Convex Optimization THL413

ETERNITY II

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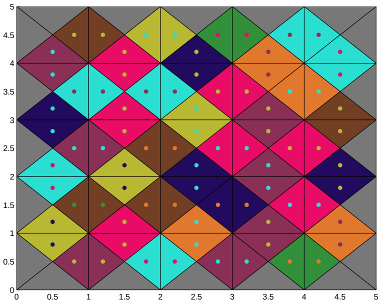
ECE

Game Description



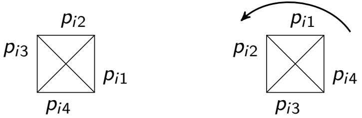
Attributes

- $M \times M$ board
- L colored symbols + grey

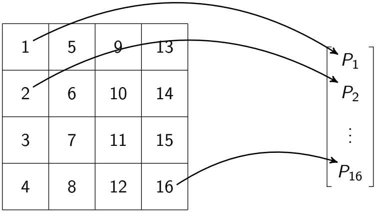
Game object

- Identical Adjacent Edges
- Grey edges on frame

Game Modeling



$P_i = [p_{i1}, p_{i2}, p_{i3}, p_{i4}]$ $\quad$ $P_i = [p_{i4}, p_{i1}, p_{i2}, p_{i3}]$



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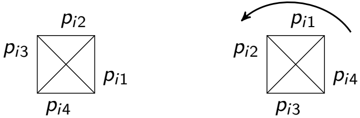

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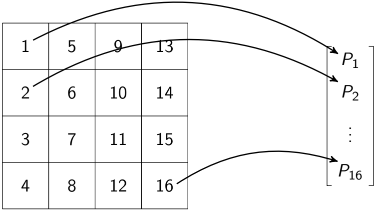

- edge $\rightarrow p_{ij} \in \mathbb{R}^L$

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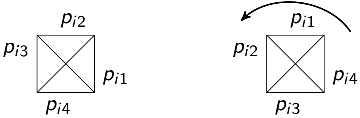

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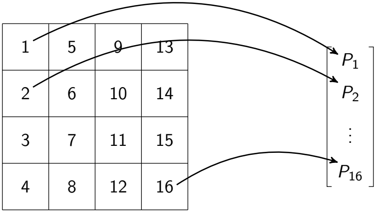

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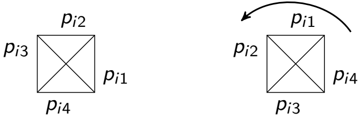

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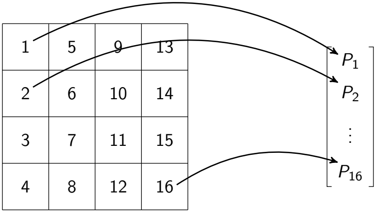

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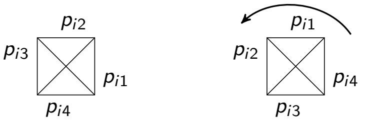

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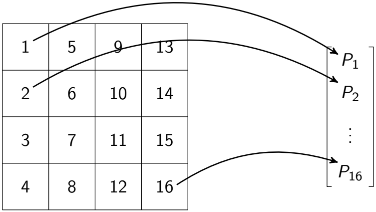

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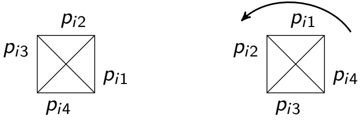
- *Piece rotation*:
  1. row circular shifting of P_i
  2. $\Pi_i P_i$, $\quad \begin{bmatrix} \Pi_1 P_1 \\ \vdots \\ \Pi_{M^2} P_{M^2} \end{bmatrix}$

Game Modeling



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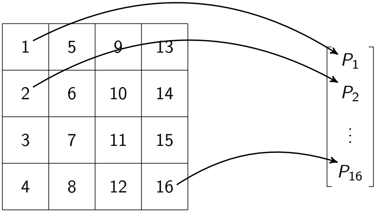

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  2. $(\Xi \otimes I_4)P$

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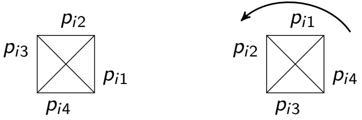

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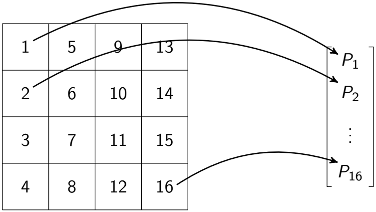

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  2. $(\Xi \otimes I_4)P$
- *Solution*: $\underbrace{(\Xi \otimes I_4)\Pi}_{\Phi} P$

Defining constraints

* Touching edges constraints

$$\Delta(\Xi \otimes I_4)\Pi P = \mathbf{0} \in \mathbb{R}^{2M(M-1)\times L}.$$

where each row Δ_i has the form $\left[0, -1 \ldots, 0, 1, 0\right] \in R^{4M^2}$

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* Boundary constraints

$$\beta^T(\Xi \otimes I_4)\Pi P\mathbf{1} = 0$$

Defining constraints

At optimality

* Each row and column of Φ has a single non zero entry $\{1\}$.

$$\Phi 1 = \mathbf{1}$$
$$\Phi^T 1 = \mathbf{1}$$
$$\Phi \text{ has minimal cardinality } 4M^2$$

* Non zero blocks ϕ_{ij} are circulant

$$\Phi \triangleq \begin{bmatrix} \phi_{11} & \cdots & \phi_{1M} \\ \vdots & \ddots & \vdots \\ \phi_{M1} & \cdots & \phi_{MM} \end{bmatrix}$$

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$$\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}, \begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}, \begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}}_{\text{4 degrees of freedom}}$$

Shrinking problem

$$\tilde{\Phi} \triangleq [\Phi(:,1), \Phi(:,5), \ldots \Phi(:,4M^2-3)] \in \mathbb{R}^{4M^2 \times M^2}$$

Reconstruction formula

$$\Phi = \sum_{i=1\ldots4} (I \otimes \pi_i)(\tilde{\Phi} \otimes e_i^T) \in \mathbb{R}^{4M^2 \times 4M^2}$$

Shrinking problem

$$\tilde{\Phi} \triangleq [\Phi(:,1), \Phi(:,5), \ldots \Phi(:,4M^2-3)] \in \mathbb{R}^{4M^2 \times M^2}$$

Reconstruction formula for vectorized variables

$$vec\Phi = \left(\sum_{i=1\ldots 4} (I \otimes e_i \otimes I \otimes \pi_i) \right) vec\tilde{\Phi} \quad \triangleq Yvec\tilde{\Phi}$$

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$$\begin{aligned}
\underset{\Phi \in 4M^2 \times 4M^2}{minimize} \quad & \| \, vec\Phi \, \|_0 \\
s.t. \quad & \Delta \Phi P = \mathbf{0} \\
& \beta^T \Phi P \mathbf{1} = 0 \\
& \Phi 1 = \mathbf{1} \\
& \Phi^T 1 = \mathbf{1} \\
& (I \otimes R_d)\Phi(I \otimes R_d^T) = (I \otimes S_d)\Phi(I \otimes S_d^T) \\
& (I \otimes R_\phi)\Phi(I \otimes S_\phi^T) = (I \otimes S_\phi)\Phi(I \otimes R_\phi^T) \\
& \Phi \geq \mathbf{0}
\end{aligned}$$

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& (P\mathbf{1} \otimes \beta)^T Yvec\tilde{\Phi} = 0 \\
& (\mathbf{1}^T \otimes I \otimes \mathbf{1}_4^T) vec\tilde{\Phi} = \mathbf{1} \\
& (I \otimes \mathbf{1}_{4M}^T) vec\tilde{\Phi} = \mathbf{1} \\
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\end{aligned}$$

Problem Requirements

$$E_1 \triangleq (P^T \otimes \Delta) Y$$

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$$\mathbf{d} \triangleq \begin{bmatrix} \mathbf{0}_{2LM(M-1)+1} \\ \mathbf{1}_{2M^2} \end{bmatrix}$$

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6	4	313	5 184	1 622 592	12 980 736
6	22	1 393	5 184	7 221 312	57 770 496
8	22	2 593	16 384	42 483 712	339 869 696
10	22	4 161	40 000	166 440 000	1 331 520 000
16	22	11 073	262 144	2 902 720 512	23 221 764 096

1. Original Problem: $>$24GB for constraints

Problem Size

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1. Original Problem: >24GB for constraints
2. Use structure of constraints to reduce size!

Problem Size

Matrix (M,L)	Values "any"	Non Zero Entries (%) (4,4)	(12,22)	(16,22)
E_1	$\{0, \pm 1\}$	2.34	0.05	0.03
E_2	$\{0, 1, 2\}$	60.9	28.2	22.0
E_3	$\{0, 1\}$	6.25	0.69	0.39
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E	$\{0, \pm 1, 2\}$	3.77	0.09	0.05

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   - All entries in $\{0, \pm 1, 2\}$

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Problem Size

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1. Original Problem: $>$24GB for constraints
2. Use structure of constraints to reduce size!
   - All entries in $\{0, \pm 1, 2\}$
     - i E_2 is the only constraint with entries "2"
   - Sparsity
     - i E_2 is has the lowest sparsity

Problem Size

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$$\text{But:} \quad \left\{ \begin{array}{c} E_2 \text{is a vector} \\ E_2 \geq \mathbf{0} \\ \text{Right hand side is 0 @ } E_2 \end{array} \right\}$$

Problem Size

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     - i E_2 is has the lowest sparsity

But: $$\left\{ \begin{array}{c} E_2 \text{is a vector} \\ E_2 \geq \mathbf{0} \\ \text{Right hand side is 0 @ } E_2 \end{array} \right\} \implies E_2 \text{ can be eliminated}$$

Problem Size 2

Matrix (M,L)	Values "any"	Non Zero Entries (%) (4,4)	(12,22)	(16,22)
E_1	$\{0, \pm 1\}$	2.34	0.05	0.03
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E_3	$\{0, 1\}$	6.25	0.69	0.39
E_4	$\{0, 1\}$	6.25	0.69	0.39
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E	$\{0, \pm 1\}$	3.33	0.08	0.05

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E	$\{0, \pm 1\}$	3.33	0.08	0.05

1. Original Problem: $>$24GB for constraints
2. Use structure of constraints to reduce size!
   - All entries in $\{0, \pm 1\}$
     - $E = E_+ - E_-$ where $E_+ \triangleq E > 0$, $E_- \triangleq E < 0$
     - represent $E_\pm$ as sparse
     - pack bits in words
     - bitwise operations
   - Sparsity

Problem Size 2

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(M,L)	"any"	(4,4)	(12,22)	(16,22)
E_1	$\{0, \pm 1\}$	2.34	0.05	0.03
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     - bitwise operations
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     - use sparse data types

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     - represent $E_\pm$ as sparse
     - pack bits in words
     - bitwise operations
   - Sparsity
     - use sparse data types $\rightarrow$ simplicity, compatibility

Problem Size 2

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E	$\{0, \pm 1\}$	3.33	0.08	0.05

1. Original Problem: >24GB for constraints
2. Use structure of constraints to reduce size! $\rightarrow$ Sparse Matrices

M	L	Eq.s	Size **x**	Entries of E	Sprs. bytes	Used%
4	4	128	1 024	131 072	77 832	7,42
6	4	312	5 184	1 617 408	437 768	3,38
6	22	1 392	5 184	7 216 128	437 768	0,76
8	22	2 592	16 384	42 467 328	1 458 184	0,43
10	22	4 160	40 000	166 400 000	3 673 608	0,28
16	22	11 072	262 144	2 902 458 368	25 231 368	0,11

Problem Size 2

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E	$\{0, \pm 1\}$	3.33	0.08	0.05

1. Original Problem: $>$24GB for constraints , Sparse: $\sim$25MB
2. Use structure of constraints to reduce size! $\rightarrow$ Sparse Matrices
3. Numerical Errors!!!

M	L	Eq.s	Size **x**	Entries of E	Sprs. bytes	Used%
4	4	128	1 024	131 072	77 832	7,42
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16	22	11 072	262 144	2 902 458 368	25 231 368	0,11

Cost function

$$E \triangleq \begin{bmatrix} E_1 \\ E_3 \\ E_4 \end{bmatrix} \qquad \begin{aligned} P1: \underset{\mathbf{x}}{minimize} \quad & \| \mathbf{x} \|_0 \\ s.t. \quad & E\mathbf{x} = \mathbf{d} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \qquad \begin{aligned} P2: \underset{\mathbf{x},\mathbf{y}}{minimize} \quad & \mathbf{y}^T \mathbf{x} \\ s.t. \quad & E\mathbf{x} = \mathbf{d} \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{1} \geq \mathbf{y} \geq \mathbf{0} \\ & \mathbf{1}^T \mathbf{y} = n - k \end{aligned}$$

- $\forall x, y \in \mathbb{R}_+^n, \theta \in [0, 1]$:

$$\| \theta \mathbf{x} + (1 - \theta)\mathbf{y} \|_0 \geq \min \{ \| \mathbf{x} \|_0, \| \mathbf{y} \|_0 \} \qquad (4)$$

- $\| \cdot \|_0$: quasiconcave function in $\mathbb{R}_+^n$
- k : minimum cardinality

Cost function

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- $\forall x, y \in \mathbb{R}^n_+, \theta \in [0, 1]$:

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- $\| \cdot \|_0$: quasiconcave function in $\mathbb{R}^n_+$
- k : minimum cardinality
- $\mathbf{x}$ minimizer of $P1 \iff (\mathbf{x}, \mathbf{y})$ minimizer of $P2$

Cost function

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- k : minimum cardinality
- $\mathbf{x}$ minimizer of $P1 \iff (\mathbf{x}, \mathbf{y})$ minimizer of $P2$
- Solve P2 instead

Cost function

$$
\begin{aligned}
P2: \underset{\mathbf{x},\mathbf{y}}{\text{minimize}} \quad & \mathbf{y}^T\mathbf{x} \\
s.t. \quad & E\mathbf{x} = \mathbf{d} \\
& \mathbf{x} \geq \mathbf{0} \\
& \mathbf{1} \geq \mathbf{y} \geq \mathbf{0} \\
& \mathbf{1}^T\mathbf{y} = n - k
\end{aligned}
$$

$$
\begin{aligned}
P3: \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{y}^T\mathbf{x} \\
s.t. \quad & E\mathbf{x} = \mathbf{d} \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}
$$

$$
\begin{aligned}
P4: \underset{\mathbf{y}}{\text{minimize}} \quad & \mathbf{y}^T\mathbf{x} \\
s.t. \quad & \mathbf{1} \geq \mathbf{y} \geq \mathbf{0} \\
& \mathbf{1}^T\mathbf{y} = n - k
\end{aligned}
$$

- P3, P4 linear (convex) problems

Cost function

$$
\begin{aligned}
P2: \underset{\mathbf{x},\mathbf{y}}{minimize} \quad & \mathbf{y}^T\mathbf{x} \\
s.t. \quad & E\mathbf{x} = \mathbf{d} \\
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$$

$$
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\end{aligned}
$$

- P3, P4 linear (convex) problems
- Solve sequence of P3, P4

Cost function

$$
\begin{aligned}
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s.t. \quad & E\mathbf{x} = \mathbf{d} \\
& \mathbf{x} \geq \mathbf{0} \\
& \mathbf{1} \geq \mathbf{y} \geq \mathbf{0} \\
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\end{aligned}
$$

$$
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\end{aligned}
$$

$$
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P4: \underset{\mathbf{y}}{minimize} \quad & \mathbf{y}^T\mathbf{x} \\
s.t. \quad & \mathbf{1} \geq \mathbf{y} \geq \mathbf{0} \\
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\end{aligned}
$$

- P3, P4 linear (convex) problems
- Solve sequence of P3, P4
- Does not always converge

Cost function

$$
\begin{aligned}
P2: \underset{\mathbf{x},\mathbf{y}}{minimize} \quad & \mathbf{y}^T\mathbf{x} \\
s.t. \quad & E\mathbf{x} = \mathbf{d} \\
& \mathbf{x} \geq \mathbf{0} \\
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\end{aligned}
$$

$$
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$$
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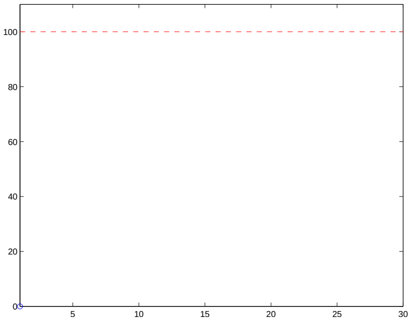
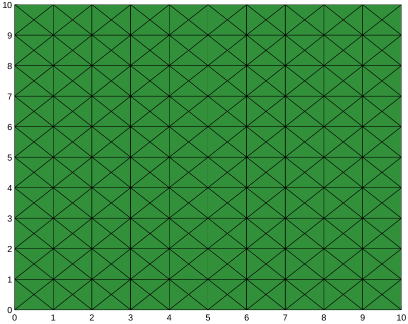
& \mathbf{1}^T\mathbf{y} = n - k
\end{aligned}
$$

- P3, P4 linear (convex) problems
- Solve sequence of P3, P4
- Does not always converge
  - Add noise to $\mathbf{y}$
  - Or even randomize, when stationary and not optimal
  - For our problem, it is common to converge

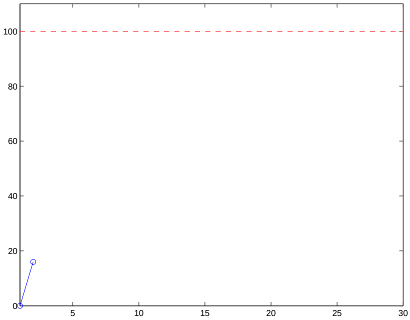
Convergence

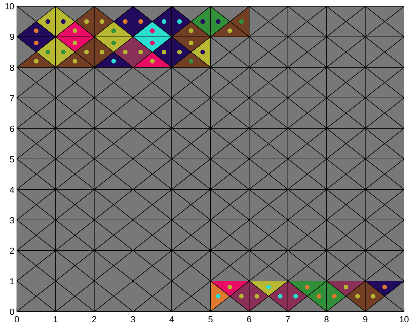




Convergence for 10x10 random puzzle
No randomization performed

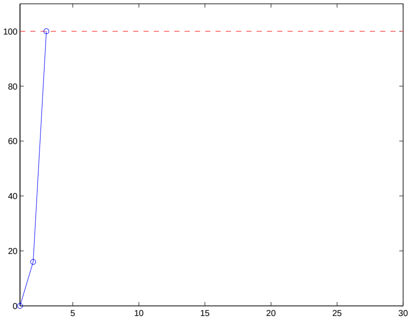
Convergence

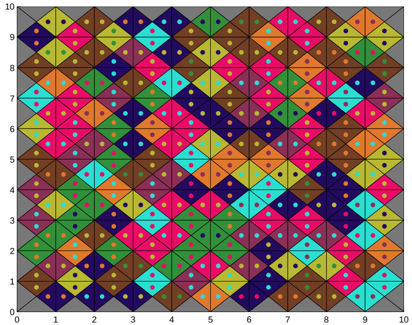




Convergence for 10x10 random puzzle
No randomization performed

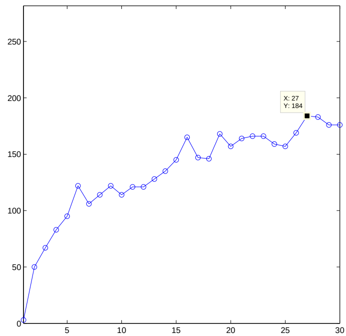
Convergence

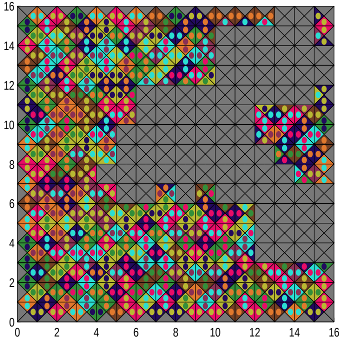




Convergence for 10x10 random puzzle
No randomization performed

Convergence





Convergence even for 16x16 original puzzle
Disabled randomization

Polyhedral Cone Theory Elimination

- Columns of E constitute a set of generators of

$$cone\mathcal{K} = \{Evec\tilde{\Phi} | \tilde{\Phi} \succeq 0\}$$

- Vector d resides on polyhedral cone's boundary

Finding smallest face generators

- Proposition. If $\exists\ z \in \mathcal{K}^*$ for which $\langle z, d\rangle = 0$ and $\langle z, E_i\rangle \neq 0$ then E_i cannot belong to the smallest face.
-

Finding smallest face generators

- Proposition. If $\exists\ z \in \mathcal{K}^*$ for which $\langle z, d\rangle = 0$ and $\langle z, E_i\rangle \neq 0$ then E_i cannot belong to the smallest face.
- $$\begin{aligned} \textit{find} \quad & z \in \mathbb{R}^n \\ s.t. \quad & d^T z = 0 \\ & E^T z \succeq 0 \\ & E_i^T z = 1 \end{aligned}$$

Finding smallest face generators

- Proposition. If $\exists\ z \in \mathcal{K}^*$ for which $\langle z, d \rangle = 0$ and $\langle z, E_i \rangle \neq 0$ then E_i cannot belong to the smallest face.
- Theorem of alternatives

$$\begin{aligned} \underset{\tilde{\Phi}, \mu \in \mathbb{R}}{find} \quad & \tilde{\Phi}, \mu \\ s.t. \quad & \mu b - E_i = Evec\tilde{\Phi} \\ & vec\Phi \succeq 0 \end{aligned}$$

Finding smallest face generators

- Proposition. If $\exists\ z \in \mathcal{K}^*$ for which $\langle z, d \rangle = 0$ and $\langle z, E_i \rangle \neq 0$ then E_i cannot belong to the smallest face.
- Saunders transformation

$$
\begin{aligned}
\underset{\tilde{\Phi},\mu \in \mathbb{R}}{find} \quad & \tilde{\Phi}, \mu \\
s.t. \quad & \mu b = E vec \tilde{\Phi} \\
& vec \Phi \succeq 0 \\
& (vec \Phi)_i \geq 0
\end{aligned}
$$

Finding smallest face generators

- Proposition. If $\exists\ z \in \mathcal{K}^*$ for which $\langle z, d \rangle = 0$ and $\langle z, E_i \rangle \neq 0$ then E_i cannot belong to the smallest face.
-
$$\begin{aligned} \underset{\tilde{\Phi}}{find} \quad & \tilde{\Phi} \\ s.t. \quad & b = Evec\tilde{\Phi} \\ & vec\Phi \succeq 0 \\ & (vec\Phi)_i = 0 \end{aligned}$$