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INEQUALITY DECOMPOSITION BY POPULATION SUBGROUPS

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This paper examines the implications of imposing a weak aggregation condition on inequality indices, so that the overall inequality value can be computed from information concerning the size, mean, and inequality value of each population subgroup. It is shown that such decomposable inequality measures must be monotonic transformations of additively decomposable indices. The general functional form of decomposable indices is derived without assuming that the measures are differentiable. The analysis is suitable for extension to the many other kinds of indices for which a similar relationship between the overall index value and subaggregates is desirable.

1. INTRODUCTION

MOST INDEX NUMBERS in common usage exhibit some kind of decomposition property that enables the overall index value to be computed from subaggregates. These subaggregates are typically based on grouping together observations which share a common characteristic. Thus, for instance, aggregate price and quantity indices are normally derived from subaggregates corresponding to commodity categories. Since the level of disaggregation and the grouping criteria are largely a matter of choice, it is clearly important that the overall index value should not depend on how the subcategories are selected. This consideration results in constraints being placed on the index numbers, and consequent restrictions on the types of functional forms that are admissible. It is not the intention of this paper to examine in full generality the question of consistent aggregation and the implications for the functional forms of index numbers. Instead, the relevant issues will be explored in detail in one specific context—that of inequality indices. At the same time, however, the framework of analysis appears suitable for extension to a wide variety of other index numbers, for which similar results may be expected to obtain.

A number of recent articles have been directed at the relationship between overall inequality values and the inequality levels corresponding to population subgroups. (See, for example, Blackorby et al. [**3**], Bourguignon [**4**], Cowell [**5**], Cowell and Kuga [**6**], Das and Parikh [**7**], Shorrocks [**12**], and Toyoda [**13**].) The central issue concerns the circumstances under which aggregate inequality can be expressed as a function of the subgroup inequality levels. If this can be done satisfactorily, the way is open to decompose the overall inequality level into the inequality contributions associated with each of the subgroups, or, alternatively, to aggregate upwards from the subgroup values to derive the composite figure. When the decomposition (or aggregation) is additive, the inequality measure I

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satisfies the constraint

$$(1) \qquad I(\boldsymbol{x}) = I(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_G) = \sum_g w_g I(\boldsymbol{x}_g) + B$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_G$ represents any partition of the distribution $\boldsymbol{x}$ into G subgroups; and where the coefficients w_g and the "between-group" term, B, depend only on subgroup means and population sizes. Shorrocks [**12**] shows that any such *additively decomposable* inequality measure, which is also differentiable, must take the form

$$(2) \qquad I(\boldsymbol{x}) = \alpha(\mu, n) \sum \{\phi(x_i) - \phi(\mu)\}$$

where μ and n respectively denote the mean and population size of $\boldsymbol{x}$. If $I(\cdot)$ is also scale invariant (homogeneous of degree zero in $\boldsymbol{x}$) and replication invariant (unchanged when the population and distribution are replicated), the only admissible indices belong to the single parameter "Generalized Entropy" family[2]

$$(3) \qquad I_c(\boldsymbol{x}) = \frac{1}{n} \frac{1}{c(c-1)} \sum_i \left\{ \left(\frac{x_i}{\mu} \right)^c - 1 \right\}, \qquad c \neq 0, 1,$$

with corresponding (limiting) expressions for $c = 0$ and $c = 1$.

Although an additive form of decomposition has particular attractions, it seems an unnecessarily severe restriction to impose as a general requirement. A weaker aggregation structure may well preserve many of the advantages of additively decomposable measures while allowing a greater variety of admissible indices, with the different perceptions of inequality that these would imply. This paper assumes a very weak aggregation condition, requiring only that the overall inequality level is some general function of the subgroup means, population sizes, and inequality values. As the additive nature of the aggregation requirement is completely relaxed, the corresponding inequality measures are described as being just *decomposable* or *aggregative*.

The following section sets out the notation and formal definitions employed in the paper. Section 3 explores the initial implications of decomposability and shows that a decomposable inequality measure must be an increasing transformation of an additively decomposable index. This result is achieved without presuming that either the inequality index or the aggregator function is differentiable. The analysis is extended in Section 4 by first demonstrating that the transformed version of the index can be chosen to satisfy equation (2), and then establishing the general form of a decomposable inequality measure that is also scale invariant and/or replication invariant. This again relies on just continuity, rather than differentiability, of the relevant functions. Section 5 summarizes the main results and indicates potential directions for further research.

For the purposes of exposition, the analysis is developed in terms of the inequality of income, but is clearly applicable to many other types of distributions.

[2] The Generalized Entropy family is also discussed in Cowell and Kuga [**6**].

2. DEFINITIONS AND PRELIMINARY RESULTS

Let $x_i \in \mathscr{R}_{++}$ represent the (positive) income of person i, and $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathscr{R}^n_{++}$ be an income distribution vector for a population containing n individuals. The set of all strictly positive income vectors of dimension at least m is denoted by $X_m = \bigcup_{n=m}^{\infty} \mathscr{R}^n_{++}$. It will be convenient to write the dimension of any vector $\boldsymbol{x}$ (i.e., the population size) as $n(\boldsymbol{x})$ and the mean of $\boldsymbol{x}$ as $\mu(\boldsymbol{x})$, so $\boldsymbol{\theta}(\boldsymbol{x}) = (\mu(\boldsymbol{x}), n(\boldsymbol{x}))$ can be regarded as a "parameter vector" for the distribution $\boldsymbol{x}$. Income distributions drawn from X_1 with a common parameter vector $\boldsymbol{\theta}$ comprise the set $S(\boldsymbol{\theta}) \equiv \{\boldsymbol{x} \in X_1 | \boldsymbol{\theta}(\boldsymbol{x}) = \boldsymbol{\theta}\}$. The notation $\bar{\boldsymbol{x}}$ is employed as shorthand for the "equalized version" of $\boldsymbol{x}$, defined by $n(\bar{\boldsymbol{x}}) = n(\boldsymbol{x})$ and $\bar{x}_i = \mu(\boldsymbol{x})$ for all i.

An inequality index is defined as a function $I: X_1 \to \mathscr{R}$ with the properties:

(4a) $I(\boldsymbol{x})$ is a continuous and symmetric function of $\boldsymbol{x}$,

(4b) $I(\bar{\boldsymbol{x}}) = 0$ (a normalization condition),

(4c) $I(\boldsymbol{Bx}) < I(\boldsymbol{x})$ for all bistochastic matrices $\boldsymbol{B}$ that are not permutation matrices (strict Schur-concavity).

Note that, strictly speaking, I refers to a countable collection of functions $I^1, I^2, I^3, \ldots$, where $I^n: \mathscr{R}^n_{++} \to \mathscr{R}$ is an inequality index constructed for n-person distributions (and satisfying (4)). However no ambiguity arises, and the notation is simplified, by writing $I(\boldsymbol{x}) \equiv I^{n(x)}(\boldsymbol{x})$. Condition (4c) is equivalent to the strict version of the Pigou–Dalton principle of transfers: any positive mean preserving transfer from one individual to a richer person increases inequality. Conditions (4b) and (4c) together ensure that $I(\boldsymbol{x}) > 0$ whenever $\boldsymbol{x} \neq \bar{\boldsymbol{x}}$.

The other principal assumption to be imposed on the inequality index is that of decomposability. An index I will be said to be *decomposable* or *aggregative*[3] iff there exists an "aggregator" function A such that

$$(5) \qquad I(\boldsymbol{x}, \boldsymbol{y}) = A(I(\boldsymbol{x}), \mu(\boldsymbol{x}), n(\boldsymbol{x}), I(\boldsymbol{y}), \mu(\boldsymbol{y}), n(\boldsymbol{y})) = A(I(\boldsymbol{x}), \boldsymbol{\theta}(\boldsymbol{x}), I(\boldsymbol{y}), \boldsymbol{\theta}(\boldsymbol{y})) \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in X_1,$$

where A is continuous and strictly increasing in the index values $I(\boldsymbol{x}), I(\boldsymbol{y})$.[4] In the decomposition sense, equation (5) implies that when any population numbering $n(\boldsymbol{x}, \boldsymbol{y}) \geq 2$ is divided into disjoint and exhaustive subgroups numbering $n(\boldsymbol{x}), n(\boldsymbol{y}) \geq 1$, so that the overall income distribution $(\boldsymbol{x}, \boldsymbol{y})$ is partitioned into the subgroup vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, then the overall inequality level $I(\boldsymbol{x}, \boldsymbol{y})$ can be expressed in terms of the subgroup means, sizes, and inequality values. Alternatively, in the aggregation sense, condition (5) states that if two groups with

[3] This is the term used by Bourguignon [4, p. 903].

[4] Although (5) is defined in terms of 2-group aggregation only, in contrast to equation (1), the aggregation can be extended by recursion to any number of subgroups. The explicit assumption that A is strictly increasing in the index argument is technically unnecessary, since it is a consequence of assumption (4c). For any change in the value of $I(\boldsymbol{x})$ can be generated by a sequence of transfers, all of which are either progressive or regressive, and the value of $I(\boldsymbol{x}, \boldsymbol{y})$ must therefore change in the same direction, respecting the movement in the subgroup index values.

distributions $\boldsymbol{x}$ and $\boldsymbol{y}$ are combined into a single population, then calculation of aggregate inequality only requires information on the size, mean income, and inequality value of the component distributions. For the purposes of this paper, either of these interpretations can be used, although the analysis tends to be framed in terms of the aggregation of groups into larger populations.

Two further assumptions frequently imposed on inequality measures concern the reaction of the index to a replication of the population and its distribution, and to a proportional change in all incomes. An inequality index I is *replication invariant*, or satisfies the principle of *population replication*, if

(6) $$I(\boldsymbol{R}_r\boldsymbol{x}) \equiv I(\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x}) = I(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in X_1 \text{ and all } r$$

where $\boldsymbol{R}_r$ is a "replicator matrix" of dimension $n(\boldsymbol{x}) \times rn(\boldsymbol{x})$, taking the form $\boldsymbol{R}_r = [\boldsymbol{E}, \boldsymbol{E}, \ldots, \boldsymbol{E}]$ for some identity matrix $\boldsymbol{E}$ of suitable dimension. An inequality index I is *scale invariant* (or *income homogeneous*, or *mean independent*) if $I(\boldsymbol{x})$ is homogeneous of degree zero in all incomes.

This section concludes with a simple result on functional equations used extensively in the subsequent analysis.

LEMMA 1: *If F, G, f_1, f_2, g_1, and g_2 are continuous and strictly monotonic functions $\mathcal{R} \to \mathcal{R}$ and if*

$$F^{-1}(f_1(u) + f_2(v)) = G^{-1}(g_1(u) + g_2(v)),$$

then there exist constants a, b, and c such that

$$f_1(u) = cg_1(u) + a,$$

$$f_2(v) = cg_2(v) + b,$$

$$F(w) = cG(w) + a + b.$$

Furthermore, if $f_k(0) = g_k(0) = 0$ for $k = 1, 2$, then $a = b = 0$.

PROOF: Setting $g_1(u) = x$, $g_2(v) = y$ we obtain

$$f_1 \circ g_1^{-1}(x) + f_2 \circ g_2^{-1}(y) = F \circ G^{-1}(x + y)$$

where $\circ$ denotes composition of functions. This is a Pexider equation whose solution [**1**, p. 142] is

$$f_1 \circ g_1^{-1}(x) = cx + a,$$

$$f_2 \circ g_2^{-1}(y) = cy + b,$$

$$F \circ G^{-1}(z) = cz + a + b.$$

The result then follows immediately.

3. PROPERTIES OF DECOMPOSABLE INEQUALITY MEASURES

It is clear from the requirements of (4) and (5) that any transformation $J = F(I)$ of any decomposable inequality index $I(\boldsymbol{x})$ will also be a decomposable inequality index, as long as F is continuous, strictly increasing, and preserves the origin. The decomposition and inequality index characteristics are also retained if F is allowed to vary continuously with the mean of x, and to depend in an arbitrary way on the dimension $n(x)$ of x. Performing such transformations will alter the structure of the corresponding aggregator function defined over the subgroup index and parameter values. The results of this section demonstrate that a suitable transformation can always be found to convert any decomposable inequality index I into an *additively* decomposable inequality index J. More specifically J can be made to satisfy the equation

$$J(x, y) = J(x) + J(y) + J(\bar{x}, \bar{y}) \quad \text{for all } x, y \in X_1$$

which will be referred to as the *normal form* equation for a decomposable inequality index. It follows that, for any decomposable inequality index "in normal form," the corresponding aggregator function is additive in the subgroup index values. Note that the original index I and the transformed measure J will be ordinally equivalent in the restricted sense of agreeing in their ranking of pairs of distributions with identical means and population sizes.

THEOREM 1: *If I is a decomposable inequality index, there exists another index $J(x) = F(I(x), \boldsymbol{\theta}(x))$ such that*

$$J(x, y) = J(x) + J(y) + J(\bar{x}, \bar{y}) \quad \textit{for all } x, y \in X_2, \tag{7}$$

where

$$F(I, \boldsymbol{\theta}) \textit{ is continuous and strictly increasing in } I, \tag{8a}$$

$$F(0, \boldsymbol{\theta}) = 0 \textit{ for all } \boldsymbol{\theta}. \tag{8b}$$

PROOF:[5] Let I' be any decomposable inequality index and define $\Theta = \{\boldsymbol{\theta}(x) | x \in X_2\}$. For all $\boldsymbol{\theta} \in \Theta$, $S(\boldsymbol{\theta})$ is a connected, open subset of X_2 containing more than one element. Hence, using the properties (4),

$$I'[S(\boldsymbol{\theta})] = \{I'(x) | x \in S(\boldsymbol{\theta})\} = [0, \xi(\boldsymbol{\theta}))$$

where $\xi(\boldsymbol{\theta})$ may be finite or infinite. Choose any function $\psi(u, \boldsymbol{\theta})$, continuous and strictly increasing in u, such that $I(x) = \psi(I'(x), \boldsymbol{\theta}(x))$ and $I[S(\boldsymbol{\theta})] = [0, 1) \equiv U$. Since I inherits the symmetry and decomposition characteristics of I', there exists an aggregator function A such that

$$I(x, y) = A(I(x), \boldsymbol{\theta}(x), I(y), \boldsymbol{\theta}(y)) \quad \text{for all } x, y \in X_2$$

where A is continuous and strictly increasing in the index arguments.

For $g = 1, 2, 3$, choose any fixed $\boldsymbol{\theta}_g^* \in \Theta$ and any $x_g \in S(\boldsymbol{\theta}_g)$. Writing $x_{gh} = (x_g, x_h)$, $\boldsymbol{\theta}_{gh}^* = \boldsymbol{\theta}(x_{gh})$ and $I_k = I(x_k)$, we obtain

$$\begin{aligned} I_{gh} &= I(x_{gh}) = I(x_g, x_h) \\ &= A(I(x_g), \boldsymbol{\theta}_g^*, I(x_h), \boldsymbol{\theta}_h^*) = A_{g,h}(I_g, I_h) \end{aligned}$$

[5] The initial part of this proof is similar to Gorman's [9] overlapping theorem for separable functions.

and

$$\begin{aligned} A_{12,3}(A_{1,2}(I_1, I_2), I_3) &= A(I_{12}, \boldsymbol{\theta}_{12}^*, I_3, \boldsymbol{\theta}_3^*) \\ &= I(\boldsymbol{x}_{12}, \boldsymbol{x}_3) = I(\boldsymbol{x}_1, \boldsymbol{x}_{23}) \\ &= A(I_1, \boldsymbol{\theta}_1^*, I_{23}, \boldsymbol{\theta}_{23}^*) \\ &= A_{1,23}(I_1, A_{2,3}(I_2, I_3)) \end{aligned}$$

where $A_{k,l}(u, v) \equiv A(u, \boldsymbol{\theta}_k^*, v, \boldsymbol{\theta}_l^*)$ is continuous and strictly increasing in u and v. Now select any continuous, strictly increasing functions ϕ_{12} and ϕ_{23} such that $\hat{A}_{g,h} \equiv \phi_{gh} \circ A_{g,h}: U \times U \to_{\text{onto}} U$. Then

$$\begin{aligned} \hat{A}_{12,3}(\hat{A}_{1,2}(I_1, I_2), I_3) &\equiv A_{12,3}(\phi_{12}^{-1} \circ \hat{A}_{1,2}(I_1, I_2), I_3) \\ &= A_{12,3}(A_{1,2}(I_1, I_2), I_3) \\ &= A_{1,23}(I_1, \phi_{23}^{-1} \circ \hat{A}_{2,3}(I_2, I_3)) \\ &\equiv \hat{A}_{1,23}(I_1, \hat{A}_{2,3}(I_2, I_3)) \end{aligned}$$

where I_1, I_2, I_3, $\hat{A}_{1,2}$ and $\hat{A}_{2,3}$ can take any value in the interval U, given a suitable choice of $\boldsymbol{x}_g \in S(\boldsymbol{\theta}_g^*)$. By an appropriate normalization of $\hat{A}_{12,3}$ and $\hat{A}_{1,23}$ onto the interval U, the prerequisites of Aczel's theorem on associative functions [**1**, p. 312, Corollary 1], are satisfied. Hence there exist continuous and strictly monotonic functions $\hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_{12}$, and $\hat{f}_{23}$ such that

$$\begin{aligned} \hat{A}_{1,2}(u, v) &= \hat{f}_{12}^{-1}(\hat{f}_1(u) + \hat{f}_2(v)), \\ \hat{A}_{2,3}(u, v) &= \hat{f}_{23}^{-1}(\hat{f}_2(u) + \hat{f}_3(v)). \end{aligned}$$

By now setting

$$\begin{aligned} f_g(u) &= \{\hat{f}_g(u) - \hat{f}_g(0)\}/\{\hat{f}_2(0.5) - \hat{f}_2(0)\} \quad \text{when } g = 1, 2, 3, \\ f_{gh}(v) &= \{\hat{f}_{gh} \circ \phi_{gh}(v) - \hat{f}_g(0) - \hat{f}_h(0)\}/\{\hat{f}_2(0.5) - f_2(0)\}, \end{aligned}$$

we obtain

(9a) $$I(\boldsymbol{x}_1, \boldsymbol{x}_2) = A(I_1, \boldsymbol{\theta}_1^*, I_2, \boldsymbol{\theta}_2^*) = f_{12}^{-1}(f_1(I_1) + f_2(I_2)),$$

(9b) $$I(\boldsymbol{x}_2, \boldsymbol{x}_3) = A(I_2, \boldsymbol{\theta}_2^*, I_3, \boldsymbol{\theta}_3^*) = f_{23}^{-1}(f_2(I_2) + f_3(I_3)),$$

where

(10a) f_k is continuous and strictly increasing $\quad (k = 1, 2, 3, 12, 23)$,

(10b) $f_g(0) = 0 \quad (g = 1, 2, 3)$,

(10c) $f_2(0.5) = 1$.

In general we may expect the functions f_k to depend on the choice of $\boldsymbol{\theta}_1^*$, $\boldsymbol{\theta}_2^*$, $\boldsymbol{\theta}_3^*$. Suppose $\boldsymbol{\theta}_3$ changes to $\tilde{\boldsymbol{\theta}}_3$ and denote the new functions f_k with a tilde superscript. Then, from (9a)

$$f_{12}^{-1}(f_1(I_1) + f_2(I_2)) = \tilde{f}_{12}^{-1}(\tilde{f}_1(I_1) + \tilde{f}_2(I_2))$$

and Lemma 1 combined with conditions (10) yields

$$\tilde{f}_k(u) = f_k(u) \quad \text{when } k = 1, 2, 12.$$

Thus f_1, f_2, and f_{12} do not depend on the choice of $\boldsymbol{\theta}_3^*$. Similar reasoning applied to equation (9b) establishes that f_2, f_3, and f_{23} are independent of the choice of $\boldsymbol{\theta}_1^*$. Making explicit the dependence of f_k on the values of $\boldsymbol{\theta}_g = \boldsymbol{\theta}(\boldsymbol{x}_g)$, and allowing $\boldsymbol{x}_g$ to be chosen freely from the set $X_2 = \{\boldsymbol{x} \in S(\boldsymbol{\theta}) | \boldsymbol{\theta} \in \Theta\}$ converts equation (9a) into

$$(11) \qquad I(\boldsymbol{x}_1, \boldsymbol{x}_2) = f_{12}^{-1}(f_1(I(\boldsymbol{x}_1), \boldsymbol{\theta}_1, \boldsymbol{\theta}_2) + f_2(I(\boldsymbol{x}_2), \boldsymbol{\theta}_2), \boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \quad \text{for all } \boldsymbol{x}_1, \boldsymbol{x}_2 \in X_2$$

where inversion of f_{12} is taken with respect to its first argument.[6] But the symmetry of I also implies that

$$I(\boldsymbol{x}_1, \boldsymbol{x}_2) = I(\boldsymbol{x}_2, \boldsymbol{x}_1) = f_{12}^{-1}(f_1(I(\boldsymbol{x}_2), \boldsymbol{\theta}_2, \boldsymbol{\theta}_1) + f_2(I(\boldsymbol{x}_1), \boldsymbol{\theta}_1), \boldsymbol{\theta}_2, \boldsymbol{\theta}_1).$$

Applying Lemma 1 to the last two equations then gives

$$f_1(u, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = cf_2(u, \boldsymbol{\theta}_1),$$

$$f_2(v, \boldsymbol{\theta}_2) = cf_1(v, \boldsymbol{\theta}_2, \boldsymbol{\theta}_1),$$

and (10c) ensures that

$$\begin{aligned} c &= f_1(0.5, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \equiv c(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) > 0 \\ &= f_1(0.5, \boldsymbol{\theta}_2, \boldsymbol{\theta}_1)^{-1} = c(\boldsymbol{\theta}_2, \boldsymbol{\theta}_1)^{-1}. \end{aligned}$$

Hence equation (11) can be rewritten in the form

$$(12) \qquad h(J(\boldsymbol{x}_1, \boldsymbol{x}_2), \boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = c(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)J(\boldsymbol{x}_1) + J(\boldsymbol{x}_2) \quad \text{for all } \boldsymbol{x}_1, \boldsymbol{x}_2 \in X_2$$

where $J(\boldsymbol{x}) \equiv f_2(I(\boldsymbol{x}), \boldsymbol{\theta}(\boldsymbol{x}))$, and h is continuous and strictly increasing in its first argument.

Now consider any $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3 \in \Theta$ and choose $\boldsymbol{x}_g \in S(\boldsymbol{\theta}_g)$. Using the earlier notation, define $\boldsymbol{x}_{gh} \equiv (\boldsymbol{x}_g, \boldsymbol{x}_h)$, $\boldsymbol{\theta}_k \equiv \boldsymbol{\theta}(\boldsymbol{x}_k)$, $J_k \equiv J(\boldsymbol{x}_k)$, and $c_{k,l} \equiv c(\boldsymbol{\theta}_k, \boldsymbol{\theta}_l)$. Then equation (12) implies

$$\begin{aligned} h^{-1}(c_{12,3}h^{-1}(J_2, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2) + J_3, \boldsymbol{\theta}_{12}, \boldsymbol{\theta}_3) &= h^{-1}(c_{12,3}J(\bar{\boldsymbol{x}}_1, \boldsymbol{x}_2) + J_3, \boldsymbol{\theta}_{12}, \boldsymbol{\theta}_3) \\ &= J(\bar{\boldsymbol{x}}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = J(\bar{\boldsymbol{x}}_1, \boldsymbol{x}_{23}) \\ &= h^{-1}(J(\boldsymbol{x}_{23}), \boldsymbol{\theta}_1, \boldsymbol{\theta}_{23}) \\ &= h^{-1}(h^{-1}(c_{2,3}J_2 + J_3, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3), \boldsymbol{\theta}_1, \boldsymbol{\theta}_{23}) \\ &\equiv H(c_{2,3}J_2 + J_3, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3). \end{aligned}$$

Applying Lemma 1 gives

$$c_{12,3}h^{-1}(u, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \gamma c_{2,3}u + \alpha,$$

$$v = \gamma v + \beta,$$

[6] This convention with respect to inversion of functions of several variables is retained throughout this paper.

and setting $u=v=0$ establishes that $\beta=0$, $\gamma=1$, $\alpha=c_{12,3}h^{-1}(0, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, and

$$h^{-1}(u, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2)-h^{-1}(0, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2)=\frac{c_{2,3}}{c_{12,3}}u=\frac{c(\boldsymbol{\theta}_2, \boldsymbol{\theta}_3)}{c(\boldsymbol{\theta}_{12}, \boldsymbol{\theta}_3)}u. \tag{13}$$

The left hand side of (13) is independent of $\boldsymbol{\theta}_3$, so for any fixed $\boldsymbol{\theta}^*$

$$\begin{aligned}\frac{c(\boldsymbol{\theta}_2, \boldsymbol{\theta}_3)}{c(\boldsymbol{\theta}_{12}, \boldsymbol{\theta}_3)}&=\frac{c(\boldsymbol{\theta}_2, \boldsymbol{\theta}^*)}{c(\boldsymbol{\theta}_{12}, \boldsymbol{\theta}^*)}=\frac{c(\boldsymbol{\theta}_2)}{c(\boldsymbol{\theta}_{12})}\\ &=\frac{c(\boldsymbol{\theta}_2, \boldsymbol{\theta}_{12})}{c(\boldsymbol{\theta}_{12}, \boldsymbol{\theta}_{12})}=c(\boldsymbol{\theta}_2, \boldsymbol{\theta}_{12})\end{aligned}$$

since $c(\boldsymbol{\theta}, \boldsymbol{\theta})=c(\boldsymbol{\theta}, \boldsymbol{\theta})^{-1}=1$. Substituting into equations (12) and (13) then yields

$$\begin{aligned}J(\boldsymbol{x}_1, \boldsymbol{x}_2)&=h^{-1}(c(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)J(\boldsymbol{x}_1)+J(\boldsymbol{x}_2), \boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\\ &=\frac{c(\boldsymbol{\theta}_2)}{c(\boldsymbol{\theta}_{12})}\left\{\frac{c(\boldsymbol{\theta}_1)}{c(\boldsymbol{\theta}_2)}J(\boldsymbol{x}_1)+J(\boldsymbol{x}_2)\right\}+h^{-1}(0, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\end{aligned}$$

and

$$\begin{aligned}J'(\boldsymbol{x}_1, \boldsymbol{x}_2)&\equiv c(\boldsymbol{\theta}_{12})J(\boldsymbol{x}_1, \boldsymbol{x}_2)=J'(\boldsymbol{x}_1)+J'(\boldsymbol{x}_2)+c(\boldsymbol{\theta}_{12})h^{-1}(0, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\\ &=J'(\boldsymbol{x}_1)+J'(\boldsymbol{x}_2)+J'(\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2).\end{aligned}$$

The proof is completed by noting that the sequence of transformations $I=\psi(I, \boldsymbol{\theta})$, $J=f_2(I, \boldsymbol{\theta})$, and $J'=c(\boldsymbol{\theta})J$ can be combined into a single transformation $J'=F(I', \boldsymbol{\theta})$ satisfying conditions (8).

Theorem 1 is a powerful result that provides the foundation for the remainder of this paper. The next theorem extends the analysis by demonstrating that the transformed index J has the properties of an inequality measure, and that its corresponding decomposition equation is applicable to any number of subgroups containing any positive number of persons.

THEOREM 2: *For any decomposable inequality index I there exists a function $F(I, \mu, n)$ continuous in I and μ, and strictly increasing in I, such that $J(\boldsymbol{x})\equiv F(I(\boldsymbol{x}), \mu(\boldsymbol{x}), n(\boldsymbol{x}))$ is another decomposable inequality index satisfying*

$$J(\boldsymbol{x}_1, \dots, \boldsymbol{x}_G)=\sum_{g=1}^{G} J(\boldsymbol{x}_g)+J(\bar{\boldsymbol{x}}_1, \dots, \bar{\boldsymbol{x}}_G) \quad \textit{for all } G\geq 2 \textit{ and all } \boldsymbol{x}_g\in X_1. \tag{14}$$

PROOF: First consider $G=2$. From Theorem 1 there exists a transformation $J=F(I, \mu, n)$ with the properties (8) such that (14) holds for all $\boldsymbol{x}_g\in X_2$. By defining $J(\boldsymbol{x})=0$ when $n(\boldsymbol{x})=1$, the index J is extended to single person distributions. Now (7) implies that

$$J(\boldsymbol{x}, \boldsymbol{x})=J(\boldsymbol{x})+J(\boldsymbol{x})+J(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}})=2J(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x}\in X_2$$

and this result also holds for all $x \in X_1$, since $J(x, x) = 0 = J(x)$ when $n(x) = 1$. Therefore, for all $x_1, x_2 \in X_1$,

$$J(x_1, x_2) = \tfrac{1}{2}J(x_1, x_2, x_1, x_2) = \tfrac{1}{2}J(x_1, x_1, x_2, x_2)$$

since J inherits the symmetry of I. Applying (7) we obtain

$$\begin{aligned} J(x_1, x_2) &= \tfrac{1}{2}J(x_1, x_1) + \tfrac{1}{2}J(x_2, x_2) + \tfrac{1}{2}J(\bar{x}_1, \bar{x}_1, \bar{x}_2, \bar{x}_2) \\ &= J(x_1) + J(x_2) + J(\bar{x}_1, \bar{x}_2) \end{aligned}$$

demonstrating that (14) holds when $G = 2$.

Extension to any $G > 2$ is accomplished by induction, since

$$\begin{aligned} J(x_1, \ldots, x_G) &= J(x_1, \ldots, x_{G-2}, (x_{G-1}, x_G)) \\ &= \sum_{g=1}^{G-2} J(x_g) + J(x_{G-1}, x_G) + B(\bar{x}_1, \ldots, \bar{x}_G) \\ &= \sum_{g=1}^{G} J(x_g) + J(\bar{x}_{G-1}, \bar{x}_G) + B(\bar{x}_1, \ldots, \bar{x}_G) \end{aligned}$$

and (14) is obtained by replacing x_g by $\bar{x}_g$ throughout.

The properties (8) of F ensure that J inherits the symmetry, normalization (4b) and Schur-concavity (4c) features of I. J is also continuous over any set $S(\mu, n)$ when μ and n are fixed. However it remains to be shown that J is continuous over $\mathscr{R}^n_{++}$. Consider any sequence $x^{(k)} \in \mathscr{R}^n_{++}$ with the limit $x^{(\infty)} \in \mathscr{R}^n_{++}$. Select any value μ such that $\mu > \mu(x^{(k)})$ for all k, and define $u^{(k)} = n(\mu - \mu(x^{(k)})) > 0$. It then follows from (14) that

$$J(x^{(k)}, u^{(k)}) = J(x^{(k)}) + J(u^{(k)}) + J(\bar{x}^{(k)}, u^{(k)})$$

and, since J is continuous over the set $S(\mu, n)$,

$$\begin{aligned} \operatorname*{Lt}_{k\to\infty} J(x^{(k)}) &= \operatorname*{Lt}_{k\to\infty} \{J(x^{(k)}, u^{(k)}) - J(\bar{x}^{(k)}, u^{(k)})\} \\ &= J(x^{(\infty)}, u^{(\infty)}) - J(\bar{x}^{(\infty)}, u^{(\infty)}) = J(x^{(\infty)}). \end{aligned}$$

Thus J is continuous over $\mathscr{R}^n_{++}$ for any n, and is therefore an inequality index.

The primary implication of Theorem 2 is that any decomposable inequality measure can be transformed, via F, into an additively decomposable inequality measure satisfying the "normal form" equation (14), which is a special case of equation (1). As additive decomposability (together with differentiability) is known to lead to indices of the form given in (2), the transition to decomposable measures can be expected to expand the class of admissible indices only by adding monotonic transformations of the type indicated. This introduces new numbering schemes for the index values, and different ranking possibilities for pairs of distributions with different means or population sizes, since the transformation F can depend on μ and n. However, any decomposable inequality index I will be constrained to agree with its corresponding additively decomposable index J when ranking distributions with similar sized populations and identical aggregate incomes. To this extent relaxing the additive element of the index aggregation provides little more than alternative, ordinally equivalent, versions of the same set of indices. This is sufficient to encompass the Atkinson [2] family of inequality measures, which are decomposable, but not additively decomposable; but does not admit the Gini coefficient, which cannot be expressed as a suitable transformation of an additively decomposable index.

The normal form index J corresponding to any decomposable index I indicates a particularly satisfactory way of assigning inequality contributions to the population subgroups. For, when equation (14) holds, we can define the subgroup contributions

$$C_g = J(\boldsymbol{x}_g) \tag{15a}$$

$$= J(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_G) - J(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{g-1}, \bar{\boldsymbol{x}}_g, \boldsymbol{x}_{g+1}, \ldots, \boldsymbol{x}_G) \tag{15b}$$

$$= J(\bar{\boldsymbol{x}}_1, \ldots, \bar{\boldsymbol{x}}_{g-1}, \boldsymbol{x}_g, \bar{\boldsymbol{x}}_{g+1}, \ldots, \bar{\boldsymbol{x}}_G) - J(\bar{\boldsymbol{x}}_1, \ldots, \bar{\boldsymbol{x}}_G) \tag{15c}$$

and

$$B = J(\bar{\boldsymbol{x}}_1, \ldots, \bar{\boldsymbol{x}}_G)$$

to obtain

$$J(\boldsymbol{x}) = J(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_G) = \sum_{g=1}^{G} C_g + B.$$

Thus the overall inequality level can be expressed as the sum of the subgroup contributions plus the "between group" term B. Furthermore, C_g is consistent with all the obvious interpretations of "the contribution to equality of subgroup g": either the inequality level of subgroup g alone (15a); the amount by which the overall J value falls if inequality within subgroup g is eliminated (15b); or the amount by which the overall J value increases if inequality within subgroup g is introduced, starting from the position of equality within each subgroup (15c). Since these interpretations will not in general coincide for the untransformed index I, there is a good argument for using the normal form of the index, and its corresponding assignment of contributions, in the absence of any strong preference for one particular cardinalization of a decomposable inequality measure.

Another interesting characteristic of the normal form index J is that it almost satisfies the replication invariance property. Specifically, by choosing r subgroups with identical distributions $\boldsymbol{x}$, equation (14) implies

$$J(\boldsymbol{R}_r\boldsymbol{x}) = J(\boldsymbol{x}, \ldots, \boldsymbol{x}) = rJ(\boldsymbol{x}) + J(\bar{\boldsymbol{x}}, \ldots, \bar{\boldsymbol{x}}) = rJ(\boldsymbol{x}). \tag{16}$$

Thus $J'(\boldsymbol{x}) = J(\boldsymbol{x})/n(\boldsymbol{x})$ is a replication invariant inequality index. Since replacing J with J' in (14) still generates an equation of the form (1), J' is also additively decomposable. It therefore follows that any decomposable inequality index is a monotonic transformation of a replication invariant, additively decomposable inequality index.

4. THE CLASS OF DECOMPOSABLE INEQUALITY MEASURES

Having established the general characteristics of decomposable inequality measures, we now turn to the question of identifying the functional representation of such indices. The following theorem demonstrates, without the assumption of differentiability, that decomposable inequality measures in normal form necessarily satisfy a restricted version of equation (2).

THEOREM 3: *If $J(\boldsymbol{x})$ is any continuous and symmetric function satisfying*

$$J(\boldsymbol{x}, \boldsymbol{y}) = J(\boldsymbol{x}) + J(\boldsymbol{y}) + J(\bar{x}, \bar{y}) \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in X_1, \tag{17}$$

then there exists a continuous function $\phi: \mathscr{R}_{++} \to \mathscr{R}$ such that

$$J(\boldsymbol{x}) = \sum_{i=1}^{n(\boldsymbol{x})} \{\phi(x_i) - \phi(\mu(\boldsymbol{x}))\} \quad \text{for all } \boldsymbol{x} \in X_1. \tag{18}$$

PROOF: Let $\boldsymbol{x}$ be any vector from X_3, and $\boldsymbol{e}_i$ be the ith standard basis vector ($e_{ii} = 1$; $e_{ij} = 0$ for $i \neq j$) of the same dimension. Then (17) implies

$$\begin{aligned} J(\boldsymbol{x} + \delta(\boldsymbol{e}_1 - \boldsymbol{e}_2)) - J(\boldsymbol{x}) &\equiv J(x_1 + \delta, x_2 - \delta, x_3, \ldots, x_n) - J(x_1, x_2, \ldots, x_n) \\ &= J(x_1 + \delta, x_2 - \delta) - J(x_1, x_2) \end{aligned}$$

and more generally, because J is symmetric,

$$J(\boldsymbol{x} + \delta(\boldsymbol{e}_i - \boldsymbol{e}_j)) - J(\boldsymbol{x}) = J(x_i + \delta, x_j - \delta) - J(x_i, x_j)$$

for all δ such that $x_i + \delta > 0$ and $x_j - \delta > 0$. Therefore

$$\begin{aligned} &J(x_i + \delta, x_k - \delta) - J(x_i, x_k) + J(x_k, x_j - \delta) - J(x_k - \delta, x_j) \\ &\quad = J(\boldsymbol{x} + \delta(\boldsymbol{e}_i - \boldsymbol{e}_k)) - J(\boldsymbol{x}) + J(\boldsymbol{x} + \delta(\boldsymbol{e}_i - \boldsymbol{e}_j)) - J(\boldsymbol{x} + \delta(\boldsymbol{e}_i - \boldsymbol{e}_k)) \\ &\quad = J(\boldsymbol{x} + \delta(\boldsymbol{e}_i - \boldsymbol{e}_j)) - J(\boldsymbol{x}) \\ &\quad = J(x_i + \delta, x_j - \delta) - J(x_i, x_j). \end{aligned}$$

Setting $x_i = u + 1$, $x_j = v + w + 1$, $x_k = v + 1$, and $\delta = v$, we obtain

$$\begin{aligned} &J(u + v + 1, 1) - J(u + 1, v + 1) + J(v + 1, w + 1) - J(1, v + w + 1) \\ &\quad = J(u + v + 1, w + 1) - J(u + 1, v + w + 1) \end{aligned}$$

and this becomes

$$\Delta(u + v, w) + \Delta(u, v) = \Delta(u, v + w) + \Delta(v, w) \tag{19}$$

when $\Delta(u, v) \equiv J(u + 1, v + 1) - J(u + 1, 1) - J(v + 1, 1)$. Note that the properties of J ensure that Δ is continuous and that $\Delta(u, v) = \Delta(v, u)$.

Equation (19) has the solution (Erdos [8])[7]

$$\Delta(u, v) = \psi(u+v) - \psi(u) - \psi(v)$$

for some continuous function ψ. Hence

$$\begin{aligned} J(u, v) &= \Delta(u-1, v-1) + J(u, 1) + J(v, 1) \\ &= \psi(u+v-2) + J(u, 1) - \psi(u) + J(v, 1) - \psi(v) \\ &= \psi(u+v-2) + \phi(u) + \phi(v) \end{aligned}$$

where $\phi(u) \equiv J(u, 1) - \psi(u)$. But $J(u, u) = 0 = \psi(2u-2) + 2\phi(u)$, so $\psi(w-2) = -2\phi(w/2)$ and

$$J(u, v) = \phi(u) + \phi(v) - 2\phi\left(\frac{u+v}{2}\right).$$

This confirms that (18) holds for all $\boldsymbol{x} \in X_1$ such that $n(\boldsymbol{x}) = 2$ (and also, trivially, when $n(\boldsymbol{x}) = 1$).

The result can be extended to all $\boldsymbol{x} \in X_1$ by induction on $n(\boldsymbol{x})$. Suppose $\mu = \mu(\boldsymbol{x})$ and $n = n(\boldsymbol{x})$. Then (17) implies

$$J(\mu, \boldsymbol{x}) = J(\boldsymbol{x}) + J(\mu, \bar{\boldsymbol{x}}) = J(\boldsymbol{x})$$

and also

$$\begin{aligned} J(\boldsymbol{x}) &= J(x_1, x_2) - J(\mu, x_1 + x_2 - \mu) + J(\mu, x_1 + x_2 - \mu, x_3, \ldots, x_n) \\ &= J(x_1, x_2) - J(\mu, x_1 + x_2 - \mu) + J(x_1 + x_2 - \mu, x_3, \ldots, x_n) \\ &= \phi(x_1) + \phi(x_2) - \phi(\mu) - \phi(x_1 + x_2 - \mu) + \phi(x_1 + x_2 - \mu) \\ &\quad + \sum_{i=3}^{n} \phi(x_i) - (n-1)\phi(\mu) \\ &= \sum_{i=1}^{n} \{\phi(x_i) - \phi(\mu)\} \end{aligned}$$

where it has been assumed that $x_1 + x_2 > \mu$ (without loss of generality, since J is symmetric). This completes the proof.

The results of Theorems 2 and 3 can be combined to provide a general statement concerning the functional form of decomposable inequality indices. Given the discussion of replication invariance following equation (16), it is also a simple matter to incorporate this property.

THEOREM 4: *I is a decomposable inequality index iff there exist functions F and ϕ such that*

$$F(I(\boldsymbol{x}), \mu(\boldsymbol{x}), n(\boldsymbol{x})) = \frac{1}{n(\boldsymbol{x})} \sum_{i=1}^{n(\boldsymbol{x})} \{\phi(x_i) - \phi(\mu(\boldsymbol{x}))\} \tag{20}$$

[7] I am grateful to Professor Aczel for bringing this reference to my attention, and to Dr. C. T. Ng for suggesting to me how the problem in hand could be transformed to make use of this result.

where $\phi(x)$ is continuous and strictly convex; and where $F(I, \mu, n)$ is continuous in I and μ, strictly increasing in I, with $F(0, \mu, n)=0$. In addition, I is replication invariant iff F is independent of n.

PROOF: From Theorem 2 there exists a function $\hat{F}$ with the stated properties, such that $J=\hat{F}(I, \mu, n)$ is an inequality index satisfying equation (14). Theorem 3 shows that J can be written as

$$J(\boldsymbol{x})=\sum_{i=1}^{n}\{\phi(x_i)-\phi(\mu)\}$$

where ϕ is continuous and strictly convex, since J satisfies (4c). Necessity of (20) then follows by defining $F(I, \mu, n)\equiv\hat{F}(I, \mu, n)/n=J/n$. Sufficiency of (20) is demonstrated by checking that $I(\boldsymbol{x})$, as given by (20), satisfies conditions (4) and (5). The equivalence between I being replication invariant and F being independent of n follows immediately from the fact that the right hand side of (20) is invariant to a replication of the distribution.

We finally consider the property of scale invariance which, when combined with decomposability, produces the following result:

THEOREM 5: *I is a scale invariant, decomposable inequality index iff there exists a parameter $c\in\mathcal{R}$ and a function $F(I, n)$, continuous and strictly increasing in I, with $F(0, n)=0$, such that*

$$F(I(\boldsymbol{x}), n(\boldsymbol{x}))=\begin{cases}\dfrac{1}{n}\dfrac{1}{c(c-1)}\displaystyle\sum_{i=1}^{n}\left\{\left(\frac{x_i}{\mu}\right)^{c}-1\right\} & \text{if } c\neq 0, 1,\\[2ex] \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\frac{x_i}{\mu}\log\left(\frac{x_i}{\mu}\right) & \text{if } c=1,\\[2ex] \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\log\left(\frac{\mu}{x_i}\right) & \text{if } c=0.\end{cases}\tag{21}$$

I is also replication invariant iff F is independent of n.

PROOF: Since I is decomposable it satisfies (20) for some suitable F and ϕ. Normalize ϕ so that $\phi(1)=0$ and define $\hat{F}(I, \mu, n)=nF(I, \mu, n)$. Choose any $\boldsymbol{x}_1, \boldsymbol{x}_2\in X_2$, and write $\boldsymbol{x}_{12}=(\boldsymbol{x}_1, \boldsymbol{x}_2)$, $I_k=I(\boldsymbol{x}_k)$, $n_k=n(\boldsymbol{x}_k)$ and $\mu_k=\mu(\boldsymbol{x}_k)$. Then

$$\begin{aligned}I(\boldsymbol{x}_{12})&=\hat{F}^{-1}\left(\sum_{i=1}^{n_1}\phi(x_{1i})+\sum_{i=1}^{n_2}\phi(x_{2i})-n_{12}\phi(\mu_{12}), \mu_{12}, n_{12}\right)\\ &=\hat{F}^{-1}(\hat{F}(I_1, \mu_1, n_1)+\hat{F}(I_2, \mu_2, n_2)+n_1\phi(\mu_1)+n_2\phi(\mu_2)\\ &\qquad -n_{12}\phi(\mu_{12}), \mu_{12}, n_{12})\\ &\equiv H(\hat{F}(I_1, \mu_1, n_1)+\hat{F}(I_2, \mu_2, n_2), \mu_1, \mu_2, n_1, n_2)\\ &=I(\lambda\boldsymbol{x}_{12})\\ &=H(\hat{F}(I_1, \lambda\mu_1, n_1)+\hat{F}(I_2, \lambda\mu_2, n_2), \lambda\mu_1, \lambda\mu_2, n_1, n_2).\end{aligned}$$

Applying Lemma 1 gives

$$\hat{F}(I_1, \lambda\mu_1, n_1) = c\hat{F}(I_1, \mu_1, n_1),$$
$$\hat{F}(I_2, \lambda\mu_2, n_2) = c\hat{F}(I_2, \mu_2, n_2),$$

from which it follows that

$$c = \frac{\hat{F}(I, \lambda\mu, n)}{\hat{F}(I, \mu, n)} = c(\lambda), \quad \text{independent of } I, \mu, \text{ and } n.$$

But

$$\hat{F}(I, \lambda\mu, n) = c(\lambda)\hat{F}(I, \mu, n) = c(\lambda)c(\mu)\hat{F}(I, 1, n)$$
$$= c(\lambda\mu)\hat{F}(I, 1, n).$$

So $c(\lambda\mu) = c(\lambda)c(\mu) > 0$ for all $\lambda, \mu > 0$, and continuity of c ensures that $c(\mu) = \mu^c$ [**1**, p. 144, Theorem 4]. Hence

$$(22) \qquad F(I(\boldsymbol{x}), 1, n) = \hat{F}(I(\boldsymbol{x}), 1, n)/n = \mu^{-c}\hat{F}(I(\boldsymbol{x}), \mu, n)/n$$
$$= \mu^{-c}F(I(\boldsymbol{x}), \mu, n) = \sum_{i=1}^{n} \{\phi(x_i) - \phi(\mu)\}/n\mu^c.$$

Now choose $\boldsymbol{x} = (u, v)$ and define $\psi(u, \lambda) \equiv \phi(\lambda u) - \lambda^c\phi(u)$. Then

$$0 = \hat{F}(I(\lambda\boldsymbol{x}), \lambda\mu(\boldsymbol{x}), 2) - \lambda^c\hat{F}(I(\boldsymbol{x}), \mu(\boldsymbol{x}), 2)$$
$$= \phi(\lambda u) + \phi(\lambda v) - 2\phi(\lambda(u+v)/2) - \lambda^c\{\phi(u) + \phi(v) - 2\phi((u+v)/2)\}$$
$$= \psi(u, \lambda) + \psi(v, \lambda) - 2\psi(\tfrac{1}{2}(u+v), \lambda).$$

This is a Pexider equation, whose solution [**1**, p. 142] is

$$(23) \qquad \psi(u, \lambda) = a(\lambda)u + b(\lambda).$$

But

$$\psi(u, \lambda\mu) = \phi(u\mu\lambda) - \lambda^c\mu^c\phi(u)$$
$$= \psi(u\mu, \lambda) + \lambda^c\psi(u, \mu) = \psi(u\lambda, \mu) + \mu^c\psi(u, \lambda).$$

Substituting (23) and equating coefficients of u produces

$$a(\lambda\mu) = a(\lambda)\mu + \lambda^c a(\mu) = a(\mu)\lambda + \mu^c a(\lambda),$$
$$b(\lambda\mu) = b(\lambda) + \lambda^c b(\mu) = b(\mu) + \mu^c b(\lambda),$$

for which the solutions are

$$a(\mu) = \begin{cases} \alpha(\mu^c - \mu), & c \neq 1, \\ \alpha\mu \log \mu, & c = 1, \end{cases}$$

$$b(\mu) = \begin{cases} \beta(\mu^c - 1), & c \neq 0, \\ \beta \log \mu, & c = 0. \end{cases}$$

Hence

$$\phi(x)=\psi(1,x)=a(x)+b(x), \quad c\neq 0,1,$$

$$=\begin{cases}(\alpha+\beta)x^c-\alpha x-\beta, & c\neq 0,1,\\ \alpha x\log x+\beta x-\beta, & c=1,\\ \beta\log x-\alpha x+\alpha, & c=0.\end{cases}$$

Substituting into (22) and incorporating a suitable constant into F produces (21).

The converse result is trivially accomplished by confirming that any function I defined by (21) satisfies scale invariance and conditions (4) and (5). Similarly, that replication invariance of I is equivalent to F being independent of n follows immediately from the fact that the Generalized Entropy index form on the right hand side of (21) is replication invariant.

Theorem 5 shows that the combination of decomposability and scale invariance for an inequality index leads us inexorably towards the Generalized Entropy family given earlier in equation (3). In fact, given the additional undemanding requirement of replication invariance, the class of admissible indices I are just simple increasing transformations $F(J)$ of a Generalized Entropy measure J. Thus I is ordinally equivalent to J and will rank any pair of distributions in the same way. There may be circumstances in which, say, the Atkinson **[2]** index or the coefficient of variation is preferred to its corresponding Generalized Entropy form. But the general conclusion appears to be that, when decomposability is desired, and scale and replication invariance are accepted, nothing substantial is lost by focussing exclusively on the Generalized Entropy indices.

5. CONCLUDING REMARKS

This paper has examined the implications for inequality measures of imposing a weak aggregation property which requires that overall inequality can always be calculated from the size, mean, and inequality value of each population subgroup. In Section 3 it was shown (Theorems 1 and 2) that this decomposability requirement implies that the index must be capable of being transformed into an *additively* decomposable index. The general form of a decomposable index was derived in Section 4 (Theorem 4) and the impact of assuming replication invariance and scale invariance (homogeneity) were examined in Theorems 4 and 5. Imposing all three restrictions leads to simple monotonic transformations of the Generalized Entropy family. Thus replacing the *additively* decomposable contraint considered in Shorrocks **[12]** with a very weak aggregation property expands the admissible set of indices by including monotonic transformations, but otherwise cannot be said to allow any new perceptions of inequality to be incorporated.

In the context of inequality measurement there are two directions in which the results may be extended. The first is to impose constraints on the permissible partitions of the population. For instance, the Gini coefficient is known to be decomposable, in the sense of equation (5), when the incomes in one subgroup are all less than those in the other subgroup.[8] So the class of inequality measures that are decomposable under all *non-overlapping* partitions of the income distribution certainly contains indices that are not covered by, say, equation (20). A second potential line of development is to relax the assumption that the subgroup *means* are relevant in the aggregation, and substitute a variable relating to a more general concept of representative income. This may not present too many difficulties. For the mean income parameter plays a relatively minor role in Theorem 1, being used primarily to ensure that the sets $S(\boldsymbol{\theta})$ have suitable properties, like connectedness. Consequently Theorems 1 and 2 may remain substantially intact.

To extend the analysis to indices other than inequality measures, assumption (4) will clearly need to be amended. However, continuity, symmetry and some kind of normalization condition are still likely to be appropriate. So it is only Schur-concavity that may have no general analogue, and this again does not appear to be critically important in the derivation of the results. It would also appear possible to replace the scalar-valued individual observations with vectors: for example, the price-quantity pairs that might be needed for a general analysis of price and quantity indices.[9]

A useful and comprehensive type of index aggregation that could be investigated along the lines of this paper takes the form of a recursive aggregation structure. The lowest level "indices" would be those for which there is a simple and obvious relationship between the subgroup index values and the population index. The subgroup sizes (number of observation vectors), or the subgroup sum (for example, total subgroup income), are two possible candidates for level 1 indices. Level 1 index values then become the subgroup parameters for level 2 aggregation. Thus, for instance, the representative income level for the population may be a function of the subgroup representative incomes *and* sizes. This would allow a generalization of the mean income parameter used earlier, while retaining its aggregation properties. Aggregation of level 3 indices would then treat level 1 and 2 indices as parameters in the same way that the aggregation of inequality measures in this paper employs the subgroup means and sizes. In principle, this structure could be extended indefinitely, with each aggregation level treating all the lower level indices as parameters in the aggregation process. But most practical interest is likely to reside in the few lowest levels, if only because the informational advantages of aggregative indices will be diluted if large numbers of parameters are required in the computation. The proof of Theorem 1 indicates an approach that might be applied to recursive aggregation structures, using an inductive argument. The results of this paper also suggest that recursively aggregable indices

[8] See, for example, Pyatt [**11**] or Mookherjee and Shorrocks [**10**].

[9] This could be viewed as a case in which the permitted partitions of the set of price and quantity observations is constrained, so that price-quantity pairs are never split up.

are likely to be transformations of additive forms of indices. However, confirmation of such a conjecture lies some way in the future.

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