Professor Um
S2C. Eason

Math 20B Lecture 10 7/20

$\int_1^\infty \frac{1}{x^p}\,dx$ if $p \le 1$ $\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \text{div}$

if $p > 1$ $= \frac{1}{p-1}$ conv

Comparisin Test

If $0 \le g(x) \le f(x)$ for $x \ge a$

$$\int_a^\infty g(x) \le \int_a^\infty f(x)\,dx$$

If $\int_a^\infty g(x)\,dx = \text{div}$, then $\int_a^\infty f(x)\,dx = \text{div}$

If $\int_a^\infty f(x)\,dx$ conv, then $\int_a^\infty g(x)\,dx = \text{conv}$.

Ex. $\int_1^\infty \frac{2+e^{-x}}{\sqrt{x}}\,dx$ conv. or div. ?

$$\underset{f(x)}{\frac{2+e^{-x}}{\sqrt{x}}} \ge \underset{g(x)}{\frac{2}{\sqrt{x}}} \text{ or } \frac{e^{-x}}{\sqrt{x}}$$

$\int_1^\infty \frac{2}{\sqrt{x}}\,dx$, $p = \frac{1}{2} < 1$, diverges

By comparison test, $\int_1^\infty \frac{2+e^{-x}}{\sqrt{x}}\,dx = \text{diverges}$.

10.1 Sequences

Def $a_1, a_2, a_3, \ldots a_n$

A sequence is a list of numbers written in some special order.

Ex. Find a formula for a_n of

$-\frac{1}{2}\ln\left(\frac{5}{7}\right), \frac{1}{4}\ln\left(\frac{5}{14}\right), -\frac{1}{8}\ln\left(\frac{5}{21}\right), \frac{1}{16}\ln\left(\frac{5}{28}\right), \ldots \ldots a_n$

$$a_n = \frac{(-1)^n}{\boxed{2^n}}\ln\left(\frac{5}{\boxed{7n}}\right) = \left(\frac{-1}{2}\right)^n \ln\left(\frac{5}{7n}\right)$$

Fibonacci Sequence

1, 1, 2, 3, 5, 8, 13, 21, 34 ...

$$a_n + a_{n+1} = a_{n+2}, \quad n \geq 1$$

$a_1 = 1$
$a_2 = 1$

$$\varphi = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.618034 \text{ (Golden Ratio)}$$

$$a_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$$

Def: A seq $\{a_n\}$ converges to L (a number) (i.e $\lim_{n\to\infty} a_n = L$)
if $|a_n - L|$ can be made arbitrarily for any sufficiently large n.

$a_n = \frac{1}{n}$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\{\frac{1}{n}\}$ converges to 0
$\{\frac{1}{n^{\oplus}}\}$ " "
$\{n^{\oplus}\} \to \infty$

If no limit exists we say $\{a_n\}$ diverges

$\{a_n\} = -1, 1, -1, 1, -1, 1 \to$ diverges

$\{a_n = n^2\}$ div. to ∞

$1, 4, 9, 16 \to \infty$

Actually a sequence is a function from positive integers to real #

$$a_n = \{f(n)\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n)$$

real # $\downarrow$ integers $\downarrow$

Ex. $a_n = \frac{1}{n} = f(n)$ $\qquad \lim_{x\to\infty} f(x) = \lim_{n\to\infty} f(n)$

Limit Laws for sequences

Assume $\{a_n\}, \{b_n\}$: conv.

(i) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$ $\qquad a_1\ a_2\ a_3\ a_4 \dots \to 5$

$b_1\ b_2\ b_3 \dots \to 2$

(ii) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ for $\lim_{n\to\infty} b_n \neq 0$. $\qquad a_n + b_n \to 7$.

const

(iii) $\lim_{n\to\infty} (c\, a_n) = c \lim_{n\to\infty} a_n$.

(iv)

If function is continuous $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$

Ex. (i) $\lim_{n\to\infty} n\left(\sqrt{n^2+1} - n\right)$

$$= \lim_{n\to\infty} \frac{n\left(\sqrt{n^2+1} - n\right)\left(\sqrt{n^2+1} + n\right)}{\sqrt{n^2+1} + n}$$

$$= \lim_{n\to\infty} \frac{n\left(\cancel{n^2}+1 - \cancel{n^2}\right)}{\sqrt{n^2+1} + n}$$

divide both top & bottom by biggest term in denominators

$$= \lim_{n\to\infty} \frac{n/n}{\sqrt{\frac{n^2+1}{n^2}} + n/n}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1} + 1}$$

$$= \frac{1}{2}$$

(i) $\lim_{n\to\infty}\left(\ln(n^2+5) - \ln(2n^2-1)\right)$

$= \lim_{n\to\infty} \ln\left(\frac{n^2+5}{2n^2-1}\right)$

$= \ln\left(\lim_{n\to\infty} \frac{n^2/n^2+5/n^2}{2n^2/n^2-1/n^2}\right)$

$= \ln \frac{1}{2}$

The squeeze theorem

If $b_n \le a_n \le c_n$ for $n > M$ ← const

and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$

then $\lim_{n\to\infty} a_n = L$,

Ex. $\lim_{n\to\infty} \frac{3^n}{n!}$ $\qquad n! \overset{def}{=} 1\cdot 2\cdot 3 \cdots n$

$0! = 1$

$$\boxed{0} \le a_n = \frac{3^n}{n!} = \frac{3\cdot 3\cdot 3\cdot 3\cdot 3\cdots 3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots n} \le \boxed{\frac{9}{2}\cdot 1\cdot \frac{3}{n}}$$

as $n\to\infty$ $\rightarrow L=0 \leftarrow$

So by squeeze thm, $\lim_{n\to\infty} \frac{3^n}{n!} = 0$

Geometric Sequences.

$$3,\ \frac{3}{2},\ \frac{3}{4},\ \frac{3}{8},\ \frac{3}{16},\ \frac{3}{32}$$

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

$r = \frac{1}{2}$

$$a_n = 3\left(\frac{1}{2}\right)^{n-1}$$

$a_n = a \cdot r^n$ (n positive integers)

$$\lim_{n\to\infty} r^n \begin{cases} 0<r<1 & r^n \to 0 \\ r=1 & r^n \to 1^n = 1 \\ r>1 & r^n \to \infty \end{cases}$$

$r = -1$ $\quad \lim_{n\to\infty}(-1)^n = \text{div.}$

$r < -1 \quad r^n \to \text{div}$

$-1<r<0 \quad r^n \to 0$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} ar^n = \begin{cases} 0 & |r|<1 \\ 1 & r=1 \\ \text{div} & r=-1 \\ \text{div} & |r|>1. \end{cases}$$

Ex. Let $a_n = \dfrac{2^n - e^n}{7+4^n}$ Is $\{a_n\}$ conv or div?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{2^n}{4^n} - e^n/4^n}{7/4^n + 4^n/4^n} = \frac{(2/4)^n - (e/4)^n}{7\left(\frac{1}{4}\right)^n + 1} = \frac{0}{1} = 0$$