Professor Um

SIL - Eason

Math 20B Lecture 12 7/25

10.3 Convergence of Series with positive terms

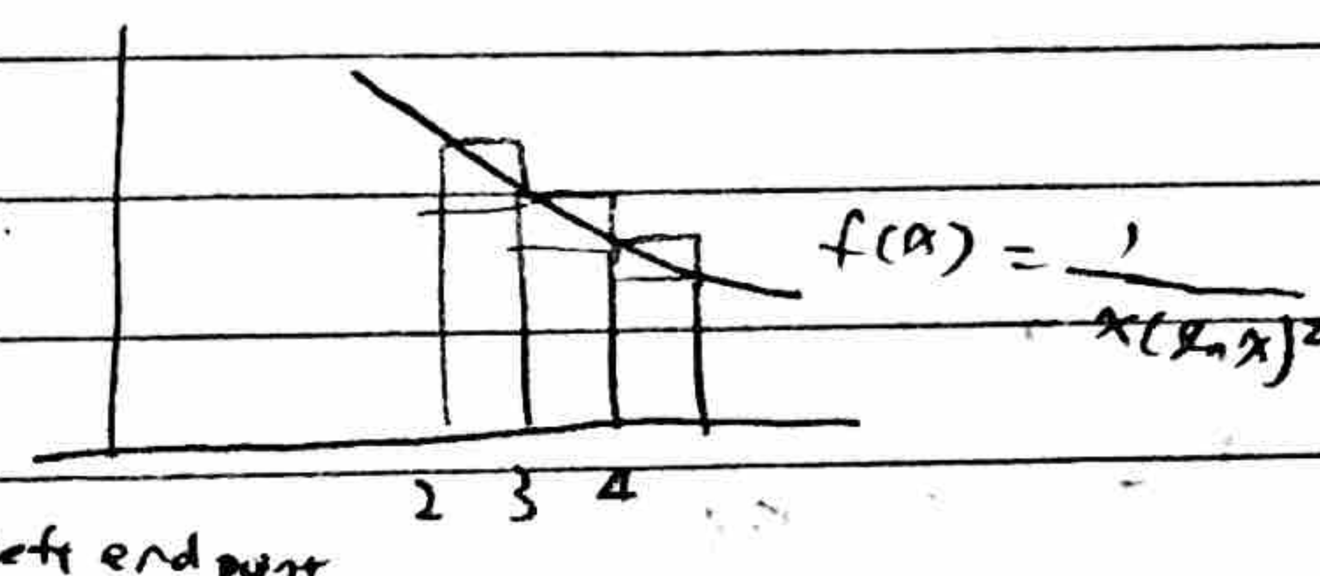
To test Σa_n conv or div, we want to develope some tests. The integral test is the first on the list.

Suppose we want to see if $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ conv. or div.

$\sum_{n=1}^{\infty} a_n \qquad \frac{1}{2(\ln 2)^2} + \frac{1}{3(\ln 3)^2} + \frac{1}{4(\ln 4)^2} + \cdots$

$= \lim_{N \to \infty} \sum_{n=1}^{N} a_n$

$= \lim_{N \to \infty} S_N$



right end point

$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2} \overset{①}{\le} \int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx \overset{②}{\le} \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

If $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \infty$ (div), then from ② $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} = \infty$: div

If $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = C$ # (conv), then from ① $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2} \le C$: (conv)

$\sum_{n=3}^{N} \frac{1}{n(\ln n)^2} \le \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2} \le C$

$S_N \le C$ and $a_n > 0$ ~~so~~ $\{S_N\}$ increasing $\xrightarrow{10.1} \lim_{N \to \infty} S_N$ exists.

$\frac{1}{n(\ln n)^2} > 0$

so $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$ conv.

Integral Test

If f is a positive, continuous decreasing function and $a_n = f(n)$. Then If $\int_M^{\infty} f(x)\,dx$: conv, then $\sum_{n=M}^{\infty} a_n$: conv.

~~If~~ " : div " div.

To check $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ conv or div, we need to check

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx \quad \text{conv. or div.}$$

$$= \lim_{t\to\infty} \left(\int_2^t \frac{1}{x(\ln x)^2}\,dx \right) \qquad u = \ln x$$

$$= \lim_{t\to\infty} \int_{u=\ln 2}^{\ln t} \frac{1}{u^2}\,du \qquad du = \frac{1}{x}dx$$

$$= \lim_{t\to\infty} \left[-u^{-1} \right]_{\ln 2}^{\ln t}$$

$$= \lim_{t\to\infty} \left(-\cancel{\frac{1}{\ln t}}^{\,0} + \frac{1}{\ln 2} \right) = \frac{1}{\ln 2} : \text{conv}$$

By integral test,

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} : \text{conv.}$$

Condition for Int. Test

$f(x) = \frac{1}{x(\ln x)^2}$ $\quad x \geq 2$

positive? ✓

conti? ✓ since only discontinuous at $x=1$.

decreasing?

1. $f' < 0$
2. ~~$f(n) > f(n+1)$ for all n.~~

$$\frac{1}{n(\ln n)^2} > \frac{1}{(n+1)(\ln(n+1))^2} \quad (\text{since } \ln : \text{inc})$$

Remark: Although this test tells us whether or not a series converges. It does not give the actual value of $\sum a_n$.

Ex. Conv. or Div?

(i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

$f(x) = \frac{1}{\sqrt[3]{x}}$ - positive $x \geq 1$
- continuous
decreasing $\frac{1}{\sqrt[3]{n}} > \frac{1}{\sqrt[3]{n+1}}$

$\int_1^{\infty} \frac{1}{\sqrt[3]{x}} dx$: div $\overset{\frac{1}{3}}{p} < 1$

$\left(\int_1^{\infty} \frac{1}{x^p} dx \right.$
conv if $p > 1$
div if $p < 1$

By Int test
$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$: div

Thm $\sum_{n=1}^{\infty} \frac{1}{n^p}$ conv if $p > 1$
div if $p \leq 1$

$p = 1 \rightarrow$ (Harmonic Series)

(ii) $\sum_{n=1}^{\infty} n e^{-n^2}$

$f(x) = x e^{-x^2}$ positive ✓
continuous ✓
decreasing ✓

$f'(x) = 1e^{-x^2} + xe^{-x^2}(-2x)$
$= \underbrace{e^{-x^2}}_{+}\underbrace{(1-2x^2)}_{-}$
< 0

$\int_1^{\infty} x e^{-x^2} dx$

$= \lim_{t \to \infty} \left(\int_1^t x e^{-x^2} dx \right)$ $u = -x^2$, $du = -2x\,dx$

$= \lim_{t \to \infty} \int_{-1}^{-t^2} e^u \frac{du}{-2}$

$= \lim_{t \to \infty} -\frac{1}{2} e^u \Big]_{-1}^{-t^2} = \lim_{t \to \infty} -\frac{1}{2} (\overset{0}{e^{-t^2}} - e^{-1}) = \frac{1}{2e}$ conv.

By the Integral test Σ

Direct Comparison Test

Suppose $0 \le a_n \le b_n$

for any $n \ge M$

(i) If $\sum_{n=M}^{\infty} b_n$: conv, then $\sum_{n=M}^{\infty} a_n$ conv.

(ii) if $\sum_{n=M}^{\infty} a_n$: div, then $\sum_{n=M}^{\infty} b_n$: div

ex.

$M=3$

~~$0 \le a_3 \le b_3$~~

$0 \le a_4 \le b_4$

$\vdots$

$0 \le a_N \le b_N$

$\lim_{N\to\infty} \sum_{3}^{N} a_n \le \lim_{N\to\infty} \sum_{3}^{N} b_n$

Ex. $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ conv or div

$0 \le \frac{1}{n^3+1} \le \frac{1}{n^3}$ $\quad \frac{\triangle}{\boxed{}} < \frac{\triangle}{\square}$

$\underline{\underline{}}$

conv by p-ser. test

By Direct Comparison test $\quad$ ($\sum \frac{1}{n^p}$ conv if $p>1$, div if $p \le 1$)

Limit Comparison Test

Suppose $\sum a_n$, $\sum b_n$ are series with positive terms

Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$, If $0 < L < \infty$, then $\sum a_n \sum b_n$: either both conv or both div

Ex. If $L=3$

$\lim_{n\to\infty} \frac{a_n}{b_n} = 3$ $\quad$ $\sum a_n = a_1 + a_2 \ldots a_{m+1} + \ldots$

$\frac{a_n}{b_n} \approx 3$ ~~for large n.~~ $\quad$ $\sum b_n = b_1 + b_2 \ldots b_{m+1}$

$a_n \approx 3b_n$

$a_n < b_n$

If $L=0$, then if $\sum b_n$ = conv, then $\sum a_n$: Conv.

If $L=\infty$, then if $\sum a_n$: conv, then $\sum b_n$: Conv.

$a_n > b_n$

Ex. (1) $\sum_{n=1}^{\infty} \frac{1}{n^3-1}$ = conv or div

$= a_n$ $\qquad \sum \frac{1}{n^3}$ b_n

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^3-1} \cdot \frac{n^3}{1} = \lim_{n\to\infty} \frac{n^3}{n^3-1} = 1 = L$$

Since $0 < L = 1 < \infty$, by L.C.T

either $\sum \frac{1}{n^3-1}$, $\sum \frac{1}{n^3}$ both conv, or both diverge

Since $\sum \frac{1}{n^3}$ converges $\rightarrow \frac{1}{n^3-1}$ converges.

Ex. (i) $\sum_{n=1}^{\infty} \frac{n^2+2n}{\sqrt{n^7+3}}$ $\qquad$ (ii) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+\ln n}$ LCT $\quad b_n = \frac{1}{\sqrt{n}}$

$a_n = \frac{n^2+2n}{\sqrt{n^7+3}}$ $\quad b_n = \frac{n^2}{\sqrt{n^7}}$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{n^2+2n}{\sqrt{n^7+3}} \cdot \frac{\sqrt{n^7}}{n^2}$$

$$= \lim \frac{\sqrt{n^7/n^7}}{\sqrt{\frac{n^7}{n^7}+\frac{3}{n^7}}} \cdot \frac{n^2+2n}{n^2}$$

$$= \frac{1}{\sqrt{1+\cancel{\frac{3}{n^7}}^{0}}} \cdot 1$$

$= 1 \cdot 1$

$= 1$

Since $0 < L = 1 < \infty$

$\sum \frac{n^2+2n}{\sqrt{n^7+3}}$ $\quad \sum \frac{n^2}{\sqrt{n^7}} = \sum \frac{1}{n^{1.5}}$ conv.

(conv)

(iii) $\sum_{n=1}^{\infty} \frac{5+\cos n}{n^2}$ $\qquad -1 \le \cos n \le 1$

$$\sum \frac{4}{n^2} \le \frac{5 \cos n}{n^2} \le \sum \frac{6}{n^2}$$

By D.C.T. $\qquad$ conv $\qquad$ conv