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Math 18 Lecture 9 9/5/2017

Thm 3 $A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ $n \times n$

(i) $\det \begin{bmatrix} a_1 \\ a_i + \alpha a_j \\ a_n \end{bmatrix} = \det A$ for all $\alpha \in \mathbb{R}$, $i \neq j$

(ii) $\det \begin{bmatrix} a_1 \\ a_i \\ \vdots \\ a_j \\ a_n \end{bmatrix} = -\det \begin{bmatrix} a_1 \\ a_j \\ \vdots \\ a_i \\ a_n \end{bmatrix}$

(iii) $\det \begin{bmatrix} a_1 \\ \alpha a_i \\ \vdots \\ a_n \end{bmatrix} = \alpha \det A.$

Remark: Since $\det A = \det A^T$ (Thm 5), then Thm 3 is also true for column operations

Ex: $A = \begin{bmatrix} 3 & 2 & 1 \\ 7 & 3 & 3 \\ 6 & 4 & 1 \end{bmatrix}$ Compute $\det A$.

Soln: $\det A \underset{\substack{c_1 - 3c_3 \\ c_2 - 2c_1}}{=} \det \begin{vmatrix} 0 & 0 & 1 \\ -6 & -3 & 3 \\ 3 & 2 & 1 \end{vmatrix} = 3 \det \begin{vmatrix} 0 & 0 & 1 \\ -2 & -1 & 1 \\ 3 & 2 & 1 \end{vmatrix}$

$= 3 \cdot 1(-1)^{1+3} \det \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix} = 3$

Remark: 1) A $n \times n$, $\alpha \in \mathbb{R} \Rightarrow \det \alpha A = \alpha \det A.$
2) A invertible $\Rightarrow \det(A^{-1}) = (\det A)^{-1}$

3.3. Cramer's rule, volume

A $n \times n$, $b \in \mathbb{R}^n$, $A_i(b) = [a_1 \ldots \overset{a_n\ i^{th}\ comp}{b} \ldots a_n]$
$= [a_1 \ldots a_n]$

Thm 7 (Cramer's Rule)

A invertible $n \times n$ $\Rightarrow$ for all $b \in \mathbb{R}^n$, there exits and is unique $x \in \mathbb{R}^n$ s.t. $A\vec{x} = b$, & if $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then $x_i = \frac{\det A_i(b)}{\det A}$, $\forall i$.

Proof: $x = A^{-1}b$ is the unique sol.

$A \cdot [e_1 \dots x, \dots e_n] = [Ae_1 \dots Ax \dots Ae_n] = [a_1 \dots b \dots a_n] = A_i(b)$

$\det A \cdot \det[e_1 \dots x \dots e_n] = \det A_i(b) = \det A \, x_i = \det A_i(b)$

$\Rightarrow x_i = \frac{\det A_i(b)}{\det A}$

A formula for A

$\left.\begin{array}{l} A A^{-1} = I_n \\ A^{-1} = [b_1 \dots b_n] \end{array}\right\} \Rightarrow A[b_1 \dots b_n] = [e_1 \dots e_n]$

$\Rightarrow Ab_j = e_j$ for all j.

$b_j = \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} \xrightarrow{\text{Thm}} b_{ij} = \det(A_i(e_j))$

$A^{-1} = [b_1, \dots b_n] = \frac{1}{\det A} \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix}$, $= \operatorname{adj} A$

Thm 8 If A is invertible, then $A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A$, $\operatorname{adj} A = [c_{ji}]$, the adjoint of A.

Ex. $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 2 & 0 \end{bmatrix}$ Calculate A^{-2} if A is invertible

Sol $\det A = (-1)(-1)^{3+1} \det\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + 2(-1)^{3+2} \det\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = (-1)(-1) + 2(-1)(-1) = 3 \neq 0$.

$\Rightarrow$ A is invertible

$c_{11} = (-1)^{1+1} \det\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} = -2$, $c_{21} = (-1)^{2+1} \det\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = 2$, $c_{31} = \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$

$c_{12} = (-1)^{1+2} \det\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = 1$, $c_{22} = (-1)^{2+2} \det\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = 1$, $c_{32} = -\det\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = 1$

$c_{13} = (-1)^{1+3} \det\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} =$ $c_{23} = (-1)^{2+3} \det\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = -2$ $c_{33} =$ 1

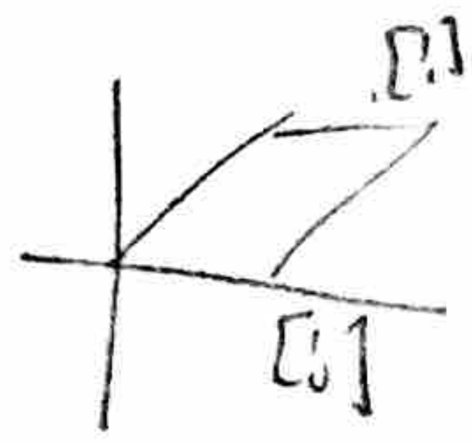
$\operatorname{adj} A = \begin{bmatrix} -2 & -1 & 5 \\ 2 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}^T \Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} -2 & 2 & -1 \\ -1 & 1 & 1 \\ 5 & -2 & \end{bmatrix}$

Thm 9

i) $A_{2\times 2} \Rightarrow$ the area of the parallelogram determined by the colums of A is $|\det A|$.

ii) $A_{3\times 3} \Rightarrow$ the vol of the parallelipped determined by the cols of A is $|\det A|$.

ex. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$



Thm 10

i) $\left.\begin{array}{l} T: \mathbb{R}^2 \to \mathbb{R}^2,\ T(x) = Ax,\ A_{2\times 2} \\ S \text{ parallelogram} \end{array}\right\} \Rightarrow \{\text{Area } T(S) = \det A \,(\text{Area } S)\}$

ii) $\left.\begin{array}{l} T: \mathbb{R}^3 \to \mathbb{R}^3,\ T(x) = A\vec{x},\ A_{3\times 3} \\ S \text{ parallelipiped} \end{array}\right\} \Rightarrow \text{Volume } T(S) = (|\det A| \cdot \text{Volume } S)$

$T\ \mathbb{R}^2 \to \mathbb{R} \quad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2^2 \\ x_1^2 \end{bmatrix}$

5.1 Eigenvalues and eigenvectors

Def. $A_{n\times n}$, A nonzero $x \in \mathbb{R}^n$ is an <u>eigenvector</u> if there exits a scalar λ s.t. $Ax = \lambda x$.

λ is called an <u>eigenvalue</u> of A associated to x.

Remark: If x is an eigenvector associated to λ, then $x \in \mathrm{Nul}(A - \lambda I_n)$ $x = 0$.

In the case $\mathrm{Nul}(A - \lambda I_n)$ is called the <u>eigenspace</u> of λ.

$\mathrm{Nul}(A - \lambda I_n)$ is a subspace

$v \neq 0$

if $v \in \mathrm{Nul}(A - \lambda I)$

$\Rightarrow Av = \lambda v$

$\Rightarrow v$ eigenvector

Ex. $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

i) Are u, v eigenvectors?
ii) Is -4 eigenvalue.

i) $Au = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{matrix} 7 \\ 7 \end{matrix} \Rightarrow 7\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 7u$

$\Rightarrow$ u is an eigenvector for A with $\lambda = 7$

$Av = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{matrix} 13 \\ 9 \end{matrix} \neq \alpha v$, for any α.

$\Rightarrow$ v is NOT an eigenvector.

ii) Solve $Ax = -4x$, $\Leftrightarrow (A + 4I_2)x = 0$

$Ax = -4x \Leftrightarrow Ax = 4I_2 x$
$\Leftrightarrow (Ax + 4I_2)x = 0$
$\Leftrightarrow (A + 4I_2)x = 0.$

$\begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{cases} 5x_1 + 6x_2 = 0 \\ x_2 \text{ free} \end{cases}$

$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-6x_2}{5} \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -6/5 \\ 1 \end{bmatrix}$

$\mathrm{Nul}(A - (-4)I_2) = \mathrm{Span}\left\{\begin{bmatrix} -6/5 \\ 1 \end{bmatrix}\right\} \neq \{0\}$

$\Rightarrow$ -4 is an eigenvalue of A with $\begin{bmatrix} -6/5 \\ 1 \end{bmatrix}$ an eigenvector associated to it.

$A\vec{x} = -4x \Rightarrow A(10x) = -4(10x).$

Remark:

1) For a given λ, if $Ax = \lambda x$ has only the trivial soln, then λ is NOT an eigenvalue
if $Ax = \lambda x$ has non-trivial soln, then λ is an eigenvalue

2) 0 is NOT eigenvalue for $A \Leftrightarrow A$ is invertible

Pf: 0 is not an eigenvalue for $A \Leftrightarrow Ax = 0x$ has only the trivial soln,
$\Leftrightarrow$ the colns are lin indep. $\overset{\text{IMT}}{\Longleftrightarrow}$ A is invertible.

3) λ is eigenvalue for $A \iff \det(A - \lambda I) = 0$.

Sol: λ is an eigenvalue for $A \iff Ax = \lambda x$ has nontrivial solutions

$\iff (A - \lambda I_n)x = 0$ has nontrivial soln.

$\overset{def}{\iff}$ 0 is an eigen value for $A - \lambda I_n$

$\overset{Remark\ 2}{\iff}$ $A - \lambda I_n$ is NOT invertible $\iff \det(A - \lambda I_n) = 0$

Thm 1 The eigenvalues of a triangular matrix are the entries on the main diagonal.

Proof: A 3×3, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

$\det(A - \lambda I_3) = \det \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{vmatrix} = (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)$

Remark 3 $\Rightarrow$ λ is an eigenv for A iff $\det(A - I_3) = 0$

iff $\lambda = a_n$, or $\lambda = 0$.

Ex. $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 3 & 1 & -8 & 0 \\ 4 & 2 & 5 & 10 \end{bmatrix}$ Thm 1 $\Rightarrow 0, 7, -8, 10$ are the eigen values of A.

Thm 2: A $n\times n$, take $v_1, \dots, v_r$, eigenvectors of A with $\lambda_1, \dots, \lambda_r$ the corresponding eigenvalues ($v_i \neq 0$, $Av_i = \lambda_i v_i$ for all $i \in \{1, 2, \dots r\}$.
Suppose $\lambda_1, \dots \lambda_r$ are distinct ($\lambda_i \neq \lambda_j$ if $i \neq j$) $\Rightarrow \{v_1, \dots v_r\}$ is linearly independent.

Corollary: If a matrix A $n\times n$ has n distinct eigen values, then A has n eigenvectors lin. independent.

ex. A 2×2, 3, 5 are eigenvalues for A, $\quad Av_1 = 3v_1$
$\{v_1, v_2\}$ is a basis for $\mathbb{R}^2$. $\quad Av_2 = 5v_2$.

Proof of Thm 2

Assume the contrary $\Rightarrow \{v_1, \dots, v_r\}$ is linearly dep.

Thm in sect 1.7 $\Rightarrow \exists\ p \le s$ s.t. $v_p \in \text{Span}\{v_1, \dots, v_{p-1}\}$ and $\{v_1, \dots, v_{p-1}\}$ is lin indep.

$\exists\ c_1 \dots c_p$ s.t. $v_p = c_1 v_1 + \dots c_{p-1} v_{p-1} \Rightarrow A v_p = c_1 A v_1 + \dots + c_{p-1} A v_{p-1}$

$\lambda_p v_p = c_1 \lambda_1 v_1 + \dots + c_{p-1} \lambda_{p-1} v_{p-1}$

$\Rightarrow \lambda_p v_p = c_1 \lambda_p v_1 + \dots + c_{p-1} \lambda_p \lambda_{p-1}$

$0 = c_1(\lambda_1 - \lambda_p) v_1 + \dots c_{p-1}(\lambda_{p-1} - \lambda_p) v_{p-1} \Rightarrow$,

$\{v_1, \dots, v_{p-1}\}$ is lin indep. $\Rightarrow c_1(\lambda_1 - \lambda_p) = \dots = c_{p-1}(\lambda_{p-1} - \lambda_p) = 0$.

$\Rightarrow c_1 = c_2 = \dots c_{p-1} = 0 \rightarrow v_p = 0 v_1 + \dots 0 v_{p-1} = 0$,

contradiction !

5.2 The characteristic equation

Remark: A $n \times n$ $\Rightarrow \lambda$ is an eigenvalue of A

$\det(A - \lambda I_n) = 0$ is called the characteristic equation.

Ex. Find the characteristic eq. for $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ -2 & 1 & 3 & 2 \end{bmatrix}$

Sol $\det(A - \lambda I_4) = \det \begin{bmatrix} 2-\lambda & 0 & 0 & 0 \\ 7 & 3-\lambda & 0 & 0 \\ 0 & 10 & -\lambda & 0 \\ 2 & 1 & 3 & 2-\lambda \end{bmatrix} \overset{\text{lower triangular}}{=} (2-\lambda)^2 \cdot (3-\lambda)(-\lambda)$.

The char eq. is $(2-\lambda)^2 (3-\lambda)(-\lambda) = 0$

Remark A $n \times n$ $\Rightarrow \det(A - \lambda I_n)$ is a polynomial degree n, called the characteristic polynomial

Def The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Ex. 2, 3, 0 are eigenvalues.

2 has multiplicity 2

3 - " - 1

$A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

$AA = \begin{bmatrix} 2^2 & 0 \\ 0 & 5^2 \end{bmatrix}$

$\vdots$

$A^n = \begin{bmatrix} 2^n & 0 \\ 0 & 5^n \end{bmatrix}$

$A = PDP^{-1}$

$D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ diagonal

$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$

$A^n = PD^nP^{-1} = P\begin{bmatrix} 2^n & 0 \\ 0 & 5^n \end{bmatrix}P^{-1}$.

<u>Def</u>: A, B $n \times n$ <u>A is similar to B</u> if $\exists$ P invertible s.t. $A = PBP^{-1}$.

<u>Remark</u> If A is similar to B, then B is similar to A
$A = PBP^{-1}$ want Q s.t. $B = QAQ^{-1}$
Take $Q = P^{-1} \Rightarrow B = P^{-1}A(P^{-1})^{-1} = P^{-1}AP$

Thm 4 A, B $n \times n$, A is similar to B
$\rightarrow$ they have the same characteristic polynomial, thus the same eigenvalue.

Remark 1) $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.