Operations Research, Spring 2021 (109-2)
Homework 1

Ling-Chieh Kung[*]

March 16, 2021

1 Rules

- This homework is due at ***23:59 am, March 21***. Those who submit their works late but are late by less than one hour gets 10 points off. Works that are late more than one hour get no point.
- For this homework, students should work individually. While discussions are encouraged, copying is prohibited.
- Please submit a ***PDF file*** through NTU COOL and make sure that the submitted work contains the student ID and name. Those who fail to do these will get 10 points off.
- You are required to ***type*** your work with LaTeX (strongly suggested) or a text processor with a formula editor. Hand-written works are not accepted. You are responsible to make your work professional in mathematical writing by following at least the following rules:[1]

  1. When there is a symbol denoted by an English letter, make it italic. For example, write $a + b = 3$ rather than a + b = 3.
  2. An operator (e.g., +) should not be italic. A function with a well-known name (e.g., log, max and sin) is considered as an operator.
  3. A number should not be italic. For example, it should be $a + b = 3$ rather than $a + b = \mathit{3}$.
  4. Superscripts or subscripts should be put in the right positions. For example, a_1 and $a1$ are completely different: The former is a variable called a_1 while the latter is actually $a \times 1$.
  5. When there is a subtraction, write $-$ rather than -. For example, write $a - b = 3$ rather than a - $b = 3$. The same thing applies to the negation operator. For example, write $a = -3$ rather than $a =$ -3.
  6. If you want to write down the multiplication operator, write $\times$ rather than *.
  7. For an exponent, write it as a superscript rather than using ^. For example, write 10^2 rather than 10^2.
  8. There should be proper space between a binary operator. For example, it should be $a + b = 3$ rather than a+b=3.

  Those who fail to follow these rules may get at most 10 points off.

- As we may see, there are many students, many problems, but only a few TAs. Therefore, when the TAs grade this homework, it is possible for only some problems to be randomly selected and graded. For all problems, detailed suggested solutions will be provided.

[*]Department of Information Management, National Taiwan University. E-mail: lckung@ntu.edu.tw.

[1]A more complete list of formatting rules is on NTU COOL.

2 Problems

1. (20 points; 5 points each) Jay owns a bakery to bake N kinds of cakes. During any day, he can bake at most K cakes. The unit production cost for cake i is C_i. The demand for cake i in day t, which must be met on time, is D_{it}. It costs H_i to hold a unit of cake i in inventory for a day. Initially there are I_i cakes in each category. We want to formulate an LP that can minimize the total cost of meeting the next T days' demands.

   While you have learned how to formulate an LP for this, let's try a different way to formulate this problem. Let x_{ijt} be the number of cake i that you produce in day j and plan to be sold in day t, $i = 1, ..., N$, $j = 1, ..., T$, and $t = j, ..., T$. For example, x_{111} is the number of cake 1 produced and sold both in day 1, x_{113} is the number of cake 1 produced in day 1, kept in your refrigerator for two days, and sold in day 3, etc. Use this decision variable to solve the following problems.

   (a) Write down a constraint (or constraints) to satisfy the demand for cake i in day t.

   (b) Write down an expression (as a function of I_i, x_{ijt}, and D_{it}) that calculates the ending inventory level for cake i in day t.

   Hint. For day 1, how to express the total production quantity of cake i? If the ending inventory level of day 1 is the sum of initial inventory and production quantity minus the demand volume, how to express the ending inventory level of day 1? How about day 2?

   (c) Write down a complete LP formulation that solves this problem.

   Hint. You do not need any decision variable other than x_{ijt}.

   (d) Now suppose that you are not allowed to sell a cake if it is kept for E or more days. For example, if $E = 2$, a cake produced in day 1 can be sold only in days 1 or 2. You may assume that all initial inventory are allowed to be sold in days 1, 2, ..., and $E - 1$ but not in day E. Write down a constraint to add this new restriction into your formulation.

   Note. Recall the formulation using x_{it} (production quantity) and y_{it} (ending inventory) as decision variables that we introduced in class. Is it easy to write down the above constraint using that formulation? Considering this, maybe you would also agree that the formulation with x_{ijt} as a decision variable is helpful in some cases.

2. (20 points; 10 points each) IEDO Oil has refineries in N locations. Currently, refinery i may refine up to K_i million barrels of oil per year, $i = 1, ..., N$. Once refined, oil is shipped to M distribution points. IEDO Oil estimates that distribution point j may sell up to D_j million barrels per year. Because of differences in shipping and refining costs, the profit earned per million barrels of oil sold depends on where the oil was refined and on the point of distribution. In particular, the profit earned per million barrels of oil refined at refinery i and sold in distribution point j is P_{ij}, $i = 1, ..., N$, $j = 1, ..., M$. IEDO Oil is now considering expanding the capacity of each refinery. Each million barrels of annual refining capacity that is added will cost C_i for refinery i, $i = 1, ..., N$. Capacity can only be added now but can be used in the future T years. Solve each of the following problems independently.

   (a) Suppose that capacity expansion at refinery i requires not only the variable cost C_i mentioned above but also a fixed cost F_i as long as the capacity is expanded by any positive amount. Formulate an IP that maximizes IEDO's profits minus expansion costs over a T-year period.

   (b) Suppose that future demand is uncertain. IEDO believes that in the future T years, there are S possible *scenarios*. In scenario k, the demand (maximum possible sales quantity) in distribution point j is D_{kj}, $k = 1, ..., S$, $j = 1, ..., M$.[2] IEDO may observe scenario realization only after capacity expansion has been done. However, it may make its production, shipping, and sales decisions after the realized scenario is observed. At this moment, it is believed that scenario k will be realized with probability Q_k. Formulate an LP that maximizes IEDO's *expected* profits minus expansion costs over a T-year period.

   Hint. Make different plans upon observing different scenarios!

[2] For example, there may be two scenarios, where the first one is with high demands and the second one is with low demands, and $D_{1,j} < D_{2,j}$ for all $j = 1, ..., M$.

3. (25 points) You are going to produce N types of products to one single machine in the next T hours. In each hour, the machine may be set up to produce only one type of product. If hour t is used to process product j, at most Q_{jt} units of product j may be produced, $t = 1, ..., T$, $j = 1, ..., N$. The demand quantity of product j is D_j, $j = 1, ..., N$. The shortage amount of product j is D_j minus the total production quantity of product j if this quantity is positive or 0 otherwise, $j = 1, ..., N$. Solve each of the following problems independently.

   (a) (5 points) Formulate an IP that minimizes the total amount of shortage of all products.

   (b) (5 points) Suppose that we earn R_j dollars by fulfilling the demand of product j completely (i.e., to make the total production quantity equal the demand). Formulate an IP that maximize the total revenue.

   (c) (5 points) Suppose that within the T hours there must be at least H hours reserved for machine maintenance. During these maintenance hours, no products may be produced. Moreover, the time gap between two maintenance hours must be at least G hours. For example, if $G = 3$, scheduling maintenance in hours 2 and 5 is not allowed where doing that in hours 2 and 6 is allowed. Formulate an IP to maximize the number of jobs which has no shortage.

   (d) (10 points) Suppose that now the starting time for product j (the first hour for the machine to produce product j) cannot be later than L_j, $j = 1, ..., N$. Note that having $L_j = T$ means for job j such a restriction does not exist. Formulate an IP to maximize the number of jobs which has no shortage.

4. (15 points) Consider the following nonlinear integer program

$$
\begin{aligned}
\max \quad & 2w + 3x + \min\{5y, 7z, 9\} \\
\text{s.t.} \quad & \max\{2w + 5z, 3x + 8y\} \leq 10 \\
& w \leq 50y \\
& w \leq 5xz \\
& w \geq 0, x \geq 0 \\
& y \in \{0, 1\}, z \in \{0, 1\}.
\end{aligned}
$$

   Please linearize it into a linear integer program. Do not attempt to solve it or simply it.

5. (20 points; 10 points each) Recall Problems 1d and 2b.

   (a) Solve Problem 1d with the data given in the spreadsheet "Cake" in the accompanying MS Excel file. Write down an optimal solution you find (or conclude that the instance is infeasible or unbounded) in a business language.

   (b) Solve Problem 2b with the data given in the spreadsheet "Refinery" in the accompanying MS Excel file. Write down an optimal solution you find (or conclude that the instance is infeasible or unbounded) in a business language.

   Note 1. I understand that there are always some students who do not own MS Excel. In this case, please use Google Spreadsheet or a similar product to open the file. You are also not required to solve the instances with MS Excel solver (though for Homework 1 this is suggested). You may write Python to invoke Gurobi Optimizer, as we will introduce in the second module of this course, or solve the instance in any way you like.

   Note 2. When we say "in a business language," we mean your boss (who know nothing about Operations Research and your mathematical programs) should be able to understand your plan. If you find it helpful, you may of course use some tables or figures to demonstrate your plan.