Professor Dr.mbe

SIL: Eason

Math 18

Professor Office Hour 6434 APM

Tu 10-11am

W 9-11am

HW Due Monday 5pm

Matlab HW

Midterm: Aug 24

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Tues 6-7pm

Wed 12-1

Fri 12-1

Professor Drimbe

SIL: Eason

Lecture 1 8/8

$2x_1 + x_2 = 3$

$x_1 - x_2 = 0$

$x_1 = x_2 = 1$

$(1,1)$ is the unique solution.

Def: A linear equation in $x_1, x_2, \dots, x_n$ is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b,$$

where $a_1, \dots, a_n, b$ are real or complex numbers
$\mathbb{R}$ $\mathbb{C}$

$x_1, \dots, x_n$ are variables

$a_1, \dots, a_n$ are coefficients.

Ex. $1 + 7x_1 - 8x_2 = x_1 + x_3 + 3$

$6x_1 - 8x_2 - x_3 = 2$

Non-Ex

$x_1 x_2 = x_3 + 1$

$x_1^2 - x_3 = 0$

Def: A system of linear equations (or linear system) is a collection of one or more linear equation.

ex.
$$\begin{cases} 7x_1 + x_2 - 8x_3 = 1 \\ x_1 \qquad\quad - x_3 = 0 \end{cases}$$

A solution of a linear system is a list $(s_1, \dots, s_n)$ of numbers that makes each equation a truth statement.

When we replace $x_1, \dots, x_n$ with $s_1, \dots, s_n$.

(s_1, s_2, s_3) is a soln if $\begin{cases} 7s_1 + s_2 - 8s_3 = 1 \\ s_1 \quad\quad\quad - s_3 = 0 \end{cases}$

$(1,2,1)$ is a soln.

$(2,3,2)$ is a soln.

Def: 1) The set of all possible soln is called the solution set of the linear system.

2) Two linear system are equivalent if they have the same solution set.

Ex. $\begin{cases} x_1 + x_2 = 2 \\ 2x_1 - 2x_2 = 0 \end{cases}$ $\qquad$ $\begin{cases} x_1 + x_2 = 2 \\ x_1 - x_2 = 0 \end{cases}$

$(1,1)$ is a soln $\qquad$ $(1,1)$ is a soln.

$\{(1,1)\}$ is the soln set

$\begin{cases} x_1 + x_2 = 2 \\ \quad\quad 2x_1 = 2 \end{cases}$

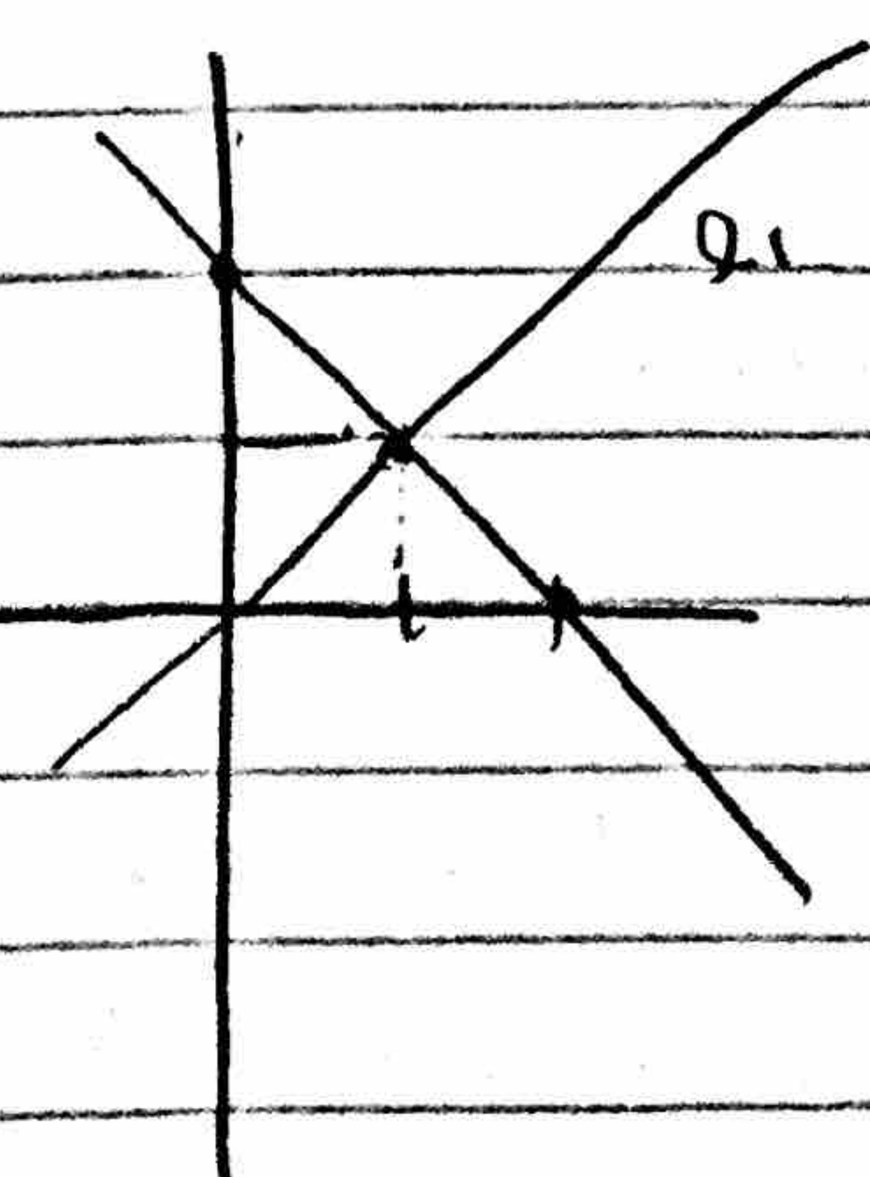
$(1,1)$

Geometric Interpretation

The soln set of a linear system with 2 eq with 2 variable. $\equiv$ Intersection of 2 lines in $\mathbb{R}^2$.

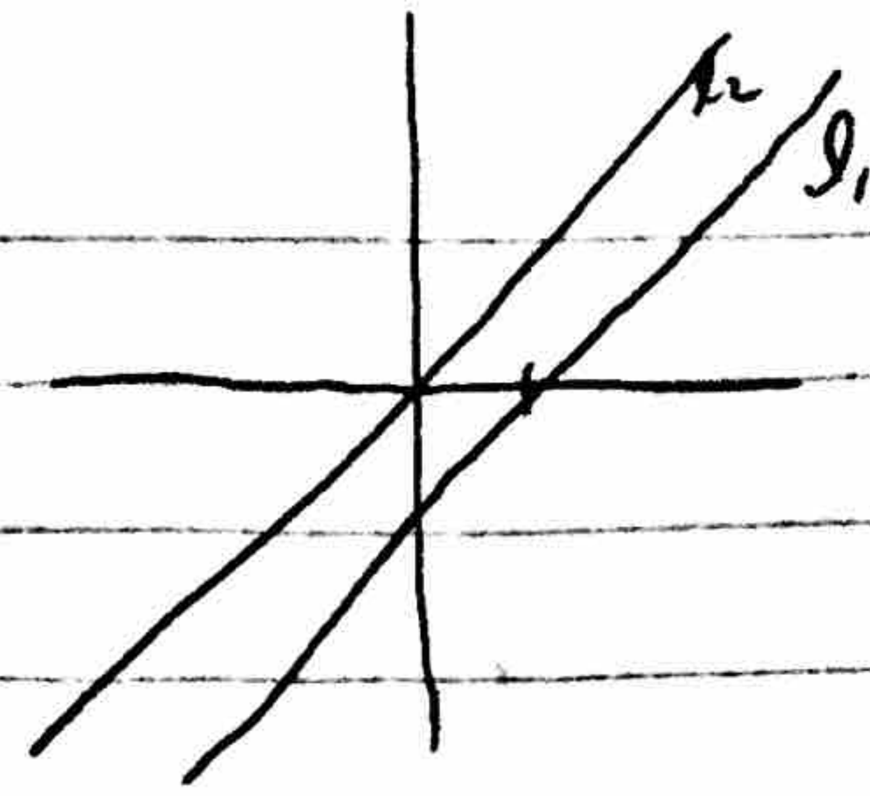
a) l_1 $x_1 + x_2 = 2$

l_2 $x_1 - x_2 = 0$



l_1 and l_2 intersect at $(1,1)$

b) $l_1 : x_1 - x_2 = 1$
$l_2 : x_1 - x_2 = 0$

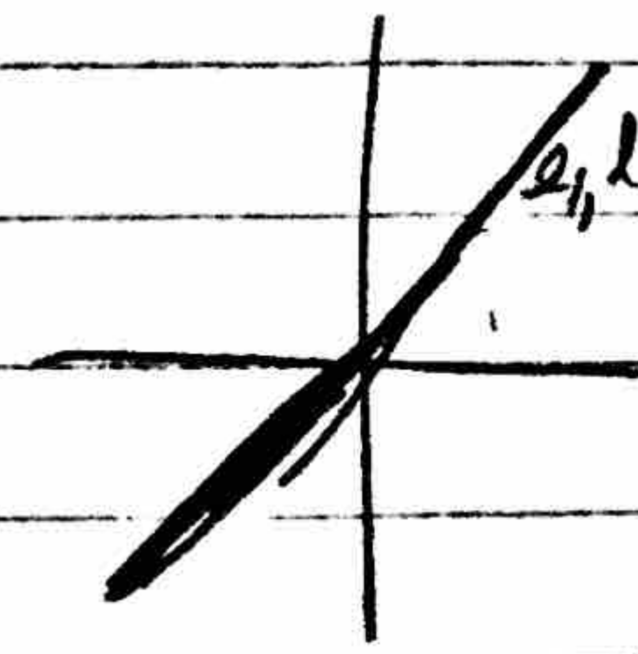


l_1, l_2 // parallel

no solution

c) $l_1 \ 2x_1 - 2x_2 = 0$
$l_2 \ x_1 - x_2 = 0$

infinitely many sol'n.



General Fact : A linear system has

1) no sol'n
or 2) exactly 1 sol'n
or 3) ∞ many sol'n.

Def: A linear system is consistent if it has at least one sol'n.
A linear system is inconsistent if it has no sol'n.

Matrix Notation.

$$\begin{cases} 2x_1 - x_3 = 7 \\ x_1 + x_2 = 0 \\ -10x_2 + \sqrt{3}x_3 = -1 \end{cases}$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -10 & \sqrt{3} \end{bmatrix}$$ — coefficient matrix

$$\left[\begin{array}{ccc:c} 2 & 0 & -1 & 7 \\ 1 & 1 & 0 & 0 \\ 0 & -10 & \sqrt{3} & -1 \end{array}\right]$$ — Augmented matrix.

(3×4) 3 rows
4 cols.

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ -1 \end{bmatrix}$$

Solving a linear system.

3 basic operation ① replace eq. by the sum of itself with a multiple of another.

② interchange eqⁿ.

③ multiply an eqⁿ. with nonzero value.

Ex. $\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 5x_1 - 5x_3 = 10 \end{cases}$ $A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$

① Keep x_1 in the 1st equation, eliminate it from equation 2 & 3.

eq3 → −5 eq1 + eq3

$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 10x_2 - 10x_3 = 10 \end{cases}$ $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$

② want coefficient of x_2 to be 1

eq2 → $\frac{1}{2}$ eq2

$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ 10x_2 - 10x_3 = 10 \end{cases}$ $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{bmatrix}$

③ Keep x_2 & eliminate it from eq3

eq3 → −10 eq2 + eq3

$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ 30x_3 = -30 \end{cases}$ $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix}$

echelon form of A

stop: $x_3 = -1$ soln: $(1, 0, -1)$

$x_2 = 0$

$x_1 = 1$

④ eliminate x_2 from eq_1

$eq_1 \to eq_1 + 2eq_2$

$$\begin{cases} x_1 \quad -7x_3 = 8 \\ x_2 - 4x_3 = 4 \\ x_3 = 1 \end{cases} \qquad \begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

⑤ eliminate x_3 from eq_1 and eq_2.

$eq_2 \to eq_2 + 4eq_3$

$eq_1 \to eq_1 + 7eq_3$

$$\begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

reduced echelon form

Elementary Row Operation

1. Replacement / replace one row by itself plus the multiple of another
2. Interchange / interchange two rows
3. Scaling / multiply all the entries in a row by a nonzero element.

Def: Two matrices are <u>row equivalent</u> if one can obtain a matrix from the other using the elementary row operations.

Remark: If the augmented matrices of two linear system are row equivalent, then the two linear systems are equivalent (they have the same soln set)

$A \overset{\text{row eqn.}}{\sim} B \xrightarrow{} \text{obtain easy sol's}$ ("nice")

Row Reduction and Echelon Form

- non-zero row (or column): row (or col'n) with at least one non-zero entry

ex. ([0 0 0 1 0])

- leading entry of a row: the first non-zero entry (left most)

ex. [0 0 0 1 0 2 0 1]

Def: A matrix is in echelon form if.

1) All the nonzero rows are above any rows of all zeros
2) Each leading entry of a row is in a col'n to the right of the leading entry of the row above it.
3) All entries in a col'n below a leading entry to be 0.

A matrix is in reduced echelon form if it satisfy 1, 2, 3, and
4) The leading entry in each non-zero row is 1.
5) Each leading entry is the only nonzero ~~element~~ in its col'n.

Ex. $\begin{bmatrix} 4 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 \end{bmatrix}$

0 0 0

echelon form

Theorem: Each matrix is row equivalent with a unique reduced echelon form.

$$A=\begin{bmatrix}1&1\\0&1\end{bmatrix}\overset{\text{row equiv}}{\sim}\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

Def: 1) If A is row equivalent to an echelon form U, we call U an echelon form of A.

2) If A is row equivalent to a reduced echelon form U, we say that U is the reduced echelon form of A.

Def: i) A <u>pivot</u> position in a matrix A is a location that corresponds to a leading 1 in the reduced echelon form of A.

pivot

$$A=\begin{bmatrix}1&1\\-1&0\end{bmatrix}\sim\begin{bmatrix}1&1\\0&1\end{bmatrix}$$

$$A=\begin{bmatrix}0&1\\1&0\end{bmatrix}\sim\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

The row reduction algorithm,

$$A=\begin{bmatrix}0&0&3&-6&6\\0&3&-7&8&-5\\0&3&-9&12&-9\end{bmatrix}\sim\text{ echelon form.}$$

① Identify the first non-zero column. This is a pivot column. The pivot position is at the top

② Select a nonzero entry in the pivot column. If necessary, interchange rows to move this entry into pivot positions.

$$R1\longleftrightarrow R3\quad\begin{bmatrix}0&3&-9&12&-9\\0&3&-7&8&-5\\0&0&3&-6&6\end{bmatrix}$$

③ Create zeros below the pivot $R_2\to R_2-R_1$

$$A \sim \begin{bmatrix} 0 & 3 & -9 & 12 & -9 \\ 0 & 0 & 2 & -4 & 4 \\ 0 & 0 & 3 & -6 & 6 \end{bmatrix}$$

(4) We find the next pivot position & obtain zeros below.

$R_3 \to R_3 - \frac{3}{2}R_2$ $\quad A \sim \begin{bmatrix} 0 & 3 & -9 & 12 & -9 \\ 0 & 0 & 2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ echelon form.

(5) Start with the right most pivot. Make it 1 & element above it zeros

$R_2 \to \frac{1}{2}R_2$

$R_1 \to \frac{1}{3}R_1$

$$A \sim \begin{bmatrix} 0 & 1 & -3 & 4 & -3 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(6) $R_1 \to R_1 + 3R_2$ $\quad A \sim \begin{bmatrix} 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ reduced echelon form

Solutions of Linear System.

Suppose the augmented matrix of the linear system has the reduced echelon form.

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & -5 & 3 & 1 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{aligned} x_1 \quad -5x_3 + 3x_4 &= 1 \\ x_2 - 2x_3 + x_4 &= 2 \\ 0 &= 0 \end{aligned}$$

x_1, x_2 = basic variables (in pivot colm)

x_3, x_4 = free variables.

$$\begin{cases} x_1 = 5x_3 - 3x_4 + 1 \\ x_2 = 2x_3 - x_4 + 2 \\ x_3, x_4 \text{ free} \end{cases}$$

Ex. Find the general soln of the linear system whose augmented matrix has been reduced to

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} -2 & -12 & -4 & 10 & 4 & 8 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \end{array}$$

soln: ① write directly

② $A \sim$ reduced echelon form

$$\begin{cases} -2x_1 - 12x_2 - 4x_3 + 10x_4 + 4x_5 = 8 \\ 2x_3 - 8x_4 - x_5 = 3 \\ x_5 = 7 \\ x_2, x_4 \text{ free} \end{cases}$$

$x_3 = 4x_4 + 5$

$-2x_1 - 12x_2 - 4(4x_4 + 5) + 10x_4 + 28 = 8$

$x_1 = -\frac{1}{2}(12x_2 + 26x_4)$

$$\begin{cases} \cancel{x_1 = -6x_2 - 13x_4} \\ \cancel{x_3 = 4x_4 + 5} \\ x_2, x_4 \text{ free} \\ x_5 = 7 \end{cases}$$

Theorem: (Existence and uniqueness theorem)

A linear system is consistent iff the right most coln of the augmented Matrix is not a pivot coln.

$[0\,0\,0\,0\,7]$ inconsistent.

Moreover, if the linear system is consistent, we have:

1) a unique soln if we have no free variable.

2) infinitely many soln if we have at least one free variable.