Math 18 Lecture 6 8/24/2017

4.2 Null spaces, columns space and linear transformation.

<u>Thm1</u>: V vector space, $v_1, \ldots, v_p \in V \Rightarrow$ Span $\{v_1, \ldots, v_p\}$ is a subspace.

Def: A $m \times n$, $\text{Nul}(A) = \{ x \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \}$ is the null space of A.

ex. $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$

$$\text{Nul}(A) = \{ x \in \mathbb{R}^3 \mid \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \}$$

$$= \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x_1 - x_2 + 2x_3 = 0, \\ 3x_1 + 2x_2 - x_3 = 0 \end{matrix} \right\}$$

<u>Thm 2</u>: A $m \times n \Rightarrow \text{Nul}(A)$ is a subspace of $\mathbb{R}^n$

Proof. WTS i) $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \text{Nul}(A)$

ii) if $u, v \in \text{Nul}(A)$, then $u + v \in \text{Nul}(A)$

iii) if $u \in \text{Nul}(A)$, $\alpha \in \mathbb{R}$, then $\alpha u \in \text{Nul}(A)$.

i) $A0 = 0 \Rightarrow 0 \in \text{Nul}(A)$.

ii) $u \in \text{Nul}(A) \Rightarrow Au = 0$
$v \in \text{Nul}(A) \Rightarrow Av = 0$ $\quad A(u+v) = 0 \Rightarrow u + v \in \text{Nul}(A)$

iii) $\left.\begin{matrix} u \in \text{Nul}(A) \\ \alpha \in \mathbb{R} \end{matrix}\right\} \Rightarrow A(\alpha u) = \alpha(Au) = \alpha 0 = 0$

Thus $\text{Nul}(A)$ is a subspace.

Ex. $H = \{(a,b,c,d) \in \mathbb{R}^4 \mid a+3b+d=0,\ a-c+d=0\}$.
Prove that H is a subspace of $\mathbb{R}^4$.

Sol 1: $H = \mathrm{Nul}\left(\begin{bmatrix} 1 & 3 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix}\right)$ is a subspace using thm 2.

Sol 2: Want to find $x \in \mathbb{R}^4$ s.t. $Ax = 0$

$$A = \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 \end{bmatrix} \underset{R_2 \to R_2 - R_1}{\sim} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & -3 & -1 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 + 3x_2 + x_4 = 0 \\ -3x_2 - x_3 = 0 \\ x_3, x_4 \text{ free} \end{cases} \Rightarrow x_2 = -\frac{x_3}{3} \Rightarrow x_1 = x_3 - x_4.$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - x_4 \\ x_3/3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1/3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \overset{\text{Thm 1}}{\Longrightarrow} \mathrm{Nul}(A) \text{ is a subspace}$$

Def A $m \times n$, $A = [a_1, \ldots, a_n]$.
$\mathrm{Col}(A) = \mathrm{Span}\{a_1, \ldots, a_n\}$ is the column space of A.

Thm 3 A $m \times n \Rightarrow \mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$

Proof: Because Thm 1.

Ex: Find A_1, A_2, A_3 three distinct matrices, s.t. $\mathrm{Col}(A_1) = \mathrm{Col}(A_2) = \mathrm{Col}(A_3)$
$= \left\{ \begin{bmatrix} 2a-b \\ 3a \\ a+b \end{bmatrix} \mid a, b \in \mathbb{R} \right\} = H.$

Sol $\begin{bmatrix} 2a-b \\ 3a \\ a+b \end{bmatrix} = a\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow H = \text{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$A_1 = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \text{col}(A_1) = H$

$A_2 = \begin{bmatrix} -2 & -1 \\ -3 & 0 \\ -1 & 1 \end{bmatrix}$, because $\text{Span}\{u, v\} = \text{Span}\{-u, 7v\}$.

$A_3 = \begin{bmatrix} 2 & -1 & 0 & 2 \\ 3 & 0 & 0 & 5 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

Remark: A $m \times n$

$\Rightarrow$ i) $\text{Nul}(A) = 0 \Leftrightarrow A\vec{x} = \vec{0}$ has only the trivial solution

ii) $\text{Col}(A) = \mathbb{R}^m \Leftrightarrow A\vec{x} = \vec{b}$ is consistent for all $b \in \mathbb{R}^m$

Ex. $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

A 3×4 $\quad$ Null (A) is a subspace of $\mathbb{R}^4$

Col (A) is a subspace o $\mathbb{R}^3$

$Au = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \neq 0 \Rightarrow u \notin \text{Nul}(A)$

$v \in \text{Col}(A) \Leftrightarrow Ax = v$ is consistent

check p. 206 Nul A vs Col A.

Kernel and range of linear transformation.

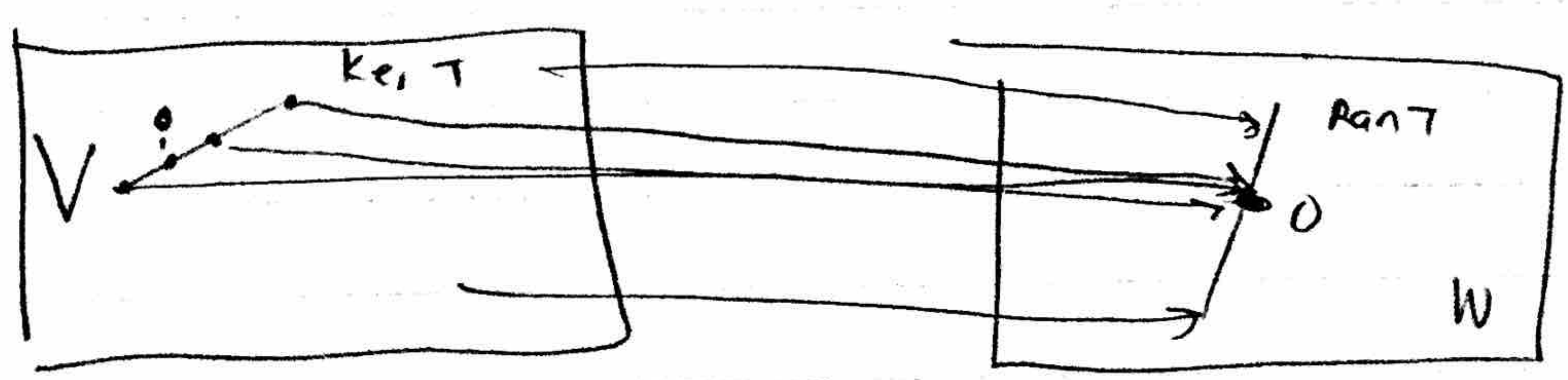
<u>Remark</u> $T: \mathbb{R}^n \to \mathbb{R}^m$ lin. tran
$T(x) = Ax$, A $m \times n$ $\Rightarrow$ Ker T = N
Ran T = Col A.

<u>Def</u> V, W are vector spaces
$T: V \to W$ is a liner transf. if
i) $T(u+v) = T(u) + T(v)$
ii) $T(cu) = cT(u)$, for all $c \in \mathbb{R}$, $u, v \in V$.

Remark 1) If $V = \mathbb{R}^n$, $W = \mathbb{R}^m \Rightarrow T$ is a liner trans. between the vector spaces, $\mathbb{R}^n$ and $\mathbb{R}^m$ iff T is a linear trans. in the sense defined before
2) $T(0) = 0$.

Def: $\operatorname{Ker} T = \{u \in V \mid T(u) = 0\}$ the <u>kernel of T</u>.
$\operatorname{Ran} T = \{y \in W \mid y = T(u)$ for some $u \in V.\}$
the range of T

Exc i) Ker T is a subspace of V
ii) Ran T is subspace of W.



$T: \mathbb{R}^n \to \mathbb{R}^m$

<u>Ex</u> $V = \{f: (0,1) \to \mathbb{R} \mid f$ differentiable & f' continuous$\}$.

$W = \{f: (0,1) \to \mathbb{R} \mid f$ cont.$\}$.

$(V \neq \mathbb{R}^n)$ $T: V \to W$, $T(f) = f'$ $\Rightarrow T$ is a linear f.

$\operatorname{Ker}(T) = \{f \in V \mid T(f) = 0\} = \{f \in V \mid f' = 0\}$.
$= \{f \in V \mid f$ is constant$\}$ $(\cong \mathbb{R})$.

$\operatorname{Ran}(T) = W$