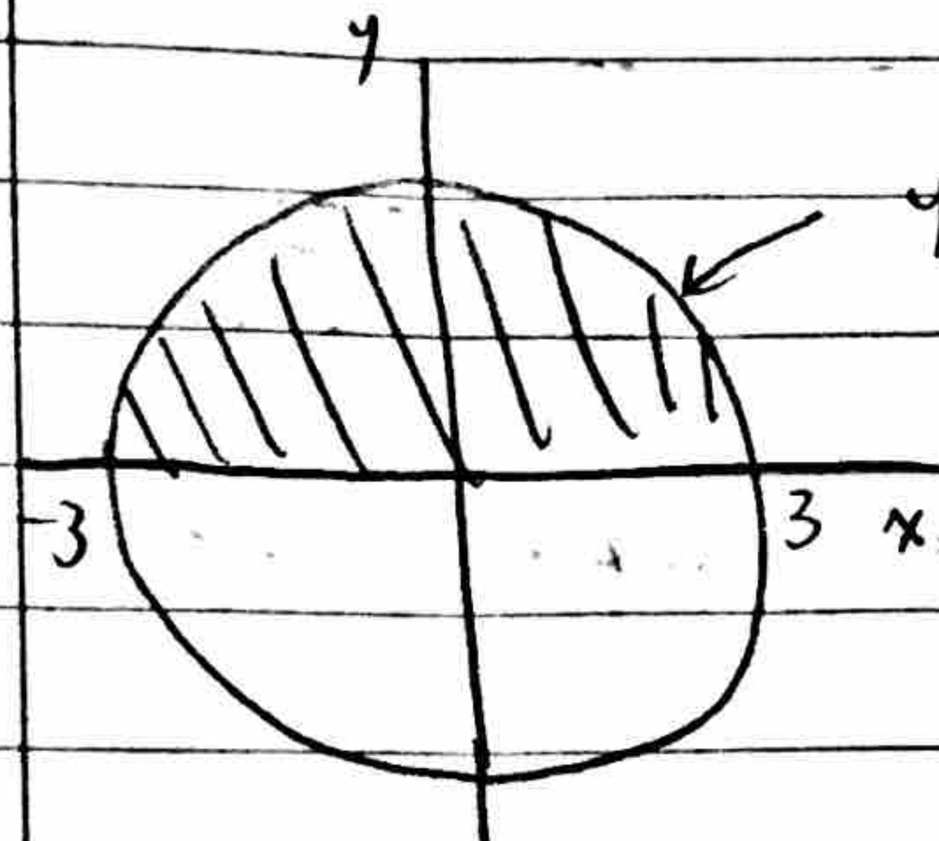
Math 20B Lecture 8 7/18

Professor Um
SIL: Eason

7.3 Trigonometric Substitution

Area of a circle whose radius is 3 $= \pi \cdot 3^2$



$y = \sqrt{9-x^2}$

$\sqrt{(x-0)^2+(y-0)^2} = 3$

$x^2 + y^2 = 3^2$

$y^2 = 9 - x^2$

$y = \pm\sqrt{9-x^2}$

Since $y > 0$ $y = \sqrt{9-x^2}$

Area of (circle) $= 4 \times$ Area of (quarter)

$= 4\int_{x=0}^{3} f(x)\,dx$ Set $x = \sin\theta$

$= 4\int_{0}^{3} \sqrt{9-x^2}\,dx$ $dx = 3\cos\theta\,d\theta$

$= 4\int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta}\; 3\cos\theta\,d\theta$ If $x=0$, $0=\sin\theta$, θ

$x=3$ $3=3\sin\theta$

$\sin\theta = 1$

$\theta = \frac{\pi}{2}$

$= 4\int_0^{\frac{\pi}{2}} \sqrt{\underbrace{9(1-\sin^2\theta)}_{}}\; 3\cos\theta\,d\theta$ ($3^2\cos^2\theta$)

cos is positive from 0 to $\frac{\pi}{2}$

$= 4\int_0^{\frac{\pi}{2}} 3|\cos\theta|\; 3\cos\theta\,d\theta$

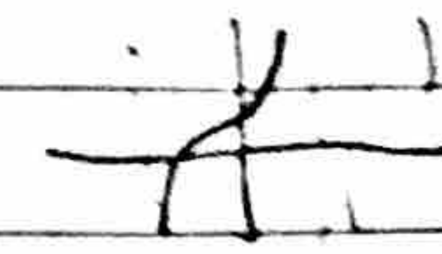
$= 4\cdot3\cdot3\int_{\theta=0}^{\pi/2} \cos\theta\cdot\cos\theta\,d\theta$

$= 36\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta$

$= 18\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{\frac{\pi}{2}}$

$= 18\left(\frac{\pi}{2} + \frac{\sin\pi}{2} - 0\right)$

$= 9\pi = \pi\cdot 3^2$

Expression	Trig Sub	Identities
$\sqrt{a^2-x^2}$	$x=a\sin\theta \quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	$1-\sin^2\theta=\cos^2\theta$ 
$\sqrt{a^2+x^2}$	$x=a\tan\theta \quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$	$1+(\tan\theta)^2=(\sec\theta)^2$ 
$\sqrt{x^2-a^2}$	$x=a\sec\theta \quad 0\le\theta<\frac{\pi}{2}$, $\pi\le\theta<\frac{3\pi}{2}$	$\tan^2\theta=\sec^2\theta-1$ 

To make 1 to 1 correspondence between θ, x

Ex (1) $\displaystyle\int\frac{1}{\sqrt{1-3x^2}}\,dx$

$1-3x^2 \qquad x=\sin\theta$

$1-(\text{var})^2 \Rightarrow \text{var}=\sin\theta$

$1-(\sqrt{3}\,x)^2 \Rightarrow \sqrt{3}\,x=\sin\theta$

$x=\frac{1}{\sqrt{3}}\sin\theta \qquad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$dx=\frac{1}{\sqrt{3}}\cos\theta\,d\theta$

$$=\int\frac{1}{\sqrt{1-\sin^2\theta}}\,\frac{1}{\sqrt{3}}\cos\theta\,d\theta$$

$(-\frac{\pi}{2},\frac{\pi}{2})$

$\cos\theta>0$

$$=\frac{1}{\sqrt{3}}\int\frac{1}{|\cos\theta|}\cos\theta\,d\theta$$

$$=\frac{1}{\sqrt{3}}\int\frac{\cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}}$$

$$=\frac{\theta}{\sqrt{3}}+C$$

$$=\frac{1}{\sqrt{3}}\sin^{-1}(\sqrt{3}\,x)+C$$

(17) $\displaystyle\int \frac{1}{x^2\sqrt{4+x^2}}\,dx$ $\qquad x = 2\tan\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = 2\sec^2\theta\, d\theta$

$$= \int \frac{1}{\cancel{4}_{2}\tan^2\theta\sqrt{4+4\tan^2\theta}}\, \cancel{2}\sec^2\theta\, d\theta$$

$\sec\theta > 0$ if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\tan^2\theta\; 2\underbrace{\sqrt{1+\tan^2\theta}}_{=\sec^2\theta}}\, d\theta$$

$$= \frac{1}{2\cdot 2}\int \frac{\sec^{\cancel{2}}\theta}{\tan^2\theta\; \cancel{2}|\cancel{\sec\theta}|}\, d\theta$$

$(+)$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

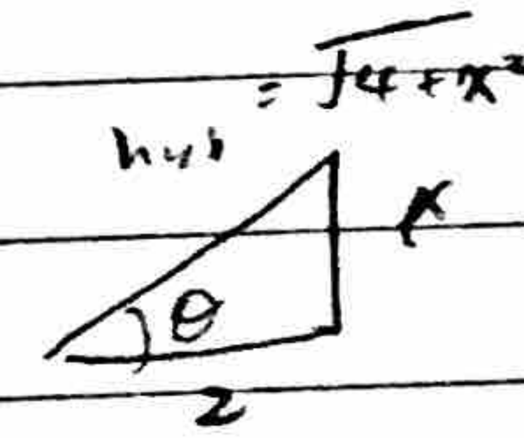
$$= \frac{1}{4}\int \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta} \cdot \frac{1}{\cancel{\cos\theta}}\, d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= \frac{1}{4}\int \frac{1}{u^2}\, du$$



$\frac{x}{2} = \tan\theta$

$= \frac{opp}{adj}$

$$= \frac{1}{4}(-u^{-1}) + C$$

$$= -\frac{1}{4}\,\frac{1}{u} + C$$

$\sin\theta = \frac{opp}{hyp} = \frac{x}{\sqrt{4+x^2}}$

$$= -\frac{1}{4}\,\frac{1}{\sin\theta} + C$$

$$= -\frac{1}{4}\,\frac{\sqrt{4+x^2}}{x} + C$$

$$\int \frac{x}{x^2-1}\,dx \qquad u = x^2-1$$
$$du = 2x\,dx$$

u-sub

$$= \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|x^2-1| + C$$

trig

$$x = \sec\theta$$
$$dx = \sec\theta\tan\theta\,d\theta$$

$$\int \frac{\sec\theta}{\sec^2\theta - 1}\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\sec\theta}{\tan^2\theta}\sec\theta\,\cancel{\tan\theta}\,d\theta$$

$$= \int \frac{\sec^2\theta}{\tan\theta}\,d\theta \qquad u = \tan\theta$$
$$du = \sec^2\theta$$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\tan\theta| + C$$

$$= \ln|\sqrt{x^2-1}| + C$$

Partial Frac

$$\int \frac{x}{(x-1)(x+1)}\,dx$$

$$\frac{x}{(x-1)(x+1)} = \frac{\frac{1}{2}}{x-1} + \frac{1/2}{x+1}$$

$$= \frac{1}{2}\ln x-1 + \frac{1}{2}\ln|x+1| + C$$

$$= \frac{1}{2}\left(\ln|x-1| + \ln|x+1|\right) + C$$

Ex. $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

$x+1 = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$= \int \frac{x}{\sqrt{\underset{+1}{3} - (x^2+2x+1)}}\,dx$

$= \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$

$= \int \frac{2\sin\theta - 1}{\sqrt{2^2 - 2^2\sin^2\theta}}\, 2\cos\theta$

$\vdots$

$= -\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$

$\int \frac{h(x)}{g(x)}\,dx$ $\quad (ax+b)(cx^2+dx+e)$

$\int \frac{A}{ax+b} + \frac{Bx+C}{cx^2+dx+e}$

$\pm(\text{var})^2 + (\text{con})^2$

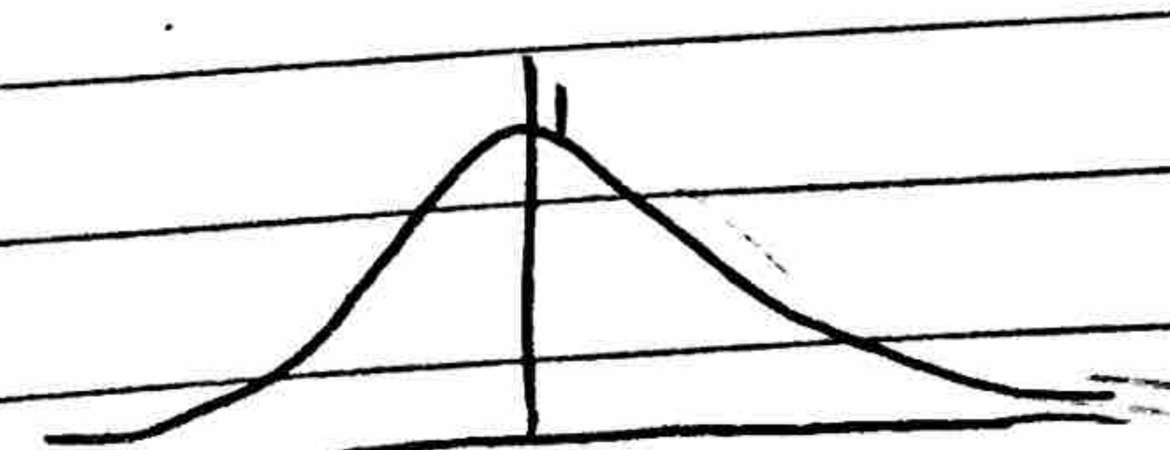
7.7 Improper Integrals

$\int_a^b f(x)\,dx \Rightarrow$ proper integrals

continuous on $[a,b]$

$p(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$

$\int_{-\infty}^{\infty} p(x)\,dx = 1$



Type I Infinite Integrals

$(\int_a^\infty \quad \int_{-\infty}^a$ or $\int_{-\infty}^\infty$

1. $\int_2^t \frac{1}{\ln x} \, dx$

$u = \ln x$

$= \int_{\ln 2}^{\ln t} \frac{1}{u} \, du$

$= \ln|u| + C \Big]_{\ln 2}^{\ln t}$

$= \ln|\ln t| - \ln|\ln 2|$

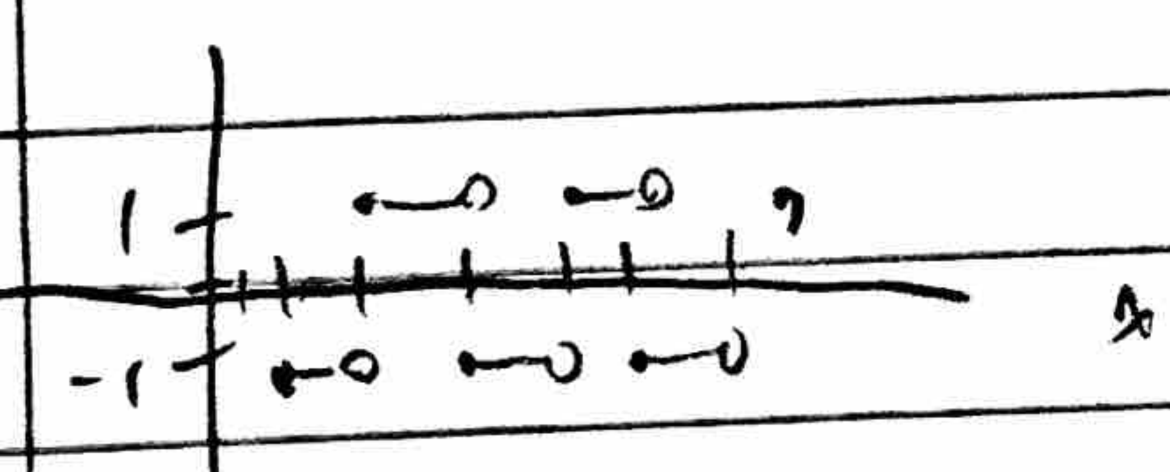
2. $\int_1^t \underset{u}{x} \, \underbrace{e^{-x} \, dx}_{dv}$

$\overset{\text{I.B.P}}{=} -xe^{-x}\Big]_1^t - \int_1^t \frac{e^{-x}}{-1} \, dx$

$= -\frac{t}{e^t} + \frac{1}{e} + \Big[-e^{-x}\Big]_1^t$

$= -\frac{t}{e^t} + \frac{1}{e} - e^{-t} + e^{-1}$

3. $\int_2^t f(x) \, dx$ where



$\int_2^{\text{even}} f(x) \, dx = 0$

$\int_2^{\text{odd}} f(x) \, dx = -1$

If $t = 3$, $A(3) = -1$

$t = 4$, $A(4) = 0$

$t = 5$, $A(5) = -1$