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Math 20B Lecture 9 7/19

7.7 Improper Integrals.

$$\int_a^b \underset{\sim}{f(x)}\, dx$$

(I) may be ∞ — may not be continuous on $[a,b]$ (II)

Type (I) Infinite Integrals

Let a be a fixed number and f be integrable over $[a,b]$ for any $b \geq a$.

$$\int_a^b f(x)\,dx = \left(\lim_{n\to\infty} \text{Riemann Sum}\right)$$

exists, f: integrable

$$\int_a^\infty f(x)\,dx \overset{\text{def}}{=\!=} \lim_{t\to\infty}\int_a^t f(x)\,dx$$

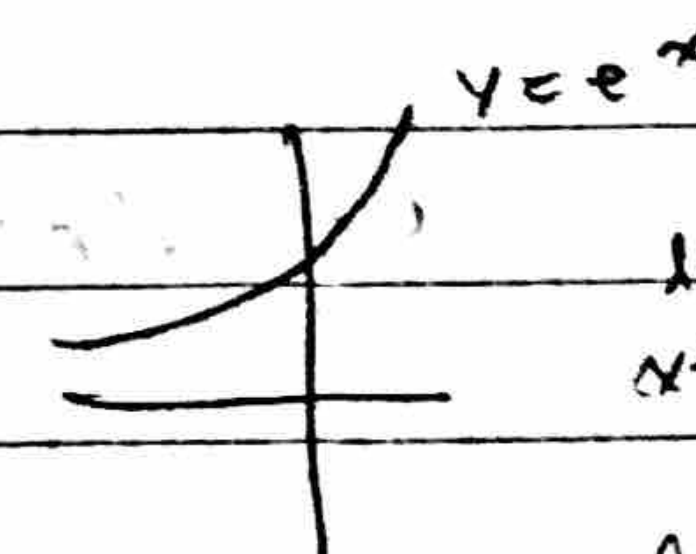
We say $\int_a^\infty f(x)\,dx$ converges if the limit exists

" diverges if " does not exist.

Similarly $\int_{-\infty}^a f(x)\,dx \overset{\text{def}}{=\!=} \lim_{t\to-\infty}\left(\int_t^a f(x)\,dx\right)$

$$\int_{-\infty}^\infty f(x)\,dx \overset{\text{def}}{=\!=} \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx \text{ for any } a.$$

if both are conv, then $\int_{-\infty}^\infty f(x)\,dx$ conv.



$\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to\infty} e^{-x} = 0$

Ex $\int_1^\infty xe^{-x}dx = \lim_{t\to\infty}\left(\int_1^t xe^{-x}dx\right)$

$$= \lim_{t\to\infty}\left(-\frac{t}{e^t} - \frac{1}{e^t} + \frac{2}{e}\right)$$

$$= \lim_{t\to\infty}\left(-\cancel{\frac{1}{e^t}}^{\,0} - 0 + \frac{2}{e}\right)$$

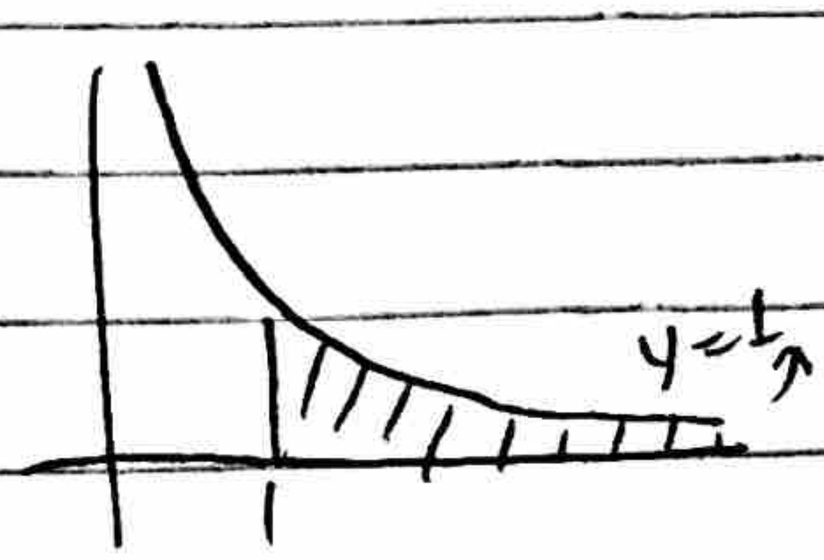
$$= \frac{2}{e} \quad \text{convergent.}$$

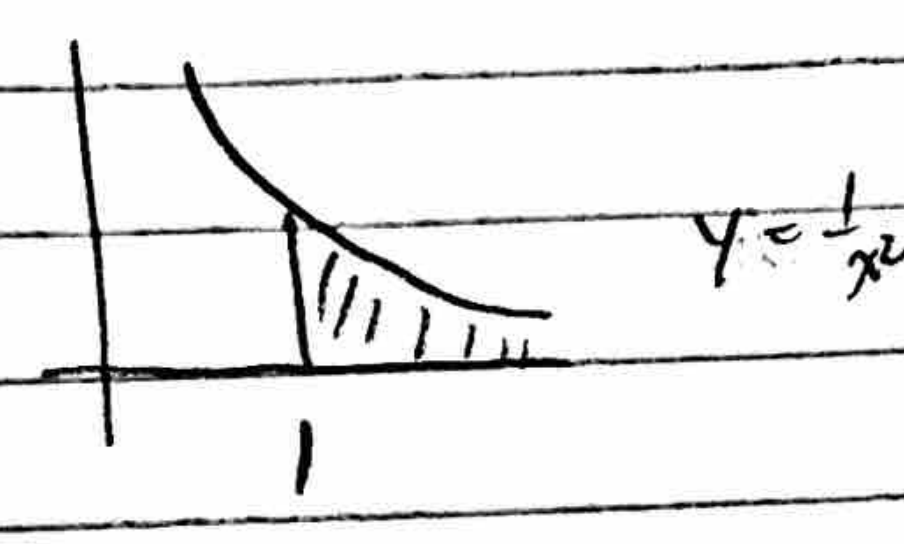
$$\int_2^\infty \frac{1}{x\ln x}\,dx$$

$$= \lim_{t\to\infty}\left(\int_2^t \frac{1}{x\ln x}\,dx\right)$$

$$= \lim_{t\to\infty}\left(\ln(\ln t) - \ln(\ln 2)\right)$$

$$= \infty \quad : \text{div}$$





$$\int_1^\infty \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \ln x\Big]_1^t$$

$$= \lim_{t\to\infty}\left(\ln t - \cancel{\ln 1}^{\,0}\right)$$

$$= \ln\infty$$

$$= \infty \quad \text{div.}$$

$$\int_1^\infty \frac{1}{x^2}\,dx$$

$$\lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty}\left[\frac{x^{-1}}{-1}\right]_1^t$$

$$= \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t$$

$$= \lim_{t\to\infty}\left(-\frac{1}{t}^{\,0}+1\right)$$

$$= 1$$

Ex. For what p is the integral convergent?

$$\int_1^\infty \frac{1}{x^p}\,dx$$

If $p=1$, integral diverges.

$$= \lim_{t\to\infty}\int_1^t x^{-p}\,dx$$

If $p \neq 1$, $\lim_{t\to\infty}\left(\int_1^t x^{-p}\,dx\right)$

power rule

$$= \lim_{t\to\infty} \int_1^t \frac{x^{-p+1}}{-p+1}\Big]_1^t$$

$$= \lim_{t\to\infty}\left(\frac{1}{-p+1}\left(t^{-p+1}-1\right)\right)$$

If $-p+1 = \oplus > 0 \qquad p<1$

$\lim_{t\to\infty} \frac{1}{-p+1}\left(t^{\oplus}-1\right) = \infty$: div

If $-p+1 = \ominus < 0 \qquad p>1$

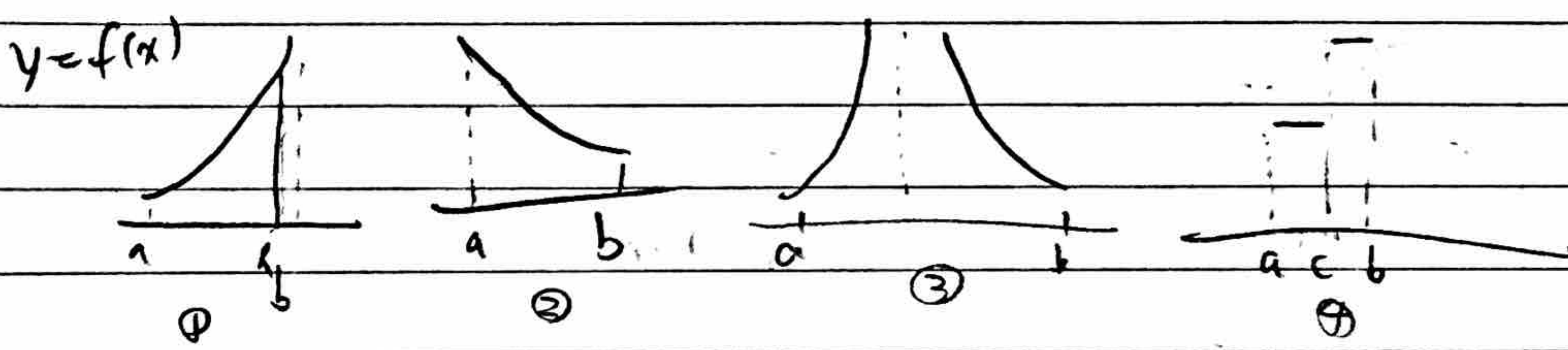
$\lim_{t\to\infty} \frac{1}{-p+1}\left(t^{\ominus}-1\right) = \frac{-1}{p+1} = \frac{1}{p-1}$ convergent

$$\int_M^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{divergent if } p\le 1 \\ \frac{1}{p-1} \text{ convergent if } p>1 \end{cases}$$

any constant >0

$\int_3^\infty \frac{1}{\sqrt{x}}\,dx$ = div.

(Type II) Unbounded function.



① If f continuous on $[a,b)$, & $\lim_{x\to b^-} f(x) = \pm\infty$ then $\int_a^b f(x)\,dx \overset{def}{=} \lim_{R\to b^-}\int_a^R f(x)\,dx$

② Similarly If " $(a,b]$ " then $\int_a^b f(x)\,dx \overset{def}{=} \lim_{R\to a^+}\int_R^b f(x)\,dx$.

In both ① and ②, we say the improper integral converges if the limit exists.

" diverges if limit doesn't exist (DNE

Ex. $\int_{-2}^4 \frac{dx}{(x+2)^{1/3}} = \lim_{R\to -2^+}\int_R^4 \frac{1}{(x+2)^{1/3}}\,dx$

$u = x+2$
$du = dx$

$= \lim_{R\to -2^+}\int_{R+2}^{u=6} \frac{1}{u^{1/3}}\,du = \frac{3}{2}(6)^{2/3}$

$= \lim_{R\to -2^+}\left[\frac{3}{2}u^{2/3}\right]_{R+2}^6$

$= \lim_{R\to -2^+} \frac{3}{2}\left(6^{2/3} - \underbrace{(R+2)^{2/3}}_{\to 0}\right)$

Ex $\int_0^1 \ln x \, dx$

$= \lim_{R\to 0^+} \int_R^1 \ln x$ $\qquad u = \ln x \quad dv = dx$

$\qquad du = \frac{1}{x} dx \quad v = x$

$= \lim_{R\to 0^+} [x \ln x]_R^1 \int \frac{1}{x} x \, dx$

$= \lim_{R\to 0^+} \left(\cancel{1 \ln 1}^{0} - R \ln R - [x]_R^1 \right)$

$= \lim_{R\to 0^+} \left(-\underset{0^+}{R} \underset{-\infty}{\ln R} - 1 + \underset{0^+}{R} \right)$

$= \lim_{R\to 0^+} \left(- \frac{\ln R}{\frac{1}{R}} - 1 \right)$

L'H

$= \lim_{R\to 0^+} \frac{R^2 \frac{1}{R}}{\cancel{R^2} \frac{1}{R^2}} - 1$

$= \lim_{R\to 0^+} R - 1$

$= -1$

- Comparison Test for improper integrals

Assume $0 \le g(x) \le f(x)$ for $x \ge a$

$\int_a^\infty g(x)\,dx \le \int_a^\infty f(x)\,dx$

If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges

Ex. (1) $\int_1^\infty \frac{x^2}{\sqrt{9+x^9}}\,dx$ Converges or diverges?

$0 \le \frac{x^2}{\sqrt{9-x^9}} < \frac{x^2}{\sqrt{x^9}}$ $\qquad \frac{1}{x^{\frac{9}{2}-2}} = \frac{1}{x^{5/2}}$

$\int_1^\infty \frac{x^2}{\sqrt{9+x^9}} \le \int_1^\infty \frac{1}{x^{5/2}}\,dx$

$\int \frac{1}{x^p}\,dx =$ conv. if $p > 1$ $\Rightarrow$ Comp Test $\int_1^\infty \frac{x^2}{\sqrt{9+x^9}}\,dx$ Converges