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Thm 4 A, B $n \times n$, A is similar to $B \Rightarrow \det(A - \lambda I_n) = \det(B - \lambda I_n)$

* $\det(CD) = \det(DC)$

$\det(CD) = \det C \det D$.

Proof: A is similar to $B \Rightarrow \exists P$ $n \times n$ invertible s.t. $A = PBP^{-1}$

$\det(A - \lambda I_n) = \det(PBP^{-1} - \lambda I_n) = \det(PBP^{-1} - \lambda P P^{-1})$

$= \det(P(BP^{-1} - \lambda P^{-1}))$

$= \det(P(B - \lambda I_n) P)$

$= \det P \det(B - \lambda I_n) \det(P^{-1}) = \det(B - \lambda I_n)$

$2P - 3P = P(2I_n - 3I_n)$.

5.3 Diagonalization

$A = P \begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1}$

$A^{10} = P \begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda^{10} \end{bmatrix} P^{-1}$

Def A $n \times n$ is diagonalizable if A is similar to n diagonal matrix.

Thm 5 The diagonalizable Thm.

A $n \times n$ is diagonalizable $\Rightarrow$ A has n linearly indep. eigenvectors.

In fact, $A = PDP^{-1}$ with D diagonal $\Leftrightarrow$ the coln of P are n lin. indep. eigen vectors.

In this case, if $D = \begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda_i \end{bmatrix}$, $P = [v_1 \dots v_n]$, then λ_i is an eigenvalue for v_i.

Proof: "$\Rightarrow$" A is diagonalizable $\Rightarrow \exists D$ diagonal, P invertible s.t.

$A = PDP^{-1}$, $P = [v_1 \dots v_n]$, $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$.

The goal is to prove that $A\vec{v}_i = \lambda_i v_i$

$AP = PDP^{-1}P = PD$

$A[v_1 \dots v_n] = [v_1 \dots v_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$[Av_1 \cdots Av_n] = [\lambda_1 v_1 \cdots \lambda_n v_n] \Rightarrow Av_i = \lambda_i v_i, \ \forall i$

$\Rightarrow v_1, \dots, v_n$ are eigenvectors for A. $P = [v_1, \dots, v_n]$ invertible $\Rightarrow \{v_1, \dots, v_n\}$ is lin. indep $\Rightarrow \{v_1, \dots, v_n\}$ lin indep. eigenvalues.

"$\Leftarrow$" Denote by $\{v_1, \dots, v_n\}$ n linear indep. eigenvectors

$\Rightarrow \exists \lambda_i$ s.t. $Av_i = \lambda_i v_i$.

Define $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, $P = [v_1, \dots, v_n]$

The goal is to show $A = PDP^{-1}$

$$AP = A[v_1 \cdots v_n] = [Av_1 \cdots Av_n] = [\lambda_1 v_1 \cdots \lambda_n v_n] = [v_1 \cdots v_n]\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

$\Rightarrow AP = PD \Rightarrow A = PDP^{-1}$.

Ex Diagonalize if possible

i) $= \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ ii) $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Sol i) Step 1 Find the eigenvalues of A. Find roots of $\det(A - \lambda I_3) = 0$.

$\det(A - \lambda I_3) = -\lambda^3 - \lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2$

$\Rightarrow$ 1 and -2 are eigenvalues of A.

ii) Step 2 Find a basis for each eigenspace (if we find 3 lin. indep vectors $\Rightarrow$ A is diagonal if not, A is not diagonalizable)

Eigenspace of 1. $\mathrm{Null}(A - I_3)$

$$A - I_3 = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{cases} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \\ x_3 \text{ free} \end{cases} \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathrm{Null}(A - 1I_3)$.

find a basis for $\mathrm{Nul}(A-(-2)I_3)$

$A+2I_3 = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = -x_2 - x_3$, x_2, x_3 free $\quad x = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 [$

$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathrm{Nul}(A-2I_3)$, $\exists$ 3 eigenvectors which are lin indep.

Thm 5 $\Rightarrow$ A is diagonalizable.

<u>Step 3</u> Construct $A = PDP^{-1}$

$D = \begin{bmatrix} 1 & & \\ & -2 & \\ & & -2 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow A = PDP^{-1}$

ii) $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

6.1 Inner products, length and orthogonality.

The distance between u and v in $\mathbb{R}^n$

is $\mathrm{dist}(u,v) \overset{\text{def}}{=} \|u-v\| = \sqrt{(u-v)\cdot(u-v)}$

$= \sqrt{(u_1-v_1)^2 + \cdots (u_n-v_n)^2}$

<u>Thm 2</u> The Pythagorean Thm

u, v are orthogonal $\iff \|u+v\|^2 = \|u\|^2 + \|v\|^2$

$\iff \|u-v\|^2 = \|u\|^2 + \|v\|^2$

<u>Thm3</u> A $m\times n$ $\Rightarrow (\mathrm{Row}\, A)^\perp = \mathrm{Nul}\, A$,

$(\mathrm{Col}\, A)^\perp = \mathrm{Nul}\, A^T$

Def : $\{u_1, \ldots, u_p\} \subseteq \mathbb{R}^n$ is an orthogonal set if $u_i \cdot u_j = 0$, for all $i \neq j$

ex. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$ is orthogonal set.

Thm 4 $S = \{u_1, \ldots u_p\}$ orthogonal set of nonzero vectors $\Rightarrow$ S is linearly independent.

Pf $c_1 u_1 + \cdots + c_p u_p = 0$

$c_1 u_1 \cdot u_1 + c_2 u_2 \cdot u_1 + \cdots + c_p u_p \cdot u_1 = 0$