Math 20B lecture 16 8/2

Professor Van
SIL Eason

A power series centered at c

$$a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots = \sum_{n=0}^{\infty} C_n(x-c)^n$$

Ex. Find a power series centered at 0 of

$\frac{1}{(1-x)^2}$

$$\left(\frac{1}{1-x}\right)' = (-1)(1-x^{-2})\cdot(-1) = \frac{1}{(1-x)^2}$$

$$\frac{1}{1-x} = \left(\frac{1}{1-x}\right)' \underset{\text{if } |x|<1}{=} (1+x+x^2+x^3+x^4 \ldots)' \text{ or } \left(=\sum_{n=0}^{\infty} x^n\right)'$$

10.7 Taylor Series

Assume $f(x)$ has a power series expansion at c

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

$$= a_0 + a_1(x-c)^1 + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

Plug in $x=c$

$$f(c) = a_0 + 0 + 0 + \cdots, \; a_0 = f(c)$$

Differentiate ①

$$f'(x) = a_1 + 2a_2(x-c)^1 + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots$$

If $x=c$,

$$f'(c) = a_1 + 0 + 0 \cdots, \; a_1 = f'(c)$$

Diff ②

$$f''(x) = 2a_2 + 3\cdot 2a_3(x-3)^1 + 4\cdot 3a_4(x-c)^2 + \cdots$$

If $x=c$;

$$f''(c) = 2a_2 + \cdots 0.$$

$$a_2 = \frac{f''(c)}{2} \Rightarrow a_n = \frac{f^{(n)}(c)}{n!}$$

<Taylor Series Expansion>

If $f(x)$ is represented by a power series centered at $x=c$ in $|x-c| < R$ then that power series is the Taylor series:

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

If $c = 0$, then $T(x)$ is called Maclaurin Series

$$T(x) = f(0) + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Ex. Find the Maclaurin Series $f(x) = \sin x$ and interval of conv.

$f(x) = \sin x \quad = f^{(4)}(x) = f^{(8)}(x)$

$f'(x) \; \cos x \quad = f^{(5)}(x)$

$f''(x) = -\sin x = f^{(6)}(x)$

$f'''(x) = -\cos x = f^{(7)}(x)$

$f(0) = \sin 0 = 0 = f^{(4k)}(x), \; k \in \mathbb{Z}^+$

$f'(0) = \cos 0 = 1 = f^{(4k+1)}(x), \; k \in \mathbb{Z}^+$

$f''(0) = -\sin 0 = 0 = f^{(4k+2)}(x), \; k \in \mathbb{Z}^+$

$f'''(0) = -\cos 0 = -1 = f^{(4k+3)}(x), \; k \in \mathbb{Z}^+$

$$T(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 \dots$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Find Interval of convergence with ratio test.

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+3)} \right|$$

$= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = 0$. Since $\rho = 0 < 1$, Ratio Test says the series converges for any $x$.

Interval $(-\infty, \infty)$.

$R = \infty$

Taylor series

$$T(x) = f(c) + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 \ldots$$

Ex. express $\int_0^1 \sin(x^2)\,dx$ as an infinite series.

$f'(x) = \cos(x^2) \cdot 2x$

$f''(x) =$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} - \frac{x^4}{4} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\int_0^1 \sin(x^2)\,dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^2)^{2n+1}\,dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{4n+3}}{4n+3} \Big]_0^1$$

$$= \frac{x^3}{3} - \frac{1}{3!} \cdot \frac{x^7}{7} + \frac{1}{5!} \cdot \frac{x^{11}}{11} - \frac{1}{7!} \cdot \frac{x^{15}}{15}$$

Ex. Find value of

$$1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \ldots$$

$A = e - 1.$

Ex. $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!}$ $= \sin \pi$

$= 0.$