▶ *Prove the expression*(18.91)

If we multiply (18.90) by $xJ_v(\alpha_m X)$ and integrate from $x = 0$ to $x = b$ them we obtain

$$\int_0^b xJ_v(\alpha_m x)f(x)dx = \sum_{n=0}^{\infty} c_n \int_0^b xJ_v(\alpha_m x)J_v(\alpha_n x)dx$$

$$= c_m \int_0^b J_v^2(\alpha_{mx})xdx$$
$$= \tfrac{1}{2}c_m b^2 J_v'^2(\alpha_m b) = \tfrac{1}{2}c_m b^2 J_{v+1}^2(\alpha_m b),$$

where in the last two lines we have used (18.88) with $\alpha_m = \alpha \neq \beta = \alpha_n,$)(18.89), the fact that $J_v(\alpha_m b) = 0$ and (18.95), which is proved below. ◀

Recurrence relations

The recurrence relations enjoyed by Bessel functions of the firs kind, $J_v(x)$, can be derived directly from the power series definition (18.79).

▶ *Prove the recurrence relation*

$$\frac{d}{dx}[x^v J_v(x)] = x^v J_{v-1}(x). \tag{18.92}$$

From the power series definition (18.79) of $J_v(x)$ we obtain

$$\frac{d}{dx}[x^v J_v(x)] = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2v+2n}}{2^{v+2n-1}n!\Gamma(v+n+1)}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2v+2n}}{2^{v+2n-1}n!\Gamma(v+n)}$$

$$= x^v \sum_{n=0}^{\infty}\frac{(-1)^n x^{v-1}}{2^{(v-1)=2n}n!\Gamma((v-1)+n+1)} = x^v J_{v-1}(x). ◀$$

It may similarly be shown that

$$\frac{d}{dx}[x^{-v} J_v(x)] = -x^{-v} J_{v+1}(x). \tag{18.93}$$

From (18.92) and (18.93) the remaining recurrence relations may be derived. Expanding out the derivative on the LHS of (18.92) and dividing through by x^{v-1}, we obtain the relation

$$xJ'(x) - vJv(x) = -xJ_{v-1}(x). \tag{18.94}$$

Similarly, by expanding out the derivative on the LHS of (18.93),and multiplying through by x^{v+1},we find

$$xJ'(x) - vJv(x) = -xJ_{v+1}(x). \tag{18.95}$$

Adding (18.94) and (18.95) and deviding through by x gives

$$xJ_{v-1}(x) - xJ_{v+1}(x) = 2J_v'(x). \tag{18.96}$$