Professor Orimbe
SIL: Eason

Math 18 lecture 3 8/15

Def: $v_1, \dots, v_p \in \mathbb{R}^n$, $\{v_1, \dots, v_p\}$ is linearly dependent if there exist $c_1, \dots, c_p \in \mathbb{R}$, not all zero, s.t. $c_1v_1 + \dots + c_pv_p = 0$.
i.e. the matrix equation $[v_1, \dots, v_p]\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = 0$ has non-trivial solution.

Ex 1) $v_1, v_2 \in \mathbb{R}^n \Rightarrow \{v_1, v_2, v_1 + 3v_2\}$ is linear dependent

$(-1)v_1 + (-3)v_2 + (v_1 + 3v_2) = 0$

2) $v_1, v_2, v_3, v_4 \in \mathbb{R}^n$, $\{v_1\ v_2\ v_3\ v_4\}$ is linearly independent
$\Rightarrow \{v_1, v_2, v_3\}$ is lin. independent.

Sol: Let c_1, c_2, c_3 scalars s.t. $c_1v_1 + c_2v_2 + c_3v_3 = 0$.
we want $c_1 = c_2 = c_3 = 0$

$c_1v_1 + c_2v_2 + c_3v_3 + 0v_4 = 0$ $\Big\}$ $- \{v_1, v_2, v_3, v_4\}$ linear indep.
$c_1 = c_2 = c_3 = 0 = 0$

3) $\{v_1, v_2\}$ is linearly dependent $\Longleftrightarrow$ one of the vectors is a multiple of the other

Sol'n
"$\Leftarrow$" By assumptions, $\exists\ \alpha \in \mathbb{R}$ st $v_1 = \alpha v_2$ or $v_2 = \alpha v_1$
suppose $v_1 = \alpha v_2 \Rightarrow v_1 + (-\alpha)v_2 = 0 \Rightarrow \{v_1, v_2\}$ is linear dep.
"$\Rightarrow$" $\exists\ c_1, c_2 \in \mathbb{R}$, not both zero s.t. $c_1v_1 + c_2v_2 = 0$

$\left.\begin{array}{r} c_1v_1 = -c_2v_2 \\ \text{suppose } c \neq 0 \end{array}\right\} \Rightarrow v_1 = \left(\frac{-c_2}{c_1}\right)v_2$, Theorem 7 (Characterization of linear dependent sets)

$S = \{v_1, \dots, v_p\}$ is linear dep $\Leftrightarrow$ one of the vectors is a linear combination of the others
$p \geq 2$

Actually, if $v \neq 0$, then S is lin. dep $\Leftrightarrow \exists\ j \in [2, \dots$

Proof: "$\Leftarrow$" Let $j \in \{2, 3, \dots, p\}$ s.t. $v_j \in \text{Span}\{v_1, \dots, v_j\}$.

"$\Rightarrow$" $\exists\ c_1, \dots, c_{j-1} \in \mathbb{R}$ s.t. $v_j = c_1 v_1 + \dots c_{j-1} v_{j-1}$.

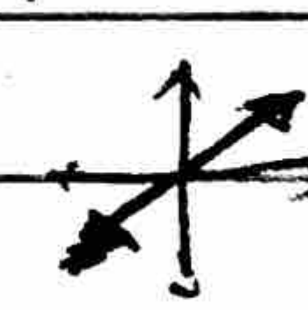
$\Rightarrow c_1 v_1 + \dots + c_{j-1} v_{j-1} + -1(v_j) = 0.$

$c_1 v_1 + \dots c_{j-1} v_{j-1} + (-1) v_j + 0 \cdot v_j + \dots 0 \cdot v_p = 0$

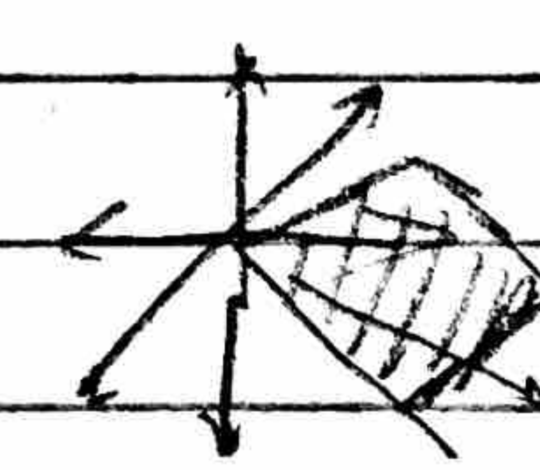
$\Rightarrow \{v_1, \dots, v_p\}$ is lin dependent

"$\Rightarrow$" exercise.

Ex: 1) $\{v_1, v_2\}$ lin. dep. $\Rightarrow$ v_1 & v_2 are on the same line.



2) $\{v_1, v_2, v_3\}$ linear dep $\Leftrightarrow$ $v_2 \in \text{Span}\{v_1\}$ or $v_3 \in \{v_1, v_2\}$
$(v_1 \neq 0)$ $\mathbb{R}^3$
$j=2$ $j=3$



Thm 8 $v_1, \dots, v_p \in \mathbb{R}^n$. $p > n \Rightarrow \{v_1, \dots, v_p\}$ is linear dependent.

proof: $A = [v_1 \dots v_p]$

Want to show $Ax = 0$ has non trivial soln

$A = n \times p = [\quad]$.

A has maximum of n pivots, and p variable $\Rightarrow$
we have free variables. $\Rightarrow Ax = 0$ has non-trivial soln.

$\left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}, \begin{bmatrix} \sqrt{2} \\ 1.8 \end{bmatrix}, \begin{bmatrix} 8 \\ -9 \end{bmatrix} \}$ is linear dep $(3 > 2)$

Remark: If $p \leq n \Rightarrow$ no clear conclusion.

$p = 2$
$n = 3$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$ is linear dependent

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is linear independent.

Theorem 9 $S = \{v_1, \dots, v_p\}$ in $\mathbb{R}^n$, $0 \in S$
$\rightarrow$ S is linear dependent

Proof: Suppose $v_1 = 0$
Take $c_1 = 1, c_2 = 0, \dots, c_p = 0$.
$1 \cdot v_1 + c_2 v_2 + \dots + c_p v_p = 0$. $\}$ S is linear dependent.

Ex. 1) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -7 \\ 8 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ is linear dep

Ex Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$ linear indep?

<u>Intro to linear transformations.</u>

Def: A transformation (or mapping or function) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule which assigns to each vector x in $\mathbb{R}^n$ to a vector $T(x)$ in $\mathbb{R}^m$.

$\mathbb{R}^n$ is the domain of T
$\mathbb{R}^m$ is the codomain of T

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$x \in \mathbb{R}^n$ $T(x) \in \mathbb{R}^m$, is called the image of x

$\{T(x) \mid x \in \mathbb{R}^n\} = \{y \in \mathbb{R}^m \mid$ there exist $x \in \mathbb{R}^n$ s.t. $T(x) = y\}$

is the range of T
(image)

Ex. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 \\ x \\ y \end{bmatrix}$

Def: A transformation T is linear if

(i) $T(u+v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$
(ii) $T(cu) = cT(u)$, for all $u \in \mathbb{R}^n$ $c \in \mathbb{R}$.

$T\ \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Remark:

1) $T(\underset{\in \mathbb{R}^n}{0}) = \underset{\in \mathbb{R}^m}{0}$

$T(0 \cdot u) = 0\, T(u) \Rightarrow T(0) = 0$.

2) $T(cu + dv) = cT(u) + dT(v)$, $\forall\ u, v \in \mathbb{R}^n$.

Proof: $T(cu + dv) = T(cu) + T(dv) = cT(u) + dT(v)$.

3) $T(c_1 v_1 + \dots + c_p v_p) = c_1 T(v_1) + \dots + c_p T(v_p)$.

(Exercise p. 3 & 4).

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x \\ y \end{bmatrix}$

$\Rightarrow$ T is a linear transf

Sol'n $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

want $T(u+v) = T(u) + T(v)$

$T(u+v) = T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ u_1+v_1 \\ v_2+v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \\ \\ \end{bmatrix}$

$= T(u) + T(v)$

(ii) $T(cu) = cT(u)$ for all $c \in \mathbb{R}$.

$T(cu) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ cu_1 \\ cu_2 \end{bmatrix} = c\begin{bmatrix} 0 \\ u_1 \\ u_2 \end{bmatrix} = cT(u).$

Ex. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$ is not a linear transformation

Sol $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$. $\Rightarrow$ T is Not a linear transf

Def: A transformation $T: \mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation if $\exists$ a matrix A $m \times n$ s.t. $T(x) = Ax$, for all $x \in \mathbb{R}^n$.

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^3$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ is a matrix transformation.

Rk If T is a matrix transformation, then T is a linear transf.

(use Thm 3 from the last lecture)

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x) = \begin{bmatrix}1 & 2\\ 2 & 3\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}x_1+2x_2\\ 2x_1+3x_2\\ -x_1\end{bmatrix}$

$u = \begin{bmatrix}1\\ 1\end{bmatrix}$, $v = \begin{bmatrix}-1\\ -1\end{bmatrix}$, $a = \begin{bmatrix}1\\ 2\\ -1\end{bmatrix}$, $b = \begin{bmatrix}1\\ 2\\ 2\end{bmatrix}$

a) Find $T(u)$

$T(u) = \begin{bmatrix}3\\ 5\\ -1\end{bmatrix}$

b) $T(u) + T(v) = \begin{bmatrix}3\\ 5\\ -1\end{bmatrix} + \begin{bmatrix}-3\\ -5\\ 1\end{bmatrix} = 0$

c) Find $x \in \mathbb{R}^2$ s.t. $T(x) = a$

$\begin{bmatrix}1 & 2\\ 2 & 3\\ -1 & 0\end{bmatrix} x = \begin{bmatrix}1\\ 2\\ -1\end{bmatrix}$

$x_1 + 2x_2 = 1$
$x_2 = 0$
$x_1 = 1$

$\begin{bmatrix}1 & 2 & 1\\ 2 & 3 & 2\\ -1 & 0 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 1\\ 0 & -1 & 0\\ 0 & 2 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$

$x = \begin{bmatrix}1\\ 0\end{bmatrix}$

d) Is x unique? yes

e) Is b in the range of T?

Solve $T(x) = b$

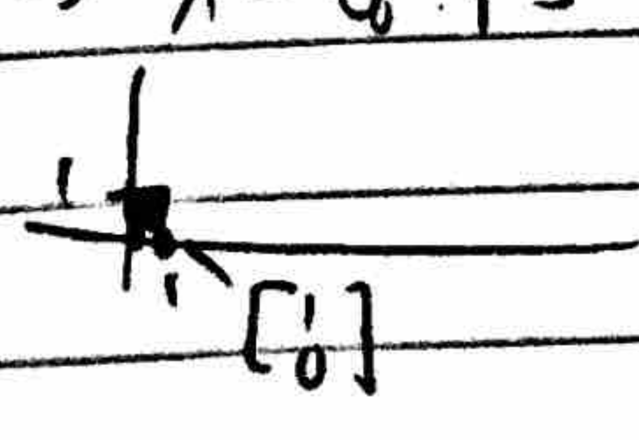
$\begin{bmatrix}1 & 2 & 1\\ 2 & 3 & 2\\ -1 & 0 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 1\\ 0 & -1 & 0\\ 0 & 0 & 3\end{bmatrix}$

Ex 1) $A = \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x) = Ax = \begin{bmatrix}0\\ x_2\\ x_3\end{bmatrix}$

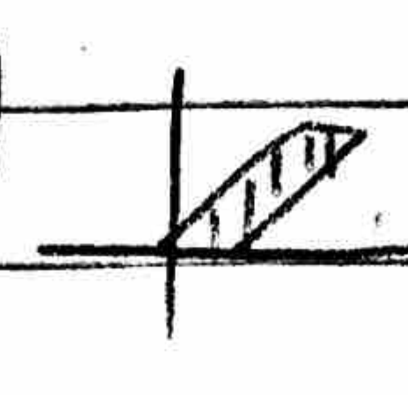
$x = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

projection onto the plane $x_2 x_3$.

2) $A = \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$, $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x) = Ax = \begin{bmatrix}x_1 + 2x_2\\ x_2\end{bmatrix}$



Shear Transf. $x = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$



$T\left(\begin{bmatrix}0\\ 1\end{bmatrix}\right) = \begin{bmatrix}2\\ 1\end{bmatrix}$

3) $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x) = rx$, $r > 0$

T — contraction if $r \le 1$

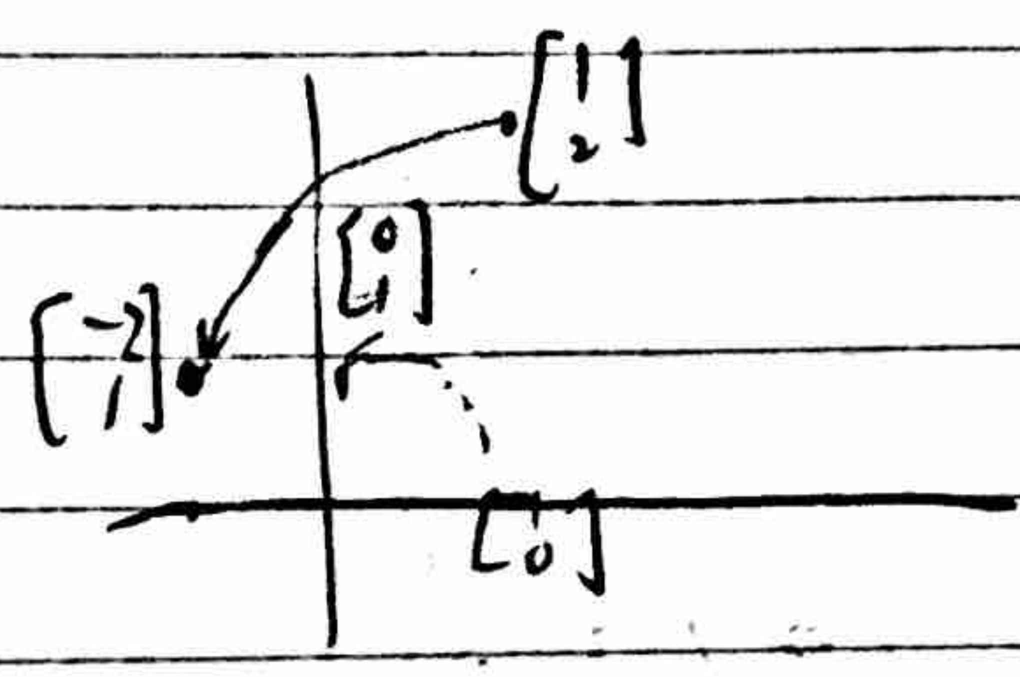
— dilation if $r > 1$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

4) $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x$ $\quad T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

rotation with $90°$ $\quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$



The matrix of a linear transformation

$$T: \mathbb{R}^2 \to \mathbb{R}^2, \; T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix}\right) =$$

$$= T\left(\begin{bmatrix} x_1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ x_2 \end{bmatrix}\right)$$

$$= x_1 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

Theorem 10 $T: \mathbb{R}^n \to \mathbb{R}^m$ a linear transformation

$\Rightarrow$ there exist and is unique a matrix A s.t. $T(x) = Ax$.

In fact, A is $m \times n$ and $A = [T(e_1) \; \cdots \; T(e_n)]$

where e_j is just $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$ — jth term

Proof: $T(x) = T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{bmatrix} + \dots + \begin{bmatrix} 0 \\ \vdots \\ x_n \end{bmatrix}\right) =$

$= T(x_1 e_1 + x_2 e_2, \dots, + x_n e_n)$

$= x_1 T(e_1) + \dots + x_n T(e_n)$

$= T(e_1)\ T(e_2) \dots T(e_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Ax.$

Suppose $T(x) = Ax = Bx$, for all $x \in \mathbb{R}^n$.

$A = [a_1 \dots a_n]$

$B = [b_1 \dots b_n]$

$\Rightarrow x_1 a_1 + \dots + x_n a_n = x_1 b_1 + \dots + x_n b_n,\ \forall x_1, \dots, x_n \in \mathbb{R}.$

Set $x_1 = 1,\ x_2 = x_3 = \dots = x_n = 0$

$\Rightarrow a_1 = b_1 \Rightarrow a_j = b_j$, for all $j \Rightarrow A = B$.

Ex. $T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x) = (x_1 + x_2, x_1 - x_2, 2x_3)$.

Find the standard matrix of T.

Sol $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} x.$

$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

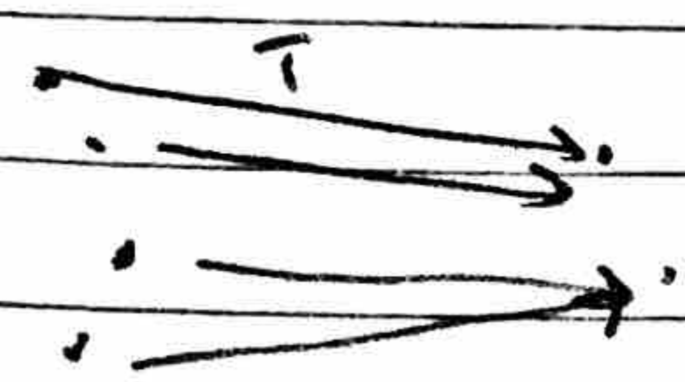
Def: 1) A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>onto</u> $\mathbb{R}^m$ if each $b \in \mathbb{R}^m$ is in the image of at least one $x \in \mathbb{R}^n$.

(i.e. the codomain of T = the range of T). $T(x) = b$ is consistent

2) A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is one to one if each $b \in \mathbb{R}^m$ is the image of at most one $x \in \mathbb{R}^n$.

($T(x) = b$ has maximum 1 sol'n).

1) $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^3$ is not onto.



T is not one-to-one

2) $T: \mathbb{R}^4 \to \mathbb{R}^3$ $T(x) = Ax$, $A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Is T onto? one-to-one?

Soln A is in echelon form.

A has a pivot in every row,
T is onto.

Thm 4 TFAE
a) $Ax = b$ is consistent for all b
b) A has a pivot in every row
c) T is onto.

3 pivots, 4 variables $\Rightarrow$ one free variable $\Rightarrow$ $Ax = 0$ has infinitely many sol's. $\Rightarrow$ 0 is the range infinitely many points in $\mathbb{R}^4$.
$\Rightarrow$ T is not one-to-one.

Theorem 11: $T: \mathbb{R}^n \to \mathbb{R}^m$ linear transformation $\Rightarrow$ T is one to one $\Leftrightarrow$ $T(x) = 0$ has a unique solution.

Proof "$\Rightarrow$" Take $x \in \mathbb{R}^n$ $\left.\begin{matrix} T(x) = 0. \\ T(0) = 0 \end{matrix}\right\} \xRightarrow{1\text{ to }1} x = 0.$

"$\Leftarrow$" Suppose T is not one to one $\Rightarrow \exists$ $u \neq v$ s.t. $T(u) = T(v)$.
$\Rightarrow T(u-v) = 0 \Rightarrow u - v = 0 \Rightarrow u = v$ $\rightarrow\!\leftarrow$.