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Math 20B lecture 14 7/31

10.5 The Ratio and Root Tests and Strategies for choosing Tests

Recall Geometric series $= a + ar + ar^2 + ar^3 + \dots = \begin{cases} \text{conv if } |r|<1 \\ \text{div otherwise.} \end{cases}$

<The ratio test>

Assume $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$ exists

(1) $\rho < 1$, then $\sum a_n$: abs conv

(2) $\rho > 1$, then $\sum a_n$: div.

(3) $\rho = 1$, Ratio test is inconclusive.

Supp. $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0.7$ $\qquad$ $\lim \left|\frac{a_{n+1}}{a_n}\right| = 1.1$

$|a_n|$ behaves like a geo. ser.

$\left|\frac{a_{n+1}}{a_n}\right| \approx 0.7$ for any large n.

$0.6 \le \left|\frac{a_{n+1}}{a_n}\right| \le 0.8$

$0.6|a_n| \le |a_{n+1}| \le 0.8|a_n|$ for $n \ge M$ large integer

$|a_{M+1}| \le 0.8|a_M|$

$|a_{M+2}| \le 0.8|a_{M+1}| \le (0.8)^2|a_M|$

$|a_{M+3}| \le (0.8)^3|a_M|$

$\vdots$

$$\sum_{n=M+1}^{\infty} |a_n| \le 0.8|a_M| + (0.8)^2 \cdot |a_M| + (0.8)^3|a_M| + \dots$$

geo ser $|r = 0.8| < 1$, converges

By Direct Comp Test,

$\sum\limits_{n=M+1}^{\infty} |a_n|$ conv

$\sum\limits_{n=1}^{\infty} |a_n|$ conv $\rightarrow \sum a_n$ abs. conv.

Ex. (i) $\sum_{n=1}^{\infty} \underset{a_n}{\underset{\shortparallel}{3^n}}$: div $\qquad \sum r^n \begin{cases} \text{con} & \text{if } |r|<1 \\ \text{div} & \text{otherwise} \end{cases}$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1}}{3^n} \right| = \lim_{n\to\infty} 3 \to 3$$

since $\rho = 3 > 1$, by ratio Test $\sum 3^n$ div

(ii) $\sum \frac{10^n}{(n+1)4^{2n+1}}$

$$a_n = \frac{10^n}{(n+1)4^{2n+1}}$$

$$a_\Delta = \frac{10^\Delta}{(\Delta+1)(4^{2\Delta+1})}$$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{10^{n+1}}{(n+2)4^{2n+3}} \cdot \frac{(n+1)4^{2n+1}}{\cancel{10^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{10\cdot(n+1)\cdot \cancel{4^{2n}}\cdot 4}{(n+2)\ \cancel{4^{2n}}\cdot 4^{2}\ 2} \right| = \lim \left| \frac{10(n+1)}{16(n+2)} \right| = \frac{5}{8}$$

$\rho < 1$, by the ratio test, series abs. conv.

(iii) $\sum \frac{n^n}{n!}$ ~~series diverges by ratio test~~

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{n+1!} \right| = \lim_{n\to\infty} \left| \frac{1}{\cancel{n+1}} \cdot \frac{(n+1)^n \cdot \cancel{(n+1)}}{n^n} \right|$$

$$= \lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right)^n \right| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$= \lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^n}$$

$$= e^{\lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right)}$$

$$= e^{\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}} \overset{LH}{=} e^{\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}\left(\frac{1}{n}\right)} = e > 1$$

<The root test>

Assume $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ exists

(i) If $L<1$, then $\sum a_n$: abs. conv.

(ii) If $L>1$ div.

(iii) $L=1$

Ex. What does the root test tell you about

$$\sum_{n=1}^{\infty} \underbrace{(-1)^{n+1}\left(\frac{5n}{8n-1}\right)^n}_{a_n}$$

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|(-1)^{n+1}\left(\frac{5n}{8n-1}\right)^n\right|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left|\frac{5n}{8n-1}\right|^{n\cdot\frac{1}{n}} = \frac{5}{8} < 1,$$ by the root test $\therefore$ abs. conv.

Strategies to test $\sum a_n$: conv or div

- If $\lim_{n\to\infty} a_n \neq 0$, Div. Test $\sum a_n$: div.
- If $\lim_{n\to\infty} = 0$, then we need other tests.

Simpler Series

- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ p series $\begin{cases} \text{conv } p>1 \\ \text{div } p\leq 1 \end{cases}$
- $\sum_{n=1}^{\infty} ar^n$ geometric series $\begin{cases} \text{con if } |r|<1 \\ \text{div otherwise} \end{cases}$

$a_n \geq 0$, $\xrightarrow{\text{Need to simplify } a_n}$ Comparison Test (compare with simpler series)

Direct Comparison Test $0 \leq a_n \leq b_n$
- If $\sum b_n$: conv, then $\sum a_n$ conv
- If $\sum a_n$: div, then $\sum b_n$ div

Limit Comparison Test

$\lim \frac{a_n}{b_n} = L$, If $0<L<\infty$, either both $\sum a_n$, $\sum b_n$ conv, or div.

$a_n \geq 0$ $\xrightarrow{\text{No need to simplify.}}$ (*)

(*) $\rightarrow$ Integral Test

(*) $\rightarrow$ Ratio test

(*) $\rightarrow$ root test

a_n is not positive. $\longrightarrow$ A.S.T

Ex. $\sum \frac{\sin(n!)}{n^2+1}$?? con or div

$$\frac{-1}{n^2+1} \le a_n \le \frac{1}{n^2+1}$$ By squeeze theorem,

$\searrow 0 \swarrow$

$\lim_{n\to\infty} a_n = 0$

But we know $\sum \frac{|\sin(n!)|}{n^2+1} \le \frac{1}{n^2+1} \le \sum \frac{1}{n^2}$

By direct comp Test

$\sum |a_n|$ conv $\to$ $\sum a_n$ conv.

Often compared inequality

$1 \le \ln x \le x$ (or $\sqrt{x}$)

$0 \le |\sin x| \le 1$

$|\tan^{-1}(\square)| < \frac{\pi}{2}$