Math 18 Lecture 8 8/31/2019

4.6 Rank

Def : A $m\times n$, $A=\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$, Row $A = \text{Span}\{v_1, \dots, v_m\} \subseteq \mathbb{R}^n$ is called the row space of A.

Remark Row A = Col A^T

ex. $A=\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \Rightarrow$ Row A = Span $\{(1,2),(3,6)\}$. = Span $\{(1,2)\}$.

Thm 13 $A \sim B \Rightarrow$ i) Row A = Row B

ii) if B is in echelon form, then the non-zero rows of B form a basis for Row B.

Proof i) Because $A \sim B$, then the rows of B are linear combinations of rows of A (and vice versa)

$\Rightarrow \left.\begin{array}{l} \text{Row } B \subset \text{Row } A \\ \& \text{ Row } A \subset \text{Row } B \end{array}\right\} \Rightarrow$ Row A = Row B.

ii) $B = \begin{bmatrix} b_1 \\ \vdots \\ b_p \\ \vdots \end{bmatrix}$ $m\times n$, $p \le m$ $b_1 \dots b_p$ are nonzeros

Row B = Span $\{b_1, \dots, b_p\}$. is lin. ind.

B is in echelon $\Rightarrow b_1 \dots b_p$

if not $\Rightarrow c_1 b_1 + \dots c_p b_p = 0$ with not all $c_1 \dots c_p$ zeros.

$\Rightarrow c_1 = 0 \dots c_p = 0$. false.

$A=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = B \Rightarrow$ Col A = Col B

Def
The rank of A is $\dim(\text{Col } A)$, denoted rank A.

Thm 14 A $m\times n$ $\Rightarrow \dim \text{Col} A = \dim \text{Row } A$

& rank A + $\dim(\text{Nul } A) = n$.

Ex A 11×13

$\dim \text{nul}(A) = 4 \Rightarrow$ rank $A = 9$.

$n = 13$

Remark A $n\times n$

rank $A=0 \iff A=0$
rank $A=n \iff A$ is invertible

rank $A=n \Rightarrow \dim \operatorname{Col} A = n \Rightarrow \dim \operatorname{Span}\{a_1, \dots, a_n\} = n$.

$S = \{a_1, \dots, a_n\}$ spans $\operatorname{Col} A \Rightarrow |S| = n$.

S is a basis for $\mathbb{R}^n \Rightarrow \operatorname{span} S = \mathbb{R}^n \Rightarrow A$ is invertible.

Proof of Thm 14

dim Col A = # of pivot columns
dim Row A = # of nonzero rows of an echelon form of A $\Rightarrow$ dim Col A = dim Row A.

rank A = # of basic var
dim Nul A = # free var
we have n variables

$\Big\} \Rightarrow$ Thm 14 ✓. $\square$.

4.4 Coordinate Systems

$$\begin{bmatrix}4\\2\end{bmatrix} = 4\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}.$$

Thm 7 (The unique representation)
$\mathcal{B} = \{b_1, \dots, b_n\}$ basis for a vector space $V \Rightarrow \forall x \in V, \exists$ unique $c_1, \dots c_n \in \mathbb{R}$ s.t. $x = c_1 b_1 + \dots c_n b_n$.

Let $d_1, \dots, d_n \in \mathbb{R}$ s.t. $x = d_1 b_1 + \dots d_n b_n$
$x = c_1 b_1 + \dots + c_n b_n$.

The goal is to prove that $c_i = d_i$,
$0 = x - x = d_1 b_1 + \dots + d_n b_n - c_1 b_1 - \dots - c_n b_n$
$= (d_1 - c_1) b_1 + \dots + (d_n - c_n) b_n$.

$\mathcal{B}$ is lin indp.

Def: The coordinates of x relative to β, are the weights $c_1, \dots, c_n$.

$$[x]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Ex. $\mathcal{B} = \{b_1, b_2\}$ basis in $\mathbb{R}^2$

$b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

i) Find $x \in \mathbb{R}^2$ s.t. $[x]_\beta = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

ii) Find $[u]_\beta$.

1) $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $x = 1\, b_1 + (-1) b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

2) Find c_1, c_2 s.t. $u = c_1 b_1 + c_2 b_2$ solve system.

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow c_1 = c_2 = 1$.

$\Rightarrow [u]_\beta = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\mathcal{B} = \{b_1, \dots, b_n\}$ basis

$x = c_1 b_1 + \dots + c_n b_n$

$[x]_\mathcal{B} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Remark: 1) $\mathcal{B} = \{e_1, \dots, e_n\}$ the standard basis in $\mathbb{R}^n$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$,

$\Rightarrow [x]_\mathcal{B} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x$.

2) $\mathcal{B} = \{b_1, \dots, b_n\}$ basis in $\mathbb{R}^n$

$x = c_1 b_1 + \dots + c_n b_n = \underbrace{[b_1 \cdots b_n]}_{P_\mathcal{B}} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$\Rightarrow x = P_\mathcal{B} [x]_\mathcal{B}$. $\mathcal{B}$ is a basis $\Rightarrow P_\mathcal{B}$ is invertible.

Thm 8 : $\mathcal{B} = \{b_1, \dots, b_n\}$ basis for V

$\Rightarrow T: V \to \mathbb{R}^n$ def by $T(x) = [x]_{\mathcal{B}}$ is a 1-1 and onto linear transformation

Proof (Exc)

$T(c_1 b_1 + \dots + c_n b_n) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

ex. $\mathbb{P}_5 \cong \mathbb{R}^6$.

4.7 Change of Basis

$\mathcal{B} = \{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{b_1}, \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b_2} \}$ $\quad x = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 0 \cdot b_1 + 2 b_2 = 1 \cdot c_1 + 2 \cdot c_2$.

$\mathcal{C} = \{ \underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{c_1} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{c_2} \}$.

Question : If you know the basis $\mathcal{B}$ and $\mathcal{C}$ and $[x]_{\mathcal{B}}$, find $[x]_{\mathcal{C}}$.

Ex. $\mathcal{B} = \{b_1, b_2\}$, $\mathcal{C} = \{c_1, c_2\}$.

Ex. $B = \{b_1, b_2\}$, $C = \{c_1, c_2\}$ $\quad b_1 = c_1 + c_2$

$b_2 = 6c_1 + 2c_2$

$x = 3b_1 + b_2$ Find $[x]_C$.

i.e. $[x]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$x = 3b_1 + b_2$ Find $[x_c]$.

Sol: $[x]_C = [3b_1 + b_2] \Rightarrow 3[b_1]_C + [b_2]_C$.

$= \begin{bmatrix} [b_1]_C & [b_2]_C \end{bmatrix} \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{[x]_{\mathcal{B}}}$

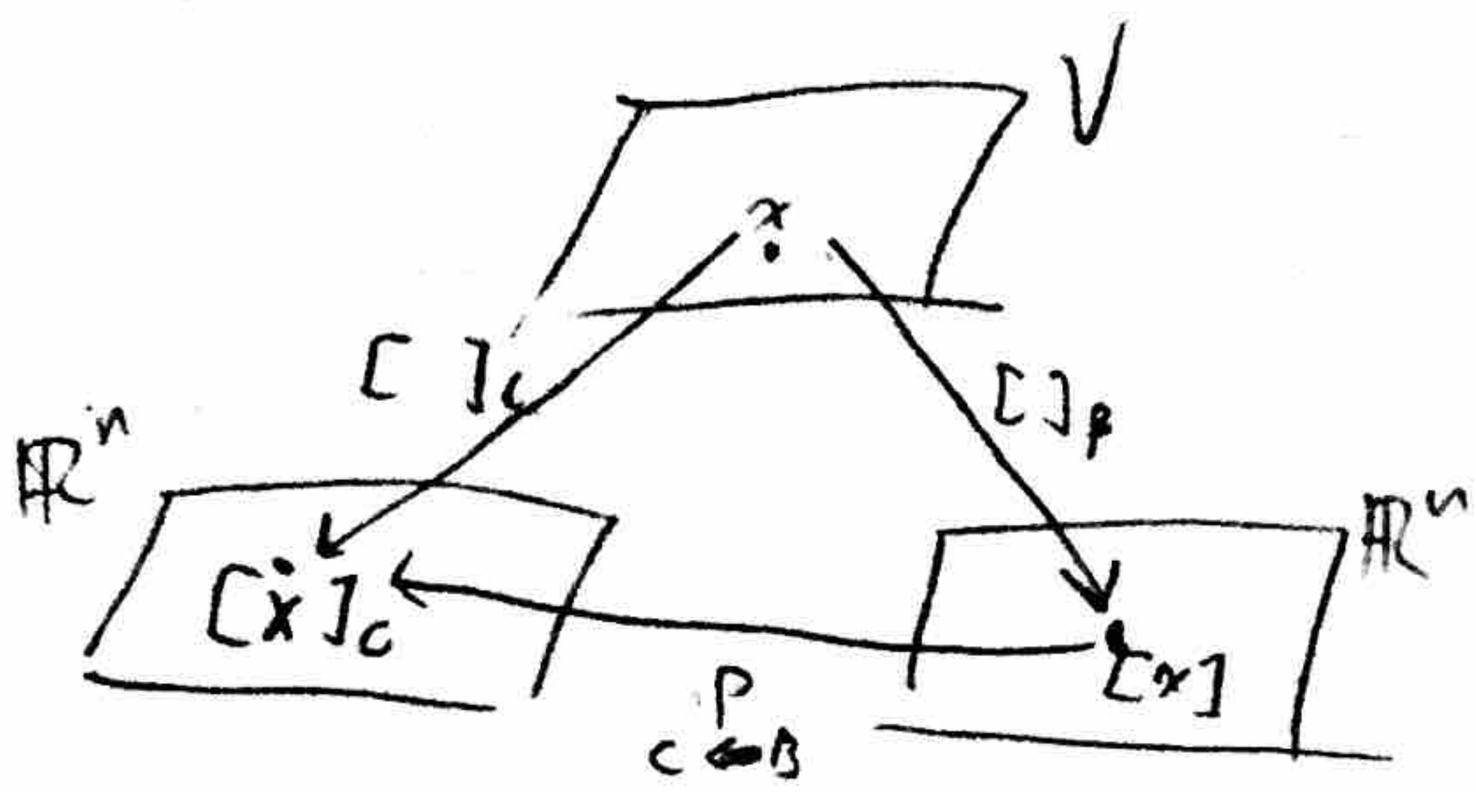
$= \begin{bmatrix} 1 & 6 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 9 \\ 2 \end{bmatrix}$.

Thm 15 $B=\{b_1,\dots,b_n\}$, $C=\{c_1,\dots,c_n\}$ bases for a vector space V $\Rightarrow$ $\exists!$ (there exists a unique) $n\times n$ matrix ${}_{C\leftarrow B}P$ s.t. $[x]_C \; {}_{C\leftarrow B}P \; [x]_B$.

Moreover, ${}_{C\leftarrow B}P = [\,[b_1]_C \cdots [b_n]_C\,]$.

${}_{C\leftarrow B}P$ is the change of coordinates matrix from B to C



Remark 1) ${}_{C\leftarrow B}P$ is invertible
2) $({}_{C\leftarrow B}P)^{-1} = {}_{B\leftarrow C}P$

Sol 2) $[x]_C = {}_{C\leftarrow B}P\,[x]_B$

$$\left({}_{C\leftarrow B}P\right)^{-1}[x]_C = [x]_B \Rightarrow \left({}_{C\leftarrow B}P\right)^{-1} = {}_{B\leftarrow C}P$$

Ex. $B=\{b_1,b_2\}$, $C=\{c_1,c_2\}$ bases for $\mathbb{R}^2$
$b_1=\begin{bmatrix}1\\0\end{bmatrix}$, $b_2=\begin{bmatrix}1\\1\end{bmatrix}$, $c_1=\begin{bmatrix}2\\0\end{bmatrix}$ $c_2=\begin{bmatrix}0\\1\end{bmatrix}$.

Find ${}_{C\leftarrow B}P$

Sol ${}_{C\leftarrow C}P = [[b_1]_C[b_2]_C]$.

$$\left[\begin{array}{cc|cc}2&0&1&1\\0&1&0&1\end{array}\right] \sim \left[\begin{array}{cc|cc}1&0&\frac12&\frac12\\0&1&0&1\end{array}\right].$$

$\uparrow\ \uparrow$
$b_1\ b_2$

$[b_1]_C = \begin{bmatrix}x_1\\x_2\end{bmatrix} \longrightarrow b_1 = x_1c_1 + x_2c_2$

$1x_1+0x_1 = \frac12 \Rightarrow x_1 = \frac12$

$[b_2] = \begin{bmatrix}y_1\\y_2\end{bmatrix} \longrightarrow b_2 = y_1c_1 + y_2c_2.$

$0x_1+1x_1 = 0 \Rightarrow x_2 = 0$

$1y_1+0y_2 = 1 \Rightarrow y_1 = \frac12$

$0y_1+1y_2 = 1 \Rightarrow y_2 = 1$

$[c_1\ c_2]\begin{bmatrix}x_1\\x_2\end{bmatrix} = b_1$

$[c_1, c_2]\begin{bmatrix}y_1\\y_2\end{bmatrix} b_2$

$\underset{C \leftarrow B}{P} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$

Find $\underset{B \leftarrow C}{P} = \left(\underset{C \leftarrow B}{P} \right)^{-1} =$ A $n \times n \Rightarrow [A \; I_n] \sim [I_n \; A^{-1}]$.

3.1 Determinents.

Thm $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ Invertible iff $\det(A) \neq 0$.

Def A $n \times n$ matrix, A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and the jth col.

Ex $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 7 & 2 \\ 3 & 0 & 1 & 2 \\ 4 & 2 & 0 & -1 \end{bmatrix}$ $A_{32} = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 7 & 2 \\ 4 & 0 & -1 \end{bmatrix}$

Def: A $n \times n$, $n \geq 2$, $A = [a_{ij}]$

$\det A \underset{=}{\text{def}}\ a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$.

$\left(= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \right)$

ex. $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$ $\det A = 1 \cdot \det \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} - 0 \det \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + (-1) \det \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

$= -2 - 6 = -8$

Def: A $n \times n$, $C_{ij} = (-1)^{i+j} \det(A_{ij})$ is called the (i,j) cofactor of A.

Thm 1 A $n \times n \Rightarrow \det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$ for all $i = 1, \ldots, n$.

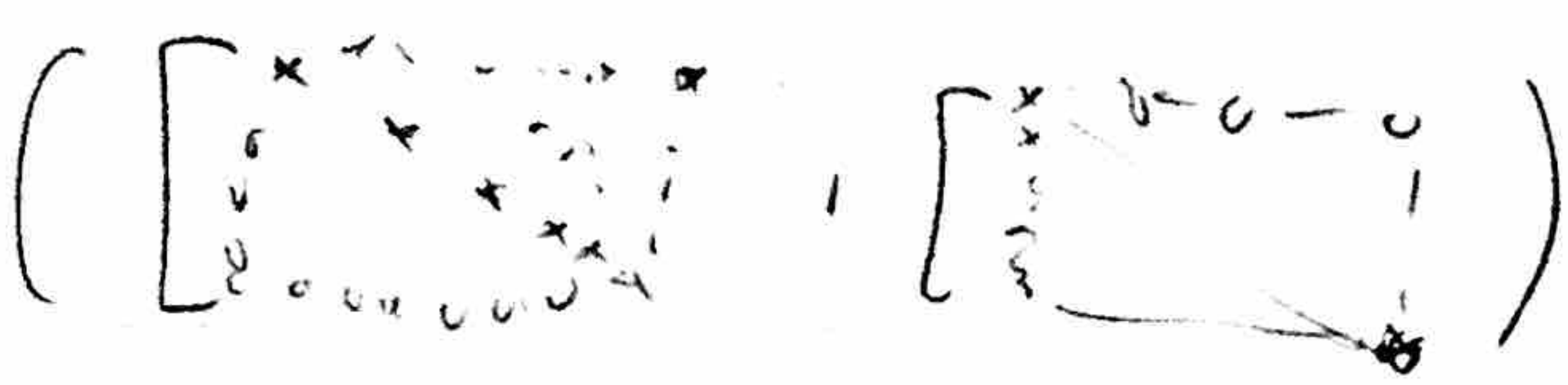
ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ $\det A = 0 \cdot C_{21} + 2 \cdot C_{22} + 0\, C_{23} = 2 (-1)^{2+2} \det \begin{pmatrix} 1 & 3 \\ 1 & 7 \end{pmatrix}$

$= 20$.

2) $A = \begin{bmatrix} 2 & 1 & 100 & 2 \\ 1 & 2 & 1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$

$$\det A = 2C_{13} + 0C_{23} + 0C_{33}$$
$$= 2(-1)^{1+2} \det \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$
$$= (2)(1) \det \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}$$
$$= (-2)(-5) = 16.$$

Thm 2 If A is a triangular matrix, then $\det A$ is the product of the entries of the main diagonal.



$7 = x$

$$\det \begin{pmatrix} x & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ -1 & -100 & 1 & 0 \\ 3 & 0 & 0 & 2 \end{pmatrix} = 4x.$$

3.2 Properties of determinants.

A, B $n \times n$ $\Rightarrow$ i) If B is obtained by adding a multiple of one row of A to another one, then $\det B = \det A$.

ii) B is obtained by interchanging two rows of A then $\det B = -\det A$

iii) if B is obtained by multiplying one row of A by k, then $\det B = k \cdot \det A$.

ex. $\det \begin{bmatrix} 2 & 2 & -8 \\ -2 & -3 & 8 \\ 1 & 1 & 0 \end{bmatrix} = \det \begin{bmatrix} 2 & 3 & -8 \\ 3 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = 0.$

$$\det \begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} = \det \begin{pmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} = \det \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -1 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$
$$= \det \begin{pmatrix} 3 & -4 & -2 \\ -12 & 10 & 10 \\ 0 & -3 & 2 \end{pmatrix}$$

thm 4. A $n \times n$ invertible $\Leftrightarrow$ $\det A \neq 0$.