Math 18 Professor Drimbe

Lecture 2 8/10 SIL: Eason

Def: Matrices $n \times 1$ (column matrices) are vectors in $\mathbb{R}^n$

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \; u_1, \dots, u_n \in \mathbb{R}.$$

The zero vector $\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

The vector addition in $\mathbb{R}^n$ and the scalor multiplicaton are def entry by entry:

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \; \alpha \in \mathbb{R}.$$

$$u+v \overset{\text{def}}{=} \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}, \quad \alpha \cdot u \overset{\text{def}}{=} \begin{bmatrix} \alpha u_1 \\ \vdots \\ \alpha u_n \end{bmatrix}$$

ex. $\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$(-2)\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 2 \end{bmatrix}$

Algebraic properties.

$u, v, w \in \mathbb{R}^n$, $c, d \in \mathbb{R}$

(i) $u+v = v+u$

(ii) $(u+v)+w = u+(v+w)$

(iii) $u+0 = 0+u = u$

(iv) $u+(-u) = 0$ where $-u = (-1)u$.

(v) $c(u+v) = cu+cv$

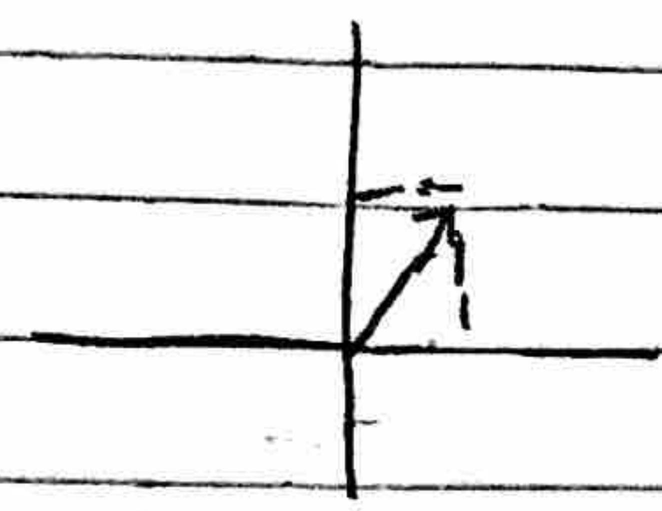
(vi) $(c+d)u = cu+du$

(vii) $c(du) = (cd)u$

(viii) $1 \cdot u = u$.

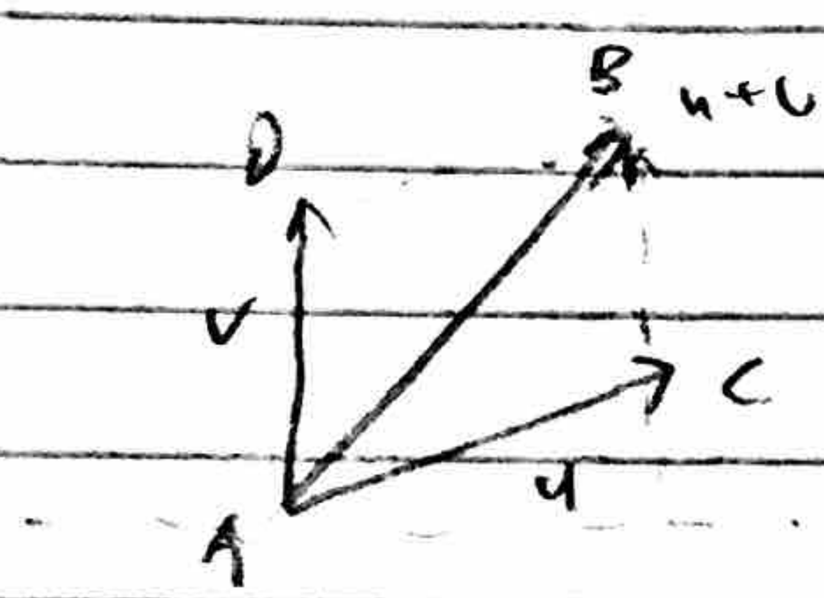
Geometric Interpretation ($\mathbb{R}^2$)

$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



Parallelogram rule for addition.

$u, v \in \mathbb{R}^2 \Rightarrow 0, u, v, u+v$ corresponds to the vertices of parallelogram.



$|AB| = |CD|$

$|AD| = |BC|$

Def: $v_1, \dots v_p \in \mathbb{R}^n$

$c_1 v_1 + \dots + c_p v_p$ is called <u>linear combination</u> of $v_1, \dots, v_p$ with weights $c_1, \dots c_p$.

Ex. $u = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Is w a lin. comb of $\{u \,\&\, v\}$. Does there exist x_1, x_2 st. $x_1 u + x_2 v = w$?

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ 0 \\ x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ -2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 + 2x_2 \\ -2x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim$$

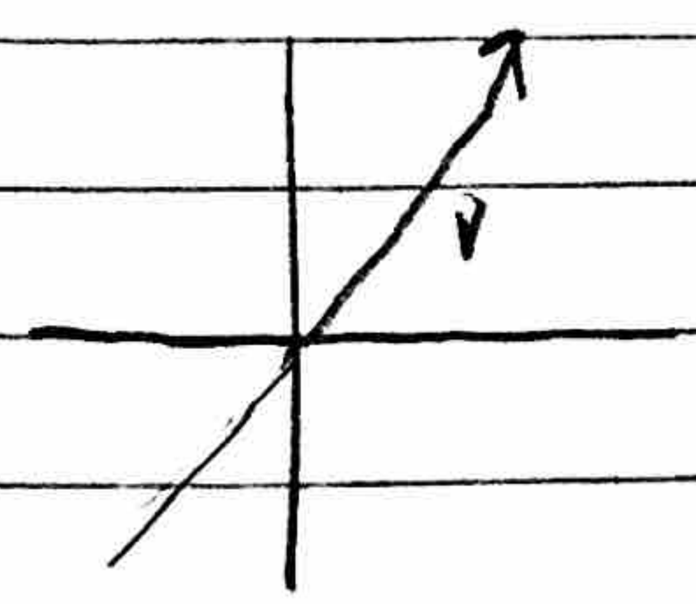
$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 = 1 \\ x_2 = -1 \end{cases}$$

Remark: A vector equation $x_1 a_1 + x_2 a_2 + \dots x_n a_n = b$ has the same soln set as the lin't system whose augmented matrix is $[a_1 \dots a_n \, b]$

Def: $v_1, \dots, v_p \in \mathbb{R}^n$

$\text{span}\{v_1, \dots, v_p\} = \{c_1 v_1 + \dots + c_p v_p \mid c_1, \dots, c_p \in \mathbb{R}\}$ subset in $\mathbb{R}^n$.

Ex. $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}$ $\quad \text{span}\{v\} = \{cv \mid c \in \mathbb{R}\}$



span $\{v\}$

Ex) $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$ $\quad \text{span}(u, v) = \{c_1 u + c_2 v \mid c_1, c_2 \in \mathbb{R}\}$

$= \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\} = \mathbb{R}^2$

check if $u \neq \alpha v \Rightarrow \text{span}(u, v) = \mathbb{R}^2$

$\text{span}\{u, v\}$ = plane that cont. u & $v \subseteq \mathbb{R}^3$.

The matrix reaction.

Def: A $m \times n$ matrix, with $a_1, \dots, a_n$ columns; $x \in \mathbb{R}^n$.

$(A = [a_1, \dots, a_n], a_1, \dots, a_n \in \mathbb{R}^m)$

$Ax \overset{\text{def}}{=} x_1 a_1 + x_2 a_2 + \dots + x_n a_n$

Ex. $\begin{bmatrix} 2 & -1 \\ 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 \\ 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ Not well defined

$v_1, v_2 \in \mathbb{R}^m$, $2v_1 - v_2 = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{cases} 2x_1 + 2x_2 - x_3 = 4 \\ x_1 \qquad + 2x_3 = 2 \end{cases}$ linear system

$\begin{bmatrix} 2x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_3 \\ 2x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ vector eq.

$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ Matrix eq.

Thm 1 A $m \times n$, $A = [a_1 \cdots a_n]$, $b \in \mathbb{R}^m$
$\Rightarrow$ the matrix eq. $Ax = b$ has the same sol set as the vector eq. $x_1 a_1 + \cdots + x_n a_n = b$.
Which has the same sol set with the linear system.
whose augmented matrix, $[a_1 \cdots a_n \ b]$.

Ex. $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 2 \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Is the equation $Ax = b$ consistent for all $b_1, b_2, b_3 \in \mathbb{R}$?

Sol. $\begin{bmatrix} 1 & 0 & 1 & b_1 \\ 2 & 2 & 1 & b_2 \\ 3 & 2 & 1 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & 2 & -1 & b_2 - 2b_1 \\ 0 & 2 & -2 & b_3 - 3b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & 2 & -1 & b_2 - 2b_1 \\ 0 & 0 & -1 & b_3 - b_1 - b_2 \end{bmatrix}$

$x_1 \quad + x_3 = b_1$

$2x_2 - x_3 = b - 3b_1$

$-x_3 = -b_1 - b_2 + b_3$

$\Rightarrow$ consistent for all b_1, b_2, b_3.

Thm 2: A $m \times n$ TFAE: (the following are equivalent,

a) for each $b \in \mathbb{R}^m$, the matrix eq. $Ax = b$ has a sol'n.

b) each $b \in \mathbb{R}^m$ is a linear combination of the columns of A

c) the columns of A span $\mathbb{R}^m$

d) A has a pivot position in every row.

Proof: a) $\Leftrightarrow$ b)

b) $\Leftrightarrow$ c)

d) $\Rightarrow$ a): Let U be an echelon form of A. Let $b \in \mathbb{R}^m$ $[A \; b] \sim [U \; d]$

Then there exist $d \in \mathbb{R}^n$ s.t. $[A \; b] \sim [U \; d] \Rightarrow$

$[U \; d]$ is consistent $\Rightarrow [A \; b]$ is consistent $\overset{\text{Thm}}{\Rightarrow}$ $Ax = b$ has sol'n

a) $\Rightarrow$ d) Assume d) is false.

$\Rightarrow$ U has a row just with zeros.

$$U = \begin{bmatrix} * & \cdots & * \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \qquad \underset{\text{inconsistent}}{\begin{bmatrix} U & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{bmatrix}} \sim \underset{\text{consistent}}{\begin{bmatrix} A & b \end{bmatrix}} \Rightarrow \times$$

$\Rightarrow$ Then $\exists$ a $b \in \mathbb{R}^m$ s.t.

$\square$.

Row-vector rule for the computing Ax:

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = x_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ -x_1 + x_3 \\ 2x_1 + x_2 + x_3 \end{bmatrix} \text{ ith entry}$$

The ith entry is the sum of the products of the corresponding entries from row i of A and from the vector x.

Thm 3 A $m \times n$, ~~$u, v \in \mathbb{R}^n$, $c \in \mathbb{R}$~~

i) $A(u+v) = Au + Av$

ii) $A(cu) = c(Au)$

Proof: i) $A = [a_1 \cdots\cdots a_n]$ $\left(A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \right)$

$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$

$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ $\quad A(u+v) = [a_1 \dots a_n] \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}$

$\overset{def}{=} (u_1+v_1)a_1 + \dots + (u_n+v_n)a_n =$

$= u_1a_1 + u_2a_2 + \dots + u_na_n + v_1a_1 + \dots + v_na_n$

$= Au + Av.$

ii) Exercise

Solution sets of linear systems

Def: A system of linear equations is said to be homogeneous if it can be written in form $Ax=0$, A $m \times n$ matrix, 0 zero vector in $\mathbb{R}^m$.

ex. $\begin{cases} x_1 - x_2 \\ 2x_1 + x_2 = 0 \end{cases}$ $\quad \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not homogeneous

$Ax = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Remarks

1) 0 is always a soln for $Ax=0$. (called the trivial soln)
if $x \neq 0$, $Ax = 0$, then x is called a non trivial soln.

2) $Ax=0$ has a non-trivial soln $\Leftrightarrow$ the linear system has at least one free variable. (using the existence and uniqueness theorem)

Ex. $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$Ax = 0$

A is echelon matrix $\Rightarrow$ x_3 is a free variable. $\Rightarrow$ we have non-trivial sol^n.

$$\begin{bmatrix} x_1 + 2x_2 + x_3 \\ 2x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 2x_2 + x_3 = 0 \\ x_3 \text{ free} \end{cases} = \begin{cases} x_2 = -\frac{1}{2}x_3 \\ x_1 = -2x_2 - x_3 = 0 \\ x_3 \text{ free.} \end{cases}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{x_3}{2} \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

(with x_3 free

or

$$x = t\begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \ t \in \mathbb{R}$$

We say that the sol^n is in parametric vector form.

Non-homogeneous Systems

Ex. $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Ax = b$.

$$\begin{cases} x_1 + 2x_2 + x_3 = 1 \Rightarrow x_1 + 2\left(-\frac{x_3}{2} + \frac{1}{2}\right) + x_3 = 1 \Rightarrow x_1 = 0 \\ 2x_2 + x_3 = 1 \Rightarrow x_2 = -\frac{x_3}{2} + \frac{1}{2} \\ x_3 \text{ free} \end{cases}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{x_3}{2} + \frac{1}{2} \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{x_3}{2} \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

$$x = x_3 \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ +\frac{1}{2} \\ 0 \end{bmatrix} \text{ with } x_3 \text{ free.}$$

$$= t\begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ +\frac{1}{2} \\ 0 \end{bmatrix}, \ t \in \mathbb{R}$$

Thm: Suppose $Ax=b$ is consistent for some given b, and let p be a particular solution

$\Rightarrow$ the solution set of $Ax=b$ is the set of all vectors of the form $w=p+v_h$, where v_h is any sol of the homogeneous eq. $Ax=0$

Proof: Suppose $w=p+v_h$ where v_h is a sol for $Ax=0$

$$Aw = A(p+v_h) = Ap + Av_h = b+0 = b.$$

Suppose w is a sol'n for $Ax=b$.

what $w = p + ?$ $\quad Ap = b$

$$Aw - Ap = b - b$$

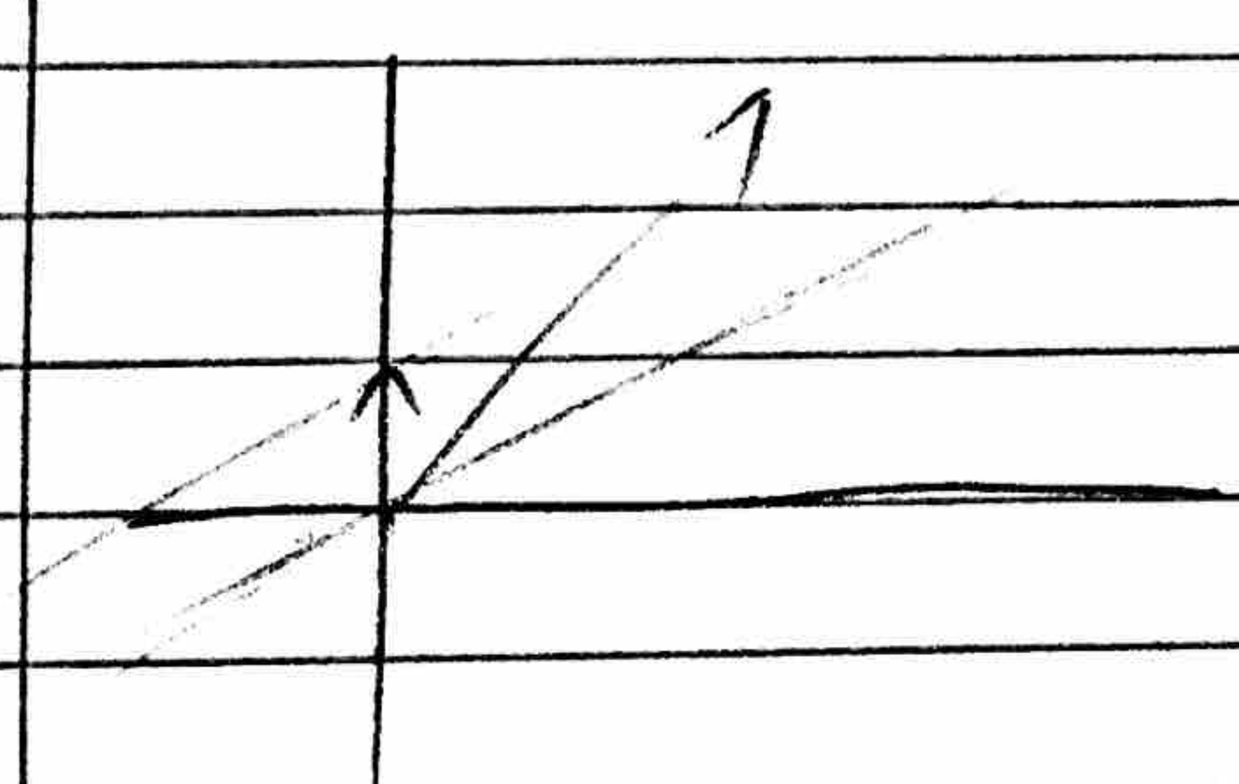
$$A(w-p) = 0$$

Take $v_h = w - p$.

Geometric Interpretation

sol set for $Ax=b$

sol set for $Ax=0$.



Linear independence.

Def: $v_1, \ldots v_p \in \mathbb{R}^n$

$\{v_1 \ldots, v_p\}$ is linearly independent if the vector equation $x_1v_1 + \ldots + x_pv_p = 0$ has only the trivial solution.)

(the matrix eq. $[v_1 \ldots v_p]\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = 0$ has only the trivial sol'n).

$\{v_1 \ldots v_p\}$ is linearly dependent if there exists weights $c_1, c_2, \ldots, c_p$, not all zero st. $c_1v_1 + \ldots c_pv_p = 0$.

Ex 1) $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $v_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$v_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = v_1 + \frac{1}{2} v_2 \Rightarrow v_1 + \frac{1}{2} v_2 + (-1) v_3 = 0$

$c_1 = 1$
$c_2 = \frac{1}{2}$
$c_3 = -1$

$\{v_1, v_2, v_3\}$ is lin dependent.

Ex: $v_1, v_2, v_3 \in \mathbb{R}^2$

$\Rightarrow \{v_1, v_2, v_3\}$ is linearly dependent

Ex 2 $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ $Ax = 0$ non trivial soln?

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

x_3 is free.

$\Rightarrow$ x_3 free variable $\Rightarrow$ non-triv. soln $\Rightarrow \{v_1, v_2, v_3\}$ is lin. dependent.

Ex. Determine if the coln of $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 3 & -2 & 0 \end{bmatrix}$ are lin indep.?