Math 18 Lecture 5 8/22/2017 SI L: Eason

Midterm 1.1 - 2.3 (skip 1.6)

Thm 7 A $n \times n$ $\Rightarrow$ A invertible $\Longleftrightarrow$ $A \sim I_n$

Proof: "$\Rightarrow$" Thm 5 $\Rightarrow$ $Ax = b$ is consistent for all $b \in \mathbb{R}^n$

$x = A^{-1}b$ $\Rightarrow$ A has a pivot in each row.

$\Rightarrow A \sim I_n$

"$\Leftarrow$" Recall: $C_1 = EC_2$, C_2 is obtained from C_1 using an element row operation & Element matrices are invertible.

$A \sim I_n \Rightarrow$ there exists p element matrices $E_1, E_2 \cdots E_p$ s.t.

$A \sim E_1 A \sim E_2(E_1 A) \sim \dots \sim (E_p(E_{p-1} \cdots E_1 A) = I_n$

$$\left.\begin{array}{l} \Rightarrow (E_p \cdots E_1) A = I_n \\ E_p \cdots E_1 \text{ is invertible} \end{array}\right\} \Rightarrow A \text{ is invertible } \& \ A^{-1} = E_p \cdots E_1$$

Remark: If A is invertible, then $\underbrace{[A \ \ I_n]}_{n \times 2n \text{ matrix}} \sim [I_n \ \ A^{-1}]$

proof:
$$\left.\begin{array}{l} (E_p \cdots E_1) = I_n \\ (E_p \cdots E_1) I_n = A^{-1} \end{array}\right\} \Rightarrow [A \ \ I_n] \underset{E_1}{\sim} \cdots \underset{E_p}{\sim} [I_n \ A^{-1}].$$

Ex. $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ Find A^{-1}!

Sol $[A \ \ I_3] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \underset{R_3 \to R_3 - 4R_1}{\overset{R_1 \leftrightarrow R_2}{\sim}} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \overset{R_3 \to R_3 + 3R_2}{\sim} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$

By Thm 7 $\Rightarrow$ A is invertible $A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$

Thm 8 (The invertible matrix theorem)

A $n\times n$ TFAE:

a) A is invertible

b) $A \sim I_n$

c) A has n pivots

d) $A\vec{x}=0$ has only the trivial sol^n

e) the column of A are lin. indep.

f) $x \to Ax$ is 1-1

g) $Ax=b$ is consistent for all $b\in\mathbb{R}^n$.

h) the col^n of A span $\mathbb{R}^n$.

i) $x \to Ax$ is onto

j) $\exists$ C $n\times n$ s.t. $CA=I_n$

k) $\exists$ D $n\times n$ s.t. $AD=I_n$

l) A^T is invertible.

Proof:

- $a) \underset{\text{def}}{\Rightarrow} j) \Rightarrow d) \Rightarrow c) \overset{\text{direct}}{\Rightarrow} b) \overset{\text{Thm 7}}{\Rightarrow} a)$

$Ax=0 \Rightarrow$
$[(CA)x=0 \Rightarrow I_n x=0 \Rightarrow x=0]$

[If A has less pivots, then we have a free variable $\Rightarrow$ ∞ many sol^n]

- $a) \Rightarrow k) \Rightarrow g) \Rightarrow c) \Rightarrow a$

Thm 4 sect 1.4

- $g) \underset{\text{def}}{\Leftrightarrow} h) \underset{\text{Thm 12 a)}}{\Leftrightarrow} i)$

$d) \Leftrightarrow e) \Leftrightarrow f)$

$a \underset{\text{Thm 6}}{\Leftrightarrow} l$

Corrolary: A, B $n \times n$, $AB = I_n \Rightarrow A, B$ are invertible & $A^{-1} = B$, $B^{-1} = A$

Ex. $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ Is A invertible?

Sol $A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ Thm 8 c) $\Rightarrow$ A is invertible.

Invertible linear transformation.

Def: A linear transf $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible if $\exists$ a function $S: \mathbb{R}^n \to \mathbb{R}^n$ s.t.

$S(T(x)) = x$
$T(S(x)) = x$, for all $x \in \mathbb{R}^n$.

Thm 9 A linear transf. $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible iff A is invertible
$T(x) = Ax$

In this case, the linear transf. $S: \mathbb{R}^n \to \mathbb{R}^n$ is the unique inverse of T.
$S(x) = A^{-1}x$.

Remark: The inverse of a linear transf. is a lin. transf.

proof of Thm 9:

"$\Rightarrow$" $S(T(x)) = x$ $T(S(x)) = x$, where $S: \mathbb{R}^n \to \mathbb{R}^n$ is a function
$S(Ax) = x$, for all $x \in \mathbb{R}^n$
Let x_0 be a sol'n of $Ax = 0$ $\Rightarrow Ax_0 = 0 \Rightarrow S(Ax_0) = S(0) = 0$
$\Rightarrow x_0 = 0$. By Thm 8, d) $\Rightarrow$ A is invertible

"$\Leftarrow$" Take $S: \mathbb{R}^n \to \mathbb{R}^n$, $S(x) = A^{-1}x$
$T(S(x)) = T(A^{-1}x) = A(A^{-1}x) = I_n x = x$.
$S(T(x)) = x \Rightarrow T$ is invertible with the inverse S.

4.1 Vector Spaces and subspaces.

$u, v, w \in \mathbb{R} \Rightarrow$ $u + v = v + u$

$(u+v) + w = (u + v + w)$

$u + 0 = u$

$7(u+v) = 7u + 7v$ ... etc.

Def: A vector space is a non empty set V on which are defined two operations, called addition and multiplication by scalar, which satisfy, for all $u, v, w \in V$, c, d scalars:

1. $u + v \in V$ (the sum between u & v)
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. there exists a zero vector 0 in V s.t. $u + 0 = u$.
5. for all $u \in V$, there exists $-u \in V$ s.t. $u + (-u) = 0$.
6. $c \cdot u \in V$ (the scalar multiple of u by c)
7. $c(u+v) = cu + cv$
8. $(c+d)u = cu + du$
9. $c(du) = (cd)u$.
10. $1 \cdot u = u$.

Elements in V are called vectors

Ex. 1) $V = \mathbb{R}^n$, with the usual addition & scalar multiple.

$0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2.) $V = \mathbb{M}_{m,n}(\mathbb{R}) = \{$ the matrices with m rows & n coln with entries real # $\}$ $\mathbb{M}_{2,3}(\mathbb{R})$ $\begin{bmatrix} 1 & -3 & 2 \\ 0 & -8 & 9 \end{bmatrix}$

$0 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$.

3) $V = \mathbb{P}_n = \{$ polynomials of degree at most n $\}$

$= \{a_0 + a_1 t + \dots + a_n t^n \mid a_0, \dots, a_n \in \mathbb{R}\}$

ex. $n = 2$ $p(t) = 3 + t^2 \in \mathbb{P}_2$

$q(t) = -1 + 3t - t^2 \in \mathbb{P}_2$

$r(t) = 1 - t \in \mathbb{P}_2$

$p(t) = a_0 + a_1 t + \dots a_n t^n$

$q(t) = b_0 + b_1 t + \dots + b_n t^n$

$\alpha \in \mathbb{R}$, $(\alpha p)(t) \overset{def}{=} \alpha \cdot a_0 + (\alpha a_1)(-t \dots + (\alpha a_n) t^n$

$(p+q)(t) \overset{def}{=} (a_0 + b_0) + (a_1 + b_1) t + \dots + $

the zero vector is $0 = 0 + 0 \cdot t + \dots 0t^n$

$(p+r)(t) = 3 + ((0-1) t + (1+0) t^2$.

4) $D \subset \mathbb{R}$ (eg $D = (0,1)$ or D $(1,2] \cup \{3\}$

$V = \{ f : D \to \mathbb{R}$

$f, g \in V$, $\alpha \in \mathbb{R}$

$(f+g)(x) \overset{def}{=} f(x) + g(x)$.

$f(x) = x + \frac{1}{x} + \sin x$, $g(x) = 2x - e^x$

$(f+g)(x) = 3x + \frac{1}{x} + \sin x - e^x$.

Define the zero vector f_0 $D \to \mathbb{R}$ $f_0(x) = 0$, for all $x \in D = (0,1)$

Remark: 1) $0 \cdot \vec{u} = \vec{0}$

2) $c \cdot \vec{0} = \vec{0}$

3. $-\vec{u} = (-1)\vec{u}$ for all $u \in V$, $c \in \mathbb{R}$

Proof: 1) 0 $(0 \cdot u) = (0\ \ 0) u$.

In 8, $c = d = 0 \Rightarrow$ $(0+0) u = 0 \cdot u + 0 \cdot u$

$\cancel{0 \cdot u} = \cancel{0 \cdot u} + 0 \cdot u$

$0 = 0 \cdot \vec{u}$

2) In 9 $c(\vec{0} + \vec{0}) = c \cdot \vec{0} + c \cdot \vec{0} \Rightarrow$ $c \cdot \vec{0} = \vec{0}$

3) In 8, $c = 1$, $d = -1$

Def: A subspace of vector space V is a subset H of V s.t.

i) $0 \in H$ (the zero vector belongs to H)

ii) $u + v \in H$, for all $u, w \in H$.

iii) $c \cdot u \in H$, for all $u \in H$, $c \in \mathbb{R}$

Remarks:

1) If H is a subspace of V, then H is a vector space.

2) If V is a vector space, then V is a subspace of itself.

Ex 1) V is vector space $\Rightarrow \{0\}$ is a sub space.

2) $V = \mathbb{R}^3$, $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} ; s, t \in \mathbb{R} \right\}$.

- $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H$
- $\begin{bmatrix} s_1 \\ t_1 \\ 0 \end{bmatrix} + \begin{bmatrix} s_2 \\ t_2 \\ 0 \end{bmatrix} = \begin{bmatrix} s_1 + s_2 \\ t_1 + t_2 \\ 0 \end{bmatrix} \in H$
- $c \in \mathbb{R}$, $u = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} \in H$

  $c \cdot u = \begin{bmatrix} cs \\ ct \\ 0 \end{bmatrix} \in H$

3) NOT a subspace

$V = \mathbb{R}^3$, $H = \left\{ \begin{bmatrix} s \\ t \\ 1 \end{bmatrix} \mid s, t \in \mathbb{R} \right.$

$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin H$.

Thm 1: V vector space ; $v_1, \dots, v_p \in V \Rightarrow$ span $\{v_1, \dots, v_p\} \overset{def}{=}$ $H. \{c_1 v_1 + \dots + c_p v_p \mid c_1, \dots c_p \in \mathbb{R}\}$. (the set of all linear comb. of $v_1, \dots v_p$) is a subspace.

Proof: $H \subset V$

i) $0 = 0 v_1 + \dots + 0 \cdot v_p \in$ Span $\{V_1, \dots V_p\} = H$

ii) $u_1 = c_1 v_1 + \dots + c_p v_p \in H$

$u_2 = d_1 v_1 + \dots d_p v_p \in H \Rightarrow u_1 + u_2 = (c_1 + d_1) v_1 + \dots + (c_p + d_p) v_p \in H$ ✓

iii) $c \cdot u_1 = (c \cdot c_1) v_1 + \dots + (c c_p) v_p \in H$.

Ex. $H = \left\{ \begin{bmatrix} a - 3b \\ 2b + a \\ c + a \\ c + b \end{bmatrix} \middle| a, b, c \right\}$ is a subspace of $\mathbb{R}^4$.

Sol: $\begin{bmatrix} a - 3b \\ 2b + a \\ c + a \\ c + b \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} -3b \\ 2b \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c \\ c \end{bmatrix} = a \underset{u_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}} + b \underset{u_2}{\begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}} + c \underset{u_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}$

$\Rightarrow H = \text{Span}\{u_1, u_2, u_3\}$ is a subspace of $\mathbb{R}^4$, by Thm 1.

Ex. For $\alpha \in \mathbb{R}$, define $H(\alpha) = \{p \in \mathbb{P}_2 \mid p(1) = \alpha\}$
Find α s.t. $H(\alpha)$ is a subspace of $\mathbb{P}_2$

$p \in H(\alpha) \Leftrightarrow a_0 + a_1 + a_2 = \alpha$

$0 = 0 + 0 \cdot t + 0 \cdot t^2$ is the zero vector in $\mathbb{P}_2$

If $H(\alpha)$ is a subspace $\Rightarrow 0 \in H(\alpha)$.

$0 \in H(\alpha) \Rightarrow 0 + 0 + 0 = \alpha \Leftrightarrow \alpha = 0$

Thus if $H(\alpha)$ is a subspace, then $\alpha = 0$ want to show: $H(0)$ is a subspace

$H(0) = \{a_0 + a_1 t + a_2 t^2 \mid a_0 + a_1 + a_2 = 0\}$

$= \{a_0 + a_1 t + (-a_0 - a_1)t^2 \mid a_0, a_1 \in \mathbb{R}\}$.

$= a_0(1 - t^2) + a_1(t - t^2) \mid a_0, a_1 \in \mathbb{R}$.

$= \text{Span}\{1 - t^2, t - t^2\}$

$\overset{\text{Thm 1}}{\Rightarrow} H(0)$ is a subspace of $\mathbb{P}_2$

$H(0)$ is a subspace: (2^{nd} sol'n)

$$\left.\begin{array}{l} 0 = 0 + 0 \cdot t + 0 \cdot t^2 \\ 0 + 0 + 0 = 0 \end{array}\right\} \Rightarrow 0 \in H(0)$$

$p, q \in H(0) \Rightarrow p(t) = a_0 + a_1 t + a_2 t^2$, $a_0 + a_1 + a_2 = 0$

$q(t) = b_0 + b_1 t + b_2 t^2$, $b_0 + b_1 + b_2 = 0$.

$(p+q)(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2$

$= a_0 + b_0 + a_1 + b_1 + a_2 + b_2 = 0 + 0 = 0.$