbrace{\qquad}_{\text{vector}}$ $\underbrace{\qquad}_{\text{matrix}}$ vector

A $m \times n$, B $n \times p$, $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$

$ [b_1 \ldots b_p]$

$A(\overrightarrow{Bx}) = A(x_1 b_1 + \ldots + x_n b_n) = A(x_1 b_1) + \ldots + A(x_n b_n)$

$= x_1 Ab_1 + \dots + x_n Ab_n.$

$= [Ab_1 \; \dots \; Ab_n] \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$

Def A $\underbrace{m\times n, \; B \; n\times p}_{\text{result}}$, $B = [b_1, \dots, b_p]$.

$$A \cdot B \overset{\text{def}}{=} [Ab_1 \; \dots \; Ab_p], \quad m\times p$$

Ex.
$$\underset{A}{\begin{bmatrix} 1 & 2 & 0 \\ -2 & 3 & 7 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 0 & 1 \end{bmatrix}} = \left[A\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \; A\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 7 & 6 \\ 7 & 9 \end{bmatrix}$$

Ex.
$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} \; \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 4 & -1 \\ 6 & -2 \end{bmatrix}$$

Row-column Rule for computing AB

A $m\times n$ $\quad$ B $n\times p$

The (i,j) entry of AB is the sum of the product of corresponding entries from row i of A and colm j of B:

$$(AB)_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}$$

$$\begin{bmatrix} \\ a_{i1} \; \cdots\cdots \; a_{in} \end{bmatrix} \begin{bmatrix} & b_{1j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix} = \begin{bmatrix} & \bigcirc & \end{bmatrix}$$

(i,j)-entry

Proof: $A = [a_1 \ldots a_n]$ $\qquad$ $A = [a_1 \ldots a_n] = [a_{ij}]$
$B = [b_1, \ldots, b_p]$

The j^{th} colm of AB is Ab_j

$$Ab_j = [a_1 \ldots a_n]\begin{bmatrix} b_{1j} \\ b_{nj} \end{bmatrix} = b_{1j}a_1 + \ldots + b_{nj}a_n$$

The ith entry of Ab_j is $b_{1j}a_{i1} + \ldots + b_{nj}a_{in}$

Ex. $\underset{3\times 2}{\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}} = \begin{bmatrix} -3 & 0 \\ 6 & 6 \\ 11 & 10 \end{bmatrix}$

Thm 2: A $m\times n$, B, C, with sizes s.t. the following operations are well defined.

a) $A(BC) = (AB)C$

b) $A(B+C) = AB + AC$

c) $(B+C)A = BA + CA$

d) $r(AB) = (rA)B = A(rB)$, for all $r \in \mathbb{R}$.

e) $I_m \cdot \underset{m\times n}{A} = A = AI_n$

Proof:)

a) 1st proof $C = [c_1 \ldots, c_p]$

$$A(BC) = A[Bc_1 \ldots Bc_p] = [A(Bc_1) \ldots A(Bc_p)]$$

$$= [(AB)c_1 \ldots (AB)c_p] = (AB)[c_1 \ldots c_p] = (AB)C.$$

2nd proof:

$A \to T_A$ $\qquad$ $T_A \circ (T_B \circ T_C) = (T_A \circ T_B) \circ T_C$ — assoc. of functions compositions.

$B \to T_B$

$C \to T_C$

Remarks:

1) $AB \neq BA$ ex. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \neq BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

2) $AB = 0 \not\Rightarrow A = 0$ or $B = 0$

ex. $$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

3). $AC = BC \not\Rightarrow A = B$

Def: A $n \times n$, k positive integer; $A^k \overset{def}{=} \underbrace{A \cdot A \cdots\cdots A}_{k \text{ times}}$

$$A^0 \overset{def}{=} I_n$$

Def: A $m \times n$, $\underline{A^T}$ is the $n \times m$ matrix whose cols. are formed from the corresponding rows of A (called the transpose of A).

ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

$\underset{m \times n}{A = [a_{ij}]}$, $A^T = \underset{n \times m}{[a_{ji}]}$.

$1 \leq i \leq m$

$1 \leq j \leq n$

__Thm 3__ A, B matrices,

a) $(A^T)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(rA)^T = rA^T$, $r \in R$.

d) $(AB)^T = B^T A^T$

Remarks

1) $(ABC)^T = C^T B^T A^T$

2) $(A_1 \cdots A_n)^T = A_n^T \cdots A_1^T$

Proof 1) $(AB)C)^T \overset{\text{Thm 3d)}}{=} C^T (AB)^T$

$\overset{\text{Thm 3 d)}}{=} C^T B^T A^T$

2) Induction

Exc $n=4$.

The inverse of a matrix

Def: A $n \times n$ matrix is invertible (non singular) if there exist C $n \times n$ s.t. $AC = CA = I_n$.

C is called an inverse of A.

ex. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \Rightarrow AC = CA = I_2$.

Remark: If A is invertible, then it has a unique inverse (called the inverse of A).

Proof: Suppose C, D are inverses for A. $AC = CA = I_2$

$AD = DA = I_2$

$C(AD) = CI_2$

$(CA)D = C \Rightarrow I_2$

$D = C \Rightarrow D = C$.

Thm 4.

a) If $ad-bc \neq 0$, then A is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

b) If $ad-bc = 0$, then A is not invertible.

$ad-bc$ is called the determinant of A, denoted $\det(A)$.

Ex. $A = \begin{bmatrix} 1 & 3 \\ -2 & 3 \end{bmatrix}$

$\det A = 1 \cdot 3 - (-2) \cdot 3 = 9 \neq 0$ A is invertible.

$A^{-1} = \frac{1}{9}\begin{bmatrix} 3 & -3 \\ 2 & 1 \end{bmatrix}$

Thm 5 A $n \times n$ invertible $\Rightarrow$ for all $b \in \mathbb{R}^n$, $Ax = b$ has a unique sol $x = A^{-1}b$.

Proof : • $A(A^{-1}b) = (AA^{-1})b = I_n b = b$.

$\Rightarrow A^{-1}b$ is a soln for $Ax = b$.

• Let $x_0 \in \mathbb{R}^n$ be a soln

$Ax_0 = b \Rightarrow A^{-1}(Ax_0) = A^{-1}b \Rightarrow I_n x_0 = A^{-1}b$

$\Rightarrow x_0 = A^{-1}b$.

Ex. $\begin{cases} x_1 + 3x_2 = 2 \\ -x_1 + 3x_2 = 1 \end{cases}$

$A = \begin{bmatrix} 1 & 3 \\ -2 & 3 \end{bmatrix} \Rightarrow A$ is invertible.

$A^{-1} = \frac{1}{9}\begin{bmatrix} 3 & -3 \\ 2 & 1 \end{bmatrix}$

$x = A^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the unique soln

$= \frac{1}{9}\begin{bmatrix} 3 & -3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3/9 \\ 5/9 \end{bmatrix}$

Thm 6:

a) A invertible $\Rightarrow A^{-1}$ invertible & $(A^{-1})^{-1} = A$.

b) A, B invert $\Rightarrow AB$ invertible & $(AB)^{-1} = B^{-1}A^{-1}$

c) A invertible $\Rightarrow A^T$ is invertible & $(A^T)^{-1} = (A^{-1})^T$

Proof:

a) $A \cdot A^{-1} = A^{-1} \cdot A = I_n$

$\overset{def}{\Rightarrow} A^{-1}$ is invertible & inverse of A^{-1} is A.

$\Longleftrightarrow (A^{-1})^{-1} = A$.

b). Find C $n \times n$ s.t. $(AB)C = I$

$C(AB) = I$.

c). $A \cdot A^{-1} = A^{-1} \cdot A = I_n$.

$(A^{-1})^T A^T = A^T (A^{-1})^T = I_n$

$\overset{def}{\Rightarrow} A^T$ is invertible & $(A^T)^{-1} = (A^{-1})^T$

$(C = (A^{-1})^T)$

Remarks:

1) A, B, C are invertible $\Rightarrow ABC$ is invertible.

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

2) $A_1 \ldots A_n$ invert $\Rightarrow A_1 \ldots A_n$ invertible.

$(A_1 \ldots A_n) = A_n^{-1} \ldots A_1^{-1}$

Elementary matrices

An elementary row operation generates an elementary matrices

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad E_1 \cdot A = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix} \quad (R_3 \to R_3 - 4R_1)$$

$$E_2 A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix} \quad (R_1 \leftrightarrow R_2$$

$$E_3 A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix} \quad (R_3 \to 5R_3)$$

Remark: Every elementary matrix is invertible.

Proof: E elementary matrix. (corresponds to an elem row op. on I_n) $\Rightarrow \exists$ F elem. matrix s.t.

$$\left.\begin{array}{l} FE = I_n \\ EF = I_n \end{array}\right\} \Rightarrow E \text{ is invertible.}$$

Thm 7 A $n \times n \Rightarrow A$ invertible $\Leftrightarrow A \sim I$.