

第一章 Main Content

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1.1 常用 Taylor 展式

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^k}{k!} + \cdots \\
\ln(1+x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^k x^{k+1}}{k+1} + \cdots, x \in (-1, 1] \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots + \frac{(-1)^k}{(2k+1)!} x^{2k+1} + \cdots \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots + \frac{(-1)^k}{(2k)!} x^{2k} + \cdots \\
\tan x &= 2 \sum_{n=1}^{\infty} \frac{(4^n - 1) \zeta(2n)}{\pi^{2n}} x^{2n-1} = x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \frac{17}{315} x^7 + \frac{62}{2835} x^9 + \frac{1382}{155925} x^{11} + o(x^{12}), x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\
\sec x &= \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} = 1 + \frac{1}{2} x^2 + \frac{5}{24} x^4 + \frac{61}{720} x^6 + \frac{277}{8064} x^8 + \frac{50521}{3628800} x^{10} + o(x^{11}), x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\
\arcsin x &= \sum_{n=0}^{\infty} \frac{C_{2n}^n}{4^n (2n+1)} x^{2n+1} = x + \frac{1}{6} x^3 + \frac{3}{40} x^5 + \frac{5}{112} x^7 + \frac{35}{1152} x^9 + \frac{63}{2816} x^{11} + o(x^{12}), x \in (-1, 1) \\
\arctan x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \cdots + \frac{(-1)^k}{2k+1} x^{2k+1} + \cdots, x \in (-1, 1) \\
\sinh x &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{1}{6} x^3 + \frac{1}{120} x^5 + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots \\
\cosh x &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{1}{2} x^2 + \frac{1}{24} x^4 + \cdots + \frac{x^{2k}}{(2k)!} + \cdots \\
\tanh x &= \sum_{n=1}^{\infty} \frac{4^n (4^n - 1) B_{2n}}{(2n)!} x^{2n-1} = x - \frac{1}{3} x^3 + \frac{2}{15} x^5 - \frac{17}{315} x^7 + \frac{62}{2835} x^9 - \frac{1382}{155925} x^{11} + o(x^{12}), x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\
\operatorname{sech} x &= \sum_{n=0}^{\infty} \frac{E_{2n} x^{2n}}{(2n)!} = 1 - \frac{1}{2} x^2 + \frac{5}{24} x^4 - \frac{61}{720} x^6 + \frac{277}{8064} x^8 - \frac{50521}{3628800} x^{10} + o(x^{11}), x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\
\operatorname{arsinh} x &= \sum_{n=0}^{\infty} \frac{(-1)^n C_{2n}^n}{4^n (2n+1)} x^{2n+1} = x - \frac{1}{6} x^3 + \frac{3}{40} x^5 - \frac{5}{112} x^7 + \frac{35}{1152} x^9 - \frac{63}{2816} x^{11} + o(x^{12}), x \in (-1, 1) \\
\operatorname{artanh} x &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = x + \frac{1}{3} x^3 + \frac{1}{5} x^5 + \frac{1}{7} x^7 + \frac{1}{9} x^9 + \frac{1}{11} x^{11} + o(x^{12}), x \in (-1, 1) \\
e^{\sin x} &= 1 + x + \frac{1}{2} x^2 - \frac{1}{8} x^4 - \frac{1}{15} x^5 - \frac{1}{240} x^6 + \frac{1}{90} x^7 + \frac{31}{5760} x^8 + \frac{1}{5670} x^9 - \frac{2951}{3628800} x^{10} + o(x^{10}) \\
e^{\tan x} &= 1 + x + \frac{1}{2} x^2 + \frac{1}{2} x^3 + \frac{3}{8} x^4 + \frac{37}{120} x^5 + \frac{59}{240} x^6 + \frac{137}{720} x^7 + \frac{871}{5760} x^8 + \frac{41641}{3628800} x^9 + o(x^9) \\
e^{\arcsin x} &= 1 + x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{5}{24} x^4 + \frac{1}{6} x^5 + \frac{17}{144} x^6 + \frac{13}{126} x^7 + \frac{629}{8064} x^8 + \frac{325}{4536} x^9 + o(x^9) \\
e^{\arctan x} &= 1 + x + \frac{1}{2} x^2 - \frac{1}{6} x^3 - \frac{7}{24} x^4 + \frac{1}{24} x^5 + \frac{29}{144} x^6 - \frac{1}{1008} x^7 - \frac{1219}{8064} x^8 - \frac{1163}{72576} x^9 + o(x^9)
\end{aligned}$$

$$\begin{aligned}\tan(\tan x) &= x + \frac{2}{3}x^3 + \frac{3}{5}x^5 + \frac{181}{315}x^7 + \frac{59}{105}x^9 + \frac{3455}{6237}x^{11} + o(x^{12}) \\ \sin(\sin x) &= x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \frac{8}{315}x^7 + \frac{13}{2520}x^9 - \frac{47}{49896}x^{11} + o(x^{12}) \\ \tan(\sin x) &= x + \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{107}{5040}x^7 - \frac{73}{24192}x^9 + \frac{41897}{39916800}x^{11} + o(x^{12}) \\ \sin(\tan x) &= x + \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{55}{1008}x^7 - \frac{143}{3456}x^9 - \frac{968167}{39916800}x^{11} + o(x^{12}) \\ (1+x)^\alpha &= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k + \cdots \\ (1+x)^{\frac{1}{x}} &= e - \frac{e}{2}x + \frac{11e}{24}x^2 - \frac{7e}{16}x^3 + \frac{2447e}{5760}x^4 - \frac{959e}{2304}x^5 + \frac{238043e}{580608}x^6 - \frac{67223e}{165888}x^7 + o(x^7) \\ (1+x^2)^{\frac{1}{x}} &= 1 + x + \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{11}{24}x^4 + \frac{11}{120}x^5 + \frac{271}{720}x^6 + \frac{53}{2520}x^7 - \frac{4069}{13440}x^8 + o(x^8) \\ (1+\sin x)^{\frac{1}{x}} &= e - \frac{e}{2}x + \frac{7e}{24}x^2 - \frac{3e}{16}x^3 + \frac{139e}{1152}x^4 - \frac{899e}{11520}x^5 + \frac{29311e}{580608}x^6 - \frac{189617e}{5806080}x^7 + o(x^7)\end{aligned}$$

1.2 cmc 决赛备考

- (1) 优先采用教材上提供的基本方法和自然的思路.
- (2) 换序问题在时间比较富裕的情况下一定要严格证明.
- (3) 中值定理一定是自然的构造, 不要想的奇奇怪怪.
- (4) 填空题是基本盘, 一定不能错, 多检查, 可以猜答案.
- (5) 遇事不绝, 分部积分.
- (6) 遇事不绝, 不妨设标准型.
- (7) 数学类高年级组误判率较高! 重代数轻分析!
- (8) 第一天考完下午和晚上是最好玩的, 抓紧时机面基各路大神, 笔者当初考完带着一堆小伙伴去撸串, 干了十瓶可乐.
- (9) 考完当天晚上就会出成绩, 误判几率不低, 可以尽力配合老师查分.

1.3 线性代数

- Jordan 分解
- 同时 (正交) 相似上三角/对角化¹
- 两个半正定矩阵 A, B 可以同时合同对角化.
- 实对称矩阵和正定矩阵显然可以同时合同对角化
- 矩阵打洞
- 摄动法, 可以不妨设可逆

¹ 正交性要求特征值全为实数, 或者实对称...

- AB 和 BA 有完全一样的非零 Jordan 块
- 与所有可逆矩阵可交换的矩阵是数量矩阵 λI .
- 对 $A \in \mathbb{R}^{m \times n}$, 有 $r(A^T A) = r(A)$.
- $r(A) = \dim \text{Im } A = n - \dim \ker A$.
- A 是半正定矩阵, 等价于 A 所有主子式²非负, 等价于 A 的所有特征值为非负实数, 等价于存在 $C \in \mathbb{R}^{n \times n}$ 使得 $A = C^T C$.
- 若 A 可逆, 则 A^{-1} 是 A 的多项式. 考虑特征多项式和哈密顿凯莱定理 (结合 $a_0 \neq 0$)
- A^* 是 A 的多项式. 注意到 $A^* = |A|A^{-1}$.
- 瑞丽商
- 反对称实矩阵的特征值实部为 0.
- 循环行列式
- 求解 $AX - XB = C$. 有唯一解的充要条件是 A, B 无相同特征值. 证明考虑分块 Jordan 爆算. 这也是对于 $AX = XB$ 交结数的证明.
- 特征多项式等于极小多项式的等价条件.
- 覆盖定理
- 实正规矩阵³的正交相似标准型.
- 相似的正规矩阵必然酉相似. 若为实正规, 则正交相似.
- 酉相似实 矩阵必然实正交相似.
- A 是复正规矩阵的充要条件是 A 的共轭转置是 A 的多项式
- 三对角矩阵的行列式递推
- Perron 判别法

引理 21.1

任何域上的可对角化矩阵限制到不变子空间上仍然是可对角化矩阵.



证明 注意到矩阵可对角化等价于极小多项式可以分解为一次式的积. 设 A 的极小多项式是 p , W 是矩阵 A 一个不变子空间且 $p|_W$ 是 $A|_W$ 极小多项式. 注意到 $p(A|_W) = 0$, 故 $p|_W$ 也是极小多项式, 故 $A|_W$ 可对角化.

² 主子式是指取任意行和对应的列交叉出来的子式的行列式, 代数主子式还要考虑逆序数.

³ $AA^T = A^T A$

命题 21.3 (一族两两可交换的可对角化矩阵可同时相似对角化)

给定域 \mathbb{F} , 设 $A_\lambda \in \mathbb{F}^{n \times n}, \lambda \in \Lambda$ 且两两可交换. 若每一个 $A_\lambda, \lambda \in \Lambda$ 都可以在 \mathbb{F} 上相似对角化, 则存在可逆矩阵 $P \in \mathbb{F}^{n \times n}$, 使得

$$P^{-1}A_\lambda P \text{ 是对角矩阵}, \forall \lambda \in \Lambda.$$

证明 设 $V = \mathbb{F}^n$ 且 A_λ 是 V 上线性变换.

Step 1 若 A_λ 都是数量矩阵, 则结果已经成立.

Step 2 取某个非数量矩阵 A_1 , 于是有 $V = \bigoplus_{i=1}^s V_i$, 这里 $s \geq 2$ 且 V_i 是属于 A_1 不同特征值的特征子空间. 显然由交换性, 对每一个 $i = 1, 2, \dots, s$, V_i 是所有 A_λ 不变子空间, 且 $A_\lambda|_{V_i}, \lambda \in \Lambda$ 是一族两两可交换的矩阵, 由引理 21.1 知它们也是可对角化的. 注意到 $1 \leq \dim V_i < n, i = 1, 2, \dots, s$, 所以我们的维度降下去了, 因此可使用归纳法.

当 $n = 1$, 命题显然成立, 假设命题对 $\leq n - 1$ 都成立, 当 n 时, 由归纳假设, 对每一个 $i = 1, 2, \dots, s$, 存在 V_i 的一个基使得 $A_\lambda|_{V_i}, \lambda \in \Lambda$ 在这个基下表示矩阵是对角矩阵. 于是把这些基合起来构成一个新的基, 我们就得到在这个新的基下 $A_\lambda, \lambda \in \Lambda$ 都是对角矩阵.

定理 21.16 (公共特征向量)

给定域 \mathbb{F} , 指标集 Λ 和 $A_\lambda \in \mathbb{F}^{n \times n}, \lambda \in \Lambda$.

1. 若 $A_\lambda, \lambda \in \Lambda$ 两两可交换;
 2. 若 $\mathbb{F} = \mathbb{R}$, n 为奇数.
- 则 A 在 \mathbb{F} 上有公共特征向量.

◎ 笔记 某种角度上说, 存在公共特征向量和可同时相似上三角化是等价的.

证明

1. 由可同时上三角化 21.1, 不妨设 $A_\lambda, \lambda \in \Lambda$ 都是上三角矩阵, 注意观察第一列知他们有公共的特征向量 e_1 ⁷.
2. 证明的关键在于奇数次实多项式必有实根, 从而奇数矩阵必有实特征值.

命题 19.8 ($r(A^T A) = r(A)$)

设 $A \in \mathbb{R}^{m \times n}$, 则 $r(A^T A) = r(A)$.

证明 设 $Ax = 0, x \in \mathbb{R}^n$, 则 $A^T Ax = 0$. 反之设 $A^T Ax = 0, x \in \mathbb{R}^n$, 则

$$0 = x^T A^T Ax = (Ax)^T (Ax) \Rightarrow Ax = 0.$$

故线性方程组 $Ax = 0$ 和 $A^T Ax = 0$ 同解. 于是有 $r(A^T A) = r(A)$.

此类技巧还有一个应用:

例题 19.5 设 $A \in \mathbb{R}^{n \times n}$ 且 $r(A^k) = r(A^{k+1}), k \in \mathbb{N}_0$, 则 $r(A^k) = r(A^{k+s}), \forall s \in \mathbb{N}_0$. 特别的 $r(A^n) = r(A^{n+1})$.

命题 19.13 (瑞丽商)

设 A, B 是 n 阶实对称矩阵, 则有

$$\min_{x \neq 0} \frac{x^T Ax}{x^T x} = \lambda_{\min}(A), \max_{x \neq 0} \frac{x^T Ax}{x^T x} = \lambda_{\max}(A). \quad (19.13)$$

以及

$$\lambda_{\min}(A) + \lambda_{\min}(B) \leq \lambda(A+B) \leq \lambda_{\max}(A) + \lambda_{\max}(B). \quad (19.14)$$

例题 19.7 设 $A, B, C, D \in \mathbb{C}^{n \times n}$ 且 $AC = CA$, 则

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|.$$

证明 不妨设 A 可逆, 否则用 $A + \lambda E_n$ 和行列式的连续性振动即可.

我们直接初等变换 $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$, 于是写出分块初等矩阵即可发现

$$\begin{aligned} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} = \det A \cdot \det (D - CA^{-1}B) \\ &= \det (AD - ACA^{-1}B) = \det (AD - CAA^{-1}B) = \det (AD - CB). \end{aligned}$$

命题 19.14

设 $A \in \mathbb{R}^{n \times n}$, 则

1. 若 A 是反对称矩阵, 则 A 的特征值是 0 或者纯虚数.

2. 若 A 是正交矩阵, 则 A 的特征值的模长都为 1.

证明

1. 设 $A\alpha = \lambda\alpha, \alpha \in \mathbb{C}^n \setminus \{0\}, \lambda \in \mathbb{C}$, 则

$$\alpha^* A^* = \bar{\lambda}\alpha^* \Rightarrow -\alpha^* A\alpha = \bar{\lambda}\alpha^*\alpha = -\lambda\alpha^*\alpha.$$

即

$$\lambda \in 0 \bigcup \{bi : b \in \mathbb{R}\}.$$

2. 设 $A\alpha = \lambda\alpha, \alpha \in \mathbb{C}^n \setminus \{0\}, \lambda \in \mathbb{C}$, 则

$$\alpha^* \alpha = \alpha^* A^* A\alpha \Rightarrow |\lambda|^2 \alpha^* \alpha = \alpha^* \alpha \Rightarrow |\lambda| = 1.$$

引理 19.1

设 A, B 是实 n 阶(半)正定矩阵,

1. 则 AB 特征值为(非负)正.
2. 则 AB 也是(半)正定的充要条件是 $AB = BA$.



证明

1. 对于正定的情况, 设可逆实矩阵 P 使得 $A = P^T P$, 则

$$(P^T)^{-1} A B P^T = (P^T)^{-1} P^T P B P^T = P B P^T \cong B.$$

故由 B 正定知 $P B P^T$ 正定, 从而特征值都是正数. 现在就知道 AB 特征值都是正数.

♣ 对于半正定的情况, 由正定的情况知道 $(A + \lambda E)(B + \lambda E)$ 在 $\lambda > 0$ 特征值都是正数. 故由特征值是矩阵元的连续函数 21.37, 我们知道如果 AB 有不是非负数的特征值 $a \in \mathbb{C}$, 我们可取 a 的小邻域使得此邻域和 $[0, +\infty)$ 无交, 这就是一个矛盾! 于是我们证明了 AB 特征值非负.

2. 当 $AB = BA$, 我们知道 AB 对称, 故由第一问知道 AB 是(半)正定矩阵. 反之若 AB 是(半)正定的, 则

$$AB = (AB)^T = B^T A^T = BA,$$

这就完成了证明.

引理 21.4 (半正定矩阵的对角线类极值性质)

设半正定矩阵 $A = (a_{ij})$ 的 (i, i) 元为 0, 则 A 的第 i 行和第 i 列都为 0.

^a可实对称可复 Hermite.



证明 不妨在 \mathbb{R} 上考虑且不妨设 $(i, i) = (1, 1)$, 否则同时交换行列使得 (i, i) 到左上角, 由合同对应的初等变换**19.2**, 我们知道这样得到的矩阵与原本的矩阵合同, 从而仍然是半正定矩阵.

现在利用半正定矩阵主子式非负我们有

$$-a_{12}^2 = a_{11} \cdot a_{22} - a_{12}^2 \geq 0,$$

定理 21.17

两个半正定矩阵 A, B ^a 可同时合同对角化.

^a可实对称可复 Hermite.



证明 不妨在 \mathbb{R} 上考虑. 注意到 $A + B$ 是半正定的, 所以存在可逆矩阵 $C \in \mathbb{R}^{n \times n}$ 使得

$$C^T (A + B) C = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} = C^T AC + C^T BC. \quad (21.8)$$

注意到 $C^T AC, C^T BC$ 都是半正定的且对角元非负. 因此 $C^T AC, C^T BC$ 的后 $n - r$ 个对角元都为 0, 结合引理**21.4**和**(21.8)**知

$$C^T AC = \begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix}, C^T BC = \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix}, A' + B' = E_r.$$

取正交矩阵 $T' \in \mathbb{R}^{r \times r}$ 使得 $(T')^{-1} B'T = \Lambda$ 为对角矩阵, 则 $(T')^{-1} A'T = E_r - \Lambda$ 也为对角矩阵. 故设 $P = C \begin{pmatrix} T' & 0 \\ 0 & E_{n-r} \end{pmatrix}$, 则有

$$P^T AP = \begin{pmatrix} E_r - \Lambda & 0 \\ 0 & 0 \end{pmatrix}, P^T BP = \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}.$$

命题 21.5

设 $A, B \in \mathbb{R}^{n \times n}$ 是半正定矩阵, 则

$$|A + B| \geq |A|. \quad (21.9)$$

且当**(21.9)**等号成立并且左右都非 0 时必有 $B = 0$.



^b **笔记** 复 Hermite 半正定矩阵也成立, 证明类似.

证明 由定理**21.17**知存在可逆矩阵 $C \in \mathbb{R}^{n \times n}$ 使得

$$C^T AC = \begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix}, C^T BC = \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix}, A' + B' = E_r,$$

这里 A', B' 都是半正定对角矩阵. 于是**(21.9)**等价于

$$|C^T AC + C^T BC| \geq |C^T AC|,$$

即等价于 $|\text{diag}\{E_r, 0\}| \geq |A'|$. 当 $0 \leq r < n$, 则上式左右都是 0. 当 $r = n$, 则上述不等式左边是 1, 右边对角线每个元素都小于等于 1, 所以行列式小于等于 1, 因此不等式成立. 此时若等号成立, 则 $B' = 0$, 即 $B = 0$

定义 21.1 (循环行列式)

证明

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1}z & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ a_{n-2}z & a_{n-1}z & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1z & a_2z & a_3z & \cdots & a_{n-1}z & a_0 \end{vmatrix} = f(w_0)f(w_1) \cdots f(w_{n-1}), \quad (21.10)$$

这里

 $f(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$, w_0, w_1, \dots, w_{n-1} 是 $w^n = z$ 的 n 个不同复根.定理 21.20 (AB 和 BA 的非 0 Jordan 完全一致)设 $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{n \times m}$, 则 AB 和 BA 的非 0 Jordan 完全一致.证明 考虑 $PAQQ^{-1}BP^{-1}, Q^{-1}BP^{-1}PAQ$ 可不妨设

$$A = \begin{pmatrix} E_r & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{pmatrix}, B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, B_1 \in \mathbb{C}^{r \times r}, B_3 \in \mathbb{C}^{(n-r) \times r},$$

这里 B 的分块和 A 对应. 于是直接计算有

$$AB = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix}, BA = \begin{pmatrix} B_1 & 0 \\ B_3 & 0 \end{pmatrix}. \quad (21.12)$$

从(21.12)可以看到 AB, BA 非 0 特征值都集中在 B_1 上, 所以非 0 特征值完全一致. 设 $\lambda \neq 0$ 是 AB, BA 特征值. 则回忆定理 19.3, 决定 C Jordan 块分布, 我们知道只需决定 $(\lambda E - C)^k, k \in \mathbb{N}_0$ 的秩即可. 于是对每个 $k \in \mathbb{N}_0$, 我们有

$$(\lambda E_m - AB)^k = \begin{pmatrix} (\lambda E_r - B_1)^k & * \\ 0 & \lambda^k E_{m-r} \end{pmatrix}, (\lambda E_n - BA)^k = \begin{pmatrix} (\lambda E_r - B_1)^k & 0 \\ * & \lambda^k E_{n-r} \end{pmatrix}.$$

作初等变换

$$\begin{pmatrix} (\lambda E_r - B_1)^k & * \\ 0 & \lambda^k E_{m-r} \end{pmatrix} \rightarrow \begin{pmatrix} (\lambda E_r - B_1)^k & 0 \\ 0 & \lambda^k E_{m-r} \end{pmatrix}, \begin{pmatrix} (\lambda E_n - B_1)^k & 0 \\ * & \lambda^k E_{n-r} \end{pmatrix} \rightarrow \begin{pmatrix} (\lambda E_r - B_1)^k & 0 \\ 0 & \lambda^k E_{n-r} \end{pmatrix}$$

于是我们证明了

$$r((\lambda E_m - AB)^k) + n = r((\lambda E_n - BA)^k) + m, \forall k \in \mathbb{N}_0,$$

这就证明了¹⁵ AB 和 BA 的非 0 Jordan 完全一致.

定理 21.23 ($F(A) = C(A)$ 的充要条件和不变子空间刻画)

设数域上 n 阶矩阵 A , 则有下述等价刻画:

1. 存在 n 维向量 v 使得 $v, Av, \dots, A^{n-1}v$ 是线性无关的;
2. A 相似于一个对角阵;
3. $F(A) = C(A)$;
4. $\dim F(A) = \dim C(A)$;
5. $\sum_{i=1}^{s_0} \max\{n_{i,1}, n_{i,2}, \dots, n_{i,m_i}\} = \sum_{i=1}^{s_0} \sum_{r=1}^{m_i} \sum_{\ell=1}^{m_i} \min\{n_{i,r}, n_{i,\ell}\}$;
6. $m_i = 1, i = 1, 2, \dots, s$;
7. 每种特征值的几何重数都是 1;
8. 对每个特征值 λ , 都有 $r(\lambda E - A) = n - 1$;
9. 特征多项式 = 极小多项式;
10. A 的不变子空间个数有限.

并当 A 的不变子空间个数有限时, 计算 A 的所有不变子空间.

**定理 21.24 (覆盖定理)**

数域上 n 维线性空间 V 不能分解为有限个非平凡子空间^a之并.

^a即不能为全空间或者空集.



笔记 我们给一个简单的方法, 比丘维声等更简单.

证明 考虑 V 的基 e_1, e_2, \dots, e_n 和一组向量

$$\alpha_i = e_1 + ie_2 + i^2e_3 + \dots + i^{n-1}e_n, i = 1, 2, \dots$$

由于计算 vandermonde 行列式, 上面这组向量任何 n 个都线性无关. 而他们必有无限个落入某个题设 V 的某个非平凡子空间中, 从而这个子空间 n 维, 这和其是非平凡子空间矛盾!

命题 21.6

1. 设 $A, B \in \mathbb{C}^{n \times n}$ 是正规矩阵. 若 A 和 B 相似, 则 A 和 B 必然酉相似.
2. 设 $A, B \in \mathbb{R}^{n \times n}$ 是实正规矩阵. 若 A 和 B 相似, 则 A 和 B 必然正交相似.



证明

1. 注意到 A 和 B 相似, 因此 A 和 B 特征值完全一样, 而 A, B 都可以酉对角化, 因此 A 和 B 必然酉相似.
2. 若 A 和 B 相似, 则 A 和 B 特征值完全一样, 由实正规矩阵的正交相似标准型21.25知 A, B 都可以正交相似于其正交相似的标准型, 因此 A 和 B 必然正交相似.

1.4 数学分析

1.4.1 利用 $\delta\epsilon$

Théorème 1

若 f 在 $[1, +\infty)$ 一致连续且广义可积, 则 $\lim_{n \rightarrow \infty} f(x) = 0$.

Note. f 的一致连续性是必要的, 反例见汪林数分反例第 4 章最后一个.

证明. $\forall \epsilon > 0, \exists \delta > 0, s.t. \forall x, y \in [1, +\infty), |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

$\exists M > 0, s.t. \forall x > M, \left| \int_x^{x+\delta} f(x) dx \right| < \delta \epsilon$. Then

$$\delta |f(x)| \leq \left| \int_x^{x+\delta} f(x) dt \right| \leq \left| \int_x^{x+\delta} f(t) dt \right| + \delta \epsilon \leq 2\delta \epsilon \Rightarrow |f(x)| \leq 2\epsilon \Rightarrow \lim_{x \rightarrow \infty} f(x) = 0.$$

□

Note. 相同的证明思想, 利用 $\delta \epsilon$, 同样出现在 $f \in \mathcal{R}[a, b], g \in C[\inf f, \sup f] \Rightarrow g \circ f \in \mathcal{R}[a, b]$ 中.

Théorème 2

$f \in \mathcal{R}[a, b], g \in C[\inf f, \sup f] \Rightarrow g \circ f \in \mathcal{R}[a, b]$

证明. $g \in C[a, b] \Rightarrow \forall \epsilon > 0, \exists \delta > 0, s.t. \forall x, y \in [a, b], |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

$\exists P = \{x_0, \dots, x_n\}$ on $[a, b], \sum_{i=0}^n (\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f)(x_i - x_{i-1}) < \delta \epsilon$. Then

$$\delta \epsilon \geq \sum_{|f(x_{i-1}) - f(x_i)| \geq \delta} \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) (x_i - x_{i-1})$$

$$\geq \delta \sum_{|f(x_{i-1}) - f(x_i)| \geq \delta} (x_i - x_{i-1})$$

$$\Rightarrow \sum_{|f(x_{i-1}) - f(x_i)| \geq \delta} (x_i - x_{i-1}) \leq \epsilon$$

$$\sum_{i=0}^n \left(\sup_{[x_{i-1}, x_i]} g \circ f - \inf_{[x_{i-1}, x_i]} g \circ f \right) (x_i - x_{i-1})$$

$$\leq \sum_{|f(x_{i-1}) - f(x_i)| < \delta} + \sum_{|f(x_{i-1}) - f(x_i)| \geq \delta} \left(\sup_{[x_{i-1}, x_i]} g \circ f - \inf_{[x_{i-1}, x_i]} g \circ f \right) (x_i - x_{i-1})$$

$$\leq (b - a)\epsilon + 2 \sup_{[x_{i-1}, x_i]} |g| \epsilon$$

Hence, $g \circ f \in \mathcal{R}[a, b]$.

□

Note. $f \in \mathcal{R}[a, b], g \in C[\inf f, \sup f] \not\Rightarrow f \circ g \in \mathcal{R}[a, b]$. 反例考虑 g 将正测集映到 f 的间断点集合 (一个零测集), 使得 $g \circ f$ 的间断点集合是正测集, 故 $f \circ g \notin \mathcal{R}[a, b]$. 只需加强 g 的连续性使得零测集的原像都是零测集.

1.4.2 利用单调性两边夹住

Exercice 1

设 f 在 $[0, +\infty)$ 上递增, 对于任何 $T > 0$, f 在 $[0, T]$ 上可积, 且

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x f(t) dt = C.$$

证明

$$\lim_{x \rightarrow +\infty} f(x) = C.$$

证明.

$$\frac{\int_0^x f(t) dt}{x} \leq f(x) \leq \frac{\int_x^{2x} f(t) dt}{x} = \frac{\int_0^{2x} f(t) dt - \int_0^x f(t) dt}{x}$$

两边取 $x \rightarrow \infty$ 得

$$C \leq \lim_{x \rightarrow \infty} f(x) \leq 2C - C = C \Rightarrow f(x) = C$$

□

Exercice 2

f 在 $[0, a]$ 单调, 瑕积分 $\int_0^a x^p f(x) dx$ 收敛, 则 $\lim_{x \rightarrow 0^+} x^{p+1} f(x) = 0$.

证明. 若 $p = -1$, 则显然 $\lim_{x \rightarrow 0^+} x^{p+1} f(x) = 0$, 否则 $\int_0^a x^p f(x) dx$ 不收敛. 若 $p > -1$, 不妨设 f 递减, 则 $\lim_{x \rightarrow 0^+} x^p f(x) = +\infty$.

$$\int_0^x t^p f(t) dt \geq \int_0^x t^p f(x) dt = \frac{1}{p+1} x^{p+1} f(x) \geq \int_x^{2x} t^p f(t) dt$$

由柯西收敛, 令 $x \rightarrow 0^+$ 得

$$0 \geq \lim_{x \rightarrow 0^+} \frac{1}{p+1} x^{p+1} f(x) \geq 0 \Rightarrow \lim_{x \rightarrow 0^+} \frac{1}{p+1} x^{p+1} f(x) = 0.$$

□

Exercice 3

若 a_n 递减, $\sum_{n=1}^{\infty} a_n$ 收敛, 证明: $\lim_{n \rightarrow \infty} na_n = 0$

证明.

$$0 \leq na_{2n} \leq \sum_{k=n+1}^{2n} a_k \rightarrow 0 \text{ (as } n \rightarrow \infty) \implies \lim_{n \rightarrow \infty} na_n = 0$$

□

Note. 注意这里还要补充 a_{2n+1} 这种奇数项的细节. $\lim_{n \rightarrow \infty} a_n = 0$ 说明这是显然的.

1.4.3 先后取极限**Exercice 4: 类似大 O Tauber 定理**

设 f 在 $[a, +\infty)$ 上恒正, 在任意有限区间 $[a, b]$ 可积, 且存在常数 $M > 0$, 使得 $\int_a^{+\infty} f(x)e^{-tx}dx \leq M$. 则有 $\int_a^{+\infty} f(x)dx$ 收敛.

证明. $\forall A > a$,

$$\begin{aligned} M &\geq \int_a^A f(x)e^{-tx}dx \geq \int_a^A f(x)(1-tx)dx \\ &= \int_a^A f(x)dx - t \int_a^A xf(x)dx \\ &\rightarrow \int_a^A f(x)dx \text{ (as } t \rightarrow 0^+) \end{aligned}$$

因此 $\int_a^{+\infty} f(x)dx \leq M$. 由单调有界可知 $\int_a^{+\infty} f(x)dx$ 收敛. □

Note. 下面几个定理见《阶的估计基础》 chap6

Théorème 3: 大 O Tauber 定理

[设](#)

$$\begin{aligned} f(x) &= O\left(\frac{1}{x}\right), \\ F(x) &= \int_0^{\infty} e^{-xt} f(t)dt \rightarrow s, \quad x \rightarrow 0+. \end{aligned}$$

则必有

$$\int_0^\infty f(t)dt = s$$

^a这是定理 Théorème 5 的推论

Théorème 4

设 $f(x)$ 为 $(0, \infty)$ 内的非负函数,

$$F(x) = \int_0^\infty e^{-xt} f(t)dt, \quad x > 0.$$

若

$$F(x) \sim \frac{s}{x^\alpha}, \quad \alpha > 0, x \rightarrow 0+$$

则必有

$$\int_0^x f(t)dt \sim \frac{sx^\alpha}{\Gamma(\alpha + 1)}, \quad x \rightarrow \infty.$$

Corollaire 1

设

$$a_n \geq 0, \quad \sum_{n=0}^{\infty} a_n x^n \sim \frac{s}{1-x}, \quad x \rightarrow 1^-.$$

则

$$\sum_{k=0}^n a_k \sim sn, \quad n \rightarrow \infty.$$

Théorème 5

设

$$f(x) > -\frac{B}{x}, \quad 0 < x < \infty,$$

$$F(x) = \int_0^\infty e^{-xt} f(t)dt \rightarrow s, \quad x \rightarrow 0+.$$

则

$$\int_0^\infty f(t)dt = s.$$

1.4.4 利用额外的参量

Exercice 5

设 $0 \leq a < b$, 是否存在 $f \in C[a, b]$ 使得

$$\int_a^b f(x)x^{2n} dx > 0, \int_a^b f(x)x^{2n+1} dx < 0, n = 0, 1, 2, \dots .(*)$$

如果存在, 请给出存在, 如果不存在, 请给出理由.

证明. 我们考虑 $F(t) = \int_a^b f(x)e^{-xt}dx$, $t \geq 0$, 则容易得到

$$|F(t)| \leq \max_{[a,b]} |f| \cdot \int_0^\infty e^{-xt} dx = \frac{\max_{[a,b]} |f|}{t},$$

即 $\lim_{t \rightarrow +\infty} F(t) = 0$.

但是对任何 $t > 0$, 我们都有

$$F(t) = \int_a^b f(x)e^{-xt}dx = \sum_{n=0}^{\infty} \int_a^b f(x) \frac{(-xt)^n}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \int_a^b f(x)x^n dx \stackrel{(*)}{\geq} \int_a^b f(x)dx > 0.$$

这就是一个矛盾! 因此我们证明了满足题目条件的 f 不存在. \square

Exercice 6: 反向洛必达

设 $f \in C[0, +\infty) \cap C^2(0, +\infty)$ 且 $f''(x) > -\frac{C}{x^2}$, 则有

$$\lim_{x \rightarrow 0^+} xf'(x) = 0.$$

证明. 不妨设 $C > 0$, 对 $h > 0$, 由 Taylor 中值定理我们知道存在 $\theta \in (x, x+h)$, 使得

$$f(x+h) = f(x) + f'(x)h + \frac{f''(\theta)}{2}h^2.$$

于是

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{f''(\theta)}{2}h.$$

于是运用条件就有

$$f'(x) \leq \frac{|f(x+h) - f(x)|}{h} + \frac{C}{2\theta^2}h \leq \frac{|f(x+h) - f(x)|}{h} + \frac{C}{2x^2}h.$$

我们取 $h = \eta x, \eta \in (0, 1)$, 就有

$$xf'(x) \leq x \frac{|f(x+\eta x) - f(x)|}{\eta x} + x \frac{C}{2x^2}\eta x = \frac{|f(x+\eta x) - f(x)|}{\eta} + \frac{C\eta}{2},$$

因此就有

$$\overline{\lim}_{x \rightarrow 0^+} [xf'(x)] \leq \frac{C\eta}{2}$$

由 η 任意性可得

$$\overline{\lim}_{x \rightarrow 0^+} [xf'(x)] \leq 0.$$

类似的由 Taylor 中值定理我们知道存在 $\vartheta \in (x-h, x)$, 使得

$$f(x-\eta x) = f(x) - f'(x)\eta x + \frac{f''(\vartheta)}{2}\eta^2 x^2.$$

于是

$$\begin{aligned} xf'(x) &= \frac{f(x) - f(x-\eta x)}{\eta} + \frac{f''(\vartheta)}{2}\eta x^2 \geq -\frac{|f(x) - f(x-\eta x)|}{\eta} - \frac{C}{2\theta^2}\eta x^2 \\ &\geq -\frac{|f(x) - f(x-\eta x)|}{\eta} - \frac{C}{2(x-\eta x)^2}\eta x^2 = -\frac{|f(x) - f(x-\eta x)|}{\eta} - \frac{C\eta}{2(1-\eta)^2}, \end{aligned}$$

于是

$$\underline{\lim}_{x \rightarrow 0^+} [xf'(x)] \geq -\frac{C\eta}{2(1-\eta)^2}.$$

由 η 任意性可得

$$\underline{\lim}_{x \rightarrow 0^+} [xf'(x)] \geq 0.$$

因此我们证明了

$$\lim_{x \rightarrow 0^+} [xf'(x)] = 0.$$

□

1.4.5 先后两次判断收敛

Théorème 6

数列 u_n, v_n 单调有界, $\lim_{n \rightarrow \infty} u_n = 0$, 证明: $\sum_{n=1}^{\infty} (-1)^{n-1} u_n v_n$ 收敛.

证明. 由狄利克雷判别法: $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ 收敛, 再由阿贝尔判别法: $\sum_{n=1}^{\infty} (-1)^{n-1} u_n v_n$ 收敛.

□

1.4.6 不同意义下的收敛

级数收敛

Définition 1

对于级数 $\sum_{n=1}^{\infty} c_n$, 记 $s_k = \sum_{n=1}^k c_n$.

- 普通收敛:

$$\lim_{k \rightarrow \infty} s_k \text{ exists}$$

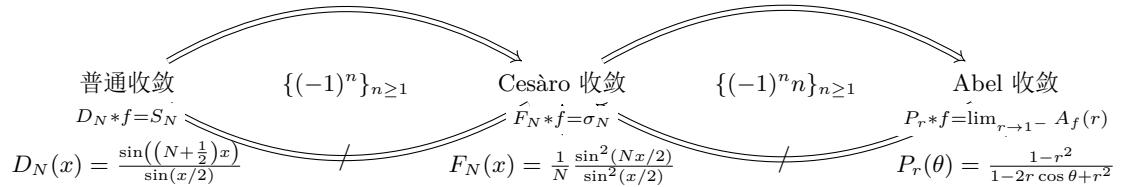
- Cesàro 收敛:

$$\lim_{N \rightarrow \infty} \sigma_N = \lim_{N \rightarrow \infty} \frac{s_1 + s_2 + \cdots + s_N}{N} \text{ exists}$$

- Abel 收敛:

$$\lim_{r \rightarrow 1^-} A(r) = \lim_{r \rightarrow 1^-} \sum_{k=0}^{\infty} c_k r^k \text{ exists}$$

三者的蕴含关系以及反例:



Note. 下面的级数 $\sum_{n=1}^{\infty} c_n$ 有时代指 f 的傅里叶级数.

Corollaire 2

设 $\lim_{n \rightarrow \infty} \sum_{k=1}^n a_k$ 存在, 则

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n k a_k}{n} = 0.$$

Note. 这题不能直接使用 Stolz 公式.

证明.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n k a_k}{n} &= \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k - \frac{\sum_{k=1}^n (n-k) a_k}{n} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k - \frac{\sum_{k=1}^n s_k}{n} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k - \text{Cesaro mean of } \sum_{k=1}^n a_k \\ &= 0 \end{aligned}$$

□

Théorème 7

f 可积, 若 f 在 x_0 处连续, 则 $\lim_{r \rightarrow 1^-} A_f(r)(x_0) = f(x_0)$. 若 f 在 x_0 处跳跃间断, 则 $\lim_{r \rightarrow 1^-} A_f(r)(x_0) = \frac{f(x_0^-) + f(x_0^+)}{2}$.

Théorème 8: Tauber's lemma

若 $\sum c_n$ 有 Abel 收敛到 s , 且 $c_n = o(1/n)^a$, 则 $\sum c_n$ 收敛到 s . 即 $\lim_{r \rightarrow 1^-} \sum c_n r^n = \sum c_n$.

^a在 Littlewood 的版本中, $c_n = O(1/n)$ 也对, 见 Stein 傅里叶分析 p63 exercise 3

Note. 这可以用来考察收敛半径为 1 的幂级数在收敛域边界上是否收敛. 若有 $c_n = o(1/n)$, 则收敛.

Théorème 9

f 可积, 且 $\hat{f}(\nu) = O(1/|\nu|)$, 则若 f 在 θ 连续, 则 $S_N(f)(\theta) \rightarrow f(\theta)$ (as $N \rightarrow \infty$), 若跳跃间断, 则 $S_N(f)(\theta) \rightarrow \frac{f(\theta-) + f(\theta+) }{2}$. 若 $f \in C[-\pi, \pi]$, 则 $S_N(f) \rightarrow f$ uniformly.

1.4.7 傅里叶变换的对称性

f 的光滑性与 \hat{f} 的阶:

$f \in C^k$	$\rightsquigarrow \hat{f} = o(1/ n ^k)$	integrate by parts and use Riemann-Lebesgue lemma
f is Lipschitz	$\rightsquigarrow \hat{f} = o(1/ n)$	使用事实1并分段估计
f is monotonic	$\rightsquigarrow \hat{f} = O(1/ n)$	使用简单函数逼近, 或者在实数情况下使用第二积分中值定理
$f \in BV$	$\rightsquigarrow \hat{f} = O(1/ n)$	强行构造分段使用积分第二积分中值定理
f $\alpha -$ Holder 连续	$\rightsquigarrow \hat{f} = O(1/ n ^\alpha)$	使用事实1并分段估计
f merely Riemann integrable	$\rightsquigarrow \hat{f} = o(1)$	由 Parseval 恒等式: $\sum \hat{f} ^2 = \int_{\mathbb{R}} f ^2 < \infty$

Propriété 1: factos

$$\hat{f}(n) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + \pi/n) e^{-inx} dx$$

hence

$$\hat{f}(n) = \frac{1}{4\pi} \int_{-\pi}^{\pi} [f(x) - f(x + \pi/n)] e^{-inx} dx.$$

1.4.8 两种方式估阶, 阶不同故矛盾

Note. 见 Stein 傅里叶分析 p117 关于 Weierstrass 函数处处连续但不可导的性质的讨论.

1.4.9 反例

对于 $[1, +\infty)$ 上的正值连续函数 f , $\sum_{n=1}^{\infty} f(n)$ 与 $\int_1^{\infty} f(x) dx$ 的敛散性互不蕴含.

Note. 但是对于非负不增函数上面的敛散性是等价的. 这说明非负不增性不能被正值连续性替代. 反例见《实分析中的反例》 p113.

级数 $\sum_{n=1}^{\infty} a_n$ 收敛蕴含 $\sum_{n=1}^N a_n$ ($\forall N \in \mathbb{N}$) 有界且 $\lim_{n \rightarrow \infty} a_n = 0$. 但是反过来不对, 有反例, 见《实分析中的反例》 p103.

级数收敛充要条件 (柯西收敛准则) 是 $\lim_{m,n \rightarrow \infty} (a_{n+1} + \cdots + a_m) = 0$, 这里 m, n 是无关的. 若 m, n 相关, 即 $\lim_{n \rightarrow \infty} (a_{n+1} + \cdots + a_{m(n)}) = 0$, 则推不出来级数 $\sum_{n=1}^{\infty} a_n$ 收敛, 反例考虑调和级数 $\sum 1/n$, 具体见《实分析中的反例》 p106.

1.4.10 隔开边界

Théorème 10: 解析函数序列收敛性

一列调和函数 $\{f_n\}_{n=1}^{\infty}$ 在 Ω 内的紧集上一致收敛到 f , 那么 $\{f'_n\}_{n=1}^{\infty}$ 在 Ω 内的紧集上一致收敛到 f' .

证明. 不妨设 $\{f_n\}_{n=1}^{\infty}$ 在 Ω 上一致收敛到 f . 记 $\Omega_{\delta} = \{z \in \Omega : \overline{D_{\delta}}(z) \in \Omega\}$. 断言对于 Ω 内的解析函数 F , 有 $\sup_{z \in \Omega_{\delta}} |F'(z)| \leq \frac{1}{\delta} \sup_{\zeta \in \Omega} |F(\zeta)|$. 于是

$$\sup_{x \in \Omega_{\delta}} |f'(z) - f'_n(z)| \leq \frac{1}{\delta} \sup_{\zeta \in \Omega} |f(z) - f_n(z)| \rightarrow 0 \quad (\text{as } n \rightarrow \infty) \quad \forall \delta > 0$$

下面证明断言:

$$\begin{aligned} |F'(z)| &= \left| \frac{1}{2\pi i} \int_{C_{\delta}(z)} \frac{F(\zeta)}{(\zeta - z)^2} d\zeta \right| \\ &\leq \frac{1}{2\pi} \int_{C_{\delta}(z)} \frac{|F(\zeta)|}{|\zeta - z|^2} d\zeta \\ &\leq \frac{1}{2\pi} \sup_{\zeta \in \Omega} |F(\zeta)| \frac{1}{\delta^2} 2\pi\delta \\ &= \frac{1}{\delta} \sup_{\zeta \in \Omega} |F(\zeta)| \end{aligned}$$

□

1.4.11 广义黎曼引理

Note. 注意到黎曼引理的证明并不依赖于 g 的周期性, 只需要 $\lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x g(t) dt$ 存在即可.

Théorème 11: 广义黎曼引理

设 $f(x)$ 在 $[0, +\infty)$ 上绝对可积. 有界函数 $g(x)$ 在任意区间上可积且 $A = \lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x g(t)dt$ 存在, 则

$$\lim_{\lambda \rightarrow \infty} \int_0^\infty f(x)g(\lambda x)dx = A \int_0^\infty f(x)dx$$

利用广义黎曼引理 Théorème 11, 我们可以解决下面问题:

Exercice 7

设 $f(x) \in C^1[0, +\infty)$, 已知 $\lim_{x \rightarrow +\infty} f(x) = 0$, $f(0) = 1$, 且 $f'(x)$ 在 $[0, +\infty)$ 上绝对可积. 已知部分和有界的序列 $\{a_n\}$ ($n \in \mathbb{N}$) 的 Cesaro 和为 A , 且满足对任意 $t > 0$, $\sum_{n=0}^{\infty} a_n f(tn)$ 都收敛. 证明: $\lim_{t \rightarrow 0^+} \sum_{n=0}^{\infty} a_n f(tn) = A$.

1.4.12 Wiener-Tauberian 定理**Théorème 12: Wiener-Tauberian 定理**

设 $f(x)$ 是 \mathbb{R} 上的绝对可积函数, 且 $f(x)$ 的 Fourier 变换没有零点, 即

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)e^{-ixt} dt \neq 0, \quad x \in \mathbb{R}.$$

若对于在任意区间上可积, 且在 \mathbb{R} 上有界的函数 $h(x)$ 有

$$\lim_{x \rightarrow +\infty} \int_{-\infty}^{+\infty} h(x-t)f(t)dt = A \int_{-\infty}^{+\infty} f(t)dt,$$

那么对任意在 \mathbb{R} 上绝对可积函数 $g(x)$, 会有以下等式

$$\lim_{x \rightarrow +\infty} \int_{-\infty}^{+\infty} h(x-t)g(t)dt = A \int_{-\infty}^{+\infty} g(t)dt.$$

Note. 这个定理可以用来证明 Riemann-Lebesgue 引理.

Remarque 1. 提示: 建议考虑先证明 $\lim_{x \rightarrow +\infty} \int_0^1 \sin(xt)dt = 0$, 然后你会发现上述积分都形如 $F(x) = \int_0^{+\infty} f(t)h(xt)dt$, 我们可以通过令 $x = e^u, t = e^{-s}$, 就可以得到卷积的形式.

1.4.13 求导证明是常数

例如平均值公式的证明??, 例如《微分几何例题详解和习题汇编 by 陈维桓》p30.

1.4.14 多项式插值

Théorème 13

若 x_0, x_1, \dots, x_n 是不同的实数, 则对任意数值 y_0, y_1, \dots, y_n , 存在唯一的次数至多是 n 次的多项式 p_n , 使得

$$p_n(x_i) = y_i, 0 \leq i \leq n.$$

证明. 唯一性: 若 p_n, q_n 都是使得插值成立的至多是 n 次的多项式, 则

$$(p_n - q_n)(x_i) = 0, 0 \leq i \leq n.$$

多项式 $p_n - q_n$ 有 $n+1$ 个零点, 根据代数基本定理, $p_n - q_n$ 平凡, 即 $p_n = q_n$.

存在性: 考虑拉格朗日插值法, 直接构造出来插值多项式. \square

Théorème 14: 插值多项式误差定理

若 $p(x)$ 是 $f(x)$ 在点 $x_0 < x_1 < \dots < x_n$ 处的插值多项式, 则

$$p(x) - f(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^n (x - x_i), \text{ 其中 } \xi_x \in [x_0, x_n]$$

证明. 证明考虑 K 值法, 直接令 $\phi(x) = p(x) - f(x) - K \prod_{i=0}^n (x - x_i)$, 两边使用 $n+1$ 次罗尔中值定理即可. \square

1.4.15 证明无穷可微

Exemple 1. 引理 4. 1 设 ζ 是这样定义的实函数:

$$\zeta(t) = \begin{cases} e^{-\frac{1}{t}}, & \text{对于 } t > 0, \\ 0, & \text{对于 } t \leq 0, \end{cases}$$

则 $\zeta \in C^\infty(\mathbf{R}, \mathbf{R})$.

证明. 运用 L' Hospital 法则很容易证明对任何多项式 $P(x)$ 都有

$$\lim_{x \rightarrow +\infty} e^{-x} P(x) = 0$$

因而

$$\lim_{t \rightarrow 0+} e^{-\frac{1}{t}} P\left(\frac{1}{t}\right) = 0.$$

为了证明 $\zeta \in C^\infty(\mathbf{R}, \mathbf{R})$, 须验证: 对任何非负整数 n , 函数 $\zeta(t)$ 在 $t > 0$ 范围和 $t \leq 0$ 范围分别求出的 n 阶导数都能在 $t = 0$ 处连续地衔接起来。通过归纳, 很容易证明 ζ 的 n 阶导数 $\zeta^{(n)}$ 可以表示为

$$\zeta^{(n)}(t) = \begin{cases} e^{-\frac{1}{t}} P_n\left(\frac{1}{t}\right), & \text{对于 } t > 0, \\ 0, & \text{对于 } t \leq 0. \end{cases}$$

这里

$$\begin{aligned} P_0\left(\frac{1}{t}\right) &= 1, \\ P_{n+1}\left(\frac{1}{t}\right) &= \left(P_n\left(\frac{1}{t}\right) - P'_n\left(\frac{1}{t}\right)\right) \frac{1}{t^2}. \end{aligned}$$

因而 P_n 是一个 $2n$ 次的多项式. 据此可知

$$\lim_{t \rightarrow 0+} e^{-\frac{1}{t}} P_n\left(\frac{1}{t}\right) = 0.$$

综上所述, 我们证明了 $\zeta \in C^\infty(\mathbf{R}, \mathbf{R})$. □

1.4.16 抽象二阶微分方程求解

Théorème 15

考虑如下微分方程:

$$y'' + ay' + by = f(x), \quad \text{其中 } a, b \text{ 是常数.} \quad (1.1)$$

它的解为 $y(x) = \text{齐通} + \int_0^x g(x-t)f(t)dt$, 其中 $g(x)$ 为 $y(0) = 0, y'(0) = 1$ 的齐次解.

Note. 这个性质可以解决一类微分方程构造问题. 比如证明:

Exercice 8

设 $f(x) \in D^2[0, 1]$ 满足 $f(0) = 2, f'(0) = 0, f'(1) = e - e^{-1}$. 证明存在 $\theta \in (0, 1)$, 使得 $f''(\theta) = f(\theta)$.

证明. • 若 f'' 具有可积性: 考虑微分方程 $f''(x) - f(x) =: h(x)$, 其特解为 $c_1 e^x + c_2 e^{-x}$, 带入定理 Théorème 15, 可知 $g(x) = \frac{e^x - e^{-x}}{2}$. 于是上述微分方程的解为

$$\begin{aligned} f(x) &= c_1 e^x + c_2 e^{-x} + \int_0^x \frac{e^{x-t} - e^{-x+t}}{2} h(t) dt \\ f'(x) &= c_1 e^x - c_2 e^{-x} + \int_0^x \frac{e^{x-t} + e^{-x+t}}{2} h(t) dt \end{aligned}$$

带入初值条件 $f(0) = 2, f'(0) = 0$, 就有

$$f(x) = e^x + e^{-x} + \int_0^x \frac{e^{x-t} - e^{-x+t}}{2} h(t) dt \quad (1.2)$$

再带入 $f'(1) = e - e^{-1}$, 得到

$$\int_0^1 \frac{e^{1-t} + e^{t-1}}{2} h(t) dt = 0, \quad \text{其中 } \frac{e^{1-t} + e^{t-1}}{2} > 0, \forall t \in [0, 1].$$

由积分中值定理: 存在 $\theta \in (0, 1)$, 使得

$$h(\theta) \int_0^1 \frac{e^{1-t} + e^{t-1}}{2} dt = 0 \Rightarrow g(\theta) = 0, \text{ 即 } f''(\theta) = f(\theta).$$

• 若 f'' 不具有可积性: ???

□

Théorème 16: 含参积分求导公式

对于有定义的积分 $f(x) = \int_{a(x)}^{b(x)} g(x, t) dt$, 其导数为

$$f'(x) = \int_{a(x)}^{b(x)} g_x(x, t) dt + g(x, b(x))b'(x) - g(x, a(x))a'(x) \quad (1.3)$$

1.4.17 等分布

Théorème 17: Weyl 等分布定理

用 $\#\{\}$ 表示集合元素个数.

设 $x_1, x_2, \dots, x_n, \dots \in [0, 1]$, 如下结果等价:

(1) 对任何整数 $k \in \mathbb{N}$, 有

$$\lim_{n \rightarrow \infty} \frac{\sum_{j=1}^n e^{2\pi i k x_j}}{n} = 0.$$

(2) 对任何 $f \in C[0, 1]$, $f(0) = f(1)$, 有

$$\lim_{n \rightarrow \infty} \frac{\sum_{j=1}^n f(x_j)}{n} = \int_0^1 f(x) dx.$$

(3) 对任何 $(a, b) \subset [0, 1]$, 有

$$\lim_{n \rightarrow \infty} \frac{\#\{1 \leq j \leq n : x_j \in (a, b)\}}{n} = b - a.$$

(4) 对任何 $I \subset [0, 1]$, I 是一个区间, 有

$$\lim_{n \rightarrow \infty} \frac{\#\{1 \leq j \leq n : x_j \in I\}}{n} = |I|.$$

(5) 对任何 $f \in R[0, 1]$, 有

$$\lim_{n \rightarrow \infty} \frac{\sum_{j=1}^n f(x_j)}{n} = \int_0^1 f(x) dx.$$

Note. 证明见 [exercises.pdf](#) 第 24 页

Théorème 18: Fejer 等分布定理

(1) 设 $\{f(n)\}_{n \in \mathbb{N}}$ 是 \mathbb{R} 上的实序列, 若存在 $k \in \mathbb{N}$, 使得:

(I) 当 n 充分大时, $\Delta^k f(n)$ ^a严格单调趋近于 0.

(II) $\lim_{n \rightarrow \infty} n |\Delta^k f(n)| = +\infty$.

则 $\{\{f(n)\}\}_{n \in \mathbb{N}}$ ^b在 $[0, 1)$ 上等分布.

(2) 设 $f : \mathbb{R} \rightarrow \mathbb{R}$ 是足够光滑的函数, 若存在 $k \in \mathbb{N}$, 使得

(I) 当 x 充分大时, $f^{(k)}(x)$ 严格单调趋近于 0.

(II) $\lim_{x \rightarrow +\infty} x |f^{(k)}(x)| = +\infty$

则 $\{f(n)\}_{n \in \mathbb{N}}$ 在 $[0, 1)$ 上等分布.

${}^a\Delta^k$ 表示 k 阶差分, $\Delta f(n) = f(n+1) - f(n)$.

${}^b\{-\}$ 表示取小数部分.

Théorème 19: Van Der Corput 差分定理

设 $\{\xi_n\}_{n \in \mathbb{N}}$ 是 $[0, 1)$ 上的实序列, 若对 $\forall h \in \mathbb{N}$ 都有 $\{\xi_{n+h} - \xi_n\}_{n \in \mathbb{N}}$ 在 $[0, 1)$ 上等分布, 则 $\{\xi_n\}_{n \in \mathbb{N}}$ 是 $[0, 1)$ 上的等分布序列.

Note. 这个定理就是模 1 等分布, 见 [exercises.pdf](#)

Corollaire 3

$\left\{ \left\{ \frac{n^\alpha}{2\pi} \right\} \right\}_{n \in \mathbb{N}}$ 在 $[0, 1)$ 上等分布, 其中 $\alpha > 0$.

Note. 用一般的 Weyl 等分布定理 Théorème 17 只能证明 $0 < \alpha < 1$ 的情况.

见 Stein 傅里叶分析 Chapter 4 Ex 8.

证明. 证明 Van Der Corput 差分定理 Théorème 19 由 Weyl 准则可知

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i k (\xi_{n+h} - \xi_n)} = 0, \quad \forall h \in \mathbb{N}, \forall k \in \mathbb{Z} \setminus \{0\}$$

设 $u_n = e^{2\pi i k \xi_n}$, 则可表示为

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N u_{n+d} \overline{u_n} = 0, \text{ 对于任意 } d \in \mathbb{Z}.$$

我们的目标是要证

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N u_n = 0$$

对于任意给定的 $D \in \mathbb{N}$, 由三角不等式:

$$\left| \frac{1}{N} \sum_{n=1}^N u_n \right| \leq \left| \frac{1}{N} \sum_{n=1}^N u_n - \frac{1}{N} \sum_{n=1}^N \frac{1}{D} \sum_{d=1}^D u_{n+d} \right| + \left| \frac{1}{ND} \sum_{n=1}^N \sum_{d=1}^D u_{n+d} \right| \quad (1.4)$$

对于第一部分:⁴

$$\left| \frac{1}{N} \sum_{n=1}^N u_n - \frac{1}{N} \sum_{n=1}^N \frac{1}{D} \sum_{d=1}^D u_{n+d} \right| \stackrel{\text{裂项相消}}{=} \frac{1}{N} \left| \sum_{k=1}^D \frac{D+1-k}{D} u_k - \sum_{k=1}^D \frac{D+1-k}{D} u_{N+k} \right| \stackrel{D \text{ 是固定的}}{=} O\left(\frac{1}{N}\right). \quad (1.5)$$

⁴这里用到了重要分析思想, 先给定 D .

对于第二部分：

$$\begin{aligned}
 \left| \frac{1}{ND} \sum_{n=1}^N \sum_{d=1}^D u_{n+d} \right| &\leq \frac{1}{ND} \sqrt{\sum_{n=1}^N 1^2 \sum_{n=1}^N \left| \sum_{d=1}^D u_{n+d} \right|^2} = \sqrt{\frac{1}{D^2} \sum_{d_1, d_2=1}^D \left(\sum_{n=1}^N \frac{u_{n+d_1} \bar{u}_{n+d_2}}{N} \right)} \\
 &= \sqrt{\frac{1}{D^2} \left[\sum_{\substack{d_1, d_2=1 \\ d_1 \neq d_2}}^D \left(\sum_{n=1}^N \frac{u_{n+d_1} \bar{u}_{n+d_2}}{N} \right) + \sum_{d=1}^D \left(\sum_{n=1}^N \frac{u_{n+d} \bar{u}_{n+d}}{N} \right) \right]} \\
 &\stackrel{u_{n+d} \bar{u}_{n+d} = |u_{n+d}|^2}{=} \sqrt{\frac{1}{D^2} \left[\sum_{\substack{d_1, d_2=1 \\ d_1 \neq d_2}}^D \left(\sum_{n=1}^N o(1) \right) + \sum_{d=1}^D \left(\sum_{n=1}^N \frac{1}{N} \right) \right]} = o(1) + \frac{1}{D}
 \end{aligned}$$

于是

$$\left| \frac{1}{N} \sum_{n=1}^N u_n \right| \leq O\left(\frac{1}{N}\right) + o(1) + \frac{1}{D} \xrightarrow{\text{当 } N \rightarrow \infty} \frac{1}{D} \quad (1.6)$$

由 D 的任意性可知：

$$\lim_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N u_n \right| = 0 \implies \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N u_n = 0 \quad (1.7)$$

由 Weyl 等分布定理 Théorème 17 可知 $\{\xi_n\}_{n \in \mathbb{N}}$ 是 $[0, 1)$ 上的等分布序列.

□

证明. 证明 Fejer 等分布定理 Théorème 18

(1) 考虑数学归纳法, 当 $k = 1$ 时, 考虑对 $\forall u, v \in \mathbb{R}$, 此时有

$$\begin{aligned}
 |e^{2\pi i u} - e^{2\pi i v} - 2\pi i(u-v)e^{2\pi i v}| &= 4\pi^2 \left| \int_0^{u-v} (u-v-\omega) e^{2\pi i \omega} d\omega \right| \\
 &\leq 4\pi^2 \left| \int_0^{u-v} (u-v-\omega) d\omega \right| = 2\pi^2(u-v)^2
 \end{aligned}$$

我们代入 $u = hf(n+1), v = hf(n)$, 其中 $h \in \mathbb{N}$, 由上式可得

$$\left| \frac{e^{2\pi i hf(n+1)}}{\Delta f(n)} - \frac{e^{2\pi i hf(n)}}{\Delta f(n)} - 2\pi i h e^{2\pi i hf(n)} \right| \leq 2\pi^2 h^2 |\Delta f(n)|, n \in \mathbb{N}$$

由绝对值的三角不等式进一步可得

$$\begin{aligned}
 &\left| \frac{e^{2\pi i hf(n+1)}}{\Delta f(n+1)} - \frac{e^{2\pi i hf(n)}}{\Delta f(n)} - 2\pi i h e^{2\pi i hf(n)} \right| \\
 &= \left| \left(\frac{e^{2\pi i hf(n+1)}}{\Delta f(n+1)} - \frac{e^{2\pi i hf(n+1)}}{\Delta f(n)} \right) + \frac{e^{2\pi i hf(n+1)}}{\Delta f(n)} - \frac{e^{2\pi i hf(n)}}{\Delta f(n)} - 2\pi i h e^{2\pi i hf(n)} \right| \\
 &\leq \left| \frac{e^{2\pi i hf(n+1)}}{\Delta f(n+1)} - \frac{e^{2\pi i hf(n+1)}}{\Delta f(n)} \right| + 2\pi^2 h^2 |\Delta f(n)|
 \end{aligned}$$

于是有

$$\begin{aligned} \left| 2\pi i h \sum_{n=1}^{N-1} e^{2\pi i h f(n)} \right| &= \left| \sum_{n=1}^{N-1} \left(2\pi i h e^{2\pi i h f(n)} - \frac{e^{2\pi i h f(n+1)}}{\Delta f(n+1)} + \frac{e^{2\pi i h f(n)}}{\Delta f(n)} \right) + \frac{e^{2\pi i h f(n)}}{\Delta f(n)} - \frac{e^{2\pi i h f(1)}}{\Delta f(1)} \right| \\ &\leq \sum_{n=1}^{N-1} \left| 2\pi i h e^{2\pi i h f(n)} - \frac{e^{2\pi i h f(n+1)}}{\Delta f(n+1)} + \frac{e^{2\pi i h f(n)}}{\Delta f(n)} \right| + \frac{1}{|\Delta f(N)|} + \frac{1}{|\Delta f(1)|} \\ &\leq \sum_{n=1}^{N-1} \left| \frac{1}{\Delta f(n)} - \frac{1}{\Delta f(n+1)} \right| + 2\pi^2 h^2 \sum_{n=1}^{N-1} |\Delta f(n)| + \frac{1}{|\Delta f(N)|} + \frac{1}{|\Delta f(1)|} \end{aligned}$$

注意到 $\Delta f(n)$ 是单调的, 利用交错相消性可知

$$\left| \frac{1}{N} \sum_{n=1}^{N-1} e^{2\pi i h f(n)} \right| \leq \frac{1}{\pi |h|} \left(\frac{1}{N |\Delta f(1)|} + \frac{1}{N |\Delta f(N)|} \right) + \frac{\pi |h|}{N} \sum_{n=1}^{N-1} |\Delta f(n)| \xrightarrow{N \rightarrow \infty} 0$$

由 Weyl 准则命题得证。其它情形下, 不妨设 $k \in \mathbb{N}$, 当 $k+1$ 情形时满足条件, 我们有

$$f(n+h) - f(n) = \sum_{j=0}^{h-1} \Delta f(n+j)$$

于是有

$$\Delta^k(f(n+h) - f(n)) = \sum_{j=0}^{h-1} \Delta^{k+1} f(n+j)$$

由 $\Delta^{k+1} f(n)$ 的单调趋近 0 性质可知, 当 n 充分大时, 于是得 $\Delta^k(f(n+h) - f(n))$ 单调趋近于 0, 且 $n |\Delta^k(f(n+h) - f(n))| \rightarrow +\infty$ 重复以上步骤, 得 $\Delta(\Delta^{k-1}(f(n+h) - f(n)))$ 单调趋近于 0, 且

$$n |\Delta(\Delta^{k-1}(f(n+h) - f(n)))| \rightarrow +\infty$$

反复重复差分运算, 将差分次数由 $k+1$ 降低至 1, 运用 Van Der Corput 差分定理 Théorème 19 最终得 $\{\{f(n)\}\}_{n \in \mathbb{N}}$ 在 $[0, 1]$ 上等分布。

(2) 注意到

$$\Delta f(n) = \int_0^1 f'(n+t) dt \quad \Delta^k f(n) = \int_0^1 \int_0^1 \cdots \int_0^1 f^{(k+1)}(n+t_1+t_2+\cdots+t_k) dt_1 dt_2 \cdots dt_k$$

当 $k=1$ 时, 由 $f'(x)$ 单调趋近 0 易知 $\Delta f(n)$ 也单调趋近 0, 同理易知 $n |\Delta f(n)| \rightarrow +\infty$, 利用 (i) 可知此时命题得证, 当 $k+1$ 情形时, 有 $\Delta^k f(n)$ 单调趋近于零, 且 $n |\Delta^k f(n)| \rightarrow +\infty$, 于是利用 (i) 的结论可知 $\{\{f(n)\}\}_{n \in \mathbb{N}}$ 在 $[0, 1]$ 上等分布.

□

Note. 断言命题对于 $k = 1$ 成立. 即若 $\Delta f(n)$ 严格递减趋于 0 (当 n 充分大), 且 $\lim_{n \rightarrow \infty} n|\Delta f(n)| = +\infty$, 就有 $\{\{f(n)\}\}_{n \in \mathbb{N}}$ 在 $[0, 1]$ 上等分布.

现在我们有: 若 $\Delta^{k+1} f(n)$ 严格递减趋于 0 (当 n 充分大), 且 $\lim_{n \rightarrow \infty} n|\Delta^{k+1} f(n)| = +\infty$, 则 $\forall h_1 \in \mathbb{N}$, 有 $\Delta^k [f(n+h_1) - f(n)] = \Delta^k f(n+h_1) - \Delta^k f(n)$ 严格递减趋于 0 (当 n 充分大), 且 $\{\Delta^k f(n+h_1)\} - \{\Delta^k f(n)\} = \{\Delta^k f(n+h_1) - \Delta^k f(n)\}$, 且 $\lim_{n \rightarrow \infty} n|\Delta^k f(n+h_1) - \Delta^k f(n)| = +\infty$. 记 $f_{h_1}(x) = f(x+h_1) - f(x)$, 于是 $\Delta^k f_{h_1}(n)$ 严格递减趋于 0 (当 n 充分大), 且 $\{\Delta^k f(n+h_1)\} - \{\Delta^k f(n)\} = \{\Delta^k f_{h_1}(n)\}$, 且 $\lim_{n \rightarrow \infty} n|\Delta^k f_{h_1}(n)| = +\infty$. 重复这个过程, 最后 $\Delta f_{h_1, h_2, \dots, h_k}(n)$ 严格递减趋于 0 (当 n 充分大), 且 $\{\Delta^k f(n+h_j)\} - \{\Delta^k f(n)\} = \{\Delta^k f_{h_j}(n)\} \forall 1 \leq j \leq k$, 且 $\lim_{n \rightarrow \infty} n|\Delta f_{h_1, h_2, \dots, h_k}(n)| = +\infty$. 于是 $\{\{f_{h_1, h_2, \dots, h_k}(n)\}\}_{n \in \mathbb{N}}$ 在 $[0, 1]$ 上等分布. 应用 k 次 Van Der Corput 差分定理 Théorème 19: $\{\{f_{h_1, h_2, \dots, h_{k-1}}(n)\}\}_{n \in \mathbb{N}}$ 在 $[0, 1]$ 上等分布 $\{\{f(n)\}\}_{n \in \mathbb{N}}$ 在 $[0, 1]$ 上等分布.

1.4.18 利用任意性

在解析数论中, 定义 $\psi(x) = \sum_{p^m \leq x} \log p = \sum_{p \leq x} \left[\frac{\log x}{\log p} \right] \log p$, $\pi(x) = \sum_{p \leq x} 1$.

Théorème 20

若 $\psi(x) \sim x (x \rightarrow \infty)$, 则 $\pi(x) \sim x / \log x (x \rightarrow \infty)$.

证明. 一方面,

$$\psi(x) = \sum_{p \leq x} \left[\frac{\log x}{\log p} \right] \log p \leq \sum_{p \leq x} \log x = \pi(x) \log x \implies \frac{\psi(x)}{x} \leq \frac{\pi(x) \log x}{x}, \forall x \in \mathbb{R} \quad (1.8)$$

另一方面, 对于任意给定的 $0 < \alpha < 1$

$$\psi(x) \geq \sum_{p \leq x} \log p \geq \sum_{x^\alpha < p \leq x} \log p \geq (\pi(x) - \pi(x^\alpha)) \log x^\alpha \implies \psi(x) + \alpha \pi(x^\alpha) \log x \geq \alpha \pi(x) \log x$$

于是

$$\frac{\psi(x)}{x} + \alpha \frac{\log x}{x^{1-\alpha}} \geq \frac{\psi(x)}{x} + \alpha \frac{\pi(x^\alpha) \log x}{x} \geq \alpha \frac{\pi(x) \log x}{x} \implies \limsup_{x \rightarrow \infty} \frac{\psi(x)}{x} \geq \alpha \limsup_{x \rightarrow \infty} \frac{\pi(x) \log x}{x}$$

由 α 任意性

$$1 = \limsup_{x \rightarrow \infty} \frac{\psi(x)}{x} \geq \limsup_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} \geq \liminf_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} \stackrel{(1.8)}{\geq} \liminf_{x \rightarrow \infty} \frac{\psi(x)}{x} = 1$$

□

1.4.19 求和下的大 O 小 o 估阶

Proposition 1

设 $b_n > 0$, 且 $a_n = O(b_n)$, $n \rightarrow \infty$, 则

$$\sum_{n=1}^N a_n = O\left(\sum_{n=1}^N b_n\right), \quad N \rightarrow \infty$$

Note. 此证明相当显然.

证明. 存在常数 $C > 0, N > 0$, 使得 $|a_n| < C \cdot b_n, \forall n \geq N$.

只需要验证存在常数 $C' > 0$, 使得 $\left|\sum_{k=1}^n a_k\right| \leq C' \sum_{k=1}^n b_k, \forall n > N$. 这只需要如下的简单放缩:

$$\begin{aligned} \left|\sum_{k=1}^n a_k\right| &= \left|\sum_{k=1}^N a_k\right| + \left|\sum_{k=N+1}^n a_k\right| \\ &= \left|\sum_{k=1}^N \frac{a_k}{b_k} b_k\right| + \left|\sum_{k=N+1}^n a_k\right| \\ &\leq \sup_{1 \leq k \leq N} \frac{|a_k|}{b_k} \sum_{k=1}^N b_k + C \sum_{k=N+1}^n b_k \\ &\leq \left(C + \sup_{1 \leq k \leq N} \frac{|a_k|}{b_k}\right) \sum_{k=1}^n b_k \end{aligned}$$

□

Proposition 2

设 $b_n > 0, \sum_{n=1}^{\infty} b_n = \infty$, 且 $a_n = o(b_n)$, $n \rightarrow \infty$, 则

$$\sum_{n=1}^N a_n = o\left(\sum_{n=1}^N b_n\right), \quad N \rightarrow \infty$$

Note. 这里的证明要在 N 上做手脚.

证明. 对于任意给定的 $\varepsilon > 0$, 存在 $N > 0$, 使得 $a_n \leq \frac{\varepsilon}{2} \cdot b_n, \forall n \geq N$, 还存在

$$N' > 0, \text{ 使得 } \sum_{k=1}^N b_k \leq \frac{\varepsilon}{2 \sup_{1 \leq k \leq N} \frac{|a_k|}{b_k}} \sum_{k=1}^n b_k, \forall n > N'.$$

只需要验证 $\forall \varepsilon > 0$, 使得 $\left| \sum_{k=1}^n a_k \right| \leq \varepsilon \sum_{k=1}^n b_k, \forall n > N'$. 这只需要如下的简单放缩:

$$\begin{aligned} \left| \sum_{k=1}^n a_k \right| &= \left| \sum_{k=1}^N a_k \right| + \left| \sum_{k=N+1}^n a_k \right| \\ &= \left| \sum_{k=1}^N \frac{a_k}{b_k} b_k \right| + \left| \sum_{k=N+1}^n a_k \right| \\ &\leq \sup_{1 \leq k \leq N} \frac{|a_k|}{b_k} \sum_{k=1}^N b_k + \frac{\varepsilon}{2} \sum_{k=N+1}^n b_k \\ &\leq \frac{\varepsilon}{2} \sum_{k=1}^n b_k + \frac{\varepsilon}{2} \sum_{k=N+1}^n b_k \\ &\leq \varepsilon \sum_{k=1}^n b_k \end{aligned}$$

□

Remarque 2. 此证明还可见于《阶的估计基础》p18 定理 2, 其中采用了大 O 小 o 的语言来书写证明, 比较抽象.

1.4.20 利用一致收敛性

定理 Théorème 2 就是一个例子, 下面还有另一个例子:

Théorème 21

Bernstein 多项式逼近 f 是 $[0, 1]$ 上的连续函数, 则其 Bernstein 多项式 $B_n(f)$ 在 $C[0, 1]$ 上一致收敛于 f . 其中

$$B_n(f)(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) C_n^k x^k (1-x)^{n-k}$$

也就是说

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, 1]} |B_n(f)(x) - f(x)| = 0$$

Note. 从概率论的角度给出证明，这样更好地给出了 Bernstein 多项式的构造思路。来源于 [Durrett](#)。

证明。考虑 n 个独立随机变量 X_i ，且 $P(X_i = 1) = p, P(X_i = 0) = 1 - p$ ，有 $EX_i = p, Var(X_i) = p(1 - p)$ ，令 $S_n = X_1 + X_2 + \dots + X_n$ ，就有

$$P(S_n = m) = C_n^m p^m (1 - p)^{n-m}$$

于是 $Ef(S_n/n) = B_n(f)(p)$ 。由 Durrett Theorem 2.2.3， S_n/n 依概率收敛于 p 。结合 Theorem 2.2.3 的证明和 $p(1 - p) \leq \frac{1}{4}, \forall p \in [0, 1]$ ，有

$$P(|S_n/n - p| \geq \delta) \leq \frac{var(S_n/n)}{\delta^2} = \frac{p(1 - p)}{n\delta^2} \leq \frac{1}{4n\delta^2}$$

为了证明 $Ef(S_n/n) \rightarrow f(p)$ ，由于 f 在 $[0, 1]$ 上一致连续，对于任意 $\varepsilon > 0$ ，存在 $\delta > 0$ ，使得当 $|x - y| < \delta$ 时， $|f(x) - f(y)| < \varepsilon$ 。使用 Jensen 不等式 Théorème 22 就有

$$\begin{aligned} |Ef(S_n/n) - f(p)| &\stackrel{\text{Théorème 22}}{\leq} E|f(S_n/n) - f(p)| = \int |f(S_n/n) - f(p)| dP \\ &= \int_{\{|S_n/n-p|\leq\delta\}} |f(S_n/n) - f(p)| dP + \int_{\{|S_n/n-p|>\delta\}} |f(S_n/n) - f(p)| dP \\ &\leq \varepsilon P(|S_n/n - p| \leq \delta) + \sup_{[0,1]} |f(x) - f(p)| P(|S_n/n - p| > \delta) \\ &\leq \varepsilon + 2 \sup_{x \in [0,1]} |f(x)| \frac{1}{4n\delta^2} \end{aligned}$$

令 $n \rightarrow \infty$ ，就有 $\limsup_{n \rightarrow \infty} |Ef(S_n/n) - f(p)| \leq \varepsilon$ 。由 ε 的任意性，即有 $\limsup_{n \rightarrow \infty} |Ef(S_n/n) - f(p)| = 0$ ，即 $Ef(S_n/n) \rightarrow f(p)$ 。 \square

Théorème 22: Jensen 不等式

若 φ 是凸函数^a，则 $E\varphi(X) \geq \varphi(EX)$ 。

^a即 $\forall x, y \in \mathbb{R}$ ，有 $\theta\varphi(x) + (1 - \theta)\varphi(y) \leq \varphi(\theta x + (1 - \theta)y)$ ，其中 $0 \leq \theta \leq 1$ 。

Théorème 23: Markov 不等式

设 X 是随机变量，则对任意 $a > 0$ ，有 $P(|X| \geq a) \leq \frac{E|X|}{a}$ 。

证明。由 X 的非负性，有 $E|X| = \int_0^\infty P(|X| \geq t) dt \geq \int_0^a P(|X| \geq t) dt \geq aP(|X| \geq a)$ ，即 $P(|X| \geq a) \leq \frac{E|X|}{a}$ 。 \square

1.4.21 多元函数

Théorème 24: 含参积分求导公式

设 $f(x, y)$ 与 $D_2 f(x, y)$ 在矩形 $[a, b] \times [c, d]$ 上连续, 设 $p(x), q(x)$ 在 $[a, b]$ 上可微, 且 $\forall y \in [c, d]$, 有 $p(y), q(y) \in [a, b]$, 则

$$\frac{d}{dx} \int_{p(y)}^{q(y)} f(x, y) dx = -f(p(y), y)p'(y) + f(q(y), y)q'(y) + \int_{p(y)}^{q(y)} D_2 f(x, y) dx$$

证明. 设 $G(x_1, x_2, x_3) = \int_{x_1}^{x_2} f(t, x_3) dt$, 考虑 $F(y) = G(p(y), q(y), y)$ 作为复合函数, 由链式法则得到

$$F'(y) = D_1 G(p(y), q(y), y)p'(y) + D_2 G(p(y), q(y), y)q'(y) + D_3 G(p(y), q(y), y)$$

我们有 $D_1 G(x_1, x_2, x_3) = -f(x_1, x_3)$, $D_2 G(x_1, x_2, x_3) = f(x_2, x_3)$, $D_3 G(x_1, x_2, x_3) = \int_{x_1}^{x_2} D_2 f(t, x_3) dt$, 于是

$$F'(y) = -f(p(y), y)p'(y) + f(q(y), y)q'(y) + \int_{p(y)}^{q(y)} D_2 f(t, y) dt$$

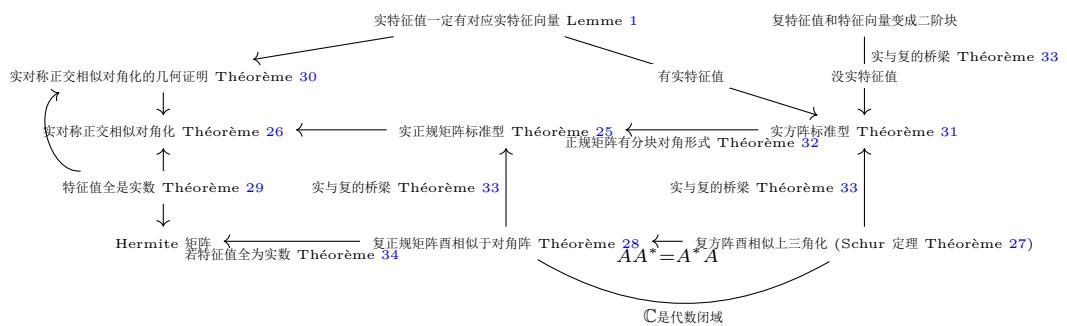
□

1.4.22 函数方程

Cauchy 函数方程

1.5 高等代数

1.5.1 实矩阵、正规、对称



Théorème 25

实正规矩阵可以正交相似为标准型.

Théorème 26

实对称矩阵可正交相似对角化.

Théorème 27: Schur 定理

设 V 是 n 维酉空间, φ 是 V 上的线性算子, 则存在 V 的一组标准正交基, 使 φ 在这组基下的表示矩阵为上三角阵.

证明. 选取 φ 的一个单位特征向量, 再扩充成 V 的一组标准正交基, 归纳得证. \square

Théorème 28

复矩阵 A 为复正规矩阵的充分必要条件是 A 酉相似于对角阵.

Note. 实对称矩阵一定是实正规矩阵, 于是考虑正交相似标准型, 这个正交相似标准型还要是对称的, 故斜对角线上的 $b_i = 0, \forall i$, 于是是对角阵.

Propriété 2: factos

在内积空间中平行四边形两对角线平方和等于四边平方和, 即

$$\|\alpha + \beta\|^2 + \|\alpha - \beta\|^2 = 2\|\alpha\|^2 + 2\|\beta\|^2.$$

Définition 2: 伴随 $(\varphi(\alpha), \beta) = (\alpha, \varphi^*(\beta))$

设 V 是 n 维内积空间, $\{e_1, e_2, \dots, e_n\}$ 是 V 的一组标准正交基. 若 V 上的线性算子 φ 在这组基下的表示矩阵为 A , 则当 V 是酉空间时, φ^* ^a 在同一组基下的表示矩阵为 \bar{A}' , 即 A 的共轭转置; 当 V 是欧氏空间时, φ^* 的表示矩阵为 A' , 即 A 的转置.

^a $\forall \alpha, \beta \in V, (\varphi(\alpha), \beta) = (\alpha, \varphi^*(\beta)).$

Définition 3: 保持内积的线性变换 $(\varphi(\alpha), \varphi(\beta)) = (\alpha, \beta)$

设 φ 是内积空间 V 上保持内积的线性变换^a, 若 V 是欧氏空间, 则称 φ 为正交变换或正交算子^b; 若 V 是酉空间, 则称 φ 为酉变换或酉算子^c.

^a $\forall \alpha, \beta \in V, (\varphi(\alpha), \varphi(\beta)) = (\alpha, \beta).$

^b在欧式空间任意一组标准正交基下表示矩阵为正交矩阵 $A, AA^T = I.$

^c在酉空间任意一组标准正交基下表示矩阵为酉矩阵 $U, UU^* = I.$

Définition 4: 自伴随算子 $\varphi^* = \varphi$

设 φ 是内积空间 V 上的线性变换, φ^* 是 φ 的伴随, 若 $\varphi^* = \varphi$, 则称 φ 是自伴随算子. 当 V 是欧氏空间时, φ 也称为对称算子或对称变换^a; 当 V 是酉空间时, φ 也称为 Hermite 算子或 Hermite 变换^b.

^a在欧式空间任意一组标准正交基下表示矩阵为对称矩阵 $A, A = A^T.$

^b在酉空间任意一组标准正交基下表示矩阵为 Hermite 矩阵 $A, A = A^*.$

Définition 5: 正规算子

设 φ 是内积空间 V 上的线性变换, φ^* 是其伴随, 若 $\varphi\varphi^* = \varphi^*\varphi$, 则称 φ 是 V 上的正规算子. 为了不引起混淆, 我们也称酉空间(欧氏空间) V 上的正规算子 φ 为复正规算子(实正规算子). 复矩阵 A 若适合 $\bar{A}'A = A\bar{A}'$, 则称其为复正规矩阵. 实矩阵 A 若适合 $A'A = AA'$, 则称其为实正规矩阵.

Théorème 29

设 V 是 n 维酉空间, φ 是 V 上的自伴随算子, 则 φ 的特征值全是实数且属于不同特征值的特征向量互相正交.

证明. 证明设 λ 是 φ 的特征值, x 是属于 λ 的特征向量, 则

$$\begin{aligned}\lambda(x, x) &= (\lambda x, x) = (\varphi(x), x) = (x, \varphi^*(x)) \\ &= (x, \varphi(x)) = (x, \lambda x) = \bar{\lambda}(x, x).\end{aligned}$$

因为 $(x, x) \neq 0$, 故 $\bar{\lambda} = \lambda$, 即 λ 是实数. 又若设 μ 是 φ 的另一个特征值, y

是属于 μ 的特征向量, 注意到 λ, μ 都是实数, 故有

$$\begin{aligned}\lambda(\mathbf{x}, \mathbf{y}) &= (\lambda\mathbf{x}, \mathbf{y}) = (\varphi(\mathbf{x}), \mathbf{y}) = (\mathbf{x}, \varphi^*(\mathbf{y})) \\ &= (\mathbf{x}, \varphi(\mathbf{y})) = (\mathbf{x}, \mu\mathbf{y}) = \mu(\mathbf{x}, \mathbf{y})\end{aligned}$$

由于 $\lambda \neq \mu$, 故 $(\mathbf{x}, \mathbf{y}) = 0$, 即 $\mathbf{x} \perp \mathbf{y}$. \square

Note. 推论是实 (复) 对称矩阵的特征值全为实数, 不同特征值的特征向量两两正交.

Théorème 30: 实对称正交相似对角化的几何证明

设 V 是 n 维内积空间, φ 是 V 上的自伴随算子, 则存在 V 的一组标准正交基, 使 φ 在这组基下的表示矩阵为实对角阵, 且这组基恰为 φ 的 n 个线性无关的特征向量.

证明. 证明首先需要说明的是, 若 V 是欧氏空间, 则由于自伴随算子 φ 的特征值都是实数, 故有实的特征向量. 不妨设 u 是 φ 的特征向量, 令 $v_1 = \frac{u}{\|u\|}$, 则 v_1 是 φ 的长度等于 1 的特征向量. 我们对维数 n 用归纳法.

若 $\dim V = 1$, 结论已成立. 设对小于 n 维的内积空间结论成立. 令 W 为由 v_1 张成的子空间, W^\perp 为 W 的正交补空间, 则 W 是 φ 的不变子空间且

$$V = W \oplus W^\perp, \quad \dim W^\perp = n - 1.$$

由命题 9.3.1 可知 W^\perp 是 $\varphi^* = \varphi$ 的不变子空间. 将 φ 限制在 W^\perp 上仍是自伴随算子. 由归纳假设, 存在 W^\perp 的一组标准正交基 $\{v_2, \dots, v_n\}$, 使 φ 在这组基下的表示矩阵为实对角阵, 且 $\{v_2, \dots, v_n\}$ 是其特征向量. 因此, $\{v_1, v_2, \dots, v_n\}$ 构成了 V 的一组标准正交基, φ 在这组基下的表示矩阵为实对角阵, 且 $\{v_1, v_2, \dots, v_n\}$ 为 φ 的 n 个线性无关的特征向量. \square

Théorème 31: 一般实方阵的正交相似上三角化

例 9.87 证明: n 阶实方阵 A 必正交相似于下列分块上三角矩阵:

$$C = \begin{pmatrix} A_1 & & & \\ & \ddots & & * \\ & & A_r & \\ & & c_1 & \\ & & & \ddots \\ & & & c_k \end{pmatrix},$$

其中 $A_i (1 \leq i \leq r)$ 是二阶实矩阵且 A_i 的特征值具有 $a_i \pm b_i i (b_i \neq 0)$ 的形状, $c_j (1 \leq j \leq k)$ 是实数.

证明. 先证明引理:

Lemme 1: 实特征值一定有实特征向量

A 是实方阵, 那么 A 的实特征值一定有实特征向量.

证明. 若 A 有实特征值 λ , 则考虑 λ 对应的特征向量 $\mathbf{u} + i\mathbf{v}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, 于是 $A\mathbf{u} + iA\mathbf{v} = A(\mathbf{u} + i\mathbf{v}) = \lambda\mathbf{u} + i\mathbf{v} = \lambda\mathbf{u} + i\lambda\mathbf{v} \Rightarrow \mathbf{u}, \mathbf{v}$ 都是 A 的实特征值 λ 对应的实特征向量. \square

若 A 有实特征值 λ , 由引理 Lemme 1 可知, 存在 $\mathbf{u} \in \mathbb{R}^n$, 使得 $A\mathbf{u} = \lambda\mathbf{u}$. 考虑 \mathbf{u} 扩充成 \mathbb{R}^n 的一组基, 则 A 有表示 $\begin{pmatrix} \lambda & \alpha^T \\ O & A_{n-1} \end{pmatrix}$. 于是对 A_{n-1} 进行归纳即可得证.

若 A 没有实特征值, 考虑 A 的复特征值 $\lambda = a + ib$, $A(\mathbf{u} + i\mathbf{v}) = (a + ib)(\mathbf{u} + i\mathbf{v})$. 由定理 Théorème 33 可知, \mathbf{u}, \mathbf{v} 线性无关. 由于 A 仅仅是实方阵, 没有实正规, 所以 \mathbf{u}, \mathbf{v} 不一定正交. 取 $L(\mathbf{u}, \mathbf{v})$ 的标准正交基 e_1, e_2 , 扩充为 \mathbb{R}^n 的一组基, 则 A 有表示 $\begin{pmatrix} A_{12} & B \\ O & A_{n-2} \end{pmatrix}$. 其中 A_{12} 相似于 $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. 于是对 A_{n-2} 进行归纳即可得证. \square

Théorème 32: 正规矩阵有分块对角形式

例 9.89 设 A, B 是实方阵且分块矩阵 $\begin{pmatrix} A & C \\ O & B \end{pmatrix}$ 是实正规矩阵, 求证:
 $C = O$ 且 A, B 也是正规矩阵.

证明. 证明由已知

$$\begin{pmatrix} A & C \\ O & B \end{pmatrix} \begin{pmatrix} A' & O \\ C' & B' \end{pmatrix} = \begin{pmatrix} A' & O \\ C' & B' \end{pmatrix} \begin{pmatrix} A & C \\ O & B \end{pmatrix},$$

从而 $AA' + CC' = A'A$. 由于 $\text{tr}(AA' + CC') = \text{tr}(A'A) = \text{tr}(AA')$, 故可得 $\text{tr}(CC') = 0$, 结合引理 Lemme 3 再由 C 是实矩阵可推出 $C = O$, 于是 $AA' = A'A, BB' = B'B$. \square

Corollaire 4

由定理 Théorème 31 和定理 Théorème 32 可知实正规矩阵相似于标准型. Théorème 25

Théorème 33: 实与复的桥梁

例 9.86 设 A 是 n 阶实矩阵, 虚数 $a + bi$ 是 A 的一个特征值, $\mathbf{u} + b\mathbf{i}\mathbf{v}$ 是对应的特征向量, 其中 \mathbf{u}, \mathbf{v} 是实列向量. 求证: \mathbf{u}, \mathbf{v} 必线性无关. 若 A 是正规矩阵, 则 \mathbf{u}, \mathbf{v} 相互正交且长度相同 (取实列向量空间的标准内积).

证明. 由假设

$$A(\mathbf{u} + b\mathbf{i}\mathbf{v}) = (a + bi)(\mathbf{u} + b\mathbf{i}\mathbf{v}) = (au - bv) + (av + bu)\mathbf{i}. \quad (1.9)$$

假设 \mathbf{u}, \mathbf{v} 线性相关, 不妨设 $\mathbf{u} \neq \mathbf{0}, \mathbf{v} = k\mathbf{u}$, 则 $(1+ki)A\mathbf{u} = (1+ki)(a+bi)\mathbf{u}$, 于是 $A\mathbf{u} = (a+bi)\mathbf{u}$, 由此可得 $A\mathbf{u} = au, bu = \mathbf{0}$, 这与 $b \neq 0$ 且 $\mathbf{u} \neq \mathbf{0}$ 相矛盾. 若 A 是正规矩阵, 在 eq. (1.9) 式中比较实部和虚部得到

$$A\mathbf{u} = au - bv, \quad A\mathbf{v} = av + bu.$$

Lemme 2

$AA^* = A^*A$, 则 $A\mathbf{u} = \lambda\mathbf{u}$ 当且仅当 $A^*\mathbf{u} = \bar{\lambda}\mathbf{u}$.

Lemme 3

证明. $A \in M_{m \times n}(\mathbb{C})^a$, 则 $\text{tr}(AA^*) \geq 0$ 取等当且仅当 $A = O$.

^a这意味着对向量也成立

证明. 设 $A = (a_{ij})$, 则 $\text{tr}(AA^*) = \sum_{1 \leq i, j \leq n} a_{ij}\bar{a}_{ij} = \sum_{1 \leq i, j \leq n} |a_{ij}|^2 \geq 0$. 取等当且仅当 $|a_{ij}| = 0, \forall i, j$, 即 $A = O$. \square

由引理 Lemme 3, $(A - \lambda I)\mathbf{u} = 0 \Leftrightarrow ((A - \lambda I)\mathbf{u})^*(A - \lambda I)\mathbf{u} = 0 \Leftrightarrow \mathbf{u}^*(A - \lambda I)^*(A - \lambda I)\mathbf{u} = 0 \Leftrightarrow \mathbf{u}^*(A - \lambda I)(A - \lambda I)^*\mathbf{u} = 0 \Leftrightarrow (A^* - \bar{\lambda}I)\mathbf{u} = (A - \lambda I)^*\mathbf{u} = 0$. \square

因为 A 正规, 故由引理 Lemme 2 可知, $\mathbf{u} + v\mathbf{i}$ 也是 A' 的属于特征值 $a - bi$ 的特征向量, 即

$$A'(\mathbf{u} + v\mathbf{i}) = (a - bi)(\mathbf{u} + v\mathbf{i}) = (a\mathbf{u} + bv) + (av - bu)\mathbf{i}.$$

比较实部和虚部得到

$$A'\mathbf{u} = a\mathbf{u} + bv, \quad A'\mathbf{v} = av - bu. \quad (1.10)$$

又 $(A\mathbf{u}, \mathbf{u}) = (\mathbf{u}, A'\mathbf{u})$, $(A\mathbf{u}, \mathbf{v}) = (\mathbf{u}, A'\mathbf{v})$, 将 $A\mathbf{u}$, $A'\mathbf{u}$ 及 $A'\mathbf{v}$ 代入得到

$$(a\mathbf{u} - bv, \mathbf{u}) = (\mathbf{u}, a\mathbf{u} + bv), \quad (a\mathbf{u} - bv, \mathbf{v}) = (\mathbf{u}, av - bu).$$

由此可得 $(\mathbf{u}, \mathbf{v}) = 0$, $(\mathbf{u}, \mathbf{u}) = (\mathbf{v}, \mathbf{v})$.

\square

Théorème 34: 特征值全为实数的正规算子是自伴随算子

定理 9.8.2 设 φ 是酉空间 V 上的正规算子. 若 φ 的特征值全是实数, 则 φ 是自伴随算子; 若 φ 的特征值全是非负实数, 则 φ 是半正定自伴随算子;

若 φ 的特征值全是正实数, 则 φ 是正定自伴随算子; 若 φ 的特征值的模长等于 1, 则 φ 是酉算子.

证明. 设 φ 的谱分解为

$$\varphi = \lambda_1 \mathbf{E}_1 + \lambda_2 \mathbf{E}_2 + \cdots + \lambda_k \mathbf{E}_k,$$

则

$$\varphi^* = \bar{\lambda}_1 \mathbf{E}_1 + \bar{\lambda}_2 \mathbf{E}_2 + \cdots + \bar{\lambda}_k \mathbf{E}_k.$$

若 φ 的特征值全是实数, 则 $\varphi^* = \varphi$, 即 φ 是自伴随算子. 若 λ_i 全是非负实数, 则对任意的非零向量 $\alpha \in V$, 有

$$\alpha = \mathbf{E}_1(\alpha) + \mathbf{E}_2(\alpha) + \cdots + \mathbf{E}_k(\alpha)$$

$$\varphi(\alpha) = \lambda_1 \mathbf{E}_1(\alpha) + \lambda_2 \mathbf{E}_2(\alpha) + \cdots + \lambda_k \mathbf{E}_k(\alpha)$$

从而

$$(\varphi(\alpha), \alpha) = \lambda_1 \|\mathbf{E}_1(\alpha)\|^2 + \lambda_2 \|\mathbf{E}_2(\alpha)\|^2 + \cdots + \lambda_k \|\mathbf{E}_k(\alpha)\|^2 \geq 0.$$

同理, 若特征值全是正实数, 则 φ 是正定自伴随算子. 最后, 若 $|\lambda_i| = 1$, 则

$$\begin{aligned} \varphi\varphi^* &= \lambda_1 \bar{\lambda}_1 \mathbf{E}_1 + \lambda_2 \bar{\lambda}_2 \mathbf{E}_2 + \cdots + \lambda_k \bar{\lambda}_k \mathbf{E}_k \\ &= |\lambda_1|^2 \mathbf{E}_1 + |\lambda_2|^2 \mathbf{E}_2 + \cdots + |\lambda_k|^2 \mathbf{E}_k \\ &= \mathbf{E}_1 + \mathbf{E}_2 + \cdots + \mathbf{E}_k = \mathbf{I}, \end{aligned}$$

即 φ 是酉算子. □

1.5.2 Jordan 标准型

Théorème 35

例 7.52 设 λ_0 是 n 阶矩阵 \mathbf{A} 的特征值, 证明: 对任意的正整数 k , 特征值为 λ_0 的 k 阶 Jordan 块 $\mathbf{J}_k(\lambda_0)$ 在 \mathbf{A} 的 Jordan 标准型 \mathbf{J} 中出现的个数为

$$r\left((\mathbf{A} - \lambda_0 \mathbf{I}_n)^{k-1}\right) + r\left((\mathbf{A} - \lambda_0 \mathbf{I}_n)^{k+1}\right) - 2r\left((\mathbf{A} - \lambda_0 \mathbf{I}_n)^k\right),$$

其中约定 $r\left((\mathbf{A} - \lambda_0 \mathbf{I}_n)^0\right) = n$.

证明. 相似化成 Jordan 标准型, 这是显然的. □

Exercice 9

例 7.55 设 $J = J_n(0)$ 是特征值为零的 n 阶 Jordan 块, 求 $J^m (m \geq 1)$ 的 Jordan 标准型.

Note. 考虑几何观点, 即循环不变子空间.

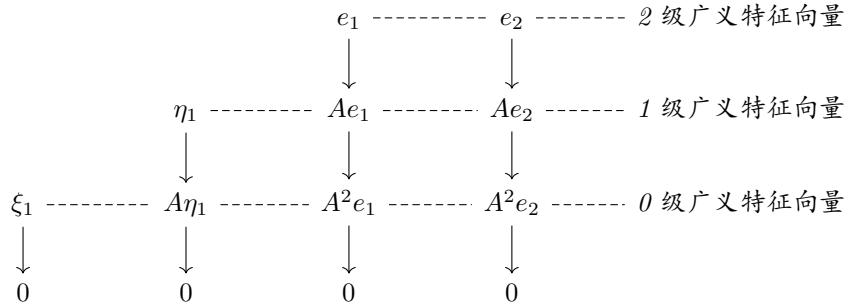
Exercice 10

例 7.62 设 9 阶幂零矩阵 A 的 Jordan 标准型 $J = \text{diag} \{0, J_2(0), J_3(0), J_3(0)\}$, 求非异阵 P , 使 $P^{-1}AP = J$.

Note. 考虑 A 在哪组基下表示矩阵为 $J = \text{diag} \{0, J_2(0), J_3(0), J_3(0)\}$.

$\mathbf{0} \quad J_2(0) \quad J_3(0) \quad J_3(0)$

轨道 1 轨道 2 轨道 3 轨道 4



于是这组基为 $\{\xi_1, \eta_1, A\eta_1, e_1, Ae_1, A^2e_1, e_2, Ae_2, A^2e_2\}$

1.5.3 可交换

Exercice 11

令 $G = GL(n, \mathbb{C})$, P 是主对角线上的元均为 1 的 $n \times n$ 上三角方阵全体形成的 G 的子群。确定 $N_G(P)$, $C_G(P)$ 和 P 的中心 $Z(P)$ 。

Solution. 由矩阵的乘法直接计算可知与任一 n 阶主对角线为 1 的上三角矩阵可换的矩阵必为上三角矩阵, 且形如

$$J_n(\lambda, \mu) = \begin{pmatrix} \lambda & 0 & \cdots & 0 & \mu \\ & \lambda & \cdots & 0 & 0 \\ & & \ddots & \vdots & \vdots \\ & & & \lambda & 0 \\ & & & & \lambda \end{pmatrix},$$

其中 $\mu \in \mathbb{C}$. 要看出这一点, 只要取主对角线为 1 且 $(i, i+1)$ 处为 1, 其余全为 0 的上三角矩阵即可, $i = 1, \dots, n-1$. 因此

$$C_G(P) = \{J_n(\lambda, \mu) \mid \lambda, \mu \in \mathbb{C}, \lambda \neq 0\}.$$

由此即知 $Z(P) = \{J_n(1, \mu) \mid \mu \in \mathbb{C}\}$ 。下面求

$$\begin{aligned} N_G(P) &:= \{A \in GL(n, \mathbb{C}) \mid AP = PA\} \\ &= \{A \in GL(n, \mathbb{C}) \mid AJA^{-1} \subseteq P, A^{-1}JA \subseteq P, \forall J \in P\}. \end{aligned}$$

首先, 所有 n 阶上三角可逆矩阵均在 $N_G(P)$ 中. 我们断言: $N_G(P)$ 恰是所有 n 阶上三角可逆矩阵作成的群。

设 A 是 n 阶可逆矩阵且 A 不是上三角的. 设 a_{ij} 是 A 中主对角线以下的非零元且 i 最大, 即

$$a_{ij} \neq 0, i > j; \text{ 且若 } a_{st} \neq 0, s > t, \text{ 则 } i \geqslant s.$$

令 J 是主对角线全为 1 且 $(j, j+1)$ 处为 1, 其余全为 0 的矩阵. 则 AJ 是将 A 的第 j 列加到第 $j+1$ 列后得到的矩阵, 从而 AJ 不是上三角的.

下证 $A \notin N_G(P)$. 否则存在 $B \in P$ 使得 $AJ = BA$, 注意到 AJ 的第 $(i, j+1)$ 处元是

$$a_{i,j} + a_{i,j+1}.$$

但 BA 是对 A 施行一系列初等行变换得到的, 这些行变换是将大数行的若干倍加到小数行. 由 A 的选取知 BA 的 $(i, j+1)$ 处元与 A 的 $(i, j+1)$ 处元相同, 均为 $a_{i,j+1}$, 从而 $AJ \neq BA$, 矛盾. 所以 $A \notin N_G(P)$. 即 $N_G(P)$ 恰是所有 n 阶上三角可逆矩阵作成的群.

Note. $N_G(P) =$ 所有上三角可逆矩阵是一个重要结论.

1.5.4 有限覆盖的思想

Exercice 12

G 有有限个子群, 证明: G 是有限群.

证明. 首先 G 的子群阶数都是有限的, 否则 G 存在同构于 \mathbb{Z} 的子群, 而 \mathbb{Z} 有无穷多个子群 $n\mathbb{Z}$. 其次, G 的循环子群的个数也是有限的, 考虑

$$G = \bigcup_{a \in G} \langle a \rangle \quad (1.11)$$

右侧的循环子群个数是有限的, 于是存在 $a_1, a_2, \dots, a_m \in G$, 使得

$$G = \bigcup_{i=1}^m \langle a_i \rangle \quad (1.12)$$

循环子群 $\langle a_i \rangle$ 是有限群, 有限群的有限并也是有限的. 于是 G 是有限的. \square

1.6 微积分

1.6.1 第 6 届八一赛的变态积分

Solution. 设 $n \geq 2, n \in \mathbb{N}$, 对 $0 < \alpha < \beta < 1$, 若

$$I(\alpha, \beta) = \int_{\alpha}^{\beta} \frac{x+1}{x^2(x-1)^2} [(2n-3)x^{n+1} - (2n-1)x^n + 3x - 1] dx.$$

求极限 $\lim_{\alpha \rightarrow 0^+} (\frac{1}{\alpha} + \lim_{\beta \rightarrow 1^-} I(\alpha, \beta))$.

注意到

$$(2n-3)x^{n+1} - (2n-1)x^n + 3x - 1 = (x-1)^2 [(2n-3)x^{n-1} + \dots + 3x^2 + x - 1],$$

而

$$(x+1) [(2n-3)x^{n-1} + \dots + 3x^2 + x - 1] = (2n-3)x^n + 4(n-2)x^{n-1} + \dots + 4x^2 - 1.$$

因此

$$\frac{x+1}{x^2(x-1)^2} [(2n-3)x^{n+1} - (2n-1)x^n + 3x - 1] = (2n-3)x^{n-2} + 4 \sum_{k=2}^{n-1} (k-1)x^{k-2} - \frac{1}{x^2}.$$

则不定积分为

$$F(x) \equiv \left(\frac{2n-3}{n-1} \right) x^{n-1} + 4 \sum_{k=2}^{n-1} x^{k-1} + \frac{1}{x}$$

故

$$I(\alpha, \beta) = F(\beta) - F(\alpha)$$

由于

$$\lim_{\beta \rightarrow 1^-} F(\beta) = \frac{2n-3}{n-1} + 4(n-2) + 1 = \frac{4n^2 - 9n + 4}{n-1}.$$

因此

$$\begin{aligned} \frac{1}{\alpha} + \lim_{\beta \rightarrow 1^-} I(\alpha, \beta) &= \frac{1}{\alpha} + \lim_{\beta \rightarrow 1^-} F(\beta) - F(\alpha) \\ &= \left[\frac{4n^2 - 9n + 4}{n-1} \right] - \left[\left(\frac{2n-3}{n-1} \right) \alpha^{n-1} + 4 \sum_{k=2}^{n-1} \alpha^{n-1} \right]. \end{aligned}$$

所以

$$\lim_{\alpha \rightarrow 0^+} \left(\frac{1}{\alpha} + \lim_{\beta \rightarrow 1^-} I(\alpha, \beta) \right) = \frac{4n^2 - 9n + 4}{n-1}$$

Solution. (1) 求二重积分 $\int_0^1 \int_0^1 x \left\{ \frac{1}{1-xy} \right\} dx dy =$, 其中 $\{x\}$ 为 x 的小数部分.

(2) 求定积分 $\int_0^{+\infty} \cos(2\pi x) \left(\frac{1}{x^2} - \frac{\pi \coth(\pi x)}{x} \right) dx =$

(1) 考虑

$$(u, v) = \left(\frac{1}{1-xy}, \frac{y}{x} \right).$$

对于 $0 < x, y < 1$, 则有 $u > 1$ 和 $v > 0$, 其逆变换可由

$$(x, y) = \left(\sqrt{\frac{u-1}{uv}} \cdot \sqrt{\frac{(u-1)v}{u}}, \frac{v}{\sqrt{\frac{u-1}{uv}}} \right).$$

其雅可比矩阵为

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2u^2v} \sqrt{\frac{uv}{u-1}} & -\frac{u-1}{2uv^2} \sqrt{\frac{uv}{u-1}} \\ \frac{v}{2u^2} \sqrt{\frac{u}{(u-1)v}} & \frac{u-1}{2u} \sqrt{\frac{u}{(u-1)v}} \end{vmatrix} = \frac{1}{2u^2v}.$$

而区域 $S = \{(x, y) : 0 < x, y < 1\}$ 被方程 $xy = \alpha \in (0, 1)$ 所覆盖, 即 $u(1 - \alpha) = 1$. 故矩阵区域 S 映射为如下区域

$$\Sigma = \left\{ (u, v) : \frac{u-1}{u} < v < \frac{u}{u-1}, 1 < u < \infty \right\}.$$

然后有

$$\begin{aligned} I &= \int_0^1 \int_0^1 x \left\{ \frac{1}{1-xy} \right\} dx dy = \int_1^\infty \int_{\frac{u-1}{u}}^{\frac{u}{u-1}} \sqrt{\frac{u-1}{uv}} \{u\} \frac{1}{2u^2v} dv du \\ &= \sum_{n=1}^{\infty} \int_n^{n+1} \int_{\frac{u-1}{u}}^{\frac{u}{u-1}} \sqrt{\frac{u-1}{u}} \{u\} \frac{1}{2u^2} v^{-3/2} dv du. \end{aligned}$$

对 $n \leq u < n+1$, 则有 $\{u\} = u-n$. 先对 v 积分, 可得到

$$\begin{aligned} I &= - \sum_{n=1}^{\infty} \int_n^{n+1} \sqrt{\frac{u-1}{u}} (u-n) \frac{1}{u^2} \left[v^{-1/2} \right]_{\frac{u-1}{u}}^{\frac{u}{u-1}} du \\ &= - \sum_{n=1}^{\infty} \int_n^{n+1} \sqrt{\frac{u-1}{u}} (u-n) \frac{1}{u^2} \left(\sqrt{\frac{u-1}{u}} - \sqrt{\frac{u}{u-1}} \right) du \\ &= - \sum_{n=1}^{\infty} \int_n^{n+1} (u-n) \frac{1}{u^2} \left(\frac{u-1}{u} - 1 \right) du = \sum_{n=1}^{\infty} \int_n^{n+1} (u^{-2} - nu^{-3}) du = \sum_{n=1}^{\infty} \frac{1}{2n(n+1)^2} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2} \right] = \frac{1}{2} \left[\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) - \left(\sum_{n=1}^{\infty} \frac{1}{n^2} - 1 \right) \right] \\ &= \frac{1}{2} \left[1 - \left(\frac{\pi^2}{6} - 1 \right) \right] = 1 - \frac{\pi^2}{12} \end{aligned}$$

(2) 首先

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = \lim_{a \rightarrow 0} \frac{\pi a \coth(\pi a) - 1}{2a^2}$$

当 $a = 0$ 时恒等式成立, 假设 $a \neq 0$, 若 $a > 0$, 由于 $\sum_{n=1}^{\infty} \frac{1}{n^2+a^2}$ 是关于 a 的偶函数. 考虑

$$\int_0^{+\infty} \sin(bx) e^{-ax} dx = \frac{b}{a^2 + b^2}$$

对任意实数 $a > 0$, 有

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{+\infty} \sin(nx) e^{-\alpha x} dx = \int_0^{+\infty} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} e^{-ax} dx \\ &= \sum_{k=0}^{\infty} \int_{2k\pi}^{(2k+2)\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} e^{-ax} dx = \sum_{k=0}^{\infty} \int_0^{2\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} e^{-a(2k\pi+x)} dx \\ &= \sum_{k=0}^{\infty} e^{-2k\pi a} \int_0^{2\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} e^{-ax} dx = \frac{1}{1 - e^{-2\pi a}} \int_0^{2\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} e^{-ax} dx \end{aligned}$$

由于

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2}, \quad x \in (0, 2\pi)$$

故

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{1}{1 - e^{-2\pi a}} \int_0^{2\pi} \left(\frac{\pi - x}{2} \right) e^{-ax} dx = \frac{\pi a \coth(\pi a) - 1}{2a^2}$$

因此可得

$$\frac{1}{a^2} - \frac{\pi \coth(\pi a)}{a} = -2 \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}$$

令 $f(t) = \int_0^{+\infty} \cos(\pi tx) \left(\frac{1}{x^2} - \frac{\pi \coth(\pi x)}{x} \right) dx$, 即

$$f(t) = -2 \int_0^{+\infty} \cos(\pi tx) \sum_{n=1}^{\infty} \frac{1}{n^2 + x^2} dx = -2 \sum_{n=1}^{\infty} \int_0^{+\infty} \frac{\cos(\pi tx)}{n^2 + x^2} dx$$

考虑 $g(t) = \int_0^{+\infty} \frac{\cos(tx)}{1+x^2} dx, t > 0$, 则 $g'(t) = - \int_0^{+\infty} \frac{x \sin(tx)}{1+x^2} dx$, 根据

$$\frac{1}{1+x^2} = \int_0^{+\infty} \sin(y) e^{-xy} dy, \int_0^{+\infty} \cos(tx) e^{-xy} dx = \frac{y}{y^2 + t^2}$$

可得到

$$\begin{aligned} g(t) &= \int_0^{+\infty} \cos(tx) \int_0^{+\infty} \sin(y) e^{-xy} dy dx \\ &= \int_0^{+\infty} \sin(y) \int_0^{+\infty} \cos(tx) e^{-xy} dx dy \\ &= \int_0^{+\infty} \frac{y \sin(y)}{y^2 + t^2} dy = \int_0^{+\infty} \frac{y \sin(ty)}{y^2 + 1} dy = -g'(t). \end{aligned}$$

因此 $g'(t) = -g(t), t > 0$, 即 $g(t) = C e^{-t}$. 由于 $0 \leq 1 - \cos(x) \leq 2\sqrt{x}, \forall x \geq 0$, 即

$$0 \leq \int_0^{+\infty} \frac{1 - \cos(tx)}{1+x^2} dx \leq 2\sqrt{t} \int_0^{+\infty} \frac{\sqrt{x}}{1+x^2} dx$$

而 $\int_0^{+\infty} \frac{\sqrt{x}}{1+x^2} dx$ 收敛, 即 $\lim_{t \rightarrow 0^+} \int_0^{+\infty} \frac{1 - \cos(tx)}{1+x^2} dx = 0$. 故可得

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{t \rightarrow 0^+} \int_0^{+\infty} \frac{\cos(tx)}{1+x^2} dx = \int_0^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$$

即 $C = \frac{\pi}{2}$, 则 $g(t) = \frac{\pi}{2} e^{-t}$. 因此

$$f(t) = -2 \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{+\infty} \frac{\cos(n\pi ty)}{1+y^2} dy = -2 \sum_{n=1}^{\infty} \frac{\pi}{2n} e^{-nt\pi} = -\pi \sum_{n=1}^{\infty} \frac{e^{-nt\pi}}{n} = \pi \ln(1 - e^{-t\pi}).$$

所以

$$\int_0^{+\infty} \cos(2\pi x) \left(\frac{1}{x^2} - \frac{\pi \coth(\pi x)}{x} \right) dx = f(2) = \pi \ln(1 - e^{-2\pi})$$

Solution. 设 x 为实数, 且 $|x| < 1$, 求下列无穷级数:

$$(1) \sum_{n=1}^{\infty} \left[\frac{1}{(1-x)^2} - 1 - 2x - \cdots - nx^{n-1} \right] = ?$$

$$(2) \sum_{n=1}^{\infty} \left[\frac{1}{(1-x)^2} - 1 - 2x - \cdots - nx^{n-1} \right]^2 = ? .$$

利用

$$\frac{1}{(1-x)^r} = \sum_{k=0}^{\infty} \binom{k+r}{r} x^k \quad (|x| < 1)$$

对 $r = 0, 1, 2$ 和

$$\begin{aligned} \sum_{k=n}^{\infty} (k+1)x^k &= x^n \sum_{k=0}^{\infty} (n+k+1)x^k = x^n \left(n \sum_{k=0}^{\infty} x^k + \sum_{k=0}^{\infty} (k+1)x^k \right) \\ &= x^n \left(\frac{n}{1-x} + \frac{1}{(1-x)^2} \right) = \frac{x^n(1+(1-x)n)}{(1-x)^2} \end{aligned}$$

故有

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{1}{(1-x)^2} - 1 - 2x - \cdots - nx^{n-1} \right) &= \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} (k+1)x^k - \sum_{k=0}^{n-1} (k+1)x^k \right) \\ &= \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} (k+1)x^k \right) = \sum_{n=1}^{\infty} \frac{x^n(1+(1-x)n)}{(1-x)^2} \\ &= \frac{x}{(1-x)^2} \sum_{n=0}^{\infty} x^n + \frac{x}{(1-x)} \sum_{n=0}^{\infty} (n+1)x^n \\ &= \frac{x}{(1-x)^3} + \frac{x}{(1-x)^3} = \frac{2x}{(1-x)^3}. \end{aligned}$$

同时可得

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{1}{(1-x)^2} - 1 - 2x - \cdots - nx^{n-1} \right)^2 &= \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} (k+1)x^k \right)^2 = \sum_{n=1}^{\infty} \frac{x^{2n}(1+(1-x)n)^2}{(1-x)^4} \\ &= \frac{x^2}{(1-x)^4} \left[\sum_{n=0}^{\infty} x^{2n} + 2(1-x) \sum_{n=0}^{\infty} (n+1)x^{2n} + (1-x)^2 \sum_{n=0}^{\infty} \left(2 \binom{n+2}{2} - (n+1) \right) x^{2n} \right] \\ &= \frac{x^2}{(1-x)^4} \left[\frac{1}{(1-x^2)} + \frac{2(1-x)}{(1-x^2)^2} + (1-x)^2 \left(\frac{2}{(1-x^2)^3} - \frac{1}{(1-x^2)^2} \right) \right] \\ &= \frac{x^2}{(1-x)^4} \left[\frac{1}{(1-x)(1+x)} + \frac{2}{(1-x)(1+x)^2} + \frac{1+y^2}{(1-x)(1+x)^3} \right] = \frac{x^2(2x^2+4x+4)}{(1-x)^5(1+x)^3} \end{aligned}$$

Solution. 求微分方程 $y'' - y = \frac{1}{x} - \frac{2}{x^3}$ 的通解

本题是一个常系数二阶线性非齐次微分方程, 首先求出齐次方程的通解. 由于特征方程为 $\lambda^2 - 1 = 0$, 即 $\lambda = \pm 1$, 所以齐次方程的通解是

$$y(x) = c_1 e^x + c_2 e^{-x}.$$

再求非齐次方程的一个特解. 这里采用常数变易法, 设特解为如下的形式

$$y_*(x) = C_1(x)e^x + C_2(x)e^{-x},$$

其中 $C_1(x), C_2(x)$ 是两个待定的 \mathbb{C}^2 函数, 对 $y_*(x)$ 求导可得

$$y'_*(x) = C_1(x)e^x - C_2(x)e^{-x} + C'_1(x)e^x + C'_2(x)e^{-x}.$$

令

$$C'_1(x)e^x + C'_2(x)e^{-x} = 0. \quad (1.13)$$

于是

$$y''_*(x) = C_1(x)e^x + C_2(x)e^{-x} + C'_1(x)e^x - C'_2(x)e^{-x}. \quad (1.14)$$

代入方程可得

$$C'_1(x)e^x - C'_2(x)e^{-x} = \frac{1}{x} - \frac{2}{x^3}.$$

然后联立 eq. (1.13) 和 eq. (1.14) 得到如下方程组

$$\begin{cases} C'_1(x)e^x + C'_2(x)e^{-x} = 0, \\ C'_1(x)e^x - C'_2(x)e^{-x} = \frac{1}{x} - \frac{2}{x^3} \end{cases} \Rightarrow \begin{cases} C'_1(x) = \frac{1}{2}e^{-x} \left(\frac{1}{x} - \frac{2}{x^3} \right), \\ C'_2(x) = \frac{1}{2}e^x \left(\frac{2}{x^3} - \frac{1}{x} \right) \end{cases}$$

于是

$$\begin{cases} C_1(x) = c_1 + \int_{+\infty}^x \frac{1}{2}e^{-t} \left(\frac{1}{t} - \frac{2}{t^3} \right) dt = c_1 + \frac{1}{2}e^{-x} \left(-\frac{1}{x} + \frac{1}{x^2} \right), \\ C_2(x) = c_2 + \int_{+\infty}^x \frac{1}{2}e^t \left(\frac{2}{t^3} - \frac{1}{t} \right) dt = c_2 - \frac{1}{2}e^x \left(-\frac{1}{x} + \frac{1}{x^2} \right). \end{cases}$$

综上, 原方程的解为

$$y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{2} \left(-\frac{1}{x} + \frac{1}{x^2} \right) - \frac{1}{2} \left(-\frac{1}{x} + \frac{1}{x^2} \right) = c_1 e^x + c_2 e^{-x} - \frac{1}{x}$$

Solution. 设 $\alpha \in [0, \frac{1}{2}]$, 定义 \mathbb{R}^2 上的区域 $D(\alpha)$ 如下:

$$D(\alpha) = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1; 0 \leq y \leq 1, x + y \leq 2(1 - \alpha)\}.$$

令 $I(\alpha) = \iint_{D(\alpha)} \frac{1}{1-xy} dx dy$, 计算 $I(\alpha)$ 并求 $\lim_{\alpha \rightarrow 0} I(\alpha)$.

(5) 首先由 $0 \leq x + y = 2u \leq 2(1 - \alpha)$ 知, $0 \leq u \leq 1 - \alpha$, 利用变量代换

$$\begin{cases} u = \frac{1}{2}(x + y) \\ v = \frac{1}{2}(y - x) \end{cases},$$

$$\text{可得 } \begin{cases} x = u + v \\ y = u - v \end{cases}, \text{ 且 } |J| = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \frac{1}{2}, \text{ 故 } \varphi(\alpha) : \begin{cases} 0 \leq u + v \leq 1 \\ 0 \leq u + v \leq 1 \\ 0 \leq u \leq 1 - \alpha \end{cases}$$

所以

$$\begin{aligned} I(\alpha) &= \iint_{D(\alpha)} \frac{dx dy}{1 - xy} = \iint_{\varphi(\alpha)} \frac{2 du dv}{1 - u^2 + v^2} \\ &= 2 \int_0^{\frac{1}{2}} du \int_{-u}^u \frac{dv}{(\sqrt{1-u^2})^2 + v^2} + 2 \int_{\frac{1}{2}}^{1-\alpha} du \int_{u-1}^{1-u} \frac{dv}{(\sqrt{1-u^2})^2 + v^2} \\ &= 4 \int_0^{\frac{1}{2}} du \int_0^u \frac{dv}{(\sqrt{1-u^2})^2 + v^2} + 4 \int_{\frac{1}{2}}^{1-\alpha} du \int_0^{1-u} \frac{dv}{(\sqrt{1-u^2})^2 + v^2} \\ &= 4 \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}} \arctan \frac{u}{\sqrt{1-u^2}} du + 4 \int_{\frac{1}{2}}^{1-\alpha} \frac{1}{\sqrt{1-u^2}} \arctan \frac{1-u}{\sqrt{1-u^2}} du \\ &= I_1(\alpha) + I_2(\alpha). \end{aligned}$$

利用分部积分得到

$$I_1(\alpha) = 4 \arcsin \frac{1}{2} \arctan \frac{1}{\sqrt{3}} - 2(\arcsin u)^2 \Big|_0^{\frac{1}{2}}$$

$$I_2(\alpha) = 4 \arcsin(1-\alpha) \arctan \frac{\alpha}{\sqrt{2\alpha-\alpha^2}} - 4 \arcsin \frac{1}{2} \arctan \frac{1}{\sqrt{3}} + (\arcsin u)^2 \Big|_{\frac{1}{2}}^{1-\alpha}$$

因此

$$I(\alpha) = 4 \arcsin(1-\alpha) \arctan \frac{\alpha}{\sqrt{2\alpha-\alpha^2}} + (\arcsin(1-\alpha))^2 - \frac{\pi^2}{12}.$$

$$\begin{aligned} \text{令 } u = 1 - \alpha \text{ 得 } 4 \arcsin(1-\alpha) \arctan \frac{\alpha}{\sqrt{2\alpha-\alpha^2}} &= 4 \arcsin u \arctan \frac{1-u}{\sqrt{1-u^2}} \\ u = \sin \theta \text{ 得 } \sqrt{1-u^2} &= \cos \theta, \text{ 所以} \end{aligned}$$

$$\begin{aligned} \frac{1-u}{\sqrt{1-u^2}} &= \frac{1-\sin \theta}{\cos \theta} = \tan \left(\frac{\pi}{4} - \frac{\pi}{2} \right) = \tan \left(\frac{\pi}{4} - \frac{1}{2} \arcsin u \right). \\ \Rightarrow \arctan \frac{1-u}{\sqrt{1-u^2}} &= \frac{\pi}{4} - \frac{1}{2} \arcsin u \end{aligned}$$

$$\begin{aligned} \text{因此 } 4 \arcsin u \arctan \frac{1-u}{\sqrt{1-u^2}} &= 4 \arcsin u \left(\frac{\pi}{4} - \frac{1}{2} \arcsin u \right) = \pi \arcsin u - \\ 2(\arcsin u)^2. \text{ 故} & \end{aligned}$$

$$I(\alpha) = \pi \arcsin(1-\alpha) - (\arcsin(1-\alpha))^2 - \frac{\pi^2}{12}$$

所以有

$$\lim_{\alpha \rightarrow 0^+} I(\alpha) = \pi \lim_{\alpha \rightarrow 0^+} \arcsin(1-\alpha) - \lim_{\alpha \rightarrow 0} (\arcsin(1-\alpha))^2 - \frac{\pi^2}{12} = \frac{\pi^2}{2} - \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}$$

第二章 表示论

2.1 FG -modules

Note. See *Representations and Characters of Groups* (Gordon James, Martin Liebeck)

We will show that there is a connection between FG -modules and representations of G over F .

Let G be a group and let F be \mathbb{R} or \mathbb{C} .

Suppose that $\rho : G \rightarrow \mathrm{GL}(n, F)$ is a representation of G . Write $V = F^n$, as the vector space. For all $v \in V$ and $g \in G$, the matrix product¹

$$v \cdot \rho_g$$

is a row vector in V .

Now we list some basic properties

$$v(\rho_{gh}) = v(\rho_g)(\rho_h)$$

$$v(\rho_1) = v$$

$$(\lambda v)(g\rho_g) = \lambda(v(\rho_g)), \quad \forall \lambda \in F$$

$$(u + v)(\rho_g) = u\rho_g + v\rho_g$$

¹ $g\rho = \rho_g = \rho(g) \in \mathrm{GL}(n, F)$

4.1 Example

Let $G = D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$, and let $\rho: G \rightarrow \mathrm{GL}(2, F)$ be the representation of G over F given in Example 3.2(1). Thus

$$a\rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b\rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $v = (\lambda_1, \lambda_2) \in F^2$ then, for example,

$$\begin{aligned} v(a\rho) &= (-\lambda_2, \lambda_1), \\ v(b\rho) &= (\lambda_1, -\lambda_2), \\ v(a^3\rho) &= (\lambda_2, -\lambda_1). \end{aligned}$$

Motivated by the above observations on the product $v(\rho_g)$, we now define an FG -modules.

Définition 1: FG -module

Let V be a vector space over F and let G be a group. Then V is an **FG -module** if a multiplication $vg(v \in V, g \in G)$ is defined, satisfying the following conditions for all $u, v \in V, \lambda \in F$ and $g, h \in G$:

- (1) $vg \in V$;
- (2) $v(gh) = (vg)h$;
- (3) $v1 = v$;
- (4) $(\lambda v)g = \lambda(vg)$;
- (5) $(u + v)g = ug + vg$

Définition 2: Matrix representation

Let V be an FG -module, and let \mathcal{B} be a basis of V . For each $g \in G$, let

$$[g]_{\mathcal{B}}$$

denote the **matrix of the endomorphism** $v \rightarrow vg$ of V , relative to the basis \mathcal{B} .

If $\rho : G \rightarrow \mathrm{GL}(n, F)$ is a representation of G over F , and $V = F^n$, then V becomes an FG -module if we define the multiplication vg by

$$vg := v(\rho_g) \quad (v \in V, g \in G)$$

Moreover, there is a basis \mathcal{B} of V such that

$$\rho_g = [g]_{\mathcal{B}} \quad \forall g \in G$$

On the other hand, assume that V is an FG -module and let \mathcal{B} be a basis of V .

Then the function

$$g \rightarrow [g]_{\mathcal{B}} \quad (g \in G)$$

is a representation of G over F .

4.5 Examples

(1) Let $G = D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ and let ρ be the representation of G over F given in Example 3.2(1), so

$$a\rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, b\rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Write $V = F^2$. By Theorem 4.4(1), V becomes an FG -module if we define

$$vg = v(g\rho) \quad (v \in V, g \in G).$$

For instance,

$$(1, 0)a = (1, 0) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (0, 1).$$

If v_1, v_2 is the basis $(1, 0), (0, 1)$ of V , then we have

$$\begin{aligned} v_1a &= v_2, & v_1b &= v_1, \\ v_2a &= -v_1, & v_2b &= -v_2. \end{aligned}$$

If \mathcal{B} denotes the basis v_1, v_2 , then the representation

$$g \rightarrow [g]_{\mathcal{B}} \quad (g \in G)$$

is just the representation ρ (see Theorem 4.4(1) again).

(2) Let $G = Q_8 = \langle a, b : a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$. In Example

1.2(4) we defined Q_8 to be the subgroup of $\mathrm{GL}(2, \mathbb{C})$ generated by

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so we already have a representation of G over \mathbb{C} . To illustrate Theorem 4.4(1) we must this time take $F = \mathbb{C}$. We then obtain a $\mathbb{C}G$ -module with basis v_1, v_2 such that

$$\begin{aligned} v_1a &= iv_1, & v_1b &= v_2, \\ v_2a &= -iv_2, & v_2b &= -v_1. \end{aligned}$$

Définition 3: faithful

An FG -module V is **faithful** if the identity element of G is the only element g for which

$$vg = v \quad \text{for all } v \in V$$

For instance, the FD_8 -module which appears in Example 4.5(1) is faithful.

Our next aim is to construct faithful FG -modules by the basis for all subgroups of symmetric groups.

Définition 4: Permutation module

Let G be a subgroup of S_n . The FG -module V with basis v_1, \dots, v_n such that

$$v_i g = v_{ig} \quad \text{for all } i, \text{ and all } g \in G,$$

is called the **permutation module** for G over F . We call v_1, \dots, v_n the natural basis of V .

4.9 Example

Let $G = S_4$ and let \mathcal{B} denote the basis v_1, v_2, v_3, v_4 of V . If $g = (1\ 2)$, then

$$v_1 g = v_2, v_2 g = v_1, v_3 g = v_3, v_4 g = v_4.$$

And if $h = (1\ 3\ 4)$, then

$$v_1 h = v_3, v_2 h = v_2, v_3 h = v_4, v_4 h = v_1.$$

We have

$$[g]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, [h]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

4.11 Example

Let $G = C_3 = \langle a: a^3 = 1 \rangle$. Then G is isomorphic to the cyclic subgroup of S_3 which is generated by the permutation $(1\ 2\ 3)$. This alerts us to the fact that if V is a 3-dimensional vector space over F , with basis v_1, v_2, v_3 , then we may make V into an FG -module in which

$$\begin{aligned}v_1 1 &= v_1, v_2 1 = v_2, v_3 1 = v_3, \\v_1 a &= v_2, v_2 a = v_3, v_3 a = v_1, \\v_1 a^2 &= v_3, v_2 a^2 = v_1, v_3 a^2 = v_2.\end{aligned}$$

Of course, we define vg , for v an arbitrary vector in V and $g = 1, a$ or a^2 , by

$$(\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3)g = \lambda_1(v_1 g) + \lambda_2(v_2 g) + \lambda_3(v_3 g)$$

for all $\lambda_1, \lambda_2, \lambda_3 \in F$. Proposition 4.6 can be used to verify that V is an FG -module, but we have been motivated by the definition of permutation modules in our construction.

2.1.1 Exercise

Exercice 1

1. Suppose that $G = S_3$, and that $V = \text{sp}(\nu_1, \nu_2, \nu_3)$ is the permutation module for G over \mathbb{C} , as in Definition 4.10. Let \mathcal{B}_1 be the basis ν_1, ν_2, ν_3 of V and let \mathcal{B}_2 be the basis $\nu_1 + \nu_2 + \nu_3, \nu_1 - \nu_2, \nu_1 - \nu_3$. Calculate the 3×3 matrices $[g]_{\mathcal{B}_1}$ and $[g]_{\mathcal{B}_2}$ for all g in S_3 . What do you notice about the matrices $[g]_{\mathcal{B}_2}$?

When $g = e$, then $[g]_{\mathcal{B}_1} = I_3$. When $g = (1 2)$, then

$$[g]_{\mathcal{B}_1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[(1 3)]_{\mathcal{B}_1} = \begin{pmatrix} & & 1 \\ & & 1 \\ 1 & & \end{pmatrix}$$

$$[(2 3)]_{\mathcal{B}_1} = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

$$[(1 2 3)]_{\mathcal{B}_1} = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$$

$$[(1 3 2)]_{\mathcal{B}_1} = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$$

Now let $u_1 = v_1 + v_2 + v_3$, $u_2 = v_1 - v_2$, $u_3 = v_1 - v_3$. Then

$$u_1 e = u_1 \quad u_2 e = u_2 \quad u_3 e = u_3$$

$$u_1(1\ 2) = u_1 \quad u_2(1\ 2) = -u_2 \quad u_3(1\ 2) = v_2 - v_3 = -u_2 + u_3$$

Then

$$[e]_{\mathcal{B}_2} = \begin{pmatrix} 1 \\ & 1 \\ & & 1 \end{pmatrix}$$

$$[(1\ 2)]_{\mathcal{B}_2} = \begin{pmatrix} 1 \\ & -1 \\ & & 1 \end{pmatrix}$$

Omitted....

Exercice 2

3. Let $Q_8 = \langle a, b : a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$, the quaternion group of order 8. Show that there is an $\mathbb{R}Q_8$ -module V of dimension 4 with basis v_1, v_2, v_3, v_4 such that

$$v_1a = v_2, \quad v_2a = -v_1, \quad v_3a = -v_4, \quad v_4a = v_3, \text{ and}$$

$$v_1b = v_3, \quad v_2b = v_4, \quad v_3b = -v_1, \quad v_4b = -v_2.$$

Let $\mathcal{B} = (v_1, v_2, v_3, v_4)$. Then

$$[a]_{\mathcal{B}} = \begin{pmatrix} & 1 \\ -1 & & \\ & & -1 \\ & & & 1 \end{pmatrix}, \quad [b]_{\mathcal{B}} = \begin{pmatrix} & & 1 \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}$$

Check that V is a $\mathbb{R}Q_8$ -module. NTS: $g \mapsto [g]_{\mathcal{B}}$ is a representation of Q_8 , i.e.

$$[a]_{\mathcal{B}}^4 = I_4, \quad [a]_{\mathcal{B}}^2 = [b]_{\mathcal{B}}^2, \quad [b]_{\mathcal{B}}^{-1}[a]_{\mathcal{B}}[b]_{\mathcal{B}} = [a]_{\mathcal{B}}^{-1}$$

which is routine.

2.2 FG -submodules and reducibility

Définition 5: FG -submodule

Let V be an FG -module. A subset W of V is said to be an FG -submodule of V if W is a subspace and $wg \in W$ for all $w \in W$ and all $g \in G$.

5.2 Examples

- (1) For every FG -module V , the zero subspace $\{0\}$, and V itself, are FG -submodules of V .
- (2) Let $G = C_3 = \langle a: a^3 = 1 \rangle$, and let V be the 3-dimensional FG -module defined in Example 4.11. Thus, V has basis v_1, v_2, v_3 , and

$$\begin{aligned} v_1 1 &= v_1, v_2 1 = v_2, v_3 1 = v_3, \\ v_1 a &= v_2, v_2 a = v_3, v_3 a = v_1, \\ v_1 a^2 &= v_3, v_2 a^2 = v_1, v_3 a^2 = v_2. \end{aligned}$$

Put $w = v_1 + v_2 + v_3$, and let $W = \text{sp}(w)$, the 1-dimensional subspace spanned by w . Since

$$w 1 = w a = w a^2 = w,$$

W is an FG -submodule of V . However, $\text{sp}(v_1 + v_2)$ is not an FG -submodule, since

$$(v_1 + v_2)a = v_2 + v_3 \notin \text{sp}(v_1 + v_2).$$

Définition 6: Irreducible FG-module

An **irreducible FG-module** V is said to be irreducible if it is non-zero and it has no FG -submodules apart from $\{0\}$ and V .

Suppose that V is a reducible FG -module, so that there is an FG -submodule W with $0 < \dim W < \dim V$. Take a basis \mathcal{B}_1 of W and extend it to a basis \mathcal{B} of V . Then for all g in G , the matrix $[g]_{\mathcal{B}}$ has the form

$$(5.4) \quad \left(\begin{array}{c|c} X_g & 0 \\ \hline Y_g & Z_g \end{array} \right)$$

for some matrices X_g , Y_g and Z_g , where X_g is $k \times k$ ($k = \dim W$).

A representation of degree n is reducible if and only if it is equivalent to a representation of the form (5.4), where X_g is $k \times k$ and $0 < k < n$.

5.5 Examples

- (1) Let $G = C_3 = \langle a: a^3 = 1 \rangle$ and let V be the 3-dimensional FG -module with basis v_1, v_2, v_3 such that

$$\nu_1 a = \nu_2, \nu_2 a = \nu_3, \nu_3 a = \nu_1,$$

as in Example 4.11. We saw in Example 5.2(2) that V is a reducible FG -module, and has an FG -submodule $W = \text{sp}(\nu_1 + \nu_2 + \nu_3)$. Let \mathcal{B} be the basis $\nu_1 + \nu_2 + \nu_3, \nu_1, \nu_2$ of V . Then

$$[1]_{\mathcal{B}} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), [a]_{\mathcal{B}} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 1 & -1 & -1 \end{array} \right),$$

$$[a^2]_{\mathcal{B}} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 1 & -1 & -1 \\ 0 & 1 & 0 \end{array} \right).$$

This reducible representation gives us two other representations: at the ‘top left’ we have the trivial representation and at the ‘bottom right’ we have the representation which is given by

$$1 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, a^2 \rightarrow \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

(2) Let $G = D_8$ and let $V = F^2$ be the 2-dimensional FG -module described in Example 4.5(1). Thus $G = \langle a, b \rangle$, and for all $(\lambda, \mu) \in V$ we have

$$(\lambda, \mu)a = (-\mu, \lambda), \quad (\lambda, \mu)b = (\lambda, -\mu).$$

We claim that V is an irreducible FG -module. To see this, suppose that there is an FG -submodule U which is not equal to V . Then $\dim U \leq 1$, so $U = \text{sp}((\alpha, \beta))$ for some $\alpha, \beta \in F$. As U is an FG -module, $(\alpha, \beta)b$ is a scalar multiple of (α, β) , and hence either $\alpha = 0$ or $\beta = 0$. Since $(\alpha, \beta)a$ is also a scalar multiple of (α, β) , this forces $\alpha = \beta = 0$, so $U = \{0\}$. Consequently V is irreducible, as claimed.

(2) Let $G = D_8$ and let $V = F^2$ be the 2-dimensional FG -module described in Example 4.5(1). Thus $G = \langle a, b \rangle$, and for all $(\lambda, \mu) \in V$ we have

$$(\lambda, \mu)a = (-\mu, \lambda), \quad (\lambda, \mu)b = (\lambda, -\mu).$$

We claim that V is an irreducible FG -module. To see this, suppose that there is an FG -submodule U which is not equal to V . Then $\dim U \leq 1$, so $U = \text{sp}((\alpha, \beta))$ for some $\alpha, \beta \in F$. As U is an FG -module, $(\alpha, \beta)b$ is a scalar multiple of (α, β) , and hence either $\alpha = 0$ or $\beta = 0$. Since $(\alpha, \beta)a$ is also a scalar multiple of (α, β) , this forces $\alpha = \beta = 0$, so $U = \{0\}$. Consequently V is irreducible, as claimed.

2.2.1 Exercises

Omitted....

2.3 Group algebra

The group algebras are the source of all you need to know about representation theory. In particular, the ultimate goal of representation theory – that of understanding all the representations of finite groups – would be achieved if group algebras could be fully analysed.

Définition 7: Definition

Let G be a finite group and F be \mathbb{R} or \mathbb{C} . The vector space FG , with the natural multiplication $vg(v \in FG, g \in G)$, is called the **regular FG -module**. The representation $g \rightarrow [g]_{\mathcal{B}}$ obtained by taking \mathcal{B} to be the natural basis of FG is called the **regular representation of G over F** .

Définition 8: group algebra

The vector space FG , with multiplication defined by

$$\left(\sum_{g \in G} \lambda_g g \right) \left(\sum_{h \in G} \mu_h h \right) = \sum_{g, h \in G} \lambda_g \mu_h (gh) \quad (\lambda_g, \mu_h \in F)$$

is called the **group algebra** of G over F .

Recall that an FG -module is a vector space over F , together with a multiplication vg for $v \in V$ and $g \in G$. Now we can expand the definition of the multiplication so that we have an element vr of V for all elements r in the group algebra FG .

Suppose that V is an FG -module, and that $v \in V$ and $r \in FG$; say

$$r = \sum_{g \in G} \mu_g g \quad (\mu_g \in F)$$

Define vr by

$$vr = \sum_{g \in G} \mu_g (vg)$$

2.3.1 Exercises

Omitted...

2.4 Examples in Dummit

G is a finite group.

2.4.1 Trivial representation

Let V be a 1-dimensional vector space over F and make V into an FG -module by letting $gv = v$ for all $g \in G$ and $v \in V$. This module affords the representation

$$\varphi : G \rightarrow \mathrm{GL}(V)$$

to be trivial, i.e. taking g to 1 (the 1×1 identity matrix). The trivial representation has degree 1 and if $|G| > 1$, it is **not faithful**.

2.4.2 Regular representation

Let $V = FG$ be a FG -module, then V affords a representation

$$\varphi : G \rightarrow \mathrm{GL}(V)$$

where $\varphi_g : g_i \mapsto g \cdot g_i$. φ has degree equal to $|G|$, and it's always faithful. The matrix R_g has a 1 in row i and column j if $gg_j = g_i$, and has 0's in all other positions.

2.4.3 Permutation representation

Let $G = S_n$, V be an n -dimensional vector space over F with basis e_1, e_2, \dots, e_n . Let S_n act on V by defining for each $\sigma \in S_n$

$$\sigma \cdot e_i = e_{\sigma(i)} \quad 1 \leq i \leq n$$

This provides an injective homomorphism of S_n into $\mathrm{GL}(V)$. Hence V is an FS_n -module. R_σ has a 1 in row i and column j if $\sigma(j) = i$.

2.4.4 Composition to create new representation

If $\psi : H \rightarrow \mathrm{GL}(V)$ is any representation of H and $\varphi : G \rightarrow H$ is any group homomorphism, then the composition $\psi \circ \varphi$ is a representation of G .

2.4.5 Degree 1 representation

Any homomorphism of G into the multiplication group $F^\times = \mathrm{GL}_1(F)$ is a degree 1 (matrix) representation.

For example, suppose $G = \langle g \rangle \cong \mathbb{Z}_n$ is the cyclic group of order n and ζ is a fixed n^{th} root of 1 in F . Let $g^i \mapsto \zeta^i$, for all $i \in \mathbb{Z}$. This representation of $\langle g \rangle$ is a faithful representation if and only if ζ is a primitive n^{th} root of 1.

2.4.6 Dihedral representation

For $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rsr = s \rangle$, it has the faithful representation

$$r \mapsto R_r = \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix} \quad s \mapsto R_s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

2.4.7 Quaternion representation

For $Q_8 = \langle i, j \mid i^4 = j^4 = 1, i^2 = j^2, jij = i \rangle$, it has the faithful \mathbb{C} -linear representation

$$R_i = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \quad R_j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

And a 4-dimensional faithful \mathbb{R} -linear representation is given by

$$R_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad R_j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Note. Let $V = \mathbb{R}^4$, $F = \mathbb{R}$, $G = Q_8$.

2.4.8 \mathbb{F}_p -linear representation (modular representations)

Suppose $H \triangleleft G$ and H is an elementary abelian p -group for some prime p . Let $V = H$ be a vector space over \mathbb{F}_p , where scalar $a \in \mathbb{F}_p$ acts on the group element $v \in V$ by $a \cdot v = v^a \in V$. The conjugation action of $g \in G$ on V is \mathbb{F}_p -linear because

$$g \cdot (a \cdot v) = g \cdot (v^a) = \text{Ad}_g(v^a) = gv^a g^{-1} = (gvg^{-1})^a = (\underbrace{\text{Ad}_g v}_{\in H})^a \in V$$

Then V is an $\mathbb{F}_p G$ -module. The kernel of this representation is $C_G(H) \supset H$.

2.5 Examples in Representations and Characters of Groups Gordon James

See Representations and Characters of Groups Gordon James.

2.5.1 Example 1

Let $G = D_{12} = \langle a, b : a^6 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. Define the matrices A, B, C, D over \mathbb{C} by

$$A = \begin{pmatrix} e^{i\pi/3} & 0 \\ 0 & e^{-i\pi/3} \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Prove that each of the functions $\rho_k : G \rightarrow \mathrm{GL}(2, \mathbb{C})$ ($k = 1, 2, 3, 4$), given by

$$\begin{aligned}\rho_1 &: a^r b^s \rightarrow A^r B^s \\ \rho_2 &: a^r b^s \rightarrow A^{3r} (-B)^s \\ \rho_3 &: a^r b^s \rightarrow (-A)^r B^s \\ \rho_4 &: a^r b^s \rightarrow C^r D^s \quad (0 \leq r \leq 5, 0 \leq s \leq 1)\end{aligned}$$

is a representation of G . Which of these representations are faithful? Which are equivalent?

证明. It suffices to check that

$$\rho(A)^6 = \rho(b)^2 = 1, \quad \rho(b)^{-1} \rho(a) \rho(b) = \rho(a)^{-1}$$

which is routine. Since

$$\ker \rho_1 = \{1\}, \ker \rho_2 = \{1, a^2, a^4\}, \ker \rho_3 = \{1\}, \ker \rho_4 = \{1\}$$

ρ_1, ρ_3, ρ_4 are faithful.

$$\rho_1 \simeq \rho_3 \simeq \rho_4$$

Since their matrix representations have the same Jordan Canonical form. \square

2.5.2 Example 2

Let G be the dihedral group $D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. Define the matrices A and B by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

And check that

$$A^4 = B^2 = I, B^{-1}AB = A^{-1}$$

It follows (see Example 1.4) that the function $\rho : G \rightarrow \mathrm{GL}(2, F)$ which is given by

$$\rho : a^i b^j \rightarrow A^i B^j \quad (0 \leq i \leq 3, 0 \leq j \leq 1)$$

is a representation of D_8 over F . The degree of ρ is 2.

The matrices $g\rho$ for g in D_8 are given in the following table:

g	1	a	a^2	a^3
$g\rho$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
g	b	ab	a^2b	a^3b
$g\rho$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Let

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

Then

$$T^{-1}AT = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad T^{-1}BT = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

So we obtain another representation σ of D_8 for which

$$a\sigma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad b\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

2.5.3 Example 3

Let ρ be a representation of the group G . Suppose that g and h are elements of G such that $(g\rho)(h\rho) = (h\rho)(g\rho)$. Does it follow that $gh = hg$?

$$(g\rho)(h\rho) = (h\rho)(g\rho) \iff (ghg^{-1}h^{-1})\rho = I \iff ghg^{-1}h^{-1} \in \ker \rho$$

If ρ is faithful then $gh = hg$. Otherwise, g, h may not commute.

2.5.4 Exercises

Exercice 3: Yau-2024

Question 1. Let G be a finite group.

- (1) Let K be a field. Show that G has a finite-dimensional faithful K -linear representation.
- (2) Show that G has a faithful one-dimensional complex representation if and only if G is cyclic.
- (3) Assume moreover that G is commutative. Let $n \geq 1$ be an integer. Show that G has a faithful n -dimensional complex representation if and only if G can be generated by n elements.
- (4) Classify all finite groups having a faithful 2-dimensional real representation.

(1) $|G| = n$, $G = \{g_1, \dots, g_n\}$. Let V be a K -vector space whose basis is $\{e_{g_1}, \dots, e_{g_n}\}$. The dimension of V is $\dim_K(V) = |G|$. More formally,

$$V = \bigoplus_{g \in G} K e_g$$

V is a finite vector space over G . Define the mapping (we will show that it's a representation),

$$\rho : G \rightarrow \text{Hom}(V, V) \quad g \mapsto \rho_g$$

where $\rho_g : e_h \mapsto e_{gh}$. We can extend the action linearly to any $v = \sum_{h \in G} \alpha_h e_h \in V$, by letting

$$\rho_g(v) = \sum_{h \in G} \alpha_h e_{gh}$$

To show that it's a homomorphism, we have

$$\rho_{gg'}(e_h) = e_{gg'h} = (\rho_g \circ \rho_{g'})(e_h) \quad \forall h \in G$$

To show that the representation is faithful, consider the kernel

$$\ker \rho = \{g \in G : \rho(g) = id\}$$

For $g \in \ker \rho$,

$$e_{gh} = \rho_g(e_h) = e_h \quad \forall h \in G$$

Then we have

$$gh = h \quad \forall h \in G \implies g = e$$

Thus $\ker \rho = \{e\}$. The representation is faithful.

(2) If G is cyclic with order n , and $G = \langle g \rangle$, then the mapping

$$\rho : G \rightarrow \text{Hom}(\mathbb{C}, \mathbb{C}) \quad g^r \mapsto e^{2\pi i r/n}$$

is a faithful representation.

If G has a faithful one-dimensional complex representation,

$$\rho : G \rightarrow \text{Hom}(\mathbb{C}, \mathbb{C}) \quad g \mapsto \rho_g$$

$|G| = n$, then $\rho(g)^n = 1, \forall g \in G$. Thus every ρ_g lies in the set of n^{th} roots of unity, which has size n . Since ρ is faithful, $\rho_g \neq \rho_h$ for $g \neq h$. Then $\rho(G) = \{\mu_n\}$ is cyclic. Hence G is cyclic.

(3) If G can be generated by g_1, \dots, g_n , then for any $g \in G$, we have

$$g = g_1^{r_1} \cdots g_n^{r_n}$$

Assume that $g_i^{\alpha_i} = e$ for every i , then the mapping

$$\rho : G \rightarrow \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) \quad g \mapsto \text{diag}\{e^{2\pi i r_1/\alpha_1}, \dots, e^{2\pi i r_n/\alpha_n}\}$$

is a faithful n -dimensional complex representation.

If G has a faithful n -dimensional complex representation,

$$\rho : G \rightarrow \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) \quad g \mapsto \rho_g$$

ρ_g can be a diagonalized matrix.

(4) 具有忠实的二维实表示的有限群的分类如下:

- (1) 循环群 C_n (n 阶循环群), 对于任意 $n \geq 1$ 。
- (2) 二面体群 D_m (2 m 阶二面体群, 即正 m 边形的对称群), 对于任意 $m \geq 1$
-

Remarque 3. Let (ρ, V) be a finite dimensional \mathbb{C} -representation of a finite group G . Consider the G -invariant subspace

$$V^G := \{v \in V \mid \rho(g)(v) = v \text{ for all } g \in G\}$$

- (1) Show that $\dim V^G$ is the same as the multiplicity of the trivial representation appearing in V .
- (2) Show that $\dim V^G = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)$.
- (3) Construct a surjective map $\phi : V \rightarrow V^G$, expressed in terms of a linear combination of linear operators $\rho(g)$ for $g \in G$, such that $\phi^2 = \phi$ (i.e. ϕ is a projection) and ϕ is a homomorphism.

证明. (1) Write V as a direct sum of irreducible subrepresentations:

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_r.$$

Then $V^G = W_1^G \oplus W_2^G \oplus \cdots \oplus W_r^G$. But W_i^G is always a subrepresentation of W_i . If some W_i is irreducible and nontrivial, W_i^G must be trivial. Yet if some W_i is trivial, $W_i^G = W_i$.

To sum up, we have V^G is the direct sum of all trivial factors of V , and thus $\dim V^G$ is the same as the multiplicity of trivial representation in V .

(2) By character formula, the multiplicity of trivial representation in V is

$$\langle V, \mathbf{1} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho}(g).$$

(3) Consider the homomorphism

$$\phi = \frac{1}{|G|} \sum_{g \in G} \rho(g).$$

For each $v \in V$ and $h \in G$,

$$\rho(h)\phi(v) = \rho(h) \left(\frac{1}{|G|} \sum_{g \in G} \rho(g)(v) \right) = \frac{1}{|G|} \sum_{g \in G} \rho(hg)(v) = \frac{1}{|G|} \sum_{k \in G} \rho(k)(v) = \phi(v).$$

So $\phi(v) \in V^G$. Yet, for $v \in V^G$, we have

$$\phi(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)(v) = \frac{1}{|G|} \sum_{g \in G} v = v.$$

Thus ϕ restricted to V^G is the identity. In particular, this says that ϕ is surjective and $\phi^2 = \phi$.

Finally, we check that ϕ is a homomorphism, i.e. for $h \in G$,

$$\phi \circ \rho(h) = \frac{1}{|G|} \sum_{g \in G} \rho(g) \rho(h) = \frac{1}{|G|} \sum_{g \in G} \rho(gh) = \frac{1}{|G|} \sum_{k \in G} \rho(hk) = \rho(h) \circ \phi,$$

where the change of variable is that $k = h^{-1}gh$. \square

2.6 Representation Theory and Character Theory

2.6.1 Linear Actions and modules over group rings

F is a ring, G is a group then the **group ring** FG is defined by

$$\sum_{g \in G} \alpha_g g, \quad \alpha_g \in F.$$

with the following properties:

$$\sum_{g \in G} \alpha_g g + \sum_{g \in G} \beta_g g = \sum_{g \in G} (\alpha_g + \beta_g) g$$

$$\left(\sum_{g \in G} \alpha_g g \right) \left(\sum_{g \in G} \beta_g g \right) = \sum_{g \in G} \left(\sum_{\substack{h, k \in G \\ hk=g}} \alpha_h \beta_k \right) g$$

FG is a commutative ring iff G is an abelian group.

FG is a vector space over F with the elements of G as a basis. ($\dim FG = |G|$)

This discussion shows there is a bijection between FG -modules and pairs (V, φ) :

$$\left\{ V \text{ an } FG\text{-module} \right\} \longleftrightarrow \left\{ \begin{array}{c} V \text{ a vector space over } F \\ \text{and} \\ \varphi : G \rightarrow GL(V) \text{ a representation} \end{array} \right\}.$$

Formally, a representation $\rho : G \rightarrow GL(V)$ is **faithful** if the only element of G that is mapped to the identity transformation is the identity element of G , i.e.,

$$\ker(\rho) = \{e\}$$

where e is the identity element of G .

Definitions of irreducible (simple), indecomposable, completely reducible, constituent

Definition. Let R be a ring and let M be a nonzero R -module.

- (1) The module M is said to be *irreducible* (or *simple*) if its only submodules are 0 and M ; otherwise M is called *reducible*.
- (2) The module M is said to be *indecomposable* if M cannot be written as $M_1 \oplus M_2$ for any nonzero submodules M_1 and M_2 ; otherwise M is called *decomposable*.
- (3) The module M is said to be *completely reducible* if it is a direct sum of irreducible submodules.
- (4) A representation is called *irreducible*, *reducible*, *indecomposable*, *decomposable* or *completely reducible* according to whether the FG -module affording it has the corresponding property.
- (5) If M is a completely reducible R -module, any direct summand of M is called a *constituent* of M (i.e., N is a constituent of M if there is a submodule N' of M such that $M = N \oplus N'$).

Theorem 1. (Maschke's Theorem) Let G be a finite group and let F be a field whose characteristic does not divide $|G|$. If V is any FG -module and U is any submodule of V , then V has a submodule W such that $V = U \oplus W$ (i.e., every submodule is a direct summand).

Remark: The hypothesis of Maschke's Theorem applies to any finite group when F has characteristic 0.

An FG -module is finitely generated iff it is finite dimensional.

2.6.2 Wedderburn's theorem and some consequences

Theorem 4. (Wedderburn's Theorem) Let R be a nonzero ring with 1 (not necessarily commutative). Then the following are equivalent:

- (1) every R -module is projective
- (2) every R -module is injective
- (3) every R -module is completely reducible
- (4) the ring R considered as a left R -module is a direct sum:

$$R = L_1 \oplus L_2 \oplus \cdots \oplus L_n,$$

where each L_i is a simple module (i.e., a simple left ideal) with $L_i = Re_i$, for some $e_i \in R$ with

- (i) $e_i e_j = 0$ if $i \neq j$
- (ii) $e_i^2 = e_i$ for all i
- (iii) $\sum_{i=1}^n e_i = 1$
- (5) as rings, R is isomorphic to a direct product of matrix rings over division rings, i.e., $R = R_1 \times R_2 \times \cdots \times R_r$, where R_j is a two-sided ideal of R and R_j is isomorphic to the ring of all $n_j \times n_j$ matrices with entries in a division ring Δ_j , $j = 1, 2, \dots, r$. The integer r , the integers n_j , and the division rings Δ_j (up to isomorphism) are uniquely determined by R .

Proof: A proof of Wedderburn's Theorem is outlined in Exercises 1 to 10

Definitions of semisimple, idempotent, orthogonal, primitive, primitive central idempotent

Definition. A ring R satisfying any of the (equivalent) properties in Theorem 4 is called *semisimple with minimum condition*.

Definition.

- (1) A nonzero element e in a ring R is called an *idempotent* if $e^2 = e$.
- (2) Idempotents e_1 and e_2 are said to be *orthogonal* if $e_1 e_2 = e_2 e_1 = 0$.
- (3) An idempotent e is said to be *primitive* if it cannot be written as a sum of two (commuting) orthogonal idempotents.
- (4) The idempotent e is called a *primitive central idempotent* if $e \in Z(R)$ and e cannot be written as a sum of two orthogonal idempotents in the ring $Z(R)$.

Propositions

Proposition 6 describes the ideal structure of a matrix ring and Proposition 8 extends these results to direct products of matrix rings.

Proposition 6. Let Δ be a division ring, let $n \in \mathbb{Z}^+$, let R be the ring of all $n \times n$ matrices with entries from Δ and let I be the identity matrix (= the 1 of R).

- (1) The only two-sided ideals of R are 0 and R .
- (2) The center of R consists of the scalar matrices αI , where α is in the center of Δ : $Z(R) = \{\alpha I \mid \alpha \in Z(\Delta)\}$, and this is a field isomorphic to $Z(\Delta)$. In particular, if Δ is a field, the center of R is the subring of all scalar matrices. The only central idempotent in R is I (in particular, I is primitive).
- (3) Let e_i be the matrix with a 1 in position i, i and zeros elsewhere. Then e_1, \dots, e_n are orthogonal primitive idempotents and $\sum_{i=1}^n e_i = I$.
- (4) $L_i = Re_i$ is the left ideal consisting of arbitrary entries in column i and zeros in all other columns. L_i is a simple left R -module. Every simple left R -module is isomorphic to L_1 (in particular, all L_i are isomorphic R -modules) and as a left R -module we have $R = L_1 \oplus \dots \oplus L_n$.

2.7 Tensor product

Let V and W be vector spaces over \mathbb{C} with bases v_1, \dots, v_m and w_1, \dots, w_n , respectively. For each i, j with $1 \leq i \leq m, 1 \leq j \leq n$, we introduce a symbol $v_i \otimes w_j$. The tensor product space $V \otimes W$ is defined to be the mn -dimensional vector space over \mathbb{C} with a basis given by

$$\{v_i \otimes w_j : 1 \leq i \leq m, 1 \leq j \leq n\}$$

Thus $V \otimes W$ consists of all expressions of the form

$$\sum_{i,j} \lambda_{ij} (v_i \otimes w_j) \quad (\lambda_{ij} \in \mathbb{C})$$

For $v \in V$ and $w \in W$ with $v = \sum_{i=1}^m \lambda_i v_i$ and $w = \sum_{j=1}^n \mu_j w_j$ ($\lambda_i, \mu_j \in \mathbb{C}$), we define $v \otimes w \in V \otimes W$ by

$$v \otimes w = \sum_{i,j} \lambda_i \mu_j (v_i \otimes w_j).$$

For example,

$$(2v_1 - v_2) \otimes (w_1 + w_2) = 2v_1 \otimes w_1 + 2v_1 \otimes w_2 - v_2 \otimes w_1 - v_2 \otimes w_2$$

Do not be misled by the notation into believing that every element of $V \otimes W$ has the form $v \otimes w$, but linear combination of $v_i \otimes w_j$.

If $v \in V, w \in W$ and $\lambda \in \mathbb{C}$, then

$$v \otimes (\lambda w) = (\lambda v) \otimes w = \lambda(v \otimes w)$$

If $x_1, \dots, x_a \in V$ and $y_1, \dots, y_b \in W$, then

$$\left(\sum_{i=1}^a x_i \right) \otimes \left(\sum_{j=1}^b y_j \right) = \sum_{i,j} x_i \otimes y_j$$

Next, we define the tensor product of two $\mathbb{C}G$ -modules.

Let G be a finite group and let V and W be $\mathbb{C}G$ -modules with bases v_1, \dots, v_m and w_1, \dots, w_n , respectively. We know that the elements

$$v_i \otimes w_j \quad (1 \leq i \leq m, 1 \leq j \leq n)$$

give a basis of $V \otimes W$. The multiplication of $v_i \otimes w_j$ by an element of G is defined in the following simple way, which is then extended linearly to a multiplication on the whole of $V \otimes W$.

Let $g \in G$. For all i, j , define

$$(v_i \otimes w_j)g = v_i g \otimes w_j g$$

and, more generally, let

$$\left(\sum_{i,j} \lambda_{ij} (v_i \otimes w_j) \right) g = \sum_{i,j} \lambda_{ij} (v_i g \otimes w_j g)$$

for arbitrary complex numbers λ_{ij} .

Remarque 4. You should be warned that $(v \otimes w)r \neq vr \otimes wr$ for most elements r in $\mathbb{C}G$, e.g. when r is a scalar multiple of g .

2.8 Tensor product of modules

See dummit&Foote Sec 10.4

Définition 9: R -module

Let R be a ring (associative with 1). An **R -module** (or left R -module) is an abelian group M together with an operation $R \times M \rightarrow M$, denoted by

$(r, m) \mapsto rm$, such that for all $r, s \in R$ and $x, y \in M$:

$$r(x + y) = rx + ry$$

$$(r + s)x = rx + sx$$

$$(rs)x = r(sx)$$

$$1_R x = x$$

A right R -module is defined similarly, with the scalar multiplication on the right $M \times R \rightarrow M$, denoted by $(m, r) \mapsto mr$, satisfying analogous axioms.

Our aim is to "extend" an R -module N to an S -module.

To satisfy the relations necessary for an S -module structure² and the compatibility relation with the action of R on N ³, we must take the quotient of this abelian group by the subgroup H generated by all elements of the form

$$(s_1 + s_2, n) - (s_1, n) - (s_2, n),$$

$$(s, n_1 + n_2) - (s, n_1) - (s, n_2), \text{ and}$$

$$(sr, n) - (s, rn),$$

The resulting quotient group is denoted by $S \otimes_R N$ (or just $S \otimes N$ if R is clear from the context) and is called the **tensor product** of S and N over R . By the definition of the quotient we have forced the relations

$$(s_1 + s_2) \otimes n = s_1 \otimes n + s_2 \otimes n,$$

$$s \otimes (n_1 + n_2) = s \otimes n_1 + s \otimes n_2, \text{ and}$$

$$sr \otimes n = s \otimes rn.$$

Définition 10: universal property for the tensor product

² $(s_1 + s_2)n = s_1n + s_2n$, $s(n_1 + n_2) = sn_1 + sn_2$.

³ $(sr)n = s(rn)$

Theorem 8. Let R be a subring of S , let N be a left R -module and let $\iota : N \rightarrow S \otimes_R N$ be the R -module homomorphism defined by $\iota(n) = 1 \otimes n$. Suppose that L is any left S -module (hence also an R -module) and that $\varphi : N \rightarrow L$ is an R -module homomorphism from N to L . Then there is a unique S -module homomorphism $\Phi : S \otimes_R N \rightarrow L$ such that φ factors through Φ , i.e., $\varphi = \Phi \circ \iota$ and the diagram

$$\begin{array}{ccc} N & \xrightarrow{\iota} & S \otimes_R N \\ & \searrow \varphi & \downarrow \Phi \\ & & L \end{array}$$

commutes. Conversely, if $\Phi : S \otimes_R N \rightarrow L$ is an S -module homomorphism then $\varphi = \Phi \circ \iota$ is an R -module homomorphism from N to L .

Examples

- (1) For any ring R and any left R -module N we have $R \otimes_R N \cong N$ (so “extending scalars from R to R ” does not change the module). This follows by taking φ to be the identity map from N to itself (and $S = R$) in Theorem 8: ι is then an isomorphism with inverse isomorphism given by Φ . In particular, if A is any abelian group (i.e., a \mathbb{Z} -module), then $\mathbb{Z} \otimes_{\mathbb{Z}} A = A$.
- (2) Let $R = \mathbb{Z}$, $S = \mathbb{Q}$ and let A be a finite abelian group of order n . In this case the \mathbb{Q} -module $\mathbb{Q} \otimes_{\mathbb{Z}} A$ obtained by extension of scalars from the \mathbb{Z} -module A is 0. To see this, observe first that in any tensor product $1 \otimes 0 = 1 \otimes (0 + 0) = 1 \otimes 0 + 1 \otimes 0$, by the second relation in (4), so

$$1 \otimes 0 = 0.$$

Now, for any simple tensor $q \otimes a$ we can write the rational number q as $(q/n)n$. Then since $na = 0$ in A by Lagrange’s Theorem, we have

$$q \otimes a = \left(\frac{q}{n} \cdot n\right) \otimes a = \frac{q}{n} \otimes (na) = (q/n) \otimes 0 = (q/n)(1 \otimes 0) = 0.$$

It follows that $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$. In particular, the map $\iota : A \rightarrow S \otimes_R A$ is the zero map. By Theorem 8, we see again that any homomorphism of a finite abelian group into a rational vector space is the zero map. In particular, if A is nontrivial, then the original \mathbb{Z} -module A is not contained in the \mathbb{Q} -module obtained by extension of scalars.

- (3) *Extension of scalars for free modules:* If $N \cong R^n$ is a free module of rank n over R then $S \otimes_R N \cong S^n$ is a free module of rank n over S . We shall prove this shortly (Corollary 18) when we discuss tensor products of direct sums. For example, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$. In this case the module obtained by extension of scalars contains (an isomorphic copy of) the original R -module N . For example, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$ and \mathbb{Z}^n is a subgroup of the abelian group \mathbb{Q}^n .
- (4) *Extension of scalars for vector spaces:* As a special case of the previous example, let F be a subfield of the field K and let V be an n -dimensional vector space over F (i.e., $V \cong F^n$). Then $K \otimes_F V \cong K^n$ is a vector space over the larger field K of the same dimension, and the original vector space V is contained in $K \otimes_F V$ as an F -vector subspace.
- (5) *Induced modules for finite groups:* Let R be a commutative ring with 1, let G be a finite group and let H be a subgroup of G . As in Section 7.2 we may form the group ring RG and its subring RH . For any RH -module N define the *induced module* $RG \otimes_{RH} N$. In this way we obtain an RG -module for each RH -module N . We shall study properties of induced modules and some of their important applications to group theory in Chapters 17 and 19.

Exemple 2.

Définition 11: R -balanced

Definition. Let M be a right R -module, let N be a left R -module and let L be an abelian group (written additively). A map $\varphi : M \times N \rightarrow L$ is called R -balanced or *middle linear with respect to R* if

$$\begin{aligned}\varphi(m_1 + m_2, n) &= \varphi(m_1, n) + \varphi(m_2, n) \\ \varphi(m, n_1 + n_2) &= \varphi(m, n_1) + \varphi(m, n_2) \\ \varphi(m, rn) &= \varphi(mr, n)\end{aligned}$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, and $r \in R$.

Définition 12: R -bilinear

Definition. Let R be a commutative ring with 1 and let M , N , and L be left R -modules. The map $\varphi : M \times N \rightarrow L$ is called R -bilinear if it is R -linear in each factor, i.e., if

$$\begin{aligned}\varphi(r_1 m_1 + r_2 m_2, n) &= r_1 \varphi(m_1, n) + r_2 \varphi(m_2, n), \quad \text{and} \\ \varphi(m, r_1 n_1 + r_2 n_2) &= r_1 \varphi(m, n_1) + r_2 \varphi(m, n_2)\end{aligned}$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r_1, r_2 \in R$.

Exemple 3. We have $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3 = 0$, since $3a = a$ for $a \in \mathbb{Z}_2$ so that

$$a \otimes b = 3a \otimes b = a \otimes (3b) = a \otimes 0 = 0$$

and every simple tensor is reduced to 0. In particular $1 \otimes 1 = 0$.

Exemple 4. We have $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2 \cong \mathbb{Z}_2$, since $0 \otimes 0 = 1 \otimes 0 = 0 \otimes 1 = 0$ and $1 \otimes 1$ generates $\mathbb{Z}_2 \otimes \mathbb{Z}_2$.

Exemple 5. $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$, where a simple tensor has the form $(a/b \bmod \mathbb{Z}) \otimes (c/d \bmod \mathbb{Z})$ for some rational numbers a/b and c/d . Then

$$\begin{aligned}\left(\frac{a}{b} \bmod \mathbb{Z}\right) \otimes \left(\frac{c}{d} \bmod \mathbb{Z}\right) &= d \left(\frac{a}{bd} \bmod \mathbb{Z}\right) \otimes \left(\frac{c}{d} \bmod \mathbb{Z}\right) \\ &= \left(\frac{a}{bd} \bmod \mathbb{Z}\right) \otimes d \left(\frac{c}{d} \bmod \mathbb{Z}\right) \\ &= \left(\frac{a}{bd} \bmod \mathbb{Z}\right) \otimes 0 \\ &= 0\end{aligned}\tag{2.1}$$

Exemple 6. Similar to eq. (2.1), $A \otimes_{\mathbb{Z}} B = 0$ for any divisible abelian group A and **torsion** abelian group B (an abelian group in which every element has finite order). For example, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

Exemple 7 (dimension). The structure of a tensor product can vary considerably depending on the ring over which the tensors are taken. $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ are isomorphic as left \mathbb{Q} -modules (both are one dimensional vector spaces over \mathbb{Q}). Pick any simple tensor $\frac{a}{b} \otimes_{\mathbb{Q}} \frac{c}{d}$ in $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$, then

$$\frac{a}{b} \otimes \frac{c}{d} = 1 \otimes \frac{ac}{bd} \rightarrow \frac{ac}{bd}$$

induces an isomorphism between $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ and \mathbb{Q} . Pick any simple tensor $\frac{a}{b} \otimes_{\mathbb{Z}} \frac{c}{d}$ in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, then

$$\frac{a}{b} \otimes \frac{c}{d} = \frac{1}{b} \otimes \frac{ac}{d} = \frac{1}{b} \otimes \left(b \cdot \frac{ac}{bd} \right) = 1 \otimes \frac{ac}{bd}$$

Thus $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$. Hence,

$$\dim_{\mathbb{Q}} (\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}) = \dim_{\mathbb{Q}} (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}) = 1$$

Exemple 8. On the other hand, $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ are not isomorphic \mathbb{C} -modules (the former is a 1-dimensional vector space over \mathbb{C} and the latter is 2-dimensional over \mathbb{C}). Every simple tensor in $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ has the form $a \otimes b$, then

$$a \otimes b = 1 \otimes ab \rightarrow ab$$

induces an isomorphism between $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ and \mathbb{C} . Every simple tensor in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ has the form $(a + ib) \otimes c$, then

$$(a + ib) \otimes c = a \otimes c + (ib) \otimes c = 1 \otimes (ac) + i \otimes (bc) \xrightarrow{\sim} (ac, bc)$$

induces an isomorphism between $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \oplus \mathbb{C}$. Hence

$$\dim_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}) = \dim_{\mathbb{C}} \mathbb{C} = 1 \quad \dim_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \dim_{\mathbb{C}} (\mathbb{C} \oplus \mathbb{C}) = 2$$

2.8.1 Dimension

abstract algebra - Dimension of a tensor product as a vector space over complex number field \mathbb{C} - Mathematics Stack Exchange

Exercice 4

Problem 2 (10 points). Consider the polynomial ring $A = \mathbb{C}[x]$ and its subring $B = \mathbb{C}[x^2] \subset A$. Consider A -modules

$$M = \mathbb{C}[x]/(x^2 + x), \quad N = \mathbb{C}[x]/(x^2 - 1)$$

What is the dimension as \mathbb{C} -vector space of each tensor product below?
No need to prove or explain.

(a) $M \otimes_{\mathbb{C}} N$;

$\mathbb{C}[x]/(x^2 + x) \otimes_{\mathbb{C}} \mathbb{C}[x]/(x^2 - 1)$ has every simple tensor of the form $(a + bx) \otimes_{\mathbb{C}} (c + dx)$, then

$$\begin{aligned} (a + bx) \otimes (c + dx) &= a \otimes c + bx \otimes c + a \otimes dx + bx \otimes dx \\ &= 1 \otimes (ac) + x \otimes (bc) + (ad) \otimes x + bd(x \otimes x) \\ &= ac(1 \otimes 1) + bc(x \otimes 1) + ad(1 \otimes x) + bd(x \otimes x) \\ &\rightarrow (ac, bc, ad, bd) \end{aligned}$$

induces an isomorphism between $\mathbb{C}[x]/(x^2 + x) \otimes_{\mathbb{C}} \mathbb{C}[x]/(x^2 - 1)$ and \mathbb{C}^4 (the construction is left as an exercise). Therefore

$$\dim_{\mathbb{C}}(M \otimes_{\mathbb{C}} N) = \dim_{\mathbb{C}}(\mathbb{C}^4) = 4$$

(b) $M \otimes_A N$;

$\mathbb{C}[x]/(x^2 + x) \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(x^2 - 1)$ has every simple tensor of the form $(a + bx) \otimes_{\mathbb{C}[x]} (c + dx)$, then

$$\begin{aligned} (a + bx) \otimes (c + dx) &= ac(1 \otimes 1) + bc(x \otimes 1) + ad(1 \otimes x) + bd(x \otimes x) \\ &= ac(1 \otimes 1) + (bc + ad)((-x^2) \otimes 1) + bd(1 \otimes x^2) \\ &= ac(1 \otimes 1) - (bc + ad)(1 \otimes x^2) + bd(1 \otimes 1) \\ &= (ac - bc - ad + bd)(1 \otimes 1) \\ &\rightarrow ac - bc - ad + bd \end{aligned}$$

induces a nature isomorphism between $M \otimes_A N$ and \mathbb{C} . Therefore,

$$\dim_{\mathbb{C}}(M \otimes_A N) = \dim_{\mathbb{C}} \mathbb{C} = 1$$

(c) $M \otimes_B N$.

$\mathbb{C}[x]/(x^2 + x) \otimes_{\mathbb{C}[x^2]} \mathbb{C}[x]/(x^2 - 1)$ has every simple tensor of the form $(a + bx) \otimes_{\mathbb{C}[x^2]} (c + dx)$, then

$$\begin{aligned} (a + bx) \otimes (c + dx) &= (a - bx^2) \otimes (c + dx) \\ &= a \otimes (c + dx) - bx^2 \otimes (c + dx) \\ &= 1 \otimes (ac + adx) - 1 \otimes (bcx^2 + cdx^3) \\ &= 1 \otimes (ac + adx) - 1 \otimes (bc + cdx) \\ &= 1 \otimes [(ac - bc) + (ad - cd)x] \\ &= (ac - bc)(1 \otimes 1) + (ad - cd)(1 \otimes x) \\ &\rightarrow (ac - bc, ad - cd) \end{aligned}$$

induces a nature isomorphism between $M \otimes_B N$ and \mathbb{C}^2 . Therefore,

$$\dim_{\mathbb{C}}(M \otimes_B N) = \dim_{\mathbb{C}}(\mathbb{C}^2) = 2$$

We are done!

2.9 有限群表示论

参见肖梁、Artin.

A matrix representation of a group G is a homomorphism

$$R : G \rightarrow \mathrm{GL}_n$$

from G to one of the complex general linear groups. The number n is the dimension of the representation.

We use the notation R_g instead of $R(g)$ for the image of a group element g . Each R_g is an invertible matrix, and the statement that R is a homomorphism reads

$$R_{gh} = R_g R_h.$$

If a group is given by generators and relations, say $\langle x_1, \dots, x_n \mid r_1, \dots, r_k \rangle$, a matrix representation can be defined by assigning matrices R_{x_1}, \dots, R_{x_n} that satisfy the relations. For example, the symmetric group S_3 can be presented as $\langle x, y \mid x^3, y^2, xyxy \rangle$, so a representation of S_3 is defined by matrices R_x and R_y such that $R_x^3 = I$, $R_y^2 = I$, and $R_x R_y R_x R_y = I$. Some relations in addition to these required ones may hold.

2.9.1 Definitions of linear representation, homomorphism of representations (G -linear map), isomorphism of representations, subrepresentation, direct sum of representations, complementary representation, irreducible, tensor product, dual representations

Definition 9.1.2. Let G be a group and V a vector space over \mathbb{C} . A **linear representation** of G on V is a homomorphism $\rho : G \rightarrow \mathrm{GL}(V)$, i.e. for all $g, h \in G$,

$$\rho(gh) = \rho(g)\rho(h).$$

(Automatically, $\rho(g^{-1}) = \rho(g)^{-1}$ and $\rho(1) = I_n$.)

This is equivalent to an action of G on V that is \mathbb{C} -linear, i.e.

$$\begin{aligned} G \times V &\longrightarrow V \\ (g, v) &\longmapsto \rho(g)(v). \end{aligned}$$

such that

- for $e \in G$, $\rho(e)(v) = v$;
- for $g, h \in G$, $\rho(gh)(v) = \rho(g)(\rho(h)(v))$;
- for $g \in G$ and $\lambda \in \mathbb{C}$, $\rho(g)(\lambda v) = \lambda\rho(g)(v)$.

We write (ρ, V) or sometimes simply ρ or V for the representation.

Definition 9.1.4. If (ρ, V) and (ρ', V') are two linear representations of G , a \mathbb{C} -linear map $\phi : V \rightarrow V'$ is called a **homomorphism of representations** of G or **G -linear map** if

$$\forall g \in G, \quad \phi \circ \rho(g) = \rho'(g) \circ \phi.$$

If moreover ϕ is a bijection, we say that ϕ is an **isomorphism of representations** of G .

Definition 9.2.1. Let $\rho : G \rightarrow \mathrm{GL}(V)$ be a linear representation of G . A **subrepresentation** is a \mathbb{C} -linear subspace $W \subseteq V$ that is “stable under G -action”, i.e.

$$\forall g \in G, \quad \rho(g)(W) \subseteq W.$$

This then induces a representation $\rho_W : G \rightarrow \mathrm{GL}(W)$.

Definition 9.2.4. Let (ρ, W) and (ρ', W') be linear representations of G . Then we can form their direct sum

$$\begin{aligned} \rho'' = \rho \oplus \rho' : G &\longrightarrow \mathrm{GL}(W \oplus W') \\ g &\longmapsto \begin{pmatrix} \rho(g) & 0 \\ 0 & \rho'(g) \end{pmatrix} \end{aligned}$$

or in terms of G -action: for $g \in G$, $w \in W$, and $w' \in W'$,

$$\rho''(g)(w, w') = (\rho(g)(w), \rho'(g)(w')).$$

Theorem 9.2.5. Let G be a finite group. If $W \subseteq V$ is a subrepresentation of G , then there exists a subrepresentation $W^\circ \subseteq V$ such that

$$V = W \oplus W^\circ.$$

We call W° a **complementary representation** of W in V .

Definition 9.2.8. Let G be a group. A linear representation V is called **irreducible** if the only subrepresentations of V are $\{0\}$ and V .

We write $\text{Irr}(G)$ for the set of irreducible representations (up to isomorphisms).

Note. If G is finite, then any finite dimensional representation V is "completely reducible".

Definition 9.3.2. If (ρ_1, V) and (ρ_2, W) be two representations of G , their **tensor product** is $V \otimes W$, on which G acts as

$$\rho(g)(v \otimes w) := \rho_1(g)(v) \otimes \rho_2(g)(w).$$

9.3.3. *Dual vector space.* For a \mathbb{C} -vector space V , set $V^* := \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$; it is the dual vector space. For each $v^* \in V^*$ and $v \in V$, we may evaluate v^* at v as: $v^*(v)$. In other terms, we may rewrite this as a natural pairing

$$(9.3.3.1) \quad \begin{aligned} V^* \times V &\longrightarrow \mathbb{C} \\ (v^*, v) &\longmapsto v^*(v). \end{aligned}$$

Definition 9.3.4. If (ρ, V) is a representation of G , we define its **dual representation** or **contragredient representations** to be $V^* := \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$, and the G -action is given by, for $v^* \in V^*$ and $v \in V$,

$$(9.3.4.1) \quad (\rho^*(g)(v^*))(v) := v^*(\rho(g^{-1})(v)).$$

In terms of (9.3.3.1), the above action in (9.3.4.1) is given by

$$\langle \rho^*(g)(v^*), v \rangle = \langle v^*, \rho^*(g^{-1})(v) \rangle.$$

Remark 9.3.5. We explain why we use g^{-1} in (9.3.4.1) of the dual representation. This is because we need $\rho^*(gh) = \rho^*(g)\rho^*(h)$. We check:

$$(\rho^*(g)(\rho^*(h)(v^*))(v) = (\rho^*(h)(v^*))(\rho(g^{-1})(v)) = v^*(\rho(h^{-1})(\rho(g^{-1})(v))) = v^*(\rho((gh)^{-1})(v)).$$

Remark 9.3.6. In terms of matrix representations, when fixing an isomorphism $V \simeq \mathbb{C}^n$, it induces an isomorphism $V^* \simeq \mathbb{C}^n$. Then for $\rho(g) \in \text{GL}_n(\mathbb{C})$, we have

$$\rho^*(g) = {}^t \rho(g)^{-1}$$

is the transpose inverse.

2.9.2 A construction of homomorphism between two linear representations of a finite group G

Construction 9.2.6. It is essential to assume that G is finite. Let (ρ_1, W_1) and (ρ_2, W_2) be two linear representations of a finite group G . If $\phi : W_1 \rightarrow W_2$ is a homomorphism of representations of G , then for any $g \in G$, $\rho_2(g) \circ \phi \circ \rho_1(g)^{-1} = \phi$.

Now if we start with just a \mathbb{C} -linear map $\phi : W_1 \rightarrow W_2$ (unnecessarily respecting the G -action), we can define a homomorphism of G -representation by “averaging over G ” as follows.

$$\tilde{\phi} := \frac{1}{|G|} \sum_{g \in G} \rho_2(g) \circ \phi \circ \rho_1(g)^{-1} \in \text{Hom}_{\mathbb{C}}(W_1, W_2).$$

Let us check the properties for being a homomorphism: for $h \in H$,

$$\begin{aligned} \tilde{\phi} \circ \rho_1(h) &= \frac{1}{|G|} \sum_{g \in G} \rho_2(g) \circ \phi \circ \rho_1(g^{-1}h) \\ &\stackrel{g=hk}{=} \frac{1}{|G|} \sum_{k \in G} \rho_2(hk) \circ \phi \circ \rho_1(k^{-1}) = \rho_2(h) \circ \tilde{\phi}. \end{aligned}$$

We also remark that if ϕ is already a homomorphism of representations, then

$$\tilde{\phi} = \frac{1}{|G|} \sum_{g \in G} \rho_2(g) \circ \phi \circ \rho_1(g)^{-1} = \frac{1}{|G|} \sum_{g \in G} \phi = \phi.$$

2.9.3 Unitary representations of finite groups

There is a natural question: can we find a “good” matrix representation for a given representation ρ of G ? Here “good” means there is a positive definite Hermitian form on V so that every $g \in G$ preserves this Hermitian form; and thus the image of $\rho_\phi(G)$ in $\text{GL}_n(\mathbb{C})$ belongs to the unitary group U_n .

Let V be a Hermitian space – a complex vector space together with a positive definite Hermitian form $\langle \cdot, \cdot \rangle$. A unitary operator T on V is a linear operator with the property

$$\langle Tv, Tw \rangle = \langle v, w \rangle \quad \forall v, w \in V$$

A representation $\rho : G \rightarrow \text{GL}(V)$ on a Hermitian space V is called **unitary** if ρ_g is a unitary operator for every g . We can write this condition as

$$\langle gw, gw \rangle = \langle v, w \rangle \quad \text{or} \quad \langle \rho_g v, \rho_g w \rangle = \langle v, w \rangle$$

Similarly, a matrix representation $R : G \rightarrow \text{GL}_n$ is **unitary** if $R_g \in \text{U}_n, \forall g \in G$.

A **unitary matrix representation** is a homomorphism from G to the unitary group:

$$R : G \rightarrow \text{U}_n$$

Lemme 1

Let ρ be a unitary representation of G on a Hermitian space V , and let W be a G -invariant subspace. The orthogonal complement W^\perp is also G -invariant, and ρ is the direct sum of its restrictions to the Hermitian spaces W and W^\perp . These restrictions are also unitary representations.

证明. It is true that $V = W \oplus W^\perp$. Since ρ is unitary, it preserves orthogonality: If W is invariant and $u \perp W$, then $gu \perp gW = W$. This means that if $u \in W^\perp$, then $gu \in W^\perp$. \square

2.9.4 Equivalence of representations

In matrix terminology, two representation φ and ψ are equivalent if there is a fixed invertible matrix P such that

$$P\varphi(g)P^{-1} = \psi(g) \quad \forall g \in G$$

In FG -module terminology, two representation φ and ψ are equivalent if there is a fixed FG -module isomorphism $T : V \xrightarrow{\sim} W$ such that

$$T \circ \varphi(g) = \psi(g) \circ T \quad \forall g \in G$$

The linear transformation T or the matrix P above is said to intertwine the representation φ and ψ .

2.9.5 Characters of representations, class function

Because they involve several matrices, each of which may have many entries, representations are notationally complicated. The secret to understanding them is to throw out most of the information that the matrices contain, keeping only one essential part, its trace, or character.

Our slogan is: characters determine the representation.

Definition 10.1.2. Let $\rho : G \rightarrow \mathrm{GL}(V)$ be a representation. Define its **character** to be

$$\chi_\rho : G \longrightarrow \mathbb{C}, \quad \chi_\rho(g) = \mathrm{Tr}(\rho(g)).$$

Properties 10.1.3. If χ is a character of a representation ρ of a finite group G of dimension n , then we have

- (1) $\chi_\rho(1) = n$,
- (2) $\chi_\rho(hgh^{-1}) = \chi_\rho(g)$ for every $g, h \in G$,
- (3) $\chi_{\rho^*}(g) = \chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$ for every $g \in G$, where the overline indicates the complex conjugation.

Definition 10.1.4. A function $f : G \rightarrow \mathbb{C}$ is a **class function** if for any $g, h \in G$, we have

$$f(ghg^{-1}) = f(h).$$

In other words, if we consider the conjugation action Ad of G on $\text{Map}(G, \mathbb{C})$ by:

$$\text{Ad}_g(f)(x) = f(g^{-1}xg).$$

Then class functions are those functions f invariant under G . We denote them by

$$\mathbb{C}[G]^{\text{Ad}G} := \{f : G \rightarrow \mathbb{C} \mid f(g^{-1}xg) = f(x) \text{ for } g, x \in G\}.$$

Proposition 10.1.6. Let (ρ_1, V_1) and (ρ_2, V_2) be two representations of G . Then for every $g \in G$,

- (1) $\chi_{\rho_1 \oplus \rho_2}(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$,
- (2) $\chi_{\rho_1 \otimes \rho_2}(g) = \chi_{\rho_1}(g) \cdot \chi_{\rho_2}(g)$.

Proof. For (1), it follows from that

$$(\rho_1 \oplus \rho_2)(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$$

For (2), we write matrix form of ρ_1 and ρ_2 by $\rho_1(g) = (A_{ij})_{1 \leq i, j \leq m} \in \text{GL}_m(\mathbb{C})$ and $\rho_2(g) = (A_{k\ell})_{1 \leq k, \ell \leq n} \in \text{GL}_n(\mathbb{C})$. Then

$$\text{Tr}(\rho_1(g)) = \sum_{i=1}^m A_{ii}, \quad \text{Tr}(\rho_2(g)) = \sum_{k=1}^n B_{kk}.$$

Then for this basis, the entry of $(\rho_1 \otimes \rho_2)(g)$ labeled by (i, k) and (j, ℓ) is $A_{ij}B_{k\ell}$. Thus

$$\text{Tr}((\rho_1 \otimes \rho_2)(g)) = \sum_{i=1}^m \sum_{k=1}^n A_{ii}B_{kk} = \text{Tr}(\rho_1(g)) \cdot \text{Tr}(\rho_2(g)).$$

This implies that $\chi_{\rho_1 \otimes \rho_2}(g) = \chi_{\rho_1}(g) \circ \chi_{\rho_2}(g)$. \square

Examples

For $S_3 = \langle x, y \mid x^3, y^3, xyxy \rangle$, we have three representations, A , Σ and T .

$$A_x = \begin{pmatrix} \cos \frac{2\pi}{3} & -\sin \frac{2\pi}{3} \\ \sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} \end{pmatrix}, \quad A_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Sigma_x = [1] \quad \Sigma_y = [-1]$$

$$T_x = [1] \quad T_y = [1]$$

Then the characters of these representations are displayed below in tabular form.

	1	x	x^2	y	xy	x^2y
χ_T	1	1	1	1	1	1
χ_Σ	1	1	1	-1	-1	-1
χ_A	2	-1	-1	0	0	0

Several interesting phenomena can be observed in this table:

- The rows form orthogonal vectors of length equal to six, which is also the order of S_3 . The columns are orthogonal too.
- $\chi_R(1)$ is the dimension of the representation, also called the dimension of the character.
 - Since a representation is a homomorphism, it sends the identity in the group to the identity matrix. So $\chi_R(1)$ is the trace of the identity matrix.
- The characters are constant on conjugacy classes.
 - The conjugacy classes in S_3 are the sets $\{1\}$, $\{x, x^2\}$, and $\{y, xy, x^2y\}$.
 - This is because conjugate matrices has the same trace.

2.9.6 Schur's orthogonality

Definition 10.2.1. For two complex valued functions χ, η on G , we put

$$\langle \chi, \eta \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g) \eta(g^{-1})$$

Theorem 10.2.3 (Schur's orthogonality). *If (ρ_1, V_1) and (ρ_2, V_2) are irreducible representations of G , then*

- (1) *when $(\rho_1, V_1) \not\simeq (\rho_2, V_2)$, then $\langle \chi_{V_1}, \chi_{V_2} \rangle = 0$;*
- (2) *when $(\rho_1, V_1) = (\rho_2, V_2)$, $\langle \chi_{V_1}, \chi_{V_1} \rangle = 1$.*

In other words, the characters of irreducible representations are orthonormal functions for the pairing $\langle -, - \rangle$.

We first establish an (somewhat surprisingly) extremely useful result, called the **Schur's lemma**. It is one of the most important tools in the study of representation theory. (We stick to the version concerning finite dimensional \mathbb{C} -vector space representations.)

Proposition 10.2.4 (Schur's lemma). *Let (ρ_1, V_1) and (ρ_2, V_2) be irreducible \mathbb{C} -representations of G and let $\phi : V_1 \rightarrow V_2$ be a homomorphism. Then*

- (1) *If (ρ_1, V_1) and (ρ_2, V_2) are not isomorphic, we have $\phi = 0$.*
- (2) *If $(\rho_1, V_1) = (\rho_2, V_2)$, ϕ is a homothety, i.e. a scalar multiple of the identity.*
位似变换

Proposition 10.2.6. *Assume that $(\rho_1, V_1) \not\simeq (\rho_2, V_2)$. Identify V_1 with \mathbb{C}^m and V_2 with \mathbb{C}^n , and thus, the two representations can be viewed as*

$$g \mapsto (\rho_1(g)_{ij})_{i,j=1,\dots,m} \quad \text{and} \quad g \mapsto (\rho_2(g)_{k\ell})_{k,\ell=1,\dots,n}.$$

Then, for any $i, j \in \{1, \dots, m\}$ and $k, \ell \in \{1, \dots, n\}$, we have

$$(10.2.6.1) \quad \langle \rho_1(-)_{ij}, \rho_2(-)_{k\ell} \rangle := \frac{1}{|G|} \sum_{g \in G} \rho_1(g^{-1})_{ij} \rho_2(g)_{k\ell} = 0.$$

Proposition 10.3.1. *For a general representation (ρ, V) of G with character χ_V , it is the direct sum of irreducible representations $V = W_1^{\oplus m_1} \oplus W_2^{\oplus m_2} \oplus \cdots \oplus W_r^{\oplus m_r}$ (where W_1, \dots, W_r are irreducible and pairwise non-isomorphic). Then*

$$\begin{aligned} \langle \chi_V, \chi_{W_i} \rangle &= \left\langle \sum_j m_j \chi_{W_j}, \chi_{W_i} \right\rangle = \sum_j m_j \cdot \begin{cases} 1 & \text{if } W_j \simeq W_i \\ 0 & \text{if } W_j \not\simeq W_i \end{cases} \\ &= m_i. \end{aligned}$$

Corollary 10.3.2. *Two representations with the same characters are isomorphic.*

Corollary 10.3.3. *For a representation (ρ, V) of G , factored as $V = W_1^{\oplus m_1} \oplus W_2^{\oplus m_2} \oplus \cdots \oplus W_r^{\oplus m_r}$, we have*

$$\langle \chi_V, \chi_V \rangle = m_1^2 + m_2^2 + \cdots + m_r^2 > 0.$$

Consequently, V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

第三章 交换代数

3.1 环和理想

参见 Atiyah, [Commutative_Algebra_Atiyah.pdf](#)

Exercice 1

设 A 是环, 令

$$f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$$

证明:

- (1) f 是 $A[x]$ 的可逆元 $\Leftrightarrow a_0$ 是 A 中可逆元且 a_1, \dots, a_n 是幂零元。
- (2) f 幂零 $\Leftrightarrow a_0, a_1, \dots, a_n$ 幂零。
- (3) f 是零因子 \Leftrightarrow 存在着环 A 的非 0 元 a 使得 $af = 0$ 。
- (4) 如果 $(a_0, \dots, a_n) = (1)$, f 就叫本原多项式。证明: 如果 $f, g \in A[x]$, 那么 fg 本原 $\Leftrightarrow f, g$ 均为本原。

证明. (1) \Leftarrow : 设 $a_i^{m_i} = 0, 1 \leq i \leq n$, 不妨 $a_0 = 1$, 否则考虑 $a_0^{-1}f$, 则 $(f - 1)^{m_1 + \cdots + m_n} = 0$, 记幂数为 M , 从而 $1 = 1 - (1 - f)^M = f((1 - f)^{M-1} + \cdots + 1)$ 。(实际上就是题 1)

\Rightarrow : 设 $fg = 1$, 且 $g(x) = b_0 + b_1x + \cdots + b_mx^m$, 令 $x = 0$, 可得 $a_0b_0 = 1$, 从而 a_0 可逆。

从而有 $\forall k, \sum_{i+j=k} a_i b_j = 0$ (线性代数的结论), 特别的 $a_n b_m = 0, a_n b_{m-1} + b_m a_{n-1} = 0$, 从而 $a_n^2 b_{m-1} = 0$, 如此往下, $a_n^{m+1} b_0 = 0$, 故 a_n 幂零。对其余同理。(由于 a_n 幂零, 则 $a_n x^n$ 幂零, 故 $f - a_n x^n = \sum_{i=0}^{n-1} a_i x^i =: f_1$ 依然可逆, 对 f_1 重复上述论证可得 a_{n-1} 幂零, 依此类推) \square

3.2 模与理想

Proposition 1

极大理想都是素理想。

Exercice 2

$$(N : P) = \text{Ann}((P + N)/N).$$

Note. 这里使用 $\text{Ann}((P + N)/N)$ 而不是 $\text{Ann}(P/N)$, 是因为后者只在 $N \subset P$ 的情况下有定义, 而前者总是良好定义的。

Exercice 3

2. 设 A 是一个环, \mathfrak{a} 是 A 的理想, M 是一个 A -模, 证明: 模 $A/\mathfrak{a} \otimes_A M$ 与 $M/\mathfrak{a}M$ 同构。

证明. 建立映射 $f: \bar{a} \otimes m \rightarrow \bar{am}$, 容易证明这是良定义的。

如果 $\bar{a} \otimes m \in \text{Ker}(f)$, 则 $am \in \mathfrak{a}M$, 即 $a \in \mathfrak{a}$, 则 $\bar{a} = 0$, 故 $\bar{a} \otimes m = 0$, 即 f 为同构。 \square

3. 设 A 是一个局部环, M 和 N 都是有限生成的 A -模, 证明: 如果 $M \otimes N = 0$, 那么 $M = 0$ 或者 $N = 0$ 。

证明. 记 A 的唯一极大理想为 \mathfrak{m} , 考虑 $B = A/\mathfrak{m}$ 为域, 有 $(B \otimes M) \otimes (B \otimes N) = 0$, 由题 2, $B \otimes M$ 同构于 $M/\mathfrak{m}M$, 记为 M_B 。

根据命题我们有 $\text{Hom}(M_B \otimes N_B, N_B) \cong \text{Hom}(M_B, \text{Hom}(N_B, N_B))$, 前者只包含平凡映射, 而后者只包含平凡映射说明 M_B 和 N_B 有一个为 0, 再根据 Nakayama 引理, 这说明 $M = 0$ 或 $N = 0$ 。 \square

解释 3 的证明:

这个证明的目的是要说明: 在一个局部环 A 上, 如果两个有限生成的 A -模 M 和 N 的张量积 $M \otimes_A N = 0$, 那么 M 和 N 中至少有一个是零模。

证明过程可以分解为以下几个步骤:

3.2.1 证明步骤详解

引入剩余域 (Residue Field)

设 A 是一个局部环, 其唯一的极大理想为 \mathfrak{m} 。

考虑 $B = A/\mathfrak{m}$ 。因为 \mathfrak{m} 是极大理想, 所以 B 是一个域, 称为 A 的剩余域。

张量积与剩余域

我们已知 $M \otimes_A N = 0$ 。

将这个等式两边都用 $B = A/\mathfrak{m}$ 在 A 上作张量积，可以得到：

$$(A/\mathfrak{m}) \otimes_A (M \otimes_A N) = (A/\mathfrak{m}) \otimes_A 0 = 0$$

利用张量积的结合律和换环定理 (change of rings, $R \otimes_A (M \otimes_A N) \cong (R \otimes_A M) \otimes_R (R \otimes_A N)$)，其中 R 是一个 A -代数)，这里 $R = B = A/\mathfrak{m}$ ，我们有：

$$((A/\mathfrak{m}) \otimes_A M) \otimes_{A/\mathfrak{m}} ((A/\mathfrak{m}) \otimes_A N) = 0$$

根据题目 2 的结论 $(A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M)$ ，我们知道：

- $B \otimes_A M \cong M/\mathfrak{m}M$ 。我们记 $M_B = M/\mathfrak{m}M$ 。
- $B \otimes_A N \cong N/\mathfrak{m}N$ 。我们记 $N_B = N/\mathfrak{m}N$ 。

因此，上面的等式变为 $M_B \otimes_B N_B = 0$ 。注意 M_B 和 N_B 都是域 B 上的向量空间。

向量空间的张量积

我们得到了 $M_B \otimes_B N_B = 0$ ，其中 M_B 和 N_B 是域 B 上的向量空间。

对于域上的向量空间 V, W ，如果 $V \otimes_B W = 0$ ，那么必然有 $V = 0$ 或者 $W = 0$ 。（如果 V, W 均非零，它们分别有非空基底，则 $V \otimes_B W$ 的基底由它们基底元素的张量积构成，因此 $V \otimes_B W$ 也非零。）

证明中使用了 Hom 函子的性质来推导这一结论：

- 利用张量-Hom 伴随 (tensor-hom adjunction):

$$\text{Hom}_B(M_B \otimes_B N_B, N_B) \cong \text{Hom}_B(M_B, \text{Hom}_B(N_B, N_B))$$

- 由于 $M_B \otimes_B N_B = 0$ ，那么左边的 $\text{Hom}_B(0, N_B)$ 只包含零映射（从零空间到 N_B 的唯一线性映射是零映射）。
- 因此，右边的 $\text{Hom}_B(M_B, \text{Hom}_B(N_B, N_B))$ 也必须为零（即只包含零映射）。
- 对于域 B 上的向量空间 X 和 Y ，如果 $\text{Hom}_B(X, Y) = 0$ ，那么 $X = 0$ 或者 $Y = 0$ 。（如果 $X \neq 0$ 且 $Y \neq 0$ ，我们可以构造一个非零的从 X 到 Y 的线性映射）。
- 所以，从 $\text{Hom}_B(M_B, \text{Hom}_B(N_B, N_B)) = 0$ 可以得出： $M_B = 0$ 或者 $\text{Hom}_B(N_B, N_B) = 0$ 。

- 如果 $\text{Hom}_B(N_B, N_B) = 0$, 这意味着从 N_B 到 N_B 的唯一线性映射是零映射。但如果 $N_B \neq 0$, 那么其上的恒等映射 id_{N_B} 就不是零映射。所以, $\text{Hom}_B(N_B, N_B) = 0$ 必然意味着 $N_B = 0$ 。
- 综上, 我们得出结论: $M_B = 0$ 或者 $N_B = 0$ 。也就是 $M/\mathfrak{m}M = 0$ 或者 $N/\mathfrak{m}N = 0$ 。

应用中山正引理 (Nakayama's Lemma)

我们已知 A 是一个局部环, \mathfrak{m} 是其唯一的极大理想。 M 和 N 是有限生成的 A -模。

中山正引理的一个形式是: 若 M 是环 A 上的有限生成模, I 是 A 的一个理想且 I 包含于 A 的 Jacobson 根 $J(A)$ 中。如果 $IM = M$, 则 $M = 0$ 。

对于局部环 A , 其 Jacobson 根就是其唯一的极大理想 \mathfrak{m} 。

所以:

- 如果 $M/\mathfrak{m}M = 0$, 这意味着 $M = \mathfrak{m}M$ 。由于 M 是有限生成的 A -模, 根据中山正引理, 可得 $M = 0$ 。
- 同理, 如果 $N/\mathfrak{m}N = 0$, 这意味着 $N = \mathfrak{m}N$ 。由于 N 是有限生成的 A -模, 根据中山正引理, 可得 $N = 0$ 。

结论

因为我们已经证明了 $M/\mathfrak{m}M = 0$ 或者 $N/\mathfrak{m}N = 0$, 所以根据中山正引理, 必然有 $M = 0$ 或者 $N = 0$ 。

这就完成了证明。这个证明巧妙地利用了局部环的性质、剩余域、张量积的性质以及关键的中山正引理。

Exercice 4

4. 设 M_i 是任意一组 A -模, M 是它们的直和, 证明: M 是平坦的 $\Leftrightarrow M_i$ 都是平坦的。

证明. M 是平坦的 \Leftrightarrow 如果 A -模同态 $f : N \rightarrow N'$ 单, 则 $id \otimes f : M \otimes N \rightarrow M \otimes N'$ 单 \Leftrightarrow 如果 A -模同态 $f : N \rightarrow N'$ 单, 则 $(\bigoplus M_i) \otimes N \rightarrow (\bigoplus M_i) \otimes N'$ 单 \Leftrightarrow 如果 A -模同态 $f : N \rightarrow N'$ 单, 则 $\bigoplus(M_i \otimes N) \rightarrow \bigoplus(M_i \otimes N')$ 单 \Leftrightarrow 每个分量映射单 $\Leftrightarrow M_i$ 平坦。 \square

5. 设 $A[x]$ 是环 A 上的一个变元的多项式环, 证明: $A[x]$ 是平坦 A -代数。

证明. 显然 $A[x]$ 是 A -代数, 只需 $A[x]$ 是平坦的。

事实上, $A[x] = \bigoplus(x^i)$, 且 $M \otimes (x^i) \cong M \otimes A \cong M$, 故 $f : M \rightarrow M'$ 单显然推出 $M \otimes (x^i) \rightarrow M' \otimes (x^i)$ 单, 从而 (x^i) 是平坦的, 所以 $A[x]$ 是平坦的。 \square

证明中指出” $A[x] = \bigoplus_{i \geq 0} (x^i)$ ”。这里 (x^i) 指的是由 x^i 生成的 A -子模，即 $A \cdot x^i = \{ax^i \mid a \in A\}$ 。

作为 A -模， $A[x]$ 是所有形如 $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ （其中 $a_j \in A$ ）的多项式的集合。它可以看作是 A -模 $A \cdot 1, A \cdot x, A \cdot x^2, \dots$ 的直和：

$$A[x] = A \cdot 1 \oplus A \cdot x \oplus A \cdot x^2 \oplus \cdots = \bigoplus_{i \geq 0} A \cdot x^i.$$

这意味着 $A[x]$ 是一个自由 A -模，其基为 $\{1, x, x^2, \dots\}$ 。

第四章 Root Systems - Chapter

3 Study Guide

4.1 Table of Contents

- (1) Overview and Motivation
- (2) Section 9: Axiomatics
- (3) Section 10: Simple Roots and Weyl Group
- (4) Section 11: Classification
- (5) Section 12: Construction and Automorphisms
- (6) Section 13: Abstract Theory of Weights
- (7) Key Relationships and Dependencies

4.2 Overview and Motivation

Root systems are fundamental combinatorial-geometric objects that capture the essence of semisimple Lie algebras. They provide a way to understand the structure of these algebras through finite sets of vectors in Euclidean space that satisfy certain axioms.

Main Goals of This Chapter:

- (1) Define root systems axiomatically
- (2) Develop the theory of simple roots and Weyl groups
- (3) Classify all possible root systems
- (4) Show how to construct each type explicitly
- (5) Develop the weight theory for representations

4.3 Section 9: Axiomatics

4.3.1 9.1 Reflections in Euclidean Space

Definition 9.1.1 (Euclidean Space)

Throughout this chapter, E is a **euclidean space**: a finite-dimensional vector space over \mathbb{R} endowed with a positive definite symmetric bilinear form (α, β) .

Definition 9.1.2 (Reflection - Geometric)

A **reflection** in E is an invertible linear transformation that:

- Leaves pointwise fixed some hyperplane (subspace of codimension one)
- Sends any vector orthogonal to that hyperplane to its negative

Note: A reflection is orthogonal (preserves the inner product on E).

Definition 9.1.3 (Reflection σ_α - Algebraic)

For any nonzero vector $\alpha \in E$, the reflection σ_α is defined by:

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$$

The reflecting hyperplane is $P_\alpha = \{\beta \in E \mid (\beta, \alpha) = 0\}$.

Notation

We define $\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$.

Important: $\langle \beta, \alpha \rangle$ is linear only in the first variable.

Lemma 9.1.3

Let Φ be a finite set which spans E . Suppose all reflections σ_α ($\alpha \in \Phi$) leave Φ invariant. If $\sigma \in GL(E)$ leaves Φ invariant, fixes pointwise a hyperplane P of E , and sends some nonzero $\alpha \in \Phi$ to its negative, then $\sigma = \sigma_\alpha$ (and $P = P_\alpha$).

Proof: Let $\tau = \sigma\sigma_\alpha$. Then:

- $\tau(\Phi) = \Phi$
- $\tau(\alpha) = \alpha$

- τ acts as identity on $\mathbb{R}\alpha$ and on $E/\mathbb{R}\alpha$

This forces all eigenvalues of τ to be 1, so the minimal polynomial divides $(T - 1)^\ell$.

Since Φ is finite, some power τ^k fixes all elements of Φ . Since Φ spans E , we have $\tau^k = 1$, so the minimal polynomial divides $T^k - 1$.

Combining these conditions, the minimal polynomial is $T - 1$, hence $\tau = 1$.

4.3.2 9.2 Root Systems

Definition 9.2.1 (Root System)

A subset Φ of the euclidean space E is called a **root system** in E if the following axioms are satisfied:

- (R1) Φ is finite, spans E , and does not contain 0.
- (R2) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.
- (R3) If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.
- (R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

Definition 9.2.2 (Weyl Group)

The **Weyl group** \mathcal{W} of Φ is the subgroup of $GL(E)$ generated by the reflections σ_α ($\alpha \in \Phi$).

Lemma 9.2.3

Let Φ be a root system in E , with Weyl group \mathcal{W} . If $\sigma \in GL(E)$ leaves Φ invariant, then:

- (1) $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ for all $\alpha \in \Phi$
- (2) $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ for all $\alpha, \beta \in \Phi$

Proof: The key observation is that $\sigma\sigma_\alpha\sigma^{-1}$ acts as a reflection with respect to $\sigma(\alpha)$. By Lemma 9.1.3, this must equal $\sigma_{\sigma(\alpha)}$.

Computing explicitly:

$$\sigma\sigma_\alpha\sigma^{-1}(\sigma(\beta)) = \sigma(\beta - \langle \beta, \alpha \rangle \alpha) = \sigma(\beta) - \langle \beta, \alpha \rangle \sigma(\alpha)$$

This gives us the second assertion by comparing with the formula for $\sigma_{\sigma(\alpha)}$.

Definition 9.2.4 (Dual Root System)

The **dual** (or inverse) of Φ is $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$, where $\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$.

4.3.3 9.3 Examples

Definition 9.3.1 (Rank of Root System)

Call $\ell = \dim E$ the **rank** of the root system Φ .

When $\ell \leq 2$, we can describe Φ by drawing pictures. In view of (R2), there is only one possibility when $\ell = 1$, labeled (A_1) .

Rank 1: Type A_1

$$\Phi = \{\alpha, -\alpha\}$$

This is the unique root system of rank 1.

Rank 2: Four Types

- (1) **Type $A_1 \times A_1$** : Two orthogonal copies of A_1
- (2) **Type A_2** : $\Phi = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3)\}$
- (3) **Type B_2** : Contains roots of two different lengths
- (4) **Type G_2** : Contains roots of two different lengths with ratio $\sqrt{3}$

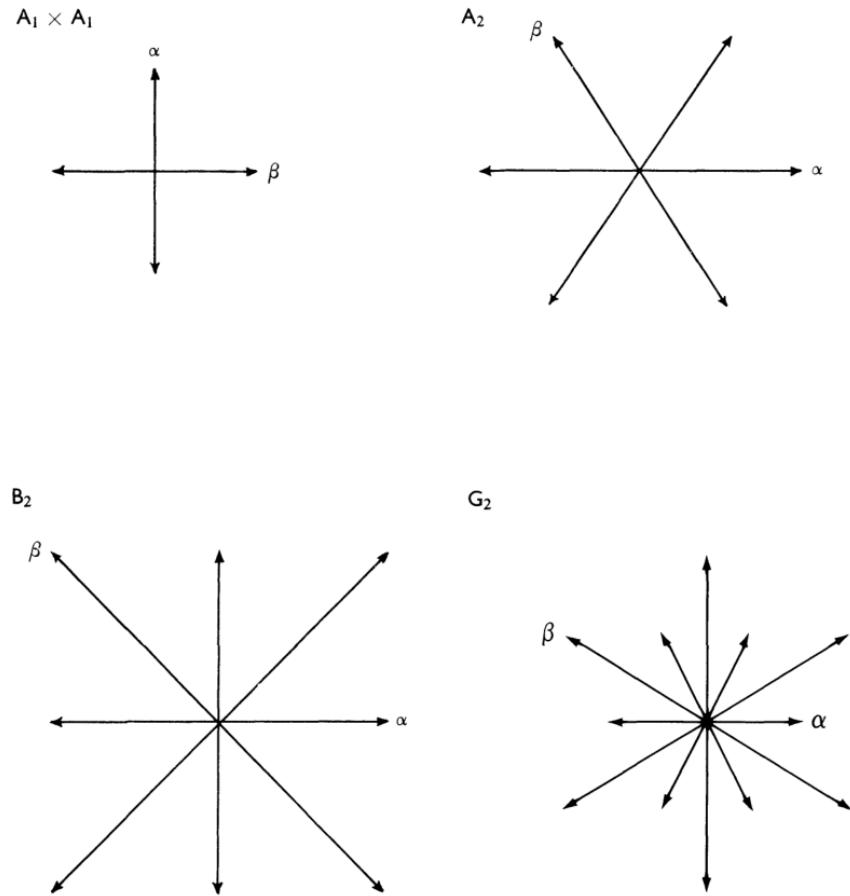


Figure 1

4.3.4 9.4 Pairs of Roots

Definition 9.4.1 (Angle Between Vectors)

The **angle** θ between vectors $\alpha, \beta \in E$ is defined by the formula:

$$\|\alpha\| \|\beta\| \cos \theta = (\alpha, \beta)$$

Equivalently, $\cos \theta = \frac{(\alpha, \beta)}{\|\alpha\| \|\beta\|}$.

This gives us:

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = 2 \frac{\|\beta\|}{\|\alpha\|} \cos \theta$$

and

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4 \cos^2 \theta$$

Lemma 9.4.1 (Angle Restrictions)

The possible values of $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ for nonproportional roots α, β are 0, 1, 2, or 3, corresponding to specific angles and length ratios.

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	$\ \beta\ ^2 / \ \alpha\ ^2$	θ
0	0	undetermined	$\pi/2$
1	1	1	$\pi/3$
-1	-1	1	$2\pi/3$
1	2	2	$\pi/4$
-1	-2	2	$3\pi/4$
1	3	3	$\pi/6$
-1	-3	3	$5\pi/6$

Lemma 9.4.2 (Root Addition)

Let α, β be nonproportional roots. If $\langle \alpha, \beta \rangle > 0$, then $\alpha - \beta$ is a root. If $\langle \alpha, \beta \rangle < 0$, then $\alpha + \beta$ is a root.

Proof: If $\langle \alpha, \beta \rangle > 0$, then $\langle \alpha, \beta \rangle > 0$. From the table above, either $\langle \alpha, \beta \rangle = 1$ or $\langle \beta, \alpha \rangle = 1$.

If $\langle \alpha, \beta \rangle = 1$, then $\sigma_\beta(\alpha) = \alpha - \beta \in \Phi$ by (R3). If $\langle \beta, \alpha \rangle = 1$, then $\beta - \alpha \in \Phi$, so $\sigma_{\beta-\alpha}(\beta - \alpha) = \alpha - \beta \in \Phi$.

Root Strings

For nonproportional roots α, β , the **α -string through β** consists of all roots of the form $\beta + i\alpha$ ($i \in \mathbb{Z}$). This string is unbroken and has the form:

$$\beta - r\alpha, \beta - (r-1)\alpha, \dots, \beta, \dots, \beta + q\alpha$$

where $r - q = \langle \beta, \alpha \rangle$.

4.4 Section 10: Simple Roots and Weyl Group

Throughout this section Φ denotes a root system of rank ℓ in a euclidean space E , with Weyl group \mathcal{W} .

4.4.1 10.1 Bases and Weyl Chambers

Definition 10.1.1 (Base/Simple Roots)

A subset Δ of Φ is called a **base** if:

(B1) Δ is a basis of E

(B2) Each root $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha \in \Delta} k_{\alpha} \alpha$ with integral coefficients k_{α} all nonnegative or all nonpositive.

The roots in Δ are called **simple roots**.

Definition 10.1.2 (Positive/Negative Roots, Height)

Relative to a base Δ :

- A root β is **positive** (written $\beta \succ 0$) if all coefficients $k_{\alpha} \geq 0$
- A root β is **negative** (written $\beta \prec 0$) if all coefficients $k_{\alpha} \leq 0$
- The **height** of β is $\text{ht}(\beta) = \sum k_{\alpha}$

The collections of positive and negative roots are denoted Φ^+ and Φ^- (clearly $\Phi^- = -\Phi^+$).

Lemma 10.1.3

If Δ is a base of Φ , then $(\alpha, \beta) \leq 0$ for $\alpha \neq \beta$ in Δ , and $\alpha - \beta$ is not a root.

Proof: If $(\alpha, \beta) > 0$ for distinct simple roots α, β , then by Lemma 9.4.2, $\alpha - \beta$ would be a root. But this contradicts (B2) since when we express $\alpha - \beta$ in terms of the base Δ , we get $\alpha - \beta = 1 \cdot \alpha + (-1) \cdot \beta + 0 \cdot \gamma$ (for other simple roots γ), which has both positive coefficient (+1) and negative coefficient (-1). However, (B2) requires that all coefficients be either all nonnegative or all nonpositive.

Definition 10.1.4 (Weyl Chambers and Related Concepts)

The hyperplanes P_{α} ($\alpha \in \Phi$) partition E into finitely many regions. The connected components of $E - \bigcup_{\alpha} P_{\alpha}$ are called **Weyl chambers**.

For each vector $\gamma \in E$, define:

$$\Phi^+(\gamma) = \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\}$$

A vector $\gamma \in E$ is **regular** if $\gamma \in E - \bigcup_{\alpha \in \Phi} P_\alpha$ and **singular** otherwise.

For regular $\gamma \in E$:

- A root $\alpha \in \Phi^+(\gamma)$ is **decomposable** if $\alpha = \beta_1 + \beta_2$ for some $\beta_i \in \Phi^+(\gamma)$
- A root is **indecomposable** otherwise
- $\Delta(\gamma) = \text{set of indecomposable roots in } \Phi^+(\gamma)$

Theorem 10.1.5 (Existence of Bases)

Φ has a base.

More precisely, for any regular $\gamma \in E$, the set $\Delta(\gamma)$ is a base of Φ , and every base is obtainable in this manner.

Proof Outline:

- (1) Show each positive root is a nonnegative \mathbb{Z} -combination of indecomposable roots
- (2) Prove indecomposable roots have pairwise obtuse angles
- (3) Establish linear independence of indecomposable roots
- (4) Show this gives a base and that every base arises this way

4.4.2 10.2 Lemmas on Simple Roots

Lemma 10.2.1 (Lemma A)

If α is positive but not simple, then $\alpha - \beta$ is a root (necessarily positive) for some $\beta \in \Delta$.

Lemma 10.2.2 (Lemma B)

Let α be simple. Then σ_α permutes the positive roots other than α .

Corollary: Setting $\delta = \frac{1}{2} \sum_{\beta > 0} \beta$, we have $\sigma_\alpha(\delta) = \delta - \alpha$ for all $\alpha \in \Delta$.

Lemma 10.2.3 (Lemma C)

Let $\alpha_1, \dots, \alpha_t \in \Delta$. Write $\sigma_i = \sigma_{\alpha_i}$. If $\sigma_1 \cdots \sigma_{t-1}(\alpha_t)$ is negative, then for some index $1 \leq s < t$:

$$\sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1}$$

Corollary: If $\sigma = \sigma_1 \cdots \sigma_t$ is a minimal expression for $\sigma \in \mathcal{W}$, then $\sigma(\alpha_t) \prec 0$.

4.4.3 10.3 The Weyl Group

Theorem 10.3.1 (Main Properties of Weyl Group)

Let Δ be a base of Φ .

- (a) \mathcal{W} acts transitively on Weyl chambers.
- (b) \mathcal{W} acts transitively on bases.
- (c) If α is any root, there exists $\sigma \in \mathcal{W}$ such that $\sigma(\alpha) \in \Delta$.
- (d) \mathcal{W} is generated by the σ_α ($\alpha \in \Delta$).
- (e) \mathcal{W} acts simply transitively on bases.

Definition 10.3.2 (Length Function)

For $\sigma \in \mathcal{W}$, the **length** $\ell(\sigma)$ is the smallest number of simple reflections needed to express σ .

Lemma 10.3.3 (Length Formula)

For all $\sigma \in \mathcal{W}$, $\ell(\sigma) = n(\sigma)$, where $n(\sigma)$ is the number of positive roots α for which $\sigma(\alpha) \prec 0$.

4.4.4 10.4 Irreducible Root Systems

Definition 10.4.1 (Irreducible Root System)

Φ is **irreducible** if it cannot be partitioned into the union of two proper subsets such that each root in one set is orthogonal to each root in the other.

Lemma 10.4.2

Φ is irreducible if and only if Δ cannot be partitioned into orthogonal subsets.

Lemma 10.4.3 (Unique Maximal Root)

Let Φ be irreducible. There is a unique maximal root β (relative to the partial ordering), and if $\beta = \sum k_\alpha \alpha$, then all $k_\alpha > 0$.

Lemma 10.4.4 (Irreducible Action)

Let Φ be irreducible. Then \mathcal{W} acts irreducibly on E .

Lemma 10.4.5 (At Most Two Root Lengths)

Let Φ be irreducible. Then at most two root lengths occur in Φ , and all roots of a given length are conjugate under \mathcal{W} .

4.5 Section 11: Classification

4.5.1 11.1 Cartan Matrix

Definition 11.1.1 (Cartan Matrix)

For a base $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$, the **Cartan matrix** is $(\langle \alpha_i, \alpha_j \rangle)$.

Proposition 11.1.2

The Cartan matrix determines Φ up to isomorphism.

Proof: Given two root systems with the same Cartan matrix, there exists a unique vector space isomorphism preserving all Cartan integers, and this extends to an isomorphism of root systems.

4.5.2 11.2 Coxeter Graphs and Dynkin Diagrams

Definition 11.2.1 (Coxeter Graph)

The **Coxeter graph** of Φ has ℓ vertices, with vertices i and j joined by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges.

Definition 11.2.2 (Dynkin Diagram)

When multiple edges occur, add an arrow pointing to the shorter root. This Dynkin diagram completely determines the Cartan matrix.

4.5.3 11.3 Irreducible Components

Proposition 11.3.1

Φ decomposes uniquely as the union of irreducible root systems Φ_i in orthogonal subspaces E_i such that $E = E_1 \oplus \dots \oplus E_t$.

4.5.4 11.4 Classification Theorem

Theorem 11.4.1 (Classification)

If Φ is an irreducible root system of rank ℓ , its Dynkin diagram is one of the following:

- **Type A_ℓ** ($\ell \geq 1$): $\circ - \circ - \dots - \circ$
- **Type B_ℓ** ($\ell \geq 2$): $\circ - \circ - \dots - \circ \Rightarrow \circ$
- **Type C_ℓ** ($\ell \geq 3$): $\circ - \circ - \dots - \circ \Leftarrow \circ$
- **Type D_ℓ** ($\ell \geq 4$): Branched diagram
- **Type E_6, E_7, E_8** : Exceptional cases
- **Type F_4** : $\circ - \circ \Rightarrow \circ - \circ$
- **Type G_2** : $\circ \gg \circ$ (triple bond)

Proof Strategy:

- (1) Classify possible Coxeter graphs using geometric constraints
- (2) Show these are the only connected graphs of "admissible" vectors
- (3) Determine which Dynkin diagrams result from each Coxeter graph

4.6 Section 12: Construction and Automorphisms

4.6.1 12.1 Construction of Types A-G

Each type can be explicitly constructed:

Type A_ℓ

- $E = \text{subspace of } \mathbb{R}^{\ell+1} \text{ orthogonal to } \varepsilon_1 + \cdots + \varepsilon_{\ell+1}$
- $\Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$
- Base: $\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_\ell - \varepsilon_{\ell+1}\}$

Type B_ℓ

- $E = \mathbb{R}^\ell$
- $\Phi = \{\pm \varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid i \neq j\}$
- Base: $\{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{\ell-1} - \varepsilon_\ell, \varepsilon_\ell\}$

[Similar constructions for other types...]

4.6.2 12.2 Automorphisms

Theorem 12.2.1

$\text{Aut}(\Phi)$ is the semidirect product of \mathcal{W} and Γ , where Γ is the group of diagram automorphisms.

For irreducible Φ :

- $\Gamma = \mathbb{Z}/2\mathbb{Z}$ for types A_ℓ ($\ell \geq 2$), D_ℓ ($\ell > 4$), E_6
- $\Gamma = S_3$ for type D_4
- $\Gamma = 1$ for all other types

4.7 Section 13: Abstract Theory of Weights

4.7.1 13.1 Weights

Definition 13.1.1 (Weight)

An element $\lambda \in E$ is called a **weight** if $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$.

Definition 13.1.2 (Weight Lattice)

The **weight lattice** is $\Lambda = \{\lambda \in E \mid \langle \lambda, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$.

Definition 13.1.3 (Root Lattice)

The **root lattice** Λ_r is the subgroup of Λ generated by Φ (i.e., the \mathbb{Z} -span of any set of simple roots).

Definition 13.1.4 (Dominant and Strongly Dominant Weights)

Fix a base $\Delta \subset \Phi$. A weight $\lambda \in \Lambda$ is:

- **dominant** if all integers $\langle \lambda, \alpha \rangle$ ($\alpha \in \Delta$) are nonnegative
- **strongly dominant** if these integers are positive

Let Λ^+ denote the set of all dominant weights.

Geometric Interpretation: Λ^+ is the set of all weights lying in the closure of the fundamental Weyl chamber $\mathcal{C}(\Delta)$, while $\Lambda \cap \mathcal{C}(\Delta)$ is the set of strongly dominant weights.

Definition 13.1.5 (Fundamental Dominant Weights)

If $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$, the **fundamental dominant weights** $\lambda_1, \dots, \lambda_\ell$ are the dual basis to $\{2\alpha_i/(\alpha_i, \alpha_i)\}$ (relative to the inner product on E):

$$\frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$$

Key Property: Λ is a lattice with basis $\{\lambda_i \mid 1 \leq i \leq \ell\}$, and $\lambda \in \Lambda^+$ if and only if $\lambda = \sum m_i \lambda_i$ with all $m_i \geq 0$.

Definition 13.1.6 (Fundamental Group)

The quotient Λ/Λ_r is finite and called the **fundamental group** of the root system. Its order equals the determinant of the Cartan matrix.

4.7.2 13.2 Dominant Weights

Definition 13.2.1 (Partial Order on Weights)

For $\lambda, \mu \in \Lambda$, write $\mu \prec \lambda$ if $\lambda - \mu$ is a sum of positive roots (or equals zero).

4.7.3 13.3 The Weight δ

Definition 13.3.1 (The Weight)

Define $\delta = \frac{1}{2} \sum_{\alpha > 0} \alpha = \sum_{i=1}^{\ell} \lambda_i$.

Key Properties:

- δ is strongly dominant
- $\sigma_i(\delta) = \delta - \alpha_i$ for all simple reflections σ_i

4.7.4 13.4 Saturated Sets of Weights

Definition 13.4.1 (Saturated Set)

A subset $\Pi \subset \Lambda$ is **saturated** if for all $\lambda \in \Pi$, $\alpha \in \Phi$, and $0 \leq i \leq \langle \lambda, \alpha \rangle$, the weight $\lambda - i\alpha$ also lies in Π .

Note: Any saturated set is automatically stable under the Weyl group \mathcal{W} .

Definition 13.4.2 (Highest Weight)

A saturated set Π has **highest weight** λ ($\lambda \in \Lambda^+$) if $\lambda \in \Pi$ and $\mu \prec \lambda$ for all $\mu \in \Pi$.

Definition 13.4.3 (Minimal Weight)

A weight $\lambda \in \Lambda^+$ is **minimal** if $\mu \in \Lambda^+$ and $\mu \prec \lambda$ implies $\mu = \lambda$.

Characterization: λ is minimal if and only if the \mathcal{W} -orbit of λ is saturated (with highest weight λ), if and only if $\lambda \in \Lambda^+$ and $\langle \lambda, \alpha \rangle \in \{-1, 0, 1\}$ for all roots α .

4.8 Key Relationships and Dependencies

4.8.1 Logical Flow of Main Results

- (1) Axioms (R1)-(R4) → Basic properties of root systems
- (2) Reflection geometry → Weyl group structure
- (3) Base existence → Positive/negative root decomposition
- (4) Simple root properties → Weyl group generation
- (5) Geometric constraints → Classification theorem

- (6) **Explicit constructions** → Verification of all types exist
- (7) **Weight lattice structure** → Representation theory foundations

4.8.2 Critical Dependencies

- **Lemma 9.4.2** (root addition) is crucial for understanding root strings
- **Theorem 10.1.5** (base existence) underlies all subsequent theory
- **Theorem 10.3.1** (Weyl group properties) is essential for classification
- **Theorem 11.4.1** (classification) is the culmination of the geometric analysis
- **Section 13** provides the foundation for representation theory

4.8.3 Key Proof Techniques

- (1) **Geometric arguments** using angles and inner products
- (2) **Induction on height** for properties of positive roots
- (3) **Minimality arguments** for Weyl group elements
- (4) **Lattice theory** for weight spaces
- (5) **Graph theory** for classification of Dynkin diagrams

This completes the comprehensive study guide for Chapter 3 on Root Systems. Each concept builds systematically on previous ones, culminating in the complete classification and the foundation for representation theory.

第五章 根系统 Root Systems - 第 三章要点总结

5.1 第 9 节：公理化理论 Axiomatics

5.1.1 9.1 欧几里得空间中的反射 Reflections in Euclidean Space

定义 9.1.1 (反射 Reflection)

设 E 是有限维实向量空间，带有正定对称双线性形式 (α, β) 。反射 (Reflection) 是可逆线性变换，满足：

- 逐点固定某个超平面（余维数 1 的子空间）
- 将该超平面的正交向量映为其负向量

定义 9.1.2 (反射 σ_α)

对任意非零向量 $\alpha \in E$ ，反射 σ_α 定义为：

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$$

反射超平面为 $P_\alpha = \{\beta \in E \mid (\beta, \alpha) = 0\}$

记号

定义 $\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$

重要： $\langle \beta, \alpha \rangle$ 仅对第一个变量线性

引理 9.1.3 (反射的唯一性)

设 Φ 是有限集合且张成 E , 所有反射 σ_α ($\alpha \in \Phi$) 保持 Φ 不变。若 $\sigma \in GL(E)$ 保持 Φ 不变, 逐点固定超平面 P , 并将某个非零 $\alpha \in \Phi$ 映为其负向量, 则 $\sigma = \sigma_\alpha$ 。

5.1.2 9.2 根系统 Root Systems

定义 9.2.1 (根系统 Root System)

欧几里得空间 E 的子集 Φ 称为 **根系统 (Root System)**, 若满足以下公理:

(R1) Φ 有限, 张成 E , 且不含 0 (R2) 若 $\alpha \in \Phi$, 则 Φ 中 α 的倍数只有 $\pm\alpha$ (R3) 若 $\alpha \in \Phi$, 则反射 σ_α 保持 Φ 不变 (R4) 若 $\alpha, \beta \in \Phi$, 则 $\langle \beta, \alpha \rangle \in \mathbb{Z}$

注记: (R2) 和 (R3) 均蕴含 $\Phi = -\Phi$

定义 9.2.2 (魏尔群 Weyl Group)

Φ 的 **魏尔群 (Weyl Group)** \mathcal{W} 是由所有反射 σ_α ($\alpha \in \Phi$) 生成的 $GL(E)$ 的子群。

由 (R3), \mathcal{W} 置换集合 Φ ; 由 (R1), \mathcal{W} 有限。

引理 9.2.3 (自同构的性质)

设 Φ 是根系统, \mathcal{W} 为其魏尔群。若 $\sigma \in GL(E)$ 保持 Φ 不变, 则:

- (1) $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ 对所有 $\alpha \in \Phi$
- (2) $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ 对所有 $\alpha, \beta \in \Phi$

定义 9.2.4 (对偶根系统 Dual Root System)

Φ 的 **对偶 (Dual)** 或 **逆 (Inverse)** 为 $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$, 其中 $\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$ 。

5.1.3 9.3 例子 Examples

秩 1: 型 A_1

$\Phi = \{\alpha, -\alpha\}$ 是唯一的秩 1 根系统

秩 2: 四种类型

- (1) 型 $A_1 \times A_1$: 两个正交的 A_1 副本
- (2) 型 A_2 : $\Phi = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3)\}$
- (3) 型 B_2 : 包含两种不同长度的根
- (4) 型 G_2 : 包含两种长度的根, 比例为 $\sqrt{3}$

5.1.4 9.4 根对 Pairs of Roots

引理 9.4.1 (角度限制)

对不成比例的根 α, β , $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ 的可能值为 0, 1, 2, 3, 对应特定的角度和长度比:

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	$\ \beta\ ^2 / \ \alpha\ ^2$	θ
0	0	不定	$\pi/2$
1	1	1	$\pi/3$
-1	-1	1	$2\pi/3$
1	2	2	$\pi/4$
-1	-2	2	$3\pi/4$
1	3	3	$\pi/6$
-1	-3	3	$5\pi/6$

引理 9.4.2 (根的加法 Root Addition)

设 α, β 是不成比例的根:

- 若 $\langle \alpha, \beta \rangle > 0$ (即角度严格锐角), 则 $\alpha - \beta$ 是根
- 若 $\langle \alpha, \beta \rangle < 0$, 则 $\alpha + \beta$ 是根

根串 Root Strings

对不成比例的根 α, β , 通过 β 的 α -串 (α -string) 由所有形如 $\beta + i\alpha$ ($i \in \mathbb{Z}$) 的根组成。设 $r, q \in \mathbb{Z}^+$ 是最大整数使得 $\beta - r\alpha \in \Phi$, $\beta + q\alpha \in \Phi$, 则:

- (1) α -串是不间断的: $\beta - r\alpha, \beta - (r-1)\alpha, \dots, \beta + q\alpha$
- (2) $r - q = \langle \beta, \alpha \rangle$
- (3) 根串长度至多为 4

5.2 第 10 节: 简单根与魏尔群 Simple Roots and Weyl Group

5.2.1 10.1 基与魏尔腔 Bases and Weyl Chambers

定义 10.1.1 (基 Base / 简单根 Simple Roots)

Φ 的子集 Δ 称为 基 (Base), 若满足:

(B1) Δ 是 E 的基 (B2) 每个根 β 可写成 $\beta = \sum k_\alpha \alpha$ ($\alpha \in \Delta$), 整系数 k_α 全非负或全非正

Δ 中的根称为 简单根 (Simple Roots)

定义 10.1.2 (正根/负根, 高度)

相对于基 Δ :

- 根 β 是 正的 (Positive) (记作 $\beta \succ 0$) 若所有 $k_\alpha \geq 0$
- 根 β 是 负的 (Negative) (记作 $\beta \prec 0$) 若所有 $k_\alpha \leq 0$
- β 的 高度 (Height) 为 $\text{ht}(\beta) = \sum k_\alpha$

记 $\Phi^+ = \{\beta \in \Phi \mid \beta \succ 0\}$, $\Phi^- = \{\beta \in \Phi \mid \beta \prec 0\}$

引理 10.1.3 (简单根的角度性质)

若 Δ 是 Φ 的基, 则对 Δ 中不同元素 $\alpha \neq \beta$:

- (1) $(\alpha, \beta) \leq 0$
- (2) $\alpha - \beta$ 不是根

定义 10.1.4 (魏尔腔 Weyl Chambers)

超平面 P_α ($\alpha \in \Phi$) 将 E 分割成有限多个区域。 $E - \bigcup_\alpha P_\alpha$ 的连通分支称为 魏尔腔 (Weyl Chambers)。

对正则向量 $\gamma \in E$ (即不在任何 P_α 上), 定义:

- $\Phi^+(\gamma) = \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\}$
- $\Delta(\gamma) = \Phi^+(\gamma)$ 中不可分解根的集合

定理 10.1.5 (基的存在性)

定理: Φ 有基。

更精确地: 对任意正则 $\gamma \in E$, 集合 $\Delta(\gamma)$ 是 Φ 的基, 且每个基都以此方式得到。

5.2.2 10.2 关于简单根的引理 Lemmas on Simple Roots

引理 10.2.1 (引理 A)

若 α 是正根但非简单根, 则对某个 $\beta \in \Delta$, $\alpha - \beta$ 是根 (必为正根)。

引理 10.2.2 (引理 B)

设 α 是简单根, 则 σ_α 置换除 α 外的正根。

推论: 设 $\delta = \frac{1}{2} \sum_{\beta > 0} \beta$, 则对所有 $\alpha \in \Delta$: $\sigma_\alpha(\delta) = \delta - \alpha$

引理 10.2.3 (引理 C)

设 $\alpha_1, \dots, \alpha_t \in \Delta$ (不必不同), 记 $\sigma_i = \sigma_{\alpha_i}$ 。若 $\sigma_1 \cdots \sigma_{t-1}(\alpha_t)$ 是负根, 则存在某个指标 $1 \leq s < t$ 使得:

$$\sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1}$$

推论: 若 $\sigma = \sigma_1 \cdots \sigma_t$ 是 $\sigma \in \mathcal{W}$ 的最小表达式, 则 $\sigma(\alpha_t) \prec 0$

5.2.3 10.3 魏尔群 The Weyl Group

定理 10.3.1 (魏尔群的主要性质)

设 Δ 是 Φ 的基, 则:

- (a) \mathcal{W} 在魏尔腔上传递作用
- (b) \mathcal{W} 在基上传递作用
- (c) 若 α 是任意根, 存在 $\sigma \in \mathcal{W}$ 使得 $\sigma(\alpha) \in \Delta$
- (d) \mathcal{W} 由 σ_α ($\alpha \in \Delta$) 生成
- (e) \mathcal{W} 在基上单传递作用

定义 10.3.2 (长度函数 Length Function)

对 $\sigma \in \mathcal{W}$, 长度 (Length) $\ell(\sigma)$ 是表示 σ 所需简单反射的最少个数。

引理 10.3.3 (长度公式 Length Formula)

对所有 $\sigma \in \mathcal{W}$: $\ell(\sigma) = n(\sigma)$, 其中 $n(\sigma)$ 是被 σ 映为负根的正根个数。

引理 10.3.4 (基本域)

设 $\lambda, \mu \in \overline{\mathfrak{C}(\Delta)}$ (基本魏尔腔的闭包)。若 $\sigma\lambda = \mu$ 对某个 $\sigma \in \mathcal{W}$, 则 σ 是固定 λ 的简单反射的乘积; 特别地, $\lambda = \mu$ 。

5.2.4 10.4 不可约根系统 Irreducible Root Systems

定义 10.4.1 (不可约根系统)

Φ 是不可约的 (Irreducible), 若不能分解为两个非平凡正交子集的并。

引理 10.4.2

Φ 不可约当且仅当 Δ 不能分解为正交子集。

引理 10.4.3 (唯一最高根)

设 Φ 不可约, 则存在唯一最高根 β (相对于偏序 \prec), 且若 $\beta = \sum k_\alpha \alpha$, 则所有 $k_\alpha > 0$ 。

引理 10.4.4 (不可约作用)

设 Φ 不可约, 则 \mathcal{W} 在 E 上不可约作用。特别地, 根的 \mathcal{W} -轨道张成 E 。

引理 10.4.5 (最多两种根长)

设 Φ 不可约, 则 Φ 中最多出现两种根长, 且给定长度的所有根在 \mathcal{W} 下共轭。

注记: 若有两种根长, 称为长根和短根。习惯上, 若所有根等长, 称为长根。

5.3 第 11 节: 分类 Classification

5.3.1 11.1 嘉当矩阵 Cartan Matrix

定义 11.1.1 (嘉当矩阵)

对基 $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$, 嘉当矩阵 (Cartan Matrix) 为 $(\langle \alpha_i, \alpha_j \rangle)$ 。

其元素称为 **嘉当整数 (Cartan Integers)**。

命题 11.1.2 (嘉当矩阵确定同构类)

嘉当矩阵完全确定根系统的同构类。

证明思路: 相同嘉当矩阵的两个根系统间存在唯一向量空间同构, 保持所有嘉当整数, 从而扩展为根系统同构。

5.3.2 11.2 考克斯特图与戴金图

定义 11.2.1 (考克斯特图 Coxeter Graph)

Φ 的 **考克斯特图 (Coxeter Graph)** 有 ℓ 个顶点, 顶点 i 和 j ($i \neq j$) 之间有 $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ 条边。

定义 11.2.2 (戴金图 Dynkin Diagram)

当出现多重边时, 加箭头指向较短根。这个 **戴金图 (Dynkin Diagram)** 完全确定嘉当矩阵。

5.3.3 11.3 不可约分解

命题 11.3.1 (不可约分解)

Φ 唯一分解为不可约根系统 Φ_i 在正交子空间 E_i 中的并, 使得 $E = E_1 \oplus \cdots \oplus E_t$ (正交直和)。

5.3.4 11.4 分类定理

定理 11.4.1 (完全分类)

若 Φ 是秩 ℓ 的不可约根系统, 其戴金图恰好是以下之一:

- A_ℓ 型 ($\ell \geq 1$): $\circ - \circ - \cdots - \circ$ (直链)
- B_ℓ 型 ($\ell \geq 2$): $\circ - \circ - \cdots - \circ \Rightarrow \circ$ (末端双边指向最后)
- C_ℓ 型 ($\ell \geq 3$): $\circ - \circ - \cdots - \circ \Leftarrow \circ$ (末端双边指向倒数第二)
- D_ℓ 型 ($\ell \geq 4$): 分叉图
- E_6, E_7, E_8 型: 例外情形
- F_4 型: $\circ - \circ \Rightarrow \circ - \circ$ (中间双边)
- G_2 型: $\circ \equiv \circ$ (三重边)

证明策略:

- (1) 用几何约束分类可能的考克斯特图
- (2) 证明这些是“可容许”向量集的唯一连通图
- (3) 确定每个考克斯特图产生的戴金图

5.4 第 12 节: 构造与自同构 Construction and Automorphisms

5.4.1 12.1 A-G 型的构造

A_ℓ 型 ($\ell \geq 1$)

- $E = \mathbb{R}^{\ell+1}$ 中正交于 $\varepsilon_1 + \cdots + \varepsilon_{\ell+1}$ 的 ℓ 维子空间
- $\Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$
- 基: $\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_\ell - \varepsilon_{\ell+1}\}$
- 魏尔群: $\mathcal{W} \cong S_{\ell+1}$ (对称群)

B_ℓ 型 ($\ell \geq 2$)

- $E = \mathbb{R}^\ell$
- $\Phi = \{\pm \varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid i \neq j\}$
- 基: $\{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{\ell-1} - \varepsilon_\ell, \varepsilon_\ell\}$
- 魏尔群: $\mathcal{W} \cong (\mathbb{Z}/2\mathbb{Z})^\ell \rtimes S_\ell$

C_ℓ 型 ($\ell \geq 3$)

可视为 B_ℓ 的对偶根系统, 或:

- $E = \mathbb{R}^\ell$
- $\Phi = \{\pm 2\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j) \mid i \neq j\}$
- 基: $\{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{\ell-1} - \varepsilon_\ell, 2\varepsilon_\ell\}$

D_ℓ 型 ($\ell \geq 4$)

- $E = \mathbb{R}^\ell$
- $\Phi = \{\pm(\varepsilon_i \pm \varepsilon_j) \mid i \neq j\}$
- 基: $\{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{\ell-1} - \varepsilon_\ell, \varepsilon_{\ell-1} + \varepsilon_\ell\}$

- 魏尔群: $\mathcal{W} \cong (\mathbb{Z}/2\mathbb{Z})^{\ell-1} \rtimes S_\ell$

例外型 $(E_6, E_7, E_8, F_4, G_2)$

通过在适当格中选取特定长度的向量构造。

5.4.2 12.2 自同构群

定理 12.2.1 (自同构群结构)

$\text{Aut}(\Phi) = \mathcal{W} \rtimes \Gamma$, 其中 Γ 是图自同构群。

对不可约 Φ :

- $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$: A_ℓ ($\ell \geq 2$), D_ℓ ($\ell > 4$), E_6
- $\Gamma \cong S_3$: D_4
- $\Gamma = 1$: 其他类型

5.5 第 13 节: 权的抽象理论 Abstract Theory of Weights

5.5.1 13.1 权 Weights

定义 13.1.1 (权格 Weight Lattice)

权格 (Weight Lattice) 为:

$$\Lambda = \{\lambda \in E \mid \langle \lambda, \alpha \rangle \in \mathbb{Z} \text{ 对所有 } \alpha \in \Phi\}$$

定义 13.1.2 (基本权 Fundamental Weights)

基本支配权 (Fundamental Dominant Weights) $\lambda_1, \dots, \lambda_\ell$ 是相对于内积对 $\{2\alpha_i/(\alpha_i, \alpha_i)\}$ 的对偶基:

$$\frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$$

命题 13.1.3 (权格的结构)

Λ 由基本权生成, 且 Λ/Λ_r 有限 (称为 基本群 (Fundamental Group)), 其中 Λ_r 是 根格 (Root Lattice)。

基本群的阶数等于嘉当矩阵的行列式。

5.5.2 13.2 支配权 Dominant Weights

定义 13.2.1 (支配权)

$\lambda \in \Lambda$ 是 支配的 (Dominant), 若对所有 $\alpha \in \Delta$: $\langle \lambda, \alpha \rangle \geq 0$ 。

λ 是 强支配的 (Strongly Dominant), 若所有不等式严格成立。

记 Λ^+ 为所有支配权的集合。

引理 13.2.2 (支配权的唯一性)

每个权都与唯一的支配权在 \mathcal{W} 下共轭。若 λ 支配, 则对所有 $\sigma \in \mathcal{W}$: $\sigma\lambda \preceq \lambda$, 且若 λ 强支配, 则 $\sigma\lambda = \lambda$ 仅当 $\sigma = 1$ 。

引理 13.2.3 (支配权的有界性)

设 $\lambda \in \Lambda^+$, 则支配权 $\mu \prec \lambda$ 的个数有限。

5.5.3 13.3 权 δ

定义与性质

权 δ 定义为:

$$\delta = \frac{1}{2} \sum_{\alpha > 0} \alpha = \sum_{i=1}^{\ell} \lambda_i$$

引理 13.3.1

δ 是强支配权, 且对所有 $\alpha_i \in \Delta$: $\sigma_i(\delta) = \delta - \alpha_i$ 。

引理 13.3.2 (重要不等式)

设 $\mu \in \Lambda^+$, $\nu = \sigma^{-1}\mu$ ($\sigma \in \mathcal{W}$), 则:

$$(\nu + \delta, \nu + \delta) \leq (\mu + \delta, \mu + \delta)$$

等号成立当且仅当 $\nu = \mu$ 。

5.5.4 13.4 饱和权集 Saturated Sets of Weights

定义 13.4.1 (饱和集)

Λ 的子集 Π 是 饱和的 (Saturated), 若对所有 $\lambda \in \Pi, \alpha \in \Phi, 0 \leq i \leq \langle \lambda, \alpha \rangle$, 都有 $\lambda - i\alpha \in \Pi$ 。

性质

- 任何饱和集自动在 \mathcal{W} 下稳定
- 饱和集有最高权 λ 若 $\lambda \in \Pi$ 且对所有 $\mu \in \Pi$: $\mu \preceq \lambda$

引理 13.4.1

有最高权 λ 的饱和集必定有限。

引理 13.4.2 (饱和集的刻画)

设 Π 饱和, 有最高权 λ 。若 $\mu \in \Lambda^+$ 且 $\mu \prec \lambda$, 则 $\mu \in \Pi$ 。

定理 13.4.3 (饱和集的存在唯一性)

对每个 $\lambda \in \Lambda^+$, 存在唯一有限饱和集以 λ 为最高权, 由所有支配权 $\mu \preceq \lambda$ 及其 \mathcal{W} -共轭组成。

引理 13.4.4 (弗罗伊登塔尔不等式)

设 Π 饱和, 有最高权 λ 。若 $\mu \in \Pi$, 则:

$$(\mu + \delta, \mu + \delta) \leq (\lambda + \delta, \lambda + \delta)$$

等号成立当且仅当 $\mu = \lambda$ 。

5.6 核心理论脉络与关键洞察

5.6.1 逻辑发展

- (1) 公理 (R1)-(R4) \rightarrow 反射几何学 \rightarrow 魏尔群的基本性质
- (2) 基的存在性 (10.1.5) \rightarrow 正负根分解 \rightarrow 简单根理论 (10.2)
- (3) 魏尔群生成 (10.3.1d) \rightarrow 长度函数 (10.3.3) \rightarrow 约化表达式理论
- (4) 不可约性概念 (10.4) \rightarrow 几何约束 \rightarrow 图论分析 \rightarrow 完全分类 (11.4.1)
- (5) 嘉当矩阵理论 (11.1) \rightarrow 戴金图 \rightarrow 同构类的完全刻画
- (6) 显式构造 (12.1) \rightarrow 自同构群结构 (12.2.1)
- (7) 权格理论 (13.1) \rightarrow 支配权 (13.2) \rightarrow 饱和集 (13.4) \rightarrow 表示论基础

5.6.2 关键洞察

根系统通过有限几何对象（反射、超平面配置、魏尔群）完全刻画了半单李代数的组合-几何结构，建立了代数结构与欧几里得几何的深刻联系，为李群表示论提供了坚实的组合基础。

核心思想：复杂的无穷代数结构可以通过有限的组合数据（戴金图）完全分类和理解。

第六章 偏微分方程

6.1 Diffusion

See Partial Differential Equations in Action, Complements and Exercises.

The **diffusion equation** (heat equation) is

$$u_t - \underbrace{D}_{\text{const.}} \Delta u = f$$

defined on a cylindrical domain $Q_T = \Omega \times (0, T)$, where Ω is a domain¹ of \mathbb{R}^n , and the Laplacian Δ is taken w.r.t. the spatial variables \mathbf{x} only.

¹connected, open subset

6.1.1 Cauchy-Dirichlet

1.2.1 The method of separation of variables

Problem 1.2.1 (Cauchy-Dirichlet). Let $D > 0$ be a constant and $g \in C^1([0, \pi])$, with $g(0) = g(\pi) = 0$. Solve, by separating the variables, the problem:

$$\begin{cases} u_t(x, t) - Du_{xx}(x, t) = 0 & 0 < x < \pi, t > 0 \\ u(x, 0) = g(x) & 0 \leq x \leq \pi \\ u(0, t) = u(\pi, t) = 0 & t > 0. \end{cases}$$

Discuss uniqueness and continuous dependence on the initial data.

Solution. We start with two preliminary observations. First of all, the choice of $[0, \pi]$ as domain for the space variable is just to keep the formulas simpler. In case the space variable x varies between 0 and $L > 0$ we can use Fourier series on suitable intervals, or reduce to $[0, \pi]$ by the change $y = x\pi/L$, $v(y, t) = u(Ly/\pi, t)$, which would give the following problem for v :

$$\begin{cases} v_t - \frac{D\pi^2}{L^2}v_{yy} = 0 & 0 < y < \pi, t > 0 \\ v(y, 0) = g(Ly/\pi) & 0 \leq y \leq \pi \\ v(0, t) = v(\pi, t) = 0 & t > 0. \end{cases}$$

Notice that the boundary condition is of Dirichlet type and *homogeneous*. The first step of the method consists in seeking *non-zero* solutions of the form

$$u(x, t) = v(x)w(t).$$

Substituting into the equation gives

$$v(x)w'(t) - Dv''(x)w(t) = 0.$$

Dividing by $v(x)w(t)$ and rearranging terms we find:

$$\frac{1}{D} \frac{w'(t)}{w(t)} = \frac{v''(x)}{v(x)}. \quad (1.2)$$

This is an identity between two members depending on different variables. Consequently they must both be equal to some constant $\lambda \in \mathbb{R}$. Thus we can split (1.2) into the two

equations

$$w'(t) - \lambda D w(t) = 0,$$

solved by

$$w(t) = C e^{\lambda D t}, \quad C \in \mathbb{R}, \quad (1.3)$$

and

$$v''(x) - \lambda v(x) = 0. \quad (1.4)$$

The Dirichlet conditions force $v(0)w(t) = v(\pi)w(t) = 0$ for any $t > 0$, i.e.

$$v(0) = v(\pi) = 0. \quad (1.5)$$

The boundary-value problem (1.4), (1.5) has non-trivial solutions only for special values of λ , called *eigenvalues*. The corresponding solutions are said *eigenfunctions*. We distinguish three cases.

Case $\lambda = \mu^2 > 0$. The general integral of (1.4) is

$$v(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}.$$

By imposing the boundary conditions we find

$$\begin{cases} C_1 + C_2 = 0 \\ e^{\mu\pi} C_1 + e^{-\mu\pi} C_2 = 0, \end{cases}$$

so $C_1 = C_2 = 0$. This gives the zero solution only.

Case $\lambda = 0$. This situations is essentially the same as the above one. From

$$v(x) = C_1 + C_2 x,$$

the Dirichlet constraints immediately force $C_1 = C_2 = 0$.

Case $\lambda = -\mu^2 < 0$. Now we have

$$v(x) = C_1 \cos \mu x + C_2 \sin \mu x, \quad v(0) = v(\pi) = 0.$$

From $v(0) = 0$ we deduce $C_1 = 0$; from $v(\pi) = 0$ we get

$$C_2 \sin \mu\pi = 0 \implies \mu = k \text{ positive integer and } C_2 \text{ arbitrary.}$$

Then, the eigenvalues are $\lambda_k = -k^2$ and the eigenfunctions $v_k(x) = \sin kx$. Recalling (1.3), we have the infinitely-many solutions

$$\varphi_k(x, t) = C e^{-k^2 D t} \sin kx, \quad k = 1, 2, \dots,$$

fulfilling $\varphi_k(0) = \varphi_k(\pi) = 0$. None of these functions satisfies the condition $u(x, 0) = g(x)$, except when $g(x) = C \sin mx$ and m is an integer. The idea is then to exploit the problem linearity by assembling the v_k into a linear combination, and trying to determine the coefficients so to satisfy the initial condition. Then our candidate solution has the form:

$$u(x, t) = \sum_{k=1}^{\infty} c_k e^{-k^2 D t} \sin kx,$$

and we seek the constants c_k by imposing

$$u(x, 0) = \sum_{k=1}^{\infty} c_k \sin kx = g(x). \quad (1.6)$$

Notice that $u(x, 0)$ is a sines-Fourier series; therefore we extend g on $[-\pi, \pi]$ as an odd function and expand it in a sines-Fourier series:

$$g(x) = \sum_{k=1}^{\infty} g_k \sin kx, \quad g_k = \frac{2}{\pi} \int_0^{\pi} g(x) \sin kx \, dx.$$

By comparison with (1.6) we have $c_k = g_k$, and thus

$$u(x, t) = \sum_{k=1}^{\infty} g_k e^{-k^2 D t} \sin kx \quad (1.7)$$

is the (formal) solution.

- *Analysis of (1.7).* The function g is $C^1([0, \pi])$ and vanishes at the endpoints, so its odd prolongation on $[-\pi, \pi]$ is $C^1([- \pi, \pi])$. The theory of Fourier series guarantees that $\sum_{k=1}^{\infty} |g_k|$ converges. Since

$$\left| g_k e^{-k^2 D t} \right| \leq |g_k|,$$

the function (1.6) converges uniformly on the entire strip $[0, \pi] \times [0, \infty)$ and we may swap the sum with the limit. This ensures that (1.7) is continuous on $[0, \pi] \times [0, \infty)$. On the other hand if $t \geq t_0 > 0$ the fast convergence rate of the exponential as $k \rightarrow \infty$ allows to differentiate term-wise (to any order), and in particular

$$u_t - Du_{xx} = \sum_{k=1}^{\infty} g_k [(u_k)_t - D(u_k)_{xx}] = 0,$$

so (1.7) solves the differential equation inside the strip.

- *Uniqueness and continuous dependence on initial data.* The uniqueness of a solution, continuous on $[0, \pi] \times [0, \infty)$, and the fact it depends in a continuous manner upon the initial data both follow from the maximum principle: indeed, if u_g is a solution corre-

sponding to the datum g , we have

$$\max_{[0, \pi] \times [0, \infty)} |u_{g1} - u_{g2}| \leq \max_{[0, \pi]} |g_1 - g_2|.$$

为了保证一个函数的傅里叶级数收敛，该函数需要满足一定的条件。以下是一些常见的条件：

(1) **狄利克雷条件 (Dirichlet Conditions)**: 一个周期为 2π 的函数 $f(x)$, 如果满足以下条件，则其傅里叶级数收敛：

- 在任何有限区间内， $f(x)$ 只有有限个不连续点。
- 在任何有限区间内， $f(x)$ 只有有限个极大值和极小值。
- $f(x)$ 在任何有限区间内是绝对可积的，即 $\int_{-\pi}^{\pi} |f(x)| dx < \infty$ 。

如果 $f(x)$ 在 x 处连续，则傅里叶级数收敛到 $f(x)$; 如果 $f(x)$ 在 x 处有跳跃不连续点，则傅里叶级数收敛到 $\frac{f(x^+) + f(x^-)}{2}$ ，其中 $f(x^+)$ 和 $f(x^-)$ 分别表示 $f(x)$ 在 x 处的右极限和左极限。

(2) **更强的条件**:

- 如果 $f(x)$ 是连续的，并且 $f'(x)$ 存在且连续，那么 $f(x)$ 的傅里叶级数一致收敛到 $f(x)$ 。
- 如果 $f(x)$ 是平方可积的，即 $\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$ ，那么 $f(x)$ 的傅里叶级数在均方意义下收敛到 $f(x)$ 。

总的来说，狄利克雷条件是最经典和常用的判断傅里叶级数收敛的条件。

6.1.2 Cauchy-Neumann

Problem 1.2.2 (Cauchy-Neumann). Let $D > 0$ be a constant and $g \in C^1([0, \pi])$ such that $g'(0) = g'(\pi) = 0$. Solve by separation of variables:

$$\begin{cases} u_t(x, t) - Du_{xx}(x, t) = 0 & 0 < x < \pi, t > 0 \\ u(x, 0) = g(x) & 0 \leq x \leq \pi \\ u_x(0, t) = 0, u_x(\pi, t) = 0 & t > 0. \end{cases}$$

Discuss uniqueness and continuous dependence on the initial data.

Solution. Since the Neumann conditions are homogeneous, we proceed by seeking non-zero solutions of the form

$$u(x, t) = v(x)w(t).$$

As in the previous problem, we obtain for w the equation

$$w'(t) - \lambda Dw(t) = 0,$$

with general solution

$$w(t) = Ce^{\lambda Dt}, \quad C \in \mathbb{R}. \quad (1.8)$$

For v we have the eigenvalue problem

$$\begin{cases} v''(x) - \lambda v(x) = 0 \\ v'(0) = v'(\pi) = 0, \end{cases}$$

where λ is a real number. As usual, we must distinguish three cases.

Case $\lambda = \mu^2 > 0$. The general integral reads

$$v(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}.$$

The Neumann conditions impose $v'(0) = v'(\pi) = 0$, so

$$\begin{cases} \mu C_1 - \mu C_2 = 0 \\ e^{\mu\pi} C_1 - e^{-\mu\pi} C_2 = 0, \end{cases}$$

and then $C_1 = C_2 = 0$ because $\mu(e^{-\mu\pi} + e^{\mu\pi}) \neq 0$. The only solution is trivial.

Case $\lambda = 0$. From

$$v(x) = C_1 + C_2 x,$$

and the Neumann conditions, we deduce immediately $C_2 = 0$ and C_1 arbitrary. Now the eigenfunctions are constant functions.

Case $\lambda = -\mu^2 < 0$. We have

$$v(x) = C_1 \cos \mu x + C_2 \sin \mu x, \quad v'(0) = v'(\pi) = 0.$$

Since

$$v'(x) = -\mu C_1 \sin \mu x + \mu C_2 \cos \mu x,$$

from $v'(0) = 0$ we deduce $C_2 = 0$; from $v'(\pi) = 0$ we infer

$$C_1 \sin \mu \pi = 0 \implies \mu = k \in \mathbb{N}, C_1 \text{ arbitrary.}$$

The eigenvalues are then $\lambda_k = -k^2$ and the eigenfunctions $v_k(x) = \cos kx$.

Recalling (1.8), we have infinitely-many solutions

$$\varphi_k(x, t) = C e^{-k^2 D t} \cos kx, \quad k \in \mathbb{N}$$

satisfying $\varphi'_k(0) = \varphi'_k(\pi) = 0$. None fulfills $u(x, 0) = g(x)$ except when $g(x) = C \cos mx$, m integer. So let us set

$$u(x, t) = \sum_{k=0}^{\infty} c_k e^{-k^2 D t} \cos kx$$

as candidate solution (in particular, for $k = 0$ we also obtain the constant solutions of case $\lambda = 0$). The coefficients c_k must be chosen so that

$$u(x, 0) = \sum_{k=0}^{\infty} c_k \cos kx = g(x). \quad (1.9)$$

Since $u(x, 0)$ is a cosines-Fourier series, we extend g as even function on $[-\pi, \pi]$ and expand it in cosines-Fourier series:

$$g(x) = \frac{g_0}{2} + \sum_{k=1}^{\infty} g_k \cos kx, \quad g_k = \frac{2}{\pi} \int_0^{\pi} g(x) \cos kx dx.$$

Notice that $g_0/2$ is the mean value of the datum g on the interval $[0, \pi]$. After comparison with (1.9) we must have $c_0 = g_0/2$, $c_k = g_k$, giving the (formal) solution

$$u(x, t) = \frac{g_0}{2} + \sum_{k=1}^{\infty} g_k u_k(x, t) = \frac{g_0}{2} + \sum_{k=1}^{\infty} g_k e^{-k^2 D t} \cos kx. \quad (1.10)$$

• *Analysis of (1.10).* The function g belongs to $C^1([0, \pi])$ and has null derivative at the endpoints, whence its even prolongation on $[-\pi, \pi]$ is in $C^1([- \pi, \pi])$. The theory of

Fourier series guarantees that $\sum_{k=1}^{\infty} |g_k|$ converges. As

$$\left| g_k e^{-k^2 D t} \right| \leq |g_k|,$$

the function (1.10) converges uniformly on the strip $[0, \pi] \times [0, \infty)$ and we may swap the limit and the sum. This makes sure that (1.10) is continuous on $[0, \pi] \times [0, \infty)$. Now let us check the Neumann conditions on the boundary. Fix $t_0 > 0$; for t close to t_0 we can differentiate term by term, so

$$u_x(x, t) = - \sum_{k=1}^{\infty} k g_k e^{-k^2 D t} \sin kx. \quad (1.11)$$

Since¹

$$\left| k g_k e^{-k^2 D t} \right| \leq \frac{1}{\sqrt{2eDt}} |g_k|,$$

the series (1.11) converges uniformly on $[0, \pi] \times [t_0, \infty)$ for any $t_0 > 0$. In particular

$$\begin{aligned} \lim_{(x,t) \rightarrow (0,t_0)} u_x(x, t) &= - \sum_{k=1}^{\infty} k g_k \lim_{(x,t) \rightarrow (0,t_0)} [e^{-k^2 D t} \sin kx] = 0 \\ \lim_{(x,t) \rightarrow (\pi,t_0)} u_x(x, t) &= - \sum_{k=1}^{\infty} k g_k \lim_{(x,t) \rightarrow (\pi,t_0)} [e^{-k^2 D t} \sin kx] = 0. \end{aligned}$$

The function is therefore C^1 on any strip $[0, \pi] \times [t_0, \infty)$. Similar computations show that if $t \geq t_0 > 0$ the fast convergence to zero of the exponential as $k \rightarrow \infty$ allows to differentiate (to any order) each term separately. In particular

$$u_t - Du_{xx} = \sum_{k=1}^{\infty} g_k [(u_k)_t - D(u_k)_{xx}] = 0,$$

so (1.10) is indeed a solution on the strip $[0, \pi] \times (0, \infty)$.

• *Uniqueness and continuous dependence on the data.* We use an *energy method*. Suppose there exist two solutions u and v of the same problem, defined on $[0, \pi] \times [0, \infty)$ and C^1 on $[0, \pi] \times (0, \infty)$. Set $w = u - v$ and

$$E(t) = \int_0^\pi w^2(x, t) dx.$$

Then $E(t) \geq 0$, $E(0) = \lim_{t \downarrow 0} E(t) = 0$, and for $t > 0$ also

$$E'(t) = 2 \int_0^\pi w w_t dx = 2D \int_0^\pi w w_{xx} dx.$$

¹ Maximise the function $f(x) = xe^{-x^2 Dt}$.

We integrate by parts and recall that w_x vanishes at the endpoints:

$$E'(t) = -2D \int_0^\pi (w_x)^2 dx \leq 0.$$

Consequently E decreases and therefore $E = 0$ for any $t \geq 0$. As w is continuous, $w(x, t) \equiv 0$.

By Bessel's equality, moreover,

$$\sup_{t>0} \|u(\cdot, t)\|_{L^2(0, \pi)}^2 = \sup_{t>0} \int_0^\pi u^2(x, t) dx \leq \pi \sum_{k=0}^{\infty} |g_k|^2 = \pi \|g\|_{L^2(0, \pi)}^2,$$

showing that the solution depends continuously (in L^2) on the initial datum.

Définition 1: Bessel's equality

令 H 是一个希尔伯特空间，并设 $\{e_k\}_{k=1}^{\infty}$ 是 H 的一个正交基。那么，对于 H 中的任何 x , Bessel 不等式成立

$$\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$$

Définition 2: Bessel's equality

Let f be a function on $[-\pi, \pi]$ such that $\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = |a_0|^2 + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2)$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

Théorème 1: Poincaré's inequality

设 $\Omega \subset \mathbb{R}^n$ 是一个有界开集。存在一个常数 C , 使得对于所有 $u \in H_0^1(\Omega)$, 有

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}.$$

证明. 设 $\Omega \subset \mathbb{R}^n$ 是一个有界开集。那么存在一个 $R > 0$, 使得 $\Omega \subset B(0, R)$ 。设 $u \in H_0^1(\Omega)$ 。我们可以将 u 延拓到 $B(0, R)$ 上, 使得 $u \in H_0^1(B(0, R))$ 。那么

u 在 $\partial B(0, R)$ 上为 0。因此，我们可以将 u 延拓到 \mathbb{R}^n 上，使得 $u \in H^1(\mathbb{R}^n)$ 。

对于 $x \in \mathbb{R}^n$ ，有

$$u(x) = \int_{-\infty}^{x_1} \frac{\partial u}{\partial x_1}(y_1, x_2, \dots, x_n) dy_1.$$

因此，

$$|u(x)| \leq \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_1}(y_1, x_2, \dots, x_n) \right| dy_1.$$

利用 Cauchy-Schwarz 不等式，我们有

$$|u(x)|^2 \leq \left(\int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_1}(y_1, x_2, \dots, x_n) \right|^2 dy_1 \right) \left(\int_{-\infty}^{\infty} 1 dy_1 \right).$$

由于 Ω 是有界的，所以 $\int_{-\infty}^{\infty} 1 dy_1 < \infty$ 。因此，

$$|u(x)|^2 \leq C \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_1}(y_1, x_2, \dots, x_n) \right|^2 dy_1.$$

在 Ω 上积分，我们有

$$\int_{\Omega} |u(x)|^2 dx \leq C \int_{\Omega} \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_1}(y_1, x_2, \dots, x_n) \right|^2 dy_1 dx.$$

因此，

$$\|u\|_{L^2(\Omega)}^2 \leq C \int_{\Omega} |\nabla u(x)|^2 dx = C \|\nabla u\|_{L^2(\Omega)}^2.$$

取平方根，我们有

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}.$$

□

Théorème 2: Poincaré 不等式 (n=1)

设 $u \in H_0^1(a, b)$ ，其中 $-\infty < a < b < \infty$ 。那么

$$\|u\|_{L^2(a,b)} \leq \frac{b-a}{\pi} \|u'\|_{L^2(a,b)}$$

证明。通过延拓 u 为 $[-\ell, \ell]$ 上的奇函数，其中 $\ell = (b-a)/2$ ，我们有 $u(-\ell) = u(\ell) = 0$ 。因此， u 的傅里叶级数为

$$u(x) = \sum_{n=1}^{\infty} a_n \sin \left(\frac{n\pi x}{\ell} \right)$$

其中

$$a_n = \frac{2}{\ell} \int_0^\ell u(x) \sin\left(\frac{n\pi x}{\ell}\right) dx.$$

Parseval 恒等式给出

$$\int_{-\ell}^\ell |u(x)|^2 dx = \frac{\ell}{2} \sum_{n=1}^{\infty} |a_n|^2.$$

此外，形式上的导数为

$$u'(x) = \sum_{n=1}^{\infty} a_n \frac{n\pi}{\ell} \cos\left(\frac{n\pi x}{\ell}\right)$$

因此

$$\int_{-\ell}^\ell |u'(x)|^2 dx = \frac{\ell}{2} \sum_{n=1}^{\infty} |a_n|^2 \frac{n^2 \pi^2}{\ell^2}.$$

由于 $n \geq 1$ ，我们有

$$\int_{-\ell}^\ell |u(x)|^2 dx \leq \frac{\ell^2}{\pi^2} \int_{-\ell}^\ell |u'(x)|^2 dx$$

或者

$$\|u\|_{L^2(-\ell, \ell)} \leq \frac{\ell}{\pi} \|u'\|_{L^2(-\ell, \ell)}.$$

由于 u 在 (a, b) 外为零，且 $\ell = (b - a)/2$ ，我们得到

$$\|u\|_{L^2(a, b)} \leq \frac{b - a}{2\pi} \|u'\|_{L^2(a, b)}.$$

□

6.1.3 Cauchy-Neumann; non-homogeneous equation

Problem 1.2.4 (Cauchy-Neumann; non-homogeneous equation). *Solve, using separation of variables, the problem:*

$$\begin{cases} u_t(x, t) - u_{xx}(x, t) = tx & 0 < x < \pi, t > 0 \\ u(x, 0) = 1 & 0 \leq x \leq \pi \\ u_x(0, t) = u_x(\pi, t) = 0 & t > 0. \end{cases}$$

Solution. This is a *non-homogeneous* Neumann problem with homogeneous boundary conditions. In order to be able to separate the variables it is convenient to consider first the homogeneous equation, and in particular the associated eigenvalue problem:

$$\begin{cases} v''(x) - \lambda v(x) = 0 \\ v'(0) = v'(\pi) = 0. \end{cases}$$

In Problem 1.2.2 we found the eigenvalues $\lambda_k = -k^2$ and the eigenfunctions $v_k(x) = \cos kx$. Let us write the candidate solution as

$$u(x, t) = \sum_{k=0}^{\infty} c_k(t) \cos kx$$

and impose (recall that $v_k'' = -k^2 v_k$):

$$u_t - u_{xx} = \sum_{k=0}^{\infty} [c'_k(t) + k^2 c_k(t)] \cos kx = tx$$

with

$$u(x, 0) = \sum_{k=0}^{\infty} c_k(0) \cos kx = 1.$$

We expand $f(x) = x$ in cosines-Fourier series:

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos[(2k+1)x]}{(2k+1)^2},$$

uniformly convergent on $[0, \pi]$. Comparing the last three equations, the coefficients $c_k(t)$ must solve the following Cauchy problems:

$$\begin{aligned} c'_0(t) &= \frac{\pi}{2}t, & c_0(0) &= 1; \\ c'_{2k}(t) + 4k^2 c_{2k}(t) &= 0, & c_{2k}(0), k \geq 1; \\ c'_{2k+1}(t) + (2k+1)^2 c_{2k+1}(t) &= -\frac{4}{\pi} \frac{1}{(2k+1)^2} t, & c_{2k+1}(0), k \geq 0. \end{aligned}$$

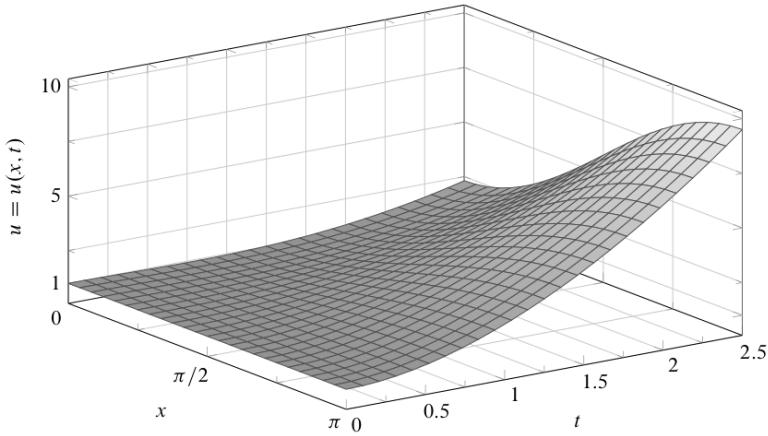


Fig. 1.1 Solution to Problem 1.2.4 for $0 < x < \pi$, $0 < t < 2.5$

Solving the ODE problem, we find

$$\begin{aligned} c_0(t) &= \frac{\pi}{4}t^2 + 1; \\ c_{2k}(t) &= 0, \quad k \geq 1; \\ c_{2k+1}(t) &= -\frac{4}{\pi(2k+1)^4} \left[t + \frac{1}{(2k+1)^2} (e^{-(2k+1)^2 t} - 1) \right], \quad k \geq 0. \end{aligned}$$

Hence the (formal) solution reads (Fig. 1.1):

$$u(x, t) = \frac{\pi}{4}t^2 + 1 + \sum_{k=0}^{\infty} c_{2k+1}(t) \cos[(2k+1)x]. \quad (1.15)$$

• *Analysis of (1.15).* As

$$\left| (e^{-(2k+1)^2 t} - 1) \cos(2k+1)x \right| \leq 2$$

we deduce that the series

$$\sum_{k=0}^{\infty} \frac{e^{-(2k+1)^2 t} - 1}{(2k+1)^6} \cos(2k+1)x,$$

the series of first and second partial derivatives in x , given by

$$-\sum_{k=0}^{\infty} \frac{e^{-(2k+1)^2 t} - 1}{(2k+1)^5} \sin(2k+1)x, \quad -\sum_{k=0}^{\infty} \frac{e^{-(2k+1)^2 t} - 1}{(2k+1)^4} \cos(2k+1)x,$$

and the series of derivatives in t , given by

$$-\sum_{k=0}^{\infty} \frac{e^{-(2k+1)^2 t}}{(2k+1)^4} \cos(2k+1)x,$$

all converge uniformly on $[0, \pi] \times [0, \infty)$. Hence the derivatives can be carried into the sum, making u a C^2 function on $[0, \pi] \times [0, \infty)$.

In particular u solves the diffusion equation on $(0, \pi) \times (0, \infty)$ and it assumes the given boundary values in pointwise sense. Within this class of functions, it can be proved that the solution is unique and it depends continuously on the data using the energy method (Problem 1.2.2 on page 6).

6.1.4 Non-homogeneous Neumann

Problem 1.2.5 (Non-homogeneous Neumann). Solve by separation of variables the following problem:

$$\begin{cases} u_t(x, t) - u_{xx}(x, t) = 0 & 0 < x < \pi, t > 0 \\ u(x, 0) = 0 & 0 \leq x \leq \pi \\ u_x(0, t) = 0, u_x(\pi, t) = U & t > 0. \end{cases}$$

If $U \neq 0$, can there be a stationary solution $u_\infty = u_\infty(x)$?

Solution. This Neumann problem has non-homogeneous boundary conditions. Let us observe immediately that $U \neq 0$ prevents the existence of stationary solutions $u_\infty = u_\infty(x)$, for otherwise we would have $u''_\infty(x) = 0$, $u'_\infty(0) = 0$, $u'_\infty(\pi) = U$, which is impossible.

To separate variables we reduce to homogeneous conditions by setting

$$w(x, t) = u(x, t) - v(x)$$

where $v_x(0) = 0$, $v_x(\pi) = U$. For example we can choose

$$v(x) = \frac{Ux^2}{2\pi}.$$

The function w solves the problem

$$\begin{cases} w_t(x, t) - w_{xx}(x, t) = U/\pi & 0 < x < \pi, t > 0 \\ w(x, 0) = -Ux^2/2\pi & 0 \leq x \leq \pi \\ w_x(0, t) = 0, w_x(\pi, t) = 0 & t > 0. \end{cases}$$

As in Problem 1.2.4, given the homogeneous Neumann conditions, we write

$$w(x, t) = \frac{c_0(t)}{2} + \sum_{k=1}^{\infty} c_k(t) \cos kx$$

so that the Neumann conditions are (formally) satisfied. We have to find $c_k(t)$ so that

$$w_t - w_{xx} = \frac{c'_0(t)}{2} + \sum_{k=1}^{\infty} [c'_k(t) + k^2 c_k(t)] \cos kx = \frac{U}{\pi}$$

and

$$w(x, 0) = \frac{c_0(0)}{2} + \sum_{k=1}^{\infty} c_k(0) \cos kx = -\frac{Ux^2}{2\pi}.$$

Let us expand $g(x) = \frac{Ux^2}{2\pi}$ in cosines Fourier series:

$$\frac{Ux^2}{2\pi} = \frac{U}{2\pi} \left\{ \frac{\pi^2}{3} + 4 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos kx \right\},$$

which is uniformly convergent on $[0, \pi]$. The comparison of the last three formulas forces the coefficients $c_k(t)$ to solve the Cauchy problems:

$$\begin{aligned} c'_0(t) &= \frac{2U}{\pi}, & c_0(0) &= -\frac{U\pi}{3}; \\ c'_k(t) + k^2 c_k(t) &= 0, & c_k(0) &= \frac{2U}{\pi} \frac{(-1)^{k+1}}{k^2}, \quad k \geq 1. \end{aligned}$$

We find

$$c_0(t) = \frac{2U}{\pi} t - \frac{U\pi}{3}, \quad c_k(t) = \frac{2U}{\pi} \frac{(-1)^{k+1}}{k^2} e^{-k^2 t}, \quad k \geq 1,$$

and the solution reads (Fig. 1.2):

$$u(x, t) = \frac{U}{\pi} t + \frac{Ux^2}{2\pi} - \frac{U\pi}{6} + \frac{2U}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} e^{-k^2 t} \cos kx. \quad (1.16)$$

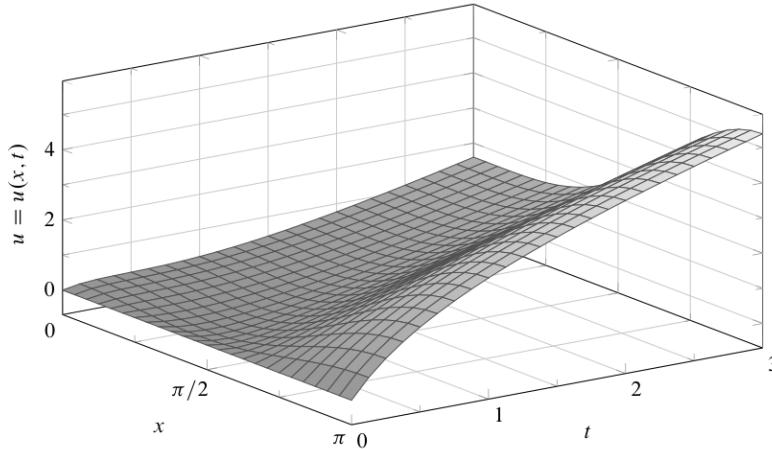


Fig. 1.2 Solution to Problem 1.2.5 ($U = \pi$)

- Analysis of (1.16). The series is uniformly convergent on $[0, \pi] \times [0, \infty)$, so u is continuous there. All derivatives pass into the sum on $[0, \pi] \times [t_0, \infty)$ for any $t_0 > 0$. Thus u solves the diffusion equation on $(0, \pi) \times (0, \infty)$.

6.1.5 Maximum principle

• *Maximum principles.* Let Ω be bounded and $w \in C^{2,1}(\bar{Q}_T) \cap C(\bar{\Omega}_T)$ a sub-solution (or super-solution), that is

$$w_t - D\Delta w = q \leq 0 \quad (\text{resp. } \geq 0) \quad \text{in } Q_T.$$

Then w reaches its maximum value (resp. minimum) on the parabolic boundary $\partial_p Q_T$ of Q_T :

$$\max_{\bar{Q}_T} w = \max_{\partial_p Q_T} w$$

(*weak maximum principle*). In particular, if w is negative (resp. positive) on $\partial_p Q_T$, then it is negative (resp. positive) over all of Q_T . If, further, $w(\mathbf{x}_0, t_0) = \max_{\bar{Q}_T} w$ with $\mathbf{x}_0 \in \Omega$, then w is constant on $\Omega \times [0, t_0]$ (*strong maximum principle*).

If $u_t - D\Delta u = 0$ on Ω , then u reaches its maximum and minimum on the boundary $\partial_p \Omega$.

1.2.2 Use of the maximum principle

Problem 1.2.8 (Maximum principle). Let u be a solution to

$$\begin{cases} u_t(x, t) - u_{xx}(x, t) = 0 & 0 < x < 1, t > 0 \\ u(x, 0) = \sin \pi x & 0 \leq x \leq 1 \\ u(0, t) = 2te^{1-t}, u(1, t) = 1 - \cos \pi t & t > 0, \end{cases}$$

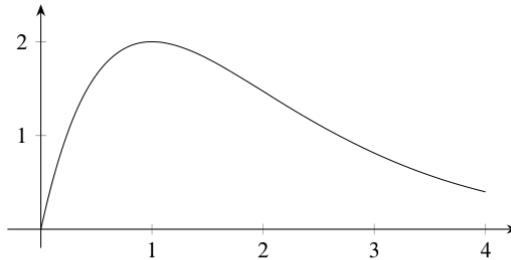
that is continuous^a on the closure of the half-strip $S = (0, 1) \times (0, \infty)$.

a) Prove that u is non-negative.

b) Determine an upper bound for $u\left(\frac{1}{2}, \frac{1}{2}\right)$ and $u\left(\frac{1}{2}, 3\right)$.

^a Note how the boundary values agree on $(0, 0)$ and $(1, 0)$.

Solution. a) The parabolic boundary $\partial_p S$ of the strip is the union of the half-lines $x = 0, x = 1, t > 0$ and the segment $0 \leq x \leq 1$ on the x -axis ($t = 0$). By the maximum principle u is non-negative on the entire strip provided $u \geq 0$ on $\partial_p S$. On the boundary

**Fig. 1.4** Graph of $t \mapsto 2te^{1-t}$

half-lines $2te^{-t} \geq 0$, and $1 - \cos \pi x \geq 0$; also the datum $\sin \pi x$ is non-negative. Hence $u \geq 0$ on S .

b) By the maximum principle, the value $u(1/2, 1/8)$ does not exceed the data on the parabolic boundary of the strip $S_{1/8} = (0, 1) \times (0, 1/8)$, which is

$$\{0 \leq x \leq 1, t = 0\} \cup \left\{x = 0, 0 \leq t \leq \frac{1}{8}\right\} \cup \left\{x = 1, 0 \leq t \leq \frac{1}{8}\right\}.$$

The maximum value of the datum and of $1 - \cos \pi t$ is 1. The graph of $2te^{1-t}$ (see Fig. 1.4) has an absolute maximum 2 at $t = 1$; on $[0, 1/8]$ the maximum is $e^{7/8}/4 \simeq 0.59972 < 1$. We can only say $u(1/2, 1/8) < 1$ (the strict inequality holds by the strong maximum principle). Similarly, $u(\frac{1}{2}, 3) < 2$. Actually, we can also say that $u(x, t) < 2$ on S .

6.1.6 Asumptotic behaviour

Problem 1.2.9 (Asymptotic behaviour). Let u be a continuous solution on the closure of $S = (0, 1) \times (0, \infty)$ to the problem

$$\begin{cases} u_t(x, t) - u_{xx}(x, t) = 0 & 0 < x < 1, t > 0 \\ u(x, 0) = x(1-x) & 0 \leq x \leq 1 \\ u(0, t) = u(1, t) = 0 & t > 0. \end{cases}$$

After proving that u is non-negative, determine positive numbers α, β , so that

$$u(x, t) \leq w(x, t) \equiv \alpha x(1-x)e^{-\beta t}.$$

Deduce that $u(x, t) \rightarrow 0$ uniformly on $[0, L]$ as $t \rightarrow +\infty$.

Solution. We begin by proving that $u \geq 0$. For this we recall that by the maximum principle it is enough to have $u \geq 0$ on the parabolic boundary $\partial_p S$ of the strip S . For $t = 0$, in fact, $u(x, 0) = x(1-x)$ is nonnegative on $[0, 1]$. Moreover $u = 0$ along the sides $x = 0, x = 1$. Hence $u \geq 0$. The idea, to have w larger than u , is to use the maximum principle for the continuous function $v = w - u$. More precisely, we look for α, β so that

$v \geq 0$ on $\partial_p S$ and v is a *super-solution* (i.e. $v_t - v_{xx} \geq 0$):

$$\begin{aligned} w(x, 0) &= \alpha x(1-x) \\ w_t(x, t) &= -\alpha \beta x(1-x)e^{-\beta t} \\ w_{xx}(x, t) &= -2\alpha e^{-\beta t}, \end{aligned}$$

so

$$\begin{cases} v_t(x, t) - v_{xx}(x, t) = \alpha(2 - \beta x(1-x))e^{-\beta t} & 0 < x < 1, t > 0 \\ v(x, 0) = (\alpha - 1)x(1-x) & 0 \leq x \leq 1 \\ v(0, t) = v(1, t) = 0 & t > 0. \end{cases}$$

Let us find $\beta > 0$ so that $2 - \beta x(1-x) \geq 0$. As $x(1-x) \leq \frac{1}{4}$,

$$2 - \beta x(1-x) \geq 2 - \frac{1}{4}\beta$$

and then it suffices to choose $0 < \beta \leq 8$. Let us find the sign of v on $\partial_p S$. Along the sides of S we have $v = 0$, while $v(x, 0) = (\alpha - 1)x(1-x) \geq 0$, for $t = 0$, provided $\alpha \geq 1$. Hence $v \geq 0$ on $\partial_p S$ if $\alpha \geq 1$. To sum up, for $\alpha \geq 1$ and $0 < \beta \leq 8$, v is a non-negative super-solution. We may then use the maximum principle on v , obtaining non-negativity, and so

$$0 \leq u(x, t) \leq \alpha x(1-x)e^{-\beta t} \leq \frac{\alpha}{4}e^{-\beta t}$$

because $x(1-x) \leq 1/4$. As $\beta > 0$, $e^{-\beta t} \rightarrow 0$ for $t \rightarrow \infty$, so $u(x, t) \rightarrow 0$ uniformly on $[0, 1]$ as $t \rightarrow +\infty$.

6.1.7 Applying the notion of fundamental solution

- *Fundamental solution and global Cauchy problem.* The function

$$\Gamma_D(\mathbf{x}, t) = \frac{1}{(4\pi D t)^{n/2}} e^{-|\mathbf{x}|^2/(4Dt)}, \quad t > 0,$$

is called *fundamental solution* to the diffusion equation; when $t > 0$ it solves

$u_t - D\Delta u = 0$ and is the unique function satisfying

$$\lim_{t \downarrow 0} \Gamma_D(\mathbf{x}, t) = \delta_n(\mathbf{x}), \quad \int_{\mathbb{R}^n} \Gamma_D(\mathbf{x}, t) d\mathbf{x} = 1 \text{ for any } t > 0,$$

where $\delta_n(\mathbf{x})$ denotes the n -dimensional Dirac's delta function.

The fundamental solution enables to construct the general solution to the global Cauchy problem

$$\begin{cases} u_t - D\Delta u = f(\mathbf{x}, t) & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(\mathbf{x}, 0) = g(\mathbf{x}) & \text{in } \mathbb{R}^n, \end{cases}$$

by means of the formula

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^n} \Gamma_D(\mathbf{x} - \mathbf{y}, t) g(\mathbf{y}) d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} \Gamma(\mathbf{x} - \mathbf{y}, t - s) f(\mathbf{y}, s) d\mathbf{y} ds.$$

The latter holds, for example, when $|g(\mathbf{x})| \leq ce^{A|\mathbf{x}|}$, f is bounded and $f, f_t, f_{x_j}, f_{x_i x_j}$ are continuous on $\mathbb{R}^n \times (0, +\infty)$. At a point \mathbf{x}_0 of continuity of g we have

$$u(\mathbf{x}, t) \rightarrow g(\mathbf{x}_0) \quad \text{as } (\mathbf{y}, t) \rightarrow (\mathbf{x}_0, 0), t > 0.$$

6.1.8 Problems on the half-line; reflection method

Problem 1.2.15 (Problems on the half-line; reflection method). Let $g : [0, +\infty) \rightarrow \mathbb{R}$ be a continuous and bounded function.

a) Find a formula for the solution to

$$\begin{cases} u_t(x, t) - Du_{xx}(x, t) = 0 & x > 0, t > 0 \\ u(x, 0) = g(x) & x > 0 \\ u(0, t) = 0 & t > 0. \end{cases}$$

Hint. Extend the initial datum, for $x < 0$, to an odd function and use the formula for the global Cauchy problem.

b) Find a formula for the solution to

$$\begin{cases} u_t(x, t) - Du_{xx}(x, t) = 0 & x > 0, t > 0 \\ u(x, 0) = g(x) & x > 0 \\ u_x(0, t) = 0 & t > 0. \end{cases}$$

Hint. Extend the initial datum, for $x < 0$, to an even function and use the formula for the global Cauchy problem.

c) Show that either formula provides the unique bounded solution to the respective problem.

Solution. a) In the first case we extend g in an odd way:

$$\tilde{g}(x) = \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x < 0 \end{cases} \quad (\text{odd reflection}).$$

The new function is continuous on \mathbb{R} only if $g(0) = 0$. Consider the global Cauchy problem

$$\begin{cases} u_t(x, t) - Du_{xx}(x, t) = 0 & x \in \mathbb{R}, t > 0 \\ u(x, 0) = \tilde{g}(x) & x \in \mathbb{R}. \end{cases}$$

For any $x \in \mathbb{R}, t > 0$, the solution reads

$$\begin{aligned}\tilde{u}(x, t) &= \int_R \Gamma_D(x - y, t) \tilde{g}(y) dy \\ &= \int_0^{+\infty} \Gamma_D(x - y, t) g(y) dy - \int_{-\infty}^0 \Gamma_D(x - y, t) g(-y) dy \\ &= \int_0^{+\infty} \Gamma_D(x - y, t) g(y) dy - \int_0^{+\infty} \Gamma_D(x + y, t) g(y) dy\end{aligned}$$

where in the last term we wrote y instead of $-y$ and swapped endpoints. Let $u(x, t)$ denote the restriction of \tilde{u} to the first quadrant. The previous computation tells

$$u(x, t) = \int_0^{+\infty} [\Gamma_D(x - y, t) - \Gamma_D(x + y, t)] g(y) dy. \quad (1.25)$$

• *Analysis of (1.25).* Clearly u is bounded and solves the heat equation on the quadrant $x > 0, t > 0$. Since Γ_D is even in the spatial variable we obtain⁵

$$u(0, t) = \int_0^{+\infty} [\Gamma_D(-y, t) - \Gamma_D(y, t)] g(y) dy = 0 \quad \text{for any } t > 0.$$

Hence u fulfills the Dirichlet condition of the half-line $x = 0$. Write

$$g^+(x) = \begin{cases} g(x) & x \geq 0 \\ 0 & x < 0 \end{cases} \quad \text{and} \quad g^-(x) = \begin{cases} 0 & x \geq 0 \\ g(-x) & x < 0, \end{cases}$$

so

$$\begin{aligned}u(x, t) &= u^+(x, t) - u^-(x, t) \\ &\equiv \int_{-\infty}^{+\infty} \Gamma_D(x - y, t) g^+(y) dy - \int_{-\infty}^{+\infty} \Gamma_D(x - y, t) g^-(y) dy.\end{aligned}$$

Therefore for every $x_0 > 0$, if $(x, t) \rightarrow (x_0, 0)$ we have

$$u^+(x, t) \rightarrow g(x_0), \text{ and } u^-(x, t) \rightarrow 0,$$

because g is continuous at x_0 . Consequently u is continuous on the closed quadrant, *except* possibly for the origin, and in particular $u(x, 0) = g(x)$, $x > 0$. Continuity at the origin holds if and only if $g(0) = 0$, because both g^+, g^- are continuous at $x = 0$. In addition, $\tilde{u} \in C^\infty$ on the half-plane $t > 0$ so $u \in C^\infty$ when $t > 0, x \geq 0$.

b) The strategy is completely similar to the previous one. Now we prolong g evenly for $x < 0$:

$$\tilde{g}(x) = \begin{cases} g(x) & x \geq 0 \\ g(-x) & x \leq 0 \end{cases} \quad (\text{even reflection})$$

and consider the global Cauchy problem with datum \tilde{g} . The solution is

$$\begin{aligned}\tilde{u}(x, t) &= \int_R \Gamma_D(x - y, t) \tilde{g}(y) dy = \\ &= \int_0^{+\infty} \Gamma_D(x - y, t) g(y) dy + \int_{-\infty}^0 \Gamma_D(x - y, t) g(-y) dy = \\ &= \int_0^{+\infty} \Gamma_D(x - y, t) g(y) dy + \int_0^{+\infty} \Gamma_D(x + y, t) g(y) dy.\end{aligned}$$

If $u(x, t)$ denotes \tilde{u} restricted to the first quadrant, we have

$$u(x, t) = \int_0^{+\infty} [\Gamma_D(x - y, t) + \Gamma_D(x + y, t)] g(y) dy. \quad (1.26)$$

• *Analysis of (1.26).* As before u is bounded and solves the heat equation on $x > 0, t > 0$. Note that \tilde{g} is continuous also at $x = 0$, so u equals the Cauchy datum continuously on the half-line $x \geq 0$. To verify the Neumann condition we have to compute $u_x(0, t)$. Observe

$$\partial_x \Gamma_D(x \pm y, t) = \partial_x \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{(x \pm y)^2}{4Dt}\right) = -\frac{x \pm y}{2Dt} \Gamma_D(x \pm y, t)$$

so at $x = 0$

$$\partial_x \Gamma_D(\pm y, t) = \mp \frac{y}{2Dt} \Gamma_D(y, t).$$

For $t > 0$ we can differentiate the integrand, obtaining

$$\begin{aligned}u_x(0, t) &= \int_0^{+\infty} [\partial_x \Gamma_D(-y, t) + \partial_x \Gamma_D(y, t)] g(y) dy \\ &= \int_0^{+\infty} \frac{y}{2Dt} [\Gamma_D(y, t) - \Gamma_D(-y, t)] g(y) dy \\ &= 0.\end{aligned}$$

In this case, too, the regularity of \tilde{u} implies $u \in C^\infty$ when $t > 0, x \geq 0$.

c) If there existed distinct bounded and regular solutions for $t > 0, x \geq 0$, the reflection would generate bounded C^2 functions for $t > 0$, solving the same global Cauchy problem, thus contradicting the general theory.

Remark. The functions

$$\Gamma_D^-(x, y, t) = \Gamma_D(x - y, t) - \Gamma_D(x + y, t)$$

and

$$\Gamma_D^+(x, y, t) = \Gamma_D(x - y, t) + \Gamma_D(x + y, t)$$

are called *fundamental solutions for the Cauchy-Dirichlet and Cauchy-Neumann problems on the quadrant* $t > 0, x > 0$, respectively (see Figs. 1.7 and 1.8).

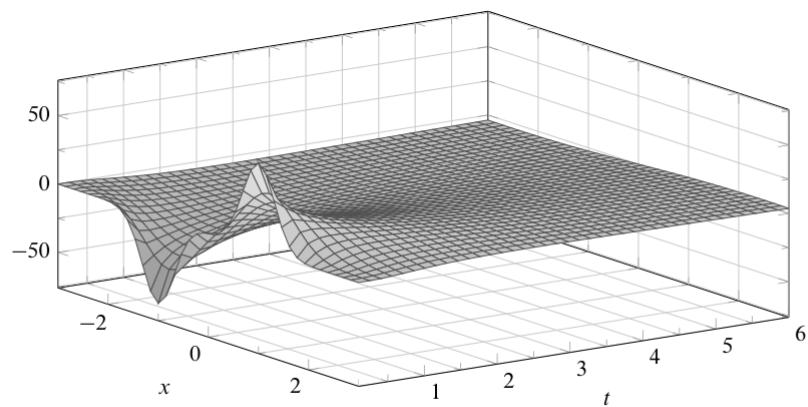


Fig. 1.7 $\Gamma_{1/4}^-(x, 1, t) = \Gamma_{1/4}(x - 1, t) - \Gamma_{1/4}(x + 1, t)$

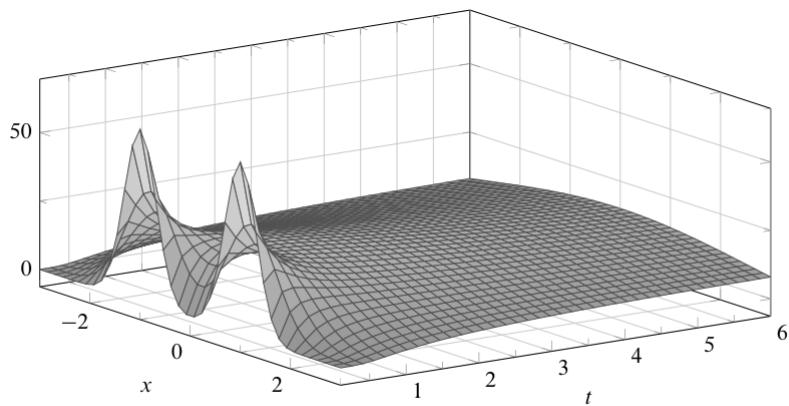


Fig. 1.8 $\Gamma_{1/4}^+(x, 1, t) = \Gamma_{1/4}(x - 1, t) + \Gamma_{1/4}(x + 1, t)$

6.1.9 Use of Fourier and Laplace transforms

Problem 1.2.19 (Fourier transform and fundamental solution). *Using the Fourier transform with respect to x recover the formula for the solution to the global Cauchy problem*

$$\begin{cases} u_t - Du_{xx} = f(x, t) & -\infty < x < \infty, t > 0 \\ u(x, 0) = g(x) & -\infty < x < \infty. \end{cases}$$

Solution. Define $\hat{u}(\xi, t) = \int_{\mathbb{R}} u(x, t) e^{-ix\xi} dx$, the partial Fourier transform of u . Then \hat{u} solves the Cauchy problem (formally)

$$\begin{cases} \hat{u}_t + D\xi^2 \hat{u} = \hat{f}(\xi, t) & -\infty < \xi < \infty, t > 0 \\ \hat{u}(\xi, 0) = \hat{g}(\xi) & -\infty < \xi < \infty, \end{cases}$$

where \hat{f} denotes the partial Fourier transform of f . We find

$$\hat{u}(\xi, t) = \hat{g}(\xi) e^{-D\xi^2 t} + \int_0^t e^{-D\xi^2(t-s)} \hat{f}(\xi, s) ds.$$

We remind that the inverse transform of the exponential $e^{-D\xi^2 t}$ is $\Gamma_D(x, t)$, and the inverse transform of a product is the convolution of the inverse transforms. This yields (at least formally)

$$u(x, t) = \int_{\mathbb{R}} \Gamma_D(x - y, t) g(y) dy + \int_0^t \int_{\mathbb{R}} \Gamma_D(x - y, t - s) f(y, s) ds.$$

6.1.10 Fourier sine transform

Problem 1.2.20 (Dirichlet conditions on the half-line). *a) Using the Fourier sine transform find a formula for the bounded solution to*

$$\begin{cases} u_t(x, t) - u_{xx}(x, t) = 0 & x > 0, t > 0 \\ u(x, 0) = 0 & x \geq 0 \\ u(0, t) = g(t) & t > 0, \end{cases} \quad (1.32)$$

where g is continuous and bounded. Show that this is the only solution with the given properties.

b) Prove that without the condition that u is bounded, problem (1.32) does not have, in general, a unique solution. Hint. Use the functions $w_1(x, t) = e^x \cos(2t + x)$ and $w_2(x, t) = e^{-x} \cos(2t - x)$.

Solution. a) The Fourier sine transform in x is defined as:

$$S(u)(\xi, t) = U(\xi, t) = \frac{2}{\pi} \int_0^\infty u(x, t) \sin(\xi x) dx,$$

with inverse formula

$$u(x, t) = \int_0^\infty U(\xi, t) \sin(\xi x) d\xi.$$

Notice that U is an odd function in ξ . Assuming that both u and u_x vanish at infinity we have

$$S(u_{xx})(\xi, t) = \frac{2}{\pi} \xi u(0, t) - \xi^2 U(\xi, t),$$

and therefore U solves the problem

$$\begin{cases} U_t(\xi, t) + \xi^2 U(\xi, t) = \frac{2}{\pi} \xi g(t) & \xi > 0, t > 0 \\ U(\xi, 0) = 0 & \xi \geq 0. \end{cases}$$

We have

$$U(\xi, t) = \frac{2}{\pi} \xi \int_0^t e^{-\xi^2(t-s)} g(s) ds.$$

Anti-transforming gives:

$$\begin{aligned} u(x, t) &= \int_0^\infty U(\xi, t) \sin(\xi x) d\xi = \frac{1}{\pi} \int_0^t g(s) \left[\int_0^\infty 2\xi e^{-\xi^2(t-s)} \sin(\xi x) d\xi \right] ds \\ &= - \int_0^t \frac{g(s)}{\pi(t-s)} \left\{ \left[\sin(\xi x) e^{-\xi^2(t-s)} \right]_0^\infty - x \int_0^\infty e^{-\xi^2(t-s)} \cos(\xi x) d\xi \right\} ds \\ &= \frac{x}{\pi} \int_0^t \frac{g(s)}{t-s} \left[\int_0^\infty e^{-\xi^2(t-s)} \cos(\xi x) d\xi \right] ds. \end{aligned}$$

Observe that

$$\int_0^\infty e^{-ax^2} \cos(\xi x) d\xi = \frac{1}{2} \int_{-\infty}^\infty e^{-a\xi^2 + i\xi x} d\xi = \sqrt{\frac{\pi}{4a}} e^{-\frac{x^2}{4a}}.$$

Substituting $a = t - s$ finally gives (at least formally)

$$u(x, t) = \frac{x}{2\sqrt{\pi}} \int_0^t \frac{g(s)}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} ds. \quad (1.33)$$

To show uniqueness, let u_1, u_2 be bounded solutions, so that also $v = u_1 - u_2$ is a bounded solution to (1.33) as well, with $g(t) \equiv 0$. But then its odd prolongation $(v(x, t) = -v(-x, t))$ on $x < 0$ is regular, bounded and it solves the global Cauchy problem for the heat equation with null initial value. The general theory says v must vanish identically, i.e. u_1 and u_2 coincide.

b) The functions w_1, w_2 are solutions of $w_t - w_{xx} = 0$ (on the whole (x, t) -plane); additionally

$$\begin{aligned} w_1(x, 0) &= e^x \cos x, & w_1(0, t) &= \cos 2t \\ w_2(x, 0) &= e^{-x} \cos x, & w_2(0, t) &= \cos 2t. \end{aligned}$$

We will modify these functions in order to have zero initial value and the same Dirichlet datum on $x = 0, t > 0$. To this end we recall from Problem 1.2.15 (page 29) that⁶

$$v_1(x, t) = \int_0^{+\infty} \Gamma^-(x, y, t) e^y \cos y dy \quad (1.34)$$

$$v_2(x, t) = \int_0^{+\infty} \Gamma^-(x, y, t) e^{-y} \cos y dy \quad (1.35)$$

solve $v_t - v_{xx} = 0$ on the quadrant $x > 0, t > 0$, with vanishing lateral datum and initial datum

$$v_1(x, 0) = e^x \cos x, \quad v_2(x, 0) = e^{-x} \cos x,$$

respectively. Then

$$u_1 = w_1 - v_1, \quad u_2 = w_2 - v_2$$

have zero initial value, Dirichlet value on $x = 0$ equal to $\cos 2t$ and it is not hard to check that they are different (for example at $(\frac{\pi}{2}, \frac{\pi}{2})$). The problem therefore has no unique solution. Note that w_1 (hence u_1) is unbounded on the quadrant.

6.1.11 Problems in dimension higher than one

We separate the variables twice.

Problem 1.2.23 (Dirichlet on the rectangle). *The surfaces of a thin rectangular plate of length a and width b are thermally insulated while the four edges are kept at zero temperature. Determine the evolution of the temperature, knowing its initial value.*

Solution. The problem is essentially two-dimensional, so that we can suppose $u = u(x, y, t)$. Then (for simplicity we fix the diffusion coefficient $D = 1$) we have the problem

$$u_t - (u_{xx} + u_{yy}) = 0, \quad 0 < x < a, 0 < y < b, t > 0$$

with initial condition

$$u(x, y, 0) = g(x, y)$$

and Dirichlet conditions

$$\begin{cases} u(0, y, t) = 0, u(a, y, t) = 0, & 0 < y < b, t > 0 \\ u(x, 0, t) = 0, u(x, b, t) = 0, & 0 < x < a, t > 0. \end{cases}$$

We can separate the variables. First we look for non-zero solutions of the form

$$u(x, y, t) = v(x, y)z(t)$$

satisfying the Dirichlet conditions. Substituting in the equation and separating variables gives:

$$\frac{v_{xx} + v_{yy}}{v} = \frac{z'}{z} = \lambda.$$

For z there are no problems; we have

$$z(t) = ce^{\lambda t}.$$

For v we obtain the *eigenvalue problem*

$$v_{xx} + v_{yy} = \lambda v \tag{1.36}$$

on the rectangle $(0, a) \times (0, b)$, with

$$\begin{cases} v(0, y) = 0, v(a, y) = 0, & 0 \leq y \leq b \\ v(x, 0) = 0, v(x, b) = 0, & 0 \leq x \leq a. \end{cases} \tag{1.37}$$

We separate again variables, setting

$$v(x, y) = X(x)Y(y).$$

Substituting in (1.36) and separating, we find

$$\frac{Y''(y)}{Y(y)} - \lambda = -\frac{X''(x)}{X(x)} = \mu$$

with μ constant. Set

$$\nu = -\lambda - \mu.$$

We have the following eigenvalue problems for X and Y :

$$\begin{cases} X'' + \mu X = 0 & \text{in } (0, a) \\ X(0) = X(a) = 0, \end{cases} \quad \begin{cases} Y'' + \nu Y = 0 & \text{in } (0, b) \\ Y(0) = Y(b) = 0. \end{cases}$$

We have already solved these problems in previous exercises. The eigenvalues and the corresponding eigenfunctions are:

$$\begin{aligned} X_m(x) &= A_m \sin\left(\frac{m\pi x}{a}\right), & \mu_m &= \frac{m^2\pi^2}{a^2}, & m &= 1, 2, \dots \\ Y_n(y) &= B_n \sin\left(\frac{n\pi y}{b}\right), & \nu_n &= \frac{n^2\pi^2}{b^2}, & n &= 1, 2, \dots \end{aligned}$$

As

$$\lambda = -(\nu + \mu),$$

we conclude that the eigenvalues for problem (1.36), (1.37) are

$$\lambda_{mn} = -\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right), \quad m, n = 1, 2, \dots$$

with eigenfunctions

$$v_{mn}(x, y) = C_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \quad m, n = 1, 2, \dots$$

Summarising, we have the solutions

$$u_{mn}(x, y, t) = C_{mn} e^{-\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) t} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right),$$

which vanish on the boundary of the rectangle. To match the initial condition we superpose the functions u_{mn} :

$$u(x, y, t) = \sum_{m,n=1}^{\infty} C_{mn} e^{-\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) t} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (1.38)$$

and impose

$$\sum_{m,n=1}^{\infty} C_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) = g(x, y).$$

If we assume g can be expanded in double sines-Fourier series, it suffices that the C_{mn} equal the corresponding Fourier coefficients of g :

$$C_{mn} = \frac{4}{ab} \int_0^a \int_0^b \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) g(x, y) dx dy.$$

As usual, if g is smooth enough, for instance of class C^1 on the closed rectangle, the series is uniformly convergent, and the fast convergence to zero of the exponentials ensures that (1.38) solve the problem.

6.1.12 Fourier transform on the half-plane

Problem 1.2.24 (Fourier transform on the half-plane). Let $g = g(x, y) : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$ be continuous and bounded. Using Fourier transforms solve the following Dirichlet problem on $S = \mathbb{R} \times (0, +\infty) \times (0, +\infty)$:

$$\begin{cases} u_t(x, y, t) - \Delta u(x, y, t) = 0 & x \in \mathbb{R}, y > 0, t > 0 \\ u(x, y, 0) = g(x, y) & x \in \mathbb{R}, y > 0 \\ u(x, 0, t) = 0 & x \in \mathbb{R}, t > 0. \end{cases}$$

Solution. Denote by $\widehat{u}(\xi, y, t) = \int_{\mathbb{R}} u(x, y, t) e^{-ix\xi} dx$ the partial Fourier transform of u in x . Since the transform of $u_{xx}(x, y, t)$ is $-\xi^2 \widehat{u}(\xi, y, t)$, $\widehat{u}(\xi, \cdot, \cdot)$ satisfies (formally) the Cauchy problem on the quadrant $y > 0, t > 0$,

$$\begin{cases} \widehat{u}_t - \widehat{u}_{yy} + \xi^2 \widehat{u} = 0 & y > 0, t > 0 \\ \widehat{u}(\xi, y, 0) = \widehat{g}(\xi, y) & y > 0 \\ \widehat{u}(\xi, 0, t) = 0 & t > 0, \end{cases}$$

where $\xi \in \mathbb{R}$. We eliminate the reaction term by setting $v(\xi, y, t) = e^{\xi^2 t} \widehat{u}(\xi, y, t)$; the function v solves $v_t - v_{yy} = 0$ with the same initial and boundary data. The reflection method used in Problem 1.2.15 on page 29 gives:

$$v(\xi, y, t) = \int_0^{+\infty} [\Gamma_1(y - z, t) - \Gamma_1(y + z, t)] \widehat{g}(\xi, z) dz$$

and then

$$u(\xi, y, t) = e^{-\xi^2 t} \int_0^{+\infty} [\Gamma_1(y - z, t) - \Gamma_1(y + z, t)] \widehat{g}(\xi, z) dz$$

where $\Gamma_1(y, t)$ is the fundamental solution for the operator $\partial_t - \partial_{yy}$. Note the inverse transform of $e^{-\xi^2 t} \widehat{g}(\xi, y)$ is

$$\int_{-\infty}^{+\infty} \Gamma_1(x - w, t) g(w, y) dw$$

and that

$$\begin{aligned} \Gamma_1(x, t) \Gamma_1(y, t) &= \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \frac{1}{2\sqrt{\pi t}} e^{-\frac{y^2}{4t}} \\ &= \frac{1}{4\pi t} e^{-\frac{x^2+y^2}{4t}} = \Gamma_1(x, y, t) \end{aligned}$$

where $\Gamma_1(x, y, t)$ is the fundamental solution for the operator $\partial_t - (\partial_{xx} + \partial_{yy})$. The final formula reads:

$$u(x, y, t) = \int_{-\infty}^{+\infty} \int_0^{+\infty} [\Gamma_1(x-w, y-z, t) - \Gamma_1(x-w, y+z, t)] g(w, z) dz dw.$$

As g is continuous and bounded, the study of the solution goes as in Problem 1.2.15. In particular, for any $x_0 \in \mathbb{R}$ and $y_0 > 0$, if $(x, y, t) \rightarrow (x_0, y_0, 0)$

$$u(x, y, t) \rightarrow g(x_0, y_0),$$

so u is continuous on the closure of S except possibly for the half-plane $y = 0$. Continuity along $y = 0$ holds precisely if $g(x, 0) = 0$.

5.1. The initial value problem for the heat equation

Consider the initial value problem for $u(x, t)$ where $x \in \mathbb{R}^n$

$$(5.2) \quad \begin{aligned} u_t &= \Delta u && \text{for } x \in \mathbb{R}^n \text{ and } t > 0, \\ u(x, 0) &= f(x) && \text{for } x \in \mathbb{R}^n. \end{aligned}$$

The irreversibility of its semiflow and the need to impose a growth condition as $|x| \rightarrow \infty$ in order to pick out a unique solution.

6.2 Schwartz solutions

The finite-dimensional phase space of the ODE is replaced by the infinite-dimensional function space \mathcal{S} ; then we think of a solution of the heat equation as a parametrized curve in the vector space \mathcal{S} .

6.2.1 Notations

By a convenient abuse of notation, we use the same symbol u to denote the scalar-valued function $u(x, t)$, where $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$, and the associated vector-valued function $u(t)$, where $u : [0, \infty) \rightarrow \mathcal{S}$. We write the vector-valued function corresponding to the associated scalar-valued function as $u(t) = u(\cdot, t)$.

In other words, u is continuous at t if

$$u(t) = \mathcal{S}\text{-}\lim_{h \rightarrow 0} u(t+h),$$

and u is differentiable at t with derivative $u_t(t)$ if

$$u_t(t) = \mathcal{S}\text{-}\lim_{h \rightarrow 0} \frac{u(t+h) - u(t)}{h}.$$

We will refer to this derivative as a strong derivative if it is understood that we are considering \mathcal{S} -valued functions and we want to emphasize that the derivative is defined as the limit of difference quotients in \mathcal{S} .

6.2.2 Spatial Fourier transform

PROOF. Since the spatial Fourier transform \mathcal{F} is a continuous linear map on \mathcal{S} with continuous inverse, the time-derivative of u exists if and only if the time derivative of $\hat{u} = \mathcal{F}u$ exists, and

$$\mathcal{F}(u_t) = (\mathcal{F}u)_t.$$

Moreover, $u \in C([0, \infty); \mathcal{S})$ if and only if $\hat{u} \in C([0, \infty); \mathcal{S})$, and $u \in C^k(0, \infty; \mathcal{S})$ if and only if $\hat{u} \in C^k(0, \infty; \mathcal{S})$.

Consider the initial value problem for $u(x, t)$ where $x \in \mathbb{R}^n$

$$(5.2) \quad \begin{aligned} u_t &= \Delta u && \text{for } x \in \mathbb{R}^n \text{ and } t > 0, \\ u(x, 0) &= f(x) && \text{for } x \in \mathbb{R}^n. \end{aligned}$$

Taking the Fourier transform of (5.2) with respect to x , we find that $u(x, t)$ is a solution with the regularity in (5.3) if and only if $\hat{u}(k, t)$ satisfies

$$(5.7) \quad \hat{u}_t = -|k|^2 \hat{u}, \quad \hat{u}(0) = \hat{f}, \quad \hat{u} \in C([0, \infty); \mathcal{S}) \cap C^1(0, \infty; \mathcal{S}).$$

Equation (5.7) has the unique solution (5.4).

$$(5.4) \quad \hat{u}(k, t) = \hat{f}(k) e^{-t|k|^2},$$

6.2.3 Green's function

$$(5.6) \quad \Gamma(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}.$$

The function $\Gamma(x, t)$ in (5.6) is called the Green's function or fundamental solution of the heat equation in \mathbb{R}^n . It is a C^∞ -function of (x, t) in $\mathbb{R}^n \times (0, \infty)$, and one can verify by direct computation that

$$(5.9) \quad \Gamma_t = \Delta \Gamma \quad \text{if } t > 0.$$

Also, since $\Gamma(\cdot, t)$ is a family of Gaussian mollifiers, we have

$$\Gamma(\cdot, t) \rightharpoonup \delta \quad \text{in } \mathcal{S}' \text{ as } t \rightarrow 0^+.$$

6.2.4 Smoothing

THEOREM 5.5. Suppose that $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^n)$. Define

$$u : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$$

by (5.5) where Γ is given in (5.6). Then $u \in C_0^\infty(\mathbb{R}^n \times (0, \infty))$ and $u_t = \Delta u$ in $t > 0$. If $1 \leq p < \infty$, then $u(\cdot, t) \rightarrow f$ in L^p as $t \rightarrow 0^+$.

PROOF. The Green's function Γ in (5.6) satisfies (5.9), and $\Gamma(\cdot, t) \in L^q$ for every $1 \leq q \leq \infty$, together with all of its derivatives. The dominated convergence theorem and Hölder's inequality imply that if $f \in L^p$ and $t > 0$, we can differentiate under the integral sign in (5.5) arbitrarily often with respect to (x, t) and that all of these derivatives approach zero as $|x| \rightarrow \infty$. Thus, u is a smooth, decaying solution of the heat equation in $t > 0$. Moreover, $\Gamma^t(x) = \Gamma(x, t)$ is a family of Gaussian mollifiers and therefore for $1 \leq p < \infty$ we have from Theorem 1.28 that $u(\cdot, t) = \Gamma^t * f \rightarrow f$ in L^p as $t \rightarrow 0^+$. \square

6.2.5 Irreversibility

then the corresponding solution

$$\hat{u}(k, t) = e^{-t|k|^2 - \sqrt{1+|k|^2}}$$

grows exponentially as $|k| \rightarrow \infty$ for every $t < 0$, and therefore $u(t)$ does not belong to \mathcal{S} (or even \mathcal{S}'). Physically, this means that the temperature distribution f cannot arise by thermal diffusion from any previous temperature distribution in \mathcal{S} (or \mathcal{S}'). The heat equation does, however, have a backward uniqueness property, meaning that if f arises from a previous temperature distribution, then (under appropriate assumptions) that distribution is unique [9].

Equivalently, making the time-reversal $t \mapsto -t$, we see that Schwartz-valued solutions of the initial value problem for the backward heat equation

$$u_t = -\Delta u \quad t > 0, \quad u(x, 0) = f(x)$$

do not exist for every $f \in \mathcal{S}$. Moreover, there is a loss of continuous dependence of the solution on the data.

6.2.6 Nonuniqueness

5.1.4. Nonuniqueness. A solution $u(x, t)$ of the initial value problem for the heat equation on \mathbb{R}^n is not unique without the imposition of a suitable growth condition as $|x| \rightarrow \infty$. In the above analysis, this was provided by the requirement that $u(\cdot, t) \in \mathcal{S}$, but the much weaker condition that u grows more slowly than $Ce^{a|x|^2}$ as $|x| \rightarrow \infty$ for some constants C, a is sufficient to imply uniqueness [9].

6.2.7 The Schrödinger equation

The initial value problem for the Schrödinger equation is

$$(5.13) \quad \begin{aligned} iu_t &= -\Delta u && \text{for } x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}, \\ u(x, 0) &= f(x) && \text{for } x \in \mathbb{R}^n, \end{aligned}$$

where $u : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{C}$ is a complex-valued function. A solution of the Schrödinger equation is the amplitude function of a quantum mechanical particle moving freely in \mathbb{R}^n . The function $|u(\cdot, t)|^2$ is proportional to the spatial probability density of the particle.

The Fourier solution of (5.13) is

$$(5.15) \quad \hat{u}(k, t) = e^{-it|k|^2} \hat{f}(k).$$

More generally, a particle moving in a potential $V : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the Schrödinger equation

$$(5.14) \quad iu_t = -\Delta u + V(x)u.$$

Unlike the free Schrödinger equation (5.13), this equation has variable coefficients and it cannot be solved explicitly for general potentials V .

Note. Formally, the Schrödinger equation is obtained by the transformation $t \mapsto -it$ of the heat equation to ‘imaginary time.’ The analytical properties of the heat and Schrödinger equations are, however, completely different and it is interesting to compare them.

6.2.8 Semigroups and groups

The solution of an $n \times n$ linear first-order system of ODEs for $\vec{u}(t) \in \mathbb{R}^n$,

$$\vec{u}_t = A\vec{u},$$

may be written as

$$\vec{u}(t) = e^{tA}\vec{u}(0) \quad -\infty < t < \infty$$

where $e^{tA} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the matrix exponential of tA . The finite-dimensionality of the phase space \mathbb{R}^n is not crucial here. As we discuss next, similar results hold for any linear ODE in a Banach space generated by a bounded linear operator.

EXAMPLE 5.21. For $1 \leq p < \infty$, let $A : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ be the bounded translation operator

$$Af(x) = f(x+1).$$

The solution $u \in C^\infty(\mathbb{R}; L^p)$ of the differential-difference equation

$$u_t(x, t) = u(x+1, t), \quad u(x, 0) = f(x)$$

is given by

$$u(x, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f(x+n).$$

Convolution operator

EXAMPLE 5.22. Suppose that $a \in L^1(\mathbb{R}^n)$ and define the bounded convolution operator $A : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ by $Af = a * f$. Consider the IVP

$$u_t(x, t) = \int_{\mathbb{R}^n} a(x-y)u(y) dy, \quad u(x, 0) = f(x) \in L^2(\mathbb{R}^n).$$

Taking the Fourier transform of this equation and using the convolution theorem, we get

$$\hat{u}_t(k, t) = (2\pi)^n \hat{a}(k) \hat{u}(k, t), \quad \hat{u}(k, 0) = \hat{f}(k).$$

The solution is

$$\hat{u}(k, t) = e^{(2\pi)^n \hat{a}(k)t} \hat{f}(k).$$

It follows that

$$u(x, t) = \int g(x-y, t) f(y) dy$$

6.3 Fourier 分析初步

参见于品数学分析讲义 65,66 节

Définition 3: L^1 函数的 Fourier 变换

对于函数 $f \in L^1(\mathbb{R}^n)$, 我们定义它的 Fourier 变换 \hat{f} (或者 $\mathcal{F}(f)$) 为如下的 \mathbb{R}^n 上的函数:

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx$$

其中 $\xi \in \mathbb{R}^n$ 。我们通常把 \hat{f} 的定义域 \mathbb{R}^n 称作是频率空间, 它的变量通常用 ξ 来表示。

我们考虑 $C(\mathbb{R}^n)$ 的线性子空间

$$C_0(\mathbb{R}^n) = \left\{ f \in C(\mathbb{R}^n) \mid \lim_{|x| \rightarrow \infty} f(x) = 0 \right\}$$

我们知道 $(C_0(\mathbb{R}^n), \|\cdot\|_\infty)$ 是 Banach 空间. (取一列 $C_0(\mathbb{R}^n)$ 内 $\{f_k\}_{k \geq 1} \rightrightarrows f$ 于 $C(\mathbb{R}^n)$, 验证 $f \in C_0(\mathbb{R}^n)$ 即可)

Lemme 1: 引理 335

给定赋范线性空间 $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$ 和它们之间的 $(\mathbb{C}-)$ 线性映射 $L: E \rightarrow F$. 那么, 如下三个论断是等价的:

- (1) L 在 $0 \in E$ 处连续;
- (2) L 是连续的;
- (3) L 是有界, 即存在 $C > 0$, 对每个 $e \in E$, 我们都有

$$\|L(e)\|_F \leq C\|e\|_E$$

Théorème 3: Theorem 4.48

对任意的 $f \in L^1(\mathbb{R}^n)$, $\widehat{f} \in C_0(\mathbb{R}^n)$. 特别地, 线性映射

$$\mathcal{F}: L^1(\mathbb{R}^n) \longrightarrow C_0(\mathbb{R}^n)$$

是连续线性映射, 即存在常数 C , 使得对任意的 $f \in L^1(\mathbb{R}^n)$, 我们有

$$\|\widehat{f}\|_{L^\infty} \leq C\|f\|_{L^1}.$$

显然

$$\|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}$$

由于 $C_0^\infty(\mathbb{R}^n)$ 在 $L^1(\mathbb{R}^n)$ 中稠密, 故只需要证明 $\widehat{\varphi} \in C_0(\mathbb{R}^n)$ 连续, 由 Lebesgue 控制收敛定理可知: 假设 \mathbb{R}^n 中的点列 $\{\xi_k\}_{k \geq 1} \rightarrow \xi$, 有

$$\widehat{f}(\xi_k) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi_k} f(x) dx \rightarrow \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(x) dx$$

控制函数为 $|f(x)|$.

对于 $\varphi \in C_0^\infty(\mathbb{R}^n)$, $k \leq n$, 我们有如下两个重要观点:

- 物理空间的求导等价于频率空间的乘法: $\widehat{\partial_k \varphi}(\xi) = i\xi_k \widehat{\varphi}(\xi)$
- 物理空间的乘法等价于频率空间的求导: $\widehat{-ix_k \varphi}(\xi) = \partial_{\xi_k} \widehat{\varphi}(\xi)$

实际上, 对任意的 $\varphi \in C_0^\infty(\mathbb{R}^n)$, 对任意的正整数 N , 利用第一个原则, 我们有

$$(1 + |\xi|^2)^N \widehat{\varphi}(\xi) = \mathcal{F}((1 - \Delta)^N \varphi)(\xi)$$

其中 Δ 是 Laplace 算子.

很明显, $(1 - \Delta)^N \varphi \in L^1(\mathbb{R}^n)$, 所以, 我们有

$$|\widehat{\varphi}(\xi)| \leq \frac{\|(1 - \Delta)^N \varphi\|_{L^1}}{(1 + |\xi|^2)^N}.$$

这表明 $\widehat{\varphi}$ 是衰减的并且我们对于 φ 求的导数越多, 那么衰减速度就越快. 特别地, 我们还证明了 $\widehat{\varphi} \in C_0(\mathbb{R}^n)$.

通常为了说明 \widehat{f} 在无穷远处趋于 0, 我们任取 $\epsilon > 0$, 再选取 $\varphi \in C_0^\infty(\mathbb{R}^n)$ 使得

$$\|f - \varphi\|_{L^1} < \frac{1}{2}\epsilon$$

从而对于任意的 ξ , 我们有

$$\begin{aligned} |\widehat{f}(\xi)| &= |\widehat{f}(\xi) - \widehat{\varphi}(\xi)| + |\widehat{\varphi}(\xi)| \\ &\leq \|\widehat{f} - \widehat{\varphi}\|_{L^\infty} + \frac{\|(1 - \Delta)^N \varphi\|_{L^1}}{(1 + |\xi|^2)^N} \\ &\leq \|f - \varphi\|_{L^1} + \frac{\|(1 - \Delta)^N \varphi\|_{L^1}}{(1 + |\xi|^2)^N} \\ &\leq \frac{\epsilon}{2} + \frac{\|(1 - \Delta)^N \varphi\|_{L^1}}{(1 + |\xi|^2)^N} \end{aligned}$$

当 $|\xi|$ 很大的时候, 有 $|\widehat{f}(\xi)| < \epsilon$.

然, 这些过程实际上都被包装在开始的命题上, 所以我们通常只对光滑有紧支集的函数 (在 L^1 中稠密) 来证明即可.

6.3.1 卷积

Théorème 4: 卷积对应于频率空间的乘积

对任意的 $f, g \in L^1(\mathbb{R}^n)$, 我们有

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$$

Proposition 1

假设 $f \in L^1(\mathbb{R}_x^n), g \in L^1(\mathbb{R}_\xi^n)$. 那么,

$$\int_{\mathbb{R}^n} \widehat{f}(\xi)g(\xi)d\xi = \int_{\mathbb{R}^n} f(x)\widehat{g}(x)dx$$

6.3.2 仿射坐标变换

Proposition 2

任给 $f \in L^1(\mathbb{R}^n), x_0 \in \mathbb{R}^n$ 和可逆线性变换

$$A : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

我们有 1) 物理空间的平移对应频率空间乘相应的频率, 即

$$\mathcal{F}(f(\cdot + x_0))(\xi) = e^{ix_0 \cdot \xi} \hat{f}(\xi).$$

2) 频率空间应该视作是余切丛, 即

$$\mathcal{F}(f \circ A)(\xi) = |\det(A)|^{-1} \hat{f}({}^t A^{-1} \xi).$$

证明: 第一部分是显然的; 为了证明第二部分, 我们直接利用换元公式:

$$\begin{aligned} \mathcal{F}(f \circ A)(\xi) &= \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(Ax) dx \\ &= |\det(A)|^{-1} \int_{\mathbb{R}^n} e^{-i(A^{-1}y) \cdot \xi} f(y) dy \\ &= |\det(A)|^{-1} \int_{\mathbb{R}^n} e^{-iy \cdot {}^t A^{-1} \xi} f(y) dy \end{aligned}$$

命题成立。

6.3.3 Gauss 函数的 Fourier 变换

Théorème 5: Gauss 函数的 Fourier 变换

对任意的正数 $\lambda > 0$, 我们有

$$\mathcal{F}\left(e^{-\lambda \frac{|x|^2}{2}}\right)(\xi) = \left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}} e^{-\frac{|\xi|^2}{2\lambda}}.$$

利上 Guass 函数的 Fourier 变换, 我们可以证明关于 Fourier 逆变换的定理。

Définition 4

对任意的 $g \in L^1(\mathbb{R}_\xi^n)$, 对任意的 $x \in \mathbb{R}^n$ 我们定义

$$(\mathcal{F}^{-1}(g))(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} g(\xi) d\xi.$$

Théorème 6: Fourier 逆变换

给定 $f \in L^1(\mathbb{R}_x^n)$, 如果 $\hat{f} \in L^1(\mathbb{R}_\xi^n)$, 那么, 我们有

$$f = \mathcal{F}^{-1}\hat{f}$$

其中, 上面等号成立是在 $L^1(\mathbb{R}^n)$ 的意义下的。

6.4 L^2 空间

注意到, $C_0^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ 是稠密的子空间。我们任意选取 $f \in C_0^\infty(\mathbb{R}^n)$, 很明显, $\hat{f} \in L^1(\mathbb{R}^n)$ (因为光滑性意味着衰减很快, 所以可积)。另外, 我们有

$$\bar{\hat{f}}(\xi) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} \overline{f(x)} dx = (2\pi)^n \mathcal{F}^{-1}(\bar{f})(\xi)$$

所以, 我们有

$$\begin{aligned} \|\hat{f}(\xi)\|_{L^2}^2 &= \int_{\mathbb{R}^n} \hat{f}(\xi) \overline{\hat{f}(\xi)} d\xi \\ &= (2\pi)^n \int_{\mathbb{R}^n} \hat{f}(\xi) \mathcal{F}^{-1}(\bar{f}(\xi)) d\xi \\ &= (2\pi)^n \int_{\mathbb{R}^n} f(x) \mathcal{F}(\mathcal{F}^{-1}(\bar{f}(\xi)))(x) d\xi \\ &= (2\pi)^n \int_{\mathbb{R}^n} f(x) \bar{f}(x) d\xi \end{aligned}$$

从而,

$$\|\hat{f}(\xi)\|_{L^2} = (2\pi)^{\frac{n}{2}} \|f\|_{L^2}$$

这表明定义在 $L^2(\mathbb{R}^n)$ 是稠密的子空间 $C_0^\infty(\mathbb{R}^n)$ 上的 Fourier 变换

$$\mathcal{F} : C_0^\infty(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n), \quad f \mapsto \hat{f}$$

是连续的。

$$\begin{array}{ccc} C_0^\infty(\mathbb{R}^n) & \xrightarrow{\iota} & L^2(\mathbb{R}^n) \\ & \searrow \mathcal{F} & \downarrow \mathcal{F} \\ & & L^2(\mathbb{R}^n) \end{array}$$

根据连续线性映射扩张的定理，我们就证明了

Théorème 7: 定理 4.55 (L^2 上的 Fourier 变换与 Planchrel 公式)

我们可以定义 Fourier 变换 \mathcal{F} :

$$\mathcal{F}: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$$

使得

$$(2\pi)^{\frac{n}{2}} \mathcal{F}: L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

是等距同构。特别地，对于 $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ ，我们有

$$\|\hat{f}\|_{L^2}^2 = (2\pi)^n \|f\|_{L^2}^2.$$

通过极化，我们有对任意的 $f, g \in L^2(\mathbb{R}^n)$ ，我们有

$$(\hat{f}, \hat{g})_{L^2} = (2\pi)^n (f, g)_{L^2}$$

证明。上述一切叙述对 $C_0^\infty(\mathbb{R}^n)$ 是成立的。对一般的 $f \in L^2(\mathbb{R}^n)$ ，用 $C_0^\infty(\mathbb{R}^n)$ 中函数逼近即可。 \square

Remarque 5. 上面的定理定义了 $f \in L^2(\mathbb{R}^n)$ 的 Fourier 变换，为了行文清楚，我们暂且把 L^2 意义下的 Fourier 变换记作 \mathcal{F}_2 。另外，对于 $g \in L^1(\mathbb{R}^n)$ ，它的 Fourier 变换是可以用 Fourier 积分表示的，我们把它记做 \mathcal{F}_1 ，也就是说

$$\mathcal{F}_1(g) = \int_{\mathbb{R}^n} g(x) e^{-ix\xi} dx$$

那么，对于 $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ ，我们有

$$\mathcal{F}_1(f) = \mathcal{F}_2(f)$$

Exemple 9. 考虑 \mathbb{R}^1 上的 L^2 一函数

$$u(x) = (1 + |x|)^{-\alpha}$$

其中, $\frac{1}{2} < \alpha \leq 1$ 。很明显, 我们知道 $e^{-ix\xi} u(x) \notin L^1(\mathbb{R}_x)$, 所以我们不能直接用 L^1 一函数的 Fourier 积分来写它的 Fourier 变换。然而, 我们知道

$$u_n(x) = u(x) \mathbf{1}_{|x| \leq n}$$

是 L^1 的, 我们可以先显式写下 u_n 的 Fourier 变换。由于序列 $\{u_n\}_{n \geq 1}$ 在 L^2 中逼近 u , 所谓, 我们有

$$\widehat{u} \stackrel{L^2}{=} \lim_{n \rightarrow \infty} \widehat{u_n}.$$

6.5 Schwartz 空间

6.5.1 Schwarz 函数（速降函数）

对任意的给定的函数 f , 对任意的多重指标 α , 我们采用如下的符号:

$$x^\alpha f(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n} f(x_1, \dots, x_n).$$

Définition 5: Schwartz 空间, Schwarz 函数（速降函数）

函数 φ 是 \mathbb{R}^n 上的光滑函数。如果 φ 满足如下的条件: 对任意的多重指标 α, β , 我们都有

$$x^\alpha \partial^\beta \varphi(x) \in L^\infty(\mathbb{R}^n)$$

那么, 我们就称 φ 是一个 **Schwartz** 函数或者是一个 **速降的函数**。我们把 \mathbb{R}^n 上所有的 Schwartz 函数所构成的线性空间称作是 **Schwartz 空间**, 并记作 $\mathcal{S}(\mathbb{R}^n)$ 。对于每个非负整数 $p \in \mathbb{Z}_{\geq 0}$, 我们定义如下的(一族)范数:

$$N_p(\varphi) = \sum_{|\alpha| \leq p, |\beta| \leq p} \|x^\alpha \partial^\beta \varphi(x)\|_{L^\infty(\mathbb{R}^n)}$$

在 $\mathcal{S}(\mathbb{R}^n)$ 上, 我们规定如下的收敛性(拓扑): 给定 Schwartz 函数的序列 $\{\varphi_k\}_{k=1,2,\dots} \subset \mathcal{S}(\mathbb{R}^n)$, 它收敛到 Schwartz 函数 $\varphi \in \mathcal{S}(\mathbb{R}^n)$, 指的是对任意的非负整数 p , 我们都有

$$\lim_{n \rightarrow \infty} N_p(\varphi_n - \varphi) = 0$$

我们把这个极限简写成

$$\varphi_k \xrightarrow{\delta(\mathbb{R}^n)} \varphi$$

Exemple 10. 我们已经见过很多 *Schwartz* 函数

- $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$;
- $e^{-x^2} \in \mathcal{S}(\mathbb{R}^n)$;
- 对于 *Schwartz* 函数 $\varphi \in \mathcal{S}(\mathbb{R}^n)$, 对它求若干次导数或者乘以一个多项式仍然是一个 *Schwartz* 函数, 即对任意的多重指标 α, β , 我们有

$$\begin{aligned} x^\alpha : \mathcal{S}(\mathbb{R}^n) &\rightarrow \mathcal{S}(\mathbb{R}^n) \\ \partial^\beta : \mathcal{S}(\mathbb{R}^n) &\rightarrow \mathcal{S}(\mathbb{R}^n) \end{aligned}$$

上面例子的验证我们留作作业。

给定一个 *Schwarz* 函数, 我们对它有如下的估计: 对于任何的多重指标 α 和 β , 其中 $|\alpha|, |\beta| \leq p$, 我们有

$$|(1+|x|)^{n+1} x^\alpha \partial^\beta \varphi(x)| \leq N_{p+n+1}(\varphi),$$

其中, n 是空间的维数。

从而, 对任意的 $x \in \mathbb{R}^n$, 我们有

$$|x^\alpha \partial^\beta \varphi(x)| \leq \frac{N_{p+n+1}(\varphi)}{(1+|x|)^{n+1}}.$$

上式右边的函数是可积的, 所以,

$$\|x^\alpha \partial^\beta \varphi(x)\|_{L^1(\mathbb{R}^n)} \leq C_n N_{p+d+1}(\varphi).$$

特别地, 我们可以对 $x^\alpha \partial^\beta \varphi(x)$ 用傅里叶积分来定义其傅里叶变换。作为推论, 我们还知道

$$\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n).$$

另外, 以上的估计是常用的技巧, 在后面的不少场合都会用到。

Théorème 8

$\mathcal{D}(\mathbb{R}^n)$ 在 $\mathcal{S}(\mathbb{R}^n)$ 中是稠密的, 即对任意的 $\varphi \in \mathcal{S}(\mathbb{R}^n)$, 存在函数序列 $\{\varphi_k\}_{k \geq 1} \subset \mathcal{D}(\mathbb{R}^n)$, 使得

$$\varphi_k \xrightarrow{\mathcal{S}(\mathbb{R}^n)} \varphi.$$

Note. $\varphi_k \xrightarrow{\mathcal{S}(\mathbb{R}^n)} \varphi$ 意味着对于任意非负整数 p , 有 $N_p(\varphi_k - \varphi) \rightarrow 0$.

Note. 光滑函数乘以一个函数, 结果仍为光滑函数, 当且仅当另一个函数也是光滑函数.

证明. 我们选取有紧支集的光滑函数 $\chi(x)$, 使得

$$\begin{cases} \chi(x) = 1, & |x| \leq 1 \\ 0 \leq \chi(x) \leq 1 \end{cases}$$

对于 $\varphi(x) \in \mathcal{S}(\mathbb{R}^n)$, 我们令

$$\varphi_k(x) = \chi\left(\frac{x}{k}\right)\varphi(x) \in \mathcal{D}(\mathbb{R}^n)$$

我们只要证明, 对任意的非负整数 p , 我们有

$$N_p(\varphi_k - \varphi) \rightarrow 0,$$

即可。对于满足 $|\alpha| \leq p, |\beta| \leq p$ 的多重指标, 我们有

$$\begin{aligned} x^\alpha \partial^\beta (\varphi - \varphi_k) &= x^\alpha \partial^\beta \left(\left(1 - \chi\left(\frac{x}{k}\right)\right) \varphi \right) \\ &= \left(1 - \chi\left(\frac{x}{k}\right)\right) x^\alpha \partial^\beta \varphi - \underbrace{\sum_{0 \neq \gamma \leq \beta} \frac{1}{k^{|\gamma|}} \frac{\beta!}{\gamma!(\beta - \gamma)!} \underbrace{x^\alpha \partial^{\beta-\gamma} \varphi(x)}_{|\cdot| \leq N_p(\varphi)} (\partial^\gamma \chi)\left(\frac{x}{k}\right)}_{\rightarrow 0, \text{ as } k \rightarrow \infty} \end{aligned}$$

上式的第二个求和部分有 k^{-1} 这样的衰减因子, 所以极限为 0。对于第一项, 由于 χ 在半径为 1 的球内部为 1, 所以然而, 我们有

$$\begin{aligned} \left| \left(1 - \chi\left(\frac{x}{k}\right)\right) x^\alpha \partial^\beta \varphi \right| &\leq 1_{|x| \geq k}(x) \cdot |x|^{-2} \cdot |x^{\alpha+2} \partial^\beta \varphi| \\ &\leq \frac{1}{k^2} N_{p+2}(\varphi) \rightarrow 0 \end{aligned}$$

这就完成了证明。 \square

6.5.2 Schwarz 函数的 Fourier 变换

Théorème 9: $\mathcal{S}(\mathbb{R}^n)$ 上的 Fourier 变换

如果 $\varphi \in \mathcal{S}(\mathbb{R}^n)$ 是 Schwartz 函数, 那么, $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ 。在 Schwartz 函数空间上的 Fourier 变换:

$$\mathcal{F}: \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n), \quad \varphi \mapsto \widehat{\varphi}(\xi)$$

满足如下的性质：对任意的 $p \in \mathbb{Z}_{\geq 0}$ ，存在常数 $C_p > 0$ ，使得对每个 $\varphi \in \mathcal{S}(\mathbb{R}^n)$ ，我们都有

$$N_p(\hat{\varphi}) \leq C_p N_{p+n+1}(\varphi)$$

特别的， $\mathcal{F}: \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ 是连续的线性同构，即对任意的在 $\mathcal{S}(\mathbb{R}^n)$ 中收敛的函数序列

$$\varphi_k \xrightarrow{s} \varphi, \quad k \rightarrow \infty,$$

我们有

$$\widehat{\varphi_k} \xrightarrow{s} \widehat{\varphi}, \quad k \rightarrow \infty.$$

另外，对任意的 $\varphi \in \mathcal{S}(\mathbb{R}^n)$ ，我们还有公式

$$\widehat{\partial_k \varphi} = i\xi_k \widehat{\varphi}, \quad \widehat{x_k \varphi} = i\partial_k \widehat{\varphi}$$

证明。我们首先证明叙述中的最后两个恒等式。对任意的 Schwartz 函数 φ ，对任意的 $k \leq n$ ，利用分部积分，我们有

$$\widehat{\partial_k \varphi}(\xi) = - \int_{\mathbb{R}^n} (-i\xi_k) e^{-ix \cdot \xi} \varphi(x) dx = i\xi_k \widehat{\varphi}(\xi)$$

第二个等式要用 Lebesgue 控制收敛的推论（积分与求导数可交换），我们有

$$\partial_{\xi_k} \widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} (-ix_k) e^{-ix \cdot \xi} \varphi(x) dx = \widehat{-ix_k \varphi}(\xi)$$

现在证明定理中的不等式（从而证明了 Fourier 变换 \mathcal{F} 的像也落在 $\mathcal{S}(\mathbb{R}^n)$ 中）。固定两个多重指标 α 和 β ，其中 $|\alpha|, |\beta| \leq p$ 。利用已经证明的公式，我们就有

$$\begin{aligned} |\xi^\alpha \partial_\xi^\beta \widehat{\varphi}(\xi)| &= |\partial^\alpha (x^\beta \varphi)| \\ &\leq C_p N_{p+n+1}(\varphi) \leq \|\partial^\alpha (x^\beta \varphi)(x)\|_{L^1} \end{aligned}$$

Fourier 变换的连续性可以通过这个不等式得到：对任意给定 p ，我们有

$$N_p(\widehat{\varphi_k} - \widehat{\varphi}) \leq C_p N_{p+n+1}(\varphi_k - \varphi) \rightarrow 0$$

按照定义，我们就有

$$\widehat{\varphi_k} \xrightarrow{s} \widehat{\varphi}, \quad k \rightarrow \infty.$$

最后，我们来说明 \mathcal{F} 是同构。实际上，我们可以直接考虑 Fourier 变换的逆 \mathcal{F}^{-1} ，因为 $\widehat{\varphi} \in L^1$ ，所以之前定义的 \mathcal{F}^{-1} 在此也是良好定义的。此时，我们已经证明了 \mathcal{F} 与 \mathcal{F}^{-1} 互为逆映射，所以命题得证（ \mathcal{F}^{-1} 也是连续的）。□

6.5.3 缓增分布

Définition 6: 缓增的分布

假设 $u \in \mathcal{D}'(\mathbb{R}^n)$ 是一个分布。如果存在非负整数 p 和常数 $C > 0$, 使得对每个 $\varphi \in \mathcal{D}(\mathbb{R}^n)$, 我们都有

$$|\langle u, \varphi \rangle| \leq C N_p(\varphi)$$

我们就说 u 是一个缓增的分布。我们用 $\mathcal{S}'(\mathbb{R}^n)$ 来表示所有缓增分布的集合, 很明显

$$\mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$$

是线性子空间。

6.6 Functional analysis in pde

6.6.1 Background of Sobolev Spaces

Let $\Omega \subset \mathbb{R}^n$ be a domain. The Sobolev spaces we will use most are: the Hilbert space

$$H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega; \mathbb{R}^n)\}$$

(all derivatives are meant in distributional sense) with the inner product

$$(u, v)_{H^1(\Omega)} = \int_{\Omega} [\nabla u \cdot \nabla v + uv] dx$$

Théorème 10

证明 $H^1(a, b) \subset C([a, b])$.^a

^aThis is false in dimension $n \geq 2$.

证明. 设 $u \in H^1(a, b)$ 。我们需要证明 u 在 $[a, b]$ 上连续。由于 $u \in H^1(a, b)$, 则 $u \in L^2(a, b)$ 且 u' (弱导数) $\in L^2(a, b)$ 。

根据 Sobolev 嵌入定理, 如果 $\Omega \subset \mathbb{R}^n$ 是一个有界开集, 且 $\partial\Omega$ 足够光滑, 那么当 $k > \frac{n}{p}$ 时, $W^{k,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ 。

在我们的例子中, $\Omega = (a, b) \subset \mathbb{R}$, 所以 $n = 1$ 。我们有 $u \in H^1(a, b) = W^{1,2}(a, b)$, 因此 $k = 1$ 且 $p = 2$ 。因为 $1 > \frac{1}{2}$, Sobolev 嵌入定理告诉我们 $W^{1,2}(a, b) \hookrightarrow C([a, b])$ 。

这意味着存在一个连续函数 $\tilde{u} \in C([a, b])$ 使得 $u = \tilde{u}$ 几乎处处成立。由于我们通常将 $H^1(a, b)$ 中的元素视为等价类 (即, 如果两个函数在 Lebesgue 测度意义下相等, 则认为它们是等价的), 我们可以选取这个连续的代表 \tilde{u} 作为 u 。

因此, u 在 $[a, b]$ 上有一个连续的代表, 这意味着 $H^1(a, b) \subset C([a, b])$ 。

□

6.6.2 Lax-Milgram Theorem

- *Abstract variational problem (stationary case):* determine $u \in V$ such that

$$B(u, v) = Fv, \quad \forall v \in V \quad (6.1)$$

where B is a bilinear form on V and $F \in V^*$.

Theorem 6.1 (Lax-Milgram). *If B is continuous and coercive with coercivity constant a , there exists a unique solution \bar{u} to problem (6.1), and the following stability estimate holds:*

$$\|\bar{u}\|_V \leq \frac{1}{a} \|F\|_{V^*}.$$

Moreover, if B is self-adjoint, then \bar{u} is the unique minimiser of the ‘energy’ functional

$$E(v) = \frac{1}{2} B(v, v) - Fv, \quad \text{for } v \in V.$$

Statement

Imagine you have:

- (1) A special kind of “space” where you can measure distances and angles (this is the **Hilbert space**, H).
- (2) A rule that takes any two items from this space and gives you a number.

This rule needs to be “nice” in two ways:

- **It doesn’t blow up:** The number you get isn’t excessively large compared to the “size” of the two items (this is **boundedness**).
- **It’s “positive” in a strong way:** When you use the rule with the same item twice, the number you get is significantly positive and related to the “size” of that item squared (this is **coercivity**). This rule is your **bilinear form**, $B(u, v)$.

- (3) Another rule that takes any one item from your space and gives you a number, and this rule is also "nice" (it doesn't blow up) (this is your **bounded linear functional**, $L(v)$).

The Lax-Milgram Theorem then says:

If your "combining rule" (B) is "nice" (bounded and coercive), then for any "measuring rule" (L) you pick, you can always find **one and only one** special item (u) in your space such that:

When you combine this special item (u) with *any* other item (v) using your rule B , you get the exact same number as when you measure that *other* item (v) using your rule L .

In short:

Under certain "good behavior" conditions for how you combine and measure things in a special space, a specific type of equation ($B(u, v) = L(v)$) is guaranteed to have exactly one solution (u) in that space.

Why it matters (the even simpler version):

It's a powerful math tool that tells us that certain types of problems (often coming from physics and engineering, like how heat spreads or structures bend) definitely have a unique answer, as long as the problem is "well-behaved" in a specific mathematical sense.

Importance of Lax-Milgram Theorem

The Big Idea: Does My Problem Even Have an Answer?

Imagine you're an engineer or a scientist. You've written down a mathematical equation (often a "partial differential equation" or PDE) that you believe describes a real-world situation. Before you spend a lot of time (and computer power) trying to *find* the answer, you'd want to know:

- (1) **Does an answer even exist?**
- (2) **If it exists, is there only one unique answer?** (If there are many, which one is the "right" one for your physical situation?)

This is where the Lax-Milgram theorem comes in. It's a powerful mathematical tool that can answer "YES!" to both questions for a specific class of problems.

How it Works (Simplified):

- (1) **Hard Problem:** The original PDE is often tricky to solve directly.
- (2) **Easier Version (Weak Formulation):** Mathematicians have a way to rephrase the hard problem into an "easier" (though still abstract) form. This easier form is an equation that looks like $B(u, v) = L(v)$.
- (3) **Lax-Milgram's Job:** If the parts of this "easier" equation (B and L) satisfy certain "good behavior" conditions (boundedness and coercivity), the Lax-Milgram theorem guarantees that there's exactly one solution u to this easier version.
- (4) **Good News:** This "weak solution" to the easier version is often good enough for practical purposes and, in many cases, can be shown to be the actual solution to the original hard problem.

Why is this "guarantee" important?

- **For Scientists/Engineers:** It tells them their mathematical model of the physical world is well-posed (meaning it has a unique solution). This gives them confidence that their model makes sense.
- **For Computer Simulations:** Many complex engineering and science problems are solved using computer simulations (e.g., Finite Element Method - FEM). These methods essentially try to find an approximate solution to the "easier" weak formulation. The Lax-Milgram theorem provides the theoretical backbone, assuring us that the problem these numerical methods are trying to solve actually *has* a unique solution to begin with. Without this, the computer might be chasing a ghost!

So, while you might not use the Lax-Milgram theorem directly in everyday calculations, it's a crucial piece of foundational mathematics that makes a lot of advanced engineering analysis and scientific computation possible and reliable. It gives a green light for many solution-seeking processes.

Application

Problem 6.2.1 (Dirichlet conditions). Write the variational formulation of the problem:

$$\begin{cases} (x^2 + 1)u'' - xu' = \sin 2\pi x & 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases}$$

Show that it has a unique solution $u \in H_0^1(0, 1)$ and find a constant C for which

$$\|u'\|_{L^2(0,1)} \leq C.$$

Solution. As the Dirichlet conditions are homogeneous, we choose test functions in $H_0^1(0, 1)$ (which is the closure, for the usual norm, of $C_0^1(0, 1)$). Multiply the equation by $v \in H_0^1(0, 1)$ and integrate. We find:

$$\int_0^1 [(x^2 + 1)u''(x) - xu'(x)] v(x) dx = \int_0^1 \sin(2\pi x)v(x) dx. \quad (6.2)$$

Integrating by parts we may write

$$\begin{aligned} & \int_0^1 (x^2 + 1)u''(x)v(x) dx \\ &= [(x^2 + 1)u'(x)v(x)]_0^1 - \int_0^1 u'(x) \frac{d}{dx} [(x^2 + 1)v(x)] dx = \\ &= - \int_0^1 [(x^2 + 1)u'(x)v'(x) + 2xu'(x)v(x)] dx \end{aligned}$$

and then substituting into (6.2),

$$\int_0^1 [(x^2 + 1)u'(x)v'(x) + 3xu'(x)v(x)] dx = - \int_0^1 \sin(2\pi x)v(x) dx.$$

Setting

$$\begin{aligned} B(u, v) &= \int_0^1 [(x^2 + 1)u'(x)v'(x) + 3xu'(x)v(x)] dx, \\ Fv &= - \int_0^1 \sin(2\pi x)v(x) dx, \end{aligned}$$

we obtain the following variational formulation: *find $u \in H_0^1(0, 1)$ such that*

$$B(u, v) = Fv \quad \text{for any } v \in H_0^1(0, 1). \quad (6.3)$$

Note that if $u \in C^2, v \in C^1$ satisfy (6.3), we can repeat the integration the other way around to obtain

$$\int_0^1 [(x^2 + 1)u''(x) - xu'(x) - \sin(2\pi x)]v(x) dx \quad \text{for all } v \in C_0^1(0, 1).$$

As v is arbitrary, we deduce

$$(x^2 + 1)u''(x) - xu'(x) - \sin(2\pi x) = 0 \quad \text{in } (0, 1)$$

so that u is the classical solution to the initial problem. This indicates that the variational formulation is coherent with the classical one for regular solutions. For the analysis we need the Lax-Milgram theorem. Clearly, B is bilinear and F is linear. As for the continuity, by the Schwarz and Poincaré inequalities (the latter holds here with $C_P = 1/\pi$, see Exercise 5.3.13, Chap. 5, page 312), we get

$$|B(u, v)| \leq \|x^2 + 1\|_{L^\infty} \|u'\|_{L^2} \|v'\|_{L^2} + \|3x\|_{L^\infty} \|u'\|_{L^2} \|v\|_{L^2} \leq \left(2 + \frac{3}{\pi}\right) \|u'\|_{L^2} \|v'\|_{L^2}$$

(where $L^p = L^p(0, 1)$) and

$$|Fv| \leq \|\sin 2\pi x\|_{L^2} \|v\|_{L^2} \leq \frac{\sqrt{2}}{2\pi} \|v'\|_{L^2}.$$

Concerning the coercivity of B we have to estimate (from below)

$$B(u, u) = \int_0^1 [(x^2 + 1)(u')^2 + 3xu'u] dx.$$

Noting that

$$uu' = (u^2)'/2,$$

and integrating by parts, using Poincaré's inequality, we can write

$$\begin{aligned} \int_0^1 3xuu' dx &= \int_0^1 \frac{3}{2}x(u^2)' dx = - \int_0^1 \frac{3}{2}u^2 dx \\ &= -\frac{3}{2}\|u\|_{L^2}^2 \geq -\frac{3}{2\pi^2}\|u'\|_{L^2}^2. \end{aligned}$$

Since $x^2 + 1 \geq 1$,

$$B(u, u) \geq a\|u'\|_{L^2}^2,$$

where $a = 1 - \frac{3}{2\pi^2} > 0$. Hence B is coercive and we can apply Lax–Milgram to obtain the existence of a unique solution u to (6.3). Moreover,

$$\|u'\|_{L^2} \leq \frac{1}{a} \|F\|_{H^{-1}} \leq \left(1 - \frac{3}{2\pi^2}\right) \frac{\sqrt{2}}{2\pi},$$

which is the required inequality.

6.6.3 例题

Exercice 1: 非线性算子的连续性; 紧性

Problem 5.2.8 (Nonlinear compact operator). Write $H = L^2(0, 1)$ and consider the nonlinear operator

$$T[f](t) = \int_0^1 (t + f(s))^2 ds.$$

- a) Show T is well defined and continuous from H to H .
- b) Using the criterium for subsets of $L^2(0, 1)$ (see [18, Chap. 6]), prove that T is compact.

一个关键的定理是：对于线性算子 $T : X \rightarrow Y$ ，以下说法是等价的：

- (1) T 在 X 上是连续的。
- (2) T 在 X 中的某一点 x_0 处是连续的。
- (3) T 在原点 $0_X \in X$ 处是连续的。
- (4) T 是有界的 (Bounded)。这是最常用和最重要的等价条件。

有界线性算子的定义：

线性算子 $T : X \rightarrow Y$ 称为有界的，如果存在一个常数 $M \geq 0$ ，使得对于所有的 $x \in X$ ，都有：

$$\|T(x)\|_Y \leq M\|x\|_X$$

这个常数 M 称为算子 T 的一个界。最小的这样的 M 称为算子 T 的范数，记为 $\|T\|$ 。

$$\|T\| = \sup_{\|x\|_X=1} \|T(x)\|_Y = \sup_{x \neq 0_X} \frac{\|T(x)\|_Y}{\|x\|_X}$$

我们可以证明：对于任意的 $f \in H$,

$$\|T[f]\|_H^2 = \int_0^1 \int_0^1 (t + f(s))^2 ds dt \leq (2 + 2\|f\|_H^2)^2$$

故 T 良定义. 接下来直接验证连续性: 对于 $f, g \in H$,

$$\begin{aligned} \|T[f] - T[g]\|_H^2 &= \int_0^1 \left[\int_0^1 |(t + f(s))^2 - (t + g(s))^2| ds \right]^2 dt \\ &= \int_0^1 \left[\int_0^1 |(f(s) - g(s))(2t + f(s) + g(s))| ds \right]^2 dt \\ &\stackrel{\text{Cauchy}}{\leq} \int_0^1 \left[\int_0^1 |f(s) - g(s)|^2 ds \cdot \int_0^1 \underbrace{|2t + f(s) + g(s)|^2}_{\leq 2[4t^2 + (f(s) + g(s))^2]} ds \right] dt \\ &= \|f - g\|_H^2 \left(\frac{8}{3} + \|f + g\|_H^2 \right) \end{aligned}$$

若 $\|f - g\|_H \rightarrow 0$, 那么 $\|T[f] - T[g]\|_H \rightarrow 0$. 故 f 连续.

接下来验证紧性

b) Write

$$\mathcal{F} = \{f \in H : \|f\|_2 \leq M\}.$$

In order to prove that $T[\mathcal{F}]$ is relatively compact in H we shall use the criterion for L^2 -compactness, and check whether $T[\mathcal{F}]$ is bounded and there exist C, α such that

$$\|T[f](\cdot + h) - T[f](\cdot)\|_2 \leq C|h|^\alpha \quad (5.9)$$

for any $f \in \mathcal{F}$ vanishing outside $(0, 1)$. As we saw in part a), from (5.8) we have, for any $f \in \mathcal{F}$,

$$\|T[f]\|_2^2 \leq [2 + 2\|f\|_2^2]^2 \leq [2 + 2M^2]^2,$$

and so $T[\mathcal{F}]$ is bounded. But we also have

$$\begin{aligned} \|T[f](\cdot + h) - T[f](\cdot)\|_2^2 &\leq \int_0^1 \left[\int_0^1 (t + h + f(s))^2 - (t + f(s))^2 ds \right]^2 dt = \\ &= \int_0^1 \left[\int_0^1 (2h(t + f(s)) + h^2) ds \right]^2 dt \leq h^2 \int_0^1 \left[\int_0^1 (h + 2t + 2f(s)) ds \right]^2 dt \end{aligned}$$

and Schwarz's inequality plus (5.8) give (5.9) with $\alpha = 2$.

Exercice 2

Problem 5.2.9 (Iterated projections^a). Let V and W be closed subspaces in a Hilbert space H . Define the sequence $\{x_n\}$ of projections as follows: $x_0 \in H$ is given, and

$$x_{2n+1} = P_W x_{2n}, \quad x_{2n+2} = P_V x_{2n+1} \quad \text{when } n \geq 0.$$

Prove the following assertions:

- a) $V \cap W = \{0\}$ implies $x_n \rightarrow 0$.
- b) $V \cap W \neq \{0\}$ implies $x_n \rightarrow P_{V \cap W} x_0$.

Hint. a) Show, in this order: $\|x_n\|$ decreases, $x_n \rightarrow 0$ and $\|x_n\|^2 = (x_{2n-1}, x_0)$; b) Reduce to previous case by subtracting $P_{V^\perp \cap W} x_0$.

^a This problem is related to Schwarz's alternating method, see Problem 6.2.15, Chap. 6 (page 361).

Note. If any subsequence of a sequence converges to x , then the sequence converges to x . (Proved by limsup and liminf)

命题 6.4 (子列极限命题)

(a): 给定 $x \in \mathbb{R} \cup \{+\infty, -\infty\}$, $\lim_{n \rightarrow \infty} x_n = x$ 的充分必要条件是对任何广义存在的 $\lim_{k \rightarrow \infty} x_{n_k}$, 都有

$$\lim_{k \rightarrow \infty} x_{n_k} = x.$$

(b): 设 $m \in \mathbb{N}$, 若 $\lim_{n \rightarrow \infty} x_{mn+r}$, $\forall r = 0, 1, 2, \dots, m-1$ 相同, 则 $\lim_{n \rightarrow \infty} x_n$ 存在且

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{mn}.$$

Solution. a) The idea – as we are looking at projections – is that the sequence of norms should decrease. Therefore $\{x_n\}$ will be bounded, with finite limit, in particular equal to zero. Let us check that $\|x_n\|$ decreases. Since

$$(x_{n+1}, x_n) = (x_{n+1}, x_{n+1}) = \|x_{n+1}\|^2,$$

for every $n \geq 0$, we have

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|x_{n+1}\|^2 - 2(x_{n+1}, x_n) + \|x_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\|x_{n+1}\|^2 + \|x_n\|^2 \\ &= -\|x_{n+1}\|^2 + \|x_n\|^2 \end{aligned}$$

so $\|x_n\| \geq \|x_{n+1}\|$ and $\|x_n\| \downarrow l \geq 0$. Moreover

$$\|x_{n+1} - x_n\| \rightarrow 0.$$

We will show that $x_n \rightharpoonup 0$ weakly and then $l = 0$ (see Problem 5.2.7 on page 286).

Let $\{x_{2n_k}\}$ be any weakly convergent subsequence of $\{x_{2n}\}$: $x_{2n_k} \rightharpoonup x$, with x in V . As also $x_{2n_k+1} \rightharpoonup x$ (in fact the distance between x_{2n_k} and x_{2n_k+1} tends to 0) one has $x \in W$, and so $x = 0$. But the subsequence is arbitrary, so $x_{2n} \rightharpoonup 0$. Similarly for $\{x_{2n+1}\}$, and altogether $x_n \rightharpoonup 0$.

Now, to fix ideas let us assume $x_n \in V$, and therefore $x_{n-1} \in W$. As orthogonal projections are symmetric operators,

$$\|x_n\|^2 = (x_n, x_{n-1}) = (x_n, P_W x_{n-2}) = (P_W x_n, x_{n-2}) = (x_{n+1}, x_{n-2}).$$

Iterating the argument gives

$$\|x_n\|^2 = (x_{n+1}, x_{n-2}) = (x_{n+2}, x_{n-3}) = \cdots = (x_{2n-1}, x_0)$$

and since $x_{2n-1} \rightharpoonup 0$ we see that $\|x_n\| \rightarrow 0$.

b) If $V \cap W \neq \{0\}$ we set

$$z_0 = x_0 - P_{V \cap W} x_0.$$

The sequence starting with z_0 and generated by projecting on V and W as before, is given by

$$z_n = x_n - P_{V \cap W} x_0.$$

It is easy to see that $P_{V \cap W} z_n = 0$, and then z_n belongs to $V \setminus (V^\perp \cap W)$ or to $W \setminus (V \cap W^\perp)$, whose intersection reduces to $\{0\}$. In this way one falls back to the previous situation, whence $z_n \rightharpoonup 0$, that is $x_n \rightharpoonup P_{V \cap W} x_0$.

Exercice 3: yau-22-problem 4

Problem 4. Let $C([0, 1])$ be the space of all continuous \mathbb{C} -valued functions equipped with L^∞ -norm. Let $\mathbf{P} \subset C([0, 1])$ be a closed linear subspace. Assume that the elements of \mathbf{P} are polynomials. Prove that $\dim \mathbf{P} < \infty$.

Main Idea:

The core strategy is to demonstrate that the closed unit ball in the subspace \mathbf{P} is compact. A fundamental theorem in functional analysis states that a normed linear space is finite-dimensional if and only if its closed unit ball is

compact. By proving the compactness of the unit ball of \mathbf{P} , the proof concludes that \mathbf{P} must be finite-dimensional.

Sketch of the Proof (Logical Flow):

- (1) **Problem Setup:** We have a closed linear subspace \mathbf{P} of $C([0, 1])$ (continuous functions on $[0, 1]$ with L^∞ -norm), and all elements of \mathbf{P} are polynomials. We want to prove $\dim \mathbf{P} < \infty$.

(2) **Introduce Slope Operators:**

- Define a family of linear operators $T_{(x,y)} : \mathbf{P} \rightarrow \mathbf{C}$ (where \mathbf{C} is the complex numbers) for distinct $x, y \in [0, 1]$:

$$T_{(x,y)}(u) = \frac{u(x) - u(y)}{x - y}$$

- This is essentially the slope of the polynomial u between x and y .

(3) **Show Pointwise Boundedness of these Operators:**

- For any fixed polynomial $u \in \mathbf{P}$, its derivative u' is continuous on $[0, 1]$ and thus bounded (i.e., $\|u'\|_{L^\infty} < \infty$).
- By the Mean Value Theorem, $|T_{(x,y)}(u)| = |u'(\xi)| \leq \|u'\|_{L^\infty}$ for some ξ between x and y .
- This means for each u , the set $\{T_{(x,y)}(u)\}$ is bounded.

(4) **Apply Banach-Steinhaus Theorem (Uniform Boundedness Principle):**

- \mathbf{P} is a closed subspace of a Banach space ($C([0, 1])$), so \mathbf{P} itself is a Banach space.
- Since the family of continuous linear operators $\{T_{(x,y)}\}$ is pointwise bounded on the Banach space \mathbf{P} , the Banach-Steinhaus Theorem implies that their operator norms are uniformly bounded.
- So, there exists a constant $C > 0$ such that $\|T_{(x,y)}\|_{\mathbf{P} \rightarrow \mathbf{C}} \leq C$ for all (x, y) .

(5) **Establish Equicontinuity of the Unit Ball of \mathbf{P} :**

- Consider the unit ball $B = \{u \in \mathbf{P} \mid \|u\|_{L^\infty} \leq 1\}$.
- For any $u \in B$:

$$\left| \frac{u(x) - u(y)}{x - y} \right| = |T_{(x,y)}(u)| \leq \|T_{(x,y)}\|_{\mathbf{P} \rightarrow \mathbf{C}} \cdot \|u\|_{L^\infty} \leq C \cdot 1 = C$$

- This implies $|u(x) - u(y)| \leq C|x - y|$ for all $u \in B$.
- This is the definition of uniform Lipschitz continuity for the functions in B , which in turn implies that the family B is equicontinuous.

(6) **Apply Arzelà-Ascoli Theorem:**

- The unit ball B is defined on the compact set $[0, 1]$.
- B is equicontinuous (from step 5).
- B is uniformly bounded (since for $u \in B$, $\|u\|_{L^\infty} \leq 1$, meaning $|u(x)| \leq 1$ for all x).
- The Arzelà-Ascoli Theorem states that a family of functions that is equicontinuous and pointwise (or uniformly) bounded on a compact set is relatively compact. Since B is also closed (it's a closed unit ball), it is compact.

(7) **Conclude Finite Dimensionality:**

- A normed linear space is finite-dimensional if and only if its closed unit ball is compact.
- Since the closed unit ball B of \mathbf{P} has been shown to be compact, \mathbf{P} must be finite-dimensional.

In essence, the proof cleverly uses the properties of polynomials (differentiability) to construct a family of operators. The Banach-Steinhaus theorem then provides a crucial uniform bound. This bound leads to the equicontinuity of the unit ball, which, combined with its boundedness, allows the Arzelà-Ascoli theorem to establish its compactness, ultimately proving finite dimensionality.

6.7 Fundamental Solution: Laplace equation in

$$\mathbb{R}^3$$

Okay, let's deduce the fundamental solution of the Laplacian more intuitively. The **fundamental solution** $\Phi(x)$ is defined as the solution to the equation:

$$\Delta\Phi(x) = \delta(x)$$

where Δ is the Laplacian operator and $\delta(x)$ is the Dirac delta function, representing a point source at the origin. The previous proof concerning the harmonic

function $u(x)$ in \mathbb{R}^3 used the representation:

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} dy \quad \text{for } \Delta u = \rho$$

This implies that the fundamental solution satisfying $\Delta\Phi(x) = \delta(x)$ is $\Phi(x) = -\frac{1}{4\pi|x|}$ in \mathbb{R}^3 . Let's see how we can arrive at this.

6.7.1 1. Intuitive Deduction using Radial Symmetry and Gauss's Law (Divergence Theorem)

This approach is very physical and geometric, especially for an operator like the Laplacian.

a. Radial Symmetry:

The Dirac delta function $\delta(x)$ is a point source at the origin. It's spherically symmetric (its value depends only on whether $x = 0$). We can expect the potential $\Phi(x)$ generated by this point source to also be spherically symmetric. Thus, $\Phi(x)$ should depend only on the distance $r = |x|$ from the origin:

$$\Phi(x) = \phi(r)$$

b. Laplacian in Spherical Coordinates:

For a radially symmetric function $\phi(r)$ in \mathbb{R}^n , the Laplacian is:

$$\Delta\phi(r) = \frac{1}{r^{n-1}} \frac{d}{dr} \left(r^{n-1} \frac{d\phi}{dr} \right)$$

In \mathbb{R}^3 (which is the context of the original problem), $n = 3$, so:

$$\Delta\phi(r) = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right)$$

c. Solution Away from the Origin ($r > 0$):

Away from the origin ($x \neq 0$, so $r > 0$), the Dirac delta is zero, $\delta(x) = 0$.

So, we must have $\Delta\Phi(x) = 0$.

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = 0 \quad \text{for } r > 0$$

This implies that $r^2 \frac{d\phi}{dr}$ must be a constant, let's call it A :

$$r^2 \frac{d\phi}{dr} = A$$

So,

$$\frac{d\phi}{dr} = \frac{A}{r^2}$$

Integrating with respect to r gives:

$$\phi(r) = -\frac{A}{r} + B$$

where B is another constant. For potentials that vanish at infinity (which is a common physical condition, and matches $u(x) \rightarrow 0$ in the problem), we typically set $B = 0$. So, $\phi(r) = -A/r$.

d. Determining the Constant A (using the source):

To find A , we use the defining equation $\Delta\Phi(x) = \delta(x)$. Integrate both sides over a small ball $B(0, \epsilon)$ of radius ϵ centered at the origin:

$$\int_{B(0, \epsilon)} \Delta\Phi(x) dV = \int_{B(0, \epsilon)} \delta(x) dV$$

The right side, by definition of the Dirac delta, is 1. For the left side, we use the Divergence Theorem (Gauss's Law):

$$\int_{B(0, \epsilon)} \Delta\Phi(x) dV = \int_{\partial B(0, \epsilon)} \nabla\Phi(x) \cdot \mathbf{n} dS$$

where $\partial B(0, \epsilon)$ is the sphere of radius ϵ , and \mathbf{n} is the outward unit normal vector.

Since $\Phi(x) = \phi(r)$, its gradient is $\nabla\Phi(x) = \frac{d\phi}{dr} \frac{x}{|x|} = \frac{d\phi}{dr} \mathbf{e}_r$, where \mathbf{e}_r is the radial unit vector. The outward normal \mathbf{n} on the sphere is also \mathbf{e}_r . So, $\nabla\Phi(x) \cdot \mathbf{n} = \frac{d\phi}{dr}$.

At $r = \epsilon$, this is $\frac{d\phi}{dr} \Big|_{r=\epsilon} = \frac{A}{\epsilon^2}$. The surface area of the sphere $\partial B(0, \epsilon)$ in \mathbb{R}^3 is $4\pi\epsilon^2$. So the surface integral becomes:

$$\int_{\partial B(0, \epsilon)} \frac{A}{\epsilon^2} dS = \frac{A}{\epsilon^2} (4\pi\epsilon^2) = 4\pi A$$

Equating this to the integral of the Dirac delta:

$$4\pi A = 1 \implies A = \frac{1}{4\pi}$$

Substituting this back into $\phi(r) = -A/r$:

$$\phi(r) = -\frac{1}{4\pi r}$$

Thus, the fundamental solution in \mathbb{R}^3 is:

$$\Phi(x) = -\frac{1}{4\pi|x|}$$

This is precisely the kernel that appeared in the integral representation for $u(x)$ in the problem you referenced.

Intuition from Flux Spreading:

Imagine a "flux" emanating from the point source at the origin.

- The term $r^{n-1} \frac{d\phi}{dr}$ (which is $r^2 \frac{d\phi}{dr}$ in \mathbb{R}^3) can be thought of as proportional to the total flux of $\nabla\phi$ through a sphere of radius r .
- The condition $\Delta\phi = 0$ for $r > 0$ means this flux is constant for any $r > 0$.
- The strength of the field (gradient) $\frac{d\phi}{dr}$ must therefore decrease as $1/r^{n-1}$ to keep the flux constant, because the surface area of the sphere is proportional to r^{n-1} .
 - In \mathbb{R}^3 , surface area $\sim r^2$, so field $\sim 1/r^2$. Potential $\phi(r) \sim 1/r$.
 - In \mathbb{R}^2 , surface area (circumference) $\sim r$, so field $\sim 1/r$. Potential $\phi(r) \sim \ln r$.
- The constant value of this flux is determined by the strength of the source $\delta(x)$, which is 1 when integrated. This directly gives the constant A .

6.7.2 2. Deduction using Fourier Transform

This method is more algebraic and general for constant-coefficient linear PDEs.

Let $\hat{f}(k) = \mathcal{F}\{f(x)\}(k) = \int_{\mathbb{R}^n} f(x)e^{-ik\cdot x} dx$ be the Fourier transform. The equation is $\Delta\Phi(x) = \delta(x)$. Taking the Fourier transform of both sides:

$$\mathcal{F}\{\Delta\Phi(x)\}(k) = \mathcal{F}\{\delta(x)\}(k)$$

- **FT of Dirac Delta:** $\mathcal{F}\{\delta(x)\}(k) = 1$.
- **FT of Laplacian:** The Fourier transform of $\frac{\partial^2}{\partial x_j^2}\Phi(x)$ is $(ik_j)^2\hat{\Phi}(k) = -k_j^2\hat{\Phi}(k)$. So, $\mathcal{F}\{\Delta\Phi(x)\}(k) = \mathcal{F}\left\{\sum_{j=1}^n \frac{\partial^2\Phi}{\partial x_j^2}\right\}(k) = \sum_{j=1}^n (-k_j^2)\hat{\Phi}(k) = -|k|^2\hat{\Phi}(k)$.

Putting these together, the equation in Fourier space becomes:

$$-|k|^2\hat{\Phi}(k) = 1$$

So, the Fourier transform of the fundamental solution is:

$$\hat{\Phi}(k) = -\frac{1}{|k|^2}$$

Now, we need to find the inverse Fourier transform $\Phi(x) = \mathcal{F}^{-1}\{-\frac{1}{|k|^2}\}(x)$:

$$\Phi(x) = - \int_{\mathbb{R}^n} \frac{1}{|k|^2} e^{ik \cdot x} \frac{dk}{(2\pi)^n}$$

This integral needs to be computed.

For $n = 3$ (as in the problem):

$$\Phi(x) = - \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{|k|^2} e^{ik \cdot x} dk$$

To evaluate this, we can use spherical coordinates for $k = (k_1, k_2, k_3)$. Let $s = |k|$. By rotating coordinates, we can align x with the z -axis, so $x = (0, 0, |x|)$, and $k \cdot x = s|x| \cos \psi$, where ψ is the angle between k and x . The volume element is $dk = s^2 \sin \psi ds d\psi d\theta_k$ (where θ_k is the azimuthal angle for k).

$$\begin{aligned} \Phi(x) &= - \frac{1}{(2\pi)^3} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{1}{s^2} e^{is|x| \cos \psi} s^2 \sin \psi d\theta_k d\psi ds \\ &= - \frac{2\pi}{(2\pi)^3} \int_0^\infty \left(\int_0^\pi e^{is|x| \cos \psi} \sin \psi d\psi \right) ds \end{aligned}$$

The inner integral is

$$\int_0^\pi e^{is|x| \cos \psi} \sin \psi d\psi = \left[-\frac{e^{is|x| \cos \psi}}{is|x|} \right]_0^\pi = -\frac{e^{-is|x|} - e^{is|x|}}{is|x|} = \frac{2 \sin(s|x|)}{s|x|}$$

So,

$$\begin{aligned} \Phi(x) &= - \frac{1}{4\pi^2} \int_0^\infty \frac{2 \sin(s|x|)}{s|x|} ds \\ &= - \frac{1}{2\pi^2|x|} \int_0^\infty \frac{\sin(s|x|)}{s} ds \end{aligned}$$

Let $t = s|x|$, then $ds = dt/|x|$. The integral becomes

$$\int_0^\infty \frac{\sin t}{t/|x|} \frac{dt}{|x|} = \int_0^\infty \frac{\sin t}{t} dt$$

This is the Dirichlet integral, which evaluates to $\pi/2$.

$$\Phi(x) = - \frac{1}{2\pi^2|x|} \cdot \frac{\pi}{2} = - \frac{1}{4\pi|x|}$$

This matches the result from the radial symmetry method.

Intuition from Fourier Transform:

- The Laplacian Δ acts as a multiplication by $-|k|^2$ in Fourier space. It heavily attenuates high frequencies.
- The Dirac delta $\delta(x)$ has a flat spectrum (all frequencies equally present): $\hat{\delta}(k) = 1$.

- $\widehat{\Phi}(k) = -1/|k|^2$ means the fundamental solution has a spectrum that decays fairly slowly for small $|k|$ (it's singular at $k = 0$) and faster for large $|k|$. This $1/|k|^2$ behavior in Fourier space is characteristic of $1/|x|$ type potentials in real space for 3D. The singularity at $k = 0$ in $\widehat{\Phi}(k)$ hints at the long-range nature of the potential $\Phi(x)$.

Both methods yield the same result. The radial symmetry/Gauss's law approach is often more physically intuitive for the Laplacian, while the Fourier transform method is a very powerful and general algebraic tool applicable to a wider range of constant-coefficient linear differential operators.

6.8 Fundamental Solution: Laplace, heat, wave equation in \mathbb{R}^n

6.8.1 1. Laplace Equation

The fundamental solution $G(x)$ for the Laplace equation satisfies

$$\Delta_x G(x) = \delta(x) \quad \text{in } \mathbb{R}^n$$

where $\Delta_x = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ is the Laplacian operator.

(1) Fourier Transform:

Applying the Fourier transform with respect to x to both sides:

$$\mathcal{F}_x\{\Delta_x G(x)\}(k) = \mathcal{F}_x\{\delta(x)\}(k)$$

We know that $\mathcal{F}_x\{\delta(x)\}(k) = 1$. For the left side, the derivative property of Fourier transforms gives $\mathcal{F}_x\{\frac{\partial}{\partial x_j} f(x)\}(k) = ik_j \widehat{f}(k)$. Thus, $\mathcal{F}_x\{\frac{\partial^2}{\partial x_j^2} f(x)\}(k) = (ik_j)^2 \widehat{f}(k) = -k_j^2 \widehat{f}(k)$. So, $\mathcal{F}_x\{\Delta_x G(x)\}(k) = \sum_{j=1}^n (-k_j^2) \widehat{G}(k) = -|k|^2 \widehat{G}(k)$.

The transformed equation is:

$$-|k|^2 \widehat{G}(k) = 1$$

Solving for $\widehat{G}(k)$:

$$\widehat{G}(k) = -\frac{1}{|k|^2}$$

(1) Inverse Fourier Transform:

Now we need to compute the inverse Fourier transform:

$$G(x) = -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{1}{|k|^2} e^{ik \cdot x} d^n k$$

- **Case $n > 2$:**

We use the Schwinger parametrization $\frac{1}{A} = \int_0^\infty e^{-sA} ds$. Here $A = |k|^2$:

$$\frac{1}{|k|^2} = \int_0^\infty e^{-s|k|^2} ds$$

Substituting this into the integral for $G(x)$:

$$G(x) = -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_0^\infty e^{-s|k|^2} ds \right) e^{ik \cdot x} d^n k$$

Assuming we can interchange the order of integration:

$$G(x) = -\frac{1}{(2\pi)^n} \int_0^\infty \left(\int_{\mathbb{R}^n} e^{-s|k|^2 + ik \cdot x} d^n k \right) ds$$

The inner integral is an n -dimensional Gaussian integral. The exponent is $-s|k|^2 + ik \cdot x = -s \sum_j k_j^2 + i \sum_j k_j x_j$. Completing the square for each component k_j : $-sk_j^2 + ik_j x_j = -s \left(k_j - \frac{ix_j}{2s} \right)^2 - \frac{x_j^2}{4s}$.

$$\int_{\mathbb{R}^n} e^{-s|k|^2 + ik \cdot x} d^n k = \prod_{j=1}^n \int_{-\infty}^\infty e^{-s \left(k_j - \frac{ix_j}{2s} \right)^2 - \frac{x_j^2}{4s}} dk_j = e^{-\frac{|x|^2}{4s}} \prod_{j=1}^n \int_{-\infty}^\infty e^{-sK_j^2} dK_j$$

Since $\int_{-\infty}^\infty e^{-sK^2} dK = \sqrt{\frac{\pi}{s}}$, the n -dimensional integral is $e^{-\frac{|x|^2}{4s}} \left(\frac{\pi}{s} \right)^{n/2}$. Substituting this back:

$$G(x) = -\frac{1}{(2\pi)^n} (\pi)^{n/2} \int_0^\infty s^{-n/2} e^{-\frac{|x|^2}{4s}} ds = -\frac{1}{2^n \pi^{n/2}} \int_0^\infty s^{-n/2} e^{-\frac{|x|^2}{4s}} ds$$

Let $t = \frac{|x|^2}{4s}$. Then $s = \frac{|x|^2}{4t}$, so $ds = -\frac{|x|^2}{4t^2} dt$. The integral becomes (for $n > 2$, so $n/2 - 1 > 0$ for convergence of $\Gamma(n/2 - 1)$):

$$\int_0^\infty s^{-n/2} e^{-\frac{|x|^2}{4s}} ds = \int_\infty^0 \left(\frac{4t}{|x|^2} \right)^{n/2} e^{-t} \left(-\frac{|x|^2}{4t^2} \right) dt = \frac{2^{n-2}}{|x|^{n-2}} \int_0^\infty t^{n/2-2} e^{-t} dt = \frac{2^{n-2}}{|x|^{n-2}} \Gamma(n/2-1)$$

So, for $n > 2$:

$$G(x) = -\frac{1}{2^n \pi^{n/2}} \frac{2^{n-2}}{|x|^{n-2}} \Gamma(n/2 - 1) = -\frac{\Gamma(n/2 - 1)}{4\pi^{n/2} |x|^{n-2}}$$

For example, if $n = 3$, $G(x) = -\frac{\Gamma(1/2)}{4\pi^{3/2} |x|} = -\frac{\sqrt{\pi}}{4\pi \sqrt{\pi} |x|} = -\frac{1}{4\pi |x|}$.

- **Case $n = 2$:**

The above derivation for the s -integral requires $n/2 - 1 > 0$, so it doesn't directly apply to $n = 2$. The direct Fourier inversion of $-1/|k|^2$ in 2D is subtle. The known result is:

$$G(x) = \frac{1}{2\pi} \ln|x|$$

This can be verified by showing $\Delta G(x) = \delta(x)$ in the distributional sense.

• **Case $n = 1$:**

Similarly, for $n = 1$, $\hat{G}(k) = -1/k^2$. Direct inversion is subtle. The known result is:

$$G(x) = \frac{1}{2}|x|$$

This satisfies $\frac{d^2}{dx^2} G(x) = \delta(x)$.

6.8.2 2. Heat Equation

The fundamental solution (or heat kernel) $G(x, t)$ satisfies the homogeneous heat equation with a Dirac delta initial condition:

$$(\partial_t - \alpha \Delta_x) G(x, t) = 0 \quad \text{for } t > 0, x \in \mathbb{R}^n$$

$$G(x, 0) = \delta(x)$$

where $\alpha > 0$ is the thermal diffusivity.

(1) **Spatial Fourier Transform:**

Let $\hat{G}(k, t) = \mathcal{F}_x\{G(x, t)\}(k)$. Applying the Fourier transform to the PDE:

$$\partial_t \hat{G}(k, t) - \alpha (-|k|^2) \hat{G}(k, t) = 0$$

$$\partial_t \hat{G}(k, t) = -\alpha |k|^2 \hat{G}(k, t)$$

The initial condition transforms to $\hat{G}(k, 0) = \mathcal{F}_x\{\delta(x)\}(k) = 1$.

(1) **Solve the ODE in Time:**

This is a first-order ODE for $\hat{G}(k, t)$ with respect to t :

$$\hat{G}(k, t) = C(k) e^{-\alpha |k|^2 t}$$

Using the initial condition $\hat{G}(k, 0) = 1$, we find $C(k) = 1$. So, for $t > 0$:

$$\hat{G}(k, t) = e^{-\alpha |k|^2 t}$$

(1) Inverse Spatial Fourier Transform:

$$G(x, t) = \mathcal{F}_k^{-1} \left\{ e^{-\alpha|k|^2 t} \right\} (x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\alpha|k|^2 t} e^{ik \cdot x} d^n k$$

The exponent is $-\alpha t|k|^2 + ik \cdot x = -\alpha t \sum_j k_j^2 + i \sum_j k_j x_j$. We complete the square for each component k_j :

$$-\alpha t k_j^2 + i k_j x_j = -\alpha t \left(k_j^2 - \frac{i x_j}{\alpha t} k_j \right) = -\alpha t \left(\left(k_j - \frac{i x_j}{2\alpha t} \right)^2 + \frac{x_j^2}{4\alpha^2 t^2} \right) = -\alpha t \left(k_j - \frac{i x_j}{2\alpha t} \right)^2 - \frac{x_j^2}{4\alpha t}$$

The integral becomes:

$$G(x, t) = \frac{1}{(2\pi)^n} e^{-\frac{|x|^2}{4\alpha t}} \prod_{j=1}^n \int_{-\infty}^{\infty} e^{-\alpha t \left(k_j - \frac{i x_j}{2\alpha t} \right)^2} dk_j$$

Each 1D Gaussian integral is $\int_{-\infty}^{\infty} e^{-\alpha t K_j^2} dK_j = \sqrt{\frac{\pi}{\alpha t}}$. Therefore:

$$G(x, t) = \frac{1}{(2\pi)^n} e^{-\frac{|x|^2}{4\alpha t}} \left(\frac{\pi}{\alpha t} \right)^{n/2} = \frac{\pi^{n/2}}{(2\pi)^n (\alpha t)^{n/2}} e^{-\frac{|x|^2}{4\alpha t}}$$

$$G(x, t) = \frac{1}{(4\pi\alpha t)^{n/2}} e^{-\frac{|x|^2}{4\alpha t}} \quad \text{for } t > 0$$

6.8.3 3. Wave Equation

The fundamental solution $G(x, t)$ for the wave equation satisfies:

$$(\partial_t^2 - c^2 \Delta_x) G(x, t) = \delta(x)\delta(t) \quad \text{in } \mathbb{R}^n \times \mathbb{R}$$

where $c > 0$ is the wave speed. We seek the retarded Green's function, satisfying $G(x, t) = 0$ for $t < 0$.

(1) Space-Time Fourier Transform:

Let $\tilde{G}(k, \omega) = \mathcal{F}_{x,t}\{G(x, t)\}(k, \omega)$.

$$\mathcal{F}\{\partial_t^2 G\} = (i\omega)^2 \tilde{G}(k, \omega) = -\omega^2 \tilde{G}(k, \omega).$$

$$\mathcal{F}\{-c^2 \Delta_x G\} = -c^2 (-|k|^2) \tilde{G}(k, \omega) = c^2 |k|^2 \tilde{G}(k, \omega).$$

$$\mathcal{F}\{\delta(x)\delta(t)\} = 1.$$

The transformed equation is:

$$(-\omega^2 + c^2 |k|^2) \tilde{G}(k, \omega) = 1$$

$$\tilde{G}(k, \omega) = \frac{1}{c^2 |k|^2 - \omega^2}$$

(1) Inverse Fourier Transform with respect to ω (Time):

$$\widehat{G}(k, t) = \mathcal{F}_\omega^{-1} \left\{ \frac{1}{c^2|k|^2 - \omega^2} \right\} (t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{c^2|k|^2 - \omega^2} d\omega$$

Let $\omega_0 = c|k|$. The integral is $\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega_0^2 - \omega^2} d\omega$. The poles are at $\omega = \pm\omega_0$. For the retarded Green's function, this integral evaluates to:

$$\widehat{G}(k, t) = \frac{\sin(\omega_0 t)}{\omega_0} H(t) = \frac{\sin(c|k|t)}{c|k|} H(t)$$

where $H(t)$ is the Heaviside step function. This ensures $G(x, t) = 0$ for $t < 0$.

Définition 7: Heaviside step function

The Heaviside step function is defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

(1) Inverse Fourier Transform with respect to k (Space):

$$G(x, t) = H(t) \mathcal{F}_k^{-1} \left\{ \frac{\sin(c|k|t)}{c|k|} \right\} (x) = \frac{H(t)}{(2\pi)^n c} \int_{\mathbb{R}^n} \frac{\sin(c|k|t)}{|k|} e^{ik \cdot x} d^n k$$

The evaluation of this integral depends heavily on the dimension n .

- For $n = 1$:

$$G(x, t) = \frac{H(t)}{(2\pi)c} \int_{-\infty}^{\infty} \frac{\sin(c|k_x|t)}{|k_x|} e^{ik_x x} dk_x$$

The integral $\int_{-\infty}^{\infty} \frac{\sin(a|k_x|)}{|k_x|} e^{ik_x x} dk_x = \pi$ if $|x| < a$, and 0 if $|x| > a$. Here $a = ct$.

$$G(x, t) = \frac{H(t)}{(2\pi)c} \cdot \pi H(ct - |x|) = \frac{H(t)H(ct - |x|)}{2c}$$

This means $G(x, t) = \frac{1}{2c}$ for $t > 0$ and $|x| < ct$, and 0 otherwise.

- For $n = 2$:

$$G(x, t) = \frac{H(t)}{(2\pi)^2 c} \int_{\mathbb{R}^2} \frac{\sin(c|k|t)}{|k|} e^{ik \cdot x} d^2 k$$

The integral evaluates to $\frac{2\pi H(ct - |x|)}{\sqrt{c^2 t^2 - |x|^2}}$.

$$G(x, t) = \frac{H(t)}{(2\pi)^2 c} \frac{2\pi H(ct - |x|)}{\sqrt{c^2 t^2 - |x|^2}} = \frac{H(t)H(ct - |x|)}{2\pi c \sqrt{c^2 t^2 - |x|^2}}$$

- For $n = 3$:

$$G(x, t) = \frac{H(t)}{(2\pi)^3 c} \int_{\mathbb{R}^3} \frac{\sin(c|k|t)}{|k|} e^{ik \cdot x} d^3 k$$

The integral can be evaluated using spherical coordinates. Let $r = |x|$.

$$\int_{\mathbb{R}^3} \frac{\sin(ct|k|)}{|k|} e^{ik \cdot x} d^3 k = \frac{2\pi^2}{r} \delta(ct - r)$$

$$G(x, t) = \frac{H(t)}{(2\pi)^3 c} \frac{2\pi^2}{|x|} \delta(ct - |x|) = \frac{H(t)}{4\pi c|x|} \delta(ct - |x|)$$

This can also be written as $\frac{H(t)}{4\pi c^2|x|} \delta(t - |x|/c)$.

These derivations show how Fourier transforms can be systematically used to find fundamental solutions. The main challenge often lies in evaluating the resulting inverse Fourier transform integrals, which can be quite involved depending on the dimension and the complexity of the function in Fourier space.

The prototypical example of a hyperbolic PDE is the wave equation

$$u_{tt} = \Delta u \tag{6.1}$$

To begin with, consider the one-dimensional wave equation on \mathbb{R}

$$u_{tt} = u_{xx}$$

The general solution is the **d'Alembert solution**

$$u(x, t) = f(x - t) + g(x + t)$$

where f, g are arbitrary functions. This solution describes a superposition of two traveling waves with arbitrary profiles, one propagating with speed one to the right, the other with speed one to the left. Let us compare this solution with the general solution of the one-dimensional heat equation

$$u_t = u_{xx}$$

which is given for $t > 0$ by

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-(x-y)^2/4t} f(y) dy$$

Some of the qualitative properties of the wave equation that differ from those of the heat equation, which are evident from these solutions.

A suitable IBVP for the wave equation with Dirichlet BCs on a bounded open set $\Omega \subset \mathbb{R}^n$ for $u : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$(7.2) \quad \begin{aligned} u_{tt} &= \Delta u && \text{for } x \in \Omega \text{ and } t \in \mathbb{R}, \\ u(x, t) &= 0 && \text{for } x \in \partial\Omega \text{ and } t \in \mathbb{R}, \\ u(x, 0) &= g(x), \quad u_t(x, 0) = h(x) && \text{for } x \in \Omega. \end{aligned}$$

To obtain the basic energy estimate for the wave equation, we multiply eq. (6.1) by u_t and write

$$\begin{aligned} u_t u_{tt} &= \left(\frac{1}{2} u_t^2 \right)_t \\ u_t \Delta u &= \operatorname{div}(u_t D u) - D u \cdot D u_t = \operatorname{div}(u_t D u) - \left(\frac{1}{2} |D u|^2 \right)_t \end{aligned}$$

to get

$$\left(\frac{1}{2} u_t^2 + \frac{1}{2} |D u|^2 \right) - \operatorname{div}(u_t D u) = 0$$

This is the differential form of conservation of energy. The quantity $\frac{1}{2} u_t^2 + \frac{1}{2} |D u|^2$ is the energy density and $-u_t D u$ is the energy flux.

If u is a solution of (7.2), then integration of (7.3) over Ω , use of the divergence theorem, and the BC $u = 0$ on $\partial\Omega$ (which implies that $u_t = 0$) gives

$$\frac{dE}{dt} = 0$$

where $E(t)$ is the total energy

$$E(t) = \int_{\Omega} \left(\frac{1}{2} u_t^2 + \frac{1}{2} |D u|^2 \right) dx.$$

Thus, the total energy remains constant. This result provides an L^2 -energy estimate for solutions of the wave equation.

6.9 The Laplace Equation

见 Evans 偏微分方程笔记

Remarque 6. 这里很多证明都非常符合正常人的思考过程，其中还用到了经典的分析学思想， \mathbb{R}^n 中既开又闭的集合平凡。非常不错。

解题参见 Partial Differential Equations in Action, Complements and Exercises

6.10 Appendix

Some results about the integration and differentiation of Banach-space valued functions of a single variable. Vector-valued integrals of integrable functions have similar properties, often with similar proofs, to scalar-valued L^1 -integrals. The existence of different topologies (such as the weak and strong topologies) in the range space of integrals that take values in an **infinite-dimensional** Banach space introduces significant new issues that do not arise in the scalar-valued case.

Suppose X a real Banach space with $\|\cdot\|$ and dual space X' .

6.10.1 Vector-Valued functions

Measurability

Définition 8

DEFINITION 6.13. A simple function $f : (0, T) \rightarrow X$ is a function of the form

$$(6.38) \quad f = \sum_{j=1}^N c_j \chi_{E_j}$$

where E_1, \dots, E_N are Lebesgue measurable subsets of $(0, T)$ and $c_1, \dots, c_N \in X$.

Définition 9: strongly measurable

DEFINITION 6.14. A function $f : (0, T) \rightarrow X$ is strongly measurable, or measurable for short, if there is a sequence $\{f_n : n \in \mathbb{N}\}$ of simple functions such that $f_n(t) \rightarrow f(t)$ strongly in X (i.e. in norm) for t a.e. in $(0, T)$.

Measurability is preserved under natural operations on functions.

- (1) If $f : (0, T) \rightarrow X$ is measurable, then $\|f\| : (0, T) \rightarrow \mathbb{R}$ is measurable.
- (2) If $f : (0, T) \rightarrow X$ is measurable and $\phi : (0, T) \rightarrow \mathbb{R}$ is measurable, then $\phi f : (0, T) \rightarrow X$ is measurable.
- (3) If $\{f_n : (0, T) \rightarrow X\}$ is a sequence of measurable functions and $f_n(t) \rightarrow f(t)$ strongly in X for t pointwise a.e. in $(0, T)$, then $f : (0, T) \rightarrow X$ is measurable.

Définition 10: weakly measurable

A function $f : (0, T) \rightarrow X$ is said to be **weakly measurable** if the real-valued function $\langle \omega, f \rangle : (0, T) \rightarrow \mathbb{R}$ is measurable for every $\omega \in X'$.

This amounts to a 'coordinatewise' definition of measurability, in which we represent a vector-valued function by its real-valued coordinate functions. For finite-dimensional, or separable, Banach spaces these definitions coincide, but for non-separable spaces a weakly measurable function need not be strongly measurable. The relationship between weak and strong measurability is given by the following Pettis theorem.

Définition 11: almost separably valued

DEFINITION 6.15. A function $f : (0, T) \rightarrow X$ taking values in a Banach space X is almost separably valued if there is a set $E \subset (0, T)$ of measure zero such that $f((0, T) \setminus E)$ is separable, meaning that it contains a countable dense subset.

Théorème 11: Pettis theorem

A function $f : (0, T) \rightarrow X$ is strongly measurable iff it is weakly measurable and almost separably valued.

Définition 12: weak continuous

DEFINITION 6.17. A function $f : [0, T] \rightarrow X$ taking values in a Banach space X is weakly continuous if $\langle \omega, f \rangle : [0, T] \rightarrow \mathbb{R}$ is continuous for every $\omega \in X'$. The space of such weakly continuous functions is denoted by $C_w([0, T]; X)$.

Since a continuous function is measurable, every almost separably valued, weakly continuous function is strongly measurable.

EXAMPLE 6.18. Suppose that \mathcal{H} is a non-separable Hilbert space whose dimension is equal to the cardinality of \mathbb{R} . Let $\{e_t : t \in (0, 1)\}$ be an orthonormal basis of \mathcal{H} , and define a function $f : (0, 1) \rightarrow \mathcal{H}$ by $f(t) = e_t$. Then f is weakly but not strongly measurable. If $K \subset [0, 1]$ is the standard middle thirds Cantor set and $\{\tilde{e}_t : t \in K\}$ is an orthonormal basis of \mathcal{H} , then $g : (0, 1) \rightarrow \mathcal{H}$ defined by $g(t) = 0$ if $t \notin K$ and $g(t) = \tilde{e}_t$ if $t \in K$ is almost separably valued since $|K| = 0$; thus, g is strongly measurable and equivalent to the zero-function.

Exemple 11 (weak but not strong measurable).

EXAMPLE 6.19. Define $f : (0, 1) \rightarrow L^\infty(0, 1)$ by $f(t) = \chi_{(0,t)}$. Then f is not almost separably valued, since $\|f(t) - f(s)\|_{L^\infty} = 1$ for $t \neq s$, so f is not strongly measurable. On the other hand, if we define $g : (0, 1) \rightarrow L^2(0, 1)$ by $g(t) = \chi_{(0,t)}$, then g is strongly measurable. To see this, note that $L^2(0, 1)$ is separable and for every $w \in L^2(0, 1)$, which is isomorphic to $L^2(0, 1)'$, we have

$$(w, g(t))_{L^2} = \int_0^1 w(x)\chi_{(0,t)}(x) dx = \int_0^t w(x) dx.$$

Thus, $(w, g)_{L^2} : (0, 1) \rightarrow \mathbb{R}$ is absolutely continuous and therefore measurable.

Exemple 12 (L^2 separable but L^∞ not).

Integration

The definition of the Lebesgue integral as a supremum of integrals of simple functions does not extend directly to vector-valued integrals because it uses the ordering properties of \mathbb{R} in an essential way.

One can use duality to define X -valued integrals $\int f dt$ in terms of the corresponding real-valued integrals $\int \langle \omega, f \rangle dt$ where $\omega \in X'$, but we will not consider such weak definitions of an integral here.

Instead, we define the integral of vector-valued functions by completing the space of simple functions with respect to the $L^1(0, T; X)$ -norm. The resulting integral is called the Bochner integral, and its properties are similar to those of the Lebesgue integral of integrable real-valued functions.

Définition 13: simple function

DEFINITION 6.20. Let

$$f = \sum_{j=1}^N c_j \chi_{E_j}$$

be the simple function in (6.38). The integral of f is defined by

$$\int_0^T f dt = \sum_{j=1}^N c_j |E_j| \in X$$

where $|E_j|$ denotes the Lebesgue measure of E_j .

Définition 14: Bochner integral

DEFINITION 6.21. A strongly measurable function $f : (0, T) \rightarrow X$ is Bochner integrable, or integrable for short, if there is a sequence of simple functions such that $f_n(t) \rightarrow f(t)$ pointwise a.e. in $(0, T)$ and

$$\lim_{n \rightarrow \infty} \int_0^T \|f - f_n\| dt = 0.$$

The integral of f is defined by

$$\int_0^T f dt = \lim_{n \rightarrow \infty} \int_0^T f_n dt,$$

where the limit exists strongly in X .

The value of the Bochner integral of f is independent of the sequence $\{f_n\}$ of approximating simple functions, and

$$\left\| \int_0^T f dt \right\| \leq \int_0^T \|f\| dt$$

Moreover, if $A : X \rightarrow Y$ is a bounded linear operator between Banach spaces X, Y and $f : (0, T) \rightarrow X$ is integrable, then $Af : (0, T) \rightarrow Y$ is integrable and

$$A \left(\int_0^T f dt \right) = \int_0^T Af dt$$

More generally, this equality holds whenever $A : \mathcal{D}(A) \subset X \rightarrow Y$ is a **closed** linear operator and $f : (0, T) \rightarrow \mathcal{D}(A)$, in which case $\int_0^T f dt \in \mathcal{D}(A)$.

Définition 15: closed linear operator

The operator T is said to be **closed** if its graph:

$$G(T) = \{(x, T(x)) \mid x \in D(T)\}$$

is a **closed set** in the product space $X \times Y$.

EXAMPLE 6.22. If $f : (0, T) \rightarrow X$ is integrable and $\omega \in X'$, then $\langle \omega, f \rangle : (0, T) \rightarrow \mathbb{R}$ is integrable and

$$\left\langle \omega, \int_0^T f dt \right\rangle = \int_0^T \langle \omega, f \rangle dt.$$

Exemple 13 ($\langle \omega, \cdot \rangle$ as an operator).

Théorème 12

THEOREM 6.24. A function $f : (0, T) \rightarrow X$ is Bochner integrable if and only if it is strongly measurable and

$$\int_0^T \|f\| dt < \infty.$$

The DCT for Bochner integrals with proof same as the scalar-valued case omitted.

Théorème 13

THEOREM 6.26. Suppose that $f_n : (0, T) \rightarrow X$ is Bochner integrable for each $n \in \mathbb{N}$,

$$f_n(t) \rightarrow f(t) \quad \text{as } n \rightarrow \infty \text{ strongly in } X \text{ for } t \text{ a.e. in } (0, T),$$

and there is an integrable function $g : (0, T) \rightarrow \mathbb{R}$ such that

$$\|f_n(t)\| \leq g(t) \quad \text{for } t \text{ a.e. in } (0, T) \text{ and every } n \in \mathbb{N}.$$

Then $f : (0, T) \rightarrow X$ is Bochner integrable and

$$\int_0^T f_n dt \rightarrow \int_0^T f dt, \quad \int_0^T \|f_n - f\| dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The definition and properties of L^p -spaces of X -valued functions are analogous to the case of real-valued functions.

Définition 16: $L^p(0, T; X)$

DEFINITION 6.27. For $1 \leq p < \infty$ the space $L^p(0, T; X)$ consists of all strongly measurable functions $f : (0, T) \rightarrow X$ such that

$$\int_0^T \|f\|^p dt < \infty$$

equipped with the norm

$$\|f\|_{L^p(0, T; X)} = \left(\int_0^T \|f\|^p dt \right)^{1/p}.$$

The space $L^\infty(0, T; X)$ consists of all strongly measurable functions $f : (0, T) \rightarrow X$ such that

$$\|f\|_{L^\infty(0, T; X)} = \sup_{t \in (0, T)} \|f(t)\| < \infty,$$

where \sup denotes the essential supremum.

As usual, functions equal pointwise a.e. are regarded equivalent.

Théorème 14

If X is a Banach space and $1 \leq p \leq \infty$ then $L^p(0, T; X)$ is a Banach space.

Simple functions of the form $f(t) = \sum_{i=1}^n c_i \chi_{E_i}(t)$ are dense in $L^p(0, T; X)$. By mollifying these functions with respect to t , we get the following density result.

Théorème 15

PROPOSITION 6.29. *If X is a Banach space and $1 \leq p < \infty$, then the collection of functions of the form*

$$f(t) = \sum_{i=1}^n c_i \phi_i(t) \quad \text{where } \phi_i \in C_c^\infty(0, T) \text{ and } c_i \in X$$

is dense in $L^p(0, T; X)$.

Théorème 16

THEOREM 6.30. Suppose that $1 \leq p < \infty$ and X is a reflexive Banach space with dual space X' . Then the dual of $L^p(0, T; X)$ is isomorphic to $L^{p'}(0, T; X')$ where

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

The action of $f \in L^{p'}(0, T; X')$ on $u \in L^p(0, T; X)$ is given by

$$\langle\langle f, u \rangle\rangle = \int_0^T \langle f(t), u(t) \rangle dt,$$

where the double brackets denote the $L^{p'}(X')$ - $L^p(X)$ duality pairing and the single brackets denote the X' - X duality pairing.

The proof is more complicated than in the scalar case and some condition on X is required. Reflexivity is sufficient.

Differentiability

Définition 17: strong continuous and strong differentiable

6.A.3. Differentiability. The definition of continuity and pointwise differentiability of vector-valued functions are the same as in the scalar case. A function $f : (0, T) \rightarrow X$ is strongly continuous at $t \in (0, T)$ if $f(s) \rightarrow f(t)$ strongly in X as $s \rightarrow t$, and f is strongly continuous in $(0, T)$ if it is strongly continuous at every point of $(0, T)$. A function f is strongly differentiable at $t \in (0, T)$, with strong pointwise derivative $f_t(t)$, if

$$f_t(t) = \lim_{h \rightarrow 0} \left[\frac{f(t+h) - f(t)}{h} \right]$$

where the limit exists strongly in X , and f is continuously differentiable in $(0, T)$ if its pointwise derivative exists for every $t \in (0, T)$ and $f_t : (0, T) \rightarrow X$ is a strongly continuously function.

6.11 Second-Order-Elliptic-Equations

We will exploit two essentially distinct techniques:

- Energy methods within Sobolev spaces (§6.1 – §6.3)
- Maximum principle methods (§6.4)

6.11.1 Definitions

We will in this chapter mostly study the boundary-value problem

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

where U is an open, bounded subset of \mathbb{R}^n , and $u : \bar{U} \rightarrow \mathbb{R}$ is the unknown, $u = u(x)$.

$$Lu = -\sum_{i,j=1}^n a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u,$$

for given coefficient functions a^{ij}, b^i, c ($i, j = 1, \dots, n$)

Définition 18: elliptic

DEFINITION. We say the partial differential operator L is *(uniformly) elliptic* if there exists a constant $\theta > 0$ such that

$$(4) \quad \sum_{i,j=1}^n a^{ij}(x)\xi_i \xi_j \geq \theta |\xi|^2$$

for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$.

Ellipticity thus means that for each point $x \in U$, the symmetric $n \times n$ matrix $\mathbf{A}(x) = ((a^{ij}(x)))$ is positive definite, with smallest eigenvalue greater than or equal to θ .

An obvious example is $a^{ij} \equiv \delta_{ij}$, $b^i \equiv 0$, $c \equiv 0$, in which case the operator L is $-\Delta$. To apply ideas of functional analysis to glean information concerning PDE, theories of Sobolov spaces is developed, which is sometimes subtle and unmotivated but ultimately will prove extremely useful.

6.12 Hölder spaces

We defined the Hölder spaces and the supremum norm $\|u\|_{C(\bar{U})} = \sup_{x \in U} |u(x)|$ and the γ^{th} -Hölder seminorm of $u : U \rightarrow \mathbb{R}$ to be

$$[u]_{C^{0,\gamma}(\bar{U})} = \sup_{x,y \in U, x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\gamma} \right\}$$

And the γ^{th} -Hölder norm to be

$$\|u\|_{C^{0,\gamma}(\bar{U})} = \|u\|_{C(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})}$$

The Hölder space $C^{k,\gamma}(\bar{U})$ consists of all functions $u \in C^k(\bar{U})$ for which the norm

$$\|u\|_{C^{k,\gamma}(\bar{U})} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})}$$

is finite. The space of functions $C^{k,\gamma}(\bar{U})$ is a Banach space. (left as an exercise)

6.12.1 Banach space

If X denotes a real linear space, then a mapping $\|\cdot\| : X \rightarrow [0, \infty)$ is called a **norm** provided

- (1) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in X$.
- (2) $\|\lambda u\| = |\lambda| \|u\|$ for all $u \in X, \lambda \in \mathbb{R}$.
- (3) $\|u\| = 0$ if and only if $u = 0$.

A norm provides us with a notion of convergence: we say a sequence $\{u_k\}_{k=1}^\infty$ converges to $u \in X$, written $u_k \rightarrow u$, if $\lim_{k \rightarrow \infty} \|u_k - u\| = 0$.

A **Banach space** is then a normed linear space which is **complete**, i.e. within which each Cauchy sequence converges.

6.13 Sobolev Spaces

Hölder spaces is not often suitable. We want other spaces containing less smooth functions. In practice we must strike a balance.

6.13.1 Weak derivatives

$C_c^\infty(U)$ denotes the space of infinitely differentiable functions $\phi : U \rightarrow \mathbb{R}$ with compact support in U . ϕ is sometimes called a test function. We have

$$\int_U u \phi_{x_i} dx = - \int_U u_{x_i} \phi dx \quad i = 1, 2, \dots, n$$

There are no boundary terms, since ϕ has compact support in U and thus vanishes near ∂U . Then we define the **weak derivative** of $u \in L^1_{\text{loc}}(U)$.

Let α be a multiindex. Say that $v \in L^1_{\text{loc}}(U)$ is the α^{th} -weak partial derivative of u , written $D^\alpha u = v$, provided

$$\int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v \phi \, dx, \quad \forall \phi \in C_c^\infty(U).$$

Note. The weak derivative is well-defined. (it's unique.)

LEMMA (Uniqueness of weak derivatives). A weak α^{th} -partial derivative of u , if it exists, is uniquely defined up to a set of measure zero.

Proof. Assume that $v, \tilde{v} \in L^1_{\text{loc}}(U)$ satisfy

$$\int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v \phi \, dx = (-1)^{|\alpha|} \int_U \tilde{v} \phi \, dx$$

for all $\phi \in C_c^\infty(U)$. Then

$$(4) \quad \int_U (v - \tilde{v}) \phi \, dx = 0$$

for all $\phi \in C_c^\infty(U)$, whence $v - \tilde{v} = 0$ a.e. \square

Example 1

Example 1. Let $n = 1$, $U = (0, 2)$, and

$$u(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } 1 \leq x < 2. \end{cases}$$

Define

$$v(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } 1 < x < 2. \end{cases}$$

Let us show $u' = v$ in the weak sense. To see this, choose any $\phi \in C_c^\infty(U)$. We must demonstrate

$$\int_0^2 u \phi' \, dx = - \int_0^2 v \phi \, dx.$$

But we easily calculate

$$\begin{aligned} \int_0^2 u \phi' \, dx &= \int_0^1 x \phi' \, dx + \int_1^2 \phi' \, dx \\ &= - \int_0^1 \phi \, dx + \phi(1) - \phi(0) = - \int_0^2 v \phi \, dx, \end{aligned}$$

as required.

u is not strongly differentiable at $x = 1$ but weakly differentiable.

Example 2

This is an example showing that the weak derivative does not always exist.

Let $n = 1$, $U = (0, 2)$ and

$$u(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 2 & \text{if } 1 < x < 2 \end{cases}$$

We assert u' does not exist in the weak sense. To check this, we must show that there does not exist any function $v \in L^1_{\text{loc}}(U)$ satisfying

$$\int_0^2 u\phi' dx = - \int_0^2 v\phi dx \quad (6.2)$$

for all $\phi \in C_c^\infty(U)$. Suppose, to the contrary, eq. (6.2) were valid for some v and all ϕ . Then

$$-\int_0^2 v\phi dx = \int_0^2 u\phi' dx = \int_0^1 x\phi' dx + 2 \int_1^2 \phi' dx = - \int_0^1 \phi dx - \phi(1) \quad (6.3)$$

Choose a sequence $\{\phi_m\}_{m=1}^\infty$ of smooth functions satisfying

$$0 \leq \phi_m \leq 1, \quad \phi_m(1) = 1, \quad \phi_m(x) \rightarrow 0 \text{ for all } x \neq 1$$

Replace ϕ by ϕ_m in eq. (6.3) and send $m \rightarrow \infty$, we discover

$$1 = \lim_{m \rightarrow \infty} \phi_m(1) = \lim_{m \rightarrow \infty} \left[\int_0^2 v\phi_m dx - \int_0^1 \phi_m dx \right] = 0$$

a contradiction.

6.13.2 Definition of Sobolev spaces

Fix $1 \leq p \leq \infty$ and let k be a nonnegative integer. The **Sobolev space** $W^{k,p}(U)$ consists of all locally summable function $u : U \rightarrow \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Remarque 7. If $p = 2$, we usually write $H^k(U) = W^{k,2}(U)$. $H^k(U)$ is a Hilbert space. Note that $H^0(U) = L^2(U)$.

If $u \in W^{k,p}(U)$, we define its **norm** to be

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p dx \right)^{1/p} & 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} \text{ess sup}_U |D^\alpha u| & p = \infty \end{cases}$$

We write

$$u_m \rightarrow u \quad \text{in } W_{\text{loc}}^{k,p}(U)$$

to mean

$$u_m \rightarrow u \quad \text{in } W^{k,p}(V)$$

for each $V \subset\subset U$. $A \subset\subset B$ 意指 \bar{A} 为紧集即有界闭集, $\bar{A} \subset B$ 且 $d(\bar{A}, \partial B) > 0$.

DEFINITION. We denote by

$$W_0^{k,p}(U)$$

the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.

Thus $u \in W_0^{k,p}(U)$ if and only if there exist functions $u_m \in C_c^\infty(U)$ such that $u_m \rightarrow u$ in $W^{k,p}(U)$. We interpret $W_0^{k,p}(U)$ as comprising those functions $u \in W^{k,p}(U)$ such that

$$\text{"}D^\alpha u = 0 \text{ on } \partial U\text{" for all } |\alpha| \leq k - 1.$$

This will all be made clearer with the discussion of traces in §5.5.

Example

The following example shows what condition the parameters should satisfy such that $u \in W^{1,p}(U)$.

Example 3. Take $U = B^0(0, 1)$, the open unit ball in \mathbb{R}^n , and

$$u(x) = |x|^{-\alpha} \quad (x \in U, x \neq 0).$$

For which values of $\alpha > 0, n, p$ does u belong to $W^{1,p}(U)$? To answer, note first that u is smooth away from 0, with

$$u_{x_i}(x) = \frac{-\alpha x_i}{|x|^{\alpha+2}} \quad (x \neq 0),$$

and so

$$|Du(x)| = \frac{|\alpha|}{|x|^{\alpha+1}} \quad (x \neq 0).$$

Let $\phi \in C_c^\infty(U)$ and fix $\varepsilon > 0$. Then

$$\int_{U-B(0,\varepsilon)} u \phi_{x_i} dx = - \int_{U-B(0,\varepsilon)} u_{x_i} \phi dx + \int_{\partial B(0,\varepsilon)} u \phi v^i dS,$$

$\nu = (\nu^1, \dots, \nu^n)$ denoting the inward pointing normal on $\partial B(0, \varepsilon)$. Now if $\alpha + 1 < n$, $|Du(x)| \in L^1(U)$. In this case

$$\left| \int_{\partial B(0, \varepsilon)} u \phi \nu^i dS \right| \leq \|\phi\|_{L^\infty} \int_{\partial B(0, \varepsilon)} \varepsilon^{-\alpha} dS \leq C \varepsilon^{n-1-\alpha} \rightarrow 0.$$

Thus

$$\int_U u \phi_{x_i} dx = - \int_U u_{x_i} \phi dx$$

for all $\phi \in C_c^\infty(U)$, provided $0 \leq \alpha < n - 1$. Furthermore $|Du(x)| = \frac{|\alpha|}{|x|^{\alpha+1}} \in L^p(U)$ if and only if $(\alpha + 1)p < n$. Consequently $u \in W^{1,p}(U)$ if and only if $\alpha < \frac{n-p}{p}$. In particular $u \notin W^{1,p}(U)$ for each $p \geq n$.

Example 4

The following example illustrates a fundamental fact of life, that although a function u belonging to a Sobolev space possesses certain smoothness properties, it can be rather badly behaved in other ways.

Example 4. Let $\{r_k\}_{k=1}^\infty$ be a countable, dense subset of $U = B^0(0, 1)$. Write

$$u(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} |x - r_k|^{-\alpha} \quad (x \in U).$$

Then $u \in W^{1,p}(U)$ for $\alpha < \frac{n-p}{p}$. If $0 < \alpha < \frac{n-p}{p}$, we see that u belongs to $W^{1,p}(U)$ and yet is unbounded on each open subset of U .

6.13.3 Elementary Properties

We verify some certain properties of weak derivatives, which is obvious for smooth functions but relies proof solely upon the definition of weak derivatives.

THEOREM 1 (Properties of weak derivatives). Assume $u, v \in W^{k,p}(U)$, $|\alpha| \leq k$. Then

- (i) $D^\alpha u \in W^{k-|\alpha|,p}(U)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$ for all multi-indices α, β with $|\alpha| + |\beta| \leq k$.
- (ii) For each $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(U)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$, $|\alpha| \leq k$.
- (iii) If V is an open subset of U , then $u \in W^{k,p}(V)$.
- (iv) If $\zeta \in C_c^\infty(U)$, then $\zeta u \in W^{k,p}(U)$ and

$$(7) \quad D^\alpha(\zeta u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u \quad (\text{Leibniz's formula}),$$

where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

We prove the Leibniz's formula by induction on $|\alpha|$. Suppose first $|\alpha| = 1$.

Choose any $\phi \in C_c^\infty(U)$. Then

$$\int_U \zeta u D^\alpha \phi \, dx = \int_U u \underbrace{D^\alpha(\zeta \phi)}_{= \phi(D^\alpha \zeta) + \zeta(D^\alpha \phi)} - u(D^\alpha \zeta) \phi \, dx = - \int_U (\zeta D^\alpha u + u D^\alpha \zeta) \phi \, dx$$

Next assume $l < k$ and formula (7) is valid for all $|\alpha| \leq l$ and all functions ζ . Choose a multiindex α with $|\alpha| = l+1$. Then $\alpha = \beta + \gamma$ for some $|\beta| = l$, $|\gamma| = 1$. Then for ϕ as above,

$$\begin{aligned} \int_U \zeta u D^\alpha \phi \, dx &= \int_U \zeta u D^\beta(D^\gamma \phi) \, dx \\ &= (-1)^{|\beta|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} D^\sigma \zeta D^{\beta-\sigma} u D^\gamma \phi \, dx \end{aligned}$$

(by the induction assumption)

$$= (-1)^{|\beta|+|\gamma|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} D^\gamma(D^\sigma \zeta D^{\beta-\sigma} u) \phi \, dx$$

(by the induction assumption again)

$$= (-1)^{|\alpha|} \int_U \sum_{\sigma \leq \beta} \binom{\beta}{\sigma} [D^\rho \zeta D^{\alpha-\rho} u + D^\sigma \zeta D^{\alpha-\sigma} u] \phi \, dx$$

(where $\rho = \sigma + \gamma$)

$$= (-1)^{|\alpha|} \int_U \left[\sum_{\sigma \leq \alpha} \binom{\alpha}{\sigma} D^\sigma \zeta D^{\alpha-\sigma} u \right] \phi \, dx,$$

since

$$\binom{\beta}{\sigma - \gamma} + \binom{\beta}{\sigma} = \binom{\alpha}{\sigma}. \quad \square$$

THEOREM 2 (Sobolev spaces as function spaces). *For each $k = 1, \dots$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(U)$ is a Banach space.*

Check $u_m \rightarrow u$ in $W^{k,p}(U)$ by checking that $D^\alpha u_m \rightarrow D^\alpha u$ in $L^p(U)$ for all $|\alpha| \leq k$.

6.14 Approximation

6.14.1 Interior approximation by smooth functions

In order to study the deeper properties of Sobolev spaces, we therefore need to develop some systematic procedures for approximating a function in a Sobolev space by smooth functions.

Fix positive integer k and $1 \leq p < \infty$. $U_\varepsilon = \{x \in U : \text{dist}(x, \partial U) > \varepsilon\}$.

THEOREM 1 (Local approximation by smooth functions). *Assume that $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$, and set*

$$u^\varepsilon = \eta_\varepsilon * u \quad \text{in } U_\varepsilon.$$

Then

- (i) $u^\varepsilon \in C^\infty(U_\varepsilon)$ for each $\varepsilon > 0$, and
- (ii) $u^\varepsilon \rightarrow u$ in $W_{\text{loc}}^{k,p}(U)$, as $\varepsilon \rightarrow 0$.

Claim that if $|\alpha| \leq k$ then

$$D^\alpha u^\varepsilon = \eta_\varepsilon * D^\alpha u \quad \text{in } U_\varepsilon$$

6.14.2 Approximation by smooth functions

Next we show that we can find smooth functions which approximate in $W^{k,p}(U)$ and not just in $W_{\text{loc}}^{k,p}(U)$. Notice that the smoothness of ∂U is not necessary.

THEOREM 2 (Global approximation by smooth functions). *Assume U is bounded, and suppose as well that $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exist functions $u_m \in C^\infty(U) \cap W^{k,p}(U)$ such that*

$$u_m \rightarrow u \quad \text{in } W^{k,p}(U).$$

使用了单位分解的思想。

6.14.3 Global approximation by smooth functions

THEOREM 3 (Global approximation by functions smooth up to the boundary). *Assume U is bounded and ∂U is C^1 . Suppose $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exist functions $u_m \in C^\infty(\bar{U})$ such that*

$$u_m \rightarrow u \quad \text{in } W^{k,p}(U).$$

6.15 Extensions

见崔尚斌

下面陈述一个重要定理, 它的证明由于比较冗长, 这里从略.

对于 \mathbf{R}^n 中的开集 Ω , 我们称它具有一致Lipschitz连续的边界, 如果存在其边界 $\partial\Omega$ 的一个至多可数的开覆盖 $\{U_j\} = \{U_j\}_{j=1}^l$ (l 为正整数) 或 $\{U_j\} = \{U_j\}_{j=1}^\infty$ 满足以下三个条件:

- (i) 存在正整数 N 使 $\{U_j\}$ 中任意 $N+1$ 个开集的交都是空集;
- (ii) 存在常数 $\delta > 0$ 使对任意 $x \in \partial\Omega$, 有某个 j 使得 $B_\delta(x) \subseteq U_j$;
- (iii) 对每个 j 存在相应的定义于 \mathbf{R}^{n-1} 上的Lipschitz连续函数 f_j , 使在 \mathbf{R}^n 的一个适当的新坐标系 $(x_1^j, x_2^j, \dots, x_n^j)$ 下, $\Omega \cap U_j$ 可表示成

$$\Omega \cap U_j = \{(x_1^j, x_2^j, \dots, x_n^j) \in \Omega : x_n^j > f_j(x_1^j, x_2^j, \dots, x_{n-1}^j)\},$$

并且存在常数 $C > 0$ 使对每个 j 都成立

$$|f_j(\xi) - f_j(\eta)| \leq C|\xi - \eta|, \quad \forall \xi, \eta \in \mathbf{R}^{n-1},$$

即 $\{f_j\}$ ($= \{f_j\}_{j=1}^l$ 或 $\{f_j\}_{j=1}^\infty$) 中的所有函数在 \mathbf{R}^{n-1} 上一致地一致 Lipschitz 连续.

容易看出, 当 Ω 是有界开集时, 它具有一致 Lipschitz 连续的边界等价于它具有 Lipschitz 连续的边界, 即“一致”一词仅是对无界开集加的.

定理 1.7.8 (延拓定理) 设 Ω 是 \mathbf{R}^n 中具有一致 Lipschitz 连续边界的开集, 则存在线性算子 E , 它把 Ω 上的函数映射成 \mathbf{R}^n 上的函数, 具有以下性质:

- (i) 对 Ω 上的每个函数 u , Eu 是 u 在 \mathbf{R}^n 上的延拓, 即 $Eu|_{\Omega} = u$;
(ii) 对每个非负整数 m 存在常数 $C(n, m, \Omega) > 0$ 使对任意 $1 \leq p < \infty$ 成立

$$\|Eu\|_{W^{m,p}(\mathbf{R}^n)} \leq C(n, m, \Omega) \|u\|_{W^{m,p}(\Omega)}, \quad \forall u \in W^{m,p}(\Omega).$$

这个定理的证明见文献 [110] 第六章定理 5. 以后将把使上述定理的结论成立的开集 $\Omega \subseteq \mathbf{R}^n$ 叫做可延拓开集. 于是上述定理可改述成: 具有一致 Lipschitz 连续边界的开集是可延拓开集. 算子 E 叫做延拓算子.

在许多应用问题中, 开集 Ω 具有一定的光滑性. 在这种情况下, 把 $W^{m,p}(\Omega)$ 中的函数延拓成 $W^{m,p}(\mathbf{R}^n)$ 中函数的问题可以用以下证明比较简单的定理来实现:

定理 1.7.9 (延拓定理) 设 Ω 是 \mathbf{R}^n 中的有界开集, 其边界属于 C^m 类, 其中 m 是正整数. 则存在线性算子 E_m , 它把 Ω 上的函数映射成 \mathbf{R}^n 上的函数, 具有以下性质:

- (i) 对 Ω 上的每个函数 u , $E_m u$ 是 u 在 \mathbf{R}^n 上的延拓, 即 $E_m u|_{\Omega} = u$;
(ii) 对任意 $1 \leq p < \infty$ 都存在相应的常数 $C(n, m, p, \Omega) > 0$ 使成立

$$\|E_m u\|_{W^{m,p}(\mathbf{R}^n)} \leq C(n, m, p, \Omega) \|u\|_{W^{m,p}(\Omega)}, \quad \forall u \in W^{m,p}(\Omega).$$

证 定理 1.4.5 的证明中作出的延拓算子 E_m 即满足这里的所有要求. \square

定理 1.7.10 设 Ω 是 \mathbf{R}^n 中的可延拓开集, 则对任意正整数 m 和每个 $1 \leq p < \infty$, 全体 $C_0^\infty(\mathbf{R}^n)$ 中的函数在 Ω 上的限制在 $W^{m,p}(\Omega)$ 中稠密.

证 令 $E : W^{m,p}(\Omega) \rightarrow W^{m,p}(\mathbf{R}^n)$ 为延拓算子. 对任意 $u \in W^{m,p}(\Omega)$ 和任意 $\delta > 0$, 根据定理 1.7.7 知存在 $v \in C_0^\infty(\mathbf{R}^n)$ 使

$$\|Eu - v\|_{W^{m,p}(\mathbf{R}^n)} < \delta.$$

令 $w = v|_{\Omega}$, 则由以上不等式得

$$\|u - w\|_{W^{m,p}(\Omega)} = \|Eu - v\|_{W^{m,p}(\Omega)} \leq \|Eu - v\|_{W^{m,p}(\mathbf{R}^n)} < \delta.$$

因此全体 $C_0^\infty(\mathbf{R}^n)$ 中的函数在 Ω 上的限制在 $W^{m,p}(\Omega)$ 中稠密. 证毕. \square

6.16 Traces

Next we discuss the possibility of assigning “boundary values” along ∂U to a function $u \in W^{1,p}(U)$, assuming that ∂U is C^1 . Now if $u \in C(\bar{U})$, then clearly u has values on ∂U in the usual sense. The problem is that a typical function $u \in W^{1,p}(U)$ is not in general continuous and, even worse, is only defined a.e. in U . Since ∂U has n -dimensional Lebesgue measure zero, there is no direct meaning we can give to the expression “ u restricted to ∂U ”. The notion of a *trace operator* resolves this problem.

For this section we take $1 \leq p < \infty$.

Théorème 17: Trace Theorem

THEOREM 1 (Trace Theorem). Assume U is bounded and ∂U is C^1 . Then there exists a bounded linear operator

$$T : W^{1,p}(U) \rightarrow L^p(\partial U)$$

such that

- (i) $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\bar{U})$ and
- (ii) $\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$,

for each $u \in W^{1,p}(U)$, with the constant C depending only on p and U .

Définition 19: trace

We call Tu the trace of u on ∂U .

We next examine more closely what it means for a function to have zero trace.

THEOREM 2 (Trace-zero functions in $W^{1,p}$). Assume U is bounded and ∂U is C^1 . Suppose furthermore that $u \in W^{1,p}(U)$. Then

$$(4) \quad u \in W_0^{1,p}(U) \quad \text{if and only if} \quad Tu = 0 \text{ on } \partial U.$$

6.17 Sobolev 嵌入定理

Sobolev 空间理论的核心部分是三个嵌入定理:

- Sobolev 嵌入定理
- Morrey 嵌入定理
- Kondrachov-Rellich 嵌入定理

Théorème 18

定理 1.8.1 设 $1 \leq p \leq q \leq \infty$ 且 $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$, 则对任意有界开集 $\Omega \subseteq \mathbf{R}^n$ 有
 $W_0^{1,p}(\Omega) \subseteq L^q(\Omega)$, 且成立不等式

$$\|u\|_{L^q(\Omega)} \leq C_1(n, p, q) |\Omega|^{\frac{1}{n} - \frac{1}{p} + \frac{1}{q}} \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega), \quad (1.8.1)$$

其中 $C_1(n, p, q) = \omega_n^{-\frac{1}{n}} \left[\left(1 - \frac{1}{p} + \frac{1}{q} \right) / n \left(\frac{1}{n} - \frac{1}{p} + \frac{1}{q} \right) \right]^{1 - \frac{1}{p} + \frac{1}{q}}$, ω_n 表示 n 维单位球的体积 (见 1.1 节). 这里及以后总记

$$\nabla u = \partial u = (\partial_1 u, \partial_2 u, \dots, \partial_n u), \quad \|\nabla u\|_{L^p(\Omega)} = \|\nabla u\|_{L^p(\Omega)}.$$

只需证明对于任意 $u \in C_0^1(\Omega)$ 都成立上述不等式, 因为由此通过取极限便可以得到这个不等式对任意 $u \in W_0^{1,p}(\Omega)$ 也都成立, 并进而得到包含关系 $W_0^{1,p}(\Omega) \subseteq L^q(\Omega)$.

当 $u \in C_0^1(\Omega)$ 时, 把 u 零延拓到 \mathbb{R}^n 上便得到 $u \in C_0^1(\mathbb{R}^n)$, 这样对于任意满足 $|\omega| = 1$ 的 $\omega \in \mathbb{R}^n$, 有

$$u(x) = - \int_0^\infty \frac{d}{dt} u(x + t\omega) dt, \quad \forall x \in \mathbb{R}^n$$

关于 ω 在单位球面上积分, 注意到 $\int_{|\omega|=1} d\omega = n\omega_n$, 就得到

$$\begin{aligned} |u(x)| &\leq \frac{1}{n\omega_n} \int_{|\omega|=1} \int_0^\infty \left| \frac{d}{dt} u(x + t\omega) \right| t^{-(n-1)} \cdot t^{n-1} dt d\omega \\ &\leq \frac{1}{n\omega_n} \int_{\mathbb{R}^n} |\nabla u(y)| |x - y|^{-(n-1)} dy \\ &= \frac{1}{n\omega_n} \int_{\Omega} |\nabla u(y)| |x - y|^{-(n-1)} dy, \quad \forall x \in \Omega \end{aligned}$$

记 $\mu = \frac{1}{p} - \frac{1}{q}$, $r = \frac{1}{1-\mu}$. 由所设条件知道 $0 \leq \mu < \frac{1}{n}$, 进而 $0 < r \leq 1$. 注意到

$$|\nabla u(y)| |x - y|^{-(n-1)} = [|\nabla u(y)|^p |x - y|^{-(n-1)r}]^{\frac{1}{q}} \cdot |\nabla u(y)|^{\mu p} \cdot |x - y|^{-(1-\frac{1}{p})(n-1)r}$$

以及 $\frac{1}{q} + \mu + \left(1 - \frac{1}{p}\right) = 1$, 应用推广的 Holder 不等式得

$$\begin{aligned} |u(x)| &\leq \frac{1}{n\omega_n} \left[\int_{\Omega} |\nabla u(y)|^p |x - y|^{-(n-1)r} dy \right]^{\frac{1}{q}} \cdot \left[\int_{\Omega} |\nabla u(y)|^p dx \right]^{\mu} \cdot \left[\int_{\Omega} |x - y|^{-(n-1)r} dy \right]^{1 - \frac{1}{p}} \\ &\leq \frac{1}{n\omega_n} [C_0(n, p, q, \Omega)]^{1 - \frac{1}{p}} \|\nabla u\|_{L^p(\Omega)}^{\mu p} \left[\int_{\Omega} |\nabla u(y)|^p |x - y|^{-(n-1)r} dy \right]^{\frac{1}{q}}, \quad \forall x \in \Omega \end{aligned}$$

其中 $C_0(n, p, q, \Omega) = \sup_{x \in \Omega} \int_{\Omega} |x - y|^{-(n-1)r} dy$. 因此

$$\begin{aligned}\|u\|_{L^q(\Omega)} &\leq \frac{1}{n\omega_n} [C_0(n, p, q, \Omega)]^{1-\frac{1}{p}} \|\nabla u\|_{L^p(\Omega)}^{\mu p} \left[\int_{\Omega} \int_{\Omega} |\nabla u(y)|^p |x - y|^{-(n-1)r} dy dx \right]^{\frac{1}{q}} \\ &\leq \frac{1}{n\omega_n} [C_0(n, p, q, \Omega)]^{1-\frac{1}{p}} \|\nabla u\|_{L^p(\Omega)}^{\mu p} \cdot [C_0(n, p, q, \Omega)]^{\frac{1}{q}} \|\nabla u\|_{L^p(\Omega)}^{\frac{p}{q}} \\ &= \frac{1}{n\omega_n} [C_0(n, p, q, \Omega)]^{1-\frac{1}{p}+\frac{1}{q}} \|\nabla u\|_{L^p(\Omega)}.\end{aligned}$$

这里用到 $\mu p + \frac{p}{q} = 1$. 选取 $R > 0$ 使得 $|\Omega| = \text{meas}B_R = \omega_n R^n$, 则对任意 $x \in \Omega$ 有

$$\begin{aligned}\int_{\Omega} |x - y|^{-(n-1)r} dy &\leq \int_{B_R(x)} |x - y|^{-(n-1)r} dy \\ &= n\omega_n \int_0^R \rho^{-(n-1)r+n-1} d\rho = \frac{n\omega_n R^{n-(n-1)r}}{n - (n-1)r}\end{aligned}$$

把 $R = \omega_n^{-\frac{1}{n}} |\Omega|^{\frac{1}{n}}$ 代入, 得

$$C_0(n, p, q, \Omega) = \sup_{x \in \Omega} \int_{\Omega} |x - y|^{-(n-1)r} dy \leq \frac{n\omega_n^{\frac{n-1}{n}r} |\Omega|^{1-\frac{n-1}{n}r}}{n - (n-1)r}.$$

注意到 $r = \frac{1}{1-\mu} = 1/\left(1 - \frac{1}{p} + \frac{1}{q}\right)$, 便从 (1. 8. 2) 得到了 (1. 8. 1). 证毕.

把 Théorème 18 应用到 $q = p$ 的特殊情况, 就有如下著名不等式

Théorème 19: Poincare 不等式

对任意 $1 \leq p < \infty$ 和有界开集 $\Omega \subseteq \mathbb{R}^n$ 成立不等式

$$\|u\|_{L^p(\Omega)} \leq \omega_n^{-\frac{1}{n}} |\Omega|^{\frac{1}{n}} \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega)$$

把 Théorème 18 应用到 $q = \infty$ 的特殊情况, 则有如下不等式

Corollaire 1

设 $p > n$, 则对任意有界开集 $\Omega \subseteq \mathbb{R}^n$ 有 $W_0^{1,p} \subseteq C(\bar{\Omega})$, 且成立不等式

$$\sup_{x \in \Omega} |u(x)| \leq C_2(n, p) |\Omega|^{\frac{1}{n}-\frac{1}{p}} \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1-p}(\Omega)$$

其中 $C_2(n, p) = \omega_n^{-\frac{1}{n}} [(p-1)/(p-n)]^{1-\frac{1}{p}}$.

Théorème 20

后面将把结论 $W_0^{1,p}(\Omega) \subseteq C(\bar{\Omega})$ ($p > n$) 推广到 $W^{1,p}(\Omega)$ ($p > n$) 并且 Ω 可以无界的情形.

定理 1.8.4 设 $1 \leq p < n$, 则对任意开集 $\Omega \subseteq \mathbf{R}^n$ (Ω 可以无界) 有 $W_0^{1,p}(\Omega) \subseteq L^{p^*}(\Omega)$, 其中 $p^* = np/(n-p)$ ($\frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}$), 且成立不等式

$$\|u\|_{L^{p^*}(\Omega)} \leq C_3(n, p) \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega), \quad (1.8.5)$$

其中 $C_3(n, p) = (n-1)p/(n-p)$.

Remarque 8. 我还是不大理解这个证明

证 先考虑 $p = 1$ 的情形. 这时 $p^* = n/(n-1)$. 对任意 $u \in C_0^1(\Omega)$, 把它看作 $C_0^1(\mathbf{R}^n)$ 中的函数, 则对每个 $1 \leq i \leq n$ 有

$$|u(x)| \leq \int_{-\infty}^{x_i} |\partial_i u(x)| dx_i \leq \int_{-\infty}^{\infty} |\partial_i u(x)| dx_i, \quad \forall x \in \mathbf{R}^n,$$

进而

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |\partial_i u(x)| dx_i \right)^{\frac{1}{n-1}}, \quad \forall x \in \mathbf{R}^n.$$

逐次对这个不等式的两端关于每个变元 x_1, x_2, \dots, x_n 积分, 并在每次积分后应用推广的 Hölder 不等式 (1.5.5) (在其中取 $q = 1, m = n-1, p_1 = p_2 = \dots = p_m = n-1$), 便得到

$$\int_{\mathbf{R}^n} |u(x)|^{\frac{n}{n-1}} dx \leq \left(\prod_{i=1}^n \int_{\mathbf{R}^n} |\partial_i u(x)| dx \right)^{\frac{1}{n-1}},$$

进而

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbf{R}^n)} \leq \left(\prod_{i=1}^n \int_{\mathbf{R}^n} |\partial_i u(x)| dx \right)^{\frac{1}{n}} \leq \|\nabla u\|_{L^1(\mathbf{R}^n)}.$$

再考虑 $1 < p < n$ 的情形. 这时令 $r = (n-1)p/(n-p)$. 则 $r > 1$. 对任意 $u \in C_0^1(\Omega)$, 由 $r > 1$ 知 $|u|^r \in C_0^1(\Omega)$, 所以应用前面已证明的结论得

$$\||u|^r\|_{L^{\frac{n}{n-1}}(\mathbf{R}^n)} \leq r \|u\|_{L^1(\mathbf{R}^n)}^{r-1} (\operatorname{sgn} u) \|\nabla u\|_{L^1(\mathbf{R}^n)} \leq r \|u\|_{L^p(\mathbf{R}^n)}^{r-1} \|\nabla u\|_{L^p(\mathbf{R}^n)}. \quad (1.8.6)$$

注意到

$$\||u|^r\|_{L^{\frac{n}{n-1}}(\mathbf{R}^n)} = \|u\|_{L^{p^*}(\mathbf{R}^n)}^{(n-1)p/(n-p)}, \quad \|u\|_{L^{p'}(\mathbf{R}^n)}^{r-1} = \|u\|_{L^{p^*}(\mathbf{R}^n)}^{n(p-1)/(n-p)},$$

所以由 (1.8.6) 即得 (1.8.5). 证毕. \square

Théorème 18 和 Théorème 20 说明, 在关于 p 和 Ω 的一定条件下, $W_0^{1,p}$ 中的函数可以有比 p 大的某些幂次 q 的可积性, 而且它们关于这些幂次 q 的

$L^q(\Omega)$ 范数可以用其一阶弱导数的 $L^p(\Omega)$ 范数界定。后一性质是 $W_0^{1,p}(\Omega)$ 所特有的；当 $W_0^{1,p}(\Omega) \neq W^{1,p}(\Omega)$ 时， $W^{1,p}(\Omega)$ 中的函数一般不具有这种性质，即 $W^{1,p}(\Omega)$ 中函数的 $L^q(\Omega)$ 范数一般不能被其一阶弱导数的 $L^p(\Omega)$ 范数界定。例如当 Ω 是有界开集时，非零的常值函数都属于 $W^{1,p}(\Omega)$ ；而对于这些函数，不等式 $\|u\|_{L^q(\Omega)} \leq C_1(n, p, q)|\Omega|^{\frac{1}{n} - \frac{1}{p} + \frac{1}{q}} \|\nabla u\|_{L^p(\Omega)}$ 和 $\|u\|_{L^{p^*}(\Omega)} \leq C_3(n, p) \|\nabla u\|_{L^p(\Omega)}$ 显然都不可能成立。至于前一性质，则不是 $W_0^{1,p}(\Omega)$ 中的函数所特有的，事实上 $W^{1,p}(\Omega)$ 中的函数也具有类似的这种性质。这一事实由以下定理所保证：

Théorème 21

定理 1.8.5 设 Ω 是 \mathbf{R}^n 中的任意可延拓开集（有界或无界均可）。又设 $1 \leq p \leq q \leq \infty$ 且 $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n}$ ，并且当 $p = n$ 时 $q \neq \infty$ 。则 $W^{1,p}(\Omega) \subseteq L^q(\Omega)$ ，且成立不等式

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega), \quad (1.8.7)$$

其中 C 是仅与 n, p, q 及 Ω 有关的常数。

反复使用 Théorème 21 便可以得到

Théorème 22: Sobolev 嵌入定理

定理 1.8.6 (Sobolev) 设 Ω 是 \mathbf{R}^n 中的任意可延拓开集 (有界或无界均可).

又设 m, k 为非负整数且 $m > k$. 再设 $1 \leq p \leq q \leq \infty$ 且 $\frac{1}{p} - \frac{1}{q} \leq \frac{m-k}{n}$, 并且当 $\frac{1}{p} = \frac{m-k}{n}$ 时 $q \neq \infty$. 则 $W^{m,p}(\Omega) \subseteq W^{k,q}(\Omega)$, 且成立不等式

$$\|u\|_{W^{k,q}(\Omega)} \leq C \|u\|_{W^{m,p}(\Omega)}, \quad \forall u \in W^{m,p}(\Omega), \quad (1.8.8)$$

其中 C 是仅与 n, m, k, p, q 及 Ω 有关的常数. \square

我们请读者注意这样两个事实 (这里总假设 Ω 为可延拓开集): ①当 $(m-k)p > n$ 时, 由 Sobolev 嵌入定理知 $W^{m,p}(\Omega) \subseteq W^{k,\infty}(\Omega)$, 且成立不等式

$$\sum_{|\alpha| \leq k} \sup_{x \in \Omega} |\partial^\alpha u(x)| \leq C \|u\|_{W^{m,p}(\Omega)}, \quad \forall u \in W^{m,p}(\Omega).$$

由于对每个函数 $u \in W^{m,p}(\Omega)$ 存在函数列 $u_k \in C^\infty(\bar{\Omega}) \cap W^{m,p}(\Omega)$ ($k = 1, 2, \dots$) 使其按 $W^{m,p}(\Omega)$ 范数收敛于 u (定理 1.7.10), 通过对函数 $u_k - u$ 应用以上不等式再令 $k \rightarrow \infty$ 取极限, 即知 $u \in C^k(\bar{\Omega})$. 这说明当 $(m-k)p > n$ 时 $W^{m,p}(\Omega) \subseteq C^k(\bar{\Omega})$. 这本来是 Sobolev 嵌入定理的一部分, 然而由于后面将要讨论的 Morrey 嵌入定理有比此更强的结论, 所以在定理 1.8.6 中我们没有写出这部分结论. ②如果 Ω 是有界开集, 那么定理 1.8.6 中的条件 $p \leq q$ 显然可以去掉. 这个事实请读者在应用定理 1.8.6 时自觉使用.

最后指出: 定理 1.8.6 的条件 $\frac{1}{p} - \frac{1}{q} \leq \frac{m-k}{n}$ 的一个易于记忆的等价形式为

$$m - \frac{n}{p} \geq k - \frac{n}{q}. \quad (1.8.9)$$

非负实数 $m - \frac{n}{p}$ 叫做 Sobolev 空间 $W^{m,p}(\Omega)$ 的**scaling 指数**. 定理 1.8.6 告诉我们: 在 $m \geq k$ 和 $p \leq q$ 的条件下, scaling 指数大的 Sobolev 空间 $W^{m,p}(\Omega)$ 可以嵌入到 scaling 指数小的 Sobolev 空间 $W^{k,q}(\Omega)$.

6.18 Morrey 嵌入定理

本节介绍 Morrey 嵌入定理, 它解释了 Sobolev 空间 $W^{m,p}(\Omega)$ 到 Holder 空间 $C^{k,\mu}(\bar{\Omega})$ 的嵌入关系。

Théorème 23

定理 1.9.1 设 $1 \leq p \leq q \leq \infty$ 且 $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$, 则对任意有界凸开集 $\Omega \subseteq \mathbf{R}^n$ 成立不等式

$$\|u - m(u)\|_{L^q(\Omega)} \leq C_4(n, p, q)(\text{diam}\Omega)^n |\Omega|^{\frac{1}{n} - \frac{1}{p} + \frac{1}{q} - 1} \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega), \quad (1.9.1)$$

其中 $m(u)$ 表示 u 的平均值, 即 $m(u) = \frac{1}{|\Omega|} \int_{\Omega} u(x) dx$, 而

$$C_4(n, p, q) = \omega_n^{1-\frac{1}{n}} \left[\left(1 - \frac{1}{p} + \frac{1}{q} \right) / n \left(\frac{1}{n} - \frac{1}{p} + \frac{1}{q} \right) \right]^{1-\frac{1}{p}+\frac{1}{q}}.$$

证明. 由于 Ω 是凸开集, 所以其边界是 Lipschitz 连续的, 进而 Ω 是可延拓开集. 因此根据定理 1.7.10 可知, 只需证明上述不等式对任意 $u \in C^1(\bar{\Omega})$ 成立即可. 这时, 由 Ω 的凸性可知对任意 $x, y \in \Omega (x \neq y)$ 有

$$u(x) - u(y) = - \int_0^{|x-y|} \frac{d}{dt} u(x+t\omega) dt, \quad \omega = -\frac{x-y}{|x-y|}$$

关于 y 在 Ω 上积分并除以 $|\Omega|$, 得

$$\begin{aligned} |u(x) - m(u)| &\leq \frac{1}{|\Omega|} \int_{\Omega} \int_0^{|x-y|} \left| \frac{d}{dt} (x+t\omega) \right| dt dy \\ &\leq \frac{1}{|\Omega|} \int_{\Omega} \int_0^{|x-y|} |\nabla u(x+t\omega)| dt dy, \quad \forall x \in \Omega \end{aligned}$$

令 $\widehat{\nabla u}$ 表示把 ∇u 的各个分量都作零延拓所得到的 \mathbf{R}^n 上的向量函数, 并记 $d = \text{diam } \Omega$, 则有

$$\begin{aligned} |u(x) - m(u)| &\leq \frac{1}{|\Omega|} \int_{B_d(x)} \int_0^{|x-y|} |\widehat{\nabla u}(x+t\omega)| dt dy \\ &= \frac{1}{|\Omega|} \int_0^\infty \int_{|\omega|=1} \int_0^d |\widehat{\nabla u}(x+t\omega)| \rho^{n-1} d\rho d\omega dt \\ &= \frac{d^n}{n|\Omega|} \int_0^\infty \int_{|\omega|=1} |\widehat{\nabla u}(x+t\omega)| d\omega dt \\ &= \frac{d^n}{n|\Omega|} \int_{\Omega} |x-y|^{-(n-1)} |\nabla u(y)| dy, \quad \forall x \in \Omega. \end{aligned}$$

据此应用与定理 1.8.1 的证明类似的方法即可得到 (1.9.1). 证毕.

□

6.19 Some Inequalities

Théorème 24: Holder 不等式

定理 1.5.4 (推广的 Hölder 不等式) 设 Ω 是可测集. 如果 $u_i \in L^{p_i}(\Omega)$ ($1 \leq p_i \leq \infty$, $i = 1, 2, \dots, m$), 且 $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} \leq 1$, 那么当令 $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m}$

时, 有 $u_1 u_2 \cdots u_m \in L^q(\Omega)$, 且成立不等式

$$\|u_1 u_2 \cdots u_m\|_{L^q(\Omega)} \leq \|u_1\|_{L^{p_1}(\Omega)} \|u_2\|_{L^{p_2}(\Omega)} \cdots \|u_m\|_{L^{p_m}(\Omega)}. \quad (1.5.5)$$

Théorème 25: 内插不等式

定理 1.5.2 (内插不等式) 设 Ω 是可测集. 则当 $1 \leq p \leq q \leq r \leq \infty$ 时, 有 $L^p(\Omega) \cap L^r(\Omega) \subseteq L^q(\Omega)$, 且成立不等式

$$\|u\|_{L^q(\Omega)} \leq \|u\|_{L^p(\Omega)}^\theta \|u\|_{L^r(\Omega)}^{1-\theta}, \quad \forall u \in L^p(\Omega) \cap L^r(\Omega), \quad (1.5.3)$$

其中 θ 是由等式 $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}$ 唯一确定的实数 (显然 $0 \leq \theta \leq 1$).

6.20 Sobolev 空间基本性质

参见于品数学分析讲义

6.20.1 分布理论与 Fourier 变换的应用: Sobolev 空间及应用

在后面的课程中, 我们会经常用所谓的 Planchrel 公式: 对任意的 $f \in L^2(\mathbb{R}^n)$, 我们有

$$\|\widehat{f}\|_{L^2}^2 = (2\pi)^n \|f\|_{L^2}^2 \Leftrightarrow \int_{\mathbb{R}^n} |f(x)|^2 dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 d\xi.$$

它的另外一个版本是说对任意的 $f, g \in L^2(\mathbb{R}^n)$, 我们有

$$(\widehat{f}, \widehat{g})_{L^2} = (2\pi)^n (f, g)_{L^2} \Leftrightarrow \int_{\mathbb{R}^n} f(x) \overline{g(x)} dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi.$$

这个公式我们之前已经证明过。

我们现在引入 \mathbb{R}^n 上的 Sobolev 空间的定义。

Définition 20: 指标为 s 的 Sobolev 空间

给定 $s \in \mathbb{R}$, 我们将把这个数称为是 Sobolev 空间的指标。我们考虑满足如下性质的缓增分布 $u \in \mathcal{S}'(\mathbb{R}^n)$:

- (1) $\hat{u} \in L^1_{\text{loc}}(\mathbb{R}^n)$ 是局部可积的函数;
- (2) $(1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi)$ 是平方可积的函数。

对于这样的函数, 我们定义其 Sobolev 范数为:

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

我们把所有满足上述条件的缓增分布的集合称作是一个指标为 s 的 Sobolev 空间, 这显然是一个复线性空间, 我们用 $H^s(\mathbb{R}^n)$ 来表示。在 $H^s(\mathbb{R}^n)$ 上所赋予的范数与下面的内积是相容的: 对任意的 $u, v \in H^s(\mathbb{R}^n)$, 令

$$(u, v)_{H^s} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi$$

所以, $(H^s(\mathbb{R}^n), (\cdot, \cdot)_{H^s})$ 是内积空间。

我们注意到, 当 $s = 0$ 时, 我们 $H^0(\mathbb{R}^n)$ 实际上就是 $L^2(\mathbb{R}^n)$, 这由 Plancherel 公式立即就可以得到:

$$u \in L^2(\mathbb{R}^n) \Leftrightarrow \hat{u} \in L^2(\mathbb{R}^n)$$

所以,

$$H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$$

类似的, 如果我们在频率空间 \mathbb{R}_ξ^n 上考虑测度

$$\mu_s = (1 + |\xi|^2)^s d\xi$$

那么, $u \in H^s(\mathbb{R}^n)$ 当且仅当 $\hat{u} \in L^2(\mathbb{R}^n, d\mu_s)$ 。利用这个观察, 我们现在证明:

Théorème 26

对任意的 $s \in \mathbb{R}$, $(H^s(\mathbb{R}^n), (\cdot, \cdot)_{H^s})$ 是 **Hilbert 空间** (即完备的内积空间)。

证明. 假设 $\{u_k\}_{k \geq 1} \subset H^s(\mathbb{R}^n)$ 是 Cauchy 列, 那么, 根据定义, $\{\widehat{u_k}\}_{k \geq 1} \subset L^2(\mathbb{R}^n, d\mu_s)$ 是 Cauchy 列。利用 L^2 一空间的完备性, 存在 $v(\xi) \in L^2(\mathbb{R}^n, d\mu_s)$ 作为上述序列的极限。我们用 $u(x) \in \mathcal{S}'(\mathbb{R}^n)$ 表示它的 Fourier 逆变换, 即

$$\widehat{u} = v.$$

那么,

$$\lim_{k \rightarrow \infty} \|\widehat{u_k} - \widehat{u}\|_{L^2(\mathbb{R}^n, d\mu_s)}^2 = 0 \Leftrightarrow \lim_{k \rightarrow \infty} \|u_k - u\|_{H^s(\mathbb{R}^n)}^2 = 0$$

这就证明了完备性。 \square

根据 Sobolev 空间的定义, 我们知道 $\{H^s(\mathbb{R}^n)\}_{s \in \mathbb{R}}$ 构成了一个下降的链, 即对任意的 $s, s' \in \mathbb{R}$

$$s \geq s' \Rightarrow H^{s'}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$$

我们观察到, Schwartz 函数生活在所有的 Sobolev 空间中:

$$\mathcal{S}(\mathbb{R}^n) \subset \bigcap_{s \in \mathbb{R}} H^s(\mathbb{R}^n)$$

Proposition 3

假设 $m \in \mathbb{Z}_{\geq 1}$ 为正整数, 那么, $H^m(\mathbb{R}^n)$ 有如下的等价刻画:

$$H^m(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}) \mid \text{对任意的多重指标 } \alpha, |\alpha| \leq m, \partial^\alpha u \in L^2(\mathbb{R}^n)\}.$$

这个证明是初等的.

6.20.2 Sobolev 空间的映射性质: 定义 Fourier 乘子

受到 $\widehat{\frac{1}{i}\partial_k u}(\xi) = \xi \widehat{u}(\xi), \forall k \leq n$ 的启发, 我们定义算子 $D_k = \frac{1}{i}\partial_k = -i\partial_k$, 简记为 $D = \frac{1}{i}\partial$. 形式上, D 对一个分布的作用在频率空间上看来就是乘以 ξ .

Définition 21: Fourier 乘子

给定频率空间上的函数 $m(\xi)$, 我们假设它是多项式增长的。对于任意的缓增分布 $u \in S'(\mathbb{R}^n)$, 我们定义

$$m(D)u = \mathcal{F}^{-1}(m(\xi)\hat{u}(\xi)) \Leftrightarrow \widehat{m(D)u} = m(\xi)\hat{u}(\xi).$$

由于 $m(\xi)$ 是多项式增长的, 所以, $m(\xi)\hat{u}(\xi)$ 仍然是缓增分布, 所以, 如下的算子是良好定义的:

$$m(D) : \delta'(\mathbb{R}^n) \rightarrow \delta'(\mathbb{R}^n)$$

Exemple 14. 我们先看几个简单的例子:

(1) 当 $m(\xi) = \xi_k$ 时, 其中 $k = 1, 2, \dots, n$, 我们有

$$m(D) = \frac{1}{i}\partial_k = D_k.$$

(2) 当 $m(\xi) = |\xi|^2$ 时, 我们有

$$m(D) = -\Delta.$$

(3) 给定线性微分算子

$$P = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha,$$

它可以被视作是一个 Fourier 乘子 $m(D)$, 其中

$$m(\xi) = \sum_{|\alpha| \leq m} i^{|\alpha|} a_\alpha \xi^\alpha,$$

(4) 算子 $(1 - \Delta)^s$ 表示的是函数 $(1 + |\xi|^2)^s$ 所对应的 Fourier 乘子。

Exemple 15. 下面的 Sobolev 空间都定义在 \mathbb{R}^n 上。

(1) 对任意的 $s < -\frac{n}{2}$, 我们有

$$\delta_0 \in H^s(\mathbb{R}^n)$$

实际上, 我们只要说明下面的积分有限即可:

$$\int_{\mathbb{R}^n} (1 + |\xi|^2)^s \cdot 1 d\xi$$

这在 $s < -\frac{n}{2}$ 时是成立的。同样的推理表明, 当 $s \geq -\frac{n}{2}$ 时, $\delta_0 \notin H^s(\mathbb{R}^n)$ 。²

常数值函数 1 不在任何的 $H^s(\mathbb{R}^n)$ 中。特别地, 这表明

$$\bigcup_{s \in \mathbb{R}} H^s(\mathbb{R}^n) \subsetneq \mathcal{S}'(\mathbb{R}^n)$$

6.20.3 Sobolev 空间在 Fourier 乘子下的映射性质

Proposition 4

给定多项式增长的乘子函数 $m(\xi)$ ，其中，我们假设存在常数 C 和 p ，使得对任意的 $\xi \in \mathbb{R}^n$ ，我们都有

$$|m(\xi)| \leq C(1 + |\xi|)^p$$

那么，对任意的 $s \in \mathbb{R}$ ，对任意的 $u \in H^s(\mathbb{R}^n)$, $m(D)u \in H^{s-p}(\mathbb{R}^n)$ 。这就定义出有界（连续）线性映射：

$$m(D) : H^s(\mathbb{R}^n) \rightarrow H^{s-p}(\mathbb{R}^n)$$

特别地，对任意的 d 一阶的微分算子 P ，对任意的 $s \in \mathbb{R}$ ，我们有连续线性映射

$$P : H^s(\mathbb{R}^n) \rightarrow H^{s-d}(\mathbb{R}^n), \quad \forall s$$

另外，对任意的 s ，我们还有连续的线性同构：

$$(1 + \Delta)^{\frac{p}{2}} : H^s(\mathbb{R}^n) \rightarrow H^{s-p}(\mathbb{R}^n)$$

其中，上述映射的逆映射是 $(1 + \Delta)^{-\frac{p}{2}}$ 。

证明. 对任意的 $u \in H^s(\mathbb{R}^n)$ ，我们首先证明 $m(D)u \in H^{s-p}(\mathbb{R}^n)$ ，其中 $m(\xi)$ 具有命题中所要求的多项式增长。根据 Planchrel 公式，我们有

$$\begin{aligned} \|m(D)u\|_{H^{s-p}}^2 &= \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s-p} |m(\xi)\hat{u}(\xi)|^2 d\xi \\ &\leq C \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s-p} (1 + |\xi|)^{2p} |\hat{u}(\xi)|^2 d\xi \\ &\leq C' \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \end{aligned}$$

所以，存在常数 C_1 ，使得

$$\|m(D)u\|_{H^{s-p}} \leq C_1 \|u\|_{H^s}$$

这表明 $m(D)$ 是从 $H^s(\mathbb{R}^n)$ 到 $H^{s-p}(\mathbb{R}^n)$ 的连续线性映射。

微分算子的情形是一个特例。为了说明 $m(D) = (1 + \Delta)^{\frac{p}{2}}$ 有逆，我们用

$$n(\xi) = (1 + |\xi|)^{-\frac{p}{2}}$$

作为乘子即可，这是因为

$$\widehat{m(D)n(D)u} = (1 + |\xi|)^{\frac{p}{2}}(1 + |\xi|)^{-\frac{p}{2}}\widehat{u}(\xi) = \widehat{u}(\xi).$$

命题得证。 \square

6.20.4 稠密性定理

Proposition 5

对每个指标 $s \in \mathbb{R}^n$ ，光滑有紧支集的函数 $C_0^\infty(\mathbb{R}^n)$ 在 $H^s(\mathbb{R}^n)$ 中是稠密的。

证明。我们首先证明 $\mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ 是稠密的，其中 $s \in \mathbb{R}$ ：这个论断对 $s = 0$ 是正确的，因为 $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ 在 $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ 中是稠密的。由于

$$(1 - \Delta)^{-\frac{s}{2}} : H^0(\mathbb{R}^n) \rightarrow H^s(\mathbb{R}^n)$$

是连续可逆的线性映射（是同胚）（因为 Proposition 4），所以 $\mathcal{S}(\mathbb{R}^n)$ 在这个算子下的像也是稠密的，然而，

$$(1 - \Delta)^{-\frac{s}{2}}(\mathcal{S}(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^n)$$

所以， $\mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ 是稠密的。

Note. 在上面的论证中， $(1 - \Delta)^{-\frac{s}{2}}$ 不一定把有紧支集的函数映射为有紧支集的函数，所以，我们的推理是对 $\mathcal{S}(\mathbb{R}^n)$ 进行的。

为了证明命题，我们只要说明在 $H^s(\mathbb{R}^n)$ 的意义下， $\mathcal{S}(\mathbb{R}^n)$ 中的任意一个函数 f 都可以被 $C_0^\infty(\mathbb{R}^n)$ 的函数逼近。我们上次证明了存在常数 C ，使得对任意的 $\psi \in \mathcal{S}(\mathbb{R}^n)$ ，我们有不等式

$$\|\psi\|_{H^s} \leq C N_{s+n+1}(\psi)$$

所以，对任意的 $f \in H^s(\mathbb{R}^n)$ ，我们先选取 $\psi \in \mathcal{S}(\mathbb{R}^n)$ ，使得

$$\|f - \psi\|_{H^s} < \frac{\varepsilon}{2}$$

再利用 $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ 的稠密性，选取 $\varphi \in C_0^\infty(\mathbb{R}^n)$ ，使得

$$N_{s+n+1}(\psi - \varphi) < \frac{\varepsilon}{2C}$$

此时，我们有

$$\|f - \varphi\|_{H^s} < \varepsilon$$

这就证明了命题。 \square

我们下 证明著名的 Sobolev 嵌 定理 (的 一种形式):

Théorème 27: Sobolev 嵌入定理

假设指标 $s > \frac{n}{2}$ ，那么，每个 $u \in H^s(\mathbb{R}^n)$ 都落在 $L^\infty(\mathbb{R}^n)$ 中。进一步，我们有连续的线性嵌入

$$\iota : H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n), \quad u \mapsto u$$

即存在 C_s ，使得对任意 $u \in H^s(\mathbb{R}^n)$ ，我们都有

$$\|u\|_{L^\infty} \leq C_s \|u\|_{H^s}$$

进一步， u 是连续函数（可以在它的代表类中选到一个连续函数）并且在无穷远处的极限为零，即 $u \in C_0(\mathbb{R}^n)$ 。

Remarque 9. 证明的想法比较简单：我们只要说明 \widehat{u} 是一个 L^1 函数即可，因为 Fourier 逆变换就把它还原成一个在 ∞ 处衰减的连续函数，从而是 L^∞ 的函数。

证明. 根据 $u = \mathcal{F}^{-1}(\widehat{u})$ ，我们知道

$$\|u\|_{L^\infty} \leq \frac{1}{(2\pi)^n} \|\widehat{u}\|_{L^1}$$

我们现在说明 $\|\widehat{u}(\xi)\|_{L^1}$ 被 $\|u\|_{H^s}$ 所控制。根据 $s > \frac{n}{2}$ ，我们可以 \widehat{u} 写成两个平方可积的函数的乘积：

$$\widehat{u}(\xi) = \underbrace{(1 + |\xi|^2)^{\frac{s}{2}}}_{L^2} \widehat{u}(\xi) \cdot \underbrace{(1 + |\xi|^2)^{-\frac{s}{2}}}_{L^2}$$

前一部分根据 $u \in H^s(\mathbb{R}^n)$ 所以是 L^2 的；后一部分根据 $s > \frac{n}{2}$ 所以是 L^2 的。

利用 Cauchy-Schwarz 不等式，我们有

$$\|\widehat{u}(\xi)\|_{L^1} \leq \|u\|_{H^s} \left\| (1 + |\xi|^2)^{-\frac{s}{2}} \right\|_{L^2} = C \|u\|_{H^s}$$

所以，

$$\|u\|_{L^\infty} \leq \frac{C}{(2\pi)^n} \|u\|_{H^s}$$

连续性的部分是明显的，因为

$$\mathcal{F}^{-1} : L^1(\mathbb{R}_\xi^n) \longrightarrow C_0(\mathbb{R}_x^n)$$

证明完毕。 \square

Corollaire 2

假设 $s > \frac{n}{2} + k$, 其中 k 为非负整数, 那么, 对任意的 $u \in H^s(\mathbb{R}^n)$, 我们都有 $u \in C^k(\mathbb{R}^n)$ 。

证明. 对任意的多重指标 α , 如果 $|\alpha| \leq k$, 那么 $\partial^\alpha u \in H^{s-|\alpha|}(\mathbb{R}^n)$ 是连续函数, 从而, $u \in C^k(\mathbb{R}^n)$ (用归纳法来证明会更严格一点)。 \square

Remarque 10. 这个版本的 Sobolev 嵌入定理说的是, 如果指标 s 足够大, 那么, 函数 u 就会非常光滑。

Remarque 11. 我们在作业中将构造函数局部可积的 $u \in H^{\frac{n}{2}}(\mathbb{R}^n)$, 使得 $u \notin L^\infty(\mathbb{R}^n)$ 。换句话说, 如下的嵌入并不成立:

$$H^{\frac{n}{2}}(\mathbb{R}^n) \not\hookrightarrow L^\infty(\mathbb{R}^n)$$

这表明 Sobolev 嵌入的指标至少是 $\frac{n}{2} + \varepsilon$, 其中 $\varepsilon > 0$ 可以任意小。

6.20.5 Sobolev 空间的代数性质

下一个定理说的是如果指标 s 足够大, 那么, 两个 H^s 的函数的乘积也是 H^s 的。这个定理在证明非线性偏微分方程的解的局部存在性时很有用。

Théorème 28

如果 $s > \frac{n}{2}$, 那么, $H^s(\mathbb{R}^n)$ 是一个代数, 即对任意的 $u, v \in H^s(\mathbb{R}^n)$, 我们有 $u \cdot v \in H^s(\mathbb{R}^n)$:

$$H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \xrightarrow{\times} H^s(\mathbb{R}^n)$$

实际上, 存在常数 C_s , 使得对任意的 $u, v \in H^s(\mathbb{R}^n)$, 我们有

$$\|u \cdot v\|_{H^s} \leq C_s \|u\|_{H^s} \|v\|_{H^s}$$

Note. Sobolev 不等式的□个重要的观点就是□函数以及它的导数的积分来控制函数的最□值。

证明。我们来计算 $u \cdot v$ 的 H^s 一范数。由于在 Fourier 变换下，乘积变化为卷积，所以按照定义，我们有

$$\begin{aligned}\|\bar{u} \cdot v\|_{H^s}^2 &= \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \left| \int_{\mathbb{R}^n} \hat{u}(\xi - \eta) \hat{v}(\eta) d\eta \right|^2 d\xi \\ &\leq \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \left(\int_{\mathbb{R}^n} |\hat{u}(\xi - \eta)| |\hat{v}(\eta)| d\eta \right)^2 d\xi\end{aligned}$$

我们要把因子 $(1 + |\xi|^2)^s$ 进行拆分。首先，对任意的 $s > 0, a, b \geq 0$ ，我们显然有

$$(a + b)^s \leq 2^s (a^s + b^s)$$

所以，对任意的 $\xi, \eta \in \mathbb{R}^n$ ，我们有如下的不等式

$$\begin{aligned}(1 + |\xi|^2)^{\frac{s}{2}} &\leq (1 + 2|\xi - \eta|^2 + 2|\eta|^2)^{\frac{s}{2}} \leq 2^{\frac{s}{2}} ((1 + |\xi - \eta|^2) + (1 + |\eta|^2))^{\frac{s}{2}} \\ &\leq 2^{2s} ((1 + |\xi - \eta|^2)^{\frac{s}{2}} + (1 + |\eta|^2)^{\frac{s}{2}})\end{aligned}$$

所以，我们就得到了

$$\begin{aligned}\|\bar{u} \cdot v\|_{H^s}^2 &\leq 2^{2s} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} (1 + |\xi - \eta|^2)^{\frac{s}{2}} \left| \hat{u}(\xi - \eta) \right| \left| \hat{v}(\eta) \right| + (1 + |\eta|^2)^{\frac{s}{2}} \left| \hat{v}(\eta) \right| \left| \hat{u}(\xi - \eta) \right| \right) d\eta d\xi \\ &\leq 2^{2s} \int_{\mathbb{R}^n} \underbrace{\left(\int_{\mathbb{R}^n} (1 + |\xi - \eta|^2)^{\frac{s}{2}} |\hat{u}(\xi - \eta)| |\hat{v}(\eta)|^2 d\eta \right)}_{f(\xi)} + \underbrace{\left(\int_{\mathbb{R}^n} (1 + |\eta|^2)^{\frac{s}{2}} |\hat{v}(\eta)| |\hat{u}(\xi - \eta)|^2 d\eta \right)}_{g(\eta)} d\xi\end{aligned}$$

上面的表达式中本质上是两项，它们的结构是类似的，我们只要处理一项就好。

我们现在利用第一项中的卷积结构来控制它。根据 H^s 的定义，我们有

$$f(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} |\hat{u}(\xi)| \in L^2(\mathbb{R}_\xi^n)$$

根据 $s > \frac{n}{2}$ ，我们在 Sobolev 不等式的证明中已经证明了 $g(\xi) = \hat{v} \in L^1(\mathbb{R}^n)$ 。

特别地，存在常数 C_1 和 C_2 ，使得

$$\|f\|_{L^2} \leq C_1 \|\bar{u}\|_{H^s}, \quad \|g\|_{L^2} \leq C_2 \|v\|_{H^s}$$

我们观察到，上面就是控制 $f * g$ 的 L^2 范数的大小。我们回忆上学期（5月9日的课程，利用 Fubini 定理）已经证明了

$$L^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \xrightarrow{*} L^2(\mathbb{R}^n)$$

其中对任意的 $\varphi \in L^1(\mathbb{R}^n), \psi \in L^2(\mathbb{R}^n)$, 我们有

$$\|\varphi * \psi\|_{L^2} \leq \|\varphi\|_{L^1} \|\psi\|_{L^2}$$

我们对 $\varphi = g$ 和 $\psi = f$ 运用这个不等式, 就得到

$$\|f * g\|_{L^2} \leq \|g\|_{L^1} \|f\|_{L^2} \leq C_1 C_2 \|\bar{u}\|_{H^s} \|v\|_{H^s}$$

□

实际上, 我们还可以证明更强的结论: 对任意的 $s > 0, H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ 是一个代数。我们之后将利用频率空间的二进分解进行证明。

6.21 傅里叶变换的一致连续性

Définition 22

设 $f \in L^1(\mathbb{R}^n)$, 即 $\int_{\mathbb{R}^n} |f(x)| dx < \infty$. 其傅里叶变换 $\hat{f}: \mathbb{R}^n \rightarrow \mathbb{C}$ 定义为:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$$

其中 $\langle x, \xi \rangle$ 表示 x 和 ξ 在 \mathbb{R}^n 中的标准内积 (点积)。

我们要证明 \hat{f} 在 \mathbb{R}^n 上是一致连续的。也就是说, 对于任意的 $\epsilon > 0$, 存在一个 $\delta > 0$, 使得对于所有 $\xi, \eta \in \mathbb{R}^n$, 若 $|\xi - \eta| < \delta$, 则 $|\hat{f}(\xi) - \hat{f}(\eta)| < \epsilon$.

证明. 考虑差值 $|\hat{f}(\xi) - \hat{f}(\eta)|$:

$$\begin{aligned} |\hat{f}(\xi) - \hat{f}(\eta)| &= \left| \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx - \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \eta \rangle} dx \right| \\ &= \left| \int_{\mathbb{R}^n} f(x) (e^{-2\pi i \langle x, \xi \rangle} - e^{-2\pi i \langle x, \eta \rangle}) dx \right| \\ &= \left| \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \eta \rangle} (e^{-2\pi i \langle x, \xi - \eta \rangle} - 1) dx \right|. \end{aligned}$$

由于 $|e^{-2\pi i \langle x, \eta \rangle}| = 1$, 我们可以得到:

$$\begin{aligned} |\hat{f}(\xi) - \hat{f}(\eta)| &\leq \int_{\mathbb{R}^n} |f(x)| e^{-2\pi i \langle x, \eta \rangle} |(e^{-2\pi i \langle x, \xi - \eta \rangle} - 1)| dx \\ |\hat{f}(\xi) - \hat{f}(\eta)| &\leq \int_{\mathbb{R}^n} |f(x)| |e^{-2\pi i \langle x, \xi - \eta \rangle} - 1| dx \end{aligned}$$

令 $h = \xi - \eta$. 我们的目标是证明当 $|h| \rightarrow 0$ 时,

$$\int_{\mathbb{R}^n} |f(x)| |e^{-2\pi i \langle x, h \rangle} - 1| dx \rightarrow 0.$$

我们使用勒贝格控制收敛定理 (Lebesgue's Dominated Convergence Theorem)。

定义函数序列 (或族) $g_h(x) = |f(x)| |e^{-2\pi i \langle x, h \rangle} - 1|$.

- (1) **逐点收敛 (Pointwise Convergence):** 对于任意固定的 $x \in \mathbb{R}^n$, 当 $|h| \rightarrow 0$ 时, $\langle x, h \rangle \rightarrow 0$. 因此, $e^{-2\pi i \langle x, h \rangle} \rightarrow e^0 = 1$. 所以, $|e^{-2\pi i \langle x, h \rangle} - 1| \rightarrow |1 - 1| = 0$. 从而, $g_h(x) = |f(x)| |e^{-2\pi i \langle x, h \rangle} - 1| \rightarrow |f(x)| \cdot 0 = 0$ 对于几乎所有的 $x \in \mathbb{R}^n$ (在 $f(x)$ 定义的意义上).
- (2) **控制函数 (Dominating Function):** 对于任意 $x \in \mathbb{R}^n$ 和任意 $h \in \mathbb{R}^n$, 我们有: $|e^{-2\pi i \langle x, h \rangle} - 1| \leq |e^{-2\pi i \langle x, h \rangle}| + |-1| = 1 + 1 = 2$. 因此, $g_h(x) = |f(x)| |e^{-2\pi i \langle x, h \rangle} - 1| \leq 2|f(x)|$. 由于 $f \in L^1(\mathbb{R}^n)$, 函数 $2|f(x)|$ 在 \mathbb{R}^n 上是可积的, 即 $\int_{\mathbb{R}^n} 2|f(x)| dx = 2\|f\|_1 < \infty$.

根据勒贝格控制收敛定理, 由于 $g_h(x)$ 逐点收敛到 0 并且被一个可积函数 $2|f(x)|$ 控制, 我们有:

$$\begin{aligned} \lim_{|h| \rightarrow 0} \int_{\mathbb{R}^n} g_h(x) dx &= \int_{\mathbb{R}^n} \lim_{|h| \rightarrow 0} g_h(x) dx \\ &= \lim_{|h| \rightarrow 0} \int_{\mathbb{R}^n} |f(x)| |e^{-2\pi i \langle x, h \rangle} - 1| dx = \int_{\mathbb{R}^n} 0 dx = 0 \end{aligned}$$

这意味着, 对于任意 $\epsilon > 0$, 存在一个 $\delta > 0$, 使得只要 $|h| < \delta$, 就有

$$\int_{\mathbb{R}^n} |f(x)| |e^{-2\pi i \langle x, h \rangle} - 1| dx < \epsilon$$

将 $h = \xi - \eta$ 代回, 如果 $|\xi - \eta| < \delta$, 则

$$|\widehat{f}(\xi) - \widehat{f}(\eta)| \leq \int_{\mathbb{R}^n} |f(x)| |e^{-2\pi i \langle x, \xi - \eta \rangle} - 1| dx < \epsilon$$

这正是函数 \widehat{f} 在 \mathbb{R}^n 上一致连续的定义。 \square

因此, \mathbb{R}^n 上的 L^1 函数的傅里叶变换是一致连续函数。这个性质也被称为 Riemann-Lebesgue 引理的一个推论或相关性质 (尽管 Riemann-Lebesgue 引理本身通常指 $\widehat{f}(\xi) \rightarrow 0$ 当 $|\xi| \rightarrow \infty$).

6.22 分布的定义与例子

参见于品数学分析讲义

6.22.1 试验函数

我们用 $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ 表示在 Ω 上定义并且有紧支集的光滑函数所组成的集合，我们把它称作是为试验函数空间。按照定义，对于任意的 $\varphi \in \mathcal{D}(\Omega)$ ，存在紧集 $K \subset \Omega$ （这是 \mathbb{R}^n 中的紧集），使得 $f|_{\Omega-K} \equiv 0$ ，即对任意的 $x \in \Omega - K, f(x) = 0$ 。

试验函数的闭包

以下是一些常见的闭包及其对应的函数空间：

- (1) 在 $L^p(\Omega)$ 空间中（使用 L^p 范数）：

试验函数空间 $\mathcal{D}(\Omega)$ 在 $L^p(\Omega)$ 空间中是稠密的，对于 $1 \leq p < \infty$ 。这意味着 $\mathcal{D}(\Omega)$ 在 L^p 范数下的闭包是整个 $L^p(\Omega)$ 空间。

简单来说，任何一个 L^p 函数都可以通过一列试验函数在该范数下逼近。

- (1) 在连续函数空间 $C(\Omega)$ 或 $C_b(\Omega)$ 中（使用一致范数 $\|f\|_\infty$ ）：

试验函数是连续的，并且具有紧支集。在一致范数下，试验函数空间 $\mathcal{D}(\Omega)$ 的闭包是具有紧支集的连续函数空间 $C_c(\Omega)$ 。这是因为任何一个紧支集连续函数都可以通过光滑的，紧支集函数（例如通过卷积光滑化）来一致逼近。

- (1) 在 Sobolev 空间 $W^{k,p}(\Omega)$ 中（使用 Sobolev 范数）：

试验函数空间 $\mathcal{D}(\Omega)$ 在 Sobolev 空间 $W^{k,p}(\Omega)$ 中是稠密的，对于 $1 \leq p < \infty$ 和非负整数 $k \geq 0$ 。这意味着 $\mathcal{D}(\Omega)$ 在 $W^{k,p}$ 范数下的闭包是整个 Sobolev 空间 $W^{k,p}(\Omega)$ 。Sobolev 空间是研究偏微分方程非常重要的函数空间，它包含了直到 k 阶的广义导数（或弱导数）存在且属于 L^p 空间的函数。

6.22.2 分布

Définition 23: 分布

所谓 Ω 上的一个分布（也称作广义函数）指的是 $\mathcal{D}(\Omega)$ 上的一个线性泛函（线性映射）：

$$u : \mathcal{D}(\Omega) \rightarrow \mathbb{C}, \quad \varphi \mapsto \langle u, \varphi \rangle$$

满足如下两个条件

- (1) 对任意的 $\varphi, \psi \in \mathcal{D}(\Omega)$ 和 $\alpha, \beta \in \mathbb{C}$ ，我们有

$$\langle u, \alpha\varphi + \beta\psi \rangle = \alpha\langle u, \varphi \rangle + \beta\langle u, \psi \rangle.$$

(1) 对任意的紧集 $K \subset \Omega$, 存在非负整数 p 和正常数 $C(p$ 和 C 依赖于 K), 使得对任意的 $\varphi \in C_K^\infty(\Omega)$, 都有

$$|\langle u, \varphi \rangle| \leq C \sup_{|\alpha| \leq p} \|\partial^\alpha \varphi\|_{L^\infty(K)}$$

如果上述的 p 的选取不依赖于紧集 K 的选取, 那么, 我们就把最小的这样的非负整数 p 称作是分布 u 的阶。

Exemple 16. 对任意的 $a \in \Omega$, 我们可以定义分布 $\delta_a \in \mathcal{D}'(\Omega)$ 。其中, 对于任意的 $\varphi \in \mathcal{D}(\Omega)$, 我们定义

$$\langle \delta_a, \varphi \rangle = \varphi(a)$$

我们来验证 δ_a 实际上是分布: 对任意的紧集 $K \subset \Omega$, 如果 $a \notin K$, 那么, 对任意的 $\varphi \in C_K^\infty(\Omega)$, 我们都有

$$\langle \delta_a, \varphi \rangle = 0$$

如果 $a \in K$, 那么, 使得对任意的 $\varphi \in C_K^\infty(\Omega)$, 我们有

$$|\langle \delta_a, \varphi \rangle| = |\varphi(a)| \leq 1 \cdot \sup_{|\alpha| \leq 0} \|\partial^\alpha \varphi\|_{L^\infty(K)}.$$

所以, 我们在分布的定义中取 $q = 0, C = 1$ 即可。特别地, 我们还知道 δ_a 的阶为 0。

Exemple 17. 给定开集 Ω (总是装配了 Borel 代数和 Lebesgue 测度), 局部可积的函数指的是在每个紧的局部上都可积的函数, 即可测函数 f (所对应的几乎处处相等的函数的等价类), 对于任意紧集 $K \subset \Omega$, 函数 $f \cdot \mathbf{1}_K \in L^1(\Omega)$ 。我们用 $L_{\text{loc}}^1(\Omega)$ 表示 Ω 上局部可积的函数。对于任意的 $f \in L_{\text{loc}}^1(\Omega)$, 我们定义 $\mathcal{D}(\Omega)$ 上的线性泛函:

$$T_f : \mathcal{D}(\Omega) \rightarrow \mathbb{C}, \varphi \mapsto \langle T_f, \varphi \rangle = \int_{\Omega} f(x)\varphi(x)dx$$

由于 φ 在它的支集 K 上有界, 所以, 上面的积分是良好定义的。我们证明 T_f 是 Ω 上的阶为 0 的分布: 对任意的紧集 $K \subset \Omega$, 对任意的 $\varphi \in C_K^\infty(\Omega)$, 我们有

$$\begin{aligned} |\langle T_f, \varphi \rangle| &= \left| \int_K f(x)\varphi(x)dx \right| \\ &\leq \|f\|_{L^1(K)} \|\varphi\|_{L^\infty(K)}. \end{aligned}$$

所以, 我们在分布的定义中取 $q = 0, \|f\|_{L^1(K)}$ 即可。

Remarque 12. 为了方便起见, 我们通常把 $\langle T_f, \varphi \rangle$ 直接写成 $\langle f, \varphi \rangle$ 。

Proposition 6

任意选定 $\chi(x) \in \mathcal{D}(\mathbb{R}^n)$ (我们通常偏爱之前所构造的那个 $\chi(x)$)，我们假定

$$\int_{\mathbb{R}^n} \chi(x) dx = 1$$

对任意的 $\varepsilon > 0$ ，我们定义

$$\chi_\varepsilon(x) = \frac{1}{\varepsilon^n} \chi\left(\frac{x}{\varepsilon}\right).$$

那么，在分布的意义下，当 $\varepsilon \rightarrow 0$ 时，我们有 $\chi_\varepsilon \xrightarrow{\mathcal{D}'} \delta_0$ 。

Exemple 18. 假设 μ 是 $(\Omega, \mathcal{B}(\Omega))$ 上的测度，其中， $\Omega \subset \mathbb{R}^n$ 是开集， $\mathcal{B}(\Omega)$ 是 Borel 代数（包含所有开集的最小 σ -代数）。如果每个紧集 $K \subset \Omega, \mu(K) < \infty$ ，我们就把这种测度称作是一个 **Radon** 测度。比如说，对任意的正函数（几乎处处） $f \in L^1_{\text{loc}}(\Omega)$ ，对任意的 $B \in \mathcal{B}(\Omega)$ ，我们可以定义

$$\mu_f(B) = \int_{\Omega} \mathbf{1}_B \cdot f(x) dx$$

这就是一个 Radon 测度。任意给定一个 Radon 测度 μ ，我们可以定义一个分布 T_μ ：对于 $\varphi \in \mathcal{D}(\Omega)$ ，我们要求

$$\langle T_\mu, \varphi \rangle = \int_{\Omega} \varphi(x) d\mu(x)$$

我们证明 T_μ 是 Ω 上阶为 0 的分布：对任意的紧集 $K \subset \Omega$ ，对任意的 $\varphi \in C_K^\infty(\Omega)$ ，我们有

$$\begin{aligned} |\langle T_\mu, \varphi \rangle| &= \left| \int_K \varphi(x) d\mu(x) \right| \\ &\leq \mu(K) \|\varphi\|_{L^\infty(K)} \end{aligned}$$

所以，我们在分布的定义中取 $q = 0, C = \mu(K)$ 即可。特别地，我们可以把 $L^1_{\text{loc}}(\Omega)$ 中的元素看作是某个 Radon 测度的密度函数，从而，定义出了同样的分布。

Remarque 13. 利用所谓的 Riesz 表示定理，我们可以证明， Ω 上所有的 0 阶分布都是（由如上方式给出的）Radon 测度。

Proposition 7

给定开集 $\Omega \subset \mathbb{R}^n$, 我们已经定义如下的线性映射 (把局部可积函数视为分布)

$$T : L_{\text{loc}}^1(\Omega) \rightarrow \mathcal{D}'(\Omega), \quad f \mapsto T_f$$

这是单射。

Remarque 14. 根据这个命题, 局部可积的函数可以看做是分布的子集合。在分析中, 我们把 $L_{\text{loc}}^1(\Omega)$ 的元素称作是 Ω 上的“函数”(这个类已经足够大了), 由于某些分布不是“函数”, 所以我们也经常把分布称作是“广义函数”。

Lemme 2

假设 $\psi(x) \in C^\infty(\mathbb{R})$ 并且 $\psi(0) = 0$, 那么, $\frac{\psi(x)}{x}$ 也是光滑函数。

所以 $\frac{\varphi(x)-\varphi(-x)}{x}$ 是光滑函数 (自然是局部可积的)。从而, 当 $n \rightarrow \infty$ 时, 上述积分的极限存在:

$$\lim_{n \rightarrow \infty} \int_{-\frac{1}{n}}^{\infty} \frac{\varphi(x) - \varphi(-x)}{x} dx = \int_0^{\infty} \frac{\varphi(x) - \varphi(-x)}{x} dx$$

我们现在定义

$$\left\langle \text{vp} \frac{1}{x}, \varphi \right\rangle = \int_0^{\infty} \frac{\varphi(x) - \varphi(-x)}{x} dx$$

为了证明这是分布, 我们利用中值定理: 对任意的紧集 $K = [-M, M] \subset \mathbb{R}$, 对任意的支集在 K 上的光滑函数 φ , 我们有

$$|\varphi(x) - \varphi(-x)| = |2x\varphi'(\xi)| \leq 2\|\varphi'\|_{L^\infty(K)} x$$

所以,

$$\left| \left\langle \text{vp} \frac{1}{x}, \varphi \right\rangle \right| \leq \int_0^M 2\|\varphi'\|_{L^\infty(K)} dx = 2M\|\varphi'\|_{L^\infty(K)}$$

所以 $\text{vp} \frac{1}{x}$ 是一个阶不超过 1 的分布。在作业中, 我们将证明 $\text{vp} \frac{1}{x}$ 的阶恰好是 1。

Note. 另外, vp 是法语 *valeur principale* 的缩略, 英文文献经常用 $\text{pv}_{\frac{1}{x}}$, 因为他们把主值写为 *principal value*。

6.23 分布的操作：限制，求导数，与微分同胚复合，链式法则。Stokes 公式的分布形式。

在微积分的学习中，我们可以对一个函数做特定的操作，比如可以把一个函数限制到比较小的定义域上、可以对一个函数求导数、两个函数可以相乘等等。我们现在讨论如何对分布做一些特定的操作。

Note. 我们通常用 φ 表示试验函数。

6.23.1 分布的限制

假设 $\Omega' \subset \Omega$ 是开子集，那么，我们可以定义限制映射

$$\text{Res} : \mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega'), u \mapsto \text{Res}(u).$$

其中，对于每个 $\varphi \in \mathcal{D}(\Omega')$ ，它自然可以看作是 $\mathcal{D}(\Omega)$ 中的元素，从而，我们可以要求

$$\langle \text{Res}(u), \varphi \rangle = \langle u, \varphi \rangle.$$

6.23.2 求偏导数

通过分部积分，我们有

$$\begin{aligned} \langle u', \varphi \rangle &= \int_{\mathbb{R}} u'(x) \varphi(x) dx = - \int_{\mathbb{R}} u(x) \varphi'(x) dx \\ &= - \langle u, \varphi' \rangle. \end{aligned}$$

这个计算启发我们对于 $u \in \mathcal{D}'(\mathbb{R})$ ，我们可以用下面的等式来定义它的导数：

$$\langle u', \varphi \rangle := - \langle u, \varphi' \rangle.$$

Définition 24

假设 $\Omega \subset \mathbb{R}^n$ 是有界开集，给定 $u \in \mathcal{D}'(\Omega)$ ，对于任意的多重指标 α ，我们定义

$$\langle \partial^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle.$$

显然根据分布的定义，Définition 23，第二条，可知分布的导数依然是分布。

Exemple 19. 我们计算 \mathbb{R} 上的 Dirac 函数 δ_a 的导数, 其中 $a \in \mathbb{R}$ 。任给 $\varphi \in \mathcal{D}(\mathbb{R})$, 我们有

$$\langle \delta'_a, \varphi \rangle = -\langle \delta_a, \varphi' \rangle = -\varphi'(a).$$

6.23.3 $C^\infty(\Omega)$ -模结构

对光滑函数 $f \in C^\infty(\Omega)$ 和分布 $u \in \mathcal{D}'(\Omega)$, 我们可以定义它们的乘积 $f \cdot u$:

$$\langle f \cdot u, \varphi \rangle := \langle u, f\varphi \rangle.$$

下一步验证 $f \cdot u \in \mathcal{D}'(\Omega)$, 进而分布具有 $C^\infty(\Omega)$ -模结构.

$$|\langle f \cdot u, \varphi \rangle| \leq C \sup_{|\alpha| \leq p} \|\partial^\alpha(f \cdot \varphi)\|_{L^\infty(K)}$$

根据 Leibniz 法则, 我们有

$$\partial^\alpha(f \cdot \varphi) = \sum_{\beta+\gamma=\alpha} \partial^\beta f \cdot \partial^\gamma \varphi$$

所以,

$$\begin{aligned} |\langle f \cdot u, \varphi \rangle| &\leq C \sup_{|\alpha| \leq p} \sum_{\beta+\gamma=\alpha} \|\partial^\beta f\|_{L^\infty(K)} \|\partial^\gamma \varphi\|_{L^\infty(K)} \\ &\leq C \underbrace{\sum_{|\beta| \leq p} \|\partial^\beta f\|_{L^\infty(K)}}_{\text{新的常数 } C'} \times \sup_{|\gamma| \leq p} \|\partial^\gamma \varphi\|_{L^\infty(K)} \end{aligned}$$

这表明 $f \cdot u \in \mathcal{D}'(\Omega)$ 。

6.23.4 分布的平移和变量替换

对于 $x_0 \in \mathbb{R}^n$, 我们有如下的平移变换:

$$\tau_{x_0} : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad x \mapsto x + x_0.$$

对于局部可积的函数 $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, 我们可以定义 (这是一种特殊的变量替换):

$$(\tau_{x_0} f)(x) := (f \circ \tau_{x_0})(x) = f(x + x_0)$$

对一般的分布 $u \in \mathcal{D}'(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$, 我们定义

$$\langle \tau_{x_0} u, \varphi \rangle := \langle u, \varphi(x - x_0) \rangle$$

容易验证 $\tau_{x_0} u \in \mathcal{D}'(x_0)$. (承自上一节中分布的 $C^\infty(\Omega)$ -模结构)

给定 \mathbb{R}^n 的两个开集 Ω_1 和 Ω_2 , 我们假定

$$\Phi : \Omega_1 \rightarrow \Omega_2$$

是微分同胚。对任意一个 Ω_2 上的局部可积的函数 f 和 Ω_1 上的试验函数 φ , 根据换元积分公式, 我们有

$$\begin{aligned}\langle \Phi^* f, \varphi(x) \rangle &= \int_{\Omega_1} f(\Phi(x)) \varphi(x) dx = \int_{\Omega_2} f(y) \varphi(\Phi^{-1}(y)) |\mathbf{J}_{\Phi^{-1}}(y)| dy \\ &= \int_{\Omega_2} f(y) \frac{\varphi(\Phi^{-1}(y))}{|\mathbf{J}_{\Phi}(\Phi^{-1}(y))|} dy \\ &= \left\langle f, \frac{\varphi(\Phi^{-1}(y))}{|\mathbf{J}_{\Phi}(\Phi^{-1}(y))|} \right\rangle.\end{aligned}$$

根据这个计算, 我们定义:

$$\Phi^* : \mathcal{D}'(\Omega_2) \rightarrow \mathcal{D}'(\Omega_1), \quad u \mapsto \Phi^* u.$$

其中, 对于 $u \in \mathcal{D}'(\Omega_2)$ 和 $\varphi \in \mathcal{D}(\Omega_1)$, 对于我们定义 $\Phi^* u$ 如下:

$$\langle \Phi^* u, \varphi(x) \rangle := \left\langle u, \varphi(\Phi^{-1}(y)) |\mathbf{J}_{\Phi^{-1}}(y)| \right\rangle = \left\langle u, \frac{\varphi(\Phi^{-1}(y))}{|\mathbf{J}_{\Phi}(\Phi^{-1}(y))|} \right\rangle.$$

简单验证可知: $\Phi^* u \in \mathcal{D}'(\Omega)$.

Remarque 15. 注记 (记号). 因为在光滑函数情况下, $\Phi^* u$ 就是函数的复合, 我们还把上面的拉回映射写成

$$u \circ \Phi = \Phi^* u.$$

给定微分同胚 $\Phi : \Omega_1 \rightarrow \Omega_2$, 它把 Ω_2 上的 Dirac 函数拉回, 得到 Ω_1 上的 Dirac 函数, 这给出了 Jacobi 行列式的一个精确的解释: 这是一点处体积的变化。

链式法则

给定微分同胚 Φ , 如果 u 是 C^1 的函数, 我们可以对求导数运算运用链式法则:

$$\partial_j(u \circ \Phi) = \sum_{k=1}^n \partial_j \Phi_k(x_1, \dots, x_n) \cdot (\partial_k u \circ \Phi)(x_1, \dots, x_n).$$

对于一般的分布 u , 我们实际上 (后来) 可以先用光滑函数逼近这个分布, 然后上面的链式法则在极限的情况下仍然成立。

我们现在给出一个直接的证明：设 $\Psi = \Phi^{-1}$ 是 Φ 的逆映射。按照分布与一个微分同胚的复合的定义，我们有

$$\begin{aligned} \left\langle \sum_{k=1}^n \frac{\partial \Phi_k}{\partial x_j} \cdot \frac{\partial u}{\partial y_k} \circ \Phi, \varphi \right\rangle &= \sum_{k=1}^n \left\langle \frac{\partial u}{\partial y_k} \circ \Phi, \frac{\partial \Phi_k}{\partial x_j} \cdot \varphi \right\rangle \\ &= \sum_{k=1}^n \left\langle \frac{\partial u}{\partial y_k}, \left(\frac{\partial \Phi_k}{\partial x_j} \circ \Psi \right) (\varphi \circ \Psi) \Big| J_\Psi \right\rangle \\ &= - \sum_{k=1}^n \left\langle u, \frac{\partial}{\partial y_k} \left[\left(\frac{\partial \Phi_k}{\partial x_j} \circ \Psi \right) (\varphi \circ \Psi) |J_\Psi| \right] \right\rangle \end{aligned}$$

另外，对任意 $g \in C_0^\infty(\Omega)$ ，我们有

$$\begin{aligned} 0 &= \int_{\Omega_1} \frac{\partial}{\partial x_j} (g \circ \Phi) dx = \sum_{k=1}^n \int_{\Omega_1} \frac{\partial \Phi_k}{\partial x_j} \cdot \frac{\partial g}{\partial y_k} \circ \Phi \\ &= \sum_{k=1}^n \int_{\Omega_2} \frac{\partial \Phi_k}{\partial x_j} \circ \Psi \cdot \frac{\partial g}{\partial y_k} \cdot |\mathbf{J}_\Psi(y)| dy \\ &= - \sum_{k=1}^n \int_{\Omega_2} g \frac{\partial}{\partial y_k} \left(\frac{\partial \Phi_k}{\partial x_j} \circ \Psi \cdot |\mathbf{J}_\Psi(y)| \right) dy \end{aligned}$$

根据 $L^1_{\text{loc}}(\Omega)$ 到 $\mathcal{D}'(\Omega)$ 嵌入的单射性，上面的等式等价于说

$$\sum_{k=1}^n \frac{\partial}{\partial y_k} \left(\frac{\partial \Phi_k}{\partial x_j} \circ \Psi \cdot |\mathbf{J}_\Psi(y)| \right) = 0$$

从而

$$\begin{aligned} \left\langle \sum_{k=1}^n \frac{\partial \Phi_k}{\partial x_j} \cdot \frac{\partial u}{\partial y_k} \circ \Phi, \varphi \right\rangle &= - \sum_{k=1}^n \left\langle u, \frac{\partial}{\partial y_k} (\varphi \circ \Psi) \left[\left(\frac{\partial \Phi_k}{\partial x_j} \circ \Psi \right) |\mathbf{J}_\Psi| \right] \right\rangle \\ &= \left\langle u, \frac{\partial \varphi}{\partial x_j} \circ \Psi \Big| \mathbf{J}_\Psi \right\rangle \\ &= - \left\langle u \circ \Phi, \frac{\partial \varphi}{\partial x_j} \right\rangle \end{aligned}$$

这就证明如下关于分布的链式法则：

$$\partial_j (\Phi^* u) = \sum_{k=1}^n \partial_j \Phi_k \cdot \Phi^* ((\partial_k u)).$$

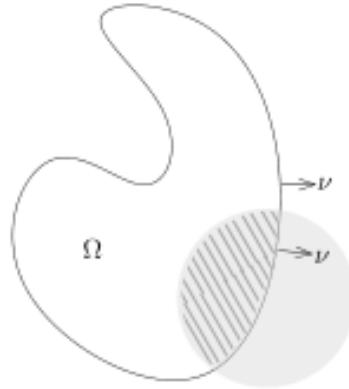
6.23.5 关于分布的 Stokes 公式

Théorème 29

(Stokes 公式) 假设 Ω 是一个有界带边光滑区域， $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ 为 $\partial\Omega$ 的单位外法向量， $d\sigma$ 为 $\partial\Omega$ 上的曲面测度。对任意的 $\varphi \in C^1(\mathbb{R}^n, \mathbb{C})$

, 我们有

$$\int_{\Omega} \frac{\partial \varphi}{\partial x_i}(x) dx = \int_{\partial\Omega} \varphi(x) \nu_i(x) d\sigma$$



给定上面 Stokes 公式中所述的 Ω , 它的边界 $\partial\Omega$ 的曲面测度 $d\sigma$ 在如下的意义下定义了 \mathbb{R}^n 上一个分布:

$$d\sigma : \mathcal{D}(\mathbb{R}^n) \rightarrow \mathbb{C}, \quad \varphi \mapsto \int_{\partial\Omega} \varphi(x) d\sigma(x)$$

这是一个 0 阶的分布, 我们把证明的细节留给不放心的同学来验证。类似地, 对每一个 $i \leq n$, 如下的公式也定义了一个分布:

$$\nu_i d\sigma : \mathcal{D}(\mathbb{R}^n) \rightarrow \mathbb{C}, \quad \varphi \mapsto \int_{\partial\Omega} \varphi(x) \nu_i(x) d\sigma(x)$$

我们可以把 Stokes 公式改写成如下的形式:

$$\langle \mathbf{1}_\Omega, \partial_i \varphi \rangle = \int_{\mathbb{R}^n} \mathbf{1}_\Omega(x) \frac{\partial \varphi}{\partial x_i}(x) dx = \int_{\partial\Omega} \varphi(x) \nu_i(x) d\sigma.$$

所以, 用分布的语言来写, 我们有

Théorème 30: Stokes 公式

假设 Ω 是一个有界带边光滑区域, $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ 为 $\partial\Omega$ 的单位外法向量, $d\sigma$ 为 $\partial\Omega$ 上的曲面测度, 作为分布, 我们有等式

$$\partial_i \mathbf{1}_\Omega \stackrel{\mathcal{D}'(\mathbb{R}^n)}{=} -\nu_i d\sigma.$$

如果用向量值分布的语言（可以望文生义地定义）来写，我们有

$$\nabla \mathbf{1}_\Omega \stackrel{\mathcal{D}'(\mathbb{R}^n)}{=} -\nu d\sigma.$$

我们现在回到 1 维的情形，此时的 Stokes 公式就是 Newton-Leibniz 公式。

Lemme 3

对于 $f(x) \in L^1((a, b))$ ，我们定义其原函数为

$$F(x) = \int_a^x f(y) dy.$$

那么， $F(x)$ 是连续函数。在分布的意义下，我们有

$$F(x)' \stackrel{\mathcal{D}'}{=} f(x).$$

证明只需照章办事。（利用 Fubini 定理）

6.24 分布的应用

这里跳过第 60 节

6.25 分布的局部刻画

6.25.1 单位分解定理

Théorème 31: 单位分解

任意给定 \mathbb{R}^n 中的紧集 K ，假设 K 被有限个开集 $\{U_1, \dots, U_N\}$ 所覆盖。

那么，对每个 $j \leq N$ ，存在光滑函数 $\chi_j \in C_0^\infty(U_j)$ ，满足

- (1) 对任意 $x \in \mathbb{R}^n$ ，有 $0 \leq \chi_j(x) \leq 1$ ；
- (2) 存在包含 K 的开集 V ，对任意 $x \in V$ ，我们有

$$\chi_1(x) + \dots + \chi_N(x) = 1.$$

Note. 单位分解的证明并没有任何启发性的意义，我们只要能够运用该结论即可。

我们现在证明一个分布 u 在（所有）小的开集上的限制决定了 u ，这表明分布是可以局部定义的：

Théorème 32

给定开集 $\Omega \subset \mathbb{R}^n$ 。任意给定一族开集 $\{\Omega_i \mid i \in I\}$ ，其中对任意的 $i \in I$, $\Omega_i \subset \Omega$ 。我们假定

$$\Omega = \bigcup_{i \in I} \Omega_i$$

对每个 $i \in I$ ，我们在 Ω_i 上指定一个分布 $u_i \in \mathcal{D}'(\Omega_i)$ 。如果这一族分布 $\{u_i\}_{i \in I}$ 满足如下的相容关系：

$$u_i|_{\Omega_i \cap \Omega_j} = u_j|_{\Omega_i \cap \Omega_j}, \quad \text{对任意的 } i, j \in I,$$

那么，存在唯一的 $u \in \mathcal{D}'(\Omega)$ ，使得对任意的 $i \in I$ ，我们都有

$$u|_{\Omega_i} = u_i$$

进一步， $u \in L^1_{\text{loc}}(\Omega)$ 当且仅当对每个 $i \in I$, $u|_{\Omega_i} \in L^1_{\text{loc}}(\Omega_i)$; $u \in C^k(\Omega)$ 当且仅当对每个 $i \in I$, $u|_{\Omega_i} \in C^k(\Omega_i)$ ，其中 $k \in \mathbb{Z}_{\geq 0}$ 。

Note. 这个定理表明，在 \mathbb{R}^n 的开集上所定义的分布可以构成 \mathbb{R}^n 上的一个层。

层

在代数几何中，仿射空间 \mathbb{R}^n 上的层通常指的是在某个拓扑空间（比如 \mathbb{R}^n 赋予通常的欧几里得拓扑）上的一个结构，它将每个开集关联到一个代数对象（比如一个环、一个模），并且这些关联满足某些相容性条件。更具体地说：

一个层 \mathcal{F} 由以下部分构成：

- (1) 对于每个开集 $U \subseteq \mathbb{R}^n$ ，有一个代数对象 $\mathcal{F}(U)$ ，例如一个交换环。我们称 $\mathcal{F}(U)$ 为 \mathcal{F} 在 U 上的截面。
- (2) 对于每一对开集 $V \subseteq U \subseteq \mathbb{R}^n$ ，有一个限制映射 $\rho_{V,U} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ ，它是一个代数同态。这个映射描述了如何将 U 上的截面限制到 V 上。

这些数据需要满足以下公理：

- (1) **恒等性：**对于每个开集 $U \subseteq \mathbb{R}^n$, $\rho_{U,U}$ 是 $\mathcal{F}(U)$ 上的恒等映射。

- (2) **传递性:** 如果 $W \subseteq V \subseteq U$ 是 \mathbb{R}^n 中的开集, 那么 $\rho_{W,V} \circ \rho_{V,U} = \rho_{W,U}$ 。换句话说, 从 U 限制到 W , 可以直接进行, 也可以先限制到 V 再限制到 W , 结果是一样的。
- (3) **粘合性:** 设 $\{U_i\}_{i \in I}$ 是开集 $U \subseteq \mathbb{R}^n$ 的一个开覆盖。如果 $s, t \in \mathcal{F}(U)$ 是两个截面, 并且对于每个 $i \in I$, 都有 $\rho_{U_i,U}(s) = \rho_{U_i,U}(t)$, 那么 $s = t$ 。也就是说, 如果两个截面在每个开覆盖的子集上都相等, 那么它们在整个集合上相等。
- (4) **整体截面的存在性:** 设 $\{U_i\}_{i \in I}$ 是开集 $U \subseteq \mathbb{R}^n$ 的一个开覆盖。假设我们有一族截面 $s_i \in \mathcal{F}(U_i)$, 使得对于每一对 $i, j \in I$, 都有 $\rho_{U_i \cap U_j, U_i}(s_i) = \rho_{U_i \cap U_j, U_j}(s_j)$ 。那么存在唯一的截面 $s \in \mathcal{F}(U)$, 使得对于每个 $i \in I$, 都有 $\rho_{U_i,U}(s) = s_i$ 。也就是说, 如果一族截面在开覆盖的交集上相容, 那么它们可以粘合成一个整体截面。

最常见的例子是连续函数层, 其中 $\mathcal{F}(U)$ 是 U 上所有连续实值函数的集合, 限制映射就是通常的函数限制。另一个重要的例子是全纯函数层, 其中 $\mathcal{F}(U)$ 是 U 上所有全纯函数的集合, 限制映射也是通常的函数限制。

6.25.2 分布的支集

给定开集上的分布 $u \in \mathcal{D}'(\Omega)$, 我们来定义它的支集。

假设 $\Omega' \subset \Omega$ 是开子集, 如果 $u|_{\Omega'} = 0$, 我们就说 u 在 Ω' 上为零, 也就是说, 对于每个 $\varphi \in \mathcal{D}(\Omega')$, 我们有

$$\langle u, \varphi \rangle = 0.$$

我们现在来说明, 存在 Ω 中使得 u 在其上为零的最大开集。为此, 我们定义

$$I = \{\Omega' \mid \Omega' \subset \Omega \text{ 为开集}, u|_{\Omega'} = 0\}.$$

令

$$U = \bigcup_{\Omega' \in I} \Omega'$$

按照定义, U 为开集。我们要证明 $U \in I$, 为此, 只要证明对任意的 $\varphi \in C_0^\infty(U)$, 我们都有

$$\langle u, \varphi \rangle = 0$$

即可。实际上，令 $K = \text{supp}(\varphi)$ ，那么存在有限个 $\Omega_1, \dots, \Omega_N \in I$ 覆盖 K 。我们取与这个覆盖相应的单位分解 χ_1, \dots, χ_N 。从而，

$$\langle u, \varphi \rangle = \sum_{j=1}^N \langle u, \underset{\substack{\text{支集在 } \Omega_i \text{ 中}}}{\widehat{\chi_i \varphi}} \rangle = 0.$$

最后一步，我们用到了 $u|_{\Omega_i} = 0$ 。很明显， U 是这种开集中最大的。

Définition 25: 分布的(紧)支集

我们把 $\Omega - U$ 称作是 u 的支集并仍然用符号 $\text{supp}(u)$ 表示。如果 $\text{supp}(u)$ 是紧集，我们就说 u 是有紧支集的分布。我们用 $\mathcal{E}'(\mathbb{R}^n)$ 来表示 \mathbb{R}^n 上有紧支集的分布的全体。

6.26 场论

6.26.1 场的基本概念

Définition 26: 等值面

设 f 是定义在空间区域 Ω 上的一个数量场. 对其值域 $f(\Omega)$ 中的任意一个实数 c , 称点集

$$f^{-1}(c) = \{M \in \Omega : f(M) = c\}$$

为 f 的一个等值面.

定义 22.1.2 设 \mathbf{F} 是定义在空间区域 Ω 上的一个向量场, C 是 Ω 中的一条有向曲线. 如果对 C 上任意一点 M , 向量 $\mathbf{F}(M)$ 都平行于曲线 C 在点 P 的单位切向量 τ , 且方向一致, 则称 C 为向量场 \mathbf{F} 的一条积分曲线.

定义 22.1.3 设 Ω 是一个空间区域, M_0 是 Ω 中一点, \mathbf{v} 是一个给定的单位向量.

(1) 对于定义在 Ω 上的数量场 f , 它在点 M_0 沿 \mathbf{v} 方向的方向导数是指极限

$$\frac{\partial f}{\partial \mathbf{v}}(M_0) = \lim_{M \rightarrow M_0} \frac{f(M) - f(M_0)}{\text{dist}(M_0, M)},$$

其中 M 是取自从 M_0 发出以 \mathbf{v} 为方向向量的射线上的点.

(2) 对于定义在 Ω 上的向量场 \mathbf{F} , 它在点 M_0 沿 \mathbf{v} 方向的方向导数是指极限

$$\frac{\partial \mathbf{F}}{\partial \mathbf{v}}(M_0) = \lim_{M \rightarrow M_0} \frac{\mathbf{F}(M) - \mathbf{F}(M_0)}{\text{dist}(M_0, M)},$$

其中 M 是取自从 M_0 发出以 \mathbf{v} 为方向向量的射线上的点.

引理 22.1.1 设 $Oxyz$ 和 $O'x'y'z'$ 是两个笛卡儿直角坐标系, 它们的三个坐标轴上的单位向量分别是 $\mathbf{i}, \mathbf{j}, \mathbf{k}$ 和 $\mathbf{i}', \mathbf{j}', \mathbf{k}'$. 又设空间区域 Ω 上的数量场 f 在这两个坐标系下的坐标表示分别是数量函数 $f(x, y, z)$ 和 $f'(x', y', z')$. 则成立等式

$$f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} = f'_{x'}(x', y', z')\mathbf{i}' + f'_{y'}(x', y', z')\mathbf{j}' + f'_{z'}(x', y', z')\mathbf{k}', \quad (22.1.1)$$

其中 (x, y, z) 和 (x', y', z') 表示 Ω 中同一个动点 M 在这两个不同坐标系下的坐标.

证明. 证明考虑使用一大堆正交变换. □

Note. 根据这个引理, 如下定义是合理的

定义 22.1.4 设 f 是空间区域 Ω 上的一个可微数量场, M_0 是 Ω 中一点. 选取一个空间直角坐标系 $Oxyz$, 设其三个坐标轴上的单位向量依次为 i, j, k . 又设数量场 f 在这个坐标系下的坐标表示为数量函数 $f(x, y, z)$, 点 M_0 的坐标是 (x_0, y_0, z_0) . 则称向量

$$f_x(x_0, y_0, z_0)i + f_y(x_0, y_0, z_0)j + f_z(x_0, y_0, z_0)k$$

为数量场 f 在点 M_0 的梯度, 记作 $\nabla f(M_0)$ 或 $\text{grad}f(M_0)$.

数量场的梯度有以下性质

定理 22.1.1 设 f 是区域 Ω 上的一个可微数量场. 则有下列结论:

(1) 对任意 $M_0 \in \Omega$ 和任意单位向量 v , f 在点 M_0 沿 v 方向的方向导数有表达式

$$\frac{\partial f}{\partial v}(M_0) = v \cdot \nabla f(M_0).$$

(2) f 在点 M_0 的梯度 $\nabla f(M_0)$ 的方向是 f 的值在点 M_0 增加最快的方向, 模等于 f 在点 M_0 的所有方向导数的最大值.

(3) f 在点 M_0 的梯度 $\nabla f(M_0)$ 正交于 f 过点 M_0 的等值面, 方向指向 f 的值增大的一侧.

定义 22.1.5 (1) 设 f 是区域 Ω 上的可微数量场. 则称向量场 $M \mapsto \nabla f(M)$, $\forall M \in \Omega$ 为 f 的梯度场, 用符号 ∇f 表示.

(2) 对区域 Ω 上的一个给定的向量场 F , 如果存在 Ω 上的可微数量场 f 使得 $F = \nabla f$, 则称该向量场 F 为 Ω 上的梯度场, 并称数量场 f 为 F 的势函数.

梯度运算的基本公式如下:

- (1) $\nabla(c) = 0$ (c 为常数);
- (2) $\nabla(cf) = c\nabla f$ (c 为常数);
- (3) $\nabla(f \pm g) = \nabla f \pm \nabla g$;
- (4) $\nabla(fg) = g\nabla f + f\nabla g$;
- (5) $\nabla\left(\frac{f}{g}\right) = \frac{1}{g^2}(g\nabla f - f\nabla g)$;
- (6) $\nabla(f \circ g) = (f' \circ g)\nabla g$;
- (7) 更一般地, 有

$$\nabla f(g_1, g_2, \dots, g_n) = \sum_{i=1}^n f_i(g_1, g_2, \dots, g_n) \nabla g_i$$

其中 f_i 表示函数 f 关于第 i 个变元的偏导数.

6.26.2 向量场的通量和散度

对于封闭区面，本章总规定其单位法向量指向其外侧，即外侧为正侧，内侧为负侧。

定义 22.2.1 设 \mathbf{F} 是区域 Ω 上的一个向量场， S 是区域 Ω 中的一张分块光滑的有向曲面， S 的单位法向量为 \mathbf{n} 。则称 \mathbf{F} 沿 S 的第二型曲面积分为

$$\iint_S \mathbf{F}(M) \cdot d\mathbf{S} = \iint_S \mathbf{F}(M) \cdot \mathbf{n}(M) d\sigma$$

为向量场 \mathbf{F} 穿过曲面 S 的通量。

以流体的速度场为例子，当流体速度场穿过 S 的通量大于零时，意味着流体从 S 所包围区域之内穿过 S 流向该区域之外的流量大于从其外部流向区域之内的流量，表明 S 所包围区域内必然有供给流体的源泉。反之亦然。

因此，向量场 \mathbf{F} 穿过封闭曲面 S 的通量反映了 S 所包围区域中产生向量场 \mathbf{F} 的物质的正源与负源的某种代数和。

但我们不能了解这种源物质在局部的分布与强弱，因此我们通过取极限来达到这样的目的。给出散度的定义。

定义 22.2.2 设 \mathbf{F} 是在开区域 Ω 上定义的一个向量场， M_0 是 Ω 中任意一点。任取一包围点 M_0 的分块光滑的封闭曲面 S ，使其所包围区域 V 完全包含于 Ω 。如前令 $|V|$ 表示区域 V 的体积。如果极限

$$\lim_{\substack{\text{diam}(V) \rightarrow 0 \\ M_0 \in V}} \frac{\iint_S \mathbf{F}(M) \cdot d\mathbf{S}}{|V|}$$

存在，则称极限值为 \mathbf{F} 在点 M_0 的散度，记作 $\operatorname{div}\mathbf{F}(M_0)$ 或 $\nabla \cdot \mathbf{F}(M_0)$ 。

当向量场 \mathbf{F} 在 M_0 的散度 $\operatorname{div}\mathbf{F}(M_0) > 0$ 时，在 M_0 必有产生向量场 \mathbf{F} 的物质的正源，并且 $\operatorname{div}\mathbf{F}(M_0)$ 越大这种正源的强度越大，反之亦然。

散度不依赖于坐标系的选取。

定理 22.2.1 设 \mathbf{F} 是定义在开区域 Ω 上的一个连续可微的向量场，在取定了一个直角坐标系 $Oxyz$ 之后，它的坐标表示式为

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}, \quad \forall (x, y, z) \in \Omega.$$

则对任意一点 $M(x, y, z) \in \Omega$ ，有

$$\operatorname{div}\mathbf{F}(M) = \frac{\partial P(x, y, z)}{\partial x} + \frac{\partial Q(x, y, z)}{\partial y} + \frac{\partial R(x, y, z)}{\partial z}. \quad (22.2.2)$$

定理 22.2.2(高斯公式) 设 \mathbf{F} 是定义在具有分块光滑边界的有界闭区域 Ω 上的一个连续可微的向量场, 则成立

$$\iint_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\Omega} \operatorname{div} \mathbf{F} dV.$$

符号 dV 表示体积微元, 即在直角坐标系下 $dV = dx dy dz$.

因此, 高斯公式也经常叫做散度定理.

散度运算具有以下基本性质:

- (1) $\operatorname{div}(c\mathbf{F}) = c \operatorname{div} \mathbf{F}$ (c 为常数);
- (2) $\operatorname{div}(\mathbf{F} \pm \mathbf{G}) = \operatorname{div} \mathbf{F} \pm \operatorname{div} \mathbf{G}$;
- (3) $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + \nabla f \cdot \mathbf{F}$.

定理 22.2.3 设 \mathbf{F} 是开区域 Ω 上的连续可微向量场. 则下面三个条件互相等价:

- (1) \mathbf{F} 是 Ω 上的无源场;
- (2) 对 Ω 中任意一个零伦的分块光滑封闭曲面 S , \mathbf{F} 穿过 S 的通量为零;
- (3) 对 Ω 中任意两个同伦的分块光滑封闭曲面 S 和 S' , \mathbf{F} 穿过 S 和 S' 的通量相等.

6.26.3 向量场的环量和旋度

定义 22.3.1 设 \mathbf{F} 是区域 Ω 上的一个向量场, C 是 Ω 中的一条分段光滑的有向封闭曲线, 其单位切向量为 τ . 则称 \mathbf{F} 沿 C 的第二型曲线积分

$$\int_C \mathbf{F}(M) \cdot d\mathbf{r} = \int_C \mathbf{F}(M) \cdot \tau(M) ds$$

为 \mathbf{F} 沿曲线 C 的环量.

由公式可以看出, 向量场 \mathbf{F} 沿封闭曲线 C 的环量反映了 \mathbf{F} 沿 C 的旋转情况.

对于流体而言, 速度场沿封闭曲线的环量是否为零反映了曲线是否包围了漩涡的涡管; 当环量非零时, 其正负号反映了曲线的方向是否与它所包围漩涡的旋转方向一致, 其大小则反映了该漩涡的旋转强度.

与散度概念类似, 为了了解向量场涡旋的局部分布情况, 我们给出方向旋量的概念

定义 22.3.2 设 Ω 是空间中的一个开区域, \mathbf{F} 是定义在 Ω 上的一个向量场. 对 Ω 中任意一点 M_0 和任意一个单位向量 \mathbf{n} , 在通过点 M_0 并以 \mathbf{n} 为法向量的平面上任取一条围绕点 M_0 右旋的分段光滑的封闭曲线 C , 设它所包围平面区域为 ΔD , 面积为 $|\Delta D|$ (图 22-3-1). 如果极限

$$\lim_{\substack{\text{diam}(\Delta D) \rightarrow 0 \\ M_0 \in \Delta D}} \frac{\int_C \mathbf{F}(M) \cdot d\mathbf{r}}{|\Delta D|}$$

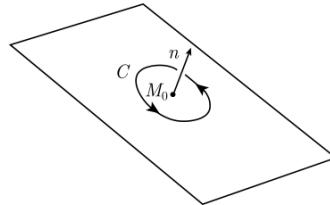


图 22-3-1 方向旋量

存在, 则称此极限值为 \mathbf{F} 在点 M_0 绕 \mathbf{n} 方向的向旋量, 记作 $\text{rot}_{\mathbf{n}} \mathbf{F}(M_0)$.

方向旋量也不依赖于坐标系的选取.

定理 22.3.1 设 \mathbf{F} 是定义在开区域 Ω 上的一个连续可微的向量场, 在取定了一个直角坐标系 $Oxyz$ 之后, 它的坐标表示式为

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}, \quad \forall (x, y, z) \in \Omega.$$

则对任意一点 $M_0 \in \Omega$ 和任意单位向量 $\mathbf{n} = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}$, 有

$$\text{rot}_{\mathbf{n}} \mathbf{F}(M_0) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \cos \alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \cos \beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cos \gamma. \quad (22.3.1)$$

等式右端的函数都在点 M_0 取值.

定义

$$\text{rot}_{\mathbf{n}} \mathbf{F}(M_0) = \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \right] \Big|_{M_0} \quad (6.4)$$

则有

$$\text{rot}_{\mathbf{n}} \mathbf{F}(M_0) = \text{rot} \mathbf{F}(M_0) \cdot \mathbf{n}$$

Définition 27: 旋度

由 eq. (6.4) 定义的向量 $\text{rot} \mathbf{F}(M_0)$ 为向量场 \mathbf{F} 在点 M_0 的**旋度**.

旋度也记作 $\nabla \times \mathbf{F}$, $\text{curl} \mathbf{F}$.

定理 22.3.3(斯托克斯公式) 设 \mathbf{F} 是定义在开区域 Ω 上的一个连续可微的向量场, S 是 Ω 中一张分块光滑的曲面, 其边界 ∂S 是 Ω 内一条分段光滑的封闭曲线. 则成立

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{rot} \mathbf{F} \cdot d\mathbf{S}.$$

∂S 的正向为按 S 的正法向右旋的方向.

因此, 也可把斯托克斯公式叫做**旋度定理**.

旋度运算具有以下基本性质:

- (1) $\text{rot}(c\mathbf{F}) = c \text{rot} \mathbf{F}$ (c 为常数);
- (2) $\text{rot}(\mathbf{F} \pm \mathbf{G}) = \text{rot} \mathbf{F} \pm \text{rot} \mathbf{G}$;
- (3) $\text{rot}(f\mathbf{F}) = f \text{rot} \mathbf{F} + \nabla f \times \mathbf{F}$.

引力场和静电场都是无旋场.

定理 22.3.4 设 \mathbf{F} 是开区域 Ω 上的连续可微向量场. 则下面三个条件互相等价:

- (1) \mathbf{F} 是 Ω 上的无旋场;
- (2) 对 Ω 中任意一条零伦的分段光滑的简单闭曲线 C , \mathbf{F} 沿 C 的环量为零;

- (3) 对 Ω 中任意两条同伦的分段光滑的简单闭曲线 C 和 C' , \mathbf{F} 沿 C 和 C' 的环量相等.

6.26.4 一些重要定理

梯度、散度和旋度联合的一些运算公式

对于梯度、散度和旋度运算，除了上面各节罗列的一些基本运算公式之外，把它们合在一起考虑，还有下面一些常用的运算公式（假定所出现的数量场与向量场都充分光滑）：

$$(1) \nabla(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times \operatorname{rot} \mathbf{G} + \mathbf{G} \times \operatorname{rot} \mathbf{F} + (\mathbf{F} \cdot \nabla) \mathbf{G} + (\mathbf{G} \cdot \nabla) \mathbf{F}.$$

其中， $(\mathbf{F} \cdot \nabla) \mathbf{G}$ 表示向量场 \mathbf{G} 沿 \mathbf{F} 方向的方向导数与 \mathbf{F} 模的乘积： $(\mathbf{F} \cdot \nabla) \mathbf{G} = |\mathbf{F}| \frac{\partial \mathbf{G}}{\partial \mathbf{F}^\circ}$ ， \mathbf{F}° 表示 \mathbf{F} 的单位化向量（如果 \mathbf{F} 在某点等于零，则规定 $(\mathbf{F} \cdot \nabla) \mathbf{G}$ 在该点也等于零）。

$$(2) \operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{rot} \mathbf{F} + \mathbf{F} \cdot \operatorname{rot} \mathbf{G}.$$

$$(3) \operatorname{rot}(\mathbf{F} \times \mathbf{G}) = (\operatorname{div} \mathbf{G}) \mathbf{F} - (\operatorname{div} \mathbf{F}) \mathbf{G} + (\mathbf{G} \cdot \nabla) \mathbf{F} - (\mathbf{F} \cdot \nabla) \mathbf{G}.$$

$$(4) \operatorname{rot}(\nabla f) = 0.$$

$$(5) \operatorname{div}(\operatorname{rot} \mathbf{F}) = 0.$$

这些公式通过一些简单的计算即可验证，留给读者完成。从最后两个公式得以下定理。

定理 22.4.1 假定所考虑的向量场都一阶连续可微，则有以下结论：

(1) 梯度场是无旋场；

(2) 旋度场是无源场。

定义

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$(6) \operatorname{div}(\nabla f) = \Delta f.$$

$$(7) \operatorname{rot}(\operatorname{rot} \mathbf{F}) = \nabla(\operatorname{div} \mathbf{F}) - \Delta \mathbf{F}.$$

保守场及其等价条件

定义 22.4.1 设 \mathbf{F} 是定义在开区域 Ω 上的一个向量场。如果对 Ω 中任意两点 M_0 和 M ， \mathbf{F} 沿以 M_0 为起点、以 M 为终点的任意一条分段光滑曲线 C 的第二型曲线积分 $\int_C \mathbf{F} \cdot d\mathbf{r}$ 都相等，即这个积分只与起点 M_0 和终点 M 有关而与连接这两点的具体曲线无关，则称 \mathbf{F} 为 Ω 上的保守场。这时积分 $\int_C \mathbf{F} \cdot d\mathbf{r}$ 也用符号 $\int_{M_0}^M \mathbf{F} \cdot d\mathbf{r}$ 表示，并称函数

$$M \mapsto \int_{M_0}^M \mathbf{F} \cdot d\mathbf{r}, \quad \forall M \in \Omega$$

为 \mathbf{F} 的势函数。

定理 22.4.2 设 \mathbf{F} 是开区域 Ω 上的连续向量场. 则下列三个条件互相等价:

- (1) \mathbf{F} 是保守场;
- (2) \mathbf{F} 是梯度场;
- (3) \mathbf{F} 沿 Ω 中任意一条分段光滑的封闭曲线的环量都是零.

亥姆霍兹分解定理

下面推导一个重要定理——亥姆霍兹分解定理. 粗略地说, 这个定理告诉我们, 每个向量场都可分解为一个无源场和一个无旋场的和.

定理 22.4.4(亥姆霍兹分解定理) 设 \mathbf{F} 是定义在全空间上的二阶连续可微向量场, 且存在常数 $p > 3$ 和 $C > 0$ 使成立

$$|\mathbf{F}(x)| + \sum_{i=1}^3 \left| \frac{\partial \mathbf{F}(x)}{\partial x_i} \right| + \sum_{i,j=1}^3 \left| \frac{\partial^2 \mathbf{F}(x)}{\partial x_i \partial x_j} \right| \leq C(1+|x|)^{-p}, \quad \forall x \in \mathbf{R}^3.$$

则存在定义在全空间上的连续可微的无旋场 \mathbf{F}_1 和无源场 \mathbf{F}_2 , 使成立:

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2.$$

结合应用以上定理和延拓定理就得到了以下定理.

定理 22.4.5 设 Ω 是 \mathbf{R}^3 中具有二阶连续可微边界的有界闭区域, \mathbf{F} 是定义在 Ω 上的二阶连续可微向量场. 则存在定义在 Ω 上的连续可微的无旋场 \mathbf{F}_1 和无源场 \mathbf{F}_2 使得

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2.$$

1. 证明以下公式:

- (1) $\nabla(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times \text{rot} \mathbf{G} + \mathbf{G} \times \text{rot} \mathbf{F} + (\mathbf{F} \cdot \nabla) \mathbf{G} + (\mathbf{G} \cdot \nabla) \mathbf{F};$
- (2) $\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{rot} \mathbf{F} + \mathbf{F} \cdot \text{rot} \mathbf{G};$
- (3) $\text{rot}(\mathbf{F} \times \mathbf{G}) = (\text{div} \mathbf{G}) \mathbf{F} - (\text{div} \mathbf{F}) \mathbf{G} + (\mathbf{G} \cdot \nabla) \mathbf{F} - (\mathbf{F} \cdot \nabla) \mathbf{G};$
- (4) $\text{rot}(\text{rot} \mathbf{F}) = \nabla(\text{div} \mathbf{F}) - \Delta \mathbf{F}.$

6.26.5 \mathbf{R}^3 上

对于给定的一组标准正交基 (e_1, e_2, e_3) , 对于标量场 $f = f(x_1, x_2, x_3)$, 定义梯度

$$\nabla f = \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \frac{\partial f}{\partial x_3} e_3$$

对于 $\mathbf{F} = F_1 e_1 + F_2 e_2 + F_3 e_3$, 定义散度

$$\text{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$$

定义旋度

$$\operatorname{rot} \mathbf{F} = \nabla \times \mathbf{F} = \operatorname{curl} \mathbf{F} = \begin{bmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

对于 $\mathbf{a} = \sum_{i=1}^3 a_i e_i, \mathbf{b} = \sum_{i=1}^3 b_i e_i$, 定义叉乘

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

6.27 常微分方程周期性

例题 12.11 设 $x : \mathbb{R} \rightarrow (-1, +\infty)$ 是下述常微分方程的解

$$x'' + \frac{1}{4(x+1)} - \frac{1}{4(x+1)^3} = 0. \quad (12.96)$$

证明 ϕ 是周期函数.

证明 首先由(12.96)知存在 $C \in \mathbb{R}$ 使得

$$(x')^2 + \frac{1}{2} \ln(1+x) + \frac{1}{4(x+1)^2} = C. \quad (12.97)$$

于是由(12.97)知

$$\frac{1}{4} \leq \frac{1}{2} \ln(1+x) + \frac{1}{4(x+1)^2} \leq C. \quad (12.98)$$

若对任何 $t_0 > 0$, 都有 $x'(t) \neq 0, \forall t > t_0$. 则 $x(t)$ 在 $(t_0, +\infty)$ 单调. 由(12.98)知 x 是有界函数. 于是

$$\lim_{t \rightarrow +\infty} x(t) = s \geq -1.$$

由结论11.1知存在

$$\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} h_n = +\infty,$$

使得

$$\lim_{n \rightarrow \infty} x'(t_n) = \lim_{n \rightarrow \infty} x''(h_n) = 0.$$

于是由(12.96)知

$$0 = \lim_{n \rightarrow \infty} \left(x''(h_n) + \frac{1}{4(1+x(h_n))} - \frac{1}{4(x(h_n)+1)^3} \right) = \frac{1}{4(s+1)} - \frac{1}{4(s+1)^3},$$

即 $s = 0$.

再由(12.97)知

$$C = \lim_{n \rightarrow \infty} \left((x'(t_n))^2 + \frac{1}{2} \ln(1+x(t_n)) + \frac{1}{4(x(t_n)+1)^2} \right) = \frac{1}{4},$$

即 $C = \frac{1}{4}$. 结合(12.98)我们知道 $x \equiv 0$.

若对任何 $t_0 > 0$, 都有 $t > t_0$, 有 $x'(t) = 0$. 则因为 $x' = 0$ 时由(12.97)必有 $\frac{1}{2} \ln(1+x) + \frac{1}{4(x+1)^2} = C$, 我们知道 x 取值只能有有限种可能. 因此必然存在 $t_2 > t_1 > 0$ 使得

$$x(t_2) = x(t_1), x'(t_1) = x'(t_2) = 0.$$

由于 $x(t+t_2-t_1)$ 和 $x(t)$ 在 t_1 有相同的初值条件. 由解的唯一性定理12.16知 $x(t+t_2-t_1) \equiv x(t)$, 故 x 有周期 $t_2 - t_1$.

Exercice 4: yau-2020

Problem 5. We consider the following ordinary differential equation:

$$\begin{cases} x''(t) + x(t) + x(t)^3 = 0, \\ (x(0), x'(0)) = (x_0, 0), \end{cases}$$

where $x(t)$ takes values in \mathbb{R} . Prove that for all $x_0 \in \mathbb{R}$, the solution of the above system is periodic.

证明. There exists constant C , s.t.

$$(x')^2 + x^2 + \frac{1}{2}x^4 = C$$

Then x is bounded since

$$0 \leq x^2 + \frac{1}{2}x^4 \leq C$$

Then $|x| < M$ for some $M > 0$.

If $\forall t_0 > 0$, we have $x'(t) \neq 0, \forall t > t_0$, then $x(t)$ is monotonic.

$$\lim_{t \rightarrow +\infty} x(t) = s \in \mathbb{R}$$

Thus there exists $\{t_n\} \rightarrow \infty, \{h_n\} \rightarrow \infty$, s.t.

$$\lim_{n \rightarrow \infty} x'(t_n) = \lim_{n \rightarrow \infty} x''(h_n) = 0$$

Then

$$\lim_{n \rightarrow \infty} (x'(t_n))^2 + x^2(t_n) + x^4(t_n) = C \Rightarrow s^2 + \frac{1}{2}s^4 = C$$

$$\lim_{n \rightarrow \infty} x''(h_n) + x(h_n) + x^3(h_n) = 0 \Rightarrow s + s^3 = 0$$

Therefore $s = 0; C = 0$; thus $x \equiv 0$.

If $\forall t_0 > 0$, there exists $t > t_0$, s.t. $x'(t) = 0$. There is a sequence $\{t_n\} \rightarrow \infty$, s.t. $x'(t_n) = 0$. Then

$$x^2(t_n) + x^4(t_n) = C$$

which has only finite solutions; thus there exists (WLOG) t_1, t_2 s.t.

$$x(t_1) = x(t_2) \quad x'(t_1) = x'(t_2) = 0$$

Then $x(t - t_1 + t_2)$ and $x(t)$ has the same initial condition at t_1 . The solution is unique since

$$x'' = f(t, x, x') = -x - x^3$$

Let $f(t, \mathbf{y}) = -y_1 - y_1^3$, then

$$\begin{aligned}|f(t, \mathbf{y}) - f(t, \mathbf{z})| &= |-y_1 - y_1^3 + z_1 + z_1^3| \\&= |y_1 - z_1||1 + y_1^2 + z_1^2 + y_1 z_1| \\&\leq (1 + 3M^2)|y_1 - z_1|\end{aligned}$$

f is Lipschitz on \mathbf{y} , which guarantees the uniqueness. Hence x is periodic with period $t_2 - t_1$. \square

6.28 高维常微分方程

参见 Arnold, 他使用了一种很高的观点.

6.28.1 Picard 近似

Consider $\dot{\mathbf{x}} = \mathbf{v}(t, \mathbf{x})$, defined by the vector field \mathbf{v} . We define the **Picard mapping**

$$(A\varphi)(t) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{v}(\tau, \varphi(\tau)) d\tau$$

Geometrically, passing from φ to $A\varphi$ means constructing w.r.t. a curve φ a new curve $A\varphi$ whose tangent for each t is parallel to a given direction field, only not on the curve $A\varphi$ itself – for then $A\varphi$ would be a solution – but at the corresponding point of the curve φ .

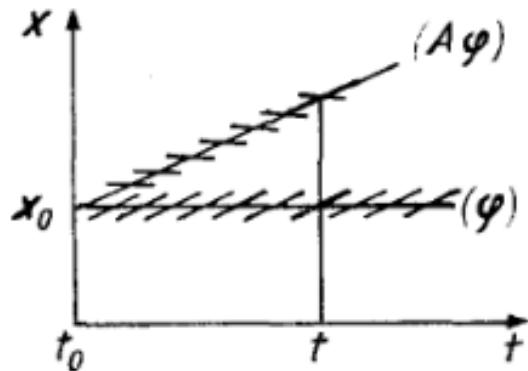


Fig. 215. The Picard mapping A

To prove convergence of the successive approximations we shall construct a complete metric space in which the Picard mapping is a contraction.

The mapping $A : (M_1, \rho_1) \rightarrow (M_2, \rho_2)$ satisfies a **Lipschitz condition** with constant L provided that

$$\rho_2(Ax, Ay) \leq L\rho_1(x, y) \quad \forall x, y \in M_1$$

Let $f : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a C^r -mapping ($r \geq 1$). Naturally, the derivative of f at $x \in U \subset \mathbb{R}^m$

$$f_{*x} : T_x \mathbb{R}^m \rightarrow T_{f(x)} \mathbb{R}^n$$

is a linear operator. If we choose a basis for each tangent space, f_{*x} has a $m \times n$ matrix representation.

Note. f_{*x} provides the best linear approximation of $f(x)$ at the neighborhood of x , i.e. $f(x + h) \approx f(x) + f_{*x}(h)$.

Theorem. A continuously differentiable mapping f satisfies a Lipschitz condition on each convex compact subset V of the domain U with constant L equal to the supremum of the derivative f on V :

$$L = \sup_{\mathbf{x} \in V} |f_{*x}|.$$

Proof. Join the points \mathbf{x} and $\mathbf{y} \in V$ with a line segment (Fig. 220): $\mathbf{z}(t) = \mathbf{x} + t(\mathbf{y} - \mathbf{x})$, $0 \leq t \leq 1$. By the Barrow formula

$$f(\mathbf{y}) - f(\mathbf{x}) = \int_0^1 \frac{d}{dt}(f(\mathbf{z}(\tau))) d\tau = \int_0^1 f_{*\mathbf{z}(\tau)} \dot{\mathbf{z}}(\tau) d\tau.$$

From formulas (1) and (2) of Sect. 2 and from the fact that $\dot{\mathbf{z}} = \mathbf{y} - \mathbf{x}$, we have

$$\left| \int_0^1 f_{*\mathbf{z}(\tau)} \dot{\mathbf{z}}(\tau) d\tau \right| \leq \int_0^1 L |\mathbf{y} - \mathbf{x}| d\tau = L |\mathbf{y} - \mathbf{x}|,$$

which was to be proved. \square

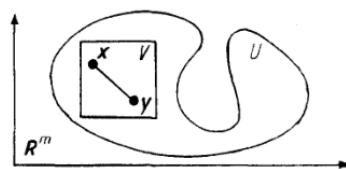


Fig. 220. Continuous differentiability implies a Lipschitz condition

Remarque 16. By hypothesis $f \in C^1$, f_{*x} is continuous; then $|f_{*x}|$ attains a maximum value L on any compact set V .

See arnold for the proof of theorem of existence and uniqueness.

Next we consider the theorem on differentiability. The motivation is as follows.

Associated with a differentiable mapping $f : U \rightarrow V$ is a linear operator on the tangent space

$$f_* : TU \rightarrow TV$$

At each point $x \in U$, $f_{*x} : T_x U \rightarrow T_{f(x)} V$.

In exactly the same way with a differential equation

$$\dot{x} = v(t, x) \quad x \in U \subset \mathbb{R}^n \quad (6.5)$$

there is associated a system of differential equations

$$\begin{cases} \dot{x} = v(t, x) & x \in U \subset \mathbb{R}^n \\ \dot{y} = v_*(t, x)y & y \in T_x U \end{cases} \quad (6.6)$$

called the system of equations of variations for eq. (6.5), and linear w.r.t. the tangent vector y .

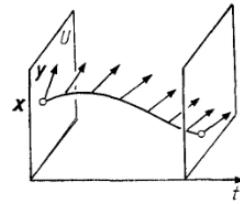


Fig. 226. The solution of the equation of variations with the initial condition (x, y)

We can correspond the unknown vector y to a unknown linear operator z .

Along with the system (2) it is convenient to consider the system

$$\dot{\mathbf{x}} = \mathbf{v}(t, \mathbf{x}), \quad \mathbf{x} \in U \subset \mathbf{R}^n, \quad \dot{z} = \mathbf{v}_*(t, \mathbf{x})z, \quad z : \mathbf{R}^n \rightarrow \mathbf{R}^n. \quad (3)$$

The system (3) is obtained from (2) by replacing the unknown vector \mathbf{y} with the unknown linear operator z . We shall also use the name *equation of variations* in reference to the system (3).

Remark. In general given a linear equation

$$\dot{\mathbf{y}} = A(t)\mathbf{y}, \quad \mathbf{y} \in \mathbf{R}^n, \quad (2')$$

it is useful to consider the associated equation

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$$\dot{z} = A(t)z, \quad z : \mathbf{R}^n \rightarrow \mathbf{R}^n, \quad (3')$$

in the linear operator z .

Knowing a solution of one of Eqs. (2') and (3'), it is easy to find a solution of the other (how?).

6.28.2 The differentiability theorem

Suppose \mathbf{v} of eq. (6.5) is C^2 in some neighborhood of (t_0, \mathbf{x}_0) . Then the solution $\mathbf{g}(t, \mathbf{x})$ of eq. (6.5) with initial condition $\mathbf{g}(t_0, \mathbf{x}) = \mathbf{x}$ depends on the condition \mathbf{x} in a continuously differentiable manner as \mathbf{x} and t vary in some neighborhood of (t_0, \mathbf{x}_0) :

$$\mathbf{v} \in C^2 \Rightarrow \mathbf{g} \in C_x^1$$

Proof. $\mathbf{v} \in C^2 \Rightarrow \mathbf{v}_* \in C^1$. Therefore the system of equations of variations (3) satisfies the hypotheses of § 31 and the sequence of Picard approximations converges uniformly to a solution of the system in a sufficiently small neighborhood of the point t_0 . We choose initial conditions $\varphi_0 = \mathbf{x}$ (sufficiently close to \mathbf{x}_0), and $\psi_0 = E$. We denote the Picard approximations by φ_n (for \mathbf{x}) and by ψ_n (for z), i.e., we set

$$\varphi_{n+1}(t, \mathbf{x}) = \mathbf{x} + \int_{t_0}^t \mathbf{v}(\tau, \varphi_n(\tau, \mathbf{x})) d\tau, \quad (4)$$

$$\psi_{n+1}(t, \mathbf{x}) = E + \int_{t_0}^t \mathbf{v}_*(\tau, \varphi_n(\tau, \mathbf{x})) \psi_n(\tau, \mathbf{x}) d\tau. \quad (5)$$

We remark that $\varphi_{0*} = \psi_0$. From the definitions (4) and (5) we conclude by induction on n that $\varphi_{n+1*} = \psi_{n+1}$. Therefore the sequence $\{\psi_n\}$ is the sequence of derivatives of the sequence $\{\varphi_n\}$. Both sequences (4) and (5) converge uniformly (being the sequences of Picard approximations of the system (3)) for $|t - t_0|$ sufficiently small. Thus the sequence $\{\varphi_n\}$ converges uniformly along with the derivatives on \mathbf{x} . Therefore the limiting function $\mathbf{g}(t, \mathbf{x}) = \lim_{n \rightarrow \infty} \varphi_n(t, \mathbf{x})$ is continuously differentiable with respect to \mathbf{x} , which was to be proved. \square

Remark. We have simultaneously proved the following theorem.

Theorem. *The derivative \mathbf{g}_* of a solution of Eq. (1) with respect to the initial condition \mathbf{x} satisfies the equation of variations (3) with the initial condition $z(t_0) = E$:*

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{g}(t, \mathbf{x}) &= \mathbf{v}(t, \mathbf{g}(t, \mathbf{x})), \quad \frac{\partial}{\partial t} \mathbf{g}_*(t, \mathbf{x}) = \mathbf{v}_*(t, \mathbf{g}(t, \mathbf{x})) \mathbf{g}_*(t, \mathbf{x}). \\ \mathbf{g}(t_0, \mathbf{x}) &= \mathbf{x}, \quad \mathbf{g}_*(t_0, \mathbf{x}) = E. \end{aligned}$$

This theorem explains the meaning of the equations of variations: they describe the action of the transformation over the time from t_0 to t on the vectors tangent to the phase space (Fig. 227).

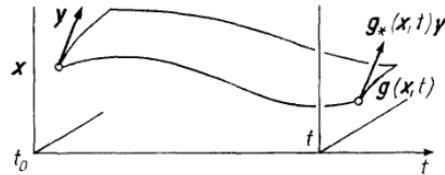


Fig. 227. The action of the transformation over the time from t_0 to t on a curve in the phase space and on a tangent vector to it

More generally,

$$\mathbf{v} \in C^r \Rightarrow \mathbf{g} \in C_x^{r-1}$$

We prove by induction

$$\mathbf{v} \in C^r \Rightarrow \mathbf{v}_* \in C^{r-1} \stackrel{\text{hypotheses}}{\Rightarrow} \mathbf{g}_* \in C_x^{r-2} \Rightarrow \mathbf{g} \in C_x^{r-1}$$

In fact, we have

$$\mathbf{v} \in C^1 \Rightarrow \mathbf{g} \in C_x^1$$

6. The Last Derivative

In the differentiability theorem (Sect. 2) we assumed the field \mathbf{v} was twice continuously differentiable. In reality it would suffice to have only continuous first-order derivatives.

Theorem. *If the right-hand side $\mathbf{v}(t, \mathbf{x})$ of the differential equation $\dot{\mathbf{x}} = \mathbf{v}(t, \mathbf{x})$ is continuously differentiable, then the solution $\mathbf{g}(t, \mathbf{x})$ with initial condition $\mathbf{g}(t_0, \mathbf{x}) = \mathbf{x}$ is a continuously differentiable function of the initial conditions:*

$$\mathbf{v} \in C^1 \Rightarrow \mathbf{g} \in C_x^1. \quad (7)$$

Corollaries.

- 1) $\mathbf{v} \in C^r \Rightarrow \mathbf{g} \in C^r$ for $r \geq 1$.
- 2) The rectifying diffeomorphisms constructed in Sect. 5 are r times continuously differentiable if $\mathbf{v} \in C^r$.

The corollaries are deduced from relation (7) by repeating verbatim the reasoning of Sects. 3, 4, and 5. The proof of relation (7) itself requires some clever tricks.

Lemme 4

The solution of a linear equation

$$\dot{\mathbf{y}} = A(t)\mathbf{y}$$

whose right-hand side depends continuously on t , exists; is unique; is determined uniquely by the initial conditions $\varphi(t_0) = \mathbf{y}_0$; is a continuous function of \mathbf{y}_0 and t ; is a **linear function** of \mathbf{y}_0 and a continuously differentiable function of t , thus C^1 in \mathbf{y}_0 and t .

Lemme 5

The solution of a linear equation

$$\dot{\mathbf{y}} = A(t, \alpha) \mathbf{y}$$

where $A(t, \alpha)$ is continuous, is a continuous function of \mathbf{y}_0 , t , and α .

We now apply Lemme 5 to the equation of variations, eq. (6.6).

The system of equations of variations

$$\dot{x} = \mathbf{v}(t, x), \quad \dot{\mathbf{y}} = \mathbf{v}_*(t, x) \mathbf{y}$$

has a solution uniquely determined by its initial data and depends continuously on them provided $\mathbf{v} \in C^1$.

第七章 复变函数

7.1 Delta-function

In complex analysis, we have the identity

$$\int_{-\infty}^{\infty} e^{itx} dx = 2\pi\delta_0(t)$$

In logic, this is not a consequence from the Fourier analysis but from the contour integration.

$$\begin{aligned} \int_{-\infty}^{\infty} e^{ixy} \underbrace{e^{-\epsilon x^2}}_{\rightarrow 1} dx &= e^{-\frac{y^2}{4\epsilon}} \int_{-\infty}^{\infty} e^{-\epsilon(x - \frac{iy}{2\epsilon})^2} dx \\ &= \underbrace{\sqrt{\frac{\pi}{\epsilon}} e^{-\frac{y^2}{4\epsilon}}}_{\rightarrow 2\pi\delta(y)} \end{aligned} \quad (1) \quad (2)$$

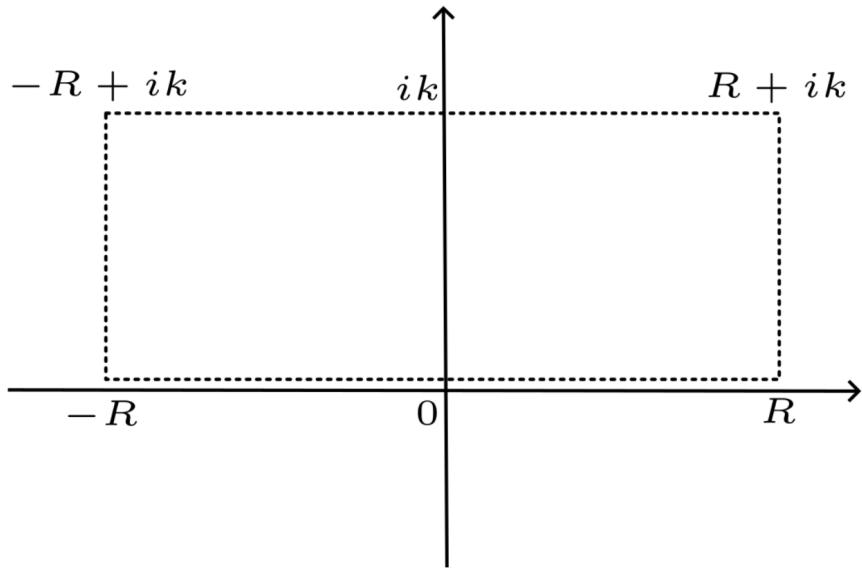
Using [Cauchy's Integral Theorem](#), the red integral in (1) is simply $\int_{-\infty}^{\infty} e^{-\epsilon x^2} dx = \sqrt{\frac{\pi}{\epsilon}}$.

As $\epsilon \rightarrow 0$, we get that (2) approximates $2\pi\delta(y)$. That is, the integral of (2) is 2π for all ϵ , and as $\epsilon \rightarrow 0$, the main mass of the function is squeezed into a very small region about 0.

It suffices to show that

$$\int_{-\infty}^{\infty} e^{-(x+ik)^2} dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Consider the toy contour:



$$\int_{-R}^R + \int_R^{R+ik} - \int_{-R+ik}^{R+ik} - \int_{-R}^{-R+ik} e^{-z^2} dz = 0$$

Since

$$\left| \int_R^{R+ik} e^{-z^2} dz \right| \leq \int_R^{R+ik} |e^{-z^2}| dz = \int_0^k |e^{-(R+ix)^2}| dx \leq ke^{-R^2} \rightarrow 0 \quad \text{as } R \rightarrow \infty$$

$$\lim_{R \rightarrow \infty} \int_{-R}^{-R+ik} e^{-z^2} dz = 0$$

We know that

$$\lim_{R \rightarrow \infty} \int_{-R}^R e^{-x^2} dx = \lim_{R \rightarrow \infty} \int_{-R+ik}^{R+ik} e^{-z^2} dz = \lim_{R \rightarrow \infty} \int_{-R}^R e^{-(x+ik)^2} dx$$

Therefore

$$\int_{-\infty}^{\infty} e^{-(x+ik)^2} dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

7.2 初等函数

See chapter 2 of A First Course in Complex Analysis with Applications.

Dennis

Exponential function

$$e^z = e^x \cos y + i e^x \sin y$$

Arg function

For $z \in \mathbb{C} \setminus \{0\}$, $w = \operatorname{Arg} z = \arg z + 2k\pi, k \in \mathbb{Z}$ has infinite many different values. $\arg z \in (-\pi, \pi]$ represents the principle value of $\operatorname{Arg} z$.

Logarithm function

$w = \operatorname{Ln} z$ is the complex number satisfying $z = e^w$.

$$w = \operatorname{Ln} z = \ln|z| + i \operatorname{Arg} z$$

$\ln z = \ln|z| + i \arg z$ is the principle value of $\operatorname{Ln} z$. Note that for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$,

$$\operatorname{Ln}(z_1 z_2) = \operatorname{Ln} z_1 + \operatorname{Ln} z_2, \quad \operatorname{Ln} \frac{z_1}{z_2} = \operatorname{Ln} z_1 - \operatorname{Ln} z_2$$

We use Ln instead of \ln for the consideration of multiple values.

原点及无穷远点对于对数函数 $w = \operatorname{Ln} z$ 有特殊的意义. 在 0 或 ∞ 的充分“小”的邻域内,任作一闭简单连续曲线 C 围绕 0 或 ∞ . 根据 $\operatorname{Arg} z$ 的连续变化情况,当一点 z 从 C 上一点 z_1 出发沿 C 连续变动一周时, $\operatorname{Ln} z$ 从它在 z_1 的任一值连续变动到其他一值. 因此我们把原点及无穷远点称为对数函数 $w = \operatorname{Ln} z$ 的支点. 这两支点还具有下列性质: 当 z 从 z_1 出发沿 C 按一定方向连续变动无论多少周时, $w = \operatorname{Ln} z$ 总不可能从它在 z_1 的任一值连续变动到同一值. 由于这一性质, 我们把原点及无穷远点称为 $w = \operatorname{Ln} z$ 的无穷阶支点, 特别称为对数支点. 对于其他多值解析函数, 我们也相应地定义支点及支点的阶数.

支点**Powers and Roots**

For $\alpha \in \mathbb{C} \setminus \{0\}$, $w = z^\alpha = e^{\alpha \operatorname{Ln} z} = e^{\alpha \ln z} \cdot e^{\alpha \cdot 2k\pi i}, k \in \mathbb{Z}$. If $\alpha = \frac{m}{n} \in \mathbb{Q}$ then z^α is n -valued. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ or $\alpha \in \mathbb{C} \setminus \mathbb{R}$, then z^α is ∞ -valued.

当 α 不是整数时, 原点及无穷远点是 $w=z^\alpha$ 的支点. 可是按照 α 是有理数或者不是有理数, 这两支点具有不同的性质. 与上段中一样, 在 0 或 ∞ 的充分小的邻域内, 任作一闭合简单连续曲线 C 围绕 0 或 ∞ . 在 C 上任取一点 z_1 , 确定 $\operatorname{Arg} z$ 在 z_1 的一值 $\arg z_1 = \theta_1$; 相应地确定

$$w = z^\alpha = e^{\alpha(\ln|z| + i\operatorname{Arg} z)}$$

在 z_1 的一值 $e^{\alpha(\ln|z_1| + i\arg z_1)} = e^{\alpha \ln z_1}$. 现考虑下列两种情况:

$n-1$ 阶支点: 转 n 圈回到最初的值。

(1) α 是有理数 $\frac{m}{n}$ (既约分数, $n \geq 2$) 当一点 z 从 z_1 出发按反时针或顺时针方向连续变动 n 周时, $\arg z$ 从 θ_1 连续变动到 $\theta_1 \pm 2n\pi$, 而 $w = z^{\frac{m}{n}}$, 则从 $e^{\frac{m}{n} \ln z_1} = e^{\frac{m}{n}(\ln|z_1| + i\theta_1)}$ 相应地连续变动到

$$e^{\frac{m}{n}(\ln z_1 + i2n\pi)} = e^{\frac{m}{n}\ln z_1},$$

亦即第一次回到了它从 z_1 出发时的值. 由于这一性质, 我们把原点及无穷远点称为 $w=z^n$ 的 $n-1$ 阶支点, 特别称为 $n-1$ 阶代数支点. 对于其他多值函数, 我们也相应地定义支点的阶数.

无穷阶支点: 转不回最初的值。

(2) α 不是有理数 这时不难验证原点及无穷远点是 $w=z^\alpha$ 的无穷阶支点.

当 α 不是整数时, 在复平面上, 任取连接 $w=z^\alpha$ 的两支点 O 及 ∞ 的一条无界简单连续曲线 K_1 作为割线, 得一区域 D_1 . 在 D_1 内, 可以把 $w=z^\alpha$ 分成解析分支. 特别地, 可取从原点出发的任何射线作为割线.

现研究 $w=z^\alpha$ 的映射性质, 其中 α 是一正实数. 设 ω 是一实数, 并且 $0 < \omega < 2\pi$. 在 z 平面上取正实轴(包括原点)作为割线, 得一区域 D^* . 考虑 D^* 内的角形 $A: 0 < \arg z < \omega$, 并且取 z^α 在 D^* 内的一个解析分支

$$w = z^\alpha \quad (1^\alpha = 1).$$

当 z 描出 A 内一条射线 $l: \arg z = \theta_0$ 时(不包括 $z=0$), w 在 w 平面上描出一条射线 $l_1: \arg w = \alpha\theta_0$. 让 θ_0 从 0 增加到 ω (不包括 0 及 ω), 那么射线 l 扫过角形 A , 而相应的射线 l_1 扫过角形 $A_1: 0 < \arg w < \alpha\omega$. 因此 $w = z^\alpha (1^\alpha = 1)$ 把夹角为 ω 的角形 A 双射成夹角为 $\alpha\omega$ 的角形 A_1 (图 7). 显然, 这一函数把 A 中以原点为心的圆弧映射成 A_1 中以原点为心的圆弧.

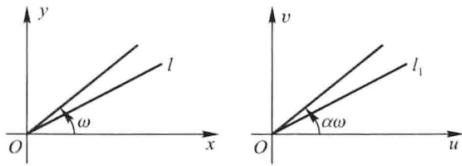


图 7

用类似的方法可以看出, 当 $n (n>1)$ 是正整数时, $w=\sqrt[n]{z}$ 的 n 个分支

$$w = \sqrt[n]{z} \quad (\sqrt[n]{1} = e^{i\frac{2k\pi}{n}})$$

$$(k = 0, 1, 2, \dots, n-1),$$

分别把区域 D^* 双射成 w 平面上的 n 个角形

$$\frac{2k\pi}{n} < \arg w < \frac{2(k+1)\pi}{n}.$$

例 1 作出一个含 i 的区域,使得函数

$$w = \sqrt{z(z-1)(z-2)}$$

在这区域内可分解成解析分支;求一个分支在点 i 的值.

我们知道

$$w = |z(z-1)(z-2)|^{\frac{1}{2}} e^{\frac{i}{2}[\operatorname{Arg} z + \operatorname{Arg}(z-1) + \operatorname{Arg}(z-2)]}.$$

先求函数 w 的支点. 由于 $z^{\frac{1}{2}}$ 的支点是 0 及 ∞ , 函数 w 的可能的支点是 0, 1, 2 及 ∞ . 任作一条简单连续闭曲线 C , 使其不经过 0, 1 及 2, 并使其内区域含 0, 但不含 1 及 2. 设 z_1 是 C 上一点. 我们确定 $\operatorname{Arg} z$, $\operatorname{Arg}(z-1)$ 及 $\operatorname{Arg}(z-2)$ 在这点的值 $\arg z_1$, $\arg(z_1-1)$ 及 $\arg(z_1-2)$ (图 8). 当 z 从 z_1 按反时针方向沿 C 连续变动一周时, 通过连续变动, $\arg z_1$ 增加了 2π , 而 $\arg(z_1-1)$ 及 $\arg(z_1-2)$ 没有变化. 于是 w 在 z_1 的值就从

$$|z_1(z_1-1)(z_1-2)|^{\frac{1}{2}} e^{\frac{i}{2}[\arg z_1 + \arg(z_1-1) + \arg(z_1-2)]} = w_1$$

连续变动到

$$|z_1(z_1-1)(z_1-2)|^{\frac{1}{2}} e^{\frac{i}{2}[\arg z_1 + 2\pi + \arg(z_1-1) + \arg(z_1-2)]} = -w_1.$$

因此 0 是函数 w 的支点; 同样可证明 1 及 2 也是它的支点. 任作一条简单连续闭曲线, 使其内区域含 0, 1 及 2, 可证明 ∞ 是函数 w 的支点^①.

在复平面上取连接 0, 1, 2 及 ∞ 的任一条无界简单连续曲线作为割线, 在所得区域内, 可把 w 分成连续分支. 例如可取 $[0, +\infty)$ 作为复平面上这样的割线, 得区域 D .

其次, 任作一条简单连续闭合曲线 C_1 , 使其不经过 0, 1 及 2, 并使其内区域含这三点中的两点, 但不含另一点. 设 z_2 是 C_1 上一点, 确定 w 在 z_2 的一个值, 与上面一样, 当 z 从 z_2 沿 C_1 连续变化一周回到 z_2 时, w 连续变化而得的值不会改变.

于是在复平面上取线段 $[0, 1]$ 以及从 2 出发且不与 $[0, 1]$ 相交的任何射线作为割线, 在所得区域内, 可把 w 分成连续分支. 例如可取 $[0, 1]$ 及 $[2, +\infty)$ 作

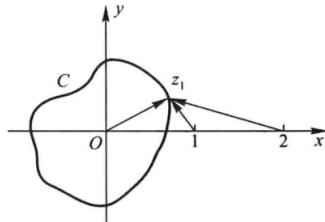


图 8

计算根式函数在某个给定的解析分支内的取值:

- (1) 先计算出这个解析分支是啥, 考虑极坐标
- (2) 再代入给定的点到这个解析分支中

为复平面上这样的割线, 得区域 D_1 .

求 w 在 D 或 D_1 内的一个解析分支

$$w = \sqrt{z(z-1)(z-2)} \quad (w(-1) = -\sqrt{6}i)$$

在 $z=i$ 的值.

在 $z=-1$, 取

$$\arg z = \pi, \arg(z-1) = \pi, \arg(z-2) = \pi.$$

于是在 D 或 D_1 内, w 可分成两个解析分支:

$$\begin{aligned} w &= |z(z-1)(z-2)|^{\frac{1}{2}} e^{\frac{i}{2}[\arg z + \arg(z-1) + \arg(z-2) + 2k\pi]} \\ &= |z(z-1)(z-2)|^{\frac{1}{2}} e^{\frac{i}{2}[\arg z + \arg(z-1) + \arg(z-2)] + ik\pi} \\ &\quad (k=0,1). \end{aligned}$$

由于所求的分支在 $z=-1$ 的值为 $-\sqrt{6}i$, 可见这一分支是

$$w = |z(z-1)(z-2)|^{\frac{1}{2}} e^{\frac{i}{2}[\arg z + \arg(z-1) + \arg(z-2)]}.$$

由图 9, 在 D 或 D_1 内 $z=i$ 处,

$$\arg z = \frac{\pi}{2}, \arg(z-1) = \frac{3\pi}{4},$$

$$\arg(z-2) = \pi - \arctan \frac{1}{2}.$$

因此 w 的所求分支在 $z=i$ 的值是

$$-\sqrt[4]{10} e^{\frac{i}{2}(\frac{\pi}{4} - \arctan \frac{1}{2})} = -\sqrt[4]{10} e^{\frac{i}{2}\arctan \frac{1}{3}}.$$

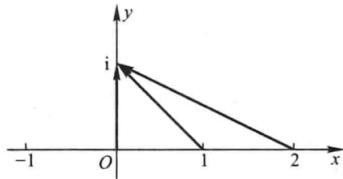


图 9

例 2 验证函数 $w = \sqrt[4]{z(1-z)^3}$ 在区域 $D = \mathbb{C} \setminus [0,1]$ 内可以分解成解析分支; 求出这函数在 $(0,1)$ 上沿取正值的一个分支在 $z=-1$ 处的值及函数在 $(0,1)$ 下沿的值.

我们有

$$w = |z(1-z)^3|^{\frac{1}{4}} e^{\frac{i}{4}[\operatorname{Arg} z + 3\operatorname{Arg}(1-z)]}.$$

不难看出, $z=0$ 及 1 是 w 的三阶支点, 可以证明 ∞ 不是它的支点.

事实上,任作一条简单连续闭曲线 C^* ,使其内区域含 0 及 1. 设 z^* 是 C^* 上的一点,取定 w 在 z^* 的一个值. 当 z 从 z^* 出发连续变化一周回到 z^* 时, w 连续变化而得的值不变.

由此可见,在区域 $D = \mathbb{C} \setminus [0, 1]$ 内,可以把 w 分成解析分支. 现在选取在 $(0, 1)$ 上沿取正实值的那一支,即在 $(0, 1)$ 的上沿, $\arg w = 0$, $w = \sqrt[4]{x(1-x)^3}$, 其中 $0 < x < 1$, 根号表示算术根. 求这一支在 $z = -1$ 的值.

在 $(0, 1)$ 上沿,可取 $\arg z = 0$, $\arg(1-z) = 0$. 于是所求的一支为

$$w = |z(1-z)^3|^{\frac{1}{4}} e^{\frac{i}{4}[\arg z + 3\arg(1-z)]}.$$

在 D 内 $z = -1$ 处,

$$\arg z = \pi, \arg(1-z) = 0.$$

于是 w 的指定的一支在 $z = -1$ 处的值是

$$\sqrt[4]{8} e^{\frac{\pi i}{4}} = \sqrt[4]{2}(1+i).$$

现在考虑上述单值支在 $(0, 1)$ 下沿取值的情况. 在区域 D 内,如图 10,当 z 沿曲线 C_1 ,从 $(0, 1)$ 的上沿变动到下沿时, $\arg z$ 没有变化,而 $\arg(1-z)$ 减少了 2π . 于是在 $(0, 1)$ 的下沿, $\arg w = -\frac{3\pi}{2}$. 如果 z 沿曲线 C_2 从 $(0, 1)$ 的上沿变动到下沿时, $\arg z$ 增加了 2π ,而 $\arg(1-z)$ 没有变化. 于是在 $(0, 1)$ 下沿, $\arg w = \frac{\pi}{2}$. 无论怎

样,当 $z = x$ 在 $(0, 1)$ 下沿时,上述单值支的值是

$$w = i\sqrt[4]{x(1-x)^3}.$$

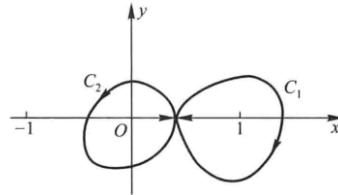


图 10

7.3 The logic of Harmoic function

Suppose f a complex function defined in a plane open set Ω , regarding f a transformation which maps Ω into \mathbf{R}^2 . Suppose f differentiable at $z_0 = 0$, WLOG $f(z_0) = 0$, then

$$f(z) = \alpha x + \beta y + \eta(z)z \quad (z = x + iy) \quad \eta(z) \rightarrow 0 \text{ as } z \rightarrow 0$$

Since $2x = z + \bar{z}$, $2iy = z - \bar{z}$, then

$$f(z) = \frac{\alpha - i\beta}{2}z + \frac{\alpha + i\beta}{2}\bar{z} + \eta(z)z$$

which suggests the introduction of the differential operators

$$\partial = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \bar{\partial} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

Now

$$f(z) = (\partial f)(0)z + (\bar{\partial}f)(0)\bar{z} + \eta(z)z \quad z \neq 0$$

If $f \in D(\Omega)$, then $f \in H(\Omega)$ iff the Cauchy-Riemann equation

$$(\bar{\partial}f)(z) = 0$$

holds for every $z \in \Omega$. In that case we have

$$f'(z) = (\partial f)(z) \quad z \in \Omega$$

If $f = u + iv$, u and v real. Then $(\bar{\partial}f)(z) = 0$ splits into $u_x = v_y, u_y = -v_x$.

The laplacian of f is defined to be

$$\Delta f = f_{xx} + f_{yy}$$

If f is continuous in Ω and if $\Delta f = 0$ then f is said to be *harmonic* in Ω . Note that $\Delta f = 4\partial\bar{\partial}f$. If f is holomorphic then $\bar{\partial}f = 0$, f has continuous derivatives of all orders and $\Delta f = 0$, which means "Holomorphic functions are harmonic." Every real harmonic function is locally the real part of a holomorphic function, and it will yield information about the boundary behavior of certain classes of holomorphic functions in open discs.

7.4 复分析知识简介-于品

假设 $\Omega \subset \mathbb{C}$ 是开集, $K \subset \Omega$ 是有界带边区域 (特别地, K 是紧的), 其边界 $\gamma = \partial K$ 是 C^1 曲线 (可以有多个连通分支)。

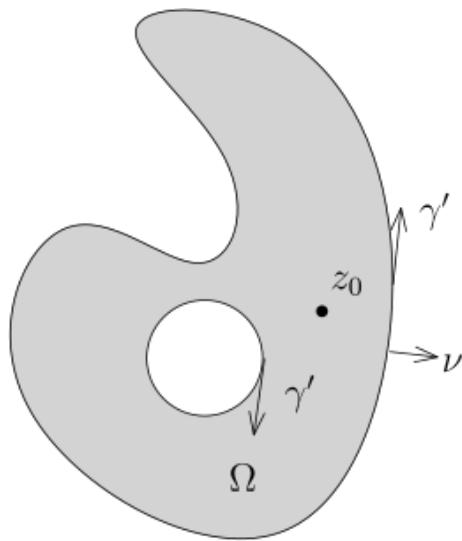


图 7.1

我们已经对复解析函数 $F(z)$ 证明了 Cauchy 积分公式：如果 $F(z)$ 在 Ω 上是复解析的，那么，对于 $z_0 \in \overset{\circ}{K}$ (K 的内部)，我们有

$$F(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{F(\xi)}{\xi - z_0} d\xi.$$

利用 Cauchy 公式，我们证明，复解析函数是“解析”的，也就是说可以在如下意义下写成幂级数的形式：

Théorème 1

假设 F 为开集 $\Omega \subset \mathbb{C}$ 上的复解析函数，并且以 z_0 为圆心以 R 为半径的开球 $B_R(z_0) \subset \Omega$ 。那么， $F(z)$ 在 z_0 处的解析半径至少是 R ，也就是说，在 $B_R(z_0)$ 上，我们有

$$F(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + a_n(z - z_0)^n + \cdots,$$

其中

$$a_k = \frac{1}{2\pi i} \int_{|\xi-z_0|=r} \frac{F(\xi)}{(\xi - z_0)^{k+1}} d\xi, \quad r < R.$$

这里，等式右边的幂级数对任意的 $z \in B_R(z_0)$ 都是收敛的。上面系数定义中的 r 可以是 $(0, R)$ 中的任意一个数。

证明的想法很简单：我们把 Cauchy 积分公式

$$F(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{F(\xi)}{\xi - z} d\xi$$

的右边的积分项强行展开即可。

证明。我们选取一个正实数 r , 使得 $|z - z_0| < r < R$ 。我们任意给定 ξ , 使得 $|\xi - z_0| = r$ 。

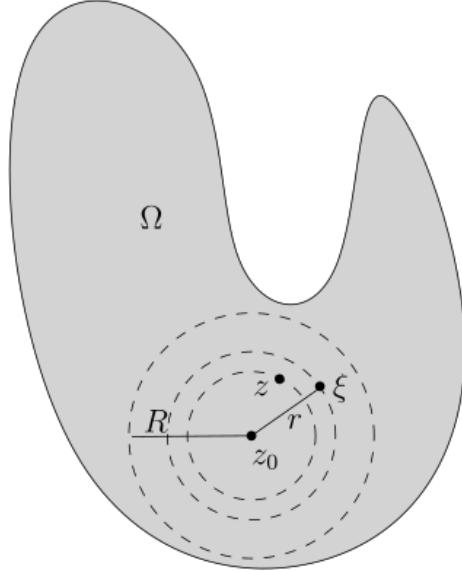


图 7.2

此时, 我们有

$$\begin{aligned} \frac{1}{\xi - z} &= \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0} \frac{1}{1 - \frac{z-z_0}{\xi-z_0}} \\ &= \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\xi - z_0)^{k+1}}. \end{aligned}$$

在上面的式子中, 由于 $\left| \frac{z - z_0}{\xi - z_0} \right| < 1$, 所以, 我们有 (级数收敛):

$$\frac{1}{1 - \frac{z-z_0}{\xi-z_0}} = \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^k.$$

这些级数显然是绝对收敛的, 从而与积分可交换 (也可以利用 Lebesgue 控制收敛定理)。所以, 代入 Cauchy 积分公式, 我们就得到

$$\begin{aligned} F(z) &= \frac{1}{2\pi i} \int_{|\xi-z_0|=r} \sum_{k=0}^{\infty} F(\xi) \frac{(z - z_0)^k}{(\xi - z_0)^{k+1}} d\xi \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|\xi-z_0|=r} \frac{F(\xi)}{(\xi - z_0)^{k+1}} d\xi \right) (z - z_0)^k \end{aligned}$$

比较系数, 这就给出了定理的证明。 \square

Corollaire 1: 零点的离散性

这里的假设与定理中是一致的。我们进一步假设 Ω 是道路连通的，即对任意的 $z_1, z_2 \in \Omega$ ，存在连续映射

$$\gamma: [0, 1] \rightarrow \Omega, \quad t \mapsto \gamma(t),$$

使得 $\gamma(0) = z_1, \gamma(1) = z_2$ 。那么，复解析函数 F 在 Ω 中的零点是离散的（即如果 z_0 是 F 的一个零点，那么，存在 $\varepsilon > 0$ ，使得对任意的 z ， $|z - z_0| < \varepsilon$ ， $F(z) \neq 0$ ），除非 $F \equiv 0$ 。

特别地，给定 Ω 上的两个复解析函数 F 和 G ，如果 F 和 G 在 $Z \subset \Omega$ 上取值相同，并且 Z 在 Ω 中有聚点，那么， $F \equiv G$ 。

证明. 假设 z_0 是 F 的一个零点，即 $F(z_0) = 0$ 。根据 F 的解析表达式，在 $B_R(z_0) \subset \Omega$ 上，我们有

$$F(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + a_n(z - z_0)^n + \cdots.$$

如果这些系数 a_i 全部为 0，那么， F 在 $B_R(z_0)$ 上恒为 0；否则，假设 a_m 是第一个不是 0 的系数，那么，我们有

$$F(z) = (z - z_0)^m (a_m + a_{m+1}(z - z_0) + a_{m+2}(z - z_0)^2 + \cdots) = (z - z_0)^m G(z).$$

根据定理的证明，级数

$$G(z) = a_m + a_{m+1}(z - z_0) + a_{m+2}(z - z_0)^2 + \cdots$$

也是绝对收敛的，特别地，这是连续的，所以，当 $z = z_0$ 时， $G(z_0) = a_m \neq 0$ 。从而，存在 δ ，使得 G 在 $B_\delta(z_0)$ 上不会等于 0，此时，我们知道， F 在 z_0 的附近 ($B_\delta(z_0)$ 上) 恰好有一个零点。¹

我们现在证明，如果 F 在 z_0 的一个邻域 $B_\delta(z_0)$ 上恒为 0，那么， F 在 Ω 上恒为 0：任意选取 $z_1 \in \Omega$ 和曲线

$$\gamma: [0, 1] \rightarrow \Omega, \quad t \mapsto \gamma(t),$$

使得 $\gamma(0) = z_0, \gamma(1) = z_1$ 。令

$$t_* = \sup \{t \in [0, 1] \mid F(\gamma(t))|_{[0, t]} \equiv 0\}.$$

¹这就是说， F 要么在 z_0 的附近恒为 0，要么在 z_0 的附近恰好只有一个零点。

由于 F 在 z_0 的一个邻域 $B_\delta(z_0)$ 上恒为 0, 所以, $t_* > 0$ 。我们现在证明 $t_* = 1$: 如果假设 $t_* < 1$, 根据连续性, 我们知道 $F(\gamma(t_*)) = 0$ 。根据前面的构造, 由于 F 在 $\gamma(t_*)$ 的任意一个小邻域中都有零点 (因为和 $\gamma^{-1}([0, t_*])$ 相交), 根据之前的推导, 存在 $\gamma(t_*)$ 一个邻域 $B_{\delta_1}(\gamma(t_*))$, 使得 F 在 $B_{\delta_1}(\gamma(t_*))$ 上恒为 0, 根据连续性, 那么, 存在 $\varepsilon > 0$, 使得 $F(\gamma(t))|_{[0, t_* + \varepsilon]} \equiv 0$, 这和 t_* 的最大性矛盾。

由于 $t_* = 1$, 所以, $F(z_1) = 0$, 这就证明了在 Ω 上, $F \equiv 0$ 。

定理中的第二个结论考虑 $F - G$ 即可。 \square

Corollaire 2

这里的假设与定理中是一致的。那么, F 的 n 次导数 $F^{(n)}$ 仍然是解析函数。我们进一步有如下的公式:

$$F^{(n)}(z) = \frac{n!}{2\pi i} \int_{|\xi-z|=r} \frac{F(\xi)}{(\xi-z)^{n+1}} d\xi.$$

特别地, 我们有如下的导数估计:

$$|F^{(n)}(z)| \leq \frac{n!}{r^n} \sup_{|\xi-z|=r} |F(\xi)|.$$

证明. 我们将 F 在 $z_0 = 0$ (不妨) 处展开为级数

$$F(z) = a_0 + a_1 z + a_2 z^2 + \cdots,$$

其中, 我们假设上面的级数的收敛半径至少是 $R > 0$, 即对于 $|z| < R$ 都是绝对收敛的。根据定理中系数的计算, 我们知道

$$|a_k| \leq \frac{1}{r^k} \sup_{|\xi|=r} |F(\xi)| \leq \frac{1}{r^k} M, \quad r < R.$$

其中, M 为 $|F|$ 在半径为 r 的圆圈上的最大值。从而, 对 $|z| \leq r' < r$, 对任意的 $k \geq 0$, 我们有

$$|ka_k z^{k-1}| \leq M \underbrace{\frac{k}{r} \left| \frac{r'}{r} \right|^{k-1}}_{b_k}.$$

当 k 足够大的时候, 比如说

$$k > k_0 = \lfloor \left(1 - \frac{r'}{r}\right)^{-1} \rfloor$$

时, 我们有

$$\frac{b_{k+1}}{b_k} = \left(1 + \frac{1}{k}\right) \frac{r'}{r} < 1,$$

所以, $\{b_k\}_{k \geq k_0}$ 可以被一个公比小于 1 几何级数来控制, 从而, 级数

$$a_1 + 2a_2 z^1 + 3a_3 z^2 + \dots$$

在 $|z| < r'$ 时是绝对收敛, 这表明可以逐项求微分 (根据 Lebesgue 控制收敛定理的推论)。这说明

$$F'(z) = a_1 + 2a_2 z^1 + 3a_3 z^2 + \dots$$

对于 $|z| < r' < r < R$ 成立。由于 r' 和 r 是任意选取的, 所以, 上面的式子对于 $|z| < R$ 都成立。特别地, 我们证明了

$$F'(z_0) = a_1.$$

由归纳法, 对任意的 k , 我们就有

$$F^{(k)}(z_0) = k! a_k.$$

再根据定理中的计算, 这就证明了这个推论叙述中的公式。导数估计是显然的。

□

Théorème 2: Liouville

假设 $F(z)$ 是在整个 \mathbb{C} 上定义的复解析函数^a。如果 F 是有界函数, 那么 F 是常值函数。

^a这样的函数称作是整函数。

证明. 我们只要证明 $F'(z) \equiv 0$ 即可: 根据 F 在一点处的展开, $F'(z) \equiv 0$, 意味着定理中的系数

$$a_1 = 2a_2 = 3a_3 = \dots = 0.$$

所以, F 在一点附近恒为 a_0 , 从而 F 为常数 (一点附近的邻域有聚点)。

我们利用导数估计:

$$|F'(z)| \leq \frac{1}{r} \sup_{|\xi - z|=r} |F(\xi)| \leq \frac{\|F\|_\infty}{r}.$$

由于 F 在整个 \mathbb{C} 上定义, 从而可以将 r 取得任意大, 这表明对任意的 $z \in \mathbb{C}$, $F'(z) = 0$ 。

□

Corollaire 3: 代数基本定理

对任意的次数非零的复系数多项式

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0,$$

它在 \mathbb{C} 上必有一个根。

证明. 我们观察到 P 在整个 \mathbb{C} 上是复解析的, 并且当 $|z| \rightarrow \infty$ 时, 我们有

$$|P(z)| \rightarrow \infty.$$

我们用反证法: 如若不然,

$$F(z) = P(z)^{-1}$$

是在全平面 \mathbb{C} 上良好定义的函数。另外, 我们有

$$\bar{\partial}(F) = -P(z)^{-2}\bar{\partial}P = 0.$$

所以, F 是复解析的。另外, 当 $|z| \rightarrow \infty$ 时, 我们还有

$$|F(z)| \rightarrow 0.$$

这说明, F 是有界的。根据 Liouville 定理, F 为常数, 从而 P 也是, 那么它的次数是 0, 矛盾。 \square

Cauchy 积分公式是复分析中最重要的公式, 除了用来证明解析性, 它还有其他众多重要的推论, 比如说关于复解析函数的极大模原理:

Théorème 3: 极大模原理

假设 $F(z)$ 是区域 $\Omega \subset \mathbb{C}$ 上的复解析函数, 那么 $|F(z)|$ 的最大值, 如果能取到的话, 一定在 Ω 的边界 $\partial\Omega$ 上取到。进一步, 如果 $|F|$ 在 Ω 的内部有最大值点, 那么 F 一定是常值函数。^a

^a按照同样的证明思路可知极小模也在边界上取到。

证明. 假设 $z_0 \in \dot{\Omega}$ 是 $|F|$ 的最大值点, 即

$$|F(z_0)| = \sup_{z \in \Omega} |F(z)|.$$

根据 Cauchy 积分公式, (选取较小的 ε , 使得 $B_\varepsilon(z_0) \subset \Omega$, 下面的的 r 只要满足 $r < \varepsilon$ 即可) 我们有

$$\begin{aligned}|F(z_0)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} F(z_0 + re^{i\theta}) d\theta \right| \\&\leq \frac{1}{2\pi} \int_0^{2\pi} \|F\|_{L^\infty} d\theta \\&= \frac{1}{2\pi} \int_0^{2\pi} |F(z_0)| d\theta \\&= |F(z_0)|.\end{aligned}$$

这表明, 上述不等式必须处处取等号, 这表明在 z_0 的整个邻域 $B_\varepsilon(z_0)$ 上, $|F(z)|$ 为常数。

我们现在证明 F 在 $B_\varepsilon(z_0)$ 上为常数, 不妨假设 $z_0 = 0$ 。如若不然, 那么, 我们当 ε 较小时, 我们有解析表达式

$$F(z) = a_0 + z^m (a_m + a_{m+1}z + a_{m+2}z^2 + \dots),$$

其中 $a_m \neq 0$ 。通过对 $F(z)$ 乘以一个常数, 我们还可以假设 $a_0 > 0$ (如果 $a_0 = 0$, 那么 $|F|$ 在 z_0 附近恒为零)。选取 ξ , 使得 $\xi^m = -\overline{a_m}$, 所以, 对于比较小的 $\delta > 0$, 我们有

$$F(\delta\xi) = a_0 - \delta^m |a_m|^2 + O(\delta^{m+1}).$$

令 $\delta \rightarrow 0$, 这就和 $|F(z_0)|$ 最大矛盾。所以, F 在 $B_\varepsilon(z_0)$ 上为常数, 所以 F 为常数。 \square

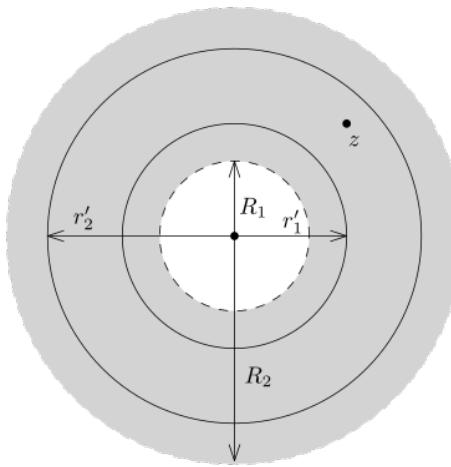


图 7.3

Théorème 4: Laurent 展开

F 是环面 $\{z \in \mathbb{C} \mid R_1 < |z| < R_2\}$ (我们经常取 $R_1 = 0$) 上的复解析函数。对每个整数 $k \in \mathbb{Z}$, 对任意的 $R_1 < r < R_2$, 我们定义

$$a_k = \frac{1}{2\pi i} \int_{|z|=r} \frac{F(z)}{z^{k+1}} dz = \frac{1}{2\pi r^k} \int_0^{2\pi} e^{-ik\theta} F(re^{i\theta}) d\theta.$$

那么, 对任意满足 $R_1 < |z| < R_2$ 的 z , 我们有 (对每个固定的 z , 以下的级数绝对收敛)

$$F(z) = \sum_{k=-\infty}^{\infty} a_k z^k.$$

证明. 根据 Cauchy 积分公式, a_k 与 r 在 (R_1, R_2) 中的选择无关。我们选取

$$R_1 < r'_1 < r_1 < |z| < r_2 < r'_2 < R_2$$

并令 C'_i 为半径为 r'_i 并且中心在原点的圆, 其中 $i = 1, 2$ 。根据 Cauchy 积分公式, 我们有

$$F(z) = \frac{1}{2\pi i} \int_{C'_2} \frac{F(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \int_{C'_1} \frac{F(\xi)}{\xi - z} d\xi.$$

我们重复之前证明复解析函数能做解析展开的做法。

对于第一项, 由于对任意的 $\xi \in C'_2$, 我们有 $|z| < |\xi|$, 我们有

$$\frac{1}{\xi - z} = \frac{1}{\xi} \frac{1}{1 - \frac{z}{\xi}} = \frac{1}{\xi} \sum_{k=0}^{\infty} \left(\frac{z}{\xi} \right)^k.$$

第二项之中, 由于 $|z| > |\xi|$, 其中 $\xi \in C'_1$, 我们有

$$\frac{1}{\xi - z} = -\frac{1}{z} \frac{1}{1 - \frac{\xi}{z}} = -\frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{\xi}{z} \right)^k.$$

将上面的两个展开代入 $F(z)$ 的表达式, 我们就有

$$F(z) = \frac{1}{2\pi i} \int_{C'_2} \frac{F(\xi)}{\xi} \sum_{k=0}^{\infty} \left(\frac{z}{\xi} \right)^k d\xi + \frac{1}{2\pi i} \int_{C'_1} \frac{F(\xi)}{z} \sum_{k=0}^{\infty} \left(\frac{\xi}{z} \right)^k d\xi.$$

将求和与积分交换就得到了要证明的结论。 \square

Définition 1

假定 $F(z)$ 在区域 $\{z \mid 0 < |z - z_0| < r\}$ 上是复解析的, 其中 $r > 0$, 它的

Laurent 展开为

$$F(z) = \sum_{k=-\infty}^{\infty} a_k z^k.$$

我们称其中的 z^{-1} 的系数 a_{-1} 为 F 在 z_0 处的留数，并记作 $\text{Res}(F; z_0)$ 。

Remarque 17. 按照定义，我们有

$$\text{Res}(F; z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} F(z) dz.$$

这是因为 Laurent 展开中其它幂次的积分都是 0。

Théorème 5: 留数定理

假定 $\Omega \subset \mathbb{C}$ 是一个紧区域， Ω 边界为分段光滑的 C^1 -曲线。除去点 $z_1, \dots, z_N \in \Omega$ 之外， F 为 $\mathbb{C} - \{z_1, \dots, z_N\}$ 上的复解析函数，那么，

$$\sum_{k=1}^N \text{Res}(F; z_k) = \frac{1}{2\pi i} \int_{\partial\Omega} F(z) dz.$$

证明。对每个 $k \leq N$ ，先将每个 z_k 附近的小圆盘 $B_\varepsilon(z_k)$ 从 Ω 上抠掉，使得 $z \in \Omega - \bigcup_{k \leq N} B_\varepsilon(z_k)$ 。我们在区域 $\Omega - \bigcup_{k \leq N} B_\varepsilon(z_k)$ 上用 Cauchy 积分公式。

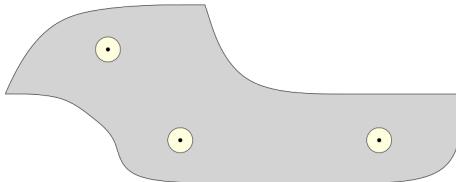


图 7.4

考虑到曲线的定向，我们有

$$\frac{1}{2\pi i} \int_{\partial\Omega} F(z) dz - \sum_{k=1}^N \frac{1}{2\pi i} \int_{\partial B_\varepsilon(z_k)} F(z) dz = 0.$$

所以，

$$\frac{1}{2\pi i} \int_{\partial\Omega} F(z) dz = \sum_{k=1}^N \frac{1}{2\pi i} \int_{\partial B_\varepsilon(z_k)} F(z) dz = \sum_{k=1}^N \text{Res}(F; z_k).$$

这就证明了留数定理。 \square

我们试举一个有趣的应用，其它在计算上的应用我们将在 Fourier 变换的一部分再做演示。

Exemple 20. 我们在第一学期已经定义了三角函数

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz}), \quad \sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

我们研究 $\sin(z)$ 的零点, 即找到 z_0 , 使得

$$e^{iz_0} - e^{-iz_0} = 0 \Leftrightarrow e^{2iz_0} = 1 \Leftrightarrow z_0 \in \pi\mathbb{Z}.$$

对任意的 $a \notin \mathbb{Z}$, 我们考虑

$$F(z) = \frac{\pi \cot(\pi z)}{(z-a)^2} = \frac{\pi \cos(\pi z)}{\sin(\pi z)(z-a)^2}.$$

那么, $F(z)$ 不解析的地方只能是 a 和 $n \in \mathbb{Z}$ 。

在 $z=a$ 处, 要想有非平凡的留数, $\cot(\pi z)$ 需要贡献一个 $z-a$ 的因子, 所以

$$\text{Res}(F; a) = (\pi \cot(\pi z))'|_{z=a} = -\frac{\pi^2}{\sin^2(\pi a)}.$$

在 $z=n$ 处, 由于 $\sin(\pi z)'|_{z=n} \neq 0$, 所以, $\sin(\pi z)$ 的零点是 1 阶的, 据此, 我们知道

$$\text{Res}(F; n) = \frac{1}{(n-a)^2}.$$

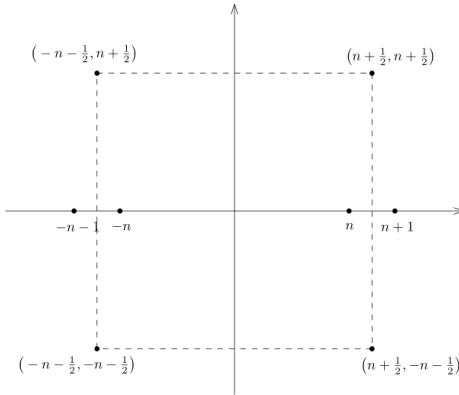


图 7.5

我们对于顶点在 $(\pm(n + \frac{1}{2}), \pm(n + \frac{1}{2}))$ 的正方形 Q_n 上用留数定理, 其中, n 足够大使得 $a \in Q_n$ 。所以,

$$\sum_{|k| \leq n} \frac{1}{(k-a)^2} - \frac{\pi^2}{\sin^2(\pi a)} = \frac{1}{2\pi i} \int_{\partial Q_n} \frac{\pi \cos(\pi z)}{\sin(\pi z)(z-a)^2} dz.$$

利用 $z = x + iy$, 我们很容易看出

$$\frac{\pi \cos(\pi z)}{\sin(\pi z)(z-a)^2} = O\left(\frac{1}{n^2}\right).$$

所以上面的积分项的贡献为

$$\frac{1}{2\pi i} \int_{\partial Q_n} \frac{\pi \cos(\pi z)}{\sin(\pi z)(z-a)^2} dz = O\left(\frac{1}{n}\right).$$

令 $n \rightarrow \infty$, 我们就得到了

$$\sum_{n \in \mathbb{Z}} \frac{1}{(n-a)^2} = \frac{\pi^2}{\sin^2(\pi a)}.$$

利用上面例子中的分析, 我们还可以证明 Euler 的著名公式。首先, 在 $\mathbb{C} - \mathbb{Z}$ 上的任意一个紧集上, 级数

$$f(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$$

是一致收敛的, 所以, $f(z)$ 是 $\mathbb{C} - \mathbb{Z}$ 上的复解析函数 (可逐项求导数)。Euler 观察到, 函数

$$g(z) = \frac{\pi^2}{\sin^2(\pi z)}$$

也是 $\mathbb{C} - \mathbb{Z}$ 上的复解析函数。进一步, 由于 $\sin(\pi z)$ 的零点是单零点, 所以, $g(z)$ 在每个 $n \in \mathbb{Z}$ 处的 Laurent 展开的负幂和 $f(z)$ 的是一致的。所以,

$$F(z) = g(z) - f(z) = \frac{\pi^2}{\sin^2(\pi z)} - \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$$

是全平面上定义的复解析函数并且具有周期性 $F(z+1) = F(z)$ 。

对于任意的 $x \in [0, 1]$, 我们很容易证明下面的极限:

$$\lim_{|y| \rightarrow \infty} F(x+iy) = 0.$$

利用周期性, 我们就知道 F 是有界的, 从而根据 Liouville 定理, 我们得到 $F \equiv 0$ 。

这表明

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}.$$

稍加变形, 我们有

$$\frac{\pi^2}{\sin^2(\pi z)} - \frac{1}{z^2} = \sum_{n \neq 0} \frac{1}{(z-n)^2}.$$

左右在 $z=0$ 处取极限, 我们就证明了著名的 Euler 公式:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{1}{6}\pi^2.$$

见 Rudin 《实分析与复分析》

7.5 解析函数基本性质

A **region** is nonempty connected open subset of \mathbb{C} . Each open set Ω in the plane is union of discs. For $z_0 \in \Omega$ if $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists then denote it $f'(z_0)$. If $f'(z_0)$ exists $\forall z_0 \in \Omega$, say $f \in H(\Omega)$. $H(\Omega)$ is the class of all holomorphic functions in Ω , which is a ring. If f is representable by power series in Ω , then $f \in H(\Omega)$ and f' is also representable. Check the convergence by evaluate

$$\frac{\sum c_n z^n}{z - w} - \underbrace{\frac{g(w)}{\sum n c_n z^{n-1}}}_{= (z-w) \sum_{k=1}^{n-1} k w^{k-1} z^{n-k-1}} = \sum_{n=1}^{\infty} c_n \underbrace{\left[\frac{z^n - w^n}{z - w} - n w^{n-1} U \right]}_{= (z-w) \sum_{k=1}^{n-1} k w^{k-1} z^{n-k-1}}$$

Fix $w \in D(a; r)$, and choose ρ s.t. $|w| < \rho < r$, where $r^{-1} = \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}$.

For $n \geq 2$, if $|z| < \rho$ then

$$\left| \sum_{k=1}^{n-1} k w^{k-1} z^{n-k-1} \right| \leq \frac{n(n-1)}{2} \rho^{n-2}$$

Then

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| \leq |z - w| \underbrace{\sum_{n=2}^{\infty} n^2 |c_n| \rho^{n-2}}_{\text{converges}} \rightarrow 0 \quad \text{as } z \rightarrow w$$

An general result is for complex (finite) measure on a measurable space X , φ is a plex measurable function on X , Ω is an open set in the plane which does not intersect $\varphi(X)$ and

$$f(z) = \int_X \frac{d\mu(\zeta)}{\varphi(\zeta) - z} \quad z \in \Omega$$

Then f is **representable by power series** in Ω .

The proof is from the fact that for $z \in D(a; r) \subset \Omega$, every $\zeta \in X$,

$$\sum_{n=0}^{\infty} \frac{(z-a)^n}{(\varphi(\zeta)-a)^{n+1}} = \frac{1}{\varphi(\zeta)-z}$$

Converges uniformly on X . Then

$$f(z) = \int_X \frac{d\mu(\zeta)}{\varphi(\zeta) - z} = \int_X \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\varphi(\zeta)-a)^{n+1}} d\mu(\zeta) = \sum_{n=0}^{\infty} \underbrace{\left[\int_X \frac{d\mu(\zeta)}{(\varphi(\zeta)-a)^{n+1}} \right]}_{:= c_n} (z-a)^n$$

Suppose $\gamma : [\alpha, \beta] \rightarrow \gamma^*$ a path, f a continuous function defined on γ^* (the range of γ), then

$$\int_{\gamma} f(z) dz = \int_{\alpha}^{\beta} f(\gamma(t)) \gamma'(t) dt$$

γ is unique under continuously differentiable one-to-one mapping.

Let γ be a closed path, Ω the complement of γ^* , and define

$$\text{Ind}_{\gamma}(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\zeta - z} \quad z \in \Omega$$

Then Ind_{γ} is an integer-valued function on Ω which is constant in each component of Ω and which is 0 in the unbounded component of Ω .

Note. We call Ind_{γ} the index of z w.r.t. γ , which is "the number of times γ winds around z ".

10.14 Cauchy's Theorem in a Convex Set Suppose Ω is a convex open set, $p \in \Omega$, f is continuous on Ω , and $f \in H(\Omega - \{p\})$. Then $f = F'$ for some $F \in H(\Omega)$. Hence

$$\int_{\gamma} f(z) dz = 0 \tag{1}$$

for every closed path γ in Ω .

PROOF Fix $a \in \Omega$. Since Ω is convex, Ω contains the straight line interval from a to z for every $z \in \Omega$, so we can define

$$F(z) = \int_{[a, z]} f(\xi) d\xi \quad (z \in \Omega). \tag{2}$$

For any z and $z_0 \in \Omega$, the triangle with vertices at a , z_0 , and z lies in Ω ; hence $F(z) - F(z_0)$ is the integral of f over $[z_0, z]$, by Theorem 10.13. Fixing z_0 , we thus obtain

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0, z]} [f(\xi) - f(z_0)] d\xi, \tag{3}$$

if $z \neq z_0$. Given $\epsilon > 0$, the continuity of f at z_0 shows that there is a $\delta > 0$ such that $|f(\xi) - f(z_0)| < \epsilon$ if $|\xi - z_0| < \delta$; hence the absolute value of the left side of (3) is less than ϵ as soon as $|z - z_0| < \delta$. This proves that $f = F'$. In particular, $F \in H(\Omega)$. Now (1) follows from Theorem 10.12. ////

10.15 Cauchy's Formula in a Convex Set Suppose γ is a closed path in a convex open set Ω , and $f \in H(\Omega)$. If $z \in \Omega$ and $z \notin \gamma^*$, then

$$f(z) \cdot \text{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi. \quad (1)$$

The case of greatest interest is, of course, $\text{Ind}_\gamma(z) = 1$.

PROOF Fix z so that the above conditions hold, and define

$$g(\xi) = \begin{cases} \frac{f(\xi) - f(z)}{\xi - z} & \text{if } \xi \in \Omega, \xi \neq z, \\ f'(z) & \text{if } \xi = z. \end{cases} \quad (2)$$

Then g satisfies the hypotheses of Theorem 10.14. Hence

$$\frac{1}{2\pi i} \int_\gamma g(\xi) d\xi = 0. \quad (3)$$

If we substitute (2) into (3) we obtain (1). ////

The easy consequence of the above theorem is the representability of holomorphic functions by power series.

The Cauchy theorem has a useful converse:

10.17 Morera's Theorem Suppose f is a continuous complex function in an open set Ω such that

$$\int_{\partial\Delta} f(z) dz = 0$$

for every closed triangle $\Delta \subset \Omega$. Then $f \in H(\Omega)$.

PROOF Let V be a convex open set in Ω . As in the proof of Theorem 10.14, we can construct $F \in H(V)$ such that $F' = f$. Since derivatives of holomorphic functions are holomorphic (Theorem 10.16), we have $f \in H(V)$, for every convex open $V \subset \Omega$, hence $f \in H(\Omega)$. ////

Next we consider the power series representation of holomorphic functions.

Suppose Ω a region, $f \in H(\Omega)$ and $Z(f) = \{a \in \Omega : f(a) = 0\}$. Then either $Z(f) = \Omega$ or $Z(f)$ has no limit point in Ω . In the later case $\exists a \mapsto m = m(a)$ such that $f(z) = (z - a)^m g(z)$ for $z \in \Omega$, where $g \in H(\Omega)$ and $g(a) \neq 0$. Furthermore, $Z(f)$ is at most countable.

Analogous results hold of course for the set of α -points of f , i.e. the zero

set of $f - \alpha$.

10.19 Definition If $a \in \Omega$ and $f \in H(\Omega - \{a\})$, then f is said to have an *isolated singularity* at the point a . If f can be so defined at a that the extended function is holomorphic in Ω , the singularity is said to be *removable*.

10.20 Theorem Suppose $f \in H(\Omega - \{a\})$ and f is bounded in $D'(a; r)$, for some $r > 0$. Then f has a removable singularity at a .

Recall that $D'(a; r) = \{z : 0 < |z - a| < r\}$.

PROOF Define $h(a) = 0$, and $h(z) = (z - a)^2 f(z)$ in $\Omega - \{a\}$. Our boundedness assumption shows that $h'(a) = 0$. Since h is evidently differentiable at every other point of Ω , we have $h \in H(\Omega)$, so

$$h(z) = \sum_{n=2}^{\infty} c_n (z - a)^n \quad (z \in D(a; r)).$$

We obtain the desired holomorphic extension of f by setting $f(a) = c_2$, for then

$$f(z) = \sum_{n=0}^{\infty} c_{n+2} (z - a)^n \quad (z \in D(a; r)). \quad //$$

10.21 Theorem If $a \in \Omega$ and $f \in H(\Omega - \{a\})$, then one of the following three cases must occur:

- (a) f has a removable singularity at a .
- (b) There are complex numbers c_1, \dots, c_m , where m is a positive integer and $c_m \neq 0$, such that

$$f(z) - \sum_{k=1}^m \frac{c_k}{(z - a)^k}$$

has a removable singularity at a .

- (c) If $r > 0$ and $D(a; r) \subset \Omega$, then $f(D'(a; r))$ is dense in the plane.

In case (b), f is said to have a *pole of order m* at a . The function

$$\sum_{k=1}^m c_k (z - a)^{-k},$$

a polynomial in $(z - a)^{-1}$, is called the *principal part* of f at a . It is clear in this situation that $|f(z)| \rightarrow \infty$ as $z \rightarrow a$.

In case (c), f is said to have an *essential singularity* at a . A statement equivalent to (c) is that to each complex number w there corresponds a sequence $\{z_n\}$ such that $z_n \rightarrow a$ and $f(z_n) \rightarrow w$ as $n \rightarrow \infty$.

PROOF Suppose (c) fails. Then there exist $r > 0$, $\delta > 0$, and a complex number w such that $|f(z) - w| > \delta$ in $D'(a; r)$. Let us write D for $D(a; r)$ and D' for $D'(a; r)$. Define

$$g(z) = \frac{1}{f(z) - w} \quad (z \in D'). \quad (1)$$

Then $g \in H(D')$ and $|g| < 1/\delta$. By Theorem 10.20, g extends to a holomorphic function in D .

If $g(a) \neq 0$, (1) shows that f is bounded in $D'(a; \rho)$ for some $\rho > 0$. Hence (a) holds, by Theorem 10.20.

If g has a zero of order $m \geq 1$ at a , Theorem 10.18 shows that

$$g(z) = (z - a)^m g_1(z) \quad (z \in D), \quad (2)$$

where $g_1 \in H(D)$ and $g_1(a) \neq 0$. Also, g_1 has no zero in D' , by (1). Put $h = 1/g_1$ in D . Then $h \in H(D)$, h has no zero in D , and

$$f(z) - w = (z - a)^{-m} h(z) \quad (z \in D'). \quad (3)$$

But h has an expansion of the form

$$h(z) = \sum_{n=0}^{\infty} b_n (z - a)^n \quad (z \in D), \quad (4)$$

with $b_0 \neq 0$. Now (3) shows that (b) holds, with $c_k = b_{m-k}$, $k = 1, \dots, m$.

This completes the proof. ////

We shall now exploit the fact that the restriction of a power series $\sum c_n(z - a)^n$ to a circle with center at a is a trigonometric series.

10.22 Theorem If

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \quad (z \in D(a; R)) \quad (1)$$

and if $0 < r < R$, then

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(a + re^{i\theta})|^2 d\theta. \quad (2)$$

The proof is routine since

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{|f(a + re^{i\theta})|^2}_{\sum_{n=0}^{\infty} c_n r^n e^{2in\theta}} d\theta = \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \int_{-\pi}^{\pi} c_n \bar{c}_{n'} r^{2n} e^{in\theta} e^{-in'\theta} d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n}$$

Here are some consequences:

Théorème 6: Liouville's theorem

Every bounded entire function is constant.

证明. Suppose f entire, $f = \sum c_n z^n$ for all z , $|f(z)| < M$ for all z . Then

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} < M^2$$

for all $r > 0$, which is possible only if $c_n = 0, \forall n$. \square

10.24 The Maximum Modulus Theorem Suppose Ω is a region, $f \in H(\Omega)$, and $D(a; r) \subset \Omega$. Then

$$|f(a)| \leq \max_{\theta} |f(a + re^{i\theta})|. \quad (1)$$

Equality occurs in (1) if and only if f is constant in Ω .

Consequently, $|f|$ has no local maximum at any point of Ω , unless f is constant.

PROOF Assume that $|f(a + re^{i\theta})| \leq |f(a)|$ for all real θ . In the notation of Theorem 10.22 it follows then that

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} \leq |f(a)|^2 = |c_0|^2.$$

Hence $c_1 = c_2 = c_3 = \dots = 0$, which implies that $f(z) = f(a)$ in $D(a; r)$. Since Ω is connected, Theorem 10.18 shows that f is constant in Ω . \square

Corollary Under the same hypotheses,

$$|f(a)| \geq \min_{\theta} |f(a + re^{i\theta})| \quad (2)$$

iff f has no zero in $D(a; r)$.

The fundamental algebra theorem

10.25 Theorem If n is a positive integer and

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0,$$

where a_0, \dots, a_{n-1} are complex numbers, then P has precisely n zeros in the plane.

Of course, these zeros are counted according to their multiplicities: A zero of order m , say, is counted as m zeros. This theorem contains the fact that the complex field is algebraically closed, i.e., that every nonconstant polynomial with complex coefficients has at least one complex zero.

PROOF Choose $r > 1 + 2|a_0| + |a_1| + \cdots + |a_{n-1}|$. Then

$$|P(re^{i\theta})| > |P(0)| \quad (0 \leq \theta \leq 2\pi).$$

If P had no zeros, then the function $f = 1/P$ would be entire and would satisfy $|f(0)| > |f(re^{i\theta})|$ for all θ , which contradicts the maximum modulus theorem. Thus $P(z_1) = 0$ for some z_1 . Consequently, there is a polynomial Q , of degree $n-1$, such that $P(z) = (z - z_1)Q(z)$. The proof is completed by induction on n . ////

Uniformly convergence

10.27 Definition A sequence $\{f_j\}$ of functions in Ω is said to converge to f uniformly on compact subsets of Ω if to every compact $K \subset \Omega$ and to every $\epsilon > 0$ there corresponds an $N = N(K, \epsilon)$ such that $|f_j(z) - f(z)| < \epsilon$ for all $z \in K$ if $j > N$.

For instance, the sequence $\{z^n\}$ converges to 0 uniformly on compact subsets of $D(0; 1)$, but the convergence is not uniform in $D(0; 1)$.

利用柯西积分公式可以给出如下定理的证明.

10.28 Theorem Suppose $f_j \in H(\Omega)$, for $j = 1, 2, 3, \dots$, and $f_j \rightarrow f$ uniformly on compact subsets of Ω . Then $f \in H(\Omega)$, and $f'_j \rightarrow f'$ uniformly on compact subsets of Ω .

Open mapping theorem

10.30 Theorem Suppose $\varphi \in H(\Omega)$, $z_0 \in \Omega$, and $\varphi'(z_0) \neq 0$. Then Ω contains a neighborhood V of z_0 such that

- (a) φ is one-to-one in V ,
- (b) $W = \varphi(V)$ is an open set, and
- (c) if $\psi: W \rightarrow V$ is defined by $\psi(\varphi(z)) = z$, then $\psi \in H(W)$.

Thus $\varphi: V \rightarrow W$ has a holomorphic inverse.

Every nonconstant holomorphic function in a region is locally of the form $\pi_m \circ \varphi$ ($\pi_m: z \mapsto z^m$), except for an additive constant.

10.32 Theorem Suppose Ω is a region, $f \in H(\Omega)$, f is not constant, $z_0 \in \Omega$, and $w_0 = f(z_0)$. Let m be the order of the zero which the function $f - w_0$ has at z_0 .

Then there exists a neighborhood V of z_0 , $V \subset \Omega$, and there exists $\varphi \in H(V)$, such that

- (a) $f(z) = w_0 + [\varphi(z)]^m$ for all $z \in V$,
- (b) φ' has no zero in V and φ is an invertible mapping of V onto a disc $D(0; r)$.

Thus $f - w_0 = \pi_m \circ \varphi$ in V . It follows that f is an exactly m -to-1 mapping of $V - \{z_0\}$ onto $D'(w_0; r^m)$, and that each $w_0 \in f(\Omega)$ is an interior point of $f(\Omega)$. Hence $f(\Omega)$ is open.

PROOF Without loss of generality we may assume that Ω is a convex neighborhood of z_0 which is so small that $f(z) \neq w_0$ if $z \in \Omega - \{z_0\}$. Then

$$f(z) - w_0 = (z - z_0)^m g(z) \quad (z \in \Omega) \quad (1)$$

for some $g \in H(\Omega)$ which has no zero in Ω . Hence $g'/g \in H(\Omega)$. By Theorem 10.14, $g'/g = h'$ for some $h \in H(\Omega)$. The derivative of $g \cdot \exp(-h)$ is 0 in Ω . If h is modified by the addition of a suitable constant, it follows that $g = \exp(h)$. Define

$$\varphi(z) = (z - z_0) \exp \frac{h(z)}{m} \quad (z \in \Omega). \quad (2)$$

Then (a) holds, for all $z \in \Omega$.

Also, $\varphi(z_0) = 0$ and $\varphi'(z_0) \neq 0$. The existence of an open set V that satisfies (b) follows now from Theorem 10.30. This completes the proof. ////

The next theorem is really contained in the preceding results, but it seems advisable to state it explicitly.

10.33 Theorem Suppose Ω is a region, $f \in H(\Omega)$, and f is one-to-one in Ω . Then $f'(z) \neq 0$ for every $z \in \Omega$, and the inverse of f is holomorphic.

PROOF If $f'(z_0)$ were 0 for some $z_0 \in \Omega$, the hypotheses of Theorem 10.32 would hold with some $m > 1$, so that f would be m -to-1 in some deleted neighborhood of z_0 . Now apply part (c) of Theorem 10.30. ////

Note that the converse of Theorem 10.33 is false: If $f(z) = e^z$, then $f'(z) \neq 0$ for every z , but f is not one-to-one in the whole complex plane.

Global Cauchy Theorem

10.34 Chains and Cycles Suppose $\gamma_1, \dots, \gamma_n$ are paths in the plane, and put $K = \gamma_1^* \cup \dots \cup \gamma_n^*$. Each γ_i induces a linear functional $\tilde{\gamma}_i$ on the vector space $C(K)$, by the formula

$$\tilde{\gamma}_i(f) = \int_{\gamma_i} f(z) dz. \quad (1)$$

Define

$$\tilde{\Gamma} = \tilde{\gamma}_1 + \dots + \tilde{\gamma}_n. \quad (2)$$

Explicitly, $\tilde{\Gamma}(f) = \tilde{\gamma}_1(f) + \dots + \tilde{\gamma}_n(f)$ for all $f \in C(K)$. The relation (2) suggests that we introduce a “formal sum”

$$\Gamma = \gamma_1 + \dots + \gamma_n \quad (3)$$

and define

$$\int_{\Gamma} f(z) dz = \tilde{\Gamma}(f). \quad (4)$$

Then (3) is merely an abbreviation for the statement

$$\int_{\Gamma} f(z) dz = \sum_{i=1}^n \int_{\gamma_i} f(z) dz \quad (f \in C(K)). \quad (5)$$

Note that (5) serves as the definition of its left side.

The objects Γ so defined are called *chains*. If each γ_j in (3) is a *closed path*, then Γ is called a *cycle*. If each γ_j in (3) is a path in some open set Ω , we say that Γ is a *chain in Ω* .

If (3) holds, we define

$$\Gamma^* = \gamma_1^* \cup \cdots \cup \gamma_n^*. \quad (6)$$

If Γ is a cycle and $\alpha \notin \Gamma^*$, we define the *index* of α with respect to Γ by

$$\text{Ind}_\Gamma(\alpha) = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z - \alpha}, \quad (7)$$

just as in Theorem 10.10. Obviously, (3) implies

$$\text{Ind}_\Gamma(\alpha) = \sum_{i=1}^n \text{Ind}_{\gamma_i}(\alpha). \quad (8)$$

If each γ_i in (3) is replaced by its opposite path (see Sec. 10.8), the resulting chain will be denoted by $-\Gamma$. Then

$$\int_{-\Gamma} f(z) dz = - \int_\Gamma f(z) dz \quad (f \in C(\Gamma^*)). \quad (9)$$

In particular, $\text{Ind}_{-\Gamma}(\alpha) = -\text{Ind}_\Gamma(\alpha)$ if Γ is a cycle and $\alpha \notin \Gamma^*$.

Chains can be added and subtracted in the obvious way, by adding or subtracting the corresponding functionals: The statement $\Gamma = \Gamma_1 + \Gamma_2$ means

$$\int_\Gamma f(z) dz = \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz \quad (10)$$

for every $f \in C(\Gamma_1^* \cup \Gamma_2^*)$.

Finally, note that a chain may be represented as a sum of paths in many ways. To say that

$$\gamma_1 \dotplus \cdots \dotplus \gamma_n = \delta_1 \dotplus \cdots \dotplus \delta_k$$

means simply that

$$\sum_i \int_{\gamma_i} f(z) dz = \sum_j \int_{\delta_j} f(z) dz$$

for every f that is continuous on $\gamma_1^* \cup \cdots \cup \gamma_n^* \cup \delta_1^* \cup \cdots \cup \delta_k^*$. In particular, a cycle may very well be represented as a sum of paths that are not closed.

Definition of Chains and its properties

10.35 Cauchy's Theorem Suppose $f \in H(\Omega)$, where Ω is an arbitrary open set in the complex plane. If Γ is a cycle in Ω that satisfies

$$\text{Ind}_\Gamma(\alpha) = 0 \quad \text{for every } \alpha \text{ not in } \Omega, \quad (1)$$

then

$$f(z) \cdot \text{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z} dw \quad \text{for } z \in \Omega - \Gamma^* \quad (2)$$

and

$$\int_\Gamma f(z) dz = 0. \quad (3)$$

If Γ_0 and Γ_1 are cycles in Ω such that

$$\text{Ind}_{\Gamma_0}(\alpha) = \text{Ind}_{\Gamma_1}(\alpha) \quad \text{for every } \alpha \text{ not in } \Omega, \quad (4)$$

then

$$\int_{\Gamma_0} f(z) dz = \int_{\Gamma_1} f(z) dz. \quad (5)$$

Cauchy's Theorem

Homotopy

10.38 Homotopy Suppose γ_0 and γ_1 are closed curves in a topological space X , both with parameter interval $I = [0, 1]$. We say that γ_0 and γ_1 are X -homotopic if there is a continuous mapping H of the unit square $I^2 = I \times I$ into X such that

$$H(s, 0) = \gamma_0(s), \quad H(s, 1) = \gamma_1(s), \quad H(0, t) = H(1, t) \quad (1)$$

for all $s \in I$ and $t \in I$. Put $\gamma_t(s) = H(s, t)$. Then (1) defines a one-parameter family of closed curves γ_t in X , which connects γ_0 and γ_1 . Intuitively, this means that γ_0 can be continuously deformed to γ_1 , within X .

If γ_0 is X -homotopic to a constant mapping γ_1 (i.e., if γ_1^* consists of just one point), we say that γ_0 is null-homotopic in X . If X is connected and if every closed curve in X is null-homotopic, X is said to be *simply connected*.

For example, every convex region Ω is simply connected. To see this, let γ_0 be a closed curve in Ω , fix $z_1 \in \Omega$, and define

$$H(s, t) = (1-t)\gamma_0(s) + tz_1 \quad (0 \leq s \leq 1, \quad 0 \leq t \leq 1). \quad (2)$$

Theorem 10.40 will show that condition (4) of Cauchy's theorem 10.35 holds whenever Γ_0 and Γ_1 are Ω -homotopic closed paths. As a special case of this, note that condition (1) of Theorem 10.35 holds for every closed path Γ in Ω if Ω is simply connected.

10.39 Lemma If γ_0 and γ_1 are closed paths with parameter interval $[0, 1]$, if α is a complex number, and if

$$|\gamma_1(s) - \gamma_0(s)| < |\alpha - \gamma_0(s)| \quad (0 \leq s \leq 1) \quad (1)$$

then $\text{Ind}_{\gamma_1}(\alpha) = \text{Ind}_{\gamma_0}(\alpha)$.

PROOF Note first that (1) implies that $\alpha \notin \gamma_0^*$ and $\alpha \notin \gamma_1^*$. Hence one can define $\gamma = (\gamma_1 - \alpha)/(\gamma_0 - \alpha)$. Then

$$\frac{\gamma'}{\gamma} = \frac{\gamma'_1}{\gamma_1 - \alpha} - \frac{\gamma'_0}{\gamma_0 - \alpha} \quad (2)$$

and $|1 - \gamma| < 1$, by (1). Hence $\gamma^* \subset D(1; 1)$, which implies that $\text{Ind}_\gamma(0) = 0$. Integration of (2) over $[0, 1]$ now gives the desired result. ////

10.40 Theorem If Γ_0 and Γ_1 are Ω -homotopic closed paths in a region Ω , and if $\alpha \notin \Omega$, then

$$\text{Ind}_{\Gamma_1}(\alpha) = \text{Ind}_{\Gamma_0}(\alpha). \quad (1)$$

Calculus of Residues

10.41 Definition A function f is said to be *meromorphic* in an open set Ω if there is a set $A \subset \Omega$ such that

- (a) A has no limit point in Ω ,
- (b) $f \in H(\Omega - A)$,
- (c) f has a pole at each point of A .

Note that the possibility $A = \emptyset$ is not excluded. Thus every $f \in H(\Omega)$ is meromorphic in Ω .

Defintion of meromorphic function

10.42 The Residue Theorem Suppose f is a meromorphic function in Ω . Let A be the set of points in Ω at which f has poles. If Γ is a cycle in $\Omega - A$ such that

$$\text{Ind}_\Gamma(\alpha) = 0 \quad \text{for all } \alpha \notin \Omega, \quad (1)$$

then

$$\frac{1}{2\pi i} \int_\Gamma f(z) dz = \sum_{a \in A} \text{Res}(f; a) \text{Ind}_\Gamma(a). \quad (2)$$

The residue theorem For $\Gamma = \gamma_1 + \cdots + \gamma_n$, $\text{Ind}_\Gamma(\alpha) = \sum_{k=1}^n \text{Ind}_{\gamma_k}(\alpha)$.

Two typical applications of the residue theorem The first one concerns zeros of holomorphic functions, the second is the evaluation of a certain integral.

10.43 Theorem Suppose γ is a closed path in a region Ω , such that $\text{Ind}_\gamma(\alpha) = 0$ for every α not in Ω . Suppose also that $\text{Ind}_\gamma(\alpha) = 0$ or 1 for every $\alpha \in \Omega - \gamma^*$, and let Ω_1 be the set of all α with $\text{Ind}_\gamma(\alpha) = 1$.

For any $f \in H(\Omega)$ let N_f be the number of zeros of f in Ω_1 , counted according to their multiplicities.

(a) If $f \in H(\Omega)$ and f has no zeros on γ^* then

$$N_f = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \text{Ind}_\Gamma(0) \quad (1)$$

where $\Gamma = f \circ \gamma$.

(b) If also $g \in H(\Omega)$ and

$$|f(z) - g(z)| < |f(z)| \quad \text{for all } z \in \gamma^* \quad (2)$$

then $N_g = N_f$.

Part (b) is usually called Rouché's theorem. It says that two holomorphic functions have the same number of zeros in Ω_1 if they are close together on the boundary of Ω_1 , as specified by (2).

PROOF Put $\varphi = f'/f$, a meromorphic function in Ω . If $a \in \Omega$ and f has a zero of order $m = m(a)$ at a , then $f(z) = (z - a)^m h(z)$, where h and $1/h$ are holomorphic in some neighborhood V of a . In $V - \{a\}$,

$$\varphi(z) = \frac{f'(z)}{f(z)} = \frac{m}{z - a} + \frac{h'(z)}{h(z)}. \quad (3)$$

Thus

$$\text{Res}(\varphi; a) = m(a). \quad (4)$$

Note. f'/f 在 a 处的极点阶数恰恰等于其在 a 处的留数.

Let $A = \{a \in \Omega_1 : f(a) = 0\}$. If our assumptions about the index of γ are combined with the residue theorem one obtains

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{a \in A} \operatorname{Res}(\varphi; a) = \sum_{a \in A} m(a) = N_f.$$

This proves one half of (1). The other half is a matter of direct computation:

$$\begin{aligned} \operatorname{Ind}_{\Gamma}(0) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{dz}{z} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\Gamma'(s)}{\Gamma(s)} ds \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f'(\gamma(s))}{f(\gamma(s))} \gamma'(s) ds = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz. \end{aligned}$$

The parameter interval of γ was here taken to be $[0, 2\pi]$.

Next, (2) shows that g has no zero on γ^* . Hence (1) holds with g in place of f . Put $\Gamma_0 = g \circ \gamma$. Then it follows from (1), (2), and Lemma 10.39 that

$$N_g = \operatorname{Ind}_{\Gamma_0}(0) = \operatorname{Ind}_{\Gamma}(0) = N_f. \quad //$$

7.6 复变函数的积分

其实就是关于某个参数 $\gamma : [0, 1] \rightarrow C$ 进行积分

$$\int_{\gamma} f(z) dz = \int_0^1 f(\gamma(t)) \cdot \gamma'(t) dt$$

柯西积分公式：若 f 在一个包含圆周 C 的区域内全纯，那么对于任意 C 内的 z 点，有

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta \quad (7.1)$$

这个公式具有极其广泛的应用场景，比如计算导数，估计不等式....

eq. (7.1) 的证明思路就是类似于调和方程的平均值公式证明

$$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{C_{\epsilon}} \frac{f(\zeta)}{\zeta - z} d\zeta \approx -f(z) \int_0^{2\pi} \frac{\epsilon e^{it}}{\epsilon e^{it}} dt = -2\pi i f(z)$$

对于 eq. (7.1) 两边同时求 n 阶导数得到

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \quad (7.2)$$

这只需要根据导数的定义归纳验证导数的存在性即可。

柯西不等式 eq. (7.3) 是 eq. (7.2) 的直接推论

$$|f^{(n)}(z_0)| \leq \frac{n! \cdot \|f\|_C}{R^n} \quad \text{where } \|f\|_C := \sup_{z \in C} |f(z)| \quad (7.3)$$

进一步我们有复变函数的幂级数表示

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n d\zeta \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) \cdot (z - z_0)^n \end{aligned} \quad (7.4)$$

同时还有 Liouville 定理：若 f 全纯且有界，那么 f 是常值函数。 (直接利用 eq. (7.2))

$$|f'(z)| = \left| \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \right| \leq \frac{1}{2\pi} \int_{C_R} \frac{M}{R^2} dz = \frac{M}{R} \rightarrow 0 \quad \text{as } R \rightarrow \infty$$

然后可以证明代数基本定理。

根据幂级数表示 eq. (7.4) 可以证明复变函数的零点集是离散的。

7.6.1 积分估计

参见《复变函数论》(第五版) 学习指导书 (钟玉泉编).

已知 Jordan 不等式

$$\frac{2\theta}{\pi} \leq \sin \theta \leq \theta \quad \theta \in \left[0, \frac{\pi}{2}\right]$$

证明：

$$\left| \int_C e^{iz} dz \right| < \pi \quad (7.5)$$

其中 C 为圆周 $|z| = R$ 的上半圆周从 R 到 $-R$.

选取参数化 $C : z = Re^{i\theta}, \theta \in [0, \pi]$. 于是

$$\begin{aligned} \left| \int_C e^{iz} dz \right| &\leq \int_C |e^{iz}| |dz| = \int_0^\pi e^{-R \sin \theta} R d\theta \\ &= 2 \int_0^{\pi/2} e^{-R \sin \theta} R d\theta \leq 2 \int_0^{\pi/2} e^{-2R\theta/\pi} R d\theta \\ &= -\pi e^{-2R\theta/\pi} \Big|_0^{\pi/2} = \pi(1 - e^{-R}) < \pi \end{aligned}$$

Remarque 18. 不等式 eq. (7.5) 是一个重要不等式。可以用来放缩很多围道积分 (用来证明 Jordan 引理)

Exercice 1

若 $I_r = \int_{C_r} \frac{e^{iz}}{z} dz$, 其中 C_r 是从 r 到 $-r$ 沿 $|z|=r$ 的上半圆周, 证明:

$$\lim_{r \rightarrow +\infty} I_r = 0 \quad \lim_{r \rightarrow 0} I_r = \pi i$$

证明. $r \rightarrow +\infty$ 时, 直接利用 eq. (7.5) 放缩

$$|I_r| \leq \frac{\pi}{r} \rightarrow 0$$

$r \rightarrow 0$ 时, 利用留数定理显然, 或者直接放缩估计 $|I_r - \pi i| \rightarrow 0$.

□

7.6.2 柯西积分定理

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta \quad z \text{ 在 } C \text{ 内部}$$

柯西积分定理对于解析函数 $f(z)$ 的实部和虚部都不成立.

对于一般的题目, 正常计算即可. 注意先将分母有理化, 再判断奇点是否在积分曲线内部.

下面介绍一种思想:

例 3.2.8 计算积分 $\int_{-i}^i \frac{dz}{z}$, 积分路径沿顶点为 $(0, -1), (1, -1), (1, 1), (0, 1)$ 的四边形的三边(图 3.2.4).

分析 如果沿着题意提出的路径计算积分, 计算量将十分繁重. 但是存在一个单连通区域 D (图 3.2.4), 被积函数 $f(z) = \frac{1}{z}$ 在 D 内解析, 并且 D 包含 $-i$ 和 i , 在 D 内积分就与路径无关, 因此, 我们可以沿另外的路径计算这个积分. 为简单起见, 我们取位于原点右边的连接 $-i$ 和 i 的单位半圆周.

解 设 $z = e^{i\theta}$, 则 $\frac{1}{z} = e^{-i\theta}$, $dz = ie^{i\theta} d\theta$. 因此,

$$\int_{-i}^i \frac{1}{z} dz = i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \pi i.$$

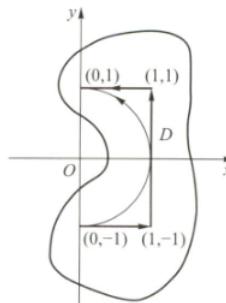


图 3.2.4

注 D 不包含原点, 这就不允许我们用位于原点左边的单位半圆周来代替给定的路径. 事实上, 在这样的半圆周上, 有

$$\int_{-i}^i \frac{1}{z} dz = i \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} d\theta = -\pi i.$$

因此, 必须小心, 确实使给定的路径与实际选作计算的路径都处于被积函数的解析区内, 并且这个区域必须是单连通的.

7.7 留数计算

参见《积分的方法与技巧》·金玉明; A first course in complex analysis with applications by Dennis Zill

留数的表示

$$\text{res}_b f = \frac{1}{2\pi i} \oint_C f(z) dz$$

留数定理由柯西积分公式得到

$$\oint_C f(z) dz = \sum_{k=1}^n \oint_{C_\epsilon(b_k)} f(z) dz = \sum_{k=1}^n 2\pi i \cdot \text{res}_{b_k} f$$

无穷远点的留数定理积分曲线方向相反 (考虑黎曼球, 这是显然的)

$$\text{res}_\infty f = \frac{1}{2\pi i} \oint_{C^-} f(z) dz = -\frac{1}{2\pi i} \oint_C f(z) dz$$

7.7.1 留数的计算方法

留数就是 $f(z)$ 的 Laurent 展开式中系数 a_{-1} , 对于极点 b , 有下面较为直接的方法, 不需要 Laurant 展开.

Remarque 19. Laurant 展开就是按照公式计算

$$\frac{1}{2\pi i} \oint_C f(z) dz$$

单极点

在 b 的邻域中有

$$f(z) = \frac{a_{-1}}{z-b} + a_0 + a_1(z-b) + a_2(z-b)^2 + \dots$$

于是

$$\text{res}_b f = \lim_{z \rightarrow b} (z-b)f(z)$$

可表为 $\varphi(z)/\psi(z)$ φ, ψ 在 b 点解析, 且 $\varphi(b) \neq 0, \psi(b) = 0$, 但是 $\psi'(b) \neq 0$, 那么

$$\text{res}_b f = \frac{\varphi(b)}{\psi'(b)}$$

n 级极点

Laurant 展开式为

$$f(z) = \frac{a_{-n}}{(z-b)^n} + \cdots + \frac{a_{-1}}{z-b} + a_0 + a_1(z-b) + \dots$$

那么

$$(z-b)^n f(z) = a_{-n} + a_{-n+1}(z-b) + \cdots + a_{-1}(z-b)^{n-1} + a_0(z-b)^n + \dots$$

两边 $\frac{d^{n-1}}{dz^{n-1}}$ 得到

$$\lim_{z \rightarrow b} \frac{d^{n-1}}{dz^{n-1}} [(z-b)^n f(z)] = (n-1)! a_{-1}$$

于是

$$\text{res}_b f = \frac{1}{(n-1)!} \cdot \lim_{z \rightarrow b} \frac{d^{n-1}}{dz^{n-1}} [(z-b)^n f(z)]$$

7.7.2 计算定积分

详细计算见《积分的方法与技巧》• 金玉明 p 328

Jordan 引理

Lemme 1: Jordan's Lemma

Let C_R be a semicircular contour in the upper half-plane defined by $z = Re^{i\theta}$ for $0 \leq \theta \leq \pi$. Suppose $f(z)$ is a function that is analytic in the upper half-plane for $|z| > R_0$ (for some constant R_0). If $f(z) \rightarrow 0$ uniformly as $|z| \rightarrow \infty$ for z in the upper half-plane (i.e., $M_R = \max_{z \in C_R} |f(z)| \rightarrow 0$ as $R \rightarrow \infty$), then for any real constant $m > 0$,

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{imz} dz = 0$$

A similar statement holds for a semicircular contour in the lower half-plane if $m < 0$.

证明. Let $m > 0$, denote

$$I_R := \int_0^\pi f(Re^{i\theta}) e^{im(Re^{i\theta})} (iRe^{i\theta}) d\theta$$

Estimate the value,

$$\begin{aligned}
 |I_R| &= \left| \int_0^\pi f(Re^{i\theta}) e^{im(Re^{i\theta})} (iRe^{i\theta}) d\theta \right| \\
 &\leq \int_0^\pi \underbrace{|f(Re^{i\theta})|}_{\leq M_R} \cdot \underbrace{|e^{im(Re^{i\theta})}|}_{=\exp\{-mR \sin \theta\}} \cdot \underbrace{|iRe^{i\theta}|}_{=R} d\theta \\
 &\leq 2RM_R \int_0^{\pi/2} e^{-mR \sin \theta} d\theta \\
 &\stackrel{\sin \theta \geq \frac{2}{\pi}\theta}{\leq} 2RM_R \int_0^{\pi/2} e^{-\frac{2}{\pi}mR\theta} d\theta \\
 &= \frac{1}{\frac{2}{\pi}mR} (1 - e^{-mR}) \\
 &= \frac{\pi M_R (1 - e^{-mR})}{m} \rightarrow 0 \quad \text{as } R \rightarrow \infty
 \end{aligned}$$

□

7.8 围道积分

参见 A first course in complex analysis with applications by Dennis Zill, Section 6.6

7.8.1 Basic Integrations

Given the integrals of the form

$$\int_0^{2\pi} F(\cos \theta, \sin \theta) d\theta \quad (7.6)$$

To convert it to a complex integral on $|z| = 1$, we parametrize the contour by $z = e^{i\theta}, 0 \leq \theta \leq 2\pi$. We can then write

$$dz = ie^{i\theta} d\theta, \quad \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

i.e.

$$d\theta = \frac{dz}{iz}, \quad \cos \theta = \frac{1}{2}(z + z^{-1}), \quad \sin \theta = \frac{1}{2i}(z - z^{-1}).$$

Then eq. (7.6) becomes

$$\oint_C F\left(\frac{1}{2}(z + z^{-1}), \frac{1}{2i}(z - z^{-1})\right) \frac{dz}{iz}$$

where C is the unit circle $|z| = 1$.

EXAMPLE 1 A Real Trigonometric Integral

$$\text{Evaluate } \int_0^{2\pi} \frac{1}{(2 + \cos \theta)^2} d\theta.$$

Solution When we use the substitutions given in (4), the given trigonometric integral becomes the contour integral

$$\oint_C \frac{1}{(2 + \frac{1}{2}(z + z^{-1}))^2} \frac{dz}{iz} = \oint_C \frac{1}{\left(2 + \frac{z^2 + 1}{2z}\right)^2} \frac{dz}{iz}.$$

Carrying out the algebraic simplification of the integrand then yields

$$\frac{4}{i} \oint_C \frac{z}{(z^2 + 4z + 1)^2} dz.$$

From the quadratic formula we can factor the polynomial $z^2 + 4z + 1$ as $z^2 + 4z + 1 = (z - z_1)(z - z_2)$, where $z_1 = -2 - \sqrt{3}$ and $z_2 = -2 + \sqrt{3}$. Thus, the integrand can be written

$$\frac{z}{(z^2 + 4z + 1)^2} = \frac{z}{(z - z_1)^2(z - z_2)^2}.$$

Because only z_2 is inside the unit circle C , we have

$$\oint_C \frac{z}{(z^2 + 4z + 1)^2} dz = 2\pi i \operatorname{Res}(f(z), z_2).$$

To calculate the residue, we first note that z_2 is a pole of order 2 and so we use (2) of Section 6.5:

$$\begin{aligned} \operatorname{Res}(f(z), z_2) &= \lim_{z \rightarrow z_2} \frac{d}{dz} (z - z_2)^2 f(z) = \lim_{z \rightarrow z_2} \frac{d}{dz} \frac{z}{(z - z_1)^2} \\ &= \lim_{z \rightarrow z_2} \frac{-z - z_1}{(z - z_1)^3} = \frac{1}{6\sqrt{3}}. \end{aligned}$$

Hence, $\frac{4}{i} \oint_C \frac{z}{(z^2 + 4z + 1)^2} dz = \frac{4}{i} \cdot 2\pi i \operatorname{Res}(f(z), z_1) = \frac{4}{i} \cdot 2\pi i \cdot \frac{1}{6\sqrt{3}}$

and, finally, $\int_0^{2\pi} \frac{1}{(2 + \cos \theta)^2} d\theta = \frac{4\pi}{3\sqrt{3}}.$

Cauchy principal value of an integral is defined by

$$\text{P.V.} \int_{-\infty}^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx.$$

If a integral diverges, it may still possess a Cauchy principal value.

EXAMPLE 2 Cauchy P.V. of an Improper Integral

Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx$.

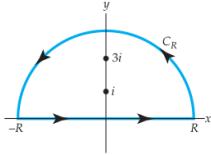


Figure 6.12 Contour for Example 2

Solution Let $f(z) = 1/(z^2+1)(z^2+9)$. Since

$$(z^2+1)(z^2+9) = (z-i)(z+i)(z-3i)(z+3i),$$

we take C be the closed contour consisting of the interval $[-R, R]$ on the x -axis and the semicircle C_R of radius $R > 3$. As seen from Figure 6.12,

$$\begin{aligned} \oint_C \frac{1}{(z^2+1)(z^2+9)} dz &= \int_{-R}^R \frac{1}{(x^2+1)(x^2+9)} dx + \int_{C_R} \frac{1}{(z^2+1)(z^2+9)} dz \\ &= I_1 + I_2 \end{aligned}$$

$$\text{and } I_1 + I_2 = 2\pi i [\operatorname{Res}(f(z), i) + \operatorname{Res}(f(z), 3i)].$$

At the simple poles $z = i$ and $z = 3i$ we find, respectively,

$$\operatorname{Res}(f(z), i) = \frac{1}{16i} \quad \text{and} \quad \operatorname{Res}(f(z), 3i) = -\frac{1}{48i},$$

$$\text{so that } I_1 + I_2 = 2\pi i \left[\frac{1}{16i} + \left(-\frac{1}{48i} \right) \right] = \frac{\pi}{12}. \quad (15)$$

We now want to let $R \rightarrow \infty$ in (15). Before doing this, we use the inequality (10) of Section 1.2 to note that on the contour C_R ,

$$|(z^2+1)(z^2+9)| = |z^2+1| \cdot |z^2+9| \geq ||z^2|-1| \cdot ||z^2|-9| = (R^2-1)(R^2-9).$$

Since the length L of the semicircle is πR , it follows from the ML -inequality, Theorem 5.3 of Section 5.2, that

$$|I_2| = \left| \int_{C_R} \frac{1}{(z^2+1)(z^2+9)} dz \right| \leq \frac{\pi R}{(R^2-1)(R^2-9)}.$$

This last result shows that $|I_2| \rightarrow 0$ as $R \rightarrow \infty$, and so we conclude that $\lim_{R \rightarrow \infty} I_2 = 0$. It follows from (15) that $\lim_{R \rightarrow \infty} I_1 = \pi/12$; in other words,

$$\lim_{R \rightarrow \infty} \int_{-R}^R \frac{1}{(x^2+1)(x^2+9)} dx = \frac{\pi}{12} \quad \text{or} \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx = \frac{\pi}{12}.$$

It is often tedious to have to show that the contour integral along C_R approaches zero as $R \rightarrow \infty$. Sufficient conditions under which this behavior is always true are summarized in the next theorem.

Theorem 6.17 Behavior of Integral as $R \rightarrow \infty$

Suppose $f(z) = \frac{p(z)}{q(z)}$ is a rational function, where the degree of $p(z)$ is n and the degree of $q(z)$ is $m \geq n+2$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, then $\int_{C_R} f(z) dz \rightarrow 0$ as $R \rightarrow \infty$.

Another powerful lemma is Lemme 1.

Next we calculate

$$\int_{-\infty}^{\infty} F(x) \sin \alpha x dx, \quad \int_{-\infty}^{\infty} F(x) \cos \alpha x dx$$

EXAMPLE 4 Using Symmetry

Evaluate the Cauchy principal value of $\int_0^\infty \frac{x \sin x}{x^2 + 9} dx$.

Solution First note that the limits of integration in the given integral are not from $-\infty$ to ∞ as required by the method just described. This can be remedied by observing that since the integrand is an even function of x (verify), we can write

$$\int_0^\infty \frac{x \sin x}{x^2 + 9} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x \sin x}{x^2 + 9} dx. \quad (17)$$

With $\alpha = 1$ we now form the contour integral

$$\oint_C \frac{z}{z^2 + 9} e^{iz} dz,$$

where C is the same contour shown in Figure 6.12. By Theorem 6.16,

$$\int_{C_R} \frac{z}{z^2 + 9} e^{iz} dz + \int_{-R}^R \frac{x}{x^2 + 9} e^{ix} dx = 2\pi i \operatorname{Res}(f(z)e^{iz}, 3i),$$

where $f(z) = z/(z^2 + 9)$, and

$$\operatorname{Res}\left(f(z)e^{iz}, 3i\right) = \left. \frac{ze^{iz}}{2z} \right|_{z=3i} = \frac{e^{-3}}{2}$$

from (4) of Section 6.5. Then, from Theorem 6.18 we conclude $\int_{C_R} f(z)e^{iz} dz \rightarrow 0$ as $R \rightarrow \infty$, and so

$$\operatorname{P.V.} \int_{-\infty}^\infty \frac{x}{x^2 + 9} e^{ix} dx = 2\pi i \left(\frac{e^{-3}}{2} \right) = \frac{\pi}{e^3} i.$$

But by (16),

$$\int_{-\infty}^\infty \frac{x}{x^2 + 9} e^{ix} dx = \int_{-\infty}^\infty \frac{x \cos x}{x^2 + 9} dx + i \int_{-\infty}^\infty \frac{x \sin x}{x^2 + 9} dx = \frac{\pi}{e^3} i.$$

Equating real and imaginary parts in the last line gives the bonus result

$$\operatorname{P.V.} \int_{-\infty}^\infty \frac{x \cos x}{x^2 + 9} dx = 0 \quad \text{along with} \quad \operatorname{P.V.} \int_{-\infty}^\infty \frac{x \sin x}{x^2 + 9} dx = \frac{\pi}{e^3}. \quad (18)$$

Finally, in view of the fact that the integrand is an even function, we obtain the value of the prescribed integral:

$$\int_0^\infty \frac{x \sin x}{x^2 + 9} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x \sin x}{x^2 + 9} dx = \frac{\pi}{2e^3}.$$

Indented contour is of great use.

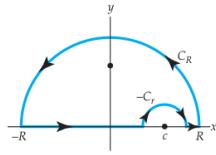


Figure 6.13 Indented contour

Indented Contours The improper integrals of forms (2) and (3) that we have considered up to this point were continuous on the interval $(-\infty, \infty)$. In other words, the complex function $f(z) = p(z)/q(z)$ did not have poles on the real axis. In the situation where f has poles on the real axis, we must modify the procedure illustrated in Examples 2–4. For example, to evaluate $\int_{-\infty}^{\infty} f(x) dx$ by residues when $f(z)$ has a pole at $z = c$, where c is a real number, we use an **indented contour** as illustrated in Figure 6.13. The symbol C_r denotes a semicircular contour centered at $z = c$ and oriented in the *positive* direction. The next theorem is important to this discussion.

Theorem 6.19 Behavior of Integral as $r \rightarrow 0$

Suppose f has a simple pole $z = c$ on the real axis. If C_r is the contour defined by $z = c + re^{i\theta}$, $0 \leq \theta \leq \pi$, then

$$\lim_{r \rightarrow 0} \int_{C_r} f(z) dz = \pi i \operatorname{Res}(f(z), c).$$

Proof Since f has a simple pole at $z = c$, its Laurent series is

$$f(z) = \frac{a_{-1}}{z - c} + g(z),$$

where $a_{-1} = \operatorname{Res}(f(z), c)$ and g is analytic at the point c . Using the Laurent series and the parametrization of C_r we have

$$\int_{C_r} f(z) dz = a_{-1} \int_0^\pi \frac{ire^{i\theta}}{re^{i\theta}} d\theta + ir \int_0^\pi g(c + re^{i\theta}) e^{i\theta} d\theta = I_1 + I_2. \quad (19)$$

First, we see that

$$I_1 = a_{-1} \int_0^\pi \frac{ire^{i\theta}}{re^{i\theta}} d\theta = a_{-1} \int_0^\pi i d\theta = \pi i a_{-1} = \pi i \operatorname{Res}(f(z), c).$$

Next, g is analytic at c , and so it is continuous at this point and bounded in a neighborhood of the point; that is, there exists an $M > 0$ for which $|g(c + re^{i\theta})| \leq M$. Hence,

$$|I_2| = \left| ir \int_0^\pi g(c + re^{i\theta}) d\theta \right| \leq r \int_0^\pi M d\theta = \pi r M.$$

It follows from this last inequality that $\lim_{r \rightarrow 0} |I_2| = 0$ and consequently $\lim_{r \rightarrow 0} I_2 = 0$. By taking the limit of (19) as $r \rightarrow 0$, the theorem is proved. 

EXAMPLE 5 Using an Indented Contour

Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 - 2x + 2)} dx$.

Solution Since the integral is of the type given in (3), we consider the contour integral

$$\oint_C \frac{e^{iz}}{z(z^2 - 2z + 2)} dz.$$

The function $f(z) = 1/z(z^2 - 2z + 2)$ has a pole at $z = 0$ and at $z = 1 + i$ in the upper half-plane. The contour C , shown in Figure 6.14, is indented at the origin. Adopting an obvious condensed notation, we have

$$\oint_C = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_r} + \int_r^R = 2\pi i \operatorname{Res}(f(z)e^{iz}, 1+i), \quad (20)$$

where $\int_{-C_r} = -\int_{C_r}$. If we take the limits of (20) as $R \rightarrow \infty$ and as $r \rightarrow 0$, it follows from Theorems 6.18 and 6.19 that

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2 - 2x + 2)} dx - \pi i \operatorname{Res}(f(z)e^{iz}, 0) = 2\pi i \operatorname{Res}(f(z)e^{iz}, 1+i).$$

Now,

$$\operatorname{Res}(f(z)e^{iz}, 0) = \frac{1}{2} \quad \text{and} \quad \operatorname{Res}(f(z)e^{iz}, 1+i) = -\frac{e^{-1+i}}{4} (1+i).$$

Therefore,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2 - 2x + 2)} dx = \pi i \left(\frac{1}{2} \right) + 2\pi i \left(-\frac{e^{-1+i}}{4} (1+i) \right).$$

Using $e^{-1+i} = e^{-1}(\cos 1 + i \sin 1)$, simplifying, and then equating real and imaginary parts, we get from the last equality

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos x}{x(x^2 - 2x + 2)} dx = \frac{\pi}{2} e^{-1} (\sin 1 + \cos 1)$$

$$\text{and} \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 - 2x + 2)} dx = \frac{\pi}{2} [1 + e^{-1} (\sin 1 - \cos 1)].$$

7.8.2 Integration along a Branch cut

In the next discussion we examine integration along a Branch cut (cf section 2.6 and 4.1 in Dennis Zill)

EXAMPLE 6 Integration along a Branch Cut

Evaluate $\int_0^\infty \frac{1}{\sqrt{x}(x+1)} dx$.

Solution First observe that the real integral is improper for two reasons. Notice an infinite discontinuity at $x = 0$ and the infinite limit of integration. Moreover, it can be argued from the facts that the integrand behaves like $x^{-1/2}$ near the origin and like $x^{-3/2}$ as $x \rightarrow \infty$, that the integral converges.

We form the integral $\oint_C \frac{1}{z^{1/2}(z+1)} dz$, where C is the closed contour shown in Figure 6.15 consisting of four components: C_r and C_R are portions of circles, and AB and ED are parallel horizontal line segments running along opposite sides of the branch cut. The integrand $f(z)$ of the contour integral is single valued and analytic on and within C , except for the simple pole at $z = -1 = e^{\pi i}$. Hence we can write

$$\oint_C \frac{1}{z^{1/2}(z+1)} dz = 2\pi i \operatorname{Res}(f(z), -1)$$

$$\text{or } \int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} = 2\pi i \operatorname{Res}(f(z), -1). \quad (23)$$

Despite what is shown in Figure 6.15, it is permissible to think that the line segments AB and ED actually rest on the positive real axis, more precisely, AB coincides with the upper side of the positive real axis for which $\theta = 0$ and ED coincides with the lower side of the positive real axis for which $\theta = 2\pi$. On AB , $z = xe^{0i}$, and on ED , $z = xe^{(0+2\pi)i} = xe^{2\pi i}$, so that

$$\int_{ED} = \int_r^R \frac{(xe^{2\pi i})^{-1/2}}{xe^{2\pi i} + 1} (e^{2\pi i} dx) = - \int_R^r \frac{x^{-1/2}}{x+1} dx = \int_r^R \frac{x^{-1/2}}{x+1} dx \quad (24)$$

$$\text{and } \int_{AB} = \int_r^R \frac{(xe^{0i})^{-1/2}}{xe^{0i} + 1} (e^{0i} dx) = \int_r^R \frac{x^{-1/2}}{x+1} dx. \quad (25)$$

Now with $z = re^{i\theta}$ and $z = Re^{i\theta}$ on C_r and C_R , respectively, it can be shown, by analysis similar to that given in Example 2 and in the proof of Theorem 6.17, that $\int_{C_r} \rightarrow 0$ as $r \rightarrow 0$ and $\int_{C_R} \rightarrow 0$ as $R \rightarrow \infty$. Thus from (23), (24), and (25) we see that

$$\lim_{R \rightarrow \infty} \left[\int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} \right] = 2\pi i \operatorname{Res}(f(z), -1)$$

is the same as

$$2 \int_0^\infty \frac{1}{\sqrt{x}(x+1)} dx = 2\pi i \operatorname{Res}(f(z), -1). \quad (26)$$

Finally, from (4) of Section 6.5,

$$\operatorname{Res}(f(z), -1) = z^{-1/2} \Big|_{z=e^{\pi i}} = e^{-\pi i/2} = -i$$

and so (26) yields the result

$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)} dx = \pi.$$

This technique also goes for $\operatorname{Ln}z$.

7.8.3 The Argument Principle and Rouche's Theorem

Argument Principle

Theorem 6.20 Argument Principle

Let C be a simple closed contour lying entirely within a domain D . Suppose f is analytic in D except at a finite number of poles inside C , and that $f(z) \neq 0$ on C . Then

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N_0 - N_p, \quad (28)$$

where N_0 is the total number of zeros of f inside C and N_p is the total number of poles of f inside C . In determining N_0 and N_p , zeros and poles are counted according to their order or multiplicities.

Proof We start with a reminder that when we use the symbol \oint_C for a contour, this signifies that we are integrating in the positive direction around the closed curve C .

The integrand $f'(z)/f(z)$ in (28) is analytic in and on the contour C except at the points in the interior of C where f has a zero or a pole. If z_0 is a zero of order n of f inside C , then by (5) of Section 6.4 we can write

$f(z) = (z - z_0)^n \phi(z)$, where ϕ is analytic at z_0 and $\phi(z_0) \neq 0$. We differentiate f by the product rule,

$$f'(z) = (z - z_0)^n \phi'(z) + n(z - z_0)^{n-1} \phi(z),$$

and divide this expression by f . In some punctured disk centered at z_0 , we have

$$\frac{f'(z)}{f(z)} = \frac{(z - z_0)^n \phi'(z) + n(z - z_0)^{n-1} \phi(z)}{(z - z_0)^n \phi(z)} = \frac{\phi'(z)}{\phi(z)} + \frac{n}{z - z_0}. \quad (29)$$

The result in (29) shows that the integrand $f'(z)/f(z)$ has a simple pole at z_0 and the residue at that point is

$$\begin{aligned} \text{Res}\left(\frac{f'(z)}{f(z)}, z_0\right) &= \lim_{z \rightarrow z_0} (z - z_0) \left[\frac{\phi'(z)}{\phi(z)} + \frac{n}{z - z_0} \right] \\ &= \lim_{z \rightarrow z_0} \left[(z - z_0) \frac{\phi'(z)}{\phi(z)} + n \right] = 0 + n = n, \end{aligned} \quad (30)$$

which is the order of the zero z_0 .

Now if z_p is a pole of order m of f within C , then by (7) of Section 6.4 we can write $f(z) = g(z)/(z - z_p)^m$, where g is analytic at z_p and $g(z_p) \neq 0$. By differentiating, in this case $f(z) = (z - z_p)^{-m} g(z)$, we have

$$f'(z) = (z - z_p)^{-m} g'(z) - m(z - z_p)^{-m-1} g(z).$$

Therefore, in some punctured disk centered at z_p ,

$$\frac{f'(z)}{f(z)} = \frac{(z - z_p)^{-m} g'(z) - m(z - z_p)^{-m-1} g(z)}{(z - z_p)^{-m} g(z)} = \frac{g'(z)}{g(z)} + \frac{-m}{z - z_p}. \quad (31)$$

We see from (31) that the integrand $f'(z)/f(z)$ has a simple pole at z_p . Proceeding as in (30), we also see that the residue at z_p is equal to $-m$, which is the negative of the order of the pole of f .

Finally, suppose that $z_{0_1}, z_{0_2}, \dots, z_{0_r}$ and $z_{p_1}, z_{p_2}, \dots, z_{p_s}$ are the zeros and poles of f within C and suppose further that the order of the zeros are n_1, n_2, \dots, n_r and that order of the poles are m_1, m_2, \dots, m_s . Then each of these points is a simple pole of the integrand $f'(z)/f(z)$ with corresponding residues n_1, n_2, \dots, n_r and $-m_1, -m_2, \dots, -m_s$. It follows from the residue theorem (Theorem 6.16) that $\oint_C f'(z) dz / f(z)$ is equal to $2\pi i$ times the sum of the residues at the poles:

$$\begin{aligned} \oint_C \frac{f'(z)}{f(z)} dz &= 2\pi i \left[\sum_{k=1}^r \text{Res}\left(\frac{f'(z)}{f(z)}, z_{0_k}\right) + \sum_{k=1}^s \text{Res}\left(\frac{f'(z)}{f(z)}, z_{p_k}\right) \right] \\ &= 2\pi i \left[\sum_{k=1}^r n_k + \sum_{k=1}^s (-m_k) \right] = 2\pi i [N_0 - N_p]. \end{aligned}$$

Dividing by $2\pi i$ establishes (28). 

Why the Name? Why is Theorem 6.20 called *the argument principle*?

This question may have occurred to you since no reference is made in the proof of the theorem to any arguments of complex quantities. But in point of fact there is a relation between the number $N_0 - N_p$ in Theorem 6.20 and $\arg(f(z))$. More precisely,

$$N_0 - N_p = \frac{1}{2\pi} [\text{change in } \arg(f(z)) \text{ as } z \text{ traverses } C \text{ once in the positive direction}].$$

This principle can be easily verified using the simple function $f(z) = z^2$ and the unit circle $|z| = 1$ as the simple closed contour C in the z -plane. Because the function f has a zero of multiplicity 2 within C and no poles, we have $N_0 - N_p = 2$. Now, if C is parametrized by $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, then its image C' in the w -plane under the mapping $w = z^2$ is $w = e^{i2\theta}$, $0 \leq \theta \leq 2\pi$, which is the unit circle $|w| = 1$. As z traverses C once starting at $z = 1$ ($\theta = 0$) and finishing at $z = 1$ ($\theta = 2\pi$), we see $\arg(f(z)) = \arg(w) = 2\theta$ increases from 0 to 4π . Put another way, w traverses or *winds* around the circle $|w| = 1$ twice. Thus,

$$\frac{1}{2\pi} [\text{change in } \arg(f(z)) \text{ as } z \text{ traverses } C \text{ once in the positive direction}] = \frac{1}{2\pi} [4\pi - 0] = 2.$$

Why the name?

Rouche's theorem

Rouche's theorem is a consequence of Argument Principle.

Rouché's Theorem The next result follows as a consequence of the argument principle. The theorem is helpful in determining the number of zeros of an analytic function.

Theorem 6.21 Rouché's Theorem

Let C be a simple closed contour lying entirely within a domain D . Suppose f and g are analytic in D . If the strict inequality $|f(z) - g(z)| < |f(z)|$ holds for all z on C , then f and g have the same number of zeros (counted according to their order or multiplicities) inside C .

Proof We start with the observation that the hypothesis “the inequality $|f(z) - g(z)| < |f(z)|$ holds for all z on C ” indicates that both f and g have

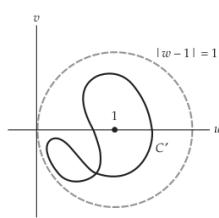


Figure 6.16 Image of C lies within the disk $|w - 1| < 1$.

no zeros on the contour C . From $|f(z) - g(z)| = |g(z) - f(z)|$, we see that by dividing the inequality by $|f(z)|$ we have, for all z on C ,

$$|F(z) - 1| < 1, \quad (32)$$

where $F(z) = g(z)/f(z)$. The inequality in (32) shows that the image C' in the w -plane of the curve C under the mapping $w = F(z)$ is a closed path and must lie within the unit open disk $|w - 1| < 1$ centered at $w = 1$. See Figure 6.16. As a consequence, the curve C' does not enclose $w = 0$, and therefore $1/w$ is analytic in and on C' . By the Cauchy-Goursat theorem,

$$\int_{C'} \frac{1}{w} dw = 0 \quad \text{or} \quad \oint_C \frac{F'(z)}{F(z)} dz = 0, \quad (33)$$

since $w = F(z)$ and $dw = F'(z) dz$. From the quotient rule,

$$F'(z) = \frac{f(z)g'(z) - g(z)f'(z)}{[f(z)]^2},$$

we get

$$\frac{F'(z)}{F(z)} = \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)}.$$

Using the last expression in the second integral in (33) then gives

$$\oint_C \left[\frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)} \right] dz = 0 \quad \text{or} \quad \oint_C \frac{g'(z)}{g(z)} dz = \oint_C \frac{f'(z)}{f(z)} dz.$$

It follows from (28) of Theorem 6.20, with $N_p = 0$, that the number of zeros of g inside C is the same as the number of zeros of f inside C . \diamond

7.8.4 Summing Infinite Series

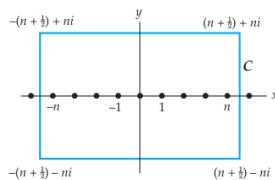


Figure 6.17 Rectangular contour C enclosing poles of (37)

Using $\cot \pi z$ In some specialized circumstances, the residues at the simple poles of the trigonometric function $\cot \pi z$ enable us to find the sum of an infinite series.

In Section 4.3 we saw that the zeros of $\sin z$ were the real numbers $z = k\pi$, $k = 0, \pm 1, \pm 2, \dots$. Thus the function $\cot \pi z$ has simple poles at the zeros of $\sin \pi z$, which are $\pi z = k\pi$ or $z = k$, $k = 0, \pm 1, \pm 2, \dots$. If a polynomial function $p(z)$ has (i) real coefficients, (ii) degree $n \geq 2$, and (iii) no integer zeros, then the function

$$f(z) = \frac{\pi \cot \pi z}{p(z)} \quad (37)$$

has an infinite number of simple poles $z = 0, \pm 1, \pm 2, \dots$ from $\cot \pi z$ and a finite number of poles $z_{p_1}, z_{p_2}, \dots, z_{p_r}$ from the zeros of $p(z)$. The closed rectangular contour C shown in Figure 6.17 has vertices $(n + \frac{1}{2}) + ni$, $-(n + \frac{1}{2}) + ni$, $-(n + \frac{1}{2}) - ni$, and $(n + \frac{1}{2}) - ni$, where n is taken large

enough so that C encloses the simple poles $z = 0, \pm 1, \pm 2, \dots, \pm n$ and *all* of the poles $z_{p_1}, z_{p_2}, \dots, z_{p_r}$. By the residue theorem,

$$\oint_C \frac{\pi \cot \pi z}{p(z)} dz = 2\pi i \left[\sum_{k=-n}^n \operatorname{Res} \left(\frac{\pi \cot \pi z}{p(z)}, k \right) + \sum_{j=1}^r \operatorname{Res} \left(\frac{\pi \cot \pi z}{p(z)}, z_{p_j} \right) \right]. \quad (38)$$

In a manner similar to that used several times in the discussion in Subsection 6.6.2, it can be shown that $\oint_C \pi \cot \pi z dz / p(z) \rightarrow 0$ as $n \rightarrow \infty$ and so (38) becomes $0 = \sum_k$ residues $+ \sum_j$ residues. That is,

$$\sum_{k=-\infty}^{\infty} \operatorname{Res} \left(\frac{\pi \cot \pi z}{p(z)}, k \right) = - \sum_{j=1}^r \operatorname{Res} \left(\frac{\pi \cot \pi z}{p(z)}, z_{p_j} \right). \quad (39)$$

Now from (4) of Section 6.5 (with the identifications $g(z) = \pi \cos \pi z / p(z)$, $h(z) = \sin \pi z$, $h'(z) = \pi \cos \pi z$), it is a straightforward task to compute the residues at the simple poles $0, \pm 1, \pm 2, \dots$:

$$\operatorname{Res} \left(\frac{\pi \cot \pi z}{p(z)}, k \right) = \frac{\pi \cos k\pi / p(k)}{\pi \cos k\pi} = \frac{1}{p(k)}. \quad (40)$$

By combining (40) and (39) we arrive at our desired result

$$\sum_{k=-\infty}^{\infty} \frac{1}{p(k)} = - \sum_{j=1}^r \operatorname{Res} \left(\frac{\pi \cot \pi z}{p(z)}, z_{p_j} \right). \quad (41)$$

Using $\csc \pi z$ There exist several more summation formulas similar to (41). If $p(z)$ is a polynomial function satisfying the same assumptions (i)–(iii) given above, then the function

$$f(z) = \frac{\pi \csc \pi z}{p(z)} \quad (42)$$

has an infinite number of simple poles $z = 0, \pm 1, \pm 2, \dots$ from $\csc \pi z$ and a finite number of poles $z_{p_1}, z_{p_2}, \dots, z_{p_r}$ from the zeros of $p(z)$. In this case it can be shown that

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{p(k)} = - \sum_{j=1}^r \operatorname{Res} \left(\frac{\pi \csc \pi z}{p(z)}, z_{p_j} \right). \quad (43)$$

In our last example we show how to use the result in (41) to find the sum of an infinite series.

EXAMPLE 8 Summing an Infinite Series

Find the sum of the series $\sum_{k=0}^{\infty} \frac{1}{k^2 + 4}$.

Solution Observe that if we identify $p(z) = z^2 + 4$, then the three assumptions (i)–(iii) preceding (37) hold true. The zeros of $p(z)$ are $\pm 2i$ and correspond to simple poles of $f(z) = \pi \cot \pi z / (z^2 + 4)$. According to the formula in (41),

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + 4} = - \left[\operatorname{Res} \left(\frac{\pi \cot \pi z}{z^2 + 4}, -2i \right) + \operatorname{Res} \left(\frac{\pi \cot \pi z}{z^2 + 4}, 2i \right) \right]. \quad (44)$$

Now again by (4) of Section 6.5 we have

$$\operatorname{Res} \left(\frac{\pi \cot \pi z}{z^2 + 4}, -2i \right) = \frac{\pi \cot 2\pi i}{4i} \quad \text{and} \quad \operatorname{Res} \left(\frac{\pi \cot \pi z}{z^2 + 4}, 2i \right) = \frac{\pi \cot 2\pi i}{4i}.$$

The sum of the residues is $(\pi/2i) \cot 2\pi i$. This sum is a real quantity because from (27) of Section 4.3:

$$\frac{\pi}{2i} \cot 2\pi i = \frac{\pi}{2i} \frac{\cosh(-2\pi)}{(-i \sinh(-2\pi))} = -\frac{\pi}{2} \coth 2\pi.$$

Hence (44) becomes

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + 4} = \frac{\pi}{2} \coth 2\pi. \quad (45)$$

This is not quite the desired result. To that end we must manipulate the summation $\sum_{k=-\infty}^{\infty}$ in order to put it in the form $\sum_{k=0}^{\infty}$. Observe

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \frac{1}{k^2 + 4} &= \sum_{k=-\infty}^{-1} \frac{1}{k^2 + 4} + \overbrace{\frac{1}{4}}^{\substack{k=0 \\ \text{term}}} + \sum_{k=1}^{\infty} \frac{1}{k^2 + 4} \\ &= \sum_{k=1}^{\infty} \frac{1}{(-k)^2 + 4} + \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{k^2 + 4} \\ &= 2 \sum_{k=1}^{\infty} \frac{1}{k^2 + 4} + \frac{1}{4} = 2 \sum_{k=0}^{\infty} \frac{1}{k^2 + 4} - \frac{1}{4}. \end{aligned} \quad (46)$$

Finally, we obtain the sum of the original series by combining (45) with (46),

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + 4} = 2 \sum_{k=0}^{\infty} \frac{1}{k^2 + 4} - \frac{1}{4} = \frac{\pi}{2} \coth 2\pi,$$

and solving for $\sum_{k=0}^{\infty}$:

$$\sum_{k=0}^{\infty} \frac{1}{k^2 + 4} = \frac{1}{8} + \frac{\pi}{4} \coth 2\pi. \quad (47)$$

With the help of calculator, we find that the right side of (47) is approximately 0.9104.

7.9 Complex Integration (Ph. D qualifying test)

7.9.1 单值性

5303

For what value of a is the function

$$f(z) = \int_1^z \left(\frac{1}{z} + \frac{a}{z^3} \right) \cos z dz$$

single-valued?

(Indiana)

Solution.

Function $F(z) = (\frac{1}{z} + \frac{a}{z^3}) \cos z$ is analytic in $\{z : 0 < |z| < +\infty\}$, and its Laurent expansion around $z = 0$ is:

$$\begin{aligned} F(z) &= \left(\frac{1}{z} + \frac{a}{z^3} \right) \cos z = \left(\frac{1}{z} + \frac{a}{z^3} \right) \left(1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \dots \right) \\ &= \frac{a}{z^3} + \left(1 - \frac{a}{2} \right) \frac{1}{z} + \left(\frac{a}{24} - \frac{1}{2} \right) z + \dots \end{aligned}$$

The necessary and sufficient condition for $f(z)$ to be single-valued is that the residue of $F(z)$ at $z = 0$ is zero, i.e., the coefficient of the term $\frac{1}{z}$ in the above development is zero. Hence we obtain $a = 2$.

7.9.2 积分恒等式

5305

Let $f(z)$ be analytic in $S = \{z \in \mathbb{C}; |z| < 2\}$. Show that

380

$$\frac{2}{\pi} \int_0^{2\pi} f(e^{it}) \cos^2 \frac{t}{2} dt = 2f(0) + f'(0).$$

(Iowa)

Solution.

It is easy to see that

$$\begin{aligned} f(0) &= \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z} dz = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) dt, \\ f'(0) &= \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^2} dz = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-it} dt. \end{aligned}$$

Note that

$$0 = \frac{1}{2\pi i} \int_{|z|=1} f(z) dz = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{it} dt.$$

It follows from the above three equalities that

$$\begin{aligned} 2f(0) + f'(0) &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})(2 + e^{it} + e^{-it}) dt \\ &= \frac{2}{\pi} \int_0^{2\pi} f(e^{it}) \cos^2 \frac{t}{2} dt. \end{aligned}$$

7.9.3 有关实部和虚部的积分不等式

5306

Suppose that the real-valued function u is harmonic in the disk $\{|z| < 2\}$, v is its harmonic conjugate and $u(0) = v(0) = 0$. Show that

$$\int_{\gamma} u^2(z)v^2(z) \frac{dz}{z} = \frac{1}{6} \int_{\gamma} (u^4(z) + v^4(z)) \frac{dz}{z},$$

where $\gamma(t) = e^{2\pi it}$, $t \in [0, 1]$.

(SUNY, Stony Brook)

Solution.

Let $f(z) = u(z) + iv(z)$. Then $f(z)$ is analytic in $\{z : |z| < 2\}$, and we have

$$\begin{aligned} \int_{\gamma} f^4(z) \frac{dz}{z} &= 2\pi i f^4(0) = 0, \\ \int_{\gamma} \bar{f}^4(z) \frac{dz}{z} &= \overline{\left(\int_{\gamma} f^4(z) \frac{dz}{\bar{z}} \right)} \\ &= \overline{\left(\int_{\gamma} z f^4(z) d\left(\frac{1}{z}\right) \right)} \\ &= \overline{\left(- \int_{\gamma} \frac{f^4(z)}{z} dz \right)} = 0. \end{aligned}$$

It follows from

$$u(z) = \frac{f(z) + \overline{f(z)}}{2}$$

and

$$v(z) = \frac{f(z) - \overline{f(z)}}{2i}$$

that

$$\begin{aligned} \int_{\gamma} u^2(z)v^2(z) \frac{dz}{z} &= -\frac{1}{16} \int_{\gamma} (f^4(z) + \bar{f}^4(z) - 2|f(z)|^4) \frac{dz}{z} \\ &= \frac{1}{8} \int_{\gamma} |f(z)|^4 \frac{dz}{z} \\ &= \frac{1}{8} \int_{\gamma} (u^4(z) + v^4(z) + 2u^2(z)v^2(z)) \frac{dz}{z}, \end{aligned}$$

which implies that

$$\int_{\gamma} u^2(z)v^2(z) \frac{dz}{z} = \frac{1}{6} \int_{\gamma} (u^4(z) + v^4(z)) \frac{dz}{z}.$$

7.9.4 关于 $\operatorname{Re} f$ 的积分恒等式

5307

Let f be an analytic function on an open set containing $\overline{D(0, 1)} = \{z; |z| \leq 1\}$.

(a) Prove that

$$\frac{d^n f}{dz^n}(0) = \frac{n!}{\pi} \int_0^{2\pi} e^{-ni\theta} [\operatorname{Re} f(e^{i\theta})] d\theta.$$

(b) If $f(0) = 1$, and if $\operatorname{Re} f(z) > 0$ for all points $z \in D(0, 1)$, prove that

$$\left| \frac{d^n f}{dz^n}(0) \right| \leq 2(n!).$$

(Indiana)

Solution.

(a) Assume that

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

we have

$$\begin{aligned} \frac{n!}{2\pi} \int_0^{2\pi} \overline{f(e^{i\theta})} e^{-ni\theta} d\theta &= \frac{n!}{2\pi} \int_0^{2\pi} \left(\sum_{k=0}^{\infty} \bar{a}_k e^{-ki\theta} \right) e^{-ni\theta} d\theta \\ &= \frac{n!}{2\pi} \sum_{k=0}^{\infty} \bar{a}_k \left(\int_0^{2\pi} e^{-(n+k)i\theta} d\theta \right) = 0. \end{aligned}$$

By Cauchy Integral Formula,

$$\frac{d^n f}{dz^n}(0) = \frac{n!}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta = \frac{n!}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ni\theta} d\theta.$$

Hence

$$\begin{aligned} \frac{d^n f}{dz^n}(0) &= \frac{n!}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ni\theta} d\theta + \frac{n!}{2\pi} \int_0^{2\pi} \overline{f(e^{i\theta})} e^{-ni\theta} d\theta \\ &= \frac{n!}{\pi} \int_0^{2\pi} e^{-ni\theta} [\operatorname{Re} f(e^{i\theta})] d\theta. \end{aligned}$$

(b) Because $\operatorname{Re}f(z)$ is harmonic on $\overline{D(0,1)}$, by the mean-value formula of harmonic functions,

$$\frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}f(e^{i\theta}) d\theta = \operatorname{Re}f(0) = 1.$$

Noting that $\operatorname{Re}f(e^{i\theta}) \geq 0$, we have

$$\begin{aligned} \left| \frac{d^n f}{dz^n}(0) \right| &= \left| \frac{n!}{\pi} \int_0^{2\pi} e^{-n i \theta} [\operatorname{Re}f(e^{i\theta})] d\theta \right| \\ &\leq \frac{n!}{\pi} \int_0^{2\pi} |e^{-n i \theta}| |\operatorname{Re}f(e^{i\theta})| d\theta \\ &= \frac{n!}{\pi} \int_0^{2\pi} \operatorname{Re}f(e^{i\theta}) d\theta \\ &= 2(n!). \end{aligned}$$

7.9.5 应用 Cauchy 积分公式

5309

Let $f = u + iv$ be an entire function.

- (a) Show that if $u^2(z) \geq v^2(z)$ for all $z \in \mathbb{C}$, then f must be a constant.
- (b) Show that if $|f(z)| \leq A + B|z|^h$ for all $z \in \mathbb{C}$ with some positive numbers A, B, h , then $f(z)$ is a polynomial of degree bounded by h .

(Stanford)

Solution.

(a) Let

$$F(z) = e^{-f^2(z)} = e^{-(u^2(z)-v^2(z))-2iu(z)v(z)}.$$

Then $F(z)$ is an entire function with

$$|F(z)| = e^{-(u^2(z)-v^2(z))} \leq 1.$$

By Liouville's theorem, $F(z)$ must be a constant, which implies that $f(z)$ is a constant.

(b) Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then

$$a_n = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{n+1}} dz.$$

For any integer $n > h$,

$$\begin{aligned} |a_n| &\leq \frac{1}{2\pi} \int_{|z|=R} \left| \frac{f(z)}{z^{n+1}} \right| \cdot |dz| = \frac{1}{2\pi R^n} \int_0^{2\pi} |f(\operatorname{Re}^{i\theta})| d\theta \\ &\leq \frac{A + BR^h}{R^n}. \end{aligned}$$

Letting $R \rightarrow +\infty$, we obtain that $a_n = 0$, which implies that $f(z)$ is a polynomial of degree bounded by h .

7.9.6 复形式 Green 公式

5311

Compute the double integral

$$\int \int_D \cos z dx dy$$

where D is the disk given by $\{z = x + iy \in \mathbb{C} : x^2 + y^2 < 1\}$.

(Iowa)

Solution.

First we have the following complex forms of Green's formula:

$$\begin{aligned} \int \int_D w_z dx dy &= \int \int_D \frac{1}{2}(w_x - iw_y) dx dy \\ &= -\frac{1}{2i} \int_{\partial D} w(dx - idy) = -\frac{1}{2i} \int_{\partial D} w d\bar{z}, \\ \int \int_D w_{\bar{z}} dx dy &= \int \int_D \frac{1}{2}(w_x + iw_y) dx dy \\ &= \frac{1}{2i} \int_{\partial D} w(dx + idy) = \frac{1}{2i} \int_{\partial D} w dz. \end{aligned}$$

The problem can be solved directly by either one of the above two forms:

$$\int \int_D \cos z dx dy = \frac{1}{2i} \int_{|z|=1} \bar{z} \cos z dz = \frac{1}{2i} \int_{|z|=1} \frac{\cos z}{z} dz = \pi;$$

or

$$\begin{aligned} \int \int_D \cos z dx dy &= -\frac{1}{2i} \int_{|z|=1} \sin z d\bar{z} = -\frac{1}{2i} \int_{|z|=1} \sin z d\left(\frac{1}{z}\right) \\ &= \frac{1}{2i} \int_{|z|=1} \frac{\sin z}{z^2} dz = \pi. \end{aligned}$$

Note. 这里利用到 $\frac{\partial}{\partial \bar{z}} \bar{z} \cos z = \cos z$. $\partial_{\bar{z}}$ 并不会影响到 $f(z)$.

7.9.7 面积公式

5312

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be analytic in $D = \{|z| < 1\}$ and assume that the integral

$$A = \int \int_D |f'(z)|^2 dx dy$$

is finite.

- (a) Express A in terms of the coefficients a_n .
- (b) Prove that

$$|f(z) - f(0)| \leq \sqrt{\frac{A}{\pi} \log \frac{1}{1 - |z|^2}}$$

for $z \in D$.

(Indiana)

Solution.

(a) By

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1},$$

we have

$$\begin{aligned} A &= \int \int_D |f'(z)|^2 dx dy = \int_0^1 r dr \int_0^{2\pi} (f'(re^{i\theta})) \overline{(f'(re^{i\theta}))} d\theta \\ &= \int_0^1 r dr \int_0^{2\pi} \left(\sum_{n=1}^{\infty} n a_n r^{n-1} e^{i(n-1)\theta} \right) \left(\sum_{n=1}^{\infty} n \bar{a}_n r^{n-1} e^{-i(n-1)\theta} \right) d\theta. \end{aligned}$$

Noting that

$$\int_0^{2\pi} e^{ik\theta} \cdot e^{-il\theta} d\theta = \begin{cases} 0 & k \neq l, \\ 2\pi & k = l, \end{cases}$$

we obtain that

$$\begin{aligned} A &= \int \int_D |f'(z)|^2 dx dy = \int_0^1 r dr \int_0^{2\pi} \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} d\theta \\ &= 2\pi \int_0^1 \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-1} dr = \pi \sum_{n=1}^{\infty} n |a_n|^2. \end{aligned}$$

(b) By Cauchy's inequality, we have

$$\begin{aligned} |f(z) - f(0)| &= \left| \sum_{n=1}^{\infty} a_n z^n \right| = \left| \sum_{n=1}^{\infty} \left(\sqrt{n} a_n \cdot \frac{1}{\sqrt{n}} z^n \right) \right| \\ &\leq \sqrt{\sum_{n=1}^{\infty} n |a_n|^2} \cdot \sqrt{\sum_{n=1}^{\infty} \frac{1}{n} |z|^{2n}} = \sqrt{\frac{A}{\pi} \log \frac{1}{1-|z|^2}}. \end{aligned}$$

7.9.8 应用留数定理

5315

Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a + \sin^2 x}, \quad |a| > 1,$$

by the method of residues.

(Columbia)

Solution.

Denote

$$I(a) = \int_0^{\frac{\pi}{2}} \frac{dx}{a + \sin^2 x}.$$

It is obvious that $I(a)$ is an analytic function in $\{a : |a| > 1\}$. Then we have

$$\begin{aligned} I(a) &= \int_0^{\frac{\pi}{2}} \frac{dx}{a + \sin^2 x} = \int_0^{\frac{\pi}{2}} \frac{2dx}{2a + 1 - \cos 2x} \\ &= \int_0^{\pi} \frac{dx}{2a + 1 - \cos x} = \frac{1}{2} \int_{-\pi}^{\pi} \frac{dx}{2a + 1 - \cos x}. \end{aligned}$$

Let $z = e^{ix}$, then

$$\begin{aligned} dx &= \frac{dz}{iz}, \\ \cos x &= \frac{z + z^{-1}}{2}, \end{aligned}$$

and

$$I(a) = \int_{|z|=1} \frac{idz}{z^2 - 2(2a+1)z + 1}.$$

Denote the two roots of $z^2 - 2(2a+1)z + 1 = 0$ by z_1 and z_2 . Since $z_1 \cdot z_2 = 1$, we may assume that $|z_1| > 1$, $|z_2| < 1$. By the residue theorem we have

$$\begin{aligned} I(a) &= \int_{|z|=1} \frac{idz}{(z - z_1)(z - z_2)} = \frac{2\pi}{z_1 - z_2} \\ &= \frac{2\pi}{\sqrt{(z_1 + z_2)^2 - 4z_1z_2}} = \frac{\pi}{2\sqrt{a(a+1)}}. \end{aligned}$$

It should be noted that $\frac{\pi}{2\sqrt{a(a+1)}}$ is also analytic in $\{a : |a| > 1\}$, and the branch of $\sqrt{a(a+1)}$ should be chosen by $\arg \sqrt{a(a+1)}|_{a>1} = 0$.

7.9.9 有关 $\log z$ 的积分恒等式

5321

Let $f(z)$ be holomorphic in the unit disk $|z| \leq 1$. Prove that

$$\int_0^1 f(x)dx = \frac{1}{2\pi i} \int_{|z|=1} f(z) \log z dz,$$

where respective integration goes along the straight line from 0 to 1 and along the positively oriented unit circle starting from the point $z = 1$. The branch of \log is chosen to be real for positive z .

(SUNY, Stony Brook)

Solution.

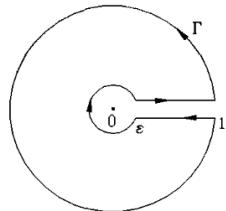


Fig.5.9

Let the contour of integration Γ be shown as in Fig.5.9, and the single-valued branch of $\log z$ be chosen by $\arg z|_{z=-1} = \pi$. Since $f(z) \log z$ is holomorphic inside the contour Γ , by Cauchy integral theorem,

$$\int_{\Gamma} f(z) \log z dz = 0,$$

where

$$\begin{aligned}\int_{\Gamma} f(z) \log z dz &= \int_{\varepsilon}^1 f(x) \log x dx + \int_{|z|=1} f(z) \log z dz \\ &\quad + \int_1^{\varepsilon} f(x)(\log x + 2\pi i) dx + \int_{2\pi}^0 f(\varepsilon e^{i\theta}) \log(\varepsilon e^{i\theta}) i\varepsilon e^{i\theta} d\theta \\ &= -2\pi i \int_{\varepsilon}^1 f(x) dx + \int_{|z|=1} f(z) \log z dz \\ &\quad - \int_0^{2\pi} f(\varepsilon e^{i\theta}) \log(\varepsilon e^{i\theta}) i\varepsilon e^{i\theta} d\theta.\end{aligned}$$

It is easy to see that

$$\lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} f(\varepsilon e^{i\theta}) \log(\varepsilon e^{i\theta}) i\varepsilon e^{i\theta} d\theta = 0.$$

Letting $\varepsilon \rightarrow 0$, we obtain

$$\int_0^1 f(x) dx = \frac{1}{2\pi i} \int_{|z|=1} f(z) \log z dz,$$

where the integration contour $|z|=1$ has starting point and end point $z=1$, and the value of $\log z$ at the starting point $z=1$ is defined as 0.

7.9.10 $\log|a + be^{i\theta}|$ 的积分

5322

Find the value of

$$\int_0^{2\pi} \log|a + be^{i\phi}| d\phi$$

where a and b are complex constants, not both equal to zero.

(Harvard)

Solution.

First we assume $|a| > |b|$, and then the multi-valued analytic function $\log(a + bz)$ has single-valued branch on $\{z : |z| \leq 1\}$. Take $e^{i\phi} = z$, then $d\phi = \frac{dz}{iz}$, and

$$\begin{aligned}\int_0^{2\pi} \log|a + be^{i\phi}| d\phi &= \operatorname{Re} \left\{ \int_0^{2\pi} \log(a + be^{i\phi}) d\phi \right\} \\ &= \operatorname{Re} \left\{ \int_{|z|=1} \frac{\log(a + bz)}{iz} dz \right\} \\ &= \operatorname{Re}\{2\pi \log a\} = 2\pi \log|a|.\end{aligned}$$

When $|a| < |b|$, we have

$$\begin{aligned}\int_0^{2\pi} \log |a + be^{i\phi}| d\phi &= \int_0^{2\pi} \log |\bar{b} + \bar{a}e^{i\phi}| d\phi \\ &= 2\pi \log |\bar{b}| = 2\pi \log |b|.\end{aligned}$$

In the case $|a| = |b|$, let $b = ae^{i\alpha}$. Then

$$\begin{aligned}\int_0^{2\pi} \log |a + be^{i\phi}| d\phi &= \int_0^{2\pi} (\log |a| + \log |1 + e^{i(\phi+\alpha)}|) d\phi \\ &= 2\pi \log |a| + \int_{-\pi}^{\pi} \log |1 + e^{i\phi}| d\phi.\end{aligned}$$

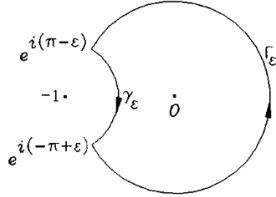


Fig.5.10

In order to evaluate the integral

$$\int_{-\pi}^{\pi} \log |1 + e^{i\phi}| d\phi,$$

we define

$$f(z) = \frac{\log(1+z)}{z},$$

where the single-valued branch is defined by $\log(1+z)|_{z=0} = 0$. Choose a contour of integration $\Gamma = \Gamma_\varepsilon \cup \gamma_\varepsilon$ as shown in Fig.5.10. Since $f(z)$ is analytic on Γ and inside Γ , by Cauchy integral theorem, $\int_{\Gamma} f(z) dz = 0$. Because

$$\left| \int_{\gamma_\varepsilon} f(z) dz \right| \leq \frac{\log \frac{1}{\varepsilon} + \frac{\pi}{2}}{1 - \varepsilon} \cdot \pi \varepsilon \rightarrow 0 \quad (\varepsilon \rightarrow 0),$$

we have

$$\int_{-\pi}^{\pi} \log |1 + e^{i\phi}| d\phi = \operatorname{Re} \int_{-\pi}^{\pi} \log(1 + e^{i\phi}) d\phi$$

$$\begin{aligned}
 &= \lim_{\epsilon \rightarrow 0} \operatorname{Re} \left\{ \int_{\Gamma_\epsilon} \log(1+z) \frac{dz}{iz} \right\} \\
 &= \lim_{\epsilon \rightarrow 0} \operatorname{Re} \left\{ \frac{1}{i} \int_{\Gamma} f(z) dz \right\} = 0.
 \end{aligned}$$

Hence we obtain

$$\int_0^{2\pi} \log |a + be^{i\phi}| d\phi = 2\pi \max\{\log |a|, \log |b|\}.$$

7.9.11 带有 $\log x$ 的积分, 考虑辅助函数带有 $\log^2 z$

Evaluate

$$\int_0^\infty \frac{\log x}{(1+x)^3} dx.$$

(Iowa)

Solution.

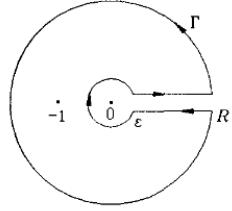


Fig.5.11

Let

$$f(z) = \frac{\log^2 z}{(1+z)^3},$$

and select the integral path Γ as shown in Fig.5.11. The single-valued branch of $\log z$ is chosen by $\arg z|_{z=-1} = \pi$. By the residue theorem, we have

$$\int_{\Gamma} f(z) dz = 2\pi i \operatorname{Res}(f, -1),$$

where

$$\begin{aligned}\int_{\Gamma} f(z) dz &= \int_{\varepsilon}^R \frac{\log^2 x}{(1+x)^3} dx + \int_0^{2\pi} iRe^{i\theta} f(Re^{i\theta}) d\theta + \int_R^{\varepsilon} \frac{(\log x + 2\pi i)^2}{(1+x)^3} dx \\ &\quad + \int_{2\pi}^0 i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta}) d\theta \\ &= \int_{\varepsilon}^R \frac{-4\pi i \log x + 4\pi^2}{(1+x)^3} dx + \int_0^{2\pi} iRe^{i\theta} f(Re^{i\theta}) d\theta \\ &\quad + \int_{2\pi}^0 i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta}) d\theta.\end{aligned}$$

It is obvious that

$$\lim_{R \rightarrow \infty} \int_0^{2\pi} iRe^{i\theta} f(Re^{i\theta}) d\theta = 0$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{2\pi}^0 i\varepsilon e^{i\theta} f(\varepsilon e^{i\theta}) d\theta = 0.$$

In order to find $\text{Res}(f, -1)$, we consider the Laurent expansion of f about $z = -1$:

$$\begin{aligned}f(z) &= \frac{\log^2[(z+1)-1]}{(z+1)^3} = \frac{(\pi i + \log[1-(z+1)])^2}{(z+1)^3} \\ &= \frac{(\pi i - (z+1) - \frac{1}{2}(z+1)^2 - \dots)^2}{(z+1)^3} \\ &= \sum_{n=-3}^{\infty} a_n (z+1)^n,\end{aligned}$$

where $a_{-1} = 1 - \pi i$. Hence

$$2\pi i \text{Res}(f, -1) = 2\pi i + 2\pi^2.$$

As $\varepsilon \rightarrow 0$ and $R \rightarrow \infty$, it turns out that

$$\int_0^{\infty} \frac{-4\pi i \log x + 4\pi^2}{(1+x)^3} dx = 2\pi i + 2\pi^2.$$

Comparing the imaginary parts on the two sides of the above identity, we obtain

$$\int_0^{\infty} \frac{\log x}{(1+x)^3} dx = -\frac{1}{2}.$$

7.10 最大模原理

参见 Rudin 《实分析与复分析》.

This chapter contains further generalizations of the maximum modulus theorem, as well as some rather striking applications of it, and it concludes with

a theorem which shows that the maximum property "almost" characterizes the class of holomorphic functions.

The Phragmen-Lindelöf Method

12.7 For a bounded region Ω , we saw in Sec. 12.1 that if f is continuous on the closure of Ω and if $f \in H(\Omega)$, the maximum modulus theorem implies

$$\|f\|_{\Omega} = \|f\|_{\partial\Omega}. \quad (1)$$

For unbounded regions, this is no longer true.

To see an example, let

$$\Omega = \left\{ z = x + iy : -\frac{\pi}{2} < y < \frac{\pi}{2} \right\}; \quad (2)$$

Ω is the open strip bounded by the parallel lines $y = \pm\pi/2$; its boundary $\partial\Omega$ is the union of these two lines. Put

$$f(z) = \exp(\exp(z)). \quad (3)$$

For real x ,

$$f\left(x \pm \frac{\pi i}{2}\right) = \exp(\pm ie^x) \quad (4)$$

since $\exp(\pi i/2) = i$, so $|f(z)| = 1$ for $z \in \partial\Omega$. But $f(z) \rightarrow \infty$ very rapidly as $x \rightarrow \infty$ along the positive real axis, which lies in Ω .

"Very" is the key word in the preceding sentence. A method developed by Phragmen and Lindelöf makes it possible to prove theorems of the following kind: If $f \in H(\Omega)$ and if $|f| < g$, where $g(z) \rightarrow \infty$ "slowly" as $z \rightarrow \infty$ in Ω (just what "slowly" means depends on Ω), then f is actually bounded in Ω , and this usually implies further conclusions about f , by the maximum modulus theorem.

Rather than describe the method by a theorem which would cover a large number of situations, we shall show how it works in two cases. In both, Ω will be a strip. In the first, f will be assumed to be bounded, and the theorem will improve the bound; in the second, a growth condition will be imposed on f which just excludes the function (3). In view of later applications, Ω will be a vertical strip in Theorem 12.8.

7.10.1 An example concerning the increasing rapid

Suppose f is an entire function and

$$|f(z)| < 1 + |z|^{1/2}$$

for all z . Then f is a constant.

This follows immediately from the Cauchy estimates, since they show that

$$f^{(n)}(0) = 0 \text{ for } n = 1, 2, 3, \dots$$

12.8 Theorem Suppose

$$\Omega = \{x + iy : a < x < b\}, \quad \bar{\Omega} = \{x + iy : a \leq x \leq b\}, \quad (1)$$

f is continuous on $\bar{\Omega}$, $f \in H(\Omega)$, and suppose that $|f(z)| < B$ for all $z \in \Omega$ and some fixed $B < \infty$. If

$$M(x) = \sup \{|f(x + iy)| : -\infty < y < \infty\} \quad (a \leq x \leq b) \quad (2)$$

then we actually have

$$M(x)^{b-a} \leq M(a)^{b-x} M(b)^{x-a} \quad (a < x < b). \quad (3)$$

Note: The conclusion (3) implies that the inequality $|f| < B$ can be replaced by $|f| \leq \max(M(a), M(b))$, so that $|f|$ is no larger in Ω than the supremum of $|f|$ on the boundary of Ω .

PROOF We assume first that $M(a) = M(b) = 1$. In this case we have to prove that $|f(z)| \leq 1$ for all $z \in \Omega$.

For each $\epsilon > 0$, we define an auxiliary function

$$h_\epsilon(z) = \frac{1}{1 + \epsilon(z - a)} \quad (z \in \bar{\Omega}). \quad (4)$$

Since $\operatorname{Re}\{1 + \epsilon(z - a)\} = 1 + \epsilon(x - a) \geq 1$ in $\bar{\Omega}$, we have $|h_\epsilon| \leq 1$ in $\bar{\Omega}$, so that

$$|f(z)h_\epsilon(z)| \leq 1 \quad (z \in \partial\Omega). \quad (5)$$

Also, $|1 + \epsilon(z - a)| \geq \epsilon|y|$, so that

$$|f(z)h_\epsilon(z)| \leq \frac{B}{\epsilon|y|} \quad (z = x + iy \in \bar{\Omega}). \quad (6)$$

Let R be the rectangle cut off from $\bar{\Omega}$ by the lines $y = \pm B/\epsilon$. By (5) and (6), $|f h_\epsilon| \leq 1$ on ∂R , hence $|f h_\epsilon| \leq 1$ on R , by the maximum modulus theorem. But (6) shows that $|f h_\epsilon| \leq 1$ on the rest of $\bar{\Omega}$. Thus $|f(z)h_\epsilon(z)| \leq 1$

for all $z \in \Omega$ and all $\epsilon > 0$. If we fix $z \in \Omega$ and then let $\epsilon \rightarrow 0$, we obtain the desired result $|f(z)| \leq 1$.

We now turn to the general case. Put

$$g(z) = M(a)^{(b-z)/(b-a)} M(b)^{(z-a)/(b-a)}, \quad (7)$$

where, for $M > 0$ and w complex, M^w is defined by

$$M^w = \exp(w \log M), \quad (8)$$

and $\log M$ is real. Then g is entire, g has no zero, $1/g$ is bounded in $\bar{\Omega}$,

$$|g(a + iy)| = M(a), \quad |g(b + iy)| = M(b), \quad (9)$$

and hence f/g satisfies our previous assumptions. Thus $|f/g| \leq 1$ in Ω , and this gives (3). (See Exercise 7.) ////

7.11 洛朗级数

Théorème 7: Laurant 定理

在圆环 $H : r < |z - a| < R(r \geq 0, R \leq +\infty)$ 内解析的函数 $f(z)$ 必可以展开成双边幂级数, 即 **Laurant 级数**

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - a)^n$$

其中 **Laurant 系数**

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta \quad (n = 0, \pm 1, \pm 2, \dots)$$

C 为圆周 $|\zeta - a| = \rho(r < \rho < R)$, 并且展开式唯一.

展开洛朗级数直接暴力展开即可.

7.11.1 奇点

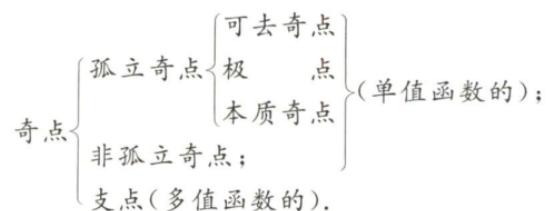
定理 5.5 函数 $f(z)$ 的孤立奇点 a 为极点 $\Leftrightarrow \lim_{z \rightarrow a} f(z) = \infty$.

注 这个定理也能说明极点的特征, 其缺点是不能指明极点的阶.

定理 5.6 函数 $f(z)$ 的孤立奇点 a 为本质奇点

$$\Leftrightarrow \lim_{z \rightarrow a} f(z) \neq \begin{cases} b \text{ (有限数),} \\ \infty, \end{cases} \text{ 即 } \lim_{z \rightarrow a} f(z) \text{ 不存在.}$$

3. 就本书所遇到的奇点情况来看, 可以列表如下:



4. 充分理解关于本质奇点的魏尔斯拉定理 5.8 和皮卡(Picard)(大)定理 5.9.

定理 5.7 若 $z=a$ 为函数 $f(z)$ 的一本质奇点, 且在点 a 的充分小去心邻域内不为零, 则 $z=a$ 亦必为 $\frac{1}{f(z)}$ 的本质奇点.

定理 5.8 如果 a 为函数 $f(z)$ 的本质奇点, 则对任何指定的数 A (有限或无限), 必存在点列 $\{z_n\} \rightarrow a$, 使得像点列 $\{f(z_n)\} \rightarrow A$ (极限等式).

定理 5.9 若 a 为函数 $f(z)$ 的本质奇点, 则对任何 $A \neq \infty$, 除可能有一个例外的有限值 A_0 , 必有点列 $\{z_n\} \rightarrow a$, 使得 $f(z_n)=A$ ($n=1, 2, \dots$) (准确等式).

这是关于函数在本质奇点附近值分布情况的一个定理, 是函数值分布理论的早期结果之一.

判断奇点特性

5. 求出下列函数的奇点, 并确定它们的类别(对于极点, 要指出它们的阶), 对于无穷远点也要加以讨论.

$$(1) \frac{z+2}{(z-1)^2 z(z+1)}; (2) e^{\frac{1}{z-2i}}; (3) \frac{z^2+1}{e^z};$$

$$(4) \frac{e^z}{z^2+4}; (5) \frac{1}{\cos z}; (6) \sec \frac{1}{z};$$

$$(7) \frac{\sin z}{(z-3)^2 z^2 (z+1)^3}; (8) \frac{\sin(z-5)}{(z-5)^2}; (9) z(e^{\frac{1}{z}} - 1).$$

(答: (1) 0, -1 为单极点, 1 为二阶极点, ∞ 为可去奇点; (2) $2i$ 为本质奇点, ∞ 为可去奇点; (3) ∞ 为本质奇点; (4) $\pm 2i$ 各为单极点, ∞ 为本质奇点; (5) $\left(n + \frac{1}{2}\right)\pi$ ($n = 0, \pm 1, \pm 2, \dots$) 均为单级点, ∞ 为非孤立奇点; (6) $1/\left[\left(n + \frac{1}{2}\right)\pi\right]$ ($n = 0, \pm 1, \pm 2, \dots$) 均为单极点, 0 为非孤立奇点; (7) 0 为单极点, 3 为二阶极点, -1 为三阶极点, ∞ 为本质奇点; (8) 5 为单极点, ∞ 为本质奇点; (9) 0 为本质奇点, ∞ 为可去奇点.)

6. 指出下列函数在指定点处的奇点特性.

$$(1) \ln \frac{z-1}{z-2}, z=\infty; (2) \sqrt{z(z-1)}, z=\infty;$$

$$(3) \frac{\ln(1+z)}{z}, z=0, z=\infty.$$

(答: (1) ∞ 为可去奇点; (2) ∞ 为单极点;
(3) 0 为可去奇点, ∞ 为本质奇点.)

7.11.2 解析函数在无穷远点处的性质

由于函数 $f(z)$ 在 ∞ 总是无意义的, 所以点 ∞ 总是 $f(z)$ 的奇点.

Example 1

对于函数

$$f(z) = \frac{z^6 + 1}{z(z^2 + 1)^2}$$

它是有理分式函数，分母的零点 $0, i, -i$ 是这个函数的极点，下面考虑它们的阶。

$z = 0$ 是分母的一阶零点，且不是分子 $z^6 + 1$ 的零点，故 $z = 0$ 是 $f(z)$ 的一阶极点。注意到 $z^6 + 1 = (z^2 + 1)(z^4 - z^2 + 1)$,

$$\frac{z^6 + 1}{z(z^2 + 1)^2} = \frac{z^4 - z^2 + 1}{z(z - i)[z - (-i)]}$$

于是 $z = i, z = -i$ 也是 $f(z)$ 的一阶极点。

接下来考虑 ∞ ，因为

$$\frac{z^6 + 1}{z(z^2 + 1)^2} = \frac{z^6 \left(1 + \frac{1}{z^6}\right)}{z^5 \left(1 + \frac{2}{z^2} + \frac{1}{z^4}\right)} = z \left(1 - \frac{2}{z^2} + \dots\right)$$

其中 $\mu(z) = 1 - \frac{2}{z^2} + \dots$ 在 $z = \infty$ 解析，且 $\mu(\infty) = 1 \neq 0$ ，因此 $z = \infty$ 是 $f(z)$ 的一阶零点。

Example 2

对于函数

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3}$$

它只有 $z = 0, z = \infty$ 作为奇点， $z = 0$ 作为 $f(z)$ 的三阶极点， $f(z)$ 在 $z = \infty$ 的主要部分（即正幂）为零，故 $z = \infty$ 是 $f(z)$ 的可去奇点。

由于 $\frac{1}{f(z)} = z^2 \left(1 - \frac{1}{z+1}\right)$ 以 $z = \infty$ 为二阶极点，故 $f(z)$ 以 $z = \infty$ 为二阶零点。

Example 3

对于函数

$$f(z) = \frac{1}{1 + e^z}$$

解 $1 + e^z = 0$ 得到 $\frac{1}{f(z)}$ 的零点

$$z_k = (2k + 1)\pi i \quad (k = 0, \pm 1, \pm 2, \dots)$$

又因为 $(1 + e^z)'|_{z=z_k} \neq 0$ ，所以 z_k 都是 $\frac{1}{f(z)}$ 的一阶零点。于是 z_k 都是 $f(z)$ 的一阶极点。

当 $k \rightarrow \infty$ 时， $z_k \rightarrow \infty$ 。故点 ∞ 是 $f(z)$ 的非孤立奇点，即极点列 $\{z_k\}$ 的聚点。

多值函数的洛朗展式

例 5.3.6 求函数 $f(z) = \sqrt{(z-1)(z-2)}$ 在 $|z| > 2$ 的洛朗展式.

分析 $z=1$ 及 $z=2$ 是 $\sqrt{(z-1)(z-2)}$ 的支点(因 $2 \neq 1$), 而 $z=\infty$ 则不是其支点(因 $2|(1+1)$). 当沿支割线 $[1,2]$ 割开 z 平面后, $\sqrt{(z-1)(z-2)}$ 就能分出两个单值解析分支, 都以 $z=\infty$ 为单值性孤立奇点. 于是在 $+\infty > |z| > 2$ 内, 就能将每一个分支展成洛朗级数.

$$\begin{aligned} \text{解 } +\infty > |z| > 2 > 1 \Rightarrow \left| \frac{1}{z} \right| < 1, \quad \left| \frac{2}{z} \right| < 1. \\ \sqrt{(z-1)(z-2)} &= \sqrt{z^2 \left(1 - \frac{1}{z} \right) \left(1 - \frac{2}{z} \right)} \\ &= \pm z \left(1 - \frac{1}{z} \right)^{\frac{1}{2}} \left(1 - \frac{2}{z} \right)^{\frac{1}{2}} \end{aligned} \quad (1)$$

$$\begin{aligned} &\xrightarrow{(4.14)} \pm z \left[1 + \frac{\frac{1}{2}}{1!} \left(-\frac{1}{z} \right) + \frac{\frac{1}{2} \left(\frac{1}{2} - 1 \right)}{2!} \left(-\frac{1}{z} \right)^2 + \dots \right] \cdot \\ &\quad \left[1 + \frac{\frac{1}{2}}{1!} \left(-\frac{2}{z} \right) + \frac{\frac{1}{2} \left(\frac{1}{2} - 1 \right)}{2!} \cdot \left(-\frac{2}{z} \right)^2 + \dots \right] \\ &= \pm z \left(1 - \frac{1}{2} \frac{1}{z} - \frac{1}{8} \frac{1}{z^2} - \frac{1}{16} \frac{1}{z^3} - \dots \right) \cdot \\ &\quad \left(1 - \frac{1}{z} - \frac{1}{2} \frac{1}{z^2} - \frac{1}{2} \frac{1}{z^3} - \dots \right) \\ &= \pm z \left(1 - \frac{3}{2} \frac{1}{z} - \frac{1}{8} \frac{1}{z^2} - \frac{3}{16} \frac{1}{z^3} - \dots \right) \\ &= \pm \left(z - \frac{3}{2} - \frac{1}{8} \frac{1}{z} - \frac{3}{16} \frac{1}{z^2} - \dots \right). \end{aligned} \quad (2)$$

从(1)式或(2)式都可以看出 $z=\infty$ 为每一个分支的一阶极点. ■

通过洛朗展式判断极点阶数

例 5.3.7 指出函数 $f(z) = \frac{1}{\ln z}$ 有什么样的奇点 ($\ln z$ 取主值支).

分析 $f(z) = \frac{1}{\ln z}$ 以 $z=0, \infty$ 为支点. 由于 $f(1)=\infty$, 故 $z=1$ 为 $f(z)$ 的极点.

余下的只需通过洛朗展式进一步判断这个极点的阶.

解 在 $0 < |z-1| < 1$ 内求 $f(z)$ 的洛朗展式.

令 $z-1=t$, 即 $z=1+t$, $0 < |t| < 1$, 则

$$\begin{aligned} f(z) &= \frac{1}{\ln z} = \frac{1}{\ln(1+t)} \stackrel{(4.13)}{=} \left(t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \dots\right)^{-1} \\ &= \frac{1}{t} \left(1 - \frac{1}{2}t + \frac{1}{3}t^2 - \frac{1}{4}t^3 + \dots\right)^{-1} \\ &= \frac{1}{t} \left(1 + \frac{1}{2}t - \frac{1}{12}t^2 + \frac{1}{24}t^3 - \dots\right) \\ &= \frac{1}{t} + \frac{1}{2} - \frac{1}{12}t + \frac{1}{24}t^2 - \dots \\ &= \frac{1}{z-1} + \frac{1}{2} - \frac{1}{12}(z-1) + \frac{1}{24}(z-1)^2 + \dots. \end{aligned}$$

故 $z=1$ 是 $f(z) = \frac{1}{\ln z}$ (主值支) 的一阶极点. ■

7.11.3 整函数与亚纯函数

定理 5.10 若 $f(z)$ 为一整函数, 则

(1) $z=\infty$ 为 $f(z)$ 的可去奇点 $\Leftrightarrow f(z) = \text{常数 } c_0$ ($f(z)$ 在点 ∞ 的主要部分为零, 即无正幂).

(2) $z=\infty$ 为 $f(z)$ 的 m 阶极点 $\Leftrightarrow f(z)$ 是一个 m 次多项式 ($f(z)$ 在点 ∞ 的主要部分为零).

$$c_0 + c_1 z + \dots + c_m z^m \quad (c_m \neq 0).$$

(3) $z=\infty$ 为 $f(z)$ 的本性奇点 \Leftrightarrow 展式 (5.14) 有无限项 ($f(z)$ 在点 ∞ 的主要部分含无限项正幂). 这时 $f(z)$ 称为超越整函数.

由此可见, 整函数按惟一奇点 $z=\infty$ 的不同类型而被分成了三类.

注 定理 5.10(1) 与刘维尔定理 (本章习题 (一) 第 9 题) 一致.

例如, e^z , $\sin z$ 及 $\cos z$ 都是超越整函数.

整函数可表示为 $f(z) = A + e^{g(z)}$

例 5.4.5 若整函数 $f(z)$ 不在 z 平面上任何点取某个值 $A (\neq \infty)$, 则 $f(z)$ 必有 $f(z) = A + e^{g(z)}$ 的形状, 此处 $g(z)$ 为整函数.

证 因函数 $f(z) - A$ 不能为零, 故

$$h(z) \stackrel{\text{def}}{=} \frac{f'(z)}{f(z) - A}$$

在 \mathbb{C} 上解析, 即为整函数. 但

$$h(z) = \frac{f'(z)}{f(z) - A} = \frac{d}{dz} \ln[f(z) - A]$$

(称为 $f(z) - A$ 的对数导数),

故

$$\ln[f(z) - A] \Big|_0^z = \int_0^z h(z) dz,$$

即

$$\ln[f(z) - A] - \ln[f(0) - A] = \int_0^z h(z) dz.$$

因而

$$\ln[f(z) - A] = \int_0^z h(z) dz + \ln[f(0) - A],$$

即

$$f(z) - A = \exp \left\{ \int_0^z h(z) dz + \ln[f(0) - A] \right\} = e^{g(z)}.$$

此处 $g(z) = \int_0^z h(z) dz + \ln[f(0) - A]$, 亦为整函数(因 \mathbb{C} 为单连通区域, $h(z)$ 在 \mathbb{C} 上解析, 由定理 3.6, $\int_0^z h(z) dz$ 在 \mathbb{C} 上亦解析). ■

证明 $O(1)$ 的整函数有零点

Exercice 2

设 $f \in H(\mathbb{C})$, 当 $z \rightarrow \infty$ 时 $\frac{f(z)}{z} \rightarrow 1$. 证明: $f(z)$ 必有一个零点.

证明. 由于 f 是整函数, 故只有 $z = \infty$ 作为孤立奇点, 于是可设

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \dots \quad 0 \leq |z| < \infty$$

又由题设

$$\lim_{z \rightarrow \infty} \frac{f(z)}{z} = 1$$

可见 $z = \infty$ 为 $\frac{f(z)}{z}$ 的可去奇点, 从而 $z = \infty$ 为 $f(z)$ 的一阶极点, 从而

$$f(z) = c_0 + z$$

于是 $f(z)$ 必然有且仅有一个零点.

□

亚纯函数在周线 C 内只有有限个零点和极点

14. 设(1) C 是一条周线, $f(z)$ 在 C 的内部是亚纯的, 且连续到 C ; (2) $f(z)$ 沿 C 不为零. 试证函数 $f(z)$ 在 C 的内部至多只有有限个零点和极点.

分析 设 $I(C)=D$. 设在区域 D 中去掉所有的极点, 得区域 D_1 , 由题设条件, 在

D_1 内 $f(z) \neq 0$.

我们要证明 $f(z)$ 在 D_1 内至多只有有限个零点. 假定不然, 从 $f(z)$ 在 D_1 内亦即在 D 内的无限多个零点中, 取出彼此不同的零点组成一个点列 $\{z_n\}$. 显然它是一个有界点列. 因而有一个收敛子点列 $\{z_{n_k}\} \rightarrow z_0$. 由题设条件可知, z_0 只能在 D_1 内. 又由惟一性定理, 在 D_1 内 $f(z) \equiv 0$, 这就得到矛盾. 于是, $f(z)$ 在 D_1 内至多只有有限个零点, 因而 $f(z)$ 在 D 内也至多只有有限个零点.

考虑函数 $\frac{1}{f(z)}$, 它必然满足题设条件. 由上一段可知 $\frac{1}{f(z)}$ 在 D 内至多只有有限个零点. 从而可知 $f(z)$ 在 D 内至多只有有限个极点.

7.11.4 Schwarz lemma

[Lemme de Schwarz —Wikipédia](#)

Statement

Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be a holomorphic function on the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

Such that:

- (1) $f(0) = 0$,
- (2) $|f(z)| \leq 1$ for all $z \in \mathbb{D}$.

Then:

- (1) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$,
- (2) $|f'(0)| \leq 1$,
- (3) Moreover, if equality holds at any point other than 0 (i.e., if $|f(z)| = |z|$ for some $z \neq 0$), or if $|f'(0)| = 1$, then $f(z) = e^{i\theta}z$ for some real constant θ .

Note. If $|f(z)| = 1$ for some $|z| = 1$, f may not be a rotation. Consider $f(z) = z^n$; it satisfies the conditions, and $|f(e^{i\alpha})| = |e^{in\alpha}| = 1 = |e^{i\alpha}|$, but $f(z) \neq e^{i\theta}z$.

Proof

The proof is a straightforward application of the maximum modulus principle on the function

$$g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0 \end{cases}$$

which is holomorphic on the whole of D , including at the origin (because f is differentiable at the origin and fixes zero). Now if $D_r = \{z : |z| \leq r\}$ denotes the closed disk of radius r centered at the origin, then the maximum modulus principle implies that, for $r < 1$, given any $z \in D_r$, there exists z_r on the boundary of D_r such that

$$|g(z)| \leq |g(z_r)| = \frac{|f(z_r)|}{|z_r|} \leq \frac{1}{r}$$

As $r \rightarrow 1$ we get $|g(z)| \leq 1$.

Moreover, suppose that $|f(z)| = |z|$ for some non-zero $z \in D$, or $|f'(0)| = 1$. Then, $|g(z)| = 1$ at some point of D . So by the maximum modulus principle, $g(z)$ is equal to a constant a such that $|a| = 1$. Therefore, $f(z) = az$, as desired.

General Schwarz lemma

If $f \in H(B_R(z_0))$, $f(z_0) = 0$, $|f(B_R(z_0))| \leq M$, then let

$$g : \mathbb{D} \rightarrow \mathbb{D} \quad z \mapsto \frac{1}{M} f(Rz + z_0)$$

Then $g(0) = 0$, $|g(\mathbb{D})| \leq 1$. Apply Schwarz lemma,

$$\left| \underbrace{g(z)}_{=M^{-1}f(Rz+z_0)} \right| \leq |z| \quad \left| \underbrace{g'(0)}_{=RM^{-1}f'(z_0)} \right| \leq 1$$

i.e.

$$|f(z)| \leq \frac{M}{R} |z - z_0|, \forall z \in B(z_0, R) \quad |f'(z_0)| \leq 1$$

7.11.5 Schwarz 引理的应用

Exercice 3: cmc 第八届高年级决赛

设函数 $f(z)$ 在单位圆 $|z| < 1$ 内解析，并且 $|f(z)| \leq M (M > 0)$, M 为常数

数. 证明:

$$|f'(0)| \leq M - \frac{|f(0)|^2}{M}$$

证明. 令 $w = w(z) = \frac{f(z)}{M}$, 做变换

$$F(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)} \quad \text{where } w(0) = \frac{f(0)}{M}$$

则该变换将单位圆 $|w| < 1$ 映射为 $|F(z)| < 1$. 由 Schwarz 引理 $|F'(0)| \leq 1$. 由于

$$\begin{aligned} F'(z)|_{z=0} &= \left. \left(\frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)} \right)' \right|_{z=0} = \lim_{z \rightarrow 0} \left(\frac{\frac{w(z) - w(0)}{z}}{1 - \overline{w(0)}w(z)} \right) = \frac{w'(0)}{1 - |w(0)|^2} \\ &= \frac{f'(0)M}{M^2 - |f(0)|^2} \end{aligned}$$

则

$$\left| \frac{f'(0)M}{M^2 - |f(0)|^2} \right| \leq 1 \implies |f'(0)|M \leq |M^2 - |f(0)||^2 = M^2 - |f(0)|^2$$

从而

$$|f'(0)| \leq M - \frac{|f(0)|^2}{M}$$

□

推广到 k 阶导数

15. 在施瓦茨引理的假设条件下, 如果原点是 $f(z)$ 的 λ 阶零点, 求证 $\left| \frac{f^{(\lambda)}(0)}{\lambda!} \right| \leqslant$

1. 要想这里的等号成立, 必须 $f(z) = e^{ia} z^\lambda$ (a 为实数, $|z| < 1$).

分析 由题条件可设

$$f(z) = c_\lambda z^\lambda + c_{\lambda+1} z^{\lambda+1} + \dots = z^\lambda (c_\lambda + c_{\lambda+1} z + \dots),$$

其中 $c_\lambda = \frac{f^{(\lambda)}(0)}{\lambda!} \neq 0$. 令

$$\varphi(z) = \begin{cases} \frac{f(z)}{z^\lambda}, & z \neq 0, \\ \frac{f^{(\lambda)}(0)}{\lambda!}, & z = 0, \end{cases}$$

则 $\varphi(z)$ 在 $|z| < 1$ 内解析 ($z=0$ 为 $\frac{f(z)}{z^\lambda}$ 的可去奇点).

证 仿照施瓦茨引理的证明, 应用最大模原理于 $\varphi(z)$.

考虑 $\varphi(z)$ 在单位圆 $|z| < 1$ 内任一点 z_0 处的值, 如果 r 满足条件 $|z_0| < r < 1$, 根据最大模原理, 有

$$|\varphi(z_0)| \leqslant \max_{|z|=r} |\varphi(z)| = \max_{|z|=r} \left| \frac{f(z)}{z^\lambda} \right| \leqslant \frac{1}{r^\lambda}.$$

令 $r \rightarrow 1$, 即得

$$|\varphi(z_0)| \leqslant 1.$$

于是

$$\left| \frac{f^{(\lambda)}(0)}{\lambda!} \right| = |\varphi(0)| \leqslant 1,$$

且当 $z_0 \neq 0$ 时, 有

$$\left| \frac{f(z_0)}{z_0^\lambda} \right| = |\varphi(z_0)| \leqslant 1,$$

即

$$|f(z_0)| \leqslant |z_0|^\lambda,$$

加上 $f(0)=0$, 就有 $|f(z)| \leqslant |z|^\lambda$ ($|z| < 1$).

如果这些关系中, 有一个取等号, 这就意味着在单位圆 $|z| < 1$ 内的某一点 z_0 , 模 $|\varphi(z_0)|$ 达到最大值, 这只有 $\varphi(z)$ 等于常数 e^{ia} (a 为实数) 时才可能, 此时必有

$$f(z) \equiv e^{ia} z^\lambda.$$

课堂练习

Exercice 4

设 $f(z)$ 在 $|z| < 1$ 内解析, $f(0) = 0$, $|Re(f(z))| < 1$, 则 $|f'(0)| \leq \frac{4}{\pi}$.

这类涉及解析函数在特定区域（通常是单位圆盘）内满足某些条件（如 $f(0) = 0$ 和其值域受限），并要求证明其导数在某点（通常是原点）的模有界的题目，通常采用以下标准解题方法：

解题步骤**(1) 确定函数的定义域和值域限制**

- **定义域 (D):** 通常是单位圆盘 $|z| < 1$ 。
- **标准化条件:** 常见的有 $f(0) = 0$ 。
- **值域 (S):** 根据题目条件确定 $f(z)$ 的取值范围。例如, $|Re(f(z))| < A$ 表示值域是带形区域 $\{-A < Re(w) < A\}$; $|f(z)| < M$ 表示值域是圆盘 $|w| < M$; $Re(f(z)) > 0$ 表示值域是右半平面。

(2) 构造辅助保形映射 (Conformal Mapping)

- 寻找一个解析函数 $H(w)$, 它能将 $f(z)$ 的值域 S 保形地映射到单位圆盘 D' ($|\zeta| < 1$)。
- 这个映射 $H(w)$ 需要将对应于 $f(0)$ 的点（在 S 中，通常是 0 点）映射到单位圆盘 D' 的原点。即，如果 $f(0) = w_0$ (在 S 中)，则需要 $H(w_0) = 0$ 。在常见的 $f(0) = 0$ 条件下，我们需要 $H(0) = 0$ 。

(3) 构造复合函数

- 定义一个新的函数 $G(z) = H(f(z))$ 。
- 验证 $G(z)$ 的性质:
 - $G(z)$ 在原定义域 D ($|z| < 1$) 内解析（因为 f 和 H 均解析）。
 - $G(0) = H(f(0)) = H(0) = 0$ (利用标准化条件和 H 的选取)。
 - $G(z)$ 将单位圆盘 D 映射到单位圆盘 D' ($|G(z)| < 1$ for $|z| < 1$)。

(4) 应用施瓦茨引理 (Schwarz's Lemma)

- 由于 $G(z)$ 满足施瓦茨引理的条件 (解析于 $|z| < 1$, $G(0) = 0$, $|G(z)| \leq 1$), 我们可以得到结论:
 - $|G(z)| \leq |z|$ for all z in D .
 - $|G'(0)| \leq 1$ (这是最关键的一步, 用于求解导数界限)。

(5) 计算导数并建立不等式

- 使用链式法则计算 $G'(z)$: $G'(z) = H'(f(z)) \cdot f'(z)$ 。
- 在 $z = 0$ 处取值: $G'(0) = H'(f(0)) \cdot f'(0)$ 。由于 $f(0) = 0$, 则 $G'(0) = H'(0) \cdot f'(0)$ 。
- 将此代入施瓦茨引理的结论: $|H'(0) \cdot f'(0)| \leq 1$ 。
- 从而得到: $|H'(0)| \cdot |f'(0)| \leq 1$ 。

(6) 求解目标界限

- 从上一步的不等式解出 $|f'(0)|$: $|f'(0)| \leq 1/|H'(0)|$ 。
- $|H'(0)|$ 是一个具体的数值, 取决于所构造的保形映射 $H(w)$ 。计算出这个值, 即可得到 $|f'(0)|$ 的上界。

(7) 讨论等号成立条件 (Sharpness)

- 等号 $|f'(0)| = 1/|H'(0)|$ 成立当且仅当施瓦茨引理中的等号 $|G'(0)| = 1$ 成立。
- 这要求 $G(z) = \lambda z$, 其中 $|\lambda| = 1$ 是一个常数。
- 因此, $H(f(z)) = \lambda z$, 即 $f(z) = H^{-1}(\lambda z)$ 。
- 这意味着使等号成立的函数 $f(z)$ 是将单位圆盘 D 保形映射 (可能经过一个旋转 λ) 到区域 S 的那些极值函数。
- 这个分析有助于判断题目要求的 $<$ (严格小于) 还是 \leq (小于等于) 是否准确, 并理解不等式的精确性。

常用保形映射示例:

- 带形 $|Re(w)| < A$ 到单位圆盘 $|\zeta| < 1$, 且 $w = 0$ 映到 $\zeta = 0$: $H(w) = \tanh(i\pi w/(4A))$ 或 $H(w) = (e^{i\pi w/(2A)} - 1)/(e^{i\pi w/(2A)} + 1)$ 。对于 $A = 1$, $H(w) = \tanh(i\pi w/4)$, 则 $|H'(0)| = \pi/4$ 。
- 右半平面 $Re(w) > 0$ 到单位圆盘 $|\zeta| < 1$, 且 $w = 1$ 映到 $\zeta = 0$ (需调整以使 $w = 0$ (如果 $f(0) = 0$ 且 0 在边界) 或某特定点映射到 0): 典型的如 $\zeta = (w - 1)/(w + 1)$ (这映射 $Re(w) > 0$ 到 $|\zeta| < 1$, 且 $w = 1$ 到 $\zeta = 0$)。如果 $f(0)$ 的像点在 S 中的某个位置 w_0 , 需要调整映射使得 $H(w_0) = 0$ 。

通过这套流程, 大部分此类问题都可以得到系统性的解决。关键在于正确识别 $f(z)$ 的值域 S 并找到合适的保形映射 $H(w)$ 。

7.11.6 Schwarz lemma and automorphisms of the disk

LectureNotes

We have seen that every rotation $z \mapsto e^{i\theta} \cdot z$, for fixed $\theta \in \mathbb{R}$, is an automorphism of the disk. For $a \in \mathbb{D}$ define

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}$$

which is also an automorphism of \mathbb{D} .

When $|z| = 1$, we have

$$|\varphi_a(z)| = \left| \frac{a - z}{1 - \bar{a}z} \right| = \left| \frac{a - z}{\bar{z} - \bar{a}} \right| \cdot |\bar{z}| = 1$$

Observe that

$$\varphi_a(a) = 0 \quad \varphi_a(0) = a$$

Then

$$\varphi_a \circ \varphi_a(0) = 0 \quad \varphi_a \circ \varphi_a(a) = a$$

By the Schwarz lemma, $\varphi_a \circ \varphi_a : z \mapsto e^{i\theta}z$, and $\theta = 0$. Thus $\varphi_a \circ \varphi_a$ is identity map of \mathbb{D} . It follows that φ_a is bijection from \mathbb{D} to \mathbb{D} .

Théorème 8: Automorphism of \mathbb{D}

A map f is an automorphism of \mathbb{D} iff there are $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$ s.t.

$$f(z) = e^{i\theta} \cdot \frac{a - z}{1 - \bar{a}z}$$

Properties of φ_a

Remarque 20. 这里的 φ_a 和前面的定义差了一个负号, 导致之前的 φ_a 是幂等映射, 而这里的 φ_a 的逆为 φ_{-a} .

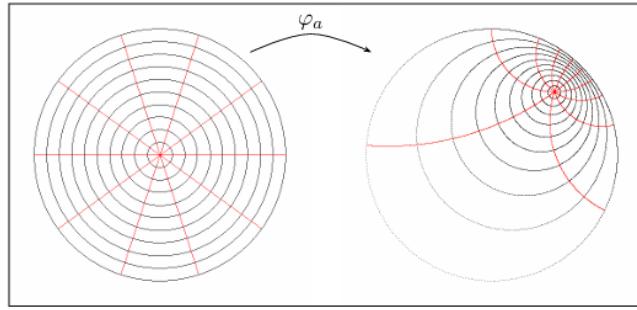


Figure 2.1: The images of the circles and rays under the map φ_a where $a = 0.5 + i0.5$.

$$\begin{aligned}\varphi_a(z) &= \frac{z - a}{1 - \bar{a}z} \\ \varphi'_a(z) &= \frac{\bar{a}z - |a|^2}{(1 - \bar{a}z)^2} + \frac{1 - \bar{a}z}{(1 - \bar{a}z)^2} = \frac{1 - |a|^2}{(1 - \bar{a}z)^2} \\ |\varphi'_a(0)| &= 1 - |a|^2 \quad |\varphi'_a(a)| = (1 - |a|^2)^{-1} \\ (\varphi_a \circ \varphi_{-a})(z) &= \frac{\varphi_{-a}(z) - a}{1 - \bar{a}\varphi_{-a}(z)} = \frac{\frac{z+a}{1+\bar{a}z} - a}{1 - \bar{a}\frac{z+a}{1+\bar{a}z}} = \frac{(1 - |a|^2)z}{1 - |a|^2} = z \Rightarrow \varphi_a \circ \varphi_{-a} = id\end{aligned}$$

For $a \in \mathbb{D}$, $f \in H(\mathbb{D})$, $\alpha := f(a)$, $|f(\mathbb{D})| < 1$, evaluate the maximum of $|f'(a)|$.

Denote $g := \varphi_\alpha \circ f \circ \varphi_{-a}$, then

$$\begin{aligned}|g'(0)| &= |(\varphi_\alpha \circ f)'| \cdot \underbrace{|\varphi_{-a}(0)|}_{=a} \cdot \underbrace{|\varphi'_{-a}(0)|}_{=1-|a|^2} \\ &= |\varphi'_\alpha(f(a))| \cdot f'(a)(1 - |a|^2) \\ &= f'(a) \frac{1 - |a|^2}{1 - |\alpha|^2}\end{aligned}$$

Apply Schwarz lemma to g , with $g(0) = 0, |g(\mathbb{D})| < 1$ then

$$\left| f'(a) \frac{1 - |a|^2}{1 - |\alpha|^2} \right| = |g'(0)| \leq 1 \implies |f'(a)| \leq \frac{1 - |f(a)|^2}{1 - |a|^2}$$

At the same time, by Schwarz lemma, for any $z \in \mathbb{D}$,

$$|g(z)| = |\varphi_\alpha(f(\varphi_{-a}(z)))| = \left| \frac{f(\varphi_{-a}(z)) - \alpha}{1 - \bar{\alpha}f(\varphi_{-a}(z))} \right| \leq |z|$$

Since $\varphi_a : \mathbb{D} \rightarrow \mathbb{D}$ is automorphism, replace z by $\varphi_a(b)$ where $b \in \mathbb{D}$, then

$$|\varphi_a(b)| \geq \left| \frac{f(\varphi_{-a}(\varphi_a(b))) - \alpha}{1 - \bar{\alpha}f(\varphi_{-a}(\varphi_a(b)))} \right| = \left| \frac{f(b) - f(a)}{1 - \bar{f(a)}f(b)} \right|$$

i.e.

$$\left| \frac{f(b) - f(a)}{1 - \bar{f(a)}f(b)} \right| \leq \left| \frac{b - a}{1 - \bar{a}b} \right| \quad \forall a, b \in \mathbb{D}$$

The above two inequalities is called **Schwarz-pick lemma**.

Schwarz-pick lemma

<http://www.webpages.ttu.edu/jengwer/notes/SchwarzPickLemmas.pdf>

NOTATION:

$$B(a; r) \equiv \{z \in \mathbb{C} : |z - a| < r\} \text{ (open ball)} \quad \bar{B}(a; r) \equiv \{z \in \mathbb{C} : |z - a| \leq r\} \text{ (closed ball)}$$

$$\mathbb{D} \equiv B(0; 1) = \{z \in \mathbb{C} : |z| < 1\}$$

$$\mathcal{A}(G) \equiv H(G) \equiv \{\text{analytic functions on region } G\} \quad \mathcal{M}(G) \equiv M(G) \equiv \{\text{meromorphic functions on region } G\}$$

$$\text{Hyperbolic Metric : } \rho_h(z, w) := 2 \operatorname{arctanh} \left| \frac{w - z}{1 - \bar{w}z} \right| \text{ where } z, w \in \mathbb{D}$$

Disc Automorphism is a one-to-one analytic map of the unit disc (\mathbb{D}) to itself.

SCHWARZ LEMMA:

$$\left\{ \begin{array}{l} f \in \mathcal{A}(\mathbb{D}) \\ f(0) = 0 \\ |f(z)| \leq 1 \quad \forall z \in \mathbb{D} \end{array} \right\} \implies \left\{ \begin{array}{l} |f'(0)| \leq 1 \\ |f(z)| \leq |z| \quad \forall z \in \mathbb{D} \end{array} \right\}$$

$$\text{Moreover, } [|f'(0)| = 1 \text{ OR } |f(z_*)| = |z_*| \text{ for some } z_* \neq 0] \implies \exists \lambda \in \mathbb{C} \text{ s.t. } \left\{ \begin{array}{l} |\lambda| = 1 \\ f(z) = \lambda z \quad \forall z \in \mathbb{D} \end{array} \right\}$$

SCHWARZ LEMMA (GENERALIZED):

$$\left\{ \begin{array}{l} f \in \mathcal{A}(B(z_0; R)) \\ f(z_0) = 0 \\ |f(z)| \leq M \quad \forall z \in B(z_0; R) \end{array} \right\} \implies \left\{ \begin{array}{l} |f'(z_0)| \leq 1 \\ |f(z)| \leq \frac{M}{R} |z - z_0| \quad \forall z \in B(z_0; R) \end{array} \right\}$$

$$\text{Moreover, } [|f'(z_0)| = 1 \text{ OR } |f(z_*)| = |z_*| \text{ for some } z_* \neq z_0] \implies \exists \lambda \in \mathbb{C} \text{ s.t. } \left\{ \begin{array}{l} |\lambda| = 1 \\ f(z) = \lambda z \quad \forall z \in B(z_0; R) \end{array} \right\}$$

SCHWARZ-PICK LEMMA:

$$\left\{ \begin{array}{l} f \in \mathcal{A}(\mathbb{D}) \\ |f(z)| < 1 \quad \forall z \in \mathbb{D} \end{array} \right\} \implies |f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2} \quad \forall z \in \mathbb{D}$$

If f is a **disc automorphism**, then equality holds. Else, strict inequality holds.

SCHWARZ-PICK LEMMA (METRIC VERSION):

$$f \in \mathcal{A}(\mathbb{D}) \text{ s.t. } f : \mathbb{D} \rightarrow \mathbb{D} \implies \rho_h[f(z), f(w)] \leq \rho_h(z, w) \quad \forall z, w \in \mathbb{D}$$

If f is a **disc automorphism**, then equality holds. Else, strict inequality holds.

7.11.7 Applications of Schwarz lemma

Exercice 5: Stanford Ph. D test

- (b) Suppose that f is analytic with $|f(z)| < 1$ in $|z| < 1$ and that $f(\pm a) = 0$ where a is a complex number with $0 < |a| < 1$. Show that $|f(0)| \leq a^2$. What can you conclude if this holds with equality.

证明. Define

$$F(z) = f(z) \cdot \frac{1 - \bar{a}z}{z - a} \cdot \frac{1 + \bar{a}z}{z + a}$$

then $F(z)$ is analytic in $\{|z| < 1\}$. When $|z| = 1$,

$$\left| \frac{1 - \bar{a}z}{z - a} \cdot \frac{1 + \bar{a}z}{z + a} \right| = 1$$

$$\lim_{|z| \rightarrow 1} |F(z)| \leq 1,$$

which implies that $|F(z)| \leq 1$ for $|z| < 1$. Take $z = 0$, we obtain

$$|f(0)| \leq |a|^2$$

When it holds with equality, we have $F(z) \equiv e^{i\theta}$, which is equivalent to

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z} \cdot \frac{z + a}{1 + \bar{a}z}$$

□

Exercice 6: Stanford

(c) Determine all entire function f that $|f'(z)| < |f(z)|$.

证明. (c) From $|f'(z)| < |f(z)|$, we know that f has no zero in \mathbb{C} , which implies that $\frac{f'(z)}{f(z)}$ is also an entire function. It follows from $\left| \frac{f'(z)}{f(z)} \right| < 1$ that $\frac{f'(z)}{f(z)} = c$, $|c| < 1$. Integrating on both sides, we obtain $\log f(z) = cz + d$. Hence $f(z) = c'e^{cz}$, where c and c' are constants and $|c| < 1$. □

7.12 解析函数

7.12.1 可微性考点

- 判定可微
- 判定不可微
- C-R 方程
- f 在 z_0 解析 $\Rightarrow f$ 在 z_0 可微
- f 在 z_0 解析 $\neq f$ 在 z_0 可微
 - 存在可微但不解析的点
 - 解析是一个局部的概念，指的是在 z_0 附近都可微
 - 可微是一个点态的概念
- 极坐标形式的 C-R 方程
- 若 $f \in H$ 则 $\bar{\partial}f = 0$.
- 洛必达法则

Exercice 7: 在 z_0 可微但不解析

例 2.1.9 考察函数 $f(z) = x^3 - y^3 + 2x^2y^2i$ 的可微性和解析性.

分析 应用推论 2.3 考察可微性; 应用定义 2.2 考察解析性.

解 因为

$$\begin{aligned} u(x, y) &= x^3 - y^3, \quad u_x = 3x^2, \quad u_y = -3y^2, \\ v(x, y) &= 2x^2y^2, \quad v_x = 4xy^2, \quad v_y = 4x^2y, \end{aligned}$$

所以此四个偏导数在 z 平面上连续. 又从

$$\begin{aligned} 3x^2 = u_x = v_y &= 4x^2y \Rightarrow x = 0 \text{ 或 } y = \frac{3}{4}, \\ -3y^2 = u_y = -v_x &= -4xy^2 \Rightarrow y = 0 \text{ 或 } x = \frac{3}{4}, \end{aligned}$$

可见, 仅当 $x = 0, y = 0$; $x = \frac{3}{4}, y = \frac{3}{4}$ 时, C.-R. 方程才成立. 故由推论 2.3 知道, 此函数仅在点 $(0, 0)$ 及 $(\frac{3}{4}, \frac{3}{4})$ 两点可微, 从而在 z 平面 \mathbf{C} 上处处不解析.

又由求导公式(2.7)得

$$\begin{aligned} f'(0) &= (u_x + iv_x)|_{(0,0)} = (3x^2 + 4xy^2i)|_{(0,0)} = 0, \\ f'\left(\frac{3}{4} + \frac{3}{4}i\right) &= (u_x + iv_x)|_{\left(\frac{3}{4}, \frac{3}{4}\right)} = (3x^2 + 4xy^2i)|_{\left(\frac{3}{4}, \frac{3}{4}\right)} \\ &= \frac{3^3}{4^2}(1+i). \end{aligned}$$

■

(2) 设 $f(z) = u(x, y) + iv(x, y)$ 在区域 D 内有定义, 则

定理 2.1 $f(z) = u(x, y) + iv(x, y)$ 在点 $z = x + iy \in D$ 可微

$$\Leftrightarrow \begin{cases} \text{偏导数 } u_x, u_y, v_x, v_y \text{ 在点 } (x, y) \text{ 存在,} \\ u(x, y), v(x, y) \text{ 在点 } (x, y) \text{ 满足 C.-R. 方程.} \end{cases}$$

定理 2.2 $f(z) = u(x, y) + iv(x, y)$ 在点 $z = x + iy \in D$ 可微

$$\Leftrightarrow \begin{cases} u(x, y), v(x, y) \text{ 在点 } (x, y) \text{ 可微,} \\ u(x, y), v(x, y) \text{ 在点 } (x, y) \text{ 满足 C.-R. 方程.} \end{cases}$$

推论 2.3 $f(z) = u(x, y) + iv(x, y)$ 在点 $z = x + iy \in D$ 可微

$$\Leftrightarrow \begin{cases} u_x, u_y, v_x, v_y \text{ 在点 } (x, y) \text{ 连续,} \\ u(x, y), v(x, y) \text{ 在点 } (x, y) \text{ 满足 C.-R. 方程.} \end{cases}$$

定理 2.4

$f(z)$ 在区域 D 内解析 $\Leftrightarrow \begin{cases} \text{在 } D \text{ 内 } u(x, y), v(x, y) \text{ 可微,} \\ \text{在 } D \text{ 内 C.-R. 方程成立.} \end{cases}$

定理 2.5 及定理 3.15

$f(z)$ 在区域 D 内解析 $\Leftrightarrow \begin{cases} u_x, u_y, v_x, v_y \text{ 在 } D \text{ 内连续,} \\ \text{在 } D \text{ 内 C.-R. 方程成立.} \end{cases}$

7. 从以上几个定理我们可以看出: C.-R. 方程是判断复变函数在一点可微或在一个区域内解析的主要条件. 在哪一点不满足它, 函数在那一点就不可微; 在哪个区域内不满足它, 函数在那个区域内就不解析.

Remarque 21.

11. 洛必达(L'Hospital)法则(一阶导数形式, 教材第二章习题(一)第2题).

若 $f(z)$ 及 $g(z)$ 在点 z_0 解析, 且

$$f(z_0) = g(z_0) = 0, \quad g'(z_0) \neq 0,$$

则

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

注 数学分析中实函数的微分中值定理不能直接推广到复函数上来.

7.12.2 初等解析函数

考点:

- $e^{z_1} = e^{z_2} \iff z_1 = z_2 + 2k\pi i, k \in \mathbb{Z}$.
- $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}), \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$
- 在 \mathbb{C} 不能断言 $|\sin z| \leq 1, |\cos z| \leq 1$.
- 证明 $f(z)$ 的一致连续性和不一致连续性
- $\tan z = \frac{\sin z}{\cos z}, \cot z = \frac{\cos z}{\sin z}$.
 - $\cos z$ 的零点 $(n + \frac{1}{2})\pi, n \in \mathbb{Z}$ 是 $\tan z$ 在 \mathbb{C} 的全部奇点
 - $\sin z$ 的零点 $n\pi, n \in \mathbb{Z}$, 是 $\cot z$ 在 \mathbb{C} 的全部奇点
- $\sinh z = \sin(\frac{z}{i}), \cosh z = \cos(\frac{z}{i})$

Exercice 8

证明

$$\lim_{n \rightarrow \infty} \left(1 + \frac{z}{n}\right)^n = e^z$$

证明. 只需证明

$$\lim_{n \rightarrow \infty} \left| \left(1 + \frac{z}{n}\right)^n \right| = e^x$$

$$\lim_{n \rightarrow \infty} \arg \left(1 + \frac{z}{n}\right)^n = \arg e^z = y$$

□

Exercice 9

例 2.2.5 函数 $f(z) = e^{-\frac{1}{z}}$ 在 \mathbf{C} 上除 $z=0$ 外都有定义, 试证明:

- (1) 在去心半圆“ $0 < |z| \leq 1, |\arg z| \leq \frac{\pi}{2}$ ”上函数 $f(z)$ 有界;
- (2) 在(1)中去心半圆上 $f(z)$ 连续, 但不一致连续;
- (3) 在去心扇形“ $0 < |z| \leq 1, |\arg z| \leq \alpha < \frac{\pi}{2}$ ”上 $f(z)$ 一致连续.

分析 (1) 证明存在 $M > 0$, 使在此去心半圆上 $|f(z)| \leq M (z \neq 0)$.

(2) 由于 $f(z)$ 在原点不连续. 证明在原点邻近总存在充分接近的两点 z', z'' 使 $|f(z') - f(z'')|$ 不能任意小.

(3) 先证明 $f(z)$ 在有界闭集“ $0 \leq |z| \leq 1, |\arg z| \leq \alpha < \frac{\pi}{2}$ ”上连续.

证 (1) 令 $z = r e^{i\varphi} = r(\cos \varphi + i \sin \varphi), r \neq 0$.

因为 $f(z) = e^{-\frac{1}{z}} = e^{-\frac{\cos \varphi - i \sin \varphi}{r}}$, 所以 $|f(z)| = e^{-\frac{\cos \varphi}{r}}$. 当 $|\varphi| = |\arg z| \leq \frac{\pi}{2}$ 时,

$\cos \varphi \geq 0$. 由于 e^x 是增函数, 故有

$$|f(z)| = e^{-\frac{\cos \varphi}{r}} \leq e^0 = 1 \quad (r \neq 0).$$

(2) 由 $z \neq 0$ 知 $-\frac{1}{z}$ 是 z 的连续函数. 因而 $f(z) = e^{-\frac{1}{z}}$ 在(1)中去心半圆上连续,

但不一致连续. 事实上, 对 $\varepsilon_0 = \frac{1}{2}$, 无论 δ 多么小, 总存在两点

$$z' = \frac{i}{(2k + \frac{1}{2})\pi} \quad \text{与} \quad z'' = -\frac{i}{(2k + \frac{1}{2})\pi},$$

虽然 $|z' - z''| = \frac{4}{(4k+1)\pi} < \delta$ (只要 k 充分大), 但

$$\begin{aligned} |e^{-\frac{1}{z}} - e^{-\frac{1}{z'}}| &= |e^{i(2k+\frac{1}{2})\pi} - e^{-i(2k+\frac{1}{2})\pi}| \\ &= 2 \left| \sin\left(2k + \frac{1}{2}\right)\pi \right| = 2 > \frac{1}{2} = \varepsilon_0. \end{aligned}$$

(3) 由(1)知 $|f(z)| = e^{-\frac{\cos \varphi}{r}}$, 而当 $|\varphi| \leq \alpha < \frac{\pi}{2}$ 时, 有 $\cos \varphi \geq \cos \alpha$, 所以

$$|f(z)| = e^{-\frac{\cos \varphi}{r}} \leq e^{-\frac{\cos \alpha}{r}} \rightarrow 0 \quad (r \rightarrow 0).$$

故

$$\lim_{z \rightarrow 0} f(z) = 0.$$

若定义当 $z=0$ 时, $f(z)=0$, 则

$$f(z) = \begin{cases} e^{-\frac{1}{z}}, & z \neq 0, \\ 0, & z = 0 \end{cases}$$

在有界闭扇形 “ $0 \leq |z| \leq 1, |\arg z| \leq \alpha < \frac{\pi}{2}$ ” 上连续, 因而一致连续. 从而 $f(z) = e^{-\frac{1}{z}}$

在去心扇形 “ $0 < |z| \leq 1, |\arg z| \leq \alpha < \frac{\pi}{2}$ ” 上一致连续. ■

证明. □

7.12.3 初等多值函数

这一节的主要内容是采用限制辐角或割破平面的方法, 来分出根式函数和对数函数的单值解析分支。

2. 根式函数 $w = \sqrt[n]{z}$ (整数 $n \geq 2$) 是整幂函数 $z = w^n$ 的反函数, 对 $z = re^{i\theta} (\neq 0, \infty)$, 其常用的 n 个单值连续解析分支为

$$w_k = (\sqrt[n]{z})_k = \sqrt[n]{r(z)} e^{\frac{i(\theta(z)+2k\pi)}{n}},$$

$k = 0, 1, 2, \dots, n-1, z \in G; -\pi < \theta < \pi$ (或 $0 < \theta < 2\pi$).

式中 $-\pi < \theta < \pi$ 等价于从原点起沿负实轴割破 z 平面, $0 < \theta < 2\pi$ 等价于从原点起沿正实轴割破 z 平面. $z=0$ 与 ∞ 是此根式函数的支点, 连接两支点的割线叫支割线.

$(\sqrt[n]{z})_0$ 为主值支. 各支的导数分别为

$$(\sqrt[n]{z})'_k = \frac{1}{n} \frac{(\sqrt[n]{z})_k}{z} \quad (k = 0, 1, 2, \dots, n-1).$$

Exercice 10

例 2.3.7 已给 $f(z) = \sqrt[3]{z-1}$. (1) 求 $f(z)$ 的支点; (2) 证明在 z 平面割去线段 $(-\infty, 1]$ 的区域 G 内 $f(z)$ 能分出三个单值解析分支; (3) 确定在点 $z=2$ 取正值那一个分支在点 $z=-1+i$ 的值.

解 (1) $f(z) = \sqrt[3]{z-1}$ 的支点为 $z=1, \infty$.

(2) 由于支割线 $(-\infty, 1]$ 是连接支点 1 和 ∞ 的射线, 所以 G 是 $f(z)$ 能分出三个单值解析分支的最大区域.

(3) 令 $z-1=re^{i\theta}$, 则

$$f_k(z) = (\sqrt[3]{z-1})_k = \sqrt[3]{r} e^{i\frac{\theta+2k\pi}{3}}, \quad k=0,1,2,$$

即

$$f_0(z) = (\sqrt[3]{z-1})_0 = \sqrt[3]{r} e^{i\frac{\theta}{3}} \quad (-\pi < \theta = \arg(z-1) < \pi),$$

$$f_1(z) = \omega f_0(z), \quad f_2(z) = \omega^2 f_0(z), \quad \omega = e^{\frac{2\pi i}{3}}.$$

当 $z=2$ 时, $|z-1|=r_0=1, \theta_0=0$. 要求 $f_k(z)$ 取正值, 必 $k=0$, 故所取分支为

$$f_0(z) = \sqrt[3]{r} e^{i\frac{\theta}{3}}, \quad -\pi < \theta < \pi.$$

由已给条件 $\arg f_0(2)=0$. 当 $z=-1+i$ 时, $|z-1|=\sqrt{5}$,

$$|f_0(-1+i)| = (\sqrt{5})^{\frac{1}{3}} = \sqrt[6]{5}.$$

又

$$\begin{aligned} \Delta_c \arg f(z) &= \frac{1}{3} \Delta_c \arg(z-1) = \frac{1}{3}(\pi - \alpha) \\ &= \frac{1}{3} \left(\pi - \arctan \frac{1}{2} \right) \quad (\text{图 2.3.5}), \end{aligned}$$

所以

$$\begin{aligned} f_0(-1+i) &\stackrel{(2.28)}{=} \sqrt[6]{5} e^{i\frac{1}{3}(\pi - \arctan \frac{1}{2})} e^{i\alpha} \\ &= \sqrt[6]{5} e^{i\frac{1}{3}(\pi - \arctan \frac{1}{2}) + i\alpha}. \end{aligned}$$

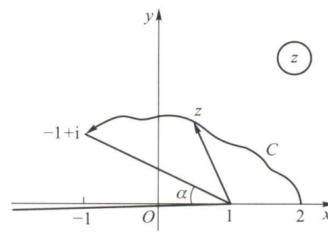


图 2.3.5

4. 对数函数 $w = \ln z$ ($z \neq 0, \infty$) 是指数函数 $z = e^w$ 的反函数.

(1) 当 $z = re^{i\theta}$ 时, 其常用的无穷多个单值连续解析分支为

$$w = \ln z = \ln |z| + i \operatorname{Arg} z,$$

即

$$w_k = (\ln z)_k = \ln r(z) + i[\theta(z) + 2k\pi] \quad (2.26)$$

$(z \in G; -\pi < \theta(z) < \pi, k = 0, \pm 1, \pm 2, \dots)$.

$z = 0, \infty$ 是此对数函数的支点, 负实轴 $(-\infty, 0]$ 是支割线.

(2) 通常把 $\ln z = \ln r + i\theta$ ($-\pi < \theta < \pi$) 称为对数函数的主值支, 即 (2.26) 式 $k = 0$ 那一支.

(3) 各支的导数为 $\frac{d}{dz}(\ln z)_k = \frac{1}{z}$ ($k = 0, \pm 1, \pm 2, \dots$).

(4) 对数函数的基本性质

$$\left. \begin{aligned} \ln(z_1 z_2) &= \ln z_1 + \ln z_2, \\ \ln \frac{z_1}{z_2} &= \ln z_1 - \ln z_2 \end{aligned} \right\} (z_1, z_2 \neq 0, \infty). \quad (2.24)$$

5. 一般幂函数 $w = z^\alpha = e^{\alpha \ln z}$ ($z \neq 0, \infty; \alpha$ 为复常数) 与一般指数函数 $w = a^z = e^{z \ln a}$ ($a \neq 0$ 为复常数).

这两种函数都可看作复合函数, 它们的性质可由其他函数的性质推导出来.

7.12.4 具有多个有限支点的多值函数

6. 具有多个有限支点的多值函数.

前面已经说过, 我们的研究对象是解析函数, 但是由于辐角的多值性, 不可避免地要涉及某些初等多值函数, 如已讨论过的单有限支点情形的根式和对数函数.

(1) 对具有多个有限支点的多值函数, 我们就不便采取限制辐角范围的办法, 而是首先求出该函数的一切交点, 然后适当连接支点以割破 z 平面. 于是, 在 z 平面上以此割线为边界的区域 G 内就能分出该函数的单值解析分支. 因为在 G 内变点不能穿过支割线, 也就不能单独绕任一个支点转一整周, 函数就不可能在 G 内同一点取不同的值了.

(2) 讨论函数

$$w = f(z) = \sqrt[n]{R(z)} \quad (2.27)'$$

的支点,其中有理函数 $R(z) = \frac{P(z)}{Q(z)}$,多项式

$$P(z) = A(z - a_1)^{\alpha_1}(z - a_2)^{\alpha_2} \cdots (z - a_m)^{\alpha_m}, \quad \alpha_1 + \alpha_2 + \cdots + \alpha_m = N,$$

$$Q(z) = B(z - b_1)^{\beta_1}(z - b_2)^{\beta_2} \cdots (z - b_l)^{\beta_l}, \quad \beta_1 + \beta_2 + \cdots + \beta_l = M.$$

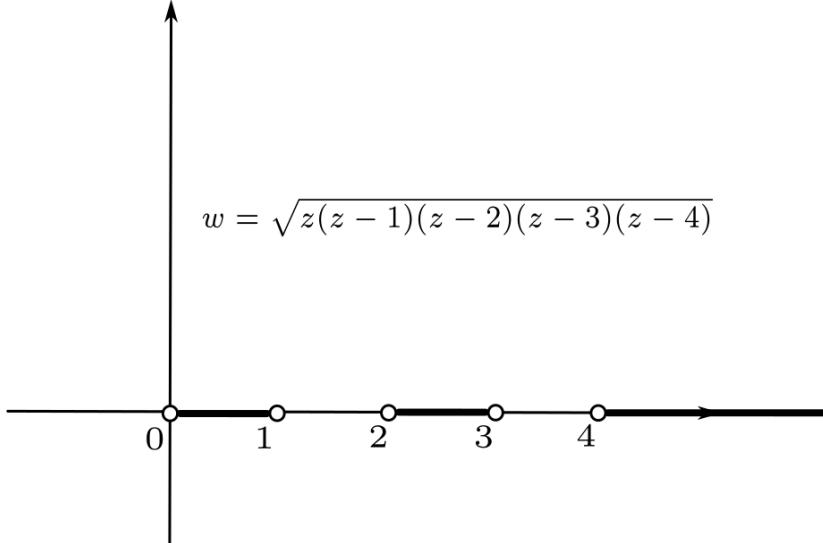
今对(2.27)'式作像教材(2.27)式的类似讨论,就能得到类似教材的下列结论:

- (a) (2.27)'式可能的支点是 $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_l$ 和 ∞ ;
- (b) 当且仅当 n 不能整除 α_i 或 β_j 时, a_i 或 b_j 是 $\sqrt[n]{R(z)}$ 的支点;
- (c) 当且仅当 n 不能整除 $N - M$ 时, ∞ 是 $\sqrt[n]{R(z)}$ 的支点;
- (d) 如果 n 能整除 $\alpha_1, \alpha_2, \dots, \alpha_m, -\beta_1, -\beta_2, \dots, -\beta_l$ 中若干个之和,则 $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_l$ 中对应的那几个就可以连接成割线,抱成团,即变点 z 沿只含它们在其内部的简单闭曲线转一整周后函数值不变. 这种抱成的团可能不止一个. 其余不入团的点则与点 ∞ 连接成一条割线.

例如,对

$$w = \sqrt{z(z-1)(z-2)(z-3)(z-4)},$$

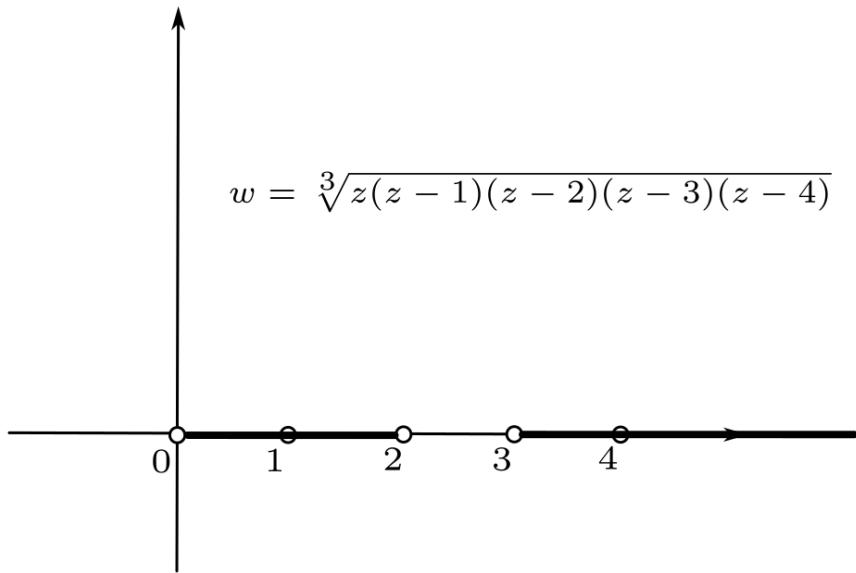
就可将 0 与 1,2 与 3 分别用直线段连接成割线,抱成两个团,再把余下的 4 与点 ∞ 连接成一条割线.



又如,对

$$w = \sqrt[3]{z(z-1)(z-2)(z-3)(z-4)},$$

就可将 0,1,2 用直线段连接成一条割线, 抱成一个团, 再把余下的 3,4 与点 ∞ 连接成一条割线.



由已给单值解析分支 $f(z)$ 的初值 $f(z_1)$, 计算终值 $f(z_2)$ 的公式为

$$f(z_2) = |f(z_2)| e^{i \Delta_C \arg f(z)} e^{i \arg f(z_1)}$$

其中 C 为连接起点 z_1 和终点 z_2 且不穿过支割线的路线.

当把 $z_2 \in G$ 换成 G 的动点 z 时, 得到的

$$f(z) = |f(z)| e^{i \Delta_C \arg f(z)} e^{i \arg f(z_1)}$$

也是此单值解析分支的解析表达式.

Exercice 11

例 2.3.14 设 $w = f(z) = \sqrt{z(z-1)(z-2)}$ 确定在沿正实轴割破的 z 平面上, 且 $f(-1) = -\sqrt{6}i$, 试求 $f(i)$ 之值.

分析 求支点, 适当割破 z 平面, 利用(2.28)式计算 $f(i)$.

解 因 $2 \nmid 1, 2 \nmid (1+1+1)$, 但 $2 \mid 2$, 故此函数的支点为 $0, 1, 2$ 和 ∞ .

可取支割线为从原点出发的正实轴, 则 $f(z)$ 在沿它割破 z 平面得到的区域 D 上就能分出两个单值解析分支. 因 z 从 -1 沿 D 内一条简单曲线 C 变动到 i 时,

$$\begin{aligned}\Delta_c \arg f(z) &= \frac{1}{2} [\Delta_c \arg z + \Delta_c \arg(z-1) + \Delta_c \arg(z-2)] \\ &= \frac{1}{2} \left(-\frac{\pi}{2} - \frac{\pi}{4} - \arctan \frac{1}{2} \right) \quad (\text{图 2.3.7}),\end{aligned}$$

故

$$\begin{aligned}f(i) &\stackrel{(2.28)}{=} |i(i-1)(i-2)|^{\frac{1}{2}} e^{-\frac{i}{2}(\frac{\pi}{2} + \frac{\pi}{4} + \arctan \frac{1}{2})} e^{-\frac{\pi}{2}i} \\ &= -\sqrt[4]{10} e^{\frac{1}{2}(\frac{\pi}{4} - \arctan \frac{1}{2})} = -\sqrt[4]{10} e^{\frac{i}{2}\arctan \frac{1}{3}},\end{aligned}$$

这是因为 $\arg f(-1) = \arg(-\sqrt{6}i) = -\frac{\pi}{2}$, 且

$$\tan\left(\frac{\pi}{4} - \arctan \frac{1}{2}\right) = \frac{\tan \frac{\pi}{4} - \tan\left(\arctan \frac{1}{2}\right)}{1 + \tan \frac{\pi}{4} \tan\left(\arctan \frac{1}{2}\right)} = \frac{1}{3}.$$

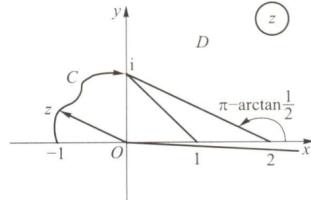


图 2.3.7

Exercice 12

例 2.3.15 试证: 在将 z 平面适当割开后, 函数

$$f(z) = \sqrt[4]{z(1-z)^3}$$

能分出四个单值解析分支. 并求出在割线上岸取正值那一支在 $z = -1$ 的值.

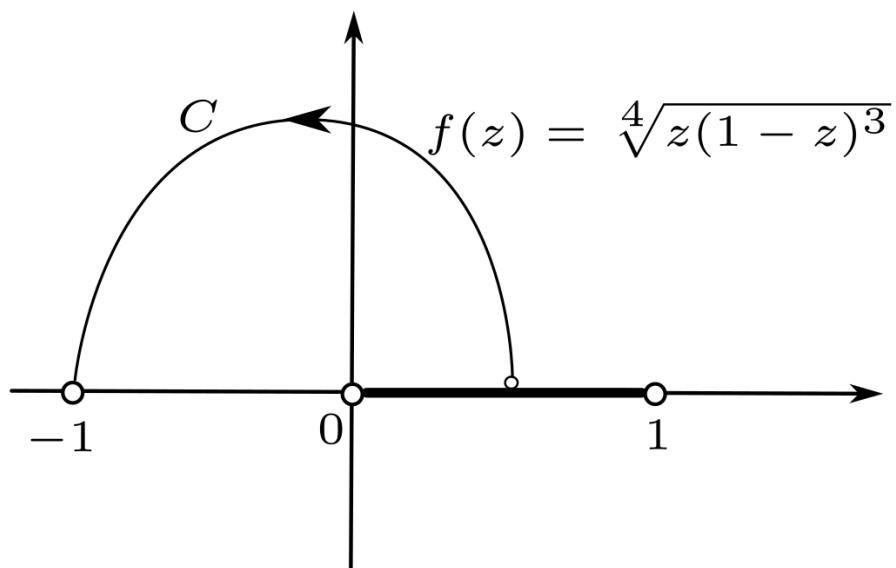
证 此函数的支点是 0 和 1 (因为 $4 \nmid 1, 4 \nmid 3$), 而 ∞ 不是支点 (因为 $4 \mid (1+3)$). 可在 z 平面上取线段 $[0, 1]$ 为支割线, 得一个以它为边界的区域 D , 在 D 内可以把 $f(z)$ 分成四个单值解析分支. 因 z 沿 D 内一条简单曲线从割线上岸变动到 -1 时,

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$$\begin{aligned}\Delta_C \arg f(z) &= \frac{1}{4} [\Delta_C \arg z + 3\Delta_C \arg(1-z)] \\ &= \frac{1}{4}(\pi + 3 \times 0) = \frac{\pi}{4} \quad (\text{图 2.3.8}),\end{aligned}$$

故

$$\begin{aligned}f(-1) &\xrightarrow{(2.28)} \sqrt[4]{(-1)[1-(-1)]^3} | e^{\frac{\pi}{4}i} e^{i \cdot 0} \\ &= \sqrt[4]{8} \cdot \frac{1}{\sqrt{2}}(1+i) = \sqrt[4]{2}(1+i).\end{aligned}$$



Exercice 13

例 2.3.16 设函数 $f(z) = \sqrt{z(1-z)}$ 的单值解析分支区域为 D .

(1) 求在支割线 $[0,1]$ 上岸取正值的那一支 $f_+(z)$ 的表达式;

(2) 求 $f_+(-1), f_+(4), f_+(\sqrt{3}i), f_+(-\sqrt{3}i), f_+(\frac{1}{2}+yi)$ 的值, 其中 y 为非零实数.

解 此函数的支点为 $0, 1$ (因为 $2 \nmid 1$), 而 ∞ 不是支点 (因为 $2 \mid (1+1)$). 在 z 平面上取线段 $[0,1]$ 为支割线, 得一个以它为边界的区域 D , $f(z)$ 在 D 内可以分成两个单值解析分支.

(1) 设 $z = r_1(z)e^{i\theta_1(z)}, 1-z = r_2(z)e^{i\theta_2(z)}$, 则

$$f_k(z) = \sqrt{r_1(z)r_2(z)} e^{\frac{\theta_1(z)+\theta_2(z)+2k\pi}{2}} \quad (z \in D, k=0,1).$$

当 z 在 $[0,1]$ 上岸时 (图 2.3.8), $\theta_1=0, \theta_2=0$,

$$0 < f_k(z) = \sqrt{r_1 r_2} e^{\frac{2k\pi i}{2}} \Leftrightarrow k=0,$$

故所求解析分支的表达式为

$$f_+(z) = \sqrt{r_1(z)r_2(z)} e^{\frac{\theta_1(z)+\theta_2(z)}{2}} \quad (z \in D),$$

或

$$f_+(z) \stackrel{(2.28)^*}{=} \sqrt{r_1(z)r_2(z)} e^{\frac{1}{2}[\Delta_C \arg z + \Delta_C \arg(1-z)]} e^{i \cdot 0} \quad (z \in D).$$

(2) 因为 $-1 \in D$, 所以 $f_+(-1) = \sqrt{1 \times 2} e^{\frac{i}{2}(\pi+0)} = \sqrt{2}i$ (图 2.3.8).

因为 $4 \in D$, 所以 $f_+(4) = \sqrt{4 \times 3} e^{\frac{i}{2}(0+(-\pi))} = -\sqrt{12}i$ (图 2.3.9).

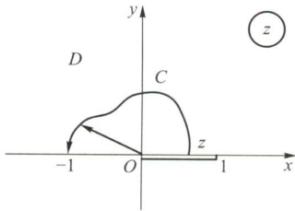


图 2.3.8

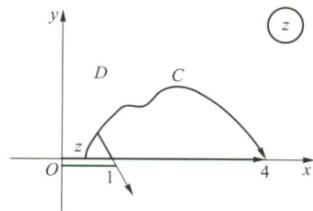


图 2.3.9

因为 $\sqrt{3}i \in D$, 所以 $f_+(\sqrt{3}i) = \sqrt[4]{12} e^{\frac{i}{2}[\frac{\pi}{3} + (-\frac{\pi}{3})]} = \sqrt[4]{12} e^{\frac{\pi}{12}i}$ (图 2.3.10).

因为 $-\sqrt{3}i \in D$, 所以 $f_+(-\sqrt{3}i) = \sqrt[4]{12} e^{\frac{i}{2}(\frac{3\pi}{2} + \frac{\pi}{3})} = \sqrt[4]{12} e^{\frac{11}{12}\pi i}$ (图 2.3.11).

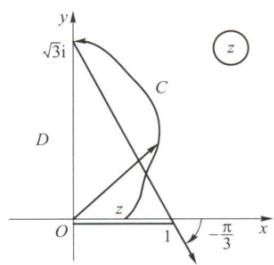


图 2.3.10

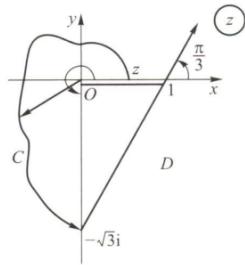


图 2.3.11

又

$$\begin{aligned}
 f_0\left(\frac{1}{2} + yi\right) &= \sqrt{\left|\frac{1}{2} + yi\right| \left|\frac{1}{2} - iy\right|} e^{i\frac{1}{2}[\Delta_C \arg z + \Delta_C \arg(1-z)]} \\
 &= \left|\frac{1}{2} + yi\right| e^{i\frac{1}{2}[\Delta_C \arg z + \Delta_C \arg(1-z)]} \\
 &= \begin{cases} \left|\frac{1}{2} + yi\right|, & y > 0 \text{ (图 2.3.12),} \\ \left|\frac{1}{2} + yi\right| e^{i\pi} = -\left|\frac{1}{2} + yi\right|, & y < 0 \text{ (图 2.3.13).} \end{cases}
 \end{aligned}$$

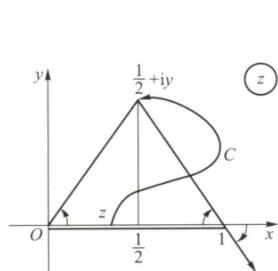


图 2.3.12

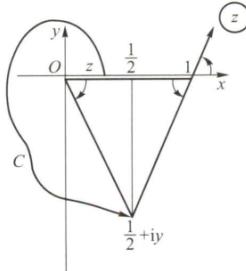


图 2.3.13

Exercice 14

例 2.3.19 求函数 $w = \ln(1 + \sin z)$ 在点 $z = \frac{i}{2}$ 的导数.

分析 应用变量代换 $\zeta = \sin z$, 可使这个无穷多个支点的情形简化为单有限支点的情形.

解 $w = \ln(1 + \sin z)$ 是 $w = \ln(1 + \zeta)$ 和 $\zeta = \sin z$ 的复合函数. 而 $w = \ln(1 + \zeta)$ 的支点是 $\zeta = -1$ 和 $\zeta = \infty$, 支割线可以取 ζ 平面上从 $\zeta = -1$ 出发的负实轴. 于是 $w = \ln(1 + \zeta)$ 在以此为边界的区域 D 内单值解析.

因为

$$\sin \frac{i}{2} = \frac{1}{2i}(e^{i\frac{1}{2}} - e^{-i\frac{1}{2}}) = -\frac{i}{2}(e^{-\frac{1}{2}} - e^{\frac{1}{2}}) \quad (\text{这是非零纯虚数}),$$

所以 $\sin \frac{i}{2} \neq -1, \infty$, 也不取小于 -1 的实数, 从而 $\zeta = \sin \frac{i}{2} \in D$. 故按复合函数求导法则,

$$\begin{aligned} [\ln(1 + \sin z)]' \Big|_{z=\frac{i}{2}} &= \frac{\cos z}{1 + \sin z} \Big|_{z=\frac{i}{2}} = \frac{\cos \frac{i}{2}}{1 + \sin \frac{i}{2}}, \\ &= \frac{\cosh \frac{1}{2}}{1 + i \sinh \frac{1}{2}} = \frac{1 - i \sinh \frac{1}{2}}{\cosh \frac{1}{2}}. \end{aligned}$$

反三角函数

7. 了解反三角函数. 反三角函数定义为三角函数的反函数, 和一般幂函数一样, 也是用对数函数表示.

$$(1) \text{ 反正弦函数 } \operatorname{Arcsin} z = \frac{1}{i} \operatorname{Ln}(iz + \sqrt{1-z^2});$$

$$(2) \text{ 反余弦函数 } \operatorname{Arccos} z = \frac{1}{i} \operatorname{Ln}(z + i\sqrt{1-z^2});$$

$$(3) \text{ 反正切函数 } \operatorname{Arctan} z = \frac{1}{2i} \operatorname{Ln} \frac{1+iz}{1-iz}.$$

这里的根式是二值的, 它们互为相反数. 每一个值取对数后又产生无穷多个值.

Note. 所有这些函数分成单值解析分支的方法, 与我们前面用过的讨论方法是类似的, 也要先讨论它们的支点, 然后适当割破平面, 只是较复杂些也较困难些. 当然也可以像例 2.3.17 和例 2.3.19 一样, 把他们视为复合函数来化简处理.

Exercice 15

例 2.3.20 若 a 为实数, 求方程 $\cos z = a$ 的解.

分析 $z = \operatorname{Arccos} a$, 按 $|a| \leq 1, a > 1$ 及 $a < -1$ 分别计算.

解 由定义,

$$\begin{aligned} z &= \operatorname{Arccos} a = -i \ln(a + \sqrt{1-a^2}i) \\ &= -i \ln(a + \sqrt{a^2-1}). \end{aligned}$$

(1) 若 $|a| \leq 1$ (即 $-1 \leq a \leq 1$), 则 $\sqrt{a^2-1}$ 为纯虚数. 所以 $|a + \sqrt{a^2-1}| = |a + \sqrt{1-a^2}i| = 1$, 故

$$\begin{aligned} z &= \operatorname{Arccos} a = -i \left[\ln 1 + i \left(\arctan \frac{\sqrt{1-a^2}}{a} + 2k\pi \right) \right] \\ &= \arctan \frac{\sqrt{1-a^2}}{a} + 2k\pi = \arccos a + 2k\pi \quad (k = 0, \pm 1, \pm 2, \dots). \end{aligned}$$

这里 $\arccos a$ 表示通常的反余弦函数的主值, 即当 $|a| \leq 1$ 时, 复数意义下 (a 为复数) 的解与通常的意义下 (a 为实数) 的解相同.

(2) 若 $|a| > 1$, 则 $\sqrt{a^2-1}$ 是实数.

当 $a > 1$ 时, $a \pm \sqrt{a^2-1}$ 是正数. 这时

$$\begin{aligned} z &= \operatorname{Arccos} a = -i \ln(a \pm \sqrt{a^2-1}) \\ &= -i [\ln(a \pm \sqrt{a^2-1}) + 2k\pi i] \end{aligned}$$

$$= 2k\pi \pm i \ln(a + \sqrt{a^2 - 1}),$$

这是因为

$$\begin{aligned}\ln(a - \sqrt{a^2 - 1}) &= \ln \frac{(a - \sqrt{a^2 - 1})(a + \sqrt{a^2 - 1})}{a + \sqrt{a^2 - 1}} \\ &= \ln \frac{a^2 - (a^2 - 1)}{a + \sqrt{a^2 - 1}} = \ln \frac{1}{a + \sqrt{a^2 - 1}} \\ &= -\ln(a + \sqrt{a^2 - 1}).\end{aligned}$$

又当 $a < -1$ 时, $a \pm \sqrt{a^2 - 1}$ 两个都是负数. 这时

$$\begin{aligned}z &= \operatorname{Arccos} a = -i[\ln|a \pm \sqrt{a^2 - 1}| + i(2k+1)\pi] \\ &= (2k+1)\pi \pm i \ln|a + \sqrt{a^2 - 1}| \\ &= (2k+1)\pi \pm i \ln(-a - \sqrt{a^2 - 1}).\end{aligned}$$

综上所述, 方程的解为

$$z = \begin{cases} 2k\pi + \operatorname{arccos} a, & |a| \leq 1, \\ 2k\pi \pm i \ln(a + \sqrt{a^2 - 1}), & a > 1, \\ (2k+1)\pi \pm i \ln(-a - \sqrt{a^2 - 1}), & a < -1 \end{cases} \quad (k = 0, \pm 1, \pm 2, \dots).$$

■

Exercice 16

例 2.3.21 求函数

$$\operatorname{Arctan} z = -\frac{i}{2} \ln \frac{1+iz}{1-iz} \quad (1)$$

的单值解析分支在 $z=1$ 的导数.

Note. 变量代换 $\zeta = \frac{1+iz}{1-iz}$, 化简, 并将 $\ln z$ 分成单值解析分支, 然后就可以按照复合函数求导。

解 引入变量代换

$$\zeta = \frac{1+iz}{1-iz}, \quad (2)$$

且 $\zeta = \xi + i\eta$, ξ 和 η 为实数. $\ln \zeta$ 的支点为 $\zeta = 0, \infty$, 支割线可以取 ζ 平面上从 $\zeta = 0$ 出发的负实轴, 于是

$$\ln \zeta + 2k\pi i \quad (k \text{ 为任一给定的整数})$$

在 ζ 平面上以此为边界的区域 D 内单值解析. 故

$$(\operatorname{arctan} z)_k = -\frac{i}{2} (\ln \zeta + 2k\pi i) \quad (3)$$

在 z 平面上 D 的原像区域 G 内单值解析.

因为 $z=1$ 时 $\zeta = \frac{1+i}{1-i} = i \in D$, $\ln \zeta$ 在 $\zeta = i$ 的邻域内有无穷多个单值解析分支

$$\ln \zeta + 2k\pi i \quad (k = 0, \pm 1, \pm 2, \dots).$$

但是它们的导数都是 $\frac{1}{\zeta}$, 所以对 $\operatorname{Arctan} z$ 在 $z=1$ 的邻域内的任一单值解析分支, 其

在 $z=1 \in G$ 的导数可由(3)式及复合函数求导法则得到:

$$\begin{aligned} \frac{d}{dz} (\operatorname{arctan} z)_k \Big|_{z=1} &= -\frac{i}{2} \frac{d}{d\zeta} (\ln \zeta + 2k\pi i) \frac{d\zeta}{dz} \Big|_{z=1} \\ &\stackrel{(2)}{=} -\frac{i}{2} \frac{1-iz}{1+iz} \left(\frac{1+iz}{1-iz} \right)' \Big|_{z=1} = \frac{1}{2} \quad (k = 0, \pm 1, \pm 2, \dots). \blacksquare \end{aligned}$$

7.13 解析函数的幂级数表示

复级数的收敛性:

$$\sum_{n=1}^{\infty} |c_n| \text{ 收敛} \implies \sum_{n=1}^{\infty} c_n \text{ 收敛}$$

收敛半径:

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} = \frac{1}{R}$$

解析函数的泰勒展开式：

$$\begin{aligned}f(z) &= \sum_{n=0}^{\infty} c_n (z-a)^n \\c_n &= \frac{1}{2\pi i} \oint_{C_\epsilon} \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta = \frac{f^{(n)}(a)}{n!} \\[\ln(1+z)]_k &= 2k\pi i + z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots + (-1)^{n-1} \frac{z^n}{n} + \dots\end{aligned}$$

7.13.1 用间接法展开为幂级数

Exercice 17

求 $f(z) = e^z \cos z$ 在 $z = 0$ 的泰勒展开式.

因为

$$e^z \cos z = \frac{1}{2} e^z (e^{iz} + e^{-iz}) = \frac{1}{2} [e^{(1+i)z} + e^{(1-i)z}]$$

于是

$$e^z \cos z = \frac{1}{2} \left[\sum_{n=0}^{\infty} \frac{(\sqrt{2})^n e^{n\pi i/4}}{n!} z^n + \sum_{n=0}^{\infty} \frac{(\sqrt{2})^n e^{-n\pi i/4}}{n!} z^n \right] = \sum_{n=0}^{\infty} \frac{(\sqrt{2})^n \cos \frac{n\pi}{4}}{n!} z^n$$

(2) 利用级数的乘、除运算.

例 4.3.4 把 $e^z \sin z$ 展成 z 的幂级数.

解 由第 4 小节的公式(3)和(5),

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

两级数均在 $|z| < +\infty$ 内绝对收敛, 故柯西积也绝对收敛.

$$\begin{array}{ccccccc} 1 & 1 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \cdots \\ \hline 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ -\frac{1}{3!} & -\frac{1}{3!} & -\frac{1}{3!2!} & -\frac{1}{3!3!} & -\frac{1}{3!4!} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{5!} & \frac{1}{5!} & \frac{1}{5!2!} & \frac{1}{5!3!} & \frac{1}{5!4!} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

由对角线方法,

$$\begin{aligned} e^z \sin z &= 0 + (1+0)z + (0+1+0)z^2 + \\ &\quad \left(\frac{1}{2!} - \frac{1}{3!} \right) z^3 + \left(\frac{1}{3!} - \frac{1}{3!} \right) z^4 + \\ &\quad \left(\frac{1}{5!} - \frac{1}{3!2!} + \frac{1}{4!} \right) z^5 + \cdots \\ &= z + z^2 + \frac{1}{3} z^3 - \frac{1}{30} z^5 + \cdots \quad (|z| < +\infty). \end{aligned}$$

■

注 比较本题与例 4.12 的解法和结果.

Exercice 18

求 $\tan z$ 在 $z = 0$ 的泰勒展式.

通过分析奇点可知, $\tan z$ 在 $|z| < \frac{\pi}{2}$ 内解析. 利用“大除法”:

$$\sin z = z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \frac{1}{7!} z^7 + \dots,$$

$$\cos z = 1 - \frac{1}{2!} z^2 + \frac{1}{4!} z^4 - \frac{1}{6!} z^6 + \dots,$$

可以像多项式按升幂排列用直式做除法那样(分离系数), 将分式的分子、分母的幂级数用直式相除, 缺项用 0 补充, 得到

$$\begin{array}{r} 0 + 1 + 0 + \frac{1}{3} + 0 + \frac{2}{15} + 0 + \dots \\ 1 + 0 - \frac{1}{2} + 0 + \frac{1}{24} + \dots \sqrt{0 + 1 + 0 - \frac{1}{6} + 0 + \frac{1}{120} + 0 + \dots} \\ \hline 1 + 0 - \frac{1}{2} + 0 + \frac{1}{24} + 0 + \dots \\ \hline \frac{1}{3} + 0 - \frac{1}{30} + 0 + \dots \\ \hline \frac{1}{3} + 0 - \frac{1}{6} + 0 + \dots \\ \hline \frac{2}{15} + 0 + \dots \\ \hline \frac{2}{15} + 0 + \dots \\ \hline \dots \end{array}$$

故

$$\tan z = \frac{\sin z}{\cos z} = z + \frac{1}{3} z^3 + \frac{2}{15} z^5 + \dots \quad \left(|z| < \frac{\pi}{2}\right).$$

■

7.13.2 待定系数法展开幂级数

Exercice 19

求 $\sec z$ 在 $z = 0$ 的泰勒展式.

解 设 $f(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n + \dots$, 其中 $c_0, c_1, c_2, \dots, c_n, \dots$ 为待定系数. 因为

$$f(-z) = f(z) = c_0 - c_1 z + c_2 z^2 - \dots,$$

由泰勒展式的惟一性, 得 $c_1 = c_3 = c_5 = \dots = 0$.

又由于 $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$, 所以

$$1 = \cos z \sec z = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\right) (c_0 + c_2 z^2 + c_4 z^4 + \dots)$$

$$\xrightarrow{\text{柯西积}} c_0 + \left(c_2 - \frac{c_0}{2!}\right) z^2 + \left(c_4 - \frac{c_2}{2!} + \frac{c_0}{4!}\right) z^4 + \dots$$

比较两端系数得

$$c_0 = 1, \quad c_2 = \frac{1}{2!}, \quad c_4 = \frac{5}{4!}, \quad \dots,$$

于是

$$\sec z = 1 + \frac{1}{2!} z^2 + \frac{5}{4!} z^4 + \dots \quad \left(|z| < \frac{\pi}{2}\right). \quad \blacksquare$$

7.13.3 微分方程法

Exercice 20展开 $e^{1/(1-z)}$.

分析 $z=1$ 是 $f(z)=e^{\frac{1}{1-z}}$ 在 \mathbf{C} 上的惟一奇点, 故 $f(z)$ 在 $|z|<1$ 内解析, 从而能展成 z 的幂级数.

解 设

$$f(z) = e^{\frac{1}{1-z}}. \quad (1)$$

求导得

$$f'(z) = e^{\frac{1}{1-z}} \cdot \frac{1}{(1-z)^2} = f(z) \cdot \frac{1}{(1-z)^2},$$

即

$$(1-z)^2 f'(z) - f(z) = 0. \quad (2)$$

对微分方程(2)逐次求导,

$$(1-z)^2 f''(z) + (2z-3)f'(z) = 0, \quad (3)$$

$$(1-z)^2 f'''(z) + (4z-5)f''(z) + 2f'(z) = 0, \quad (4)$$

...

由于 $f(0) = e^{\frac{1}{1-0}} = e$, 由上列各微分方程可得

$$f'(0) = e, \quad f''(0) = 3e, \quad f'''(0) = 13e, \quad \dots,$$

从而有

$$e^{\frac{1}{1-z}} = e \left(1 + z + \frac{3}{2!} z^2 + \frac{13}{3!} z^3 + \dots \right) \quad (|z| < 1). \quad \blacksquare$$

7.13.4 逐项求导法

例 4.3.8 用逐项求导法求函数 $\frac{1}{(1-z)^3}$ 在 $|z|<1$ 内的泰勒展式.

解 因为 $\frac{1}{(1-z)^3} = \frac{1}{2} [(1-z)^{-1}]''$ ($|z|<1$), 所以, 用逐项求导法算得

$$\begin{aligned} \frac{1}{(1-z)^3} &= \frac{1}{2} \left(\sum_{n=0}^{\infty} z^n \right)'' = \frac{1}{2} \sum_{n=2}^{\infty} n(n-1)z^{n-2} \\ &= \frac{1}{2} \sum_{m=0}^{\infty} (m+2)(m+1)z^m \quad (|z| < 1). \end{aligned} \quad \blacksquare$$

7.13.5 逐项积分法

例 4.3.9 求 $f(z) = \ln \frac{z-1}{z+1}$ 在 $z=0$ 的泰勒展式, 其中 $f(z)$ 是满足 $f(0)=\pi i$ 的那个单值解析分支.

分析 $\ln \frac{z-1}{z+1}$ 的支点为 -1 及 $+1$, 故在 $|z|<1$ 内能分出所求的单值解析分支, 它在 $|z|<1$ 内能展成 z 的幂级数.

解 因为

$$\begin{aligned} f'(z) &= \left(\ln \frac{z-1}{z+1}\right)' = \frac{z+1}{z-1} \left(\frac{z-1}{z+1}\right)' \\ &= \frac{2}{(z-1)(z+1)} = \frac{1}{z-1} - \frac{1}{z+1} \\ &= -\frac{1}{1-z} - \frac{1}{1+z} = -\sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} (-1)^n z^n \\ &= \sum_{n=0}^{\infty} [(-1)^{n+1} - 1] z^n, \end{aligned}$$

上式两端在 $|z|<1$ 内沿 0 到 z 积分, 得

$$\begin{aligned} \ln \frac{z-1}{z+1} - \pi i &= \int_0^z \left(\ln \frac{z-1}{z+1}\right)' dz \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} [(-1)^{n+1} - 1] z^{n+1}, \end{aligned}$$

因而

$$\ln \frac{z-1}{z+1} = \pi i + \sum_{n=1}^{\infty} \frac{1}{n} [(-1)^n - 1] z^n \quad (|z|<1).$$

例 4.3.14 将 $f(z) = \arctan z$ 展成 z 的幂级数, 使满足条件 $f(0)=0$ (取主值支).

解 由 $f(z) = \arctan z$, 则

$$f'(z) = \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \quad (|z|<1),$$

逐项积分得

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1} + C \quad (C \text{ 是常数}).$$

根据条件 $f(0)=0$ 得 $C=0$, 故

$$f(z) = \arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1} \quad (|z|<1). \quad \blacksquare$$

7.14 通过幂级数法来证明不等式

Exercice 21

若 $1 + w = (1 - a)e^a$, 且 $|a| < 1$, 则

$$|w| \leq \frac{|a|^2}{1 - |a|}$$

证明. 因为

$$\begin{aligned} (1 - a)e^a &= (1 - a) \left(1 + a + \frac{a^2}{2!} + \cdots + \frac{a^n}{n!} + \cdots \right) \\ &= 1 + a + \frac{1}{2!}a^2 + \cdots + \frac{a^n}{n!} + \cdots - a - a^2 - \cdots - \frac{a^n}{(n-1)!} - \cdots \\ &= 1 - \frac{a^2}{2} - \left(1 - \frac{1}{n}\right) \frac{a^n}{(n-1)!} - \cdots \quad (|a| < 1) \end{aligned}$$

所以

$$\begin{aligned} |(1 - a)e^a - 1| &= |w| \leq \frac{|a|^2}{2} + \cdots + \frac{n-1}{n!}|a|^n + \cdots \\ &= |a|^2 + |a|^3 + \cdots + |a|^n + \cdots \\ &= \frac{|a|^2}{1 - |a|} \end{aligned}$$

□

例 4.3.19 证明: 若 $|z| \leq \frac{1}{2}$, 则 $|\ln(1+z) - z| \leq |z|^2$, 这里 $\ln(1+z)$ 表主值.

证 因为

$$\ln(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n} \quad (|z| < 1),$$

所以

$$\begin{aligned} |\ln(1+z) - z| &= \left| \sum_{n=2}^{\infty} (-1)^{n+1} \frac{z^n}{n} \right| = |z|^2 \left| \sum_{n=2}^{\infty} (-1)^{n+1} \frac{z^{n-2}}{n} \right| \\ &\stackrel{\substack{\text{(例 4.3.8 注)} \\ \text{(题设)}}}{\leq} |z|^2 \sum_{n=2}^{\infty} \frac{1}{2^{n-1}} = |z|^2 \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= |z|^2 \frac{\frac{1}{2}}{1 - \frac{1}{2}} = |z|^2. \end{aligned}$$

■

例 4.3.21 若 $f(z)$ 为整函数,且

$$\lim_{r \rightarrow +\infty} \frac{M(r)}{r^n} < +\infty \quad (M(r) = \max_{|z|=r} |f(z)|),$$

则 $f(z)$ 是不高于 n 次的多项式.

分析 因 $f(z)$ 为整函数,则其可展成 z 的幂级数,收敛半径 $R = +\infty$.

证 因

$$f(z) = \sum_{k=0}^{\infty} c_k z^k, \quad |z| < +\infty,$$

其系数的柯西不等式为

$$|c_k| \leq \frac{M(r)}{r^k} \quad (k = 0, 1, 2, \dots).$$

当 $k \geq n+1$ 时,令 $k = n+p$ ($p \geq 1$),则

$$\lim_{r \rightarrow +\infty} \frac{M(r)}{r^k} = \lim_{r \rightarrow +\infty} \frac{1}{r^p} \cdot \frac{M(r)}{r^n} = 0 \quad (k \geq n+1).$$

所以,当 $k \geq n+1$ 时, $c_k = 0$. 故 $f(z)$ 是不高于 n 次的多项式. ■

注 当 $n=0$ 时,这个题就是刘维尔定理. 所以它是刘维尔定理的推广.

7.14.1 最大模定理的应用

例 4.4.11 设 $f(z) = \cos z$. 证明在任何圆周 $|z|=r$ 上,都有点 z 使 $|\cos z| > 1$.

证 因为 $f(z) = \cos z$ 在 z 平面上解析,且不为常数,又 $|f(0)| = |\cos 0| = 1$,则由最大模原理,在任何圆周 $|z|=r$ 上,都有点 z 使 $|\cos z| > 1$. ■

例 4.4.12 若 $\varphi(r) > 0$ 在 $0 \leq r < 1$ 内为增函数, $f(z)$ 在 $|z| < 1$ 内解析, $f(0) = 0$, $|f(z)| \leq \varphi(|z|)$, 则

$$|f(z)| \leq k |z| \varphi(|z|),$$

$$\text{其中 } k \text{ 可以取为 } k = 2 \frac{\varphi\left(\frac{1}{2}\right)}{\varphi(0)} (\geq 2).$$

分析 由要证明的结果看出, 我们从考虑辅助函数 $\frac{f(z)}{z}$ 着手, 并对它应用最大模原理.

证 (1) 因 $f(z)$ 在 $|z| < 1$ 内解析, 且 $f(0) = 0$, 故由泰勒定理,

$$\frac{f(z)}{z} = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} z^{n-1} \quad (0 < |z| < 1),$$

即 $\frac{f(z)}{z}$ 在 $|z| < 1$ 内解析 ($z=0$ 为其可去奇点).

(2) 由题设, $\varphi(|z|)$ 在 $|z| < 1$ 内是 $|z|$ 的增函数. 再由最大模原理知, 当 $|z| \leq \frac{1}{2}$ 时,

$$\begin{aligned} \left| \frac{f(z)}{z} \right| &\leq \max_{|z|=\frac{1}{2}} \left| \frac{f(z)}{z} \right| = 2 \max_{|z|=\frac{1}{2}} |f(z)| \stackrel{(题设)}{\leq} 2\varphi\left(\frac{1}{2}\right) \\ &\leq 2\varphi\left(\frac{1}{2}\right) \frac{\varphi\left(\frac{1}{2}\right)}{\varphi(0)} \quad \left(\text{由题设 } \varphi\left(\frac{1}{2}\right) \geq \varphi(0) \right), \end{aligned}$$

即

$$|f(z)| \leq 2 \frac{\varphi\left(\frac{1}{2}\right)}{\varphi(0)} |z| \varphi(|z|) = k |z| \varphi(|z|),$$

$$\text{这里 } k = 2 \frac{\varphi\left(\frac{1}{2}\right)}{\varphi(0)} \geq 2.$$

(3) 当 $1 > |z| > \frac{1}{2}$ 时, $2|z| > 1$,

$$|f(z)| \stackrel{(题设)}{\leq} \varphi(|z|) < 2|z| \varphi(|z|) \stackrel{(2)}{\leq} k |z| \varphi(|z|).$$

合并(2)和(3)即得: 当 $0 \leq |z| < 1$ 时,

$$|f(z)| \leq k |z| \varphi(|z|),$$

$$\text{其中 } k = 2 \frac{\varphi\left(\frac{1}{2}\right)}{\varphi(0)} \geq 2.$$

■

例 4.4.13 设 $f(z)$ 在 $|z| \leq R$ 上解析, 且 $|f(z)| \leq M, f(0) \neq 0$, 则当 $|z| \leq \frac{R}{3}$ 时, $f(z)$ 的零点个数不超过

$$\frac{1}{\ln 2} \ln \frac{M}{|f(0)|}.$$

证 设 $z_j (j=1, 2, \dots, n)$ 为 $f(z)$ 在 $|z| \leq \frac{R}{3}$ 上的零点, 则函数

$$g(z) = \frac{f(z)}{\prod_{j=1}^n \left(1 - \frac{z}{z_j}\right)} \quad (1)$$

在 $|z| \leq R$ 上解析. 而在 $|z| = R$ 上, $|z_j| \leq \frac{R}{3} = \frac{|z|}{3}$, 即 $\left|\frac{z}{z_j}\right| \geq 3$, 故由

$$|g(R e^{i\theta})| \leq \frac{|f(z)|}{\prod_{j=1}^n \left|1 - \frac{z}{z_j}\right|},$$

有

$$|g(R e^{i\theta})| \leq \frac{M}{2^n}.$$

于是由最大模原理, $|g(0)| \leq \frac{M}{2^n}$. 但由(1)式, $f(0) = g(0)$, 故 $|f(0)| \leq \frac{M}{2^n}$. 于是

$$n \leq \frac{1}{\ln 2} \ln \frac{M}{|f(0)|}. \quad \blacksquare$$

13. (最小模原理) 若区域 D 内不恒为常数的解析函数 $f(z)$, 在 D 内的点 z_0 有 $f(z_0) \neq 0$, 则 $|f(z_0)|$ 不可能是 $|f(z)|$ 在 D 内的最小值. 试证之.

提示 反证法, 应用最大模原理于函数 $\frac{1}{f(z)}$.

7.15 一些习题

4. 设 $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($a_0 \neq 0$) 的收敛半径 $R > 0$, 且

$$M = \max_{|z| \leq \rho} |f(z)| \quad (\rho < R).$$

试证在圆

$$|z| < \frac{|a_0|}{|a_0| + M} \rho$$

内 $f(z)$ 无零点.

证 由柯西不等式 $|a_n| \leq \frac{M}{\rho^n}$, 在圆 $|z| < \rho$ 内可证

$$|f(z) - a_0| \leq M \frac{|z|}{\rho - |z|}.$$

因此, 在圆

$$|z| < \frac{|a_0|}{|a_0| + M} \rho \quad (< \rho)$$

内就有

$$|f(z) - a_0| \leq M \frac{|z|}{\rho - |z|} < |a_0|,$$

从而可证得 $|f(z)| > 0$.

Exercice 22

设函数列 $\{f_n(z)\}$ 在区域 G 上解析, 且在 G 中内闭一致收敛于函数 $f(z)$.

证明:

- (1) 若 $f(z)$ 不恒为零, l 是 G 内可求长的简单闭曲线, 其内部属于 G , 且不经过 $f(z)$ 的零点, 则存在正整数 N , 使得当 $n \geq N$ 时, 在 l 的内部 $f_n(z)$ 和 $f(z)$ 有相同个数的零点.
- (2) 若 $\{f_n(z)\}$ 在区域 G 内部时单叶的, $f(z)$ 不为常数, 则 $f(z)$ 在 G 内单叶解析.

证明. 由 Weierstrass 定理, $f(z)$ 在 G 内解析. 因为 $f(z)$ 在 l 上不为零, 则

$$\min_{z \in l} |f(z)| = m > 0.$$

又 $\{f_n(z)\}$ 在 l 上一致收敛到 $f(z)$, 存在正整数 N , 使得当 $n \geq N$ 时, 在 l 上有 $|f(z) - f_n(z)| < m$. 即当 $n \geq N$ 时, 在 l 上有 $|f(z) - f_n(z)| < |f(z)|$. 由 Rouche 定理, 在 l 的内部, $f_n(z)$ 和 $f(z)$ 有相同个数的零点.

对于第二问采用反证法，若 f 在 G 内不是单叶的，则存在 $z_1 \neq z_2$ 使得 $f(z_1) = f(z_2)$ ，考虑函数 $g(z) = f(z) - f(z_1)$ ，圆周 $C_\epsilon(z_1), C_\epsilon(z_2)$ ，其中 ϵ 充分小使得其包含于 G 内，而且 $g(z)$ 在圆周内有唯一的零点。于是存在 N ，使得 $n \geq N$ 时， $f_n(z)$ 和 $f(z)$ 在 $C_\epsilon(z_1), C_\epsilon(z_2)$ 内有相同个数（1个）零点 z_1^*, z_2^* ，从而 $f_n(z_2^*) = f(z_1) = f_n(z_1^*)$ 。这与 f_n 的单叶性矛盾！□

Exercice 23: cmc 高年级第 10 届

设 z_0 是复函数 $w = f(z)$ 的 n 阶极点。证明：一定存在 $\rho > 0, R > 0$ 使得对于任意 $w \in \{w \in \mathbb{C} : |w| > R\}$ ，函数 $f(z) - w$ 在 $|z - z_0| < \rho$ 中必有 n 个零点。

证明。定义

$$\varphi(z) = (z - z_0)^n f(z)$$

且在 z_0 处补充定义（因为是可去奇点），于是 φ 在 z_0 的一个小邻域内解析，故存在 $\rho > 0$ 使得 $\varphi \in H(B_\rho(z_0))$ 。再令

$$R := \max_{|z-z_0|=\rho} \left| \frac{\varphi(z)}{(z-z_0)^n} \right|$$

由最大模原理，对于 $|z - z_0| < \rho$ 有

$$|\varphi(z)(z - z_0)^{-n}| \leq R < |w| \implies |\varphi(z)| < |w(z - z_0)^n|$$

由 Rouche 定理，函数 $F(z) := \varphi(z) - w(z - z_0)^n$ 在 $|z - z_0| < \rho$ 内零点个数与函数 $w(z - z_0)^n$ 相等，即为 n 个。由于 n 是 f 在 z_0 处的极点阶数，故 $\varphi(z_0) \neq 0$ ，于是 $F(z_0) \neq 0$ 。故 $F(z)$ 在 $0 < |z - z_0| < \rho$ 存在 n 个零点，故 $f(z) - w = \frac{F(z)}{(z - z_0)^n}$ 在 $|z - z_0| < \rho$ 中必有 n 个零点。□

Théorème 9: Picard's Big Theorem

Let f be an analytic function on a punctured neighborhood of w_0 , say on $D = \{z : 0 < |z - w_0| < r\}$. If f has an essential singularity at w_0 , then f takes on all possible complex values, with at most a single exception, infinitely often in any neighborhood of w_0 .

Théorème 10: Picard's Little Theorem

If $f : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function which is not constant, then the image of f is either the whole complex plane or the complex plane minus a single point.

7.16 共形映射

See Dennis Zill. Appendix III for [Table of Conformal Mappings](#)

Definition 7.1 Conformal Mapping

Let $w = f(z)$ be a complex mapping defined in a domain D and let z_0 be a point in D . Then we say that $w = f(z)$ is **conformal** at z_0 if for every pair of smooth oriented curves C_1 and C_2 in D intersecting at z_0 the angle between C_1 and C_2 at z_0 is equal to the angle between the image curves C'_1 and C'_2 at $f(z_0)$ in both magnitude and sense.

Some examples of conformal mapping:

- (1) Translations: $f(z) = z + b$, where b is a complex number.
- (2) Rotations: $f(z) = e^{i\phi}z$, where ϕ is a real number.
- (3) Dilations (or scalings): $f(z) = rz$, where $r > 0$ is a real number.
- (4) Inversion: $f(z) = 1/z$, $z \neq 0$.
- (5) Linear transformations: $f(z) = az + b$, where a, b are complex numbers and $a \neq 0$.
- (6) Möbius transformations (or linear fractional transformations): $f(z) = \frac{az+b}{cz+d}$, where a, b, c, d are complex numbers and $ad - bc \neq 0$.

Theorem 7.1 Conformal Mapping

If f is an analytic function in a domain D containing z_0 , and if $f'(z_0) \neq 0$, then $w = f(z)$ is a conformal mapping at z_0 .

Proof Suppose that f is analytic in a domain D containing z_0 , and that $f'(z_0) \neq 0$. Let C_1 and C_2 be two smooth curves in D parametrized by $z_1(t)$ and $z_2(t)$, respectively, with $z_1(t_0) = z_2(t_0) = z_0$. In addition, assume that $w = f(z)$ maps the curves C_1 and C_2 onto the curves C'_1 and C'_2 . We wish to show that the angle θ between C_1 and C_2 at z_0 is equal to the angle ϕ between C'_1 and C'_2 at $f(z_0)$ in both magnitude and sense. We may assume, by renumbering C_1 and C_2 if necessary, that $z'_1 = z'_1(t_0)$ can be rotated counterclockwise about 0 through the angle θ onto $z'_2 = z'_2(t_0)$. Thus, by (1), the angle θ is the unique value of $\arg(z'_2) - \arg(z'_1)$ in the interval $[0, \pi]$. From (11) of Section 2.2, C'_1 and C'_2 are parametrized by $w_1(t) = f(z_1(t))$ and $w_2(t) = f(z_2(t))$. In order to compute the tangent vectors w'_1 and w'_2 to C'_1 and C'_2 at $f(z_0) = f(z_1(t_0)) = f(z_2(t_0))$ we use the chain rule

$$w'_1 = w'_1(t_0) = f'(z_1(t_0)) \cdot z'_1(t_0) = f'(z_0) \cdot z'_1,$$

and

$$w'_2 = w'_2(t_0) = f'(z_2(t_0)) \cdot z'_2(t_0) = f'(z_0) \cdot z'_2.$$

Since C_1 and C_2 are smooth, both z'_1 and z'_2 are nonzero. Furthermore, by our hypothesis, we have $f'(z_0) \neq 0$. Therefore, both w'_1 and w'_2 are nonzero, and the angle ϕ between C'_1 and C'_2 at $f(z_0)$ is a value of

$$\arg(w'_2) - \arg(w'_1) = \arg(f'(z_0) \cdot z'_2) - \arg(f'(z_0) \cdot z'_1).$$

Now by two applications of (8) from Section 1.3 we obtain:

$$\begin{aligned} \arg(f'(z_0) \cdot z'_2) - \arg(f'(z_0) \cdot z'_1) &= \arg(f'(z_0)) + \arg(z'_2) - [\arg(f'(z_0)) + \arg(z'_1)] \\ &= \arg(z'_2) - \arg(z'_1). \end{aligned}$$

This expression has a unique value in $[0, \pi]$, namely θ . Therefore, $\theta = \phi$ in both magnitude and sense, and consequently the $w = f(z)$ is a conformal mapping at z_0 .



Définition 2: Simply Connected Domain

A domain D is **simply connected** if it is connected and every closed curve in D can be continuously deformed to a point in D .

Then we introduce the Riemann Mapping Theorem:

Remarks

In the foregoing discussion regarding conformal mappings using tables we alluded to the fact that in many applications one needs to find a conformal mapping of a domain D onto a simpler domain D' . A natural question to ask is whether such a mapping always exists. That is, given domains D and D' , does there exist a conformal mapping of D onto D' ? An answer to this question was given by the mathematician Bernhard Riemann (1826–1866). Although there was a gap in Riemann's original proof (which was subsequently filled), this amazing theorem still bears his name.

The Riemann Mapping Theorem Let D be a simply connected domain in the z -plane such that D is not all of \mathbf{C} . Then there exists a one-to-one conformal mapping $w = f(z)$ from D onto the open unit disk $|w| < 1$ in the w -plane.

It is not immediately clear that this theorem answers our question of the existence of a mapping from D onto D' . To see that it does, we first use the theorem to find a conformal mapping f from D onto the open unit disk $|w| < 1$. We then apply the theorem a second time to obtain a mapping g from D' onto the open unit disk $|w| < 1$. Since the theorem ensures that g is one-to-one, it has a well defined inverse function g^{-1} that maps the open unit disk onto D' . The desired mapping from D onto D' is then given by the composition $w = g^{-1} \circ f(z)$.

Riemann's theorem is of critical theoretical importance, but its proof is not *constructive*. This means that the theorem establishes the *existence* of the mapping f but offers no method of actually finding a formula for f . A proof of the Riemann mapping theorem is well beyond the scope of this text. The interested reader is encouraged to refer to the text *Complex Analysis* by Lars V. Alfsen, McGraw-Hill, 1979.

Definition 7.2 Linear Fractional Transformation

If a, b, c , and d are complex constants with $ad - bc \neq 0$, then the complex function defined by:

$$T(z) = \frac{az + b}{cz + d} \quad (1)$$

is called a **linear fractional transformation**.

Linear fractional transformations are also called **Möbius transformations** or **bilinear transformations**.

The values of these two limits indicate how to extend the definition of T . In particular, if $c \neq 0$, then we regard T as a one-to-one mapping of the extended complex plane defined by:

$$T(z) = \begin{cases} \frac{az + b}{cz + d}, & z \neq -\frac{d}{c}, z \neq \infty \\ \infty, & z = -\frac{d}{c} \\ \frac{a}{c}, & z = \infty. \end{cases} \quad (3)$$

Theorem 7.3 Circle-Preserving Property

If C is a circle in the z -plane and if T is a linear fractional transformation given by (3), then the image of C under T is either a circle or a line in the extended w -plane. The image is a line if and only if $c \neq 0$ and the pole $z = -d/c$ is on the circle C .

Proof When $c = 0$, T is a linear function, and we saw in Section 2.3 that linear functions map circles onto circles. It remains to be seen that the theorem still holds for $c \neq 0$. Assume then that $c \neq 0$. From (2) we have that $T(z) = f \circ g \circ h(z)$, where $f(z) = Az + B$ and $h(z) = cz + d$ are linear functions and $g(z) = 1/z$ is the reciprocal function. Observe that since h is a linear mapping, the image C' of the circle C under h is a circle. We now examine two cases:

Case 1 Assume that the origin $w = 0$ is on the circle C' . This occurs if and only if the pole $z = -d/c$ is on the circle C . From the Remarks in Section 2.5, if $w = 0$ is on C' , then the image of C' under $g(z) = 1/z$ is either a horizontal or vertical line L . Furthermore, because f is a linear function, the image of the line L under f is also a line. Thus, we have shown that if the pole $z = -d/c$ is on the circle C , then the image of C under T is a line.

Case 2 Assume that the point $w = 0$ is not on C' . That is, the pole $z = -d/c$ is not on the circle C . Let C' be the circle given by $|w - w_0| = \rho$. If we set $\xi = f(w) = 1/w$ and $\xi_0 = f(w_0) = 1/w_0$, then for any point w on C' we have

$$|\xi - \xi_0| = \left| \frac{1}{w} - \frac{1}{w_0} \right| = \frac{|w - w_0|}{|w| |w_0|} = \rho |\xi_0| |\xi|. \quad (4)$$

It can be shown that the set of points satisfying the equation

$$|\xi - a| = \lambda |\xi - b| \quad (5)$$

is a line if $\lambda = 1$ and is a circle if $\lambda > 0$ and $\lambda \neq 1$. See Problem 28 in Exercises 7.2. Thus, with the identifications $a = \xi_0$, $b = 0$, and $\lambda = \rho |\xi_0|$ we see that (4) can be put into the form (5). Since $w = 0$ is not on C' , we have $|w_0| \neq \rho$, or, equivalently, $\lambda = \rho |\xi_0| \neq 1$. This implies that the set of points given by (4) is a circle. Finally, since f is a linear function, the image of this circle under f is again a circle, and so we conclude that the image of C under T is a circle. 

7.16.1 Linear Fractional Transformations as Matrices

Linear Fractional Transformations as Matrices Matrices can be used to simplify many of the computations associated with linear fractional transformations. In order to do so, we associate the matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (6)$$

with the linear fractional transformation

$$T(z) = \frac{az + b}{cz + d} \quad (7)$$

The assignment in (6) is not unique because if e is a nonzero complex number, then the linear fractional transformation $T(z) = (az + b)/(cz + d)$ is also given by $T(z) = (eaz + eb)/(ecz + ed)$. However, if $e \neq 1$, then the two matrices

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} ea & eb \\ ec & ed \end{pmatrix} = e\mathbf{A} \quad (8)$$

are not equal even though they represent the same linear fractional transformation.

It is easy to verify that the composition $T_2 \circ T_1$ of two linear fractional transformations

$$T_1(z) = (a_1 z + b_1)/(c_1 z + d_1) \quad \text{and} \quad T_2(z) = (a_2 z + b_2)/(c_2 z + d_2)$$

is represented by the product of matrices

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a_2 a_1 + b_2 c_1 & a_2 b_1 + b_2 d_1 \\ c_2 a_1 + d_2 c_1 & c_2 b_1 + d_2 d_1 \end{pmatrix}. \quad (9)$$

In Problem 27 of Exercises 7.2 you are asked to find the formula for $T^{-1}(z)$ by solving the equation $w = T(z)$ for z . The formula for the inverse function $T^{-1}(z)$ of a linear fractional transformation T of (7) is represented by the inverse of the matrix \mathbf{A} in (6)

$$\mathbf{A}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

By identifying $e = \frac{1}{ad - bc}$ in (8) we can also represent $T^{-1}(z)$ by the matrix

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \cdot^* \quad (10)$$

This matrix is called the adjoint matrix of A .

EXAMPLE 4 Using Matrices

Suppose $S(z) = (z - i)/(iz - 1)$ and $T(z) = (2z - 1)/(z + 2)$. Use matrices to find $S^{-1}(T(z))$.

Solution We represent the linear fractional transformations S and T by the matrices

$$\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix},$$

respectively. By (10), the transformation S^{-1} is given by

$$\begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix},$$

and so, from (9), the composition $S^{-1} \circ T$ is given by

$$\begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -2 + i & 1 + 2i \\ 1 - 2i & 2 + i \end{pmatrix}$$

Therefore,

$$S^{-1}(T(z)) = \frac{(-2 + i)z + 1 + 2i}{(1 - 2i)z + 2 + i}.$$

7.16.2 Cross-ratio

Definition 7.3 Cross-Ratio

The **cross-ratio** of the complex numbers z, z_1, z_2 , and z_3 is the complex number

$$\frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1}. \quad (11)$$

When computing a cross-ratio, we must be careful with the order of the complex numbers. For example, you should verify that the cross-ratio of 0, 1, i , and 2 is $\frac{3}{4} + \frac{1}{4}i$, whereas the cross-ratio of 0, i , 1, and 2 is $\frac{1}{4} - \frac{1}{4}i$.

Theorem 7.4 Cross-Ratios and Linear Fractional Transformations

If $w = T(z)$ is a linear fractional transformation that maps the distinct points z_1, z_2 , and z_3 onto the distinct points w_1, w_2 , and w_3 , respectively, then

$$\frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1} = \frac{w - w_1}{w - w_3} \frac{w_2 - w_3}{w_2 - w_1} \quad (12)$$

for all z .

Constructing a Linear Fractional Transform mapping given points to given points

EXAMPLE 5 Constructing a Linear Fractional Transformation

Construct a linear fractional transformation that maps the points 1, i , and -1 on the unit circle $|z| = 1$ onto the points $-1, 0, 1$ on the real axis. Determine the image of the interior $|z| < 1$ under this transformation.

Solution Identifying $z_1 = 1, z_2 = i, z_3 = -1, w_1 = -1, w_2 = 0$, and $w_3 = 1$, in (12) we see from Theorem 7.4 that the desired mapping $w = T(z)$ must satisfy

$$\frac{z - 1}{z - (-1)} \frac{i - (-1)}{i - 1} = \frac{w - (-1)}{w - 1} \frac{0 - 1}{0 - (-1)}.$$

After solving for w and simplifying we obtain

Note: A linear fractional transformation can have many equivalent forms.

$$w = T(z) = \frac{z - i}{iz - 1}.$$

Using the test point $z = 0$, we obtain $T(0) = i$. Therefore, the image of the interior $|z| < 1$ is the upper half-plane $v > 0$.

EXAMPLE 6 Constructing a Linear Fractional Transformation

Construct a linear fractional transformation that maps the points $-i$, 1 , and ∞ on the line $y = x - 1$ onto the points 1 , i , and -1 on the unit circle $|w| = 1$.

Solution We proceed as in Example 5. Using (24) of Section 2.6, we find that the cross-ratio of z , $z_1 = -i$, $z_2 = 1$, and $z_3 = \infty$ is

$$\lim_{z_3 \rightarrow \infty} \frac{z+i}{z-z_3} \frac{1-z_3}{1+i} = \lim_{z_3 \rightarrow 0} \frac{z+i}{z-1/z_3} \frac{1-1/z_3}{1+i} = \lim_{z_3 \rightarrow 0} \frac{z+i}{zz_3-1} \frac{z_3-1}{1+i} = \frac{z+i}{1+i}.$$

Now from (12) of Theorem 7.4 with $w_1 = 1$, $w_2 = i$, and $w_3 = -1$, the desired mapping $w = T(z)$ must satisfy

$$\frac{z+i}{1+i} = \frac{w-1}{w+1} \frac{i+1}{i-1}.$$

After solving for w and simplifying we obtain

$$w = T(z) = \frac{z+1}{-z+1-2i}.$$

7.16.3 Schwarz-Christoffel Transformations

The motivation is from $w = f(z) = (z - x_1)^{\alpha/\pi}$ with $f'(z) = \frac{\alpha}{\pi}(z - x_1)^{(\alpha/\pi)-1}$.

Theorem 7.5 Schwarz-Christoffel Formula

Let f be a function that is analytic in the domain $y > 0$ and has the derivative

$$f'(z) = A(z - x_1)^{(\alpha_1/\pi)-1} (z - x_2)^{(\alpha_2/\pi)-1} \cdots (z - x_n)^{(\alpha_n/\pi)-1}, \quad (6)$$

where $x_1 < x_2 < \cdots < x_n$, $0 < \alpha_i < 2\pi$ for $1 \leq i \leq n$, and A is a complex constant. Then the upper half-plane $y \geq 0$ is mapped by $w = f(z)$ onto an unbounded polygonal region with interior angles $\alpha_1, \alpha_2, \dots, \alpha_n$.

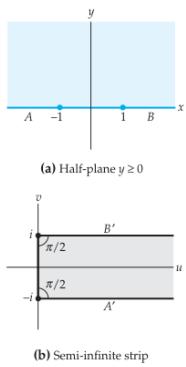


Figure 7.23 Figure for Example 1

apply (6) using only $n - 1$ of the n interior angles of the bounded polygonal region.[†] We illustrate these ideas in the following examples.

EXAMPLE 1 Using the Schwarz-Christoffel Formula

Use the Schwarz-Christoffel formula (6) to construct a conformal mapping from the upper half-plane onto the polygonal region defined by $u \geq 0$, $-1 \leq v \leq 1$.

Solution Observe that the polygonal region defined by $u \geq 0$, $-1 \leq v \leq 1$, is the semi-infinite strip shown in gray in Figure 7.23(b). The interior angles of this unbounded polygonal region are $\alpha_1 = \alpha_2 = \pi/2$, and the vertices are $w_1 = -i$ and $w_2 = i$. To find the desired mapping, we apply Theorem 7.4 with $x_1 = -1$ and $x_2 = 1$. With these identifications, (6) gives

$$f'(z) = A(z+1)^{-1/2}(z-1)^{-1/2}. \quad (7)$$

From Theorem 7.4, $w = f(z)$ is a conformal mapping from the half-plane $y \geq 0$ onto the polygonal region $u \geq 0$, $-1 \leq v \leq 1$. A formula for $f(z)$ is found by integrating (7). Since z is in the upper half-plane $y \geq 0$, we first use

[†]For a bounded polygon in the plane, any $n - 1$ of its interior angles uniquely determine the remaining one.

the principal square root to rewrite (7) as

$$f'(z) = \frac{A}{(z^2 - 1)^{1/2}}.$$

Furthermore, since the principal value of $(-1)^{1/2} = i$, we have

$$f'(z) = \frac{A}{(z^2 - 1)^{1/2}} = \frac{A}{[-1(1-z^2)]^{1/2}} = \frac{A}{i(1-z^2)^{1/2}} = -Ai \frac{1}{(1-z^2)^{1/2}}. \quad (8)$$

From (7) of Section 4.4 we recognize that an antiderivative of (8) is given by

$$f(z) = -Ai \sin^{-1} z + B, \quad (9)$$

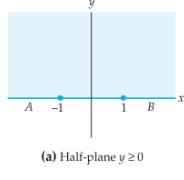
where $\sin^{-1} z$ is the single-valued function obtained by using the principal square root and principal value of the logarithm and where A and B are complex constants. If we choose $f(-1) = -i$ and $f(1) = i$, then the constants A and B must satisfy the system of equations

$$\begin{aligned} -Ai \sin^{-1}(-1) + B &= Ai \frac{\pi}{2} + B = -i \\ -Ai \sin^{-1}(1) + B &= -Ai \frac{\pi}{2} + B = i. \end{aligned}$$

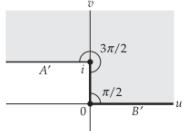
By adding these two equations we see that $2B = 0$, or, $B = 0$. Now by substituting $B = 0$ into either the first or second equation we obtain $A = -2/\pi$. Therefore, the desired mapping is given by

$$f(z) = i \frac{2}{\pi} \sin^{-1} z.$$

This mapping is shown in Figure 7.23. The line segments labeled A and B shown in color in Figure 7.23(a) are mapped by $w = i \frac{2}{\pi} \sin^{-1} z$ onto the line segments labeled A' and B' shown in black in Figure 7.23(b).



(a) Half-plane $y \geq 0$



(b) Polygonal region for Example 2

Figure 7.24 Figure for Example 2

EXAMPLE 2 Using the Schwarz-Christoffel Formula

Use the Schwarz-Christoffel formula (6) to construct a conformal mapping from the upper half-plane onto the polygonal region shown in gray in Figure 7.24(b).

Solution We proceed as in Example 1. The region shown in gray in Figure 7.24(b) is an unbounded polygonal region with interior angles $\alpha_1 = 3\pi/2$ and $\alpha_2 = \pi/2$ at the vertices $w_1 = i$ and $w_2 = 0$, respectively. If we select $x_1 = -1$ and $x_2 = 1$ to map onto w_1 and w_2 , respectively, then (6) gives

$$f'(z) = A(z+1)^{1/2}(z-1)^{-1/2}. \quad (10)$$

Since

$$(z+1)^{1/2}(z-1)^{-1/2} = \left(\frac{z+1}{z-1}\right)^{1/2} \left(\frac{z+1}{z+1}\right)^{1/2} = \frac{z+1}{(z^2-1)^{1/2}},$$

we can rewrite (10) as

$$f'(z) = A \left[\frac{z}{(z^2-1)^{1/2}} + \frac{1}{(z^2-1)^{1/2}} \right]. \quad (11)$$

An antiderivative of (11) is given by

$$f(z) = A \left[(z^2-1)^{1/2} + \cosh^{-1} z \right] + B,$$

where A and B are complex constants, and where $(z^2-1)^{1/2}$ and $\cosh^{-1} z$ represent branches of the square root and inverse hyperbolic cosine functions defined on the domain $y > 0$. Because $f(-1) = i$ and $f(1) = 0$, the constants A and B must satisfy the system of equations

$$\begin{aligned} A(0 + \cosh^{-1}(-1)) + B &= A\pi i + B = i \\ A(0 + \cosh^{-1}1) + B &= B = 0. \end{aligned}$$

Therefore, $A = 1/\pi$, $B = 0$, and the desired mapping is given by

$$f(z) = \frac{1}{\pi} \left[(z^2-1)^{1/2} + \cosh^{-1} z \right].$$

The mapping is illustrated in Figure 7.24. The line segments labeled A and B shown in color in Figure 7.24(a) are mapped by $w = f(z)$ onto the line segments labeled A' and B' shown in black in Figure 7.24(b).

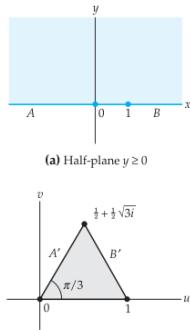


Figure 7.25 Figure for Example 3

When using the Schwarz-Christoffel formula, it is not always possible to express $f(z)$ in terms of elementary functions. In such cases, however, numerical techniques can be used to approximate f with great accuracy. The following example illustrates that even relatively simple polygonal regions can lead to integrals that cannot be expressed in terms of elementary functions.

EXAMPLE 3 Using the Schwarz-Christoffel Formula

Use the Schwarz-Christoffel formula (6) to construct a conformal mapping from the upper half-plane onto the polygonal region bounded by the equilateral triangle with vertices $w_1 = 0$, $w_2 = 1$, and $w_3 = \frac{1}{2} + \frac{1}{2}\sqrt{3}i$. See Figure 7.25.

Solution The region bounded by the equilateral triangle is a bounded polygonal region with interior angles $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$. As mentioned on page 413, we can find a desired mapping by using the Schwarz-Christoffel

formula (6) with $n - 1 = 2$ of the interior angles. After selecting $x_1 = 0$ and $x_2 = 1$, (6) gives

$$f'(z) = Az^{-2/3}(z-1)^{-2/3}. \quad (12)$$

There is no antiderivative of the function in (12) that can be expressed in terms of elementary functions. However, f' is analytic in the simply connected domain $y > 0$, and so, from Theorem 5.8 of Section 5.4, an antiderivative f does exist in this domain. The antiderivative is given by the integral formula

$$f(z) = A \int_0^z \frac{1}{s^{2/3}(s-1)^{2/3}} ds + B, \quad (13)$$

where A and B are complex constants. Requiring that $f(0) = 0$ allows us to solve for the constant B . Since $\int_0^0 = 0$, we have

$$f(0) = A \int_0^0 \frac{1}{s^{2/3}(s-1)^{2/3}} ds + B = 0 + B = B,$$

and so $f(0) = 0$ implies that $B = 0$. If we also require that $f(1) = 1$, then

$$f(1) = A \int_0^1 \frac{1}{s^{2/3}(s-1)^{2/3}} ds = 1.$$

Let Γ denote value of the integral

$$\Gamma = \int_0^1 \frac{1}{s^{2/3}(s-1)^{2/3}} ds.$$

Then $A = 1/\Gamma$ and f can be written as

$$f(z) = \frac{1}{\Gamma} \int_0^z \frac{1}{s^{2/3}(s-1)^{2/3}} ds.$$

Values of f can be approximated using a CAS. For example, using the **NIntegrate** command in *Mathematica* we find that

$$f(i) \approx 0.4244 + 0.3323i \quad \text{and} \quad f(1+i) \approx 0.5756 + 0.3323i.$$

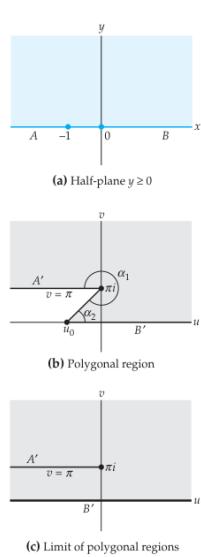


Figure 7.26 Figure for Example 4

EXAMPLE 4 Using the Schwarz-Christoffel Formula

Use the Schwarz-Christoffel formula (6) to construct a conformal mapping from the upper half-plane onto the nonpolygonal region defined by $v \geq 0$, with the horizontal half-line $v = \pi$, $-\infty < u \leq 0$, deleted. See Figure 7.26(c).

Solution Let u_0 be a point on the nonpositive u -axis in the w -plane. We can approximate the non-polygonal region defined by $v \geq 0$, with the half-line $v = \pi$, $-\infty < u \leq 0$, deleted by the polygonal region whose boundary consists of the horizontal half-line $v = \pi$, $-\infty < u \leq 0$, the line segment from πi to u_0 , and the horizontal half-line $v = 0$, $u_0 \leq u \leq \infty$. The vertices of this polygonal region are $w_1 = \pi i$ and $w_2 = u_0$, with corresponding interior angles α_1 and α_2 . See Figure 7.26(b). If we choose the points $z_1 = -1$ and $z_2 = 0$ to map onto the vertices $w_1 = \pi i$ and $w_2 = u_0$, respectively, then (6) gives the derivative

$$A(z+1)^{(\alpha_1/\pi)-1} z^{(\alpha_2/\pi)-1}. \quad (14)$$

Observe in Figure 7.26(b) that as u_0 approaches $-\infty$ along the u -axis, the interior angle α_1 approaches 2π and the interior angle α_2 approaches 0. With these limiting values, (14) suggests that our desired mapping f has derivative

$$f'(z) = A(z+1)^1 z^{-1} = A \left(1 + \frac{1}{z}\right). \quad (15)$$

An antiderivative of the function in (15) is given by

$$f(z) = A(z + \ln z) + B, \quad (16)$$

where A and B are complex constants.

In order to determine the appropriate values of the constants A and B , we first consider the mapping $g(z) = z + \ln z$ on the upper half-plane $y \geq 0$. The function g has a point of discontinuity at $z = 0$; thus, we will consider separately the boundary half-lines $y = 0$, $-\infty < x < 0$, and $y = 0$, $0 < x < \infty$, of the half-plane $y \geq 0$. If $z = x + 0i$ is on the half-line $y = 0$, $-\infty < x < 0$, then $\operatorname{Arg}(z) = \pi$, and so $g(z) = x + \log_e |x| + i\pi$.

When $x < 0$, $x + \log_e |x|$ takes on all values from $-\infty$ to -1 . Thus, the image of the negative x -axis under g is the horizontal half-line $v = \pi$, $-\infty < u < -1$. On the other hand, if $z = x + 0i$ is on the half-line $y = 0$, $0 < x < \infty$, then $\operatorname{Arg}(z) = 0$, and so $g(z) = x + \log_e |x|$. When $x > 0$, $x + \log_e |x|$ takes on all values from $-\infty$ to ∞ . Therefore, the image of the positive x -axis under g is the u -axis. It follows that the image of the half-plane $y \geq 0$ under $g(z) = z + \ln z$ is the region defined by $v \geq 0$, with the horizontal half-line $v = \pi$, $-\infty < u < -1$ deleted. In order to obtain the region shown in Figure 7.26(c), we should compose g with a translation by 1. Therefore, the desired mapping is given by

$$f(z) = z + \ln(z) + 1.$$

7.16.4 Poisson Integral Formulas

See Section 7.4....

7.16.5 Applications

See Section 7.5....

7.16.6 映射 $\frac{z-z_0}{1-\bar{z}_0 z}$, $|z_0| < 1$

The expression $\frac{z-z_0}{1-\bar{z}_0 z}$ has several important special properties in complex analysis, especially when z_0 is a point within the unit disk (i.e., $|z_0| < 1$). This

expression is commonly known as a **Blaschke factor** (or a component thereof) and is also a type of **Möbius transformation**.

Here are some of its key properties:

- (1) **Maps the unit disk to itself:** If $|z_0| < 1$, this transformation maps the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ to itself. That is, if $|z| < 1$, then $\left| \frac{z-z_0}{1-\bar{z}_0 z} \right| < 1$.
- (2) **Maps the unit circle to itself:** If $|z_0| < 1$, this transformation also maps the unit circle $C = \{z \in \mathbb{C} : |z| = 1\}$ to itself. That is, if $|z| = 1$, then $\left| \frac{z-z_0}{1-\bar{z}_0 z} \right| = 1$.
- (3) **Zero:** The transformation has a zero at $z = z_0$. That is, when $z = z_0$, $\frac{z_0-z_0}{1-\bar{z}_0 z_0} = 0$.
- (4) **Pole:** The transformation has a pole at $z = 1/\bar{z}_0$. If $|z_0| < 1$, then $1/|\bar{z}_0| = 1/|z_0| > 1$, so this pole lies outside the unit disk.
- (5) **Conformal automorphism:** When $|z_0| < 1$, this expression (or when multiplied by a constant $e^{i\phi}$ with modulus 1) is a conformal automorphism of the unit disk onto itself (i.e., a biholomorphic map). This means it preserves angles and is a one-to-one and onto mapping.
- (6) **Inverse Transformation:** If $f(z) = \frac{z-z_0}{1-\bar{z}_0 z}$, its inverse transformation is $f^{-1}(w) = \frac{w+z_0}{1+w\bar{z}_0}$. This can be seen as a transformation of the same form, $f_{-z_0}(w)$, where the parameter $-z_0$ replaces z_0 and the sign in the denominator is adjusted (or more precisely, it is $\frac{w-(-z_0)}{1-w(-z_0)}$ if we rewrite $1+w\bar{z}_0$ as $1-w(-\bar{z}_0)$).
- (7) **Derivative at z_0 :** Let $\phi_{z_0}(z) = \frac{z-z_0}{1-\bar{z}_0 z}$. Its derivative at $z = z_0$ is $\phi'_{z_0}(z_0) = \frac{1}{1-|z_0|^2}$. This property is relevant in contexts such as Schwarz's Lemma.

In summary, this expression defines a very important complex transformation that plays a central role in the geometry and function theory of the unit disk. It allows for a standardized way to map any point z_0 inside the unit disk to the origin, while preserving the disk and its boundary, which is fundamental in many areas of complex analysis.

7.16.7 例题

Exercice 24

设 $D = \{z \in \mathbb{C} : |z| < 1\}$, 若 $f \in H(D) \cap C(\overline{D})$ 使得 $|f(z)| = 1, \forall z \in \partial D$, 求全部 f .

证明. 若 f 在 D 有无穷个零点, 则若零点在 D 内部有聚点, 则解析函数唯一性告诉我们 $f \equiv 0$, 这不可能, 若这些零点的聚点只在 D 边界, 那么 f 在聚点处应该为 0, 这也不可能. 所以我们知道 f 在 D 只有有限个零点 z_1, z_2, \dots, z_n , 这里 $\{z_i\}_{i=1}^n$ 允许有相同的项出现.

考虑无零点函数

$$g(z) = \frac{f(z)}{\prod_{i=1}^n \frac{z-z_i}{1-\bar{z}_i z}} \in H(D) \cap C(\overline{D}),$$

显然 $|g(z)| = 1, \forall z \in \partial D$ 且 $\frac{1}{g(z)} \in H(D) \cap C(\overline{D})$. 故由最大模定理知 $|g(z)| \equiv 1, \forall z \in \overline{D}$. 因此我们知道 g 为常数, 所以

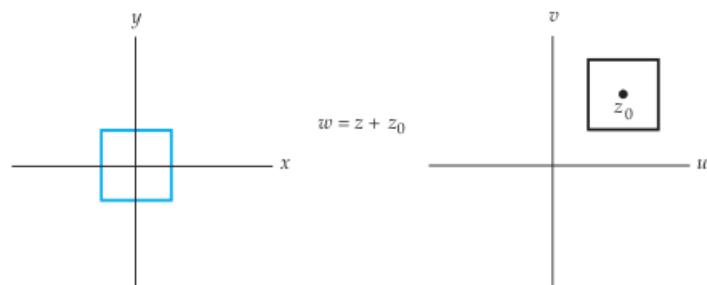
$$f(z) = c \prod_{i=1}^n \frac{z-z_i}{1-\bar{z}_i z}, c \in \mathbb{C} \setminus \{0\}.$$

□

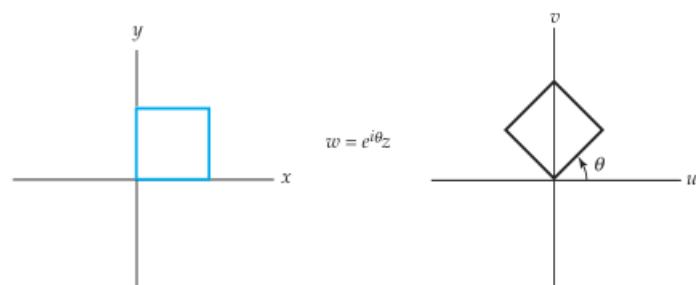
7.17 Table of Conformal Mappings

7.17.1 初等映射

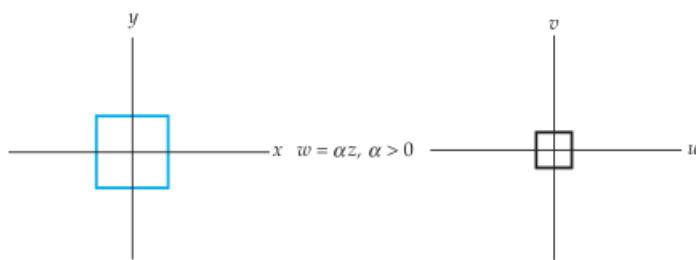
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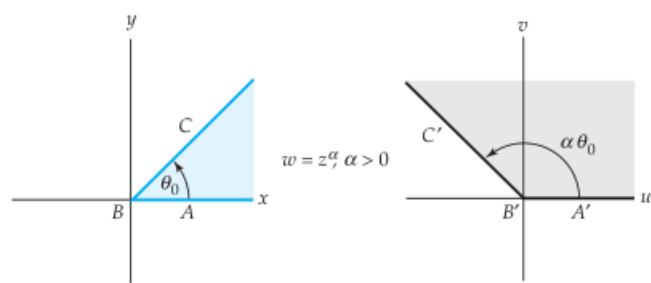
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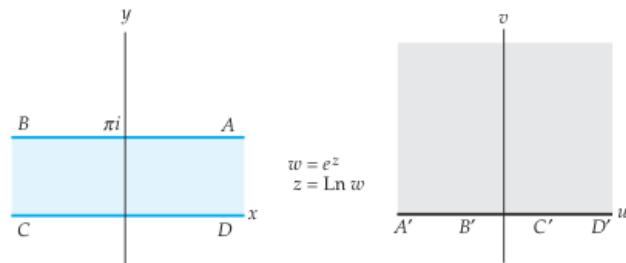
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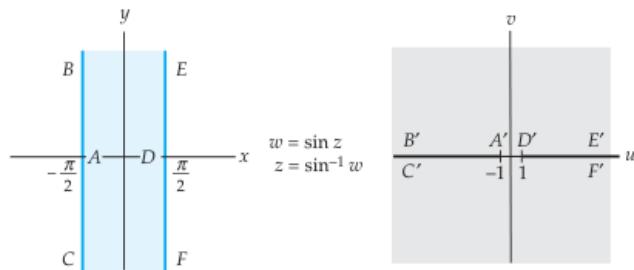
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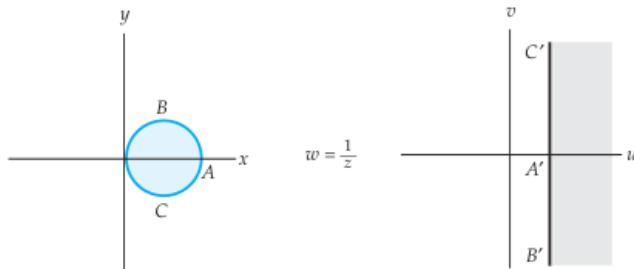
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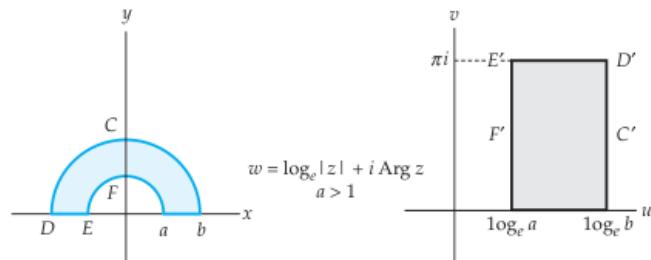
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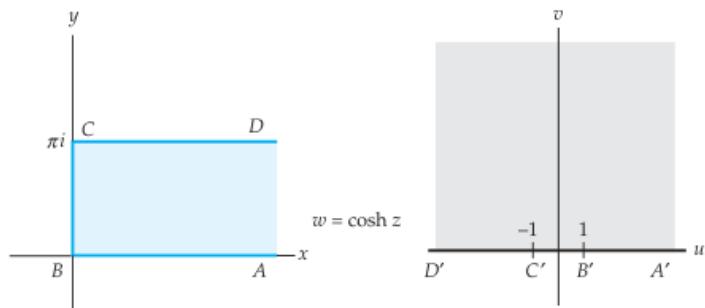
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E-8

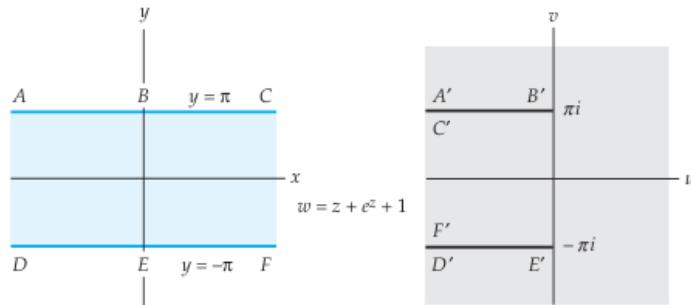


E-9

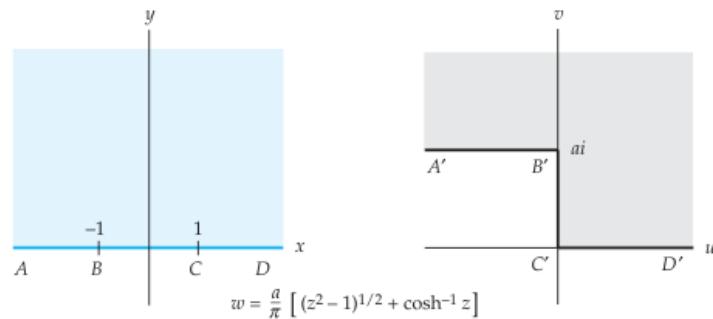


7.17.2 Miscellaneous Mappings

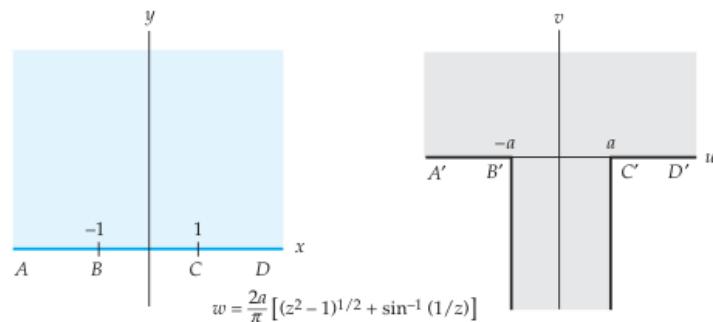
M-1



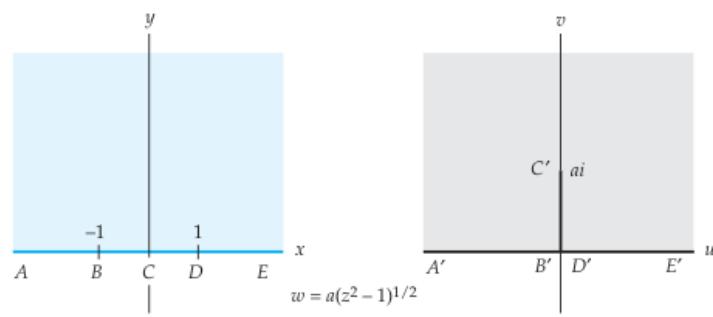
M-2

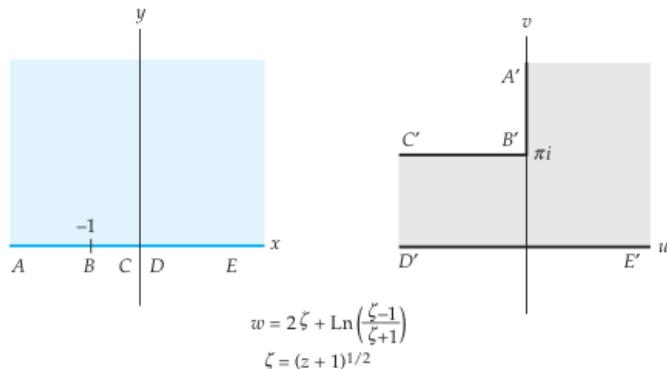
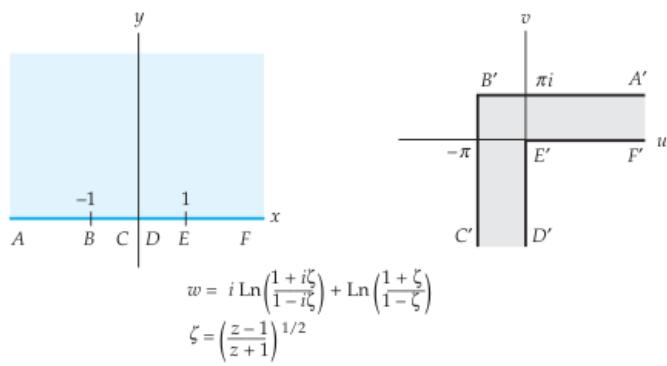
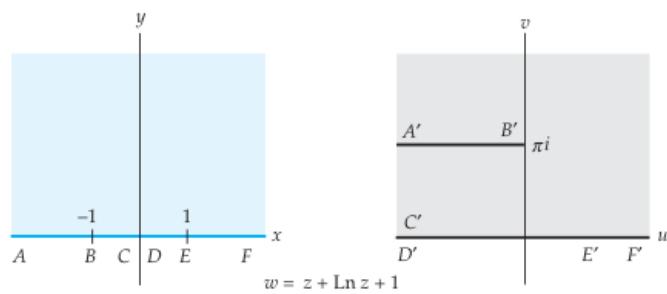
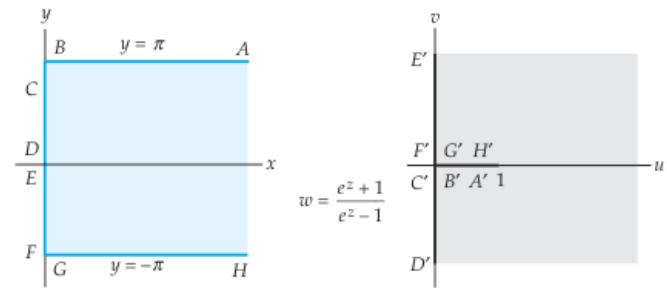


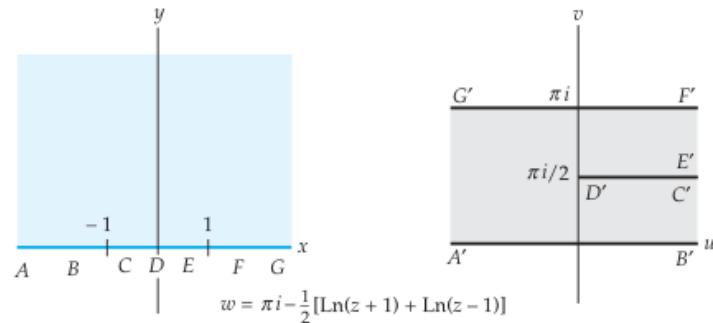
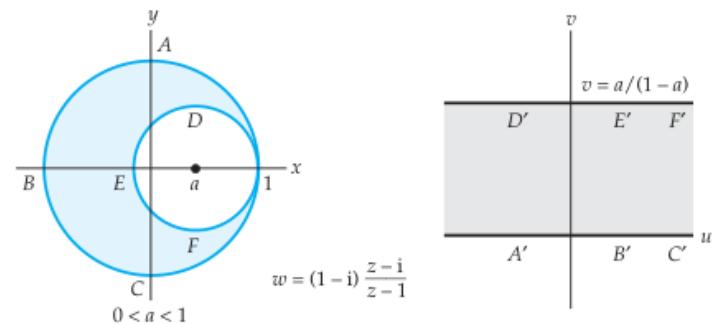
M-3



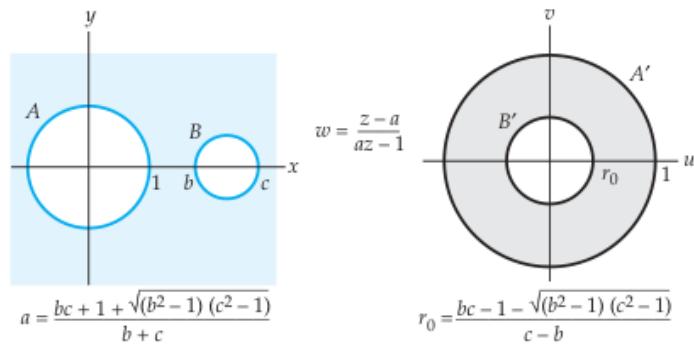
M-4



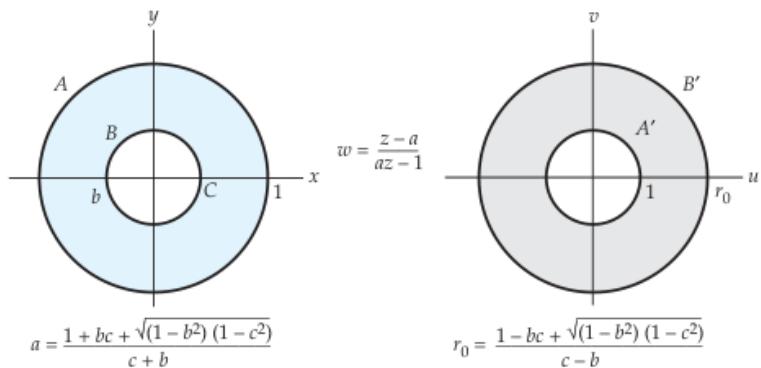
M-5**M-6****M-7****M-8**

M-9**M-10**

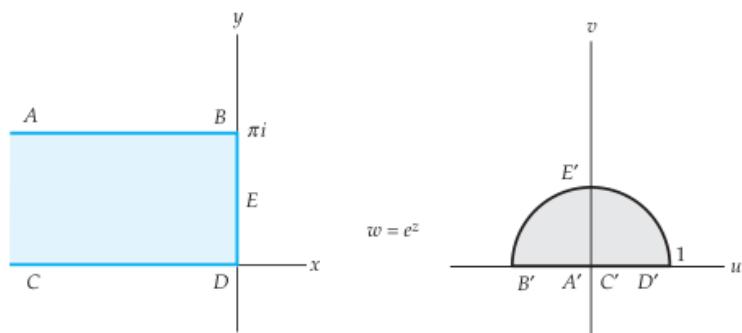
7.17.3 Mappings to Circular Regions

C-1

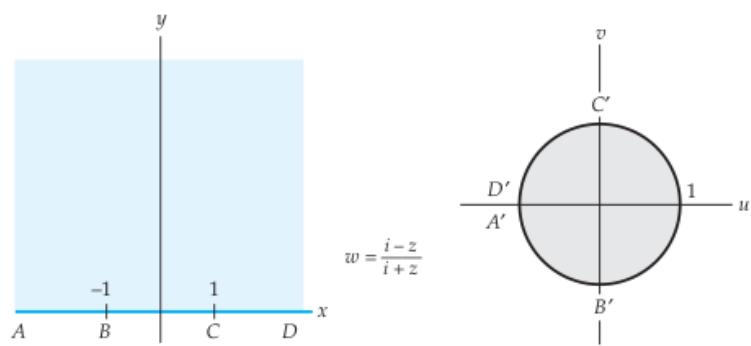
C-2



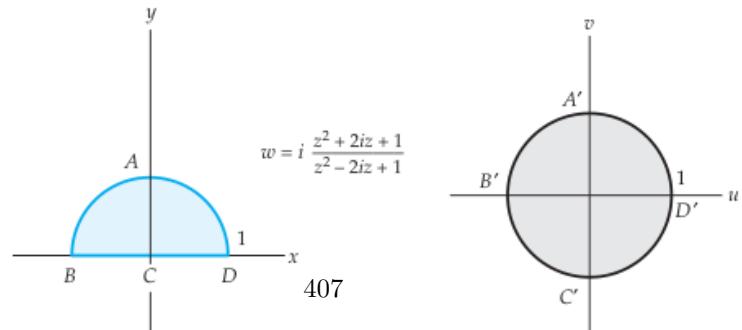
C-3



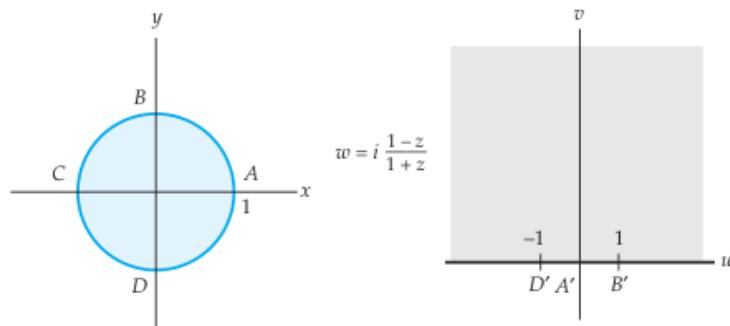
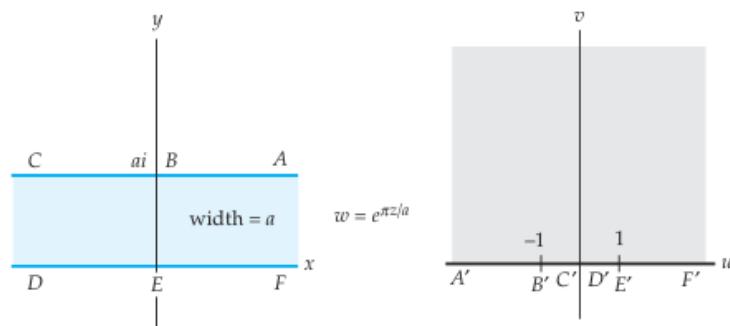
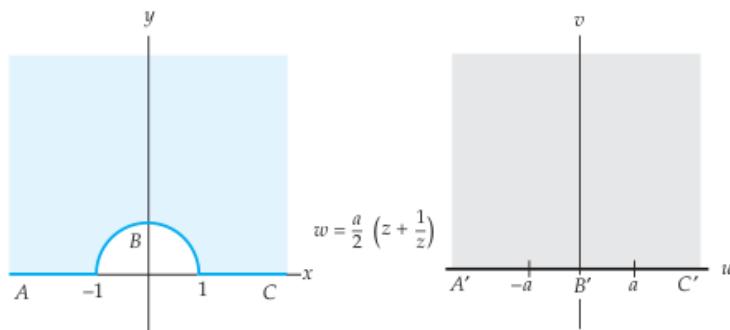
C-4

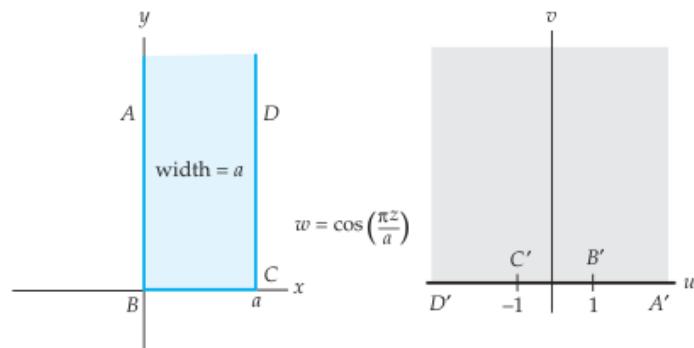
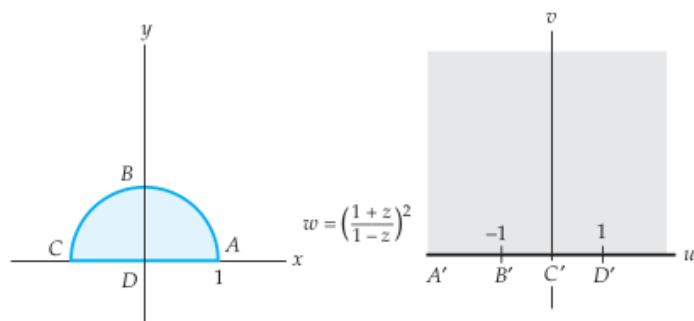
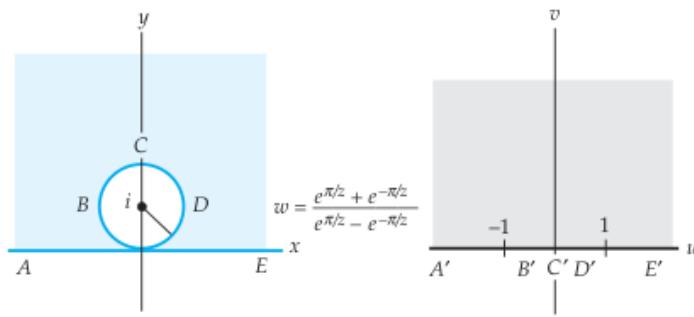


C-5



7.17.4 Mappings of Half-Planes

H-1**H-2****H-3**

H-4**H-5****H-6**

7.18 共形映射-刷题

见复变函数论(第五版)学习指导书(钟玉泉编)(Z-Library)

7.18.1 解析变换的保域性

Théorème 11

不恒为常数的解析变换 $w = f(z)$ 是保域的，即将区域 D ($f(z)$ 在 D 内解析) 变成区域 $f(D)$ 。

Corollaire 4

单叶解析变换是保域的。

Théorème 12: (定理 7.1 的推广)

设 $w = f(z)$ 在扩充 z 平面的区域 D 内除可能有极点外处处解析，且不恒为常数，则 D 的像 $G = f(D)$ 为扩充 w 平面上的区域。

7.18.2 解析变换的保角性

导数的几何意义

设 $f : D \rightarrow f(D)$ 解析， $z_0 \in D$, $f'(z_0) \neq 0$ ，于是 $\arg f'(z_0)$ 只与 z_0 有关，与过 z_0 的曲线 C 无关，称为变换 $w = f(z)$ 在 z_0 的旋转角。类似的， $|f'(z_0)|$ 称为变换 $w = f(z)$ 在 z_0 的伸缩率。

Théorème 13: 定理 7.4

如 $w = f(z)$ 在区域 D 内解析, $z_0 \in D, f'(z_0) \neq 0$, 则变换 $w = f(z)$ 在点 z_0 处是保角的 (即过 z_0 的一对曲线的交角与过像点 w_0 的一对像曲线的交角大小相等, 方向一致); 在 D 内单叶解析的变换是保角的.

7.18.3 圆在分式线性映射下的像

定理 7.10 在分式线性变换 $w=L(z)$ 下, 扩充 z 平面上的圆周 γ 变为扩充 w 平面上的圆周 $\Gamma=L(\gamma)$. 同时, γ 所界的圆 (d_1 或 d_2) 共形映射成 Γ 所界的圆 (D_1 或 D_2) (图 7.6).

注 (1) 在扩充平面上, 直线可视为经过无穷远点的圆周; 当 γ 或 $\Gamma=L(\gamma)$ 为直线时, 其所界的圆是以它为界的两个半平面.

(2) 确定圆周所界区域在分式线性变换 $w=L(z)$ 下的对应区域(圆)有如下两个方法(图 7.6):

(a) 在一个区域, 例如 d_1 中, 取一点 z_0 , 如果 $w_0=L(z_0) \in D_1$, 则可断定 $D_1=L(d_1)$, 否则 $D_2=L(d_1)$.

(b) 在 γ 上任取三点 z_1, z_2, z_3 , 当沿 z_1, z_2, z_3 顺次绕行时, d_1 在观察者前进方向的左侧, 对应地沿 w_1, w_2, w_3 顺次绕行 Γ 时, 在观察者前进方向左侧的区域就是 d_1 的像.

(3) 要使分式线性变换 $w=L(z)$ 把有限圆周 C 变成直线, 其条件是: C 上的某点 z_0 变成 ∞ .

例 7.2.3 问分式线性变换 $w=\frac{z}{z-1}$ 将闭单位圆 $|z| \leq 1$ 映成 w 平面上的什么区域?

分析 为了便于利用题设条件 $|z| \leq 1$, 我们先从已给分式线性变换写出其逆变换.

解 由分式线性变换 $w=\frac{z}{z-1}$, 利用公式(7.4)得到它的逆变换为 $z=\frac{w}{w-1}$, 所以有

$$\begin{aligned} \left| \frac{w}{w-1} \right| &= |z| \leq 1 \\ \Rightarrow |w|^2 &\leq |w-1|^2 = (w-1)(\bar{w}-1) \\ &= |w|^2 - (w+\bar{w}) + 1 \\ \Rightarrow w + \bar{w} &\leq 1 \Rightarrow \frac{1}{2}(w+\bar{w}) \leq \frac{1}{2} \\ \Rightarrow \operatorname{Re}(w) &\leq \frac{1}{2}. \end{aligned}$$

故所给分式线性变换 $w = \frac{z}{z-1}$ 将单位闭圆 $|z| \leq 1$ 映成 w 平面上的闭半平面 $\operatorname{Re}(w) \leq \frac{1}{2}$, 即它将单位圆 $|z| < 1$ 共形映射成半平面 $\operatorname{Re}(w) < \frac{1}{2}$, 并将单位圆周 $|z| = 1$ 一一地变为直线 $\operatorname{Re}(w) = \frac{1}{2}$. ■

7.18.4 圆周到圆周的分式线性映射

例 7.2.4 试证: 分式线性变换 $w = \frac{2z+3}{z-4}$ 把圆周

$$x^2 + y^2 - 4y = 0 \quad (z = x + iy)$$

变成圆周

$$16u^2 + 16v^2 + 24u + 44v + 9 = 0 \quad (w = u + iv).$$

分析 先将已给分式线性变换分解成几个简单分式线性变换, 并依次利用它们将已给圆周逐次变形.

证 因为已给分式线性变换可变形为 $w = 2 + \frac{11}{z-4}$, 从而可分解成:

$$(a) z_1 = z - 4; \quad (b) z_2 = \frac{1}{z_1}; \quad (c) z_3 = 11z_2; \quad (d) w = z_3 + 2.$$

由(a) 得 $x = x_1 + 4, y = y_1 (z_1 = x_1 + iy_1)$ 代入圆周

$$x^2 + y^2 - 4y = 0, \text{ 即 } x^2 + (y - 2)^2 = 2^2, \quad (1)$$

得圆周

$$(x_1 + 4)^2 + (y_1 - 2)^2 = 2^2. \quad (2)$$

由(b) 给出

$$z_1 = \frac{1}{z_2} \Rightarrow x_1 = \frac{x_2}{x_2^2 + y_2^2}, y_1 = \frac{-y_2}{x_2^2 + y_2^2} \quad (z_2 = x_2 + iy_2),$$

代入(2)式得

$$\left(\frac{x_2}{x_2^2 + y_2^2} + 4 \right)^2 + \left(\frac{-y_2}{x_2^2 + y_2^2} - 2 \right)^2 = 4,$$

化简后得圆周

$$(4x_2 + 1)^2 + \left(4y_2 + \frac{1}{2} \right)^2 = \left(\frac{1}{2} \right)^2. \quad (3)$$

由(c) 给出

$$z_2 = \frac{z_3}{11} \Rightarrow x_2 = \frac{x_3}{11}, y_2 = \frac{y_3}{11} \quad (z_3 = x_3 + iy_3),$$

代入(3)式化简后得圆周

$$\left(x_3 + \frac{11}{4}\right)^2 + \left(y_3 + \frac{11}{8}\right)^2 = \frac{1}{4} \left(\frac{11}{4}\right)^2. \quad (4)$$

由(d)给出

$$z_3 = w - 2 \Rightarrow x_3 = u - 2, y_3 = v,$$

代入(4)式化简后得 w 平面上的圆周

$$\left(u + \frac{3}{4}\right)^2 + \left(v + \frac{11}{8}\right)^2 = \left(\frac{11}{8}\right)^2,$$

即

$$16u^2 + 16v^2 + 24u + 44v + 9 = 0.$$

■

7.18.5 给定三个对应点的分式线性映射

使用交比，或者联立求解

交比：

$$(z_1, z_2, z_3, z_4) = \frac{z_4 - z_1}{z_4 - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}$$

例 7.2.6 求把点 $z_1 = 0, z_2 = 1, z_3 = \infty$ 变成 $w_1 = -1, w_2 = -i, w_3 = 1$ 的分式线性变换。

解一 由分式线性变换的保交比性，

$$(-1, -i, 1, w) = (0, 1, \infty, z),$$

即

$$\begin{aligned} \frac{w+1}{w+i} : \frac{1+1}{1+i} &= \frac{z-0}{z-1} : \frac{1}{1}, \\ \frac{w+1}{w+i} &= \frac{2}{1+i} \cdot \frac{z}{z-1}. \end{aligned}$$

于是

$$\frac{w+1}{w+1-(w+i)} = \frac{2z}{2z-(1+i)(z-1)},$$

所以

$$w = \frac{z-i}{z+i}.$$

解二 设所求分式线性变换为

$$w = \frac{az+b}{cz+d}, \quad ad-bc \neq 0.$$

代入对应点，得 $-1 = \frac{b}{d}$ (可见 $d \neq 0$)，

$$\begin{aligned} -i &= \frac{a+b}{c+d} = \frac{\frac{a}{d} + \frac{b}{d}}{\frac{c}{d} + 1}, \\ 1 &= \frac{a}{c} = \frac{a}{d} / \frac{c}{d}. \end{aligned}$$

解此方程组得 $\frac{a}{d} = \frac{c}{d} = -i$, 所以

$$w = \frac{\frac{a}{d}z + \frac{b}{d}}{\frac{c}{d}z + 1} = \frac{-iz - 1}{-iz + 1} = \frac{z - i}{z + i}.$$

所求分式线性变换将上半 z 平面 $\operatorname{Im} z > 0$ 共形映射成单位圆 $|w| < 1$, 并把实轴 $\operatorname{Im} z = 0$ 变成圆周 $|w| = 1$. ■

注 前两题的解法一当对应点不出现 $0, 1, \infty$ 时, 交比化简过程较繁, 不如使用解法二简捷.

7.18.6 分式线性映射保持对称点

在前面, 我们曾经讲过关于单位圆周的对称点, 现推广如下:

定义 7.5 z_1, z_2 关于圆周 $\gamma: |z - a| = R$ 对称是指 z_1, z_2 都在过圆心 a 的同一条射线上, 且满足

$$|z_1 - a| |z_2 - a| = R^2. \quad (7.6)'$$

此外, 还规定圆心 a 与点 ∞ 也是关于 γ 对称的.

由定义即知,

$$z_1, z_2 \text{ 关于 } \gamma \text{ 对称} \Leftrightarrow z_2 - a = \frac{R^2}{z_1 - a}. \quad (7.5)'$$

下述定理就是分式线性变换的保对称点性:

定理 7.12 设扩充 z 平面上两点 z_1, z_2 关于圆周 γ 对称, $w = L(z)$ 为一个分式线性变换, 则 $w_1 = L(z_1), w_2 = L(z_2)$ 两个像点关于像圆周 $\Gamma = L(\gamma)$ 为对称.

例 7.2.8 求点 $2+i$ 分别关于圆周:(1) $|z|=1$; (2) $|z-i|=3$ 的对称点.

分析 直接应用公式(7.5)及公式(7.5)'.

解 (1) 由公式(7.5), $2+i$ 关于 $|z|=1$ 的对称点为

$$w = \frac{1}{\overline{2+i}} = \frac{1}{2-i} = \frac{1}{5}(2+i).$$

(2) 由公式(7.5)', $z_1=2+i$ 关于圆周 $|z-i|=3$ 的对称点 z_2 满足

$$z_2 - i = \frac{3^2}{\overline{z_1 - i}}.$$

于是

$$\begin{aligned} z_2 &= i + \frac{9}{\overline{2+i-i}} = i + \frac{9}{2-i+i} \\ &= i + \frac{9}{2} = \frac{9}{2} + i. \end{aligned}$$

■

例 7.2.9 试写出在分式线性变换

$$w = \frac{z-i}{(2-i)z-1} \quad (1)$$

下, 直线 $C: \operatorname{Im} z=0$ (实轴) 的像 Γ 的方程, 并求 $w_0=2$ 关于 Γ 的对称点.

分析 因 $C: \operatorname{Im} z=0 \Leftrightarrow z-\bar{z}=0$, 故从(1)式的逆变换着手.

解 因为(1)式的逆变换为(由公式(7.4))

$$z = \frac{w-i}{(2-i)w-1}, \quad (2)$$

故 $C: z-\bar{z}=0$ 的像曲线 Γ 的方程为

$$\frac{w-i}{(2-i)w-1} - \frac{\overline{w}+i}{(2+i)\overline{w}-1} = 0.$$

化简得

$$w\overline{w} - (1-i)w - (1+i)\overline{w} + 1 = 0,$$

即

$$|w - (1+i)| = 1. \quad (3)$$

又因为 $w_0=2$ 的原像为(代入(2)式)

$$z_0 = \frac{2-i}{2(2-i)-1} = \frac{8+i}{13},$$

而 z_0 关于 $C: \operatorname{Im} z=0$ 的对称点为 $z_1 = \overline{z}_0 = \frac{8-i}{13}$; 所以由分式线性变换的保对称点性质

知, w_0 关于 Γ 的对称点 w_1 就是 z_1 的像, 即由(1)式

$$w_1 = \frac{z_1-i}{(2-i)z_1-1} = \frac{8-i-13i}{(2-i)(8-i)-13} = \frac{3+i}{2}.$$

或直接由公式(7.5)',

$$w_1 - (1+i) = \frac{1^2}{2-(1+i)} = \frac{1}{1+i} = \frac{1-i}{2},$$

同样得到

$$w_1 = \frac{3+i}{2}. \quad \blacksquare$$

7.18.7 特殊的分式线性映射

上半平面到上半平面

例 7.5 把上半 z 平面共形映射成上半 w 平面的分式线性变换为

$$w = \frac{az + b}{cz + d}, \quad (7.12)$$

其中 a, b, c, d 是实数, 且 $ad - bc > 0$.

注 线性变换(7.12)同时把 z 平面上的实轴变成 w 平面上的实轴, 且把下半 z 平面共形映射成下半 w 平面.

上半平面到单位圆

例 7.6 把上半 z 平面 $\operatorname{Im} z > 0$ 共形映射成单位圆 $|w| < 1$, 并使一点 $z = a (\operatorname{Im} a > 0)$ 变为 $w = 0$ 的分式线性变换为

$$w = k \frac{z - a}{z + \bar{a}} \quad (\text{复数 } k \text{ 待定}, |k| = 1). \quad (7.13)'$$

或

$$w = e^{i\beta} \frac{z - a}{z + \bar{a}} \quad (\text{实参数 } \beta \text{ 待定}). \quad (7.13)$$

单位圆到单位圆

例 7.7 把单位圆 $|z| < 1$ 共形映射成单位圆 $|w| < 1$, 并使一点 $z = a (|a| < 1)$ 变为 $w = 0$ 的分式线性变换为

$$w = k \frac{z - a}{z - \frac{1}{\bar{a}}} \quad (\text{复数 } k \text{ 待定}), \quad (7.14)'$$

或

$$w = e^{i\beta} \frac{z - a}{1 - \bar{a}z} \quad (\text{实参数 } \beta \text{ 待定}). \quad (7.14)$$

7.18.8 固定两个点和图形的共形映射

例 7.2.13 求把圆 $D: |z| < 2$ 共形映射成圆 $G: |w-1| < 1$ 的分式线性变换 $w = L(z)$, 使满足条件

$$\underbrace{L(2)}_{\cdot} = 0, \quad L(0) = \frac{1}{2}.$$

分析 先分别求把 D 共形映射成 z_1 平面上的单位圆及把 G 共形映射成 w_1 平面上的单位圆的分式线性变换. 再利用公式(7.14)求此二单位圆间的共形映射, 中间插入两个平面(图 7.2.5). 最后, 复合起来, 即得所求共形映射.

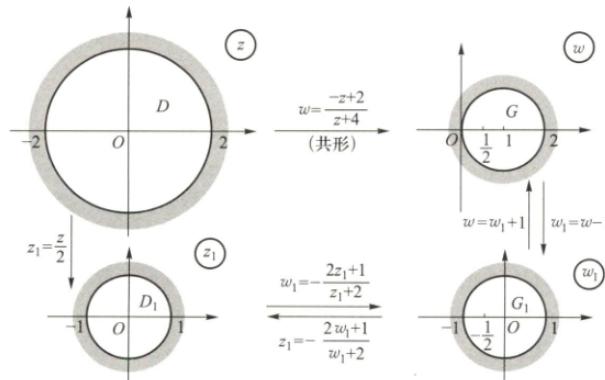


图 7.2.5

例 7.2.14 求分式线性变换 $w=L(z)$ 将圆 $|z-(1+i)| < 2$ 共形映射成上半 w 平面, 且将 $z=2+i$ 变为 $w=i$, $z=1+3i$ 变为 $w=1$ (图 7.2.6).

分析 已知

$$z=2+i \leftrightarrow w=i \quad (\text{一对内点对应点}),$$

$$z=1+3i \leftrightarrow w=1 \quad (\text{一对边界对应点}).$$

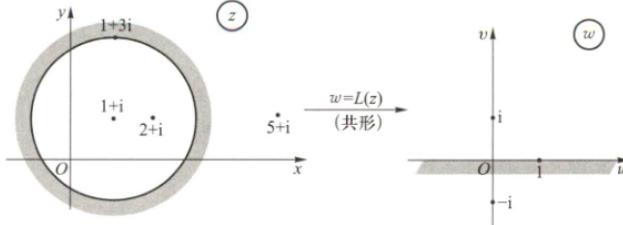


图 7.2.6

再由 $w=L(z)$ 的保对称性, 又得一对对应点. 此三对对应点就可惟一确定所求分式线性变换 $w=L(z)$.

7.18.9 初等函数的共形映射

幂函数：角形到角形

初等函数构成的共形映射对今后研究较复杂的共形映射大有作用.

1. 掌握幂函数与根式函数所构成的共形映射及其作用.

对幂函数

$$w = z^n \quad (\text{整数 } n \geq 2), \quad (7.15)$$

有

$$d : 0 < \arg z < \alpha \left(0 < \alpha \leq \frac{2\pi}{n} \right) \xrightarrow[\text{(共形)}]{w = z^n} D : 0 < \arg w < n\alpha.$$

对根式函数

$$z = \sqrt[n]{w}, \quad (7.16)$$

有

$$d : 0 < \arg z < \alpha \left(0 < \alpha \leq \frac{2\pi}{n} \right) \xleftarrow[\text{(共形)}]{z = \sqrt[n]{w}} D : 0 < \arg w < n\alpha.$$

指数函数：带形到角形

对指数函数

$$w = e^z, \quad (7.17)$$

有

$$\text{带形 } g : 0 < \operatorname{Im} z < h \left(0 < h \leq 2\pi \right) \xrightarrow[\text{(共形)}]{w = e^z}$$

$$\text{角形 } G : 0 < \arg w < h.$$

对数函数：角形到带形

对对数函数

$$z = \ln w, \quad (7.17)'$$

有

$$g : 0 < \operatorname{Im} z < h \left(0 < h \leq 2\pi \right) \xleftarrow[\text{(共形)}]{z = \ln w} G : 0 < \arg w < h.$$

这里 $\ln w$ 是 G 内的一个单值解析分支, 它的值完全由带形 g 确定.

7.18.10 带有割痕的共形映射

例 7.3.5 求把具有割痕: $-\infty < \operatorname{Re}(z) \leq a$, $\operatorname{Im} z = H$ 的带形区域 $0 < \operatorname{Im} z < 2H$ 变成带形区域 $0 < \operatorname{Im} w < 2H$ 的一个共形映射.

分析 首先要经伸缩变换将已给的带形区域共形映射成标准带形, 再经指数变换将其共形映射成上半平面(图 7.3.4). 在这样的变换过程中割线的位置长短也会随之变化.

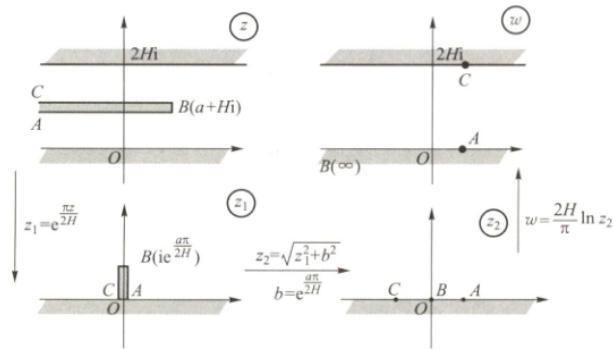


图 7.3.4

7.18.11 两角形区域到角形

(1) 把过 a, b 两点(两个顶点)的两个圆弧(其中一个可以是直线段)围成的区域
(称为两角形区域)共形映射成角形区域的分式线性变换为

$$w = k \frac{z-a}{z-b}, \quad (7.3)^*$$

它将 $z=a$ 变为 $w=0$, $z=b$ 变为 $w=\infty$.

由于 $\frac{dw}{dz} \Big|_{z=a} = \frac{k}{a-b} \neq 0 (a \neq b, k \neq 0)$, 可见 $(7.3)^*$ 式在 $z=a$ 是保角的, 即将交角

为 α 的两角形区域共形映射成以原点为顶点、张角为 α 的角形区域.

如要变成起边是正实轴的角形, 只需在 C_1 和正实轴上指定一对对应点, 即可确定 k , 从而得到所求的共形映射. 当然也可像图 7.3.5 一样, 先经 $(7.3)^*$ 式再适当旋转.

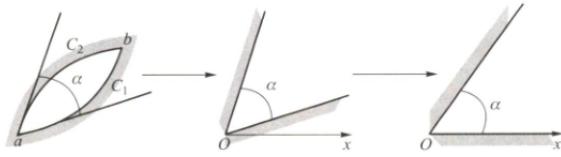


图 7.3.5

(2) 两圆周内切于点 a , 其所界的月牙形区域也是个两角形区域(两个顶点合二为一). 只需取分式线性变换

$$\xi = \frac{cz+d}{z-a},$$

切点 a 就变成点 ∞ , 月牙形就形成一个带形, 适当地选取 c, d , 就会变成标准带形 $0 < \text{Im } \xi < \pi$. 再经指数函数 $w = e^\xi$ 就变成标准区域上半 w 平面.

(3) 通过一些例题, 我们还会认识一些特殊的两角形区域. 借此, 可以解决许多较

可以利用两角形区域映射来理解

繁难的共形映射问题.

例 7.3.6 求一个把第一象限内的四分之一圆:

$$0 < \arg z < \frac{\pi}{2}, 0 < |z| < 1$$

变成单位圆的共形映射.

分析 不能认为 $w = z^4$ 就是所求的变换, 因为它把四分之一圆变成去掉了沿正实轴的半径的单位圆.

解 首先, 把四分之一圆通过变换

$$\zeta = z^2$$

共形映射成上半单位圆(这是个两角形);

其次, 把上半单位圆通过分式线性变换(例 7.14)

$$t = -\frac{\zeta + 1}{\zeta - 1}$$

(只需在前面第 3 小节公式(7.3)*中令 $a = -1, b = 1$ 并要求 $z = 0 \leftrightarrow w = 1$, 即得 $k = -1$) 共形映射成第一象限;

再把第一象限通过 $t_1 = t^2$ 共形映射成上半平面;

最后, 通过分式线性变换(7.13) $w = \frac{t_1 - i}{t_1 + i}$, 把上半平面共形映射成单位圆.

因此, 所求的一个共形映射(图 7.3.6)为

$$w = \frac{\left(\frac{z^2+1}{z^2-1}\right)^2 - i}{\left(\frac{z^2+1}{z^2-1}\right)^2 + i} = \frac{(z^2+1)^2 - i(z^2-1)^2}{(z^2+1)^2 + i(z^2-1)^2}.$$

■

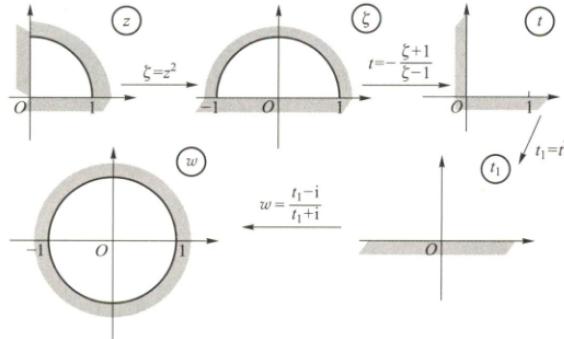


图 7.3.6

例 7.3.7 求将两角形区域 $D: |z+i| < 2, \operatorname{Im} z > 0$ 变成带形区域 $\Omega: 0 < \operatorname{Im} w < \pi$ 的共形映射.

分析 指定的区域一个是两角形 D , 另一个是带形 Ω , 两角形可通过分式线性变

换(7.3)*变成角形,且不难用幂函数变成上半平面. 带形经指数函数可变成角形,也可变成上半平面. 因而两角形 D 经中间的“跳板”上半平面,可变成要求的带形 Ω .

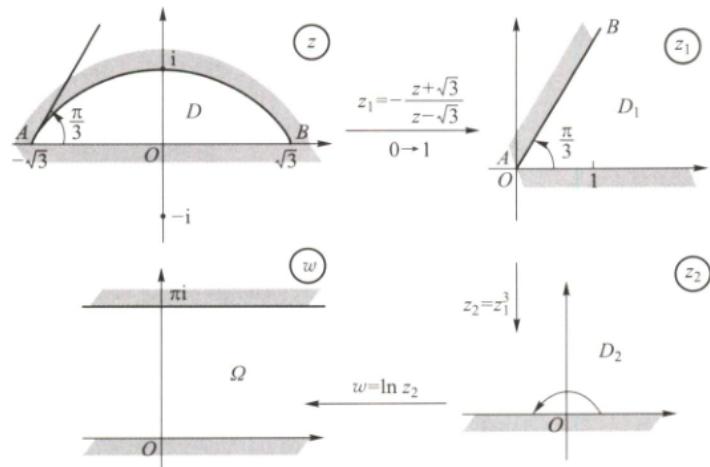


图 7.3.7

7.18.12 月牙形的共形映射

例 7.3.8 求把区域 $D: |z| < 1, \left|z - \frac{i}{2}\right| > \frac{1}{2}$ 变成上半平面的共形映射.

分析 区域 D 是两圆周 $|z| = 1$ 及 $\left|z - \frac{i}{2}\right| = \frac{1}{2}$ 相切于 $z = i$ 所围成的月牙形区域(图 7.3.8).

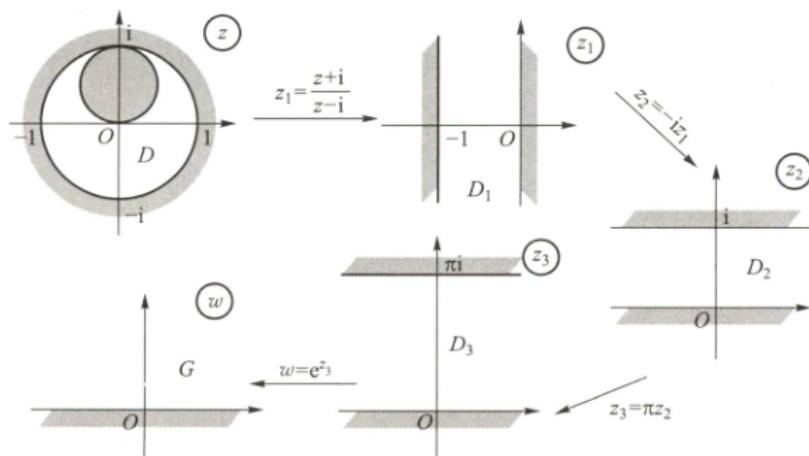


图 7.3.8

例 7.3.9 求把区域 $D: (|z|>1) \cap (|z-\sqrt{3}i|<2)$ 共形映射成单位圆 $|w|<1$ 的函数 $w=f(z)$, 使满足条件

$$f(\sqrt{3}i) = 0, \quad f'(\sqrt{3}i) > 0.$$

分析 D 为两角形, 其边界为两段圆弧 L_1, L_2 (图 7.3.9), 交点为 ± 1 . 所给条件是惟一性条件.

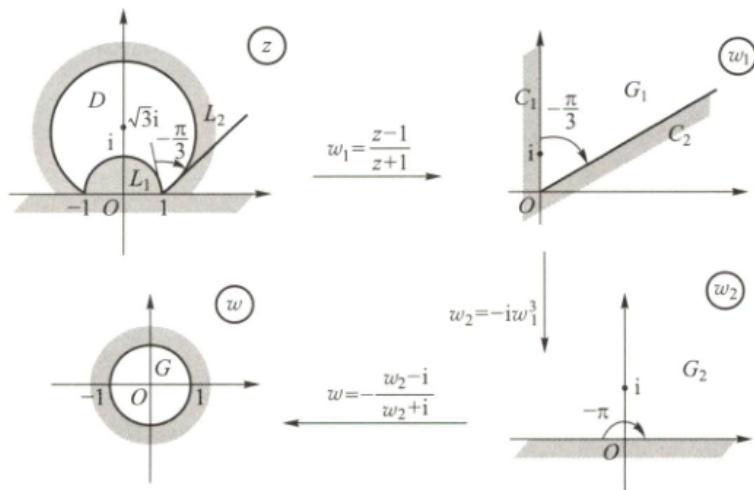


图 7.3.9

7.18.13 茹科夫斯基变换

$$w = \frac{1}{2} \left(z + \frac{1}{z} \right)$$

它把 $|z| = 1$ 内部和外部都共形映射为扩充 w 平面上去掉 $-1 \leq \operatorname{Re} w \leq 1, \operatorname{Im} w = 0$ 的区域，把上（下）半单位圆共形映射成下（上）半 w 平面，把上（下）半单位圆周变成线段 $[-1, 1]$ 的下（上）岸。

例 7.3.10 求将半带形区域 $D: -\frac{\pi}{2} < x < \frac{\pi}{2}, y > 0$ 变为上半 w 平面 $G: \operatorname{Im} w > 0$ 的共形映射 $w = f(z)$ ，使满足条件

$$f(0) = 0, \quad f\left(\pm \frac{\pi}{2}\right) = \pm 1.$$

分析 所给条件为三对边界对应点，是惟一性条件。

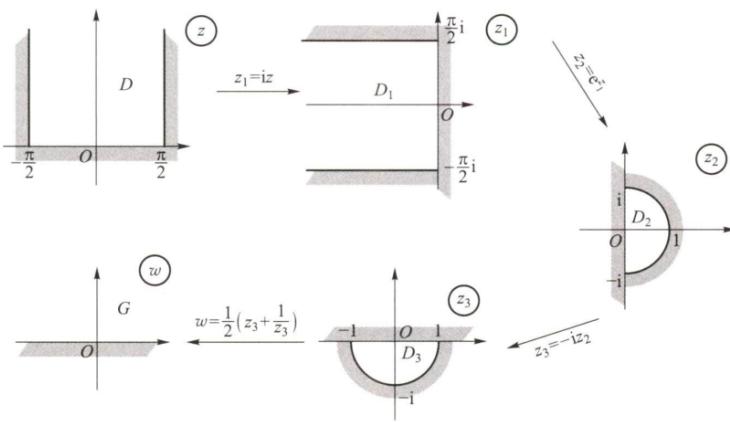


图 7.3.10

于是，复合上述变换，即得所求的共形映射为

$$w = \frac{1}{2} \left(-iz_2 + \frac{1}{iz_2} \right) = \frac{1}{2i} (e^{iz_2} - e^{-iz_2}) = \sin z.$$

这是惟一满足条件的变换。 ■

Remarque 22. 从上面的例题，我们易于想到，要求把一个区域 D 变成区域 G 的共形映射，如果能直接把函数求出最好；一般则是把 D 或 G ，或把 D 与 G 都设法变成上半平面或单位圆，然后求出共形映射。这里上半平面或单位圆起着中间或桥梁的作用，也称为标准区域。

逆变换 $\sqrt{\zeta^2 - 1} + \zeta$

$\sqrt{\zeta^2 - 1} \Big|_{\zeta=i} = \sqrt{2}i$ 那一支将 $\mathbb{C} \setminus [-1, 1]$ 映射到 $|w| > 1$. $\sqrt{\zeta^2 - 1} \Big|_{\zeta=-i} = -\sqrt{2}i$ 那一支将 $\mathbb{C} \setminus [-1, 1]$ 映射到 $|w| < 1$.

7.19 复杂的图形之间的映射

See Problems and Solutions in Mathematics (Jixiu Chen, et al.)

7.19.1 单位圆盘到去掉 $[0, 1)$ 的圆盘

Exercice 25

Find a one-to-one holomorphic map from the unit disk $\{|z| < 1\}$ onto slit disk $\{|w| < 1\} - \{[0, 1)\}$.

We construct the map by the following steps:

$$z_1 = \phi_1(z) = i \frac{z+1}{z-1} : \{z : |z| < 1\} \rightarrow \{z_1 : \operatorname{Im} z_1 < 0\}$$

$$z_2 = \phi_2(z_1) = \sqrt{z_1^2 - 1} + z_1 \quad \left(\sqrt{z_1^2 - 1} \Big|_{z_1=-i} = \sqrt{2}i \right) :$$

$$\{z_1 : \operatorname{Im} z_1 < 0\} \rightarrow \{z_2 : |z_2| < 1 \text{ and } \operatorname{Im} z_2 > 0\}$$

$$w = \phi_3(z_2) = z_2^2 : \{z_2 : |z_2| < 1 \text{ and } \operatorname{Im} z_2 > 0\} \rightarrow \{w : |w| < 1\} \setminus \{w : \operatorname{Im} w = 0, 0 \leq \operatorname{Re} w < 1\}.$$

Then $w = \phi_3 \circ \phi_2 \circ \phi_1(z) = f(z)$ is a one-to-one holomorphic map from the unit disk $\{|z| < 1\}$ onto the slit disk $\{|w| < 1\} \setminus \{[0, 1)\}$.

7.19.2 两区域之间的共形映射

Exercice 26: Illinois

- (1) Find a function f that conformally maps the region $\{z : |\arg z| < 1\}$ one-to-one onto the region $\{w : |w| < 1\}$. Show that the function you have found satisfies the required conditions.
- (2) Is it possible to require that $f(1) = 0$ and $f(2) = \frac{1}{2}$? If yes, give an explicit map; if No, explain why not.

证明. (a) $\zeta = f_1(z) = z^{\frac{\pi}{2}} = e^{\frac{\pi}{2} \log z} (\log 1 = 0)$ is a conformal map of $\{z : |\arg z| < 1\}$ onto $\{\zeta : \operatorname{Re} \zeta > 0\}$, and $w = f_2(\zeta) = \frac{\zeta - 1}{\zeta + 1}$ is a conformal map of $\{\zeta : \operatorname{Re} \zeta > 0\}$ onto $\{w : |w| < 1\}$.

Hence

$$w = f(z) = f_2 \circ f_1(z) = \frac{z^{\frac{\pi}{2}} - 1}{z^{\frac{\pi}{2}} + 1}$$

is a conformal map of $\{z : |\arg z| < 1\}$ onto $\{w : |w| < 1\}$ with $f(1) = 0$ and $f(2) = \frac{2^{\frac{\pi}{2}} - 1}{2^{\frac{\pi}{2}} + 1}$.

(b) Suppose $\tilde{w} = \tilde{f}(z)$ is an arbitrary conformal map of $\{z : |\arg z| < 1\}$ onto $\{\tilde{w} : |\tilde{w}| < 1\}$ with $\tilde{f}(1) = 0$.

Then $w = F(\tilde{w}) = f \circ \tilde{f}^{-1}(\tilde{w})$ is a conformal map of $\{\tilde{w} : |\tilde{w}| < 1\}$ onto $\{w : |w| < 1\}$ with $F(0) = 0$, and $\tilde{w} = \tilde{F}(w) = \tilde{f} \circ f^{-1}(w)$ is a conformal map of $\{w : |w| < 1\}$ onto $\{\tilde{w} : |\tilde{w}| < 1\}$ with $\tilde{F}(0) = 0$.

By Schwarz's lemma, we have both $|F(\tilde{w})| \leq |\tilde{w}|$ and $|\tilde{F}(w)| \leq |w|$, which implies that $|f(z)| = |\tilde{f}(z)|$ for every $z \in \{z : |\arg z| < 1\}$.

Since

$$f(2) = \frac{2^{\frac{\pi}{2}} - 1}{2^{\frac{\pi}{2}} + 1}$$

we cannot require that $\tilde{f}(2) = \frac{1}{2}$. □

7.19.3 共形映射类

Exercice 27: Toronto

- (1) Find one 1-1 onto conformal map f that sends the open quadrant $\{(x, y) : x > 0 \text{ and } y > 0\}$ onto the open lower half disc $\{(x, y) : x^2 + y^2 < 1 \text{ and } y < 0\}$.
- (2) Find all such f .

证明. (1) Let $\zeta = \phi_1(z) = z^2$. It is a conformal map of $\{z = x + iy : x > 0 \text{ and } y > 0\}$ onto $\{\zeta = \xi + i\eta : \eta > 0\}$. Let $w = \phi_2(\zeta) = \sqrt{\zeta^2 - 1} + \zeta$, where $\sqrt{\zeta^2 - 1} \Big|_{\zeta=i} = -\sqrt{2}i$. It is a conformal map of $\{\zeta = \xi + i\eta : \eta > 0\}$ onto $\{w = u + iv : u^2 + v^2 < 1 \text{ and } v < 0\}$. Then $w = \phi_2 \circ \phi_1(z) = \sqrt{z^4 - 1} + z^2$, where $\sqrt{z^4 - 1} \Big|_{z=e^{\frac{\pi}{4}i}} = -\sqrt{2}i$ is a required conformal map.

(2) If f is an arbitrary conformal map satisfying the condition of (1), then $\phi_2^{-1} \circ f \circ \phi_1^{-1}(\zeta)$ is a conformal map of the upper half plane onto itself,

which can be represented by $\psi(\zeta) = \frac{a\zeta+b}{c\zeta+d}$, where $a, b, c, d \in \mathbb{R}$, $ad - bc > 0$. Hence f can be written as $\phi_2 \circ \psi \circ \phi_1(z)$.

□

7.19.4 不等式估计 $|f'(0)|$ (利用 Schwarz 引理)

Exercice 28

If $f : D(0, 1) = \{z : |z| < 1\} \rightarrow \mathbb{C}$ is an analytic function which satisfies $f(0) = 0$, and if

$$|\operatorname{Re} f(z)| < 1 \text{ for all } z \in D(0, 1),$$

prove that

$$|f'(0)| \leq \frac{4}{\pi}.$$

(Indiana)

Solution.

It is easy to know that

$$w = g(\zeta) = \frac{e^{\frac{\pi i}{2}\zeta} - 1}{e^{\frac{\pi i}{2}\zeta} + 1}$$

is a conformal mapping of the domain $\{\zeta : |\operatorname{Re} \zeta| < 1\}$ onto the unit disk $\{w : |w| < 1\}$ with $g(0) = 0$. Hence $w = F(z) = g \circ f(z)$ is analytic in

$D(0, 1)$ and satisfies $F(0) = 0$ and $|F(z)| < 1$. By Schwarz's lemma, we have $|F'(0)| \leq 1$. Because

$$F'(z) = g'(f(z)) \cdot f'(z) = \frac{\pi i e^{\frac{\pi i}{2}f(z)} \cdot f'(z)}{(e^{\frac{\pi i}{2}f(z)} + 1)^2},$$

it follows from $f(0) = 0$ that

$$|f'(0)| \leq \frac{4}{\pi}.$$

7.19.5 给定约束条件下的函数极值估计

Exercice 29

Let $\Omega = \{z \in \mathbb{C}; -1 < \operatorname{Im} z < 1\}$, and let \mathcal{F} be the family of all analytic functions $f : \Omega \rightarrow \mathbb{C}$ such that $|f| < 1$ on Ω and $f(0) = 0$. Find

$$\sup_{f \in \mathcal{F}} |f(1)|.$$

(Indiana)

Solution.

It is obvious that

$$\zeta = f_0(z) = \frac{e^{\frac{\pi}{2}z} - 1}{e^{\frac{\pi}{2}z} + 1}$$

is a conformal mapping of Ω onto the unit disk with the origin fixed. For any analytic function $w = f(z) : \Omega \rightarrow \mathbb{C}$ such that $|f| < 1$ and $f(0) = 0$, we consider the composite function $w = F(\zeta) = f \circ f_0^{-1}(\zeta)$. $F(\zeta)$ is analytic in the unit disk such that $|F(\zeta)| < 1$ and $F(0) = 0$. By Schwarz's lemma,

$$|F(\zeta)| \leq |\zeta|.$$

Choose $\zeta_0 = \frac{e^{\frac{\pi}{2}} - 1}{e^{\frac{\pi}{2}} + 1}$, we have

$$|F(\zeta_0)| = |f(1)| \leq |\zeta_0| = \frac{e^{\frac{\pi}{2}} - 1}{e^{\frac{\pi}{2}} + 1}.$$

The equality holds if and only if $F(\zeta) = e^{i\theta}\zeta$, which implies

$$\sup_{f \in \mathcal{F}} |f(1)| = \frac{e^{\frac{\pi}{2}} - 1}{e^{\frac{\pi}{2}} + 1},$$

and the supremum is attained by $f(z) = e^{i\theta}f_0(z)$, where θ is a real number.

Exercice 30

Let f be an analytic function on $D = \{z; |z| < 1\}$ such that $f(0) = -1$, and suppose that $|1 + f(z)| < 1 + |f(z)|$ whenever $|z| < 1$. Prove that $|f'(0)| \leq 4$.
(Indiana)

Solution.

Let $\Omega = \mathbb{C} \setminus \{w = u + iv : u \geq 0 \text{ and } v = 0\}$. It follows from $|1 + f(z)| < 1 + |f(z)|$ that $f(D) \subset \Omega$.

Set $g(w) = \frac{\sqrt{w}-i}{\sqrt{w+i}}$, $(\sqrt{w} \Big|_{w=-1} = i)$. Then $g \circ f(z)$ is an analytic function on D with $g \circ f(0) = 0$ and $|g \circ f(z)| < 1$. By Schwarz's lemma,

$$|(g \circ f)'(0)| \leq 1.$$

Since

$$g'(w) = \frac{i}{\sqrt{w}(\sqrt{w}+i)^2},$$

we have $g'(-1) = -\frac{1}{4}$. From

$$(g \circ f)'(0) = g'(-1)f'(0),$$

we obtain

$$|f'(0)| \leq 4.$$

Exercice 31

Let P be the set of holomorphic function f on the open unit disc so that
 (i) Both the real and imaginary parts of $f(z)$ are positive for $|z| < 1$, (ii)
 $f(0) = 1 + i$. Let $E = \{f(\frac{1}{2}) : f \in P\}$. Describe E explicitly.

(Minnesota)

Solution.

Let $f \in P$ and define

$$\zeta = F(z) = \frac{f^2(z) - 2i}{f^2(z) + 2i}.$$

Then F is a holomorphic function on the unit disc with $F(0) = 0$ and $|F(z)| < 1$. By Schwarz's lemma, we have $|F(z)| \leq |z|$, which implies $|F(\frac{1}{2})| \leq \frac{1}{2}$. It should be noted that when f changes in P , $F(\frac{1}{2})$ can take any value in the disc $\{\zeta : |\zeta| \leq \frac{1}{2}\}$. Because $w = \frac{2i(1+\zeta)}{1-\zeta}$ (that is the inverse of $\zeta = \frac{w-2i}{w+2i}$) is a

conformal mapping of $\{\zeta : |\zeta| \leq \frac{1}{2}\}$ onto $\{w : |w - \frac{10}{3}i| \leq \frac{8}{3}\}$, we obtain that the set $\{f^2(\frac{1}{2}) : f \in P\}$ is equal to

$$\{w : |w - \frac{10}{3}i| \leq \frac{8}{3}\} = \{w = \rho e^{i\phi} : |\phi - \frac{\pi}{2}| \leq \arcsin \frac{4}{5}, \rho^2 - \frac{20}{3}\rho \sin \phi + 4 \leq 0\}.$$

Hence

$$E = \{f(\frac{1}{2}) : f \in P\} = \{re^{i\theta} : |\theta - \frac{\pi}{4}| \leq \frac{1}{2} \arcsin \frac{4}{5}, r^4 - \frac{20}{3}r^2 \sin 2\theta + 4 \leq 0\}.$$

If we denote the two roots of $\rho^2 - \frac{20}{3}\rho \sin \phi + 4 = 0$ by $\rho_1(\phi)$, $\rho_2(\phi)$ where $\rho_1(\phi) \leq \rho_2(\phi)$ and $|\phi - \frac{\pi}{2}| \leq \arcsin \frac{4}{5}$, the set E can also be represented by

$$\left\{ re^{i\theta} : |\theta - \frac{\pi}{4}| \leq \frac{1}{2} \arcsin \frac{4}{5}, \sqrt{\rho_1(2\theta)} \leq r \leq \sqrt{\rho_2(2\theta)} \right\}.$$

7.19.6 幂等共形映射存在唯一不动点**Exercice 32**

Let D be the upper-half and let $f \neq id$ be a conformal map of D onto itself such that $f \circ f = id$. Prove that f has a unique fixed point inside D .

(SUNY, Stony Brook)

Solution.

Since f is a conformal map of D onto itself, it can be written as $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. Then

$$f \circ f(z) = \frac{(a^2 + bc)z + b(a + d)}{c(a + d)z + d^2 + bc}.$$

It follows from $f \circ f = id$ that $b(a + d) = c(a + d) = 0$ and $a^2 + bc = d^2 + bc \neq 0$.

If $a + d \neq 0$, then $b = c = 0$. Hence $ad - bc > 0$ and $a^2 + bc = d^2 + bc$ implies $f = id$, which contradicts the condition $f \neq id$. Thus we have $a + d = 0$ and the inequality $ad - bc > 0$ can be written as $bc + a^2 < 0$.

Now we consider the equation $f(z) = \frac{az+b}{cz+d} = z$, which is equivalent to $cz^2 + (d - a)z - b = 0$. Since $\Delta = (d - a)^2 + 4bc$ is equal to $4bc + 4a^2 < 0$, we know that $f(z) = z$ has two conjugate roots, one in the upper-half plane and the other in the lower-half plane. So f has a unique fixed point inside D .

7.19.7 证明满足 $\operatorname{Re} f'(z) > 0$ 的 f 为单射

Exercice 33

Let Ω be a convex, open subset of \mathbb{C} and let $f : \Omega \rightarrow \mathbb{C}$ be an analytic function satisfying $\operatorname{Re} f'(z) > 0$, $z \in \Omega$. Prove that f is one-to-one in Ω (i.e., f is injective).

(Indiana)

Solution.

Let $z_1 \neq z_2$ be two arbitrary points in Ω . $L : z(t) = z_1 + t(z_2 - z_1)$, $t \in [0, 1]$ is the line segment connecting z_1 and z_2 . Since Ω is convex, $L \subset \Omega$, we have

$$f(z_2) - f(z_1) = \int_L f'(z) dz = \int_0^1 f'(z(t))(z_2 - z_1) dt.$$

Hence

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} = \int_0^1 f'(z(t)) dt.$$

Since $\operatorname{Re} f'(z) > 0$ for $z \in \Omega$, we know that $\int_0^1 f'(z(t)) dt \neq 0$, which implies $f(z_1) \neq f(z_2)$ whenever $z_1 \neq z_2$.

7.19.8 考虑多项式的根

Exercice 34

Show that if the polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, $n > 1$, is one-to-one in the unit disk $|z| < 1$ and $a_1 = 1$, then $|na_n| \leq 1$.

(SUNY, Stony Brook)

Solution.

It follows from the univalence of $P(z)$ in $\{|z| < 1\}$ that $P'(z) = na_n z^{n-1} + (n-1)a_{n-1} z^{n-2} + \cdots + 2a_2 z + a_1 \neq 0$ for all $z \in \{|z| < 1\}$. In other words, the roots of $P'(z)$ are all situated outside the open unit disk. Let z_1, z_2, \dots, z_{n-1} be the roots of $P'(z)$, then $|z_j| \geq 1$ for $j = 1, 2, \dots, n-1$. Because $P'(z)$ can also be written as $na_n(z - z_1)(z - z_2) \cdots (z - z_{n-1})$, by comparing the constant terms, we have

$$(-1)^{n-1}na_n \prod_{j=1}^{n-1} z_j = a_1.$$

Since $a_1 = 1$, we obtain

$$|na_n| = \frac{|a_1|}{\prod_{j=1}^{n-1} |z_j|} \leq 1.$$

7.20 解析延拓-刷题

Définition 3: 定义 8.2

若函数 $f(z)$ 在区域 D 内单值解析，则函数 $f(z)$ 与区域 D 一起称为解析函数元素，记作 $\{D, f(z)\}$.

Définition 4: 解析延拓的基础定义

设 $\{D, f(z)\}, \{G, F(z)\}$ 满足

- (1) $D \subset G, D \neq G$;
- (2) 当 $z \in D$ 时， $F(z) = f(z)$.

则称 $\{G, F(z)\}$ 是 $\{D, f(z)\}$ (向外) 的解析延拓.

(3) 相交区域的解析延拓原理.

Théorème 14: 定理 8.1

设 $\{D_1, f_1(z)\}, \{D_2, f_2(z)\}$ 为两个解析函数元素, 满足

- (1) 区域 D_1 与 D_2 有一公共区域 d_{12} ;
- (2) $f_1(z) = f_2(z) (z \in d_{12})$,

则 $\{D_1 + D_2, F(z)\}$ 也是一个解析函数元素, 其中

$$F(z) = \begin{cases} f_1(z), & z \in D_1 - d_{12}, \\ f_2(z), & z \in D_2 - d_{12}, \\ f_1(z) = f_2(z), & z \in d_{12}. \end{cases}$$

(4) 直接解析延拓.

Définition 5: 定义 8.3

如果

- (1) $D_1 \cap D_2 = d_{12}$ 为一区域;
- (2) $f_1(z) = f_2(z) (z \in d_{12})$,

则称两个解析函数元素 $\{D_1, f_1(z)\}$ 及 $\{D_2, f_2(z)\}$ 互为直接解析延拓.

Exemple 21. $\{\operatorname{Re} z > 0, e^z\}$ 与 $\{\operatorname{Im} z > 0, e^z\}$ 互为直接解析延拓. 因为它们在公共部分第一象限内等值.

Remarque 23. e^z 是一个完全解析函数.

Remarque 24. $\frac{1}{1-z}$ 是一个完全解析函数.

Définition 6: 定义 8.7

一个完全解析函数 $F(z)$ 是一个一般解析函数, 它包含其任一元素的所有解析延拓. $F(z)$ 的定义区域 G 称为它的存在区域. G 的边界称为 $F(z)$ 的自然边界. 自然边界点就是 $F(z)$ 的奇点. 一个完全解析函数的任意两个解析函数元素是互为解析延拓的.

一个完全解析函数 $F(z)$ 显然是不能再扩大的, 它可能是单值的(这时, 它的存在区域就是通常 z 平面上的区域), 也可能是多值的. 还可看出, 每一个解析函数元素必属于惟一的完全解析函数.

过去讲的孤立奇点(单值性或多值性的)就是函数自然边界上的孤立点.

7.20.1 证明互为直接解析延拓

例 8.2 设 $f_1(z) = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$ ($z \in D_1; |z-1| < 1$),
 $f_2(z) = \frac{1}{i} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-i}{i}\right)^n$ ($z \in D_2; |z-i| < 1$),

则易知:

(1) $\{D_1, f_1(z)\}$ 及 $\{D_2, f_2(z)\}$ 均是解析函数元素.

(2) 圆 D_1 及 D_2 的公共部分 d_{12} (图 8.2) 是一个区域.

(3) 根据等比级数求和公式, 当 $z \in D_1 \cap D_2$ 时,

$$f_1(z) = f_2(z) \left(= \frac{1}{z} \right),$$

因此, $\{D_1, f_1(z)\}$ 及 $\{D_2, f_2(z)\}$ 互为直接解析延拓.

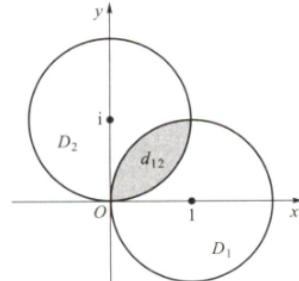


图 8.2

例 8.1.8 设

$$f_1(z) = -1 - z^2 - z^4 - \cdots - z^{2n} - \cdots \quad (D_1: |z| < 1),$$

$$f_2(z) = \frac{1}{z^2} + \frac{1}{z^4} + \frac{1}{z^6} + \cdots + \frac{1}{z^{2n}} + \cdots \quad (D_2: |z| > 1).$$

证明 $\{D_1, f_1(z)\}$ 与 $\{D_2, f_2(z)\}$ 互为(间接)解析延拓.

证 记 $g(z) = \frac{1}{z^2 - 1}$ ($z \neq \pm 1$), D 表示除 $z = \pm 1$ 外的 z 平面, 则 $g(z)$ 在 D 内解析. 因

$$f_1(z) = \frac{-1}{1 - z^2} = \frac{1}{z^2 - 1} = g(z) \quad (z \in D_1),$$

$$f_2(z) = \frac{\frac{1}{z^2}}{1 - \frac{1}{z^2}} = \frac{1}{z^2 - 1} = g(z) \quad (z \in D_2),$$

而 $D_1 \subset D$, $D_2 \subset D$, 所以完全解析函数 $\{D, g(z)\}$ 同是 $\{D_1, f_1(z)\}$ 及 $\{D_2, f_2(z)\}$ 的解析延拓. 于是 $\{D_1, f_1(z)\}$ 与 $\{D_2, f_2(z)\}$ 互为解析延拓. ■

7.20.2 构造解析延拓链

例 8.3 紧接例 8.2, 我们再设

$$f_3(z) = -\sum_{n=0}^{\infty} (z+1)^n \quad (z \in D_3; |z+1| < 1),$$

$$f_4(z) = -\frac{1}{i} \sum_{n=0}^{\infty} \left(\frac{z+i}{i}\right)^n \quad (z \in D_4; |z+i| < 1),$$

则易知 $\{D_2, f_2(z)\}$ 及 $\{D_3, f_3(z)\}$ 互为直接解析延拓;

$\{D_3, f_3(z)\}$ 及 $\{D_4, f_4(z)\}$ 互为直接解析延拓;

$\{D_4, f_4(z)\}$ 及 $\{D_1, f_1(z)\}$ 互为直接解析延拓.

于是, 解析函数元素集 $\{(D_1, f_1(z)), (D_2, f_2(z)), (D_3, f_3(z)), (D_4, f_4(z))\}$ 就能用一个单值解析函数元素 $\{D, F(z)\}$ 来代替. 其中 D 是区域 $D_1 + D_2 + D_3 + D_4$, 即以闭曲线 $ABCEFGHIA$ 及原点 $z=0$ 为边界的区域(如图 8.3); $F(z)$ 为

$$F(z) = \begin{cases} f_1(z), & z \in D_1, \\ f_2(z), & z \in D_2, \\ f_3(z), & z \in D_3, \\ f_4(z), & z \in D_4, \\ f_1(z) = f_2(z), & z \in d_{12}, \\ f_2(z) = f_3(z), & z \in d_{23}, \\ f_3(z) = f_4(z), & z \in d_{34}, \\ f_4(z) = f_1(z), & z \in d_{41}. \end{cases}$$

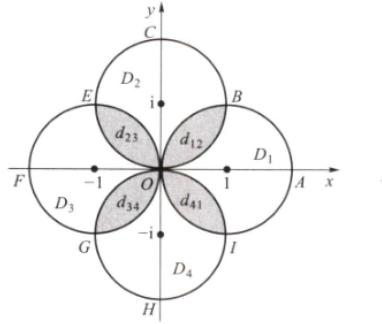


图 8.3

例 8.1.9 设

$$\begin{aligned} f_1(z) &= \ln|z| + i\arg z, \quad D_1 = \{z \mid 0 < \arg z < \pi\}; \\ f_4(z) &= \ln|z| + i\arg z, \quad D_4 = \left\{ z \mid \frac{3}{2}\pi < \arg z < \frac{5}{2}\pi \right\}. \end{aligned}$$

试证 $\{D_1, f_1(z)\}, \{D_4, f_4(z)\}$ 互为解析延拓.

证 显然 $D_1 \cap D_4 \neq \emptyset$, 但在 $D_1 \cap D_4$ 内,

$$f_1(z) = \ln|z| + i\arg z \quad (0 < \arg z < \pi),$$

$$f_4(z) = \ln|z| + i\arg z \quad \left(\frac{3}{2}\pi < \arg z < \frac{5}{2}\pi \right),$$

所以 $f_1(z) \neq f_4(z)$, 且 $\{D_1, f_1(z)\}, \{D_4, f_4(z)\}$ 不互为直接解析延拓. 然而, 作

$$f_2(z) = \ln|z| + i\arg z, \quad D_2 = \left\{ z \mid \frac{\pi}{2} < \arg z < \frac{3}{2}\pi \right\},$$

$$f_3(z) = \ln|z| + i\arg z, \quad D_3 = \{z \mid \pi < \arg z < 2\pi\}.$$

这时, $\{D_j, f_j(z)\} (j=1, 2, 3, 4)$ 构成解析延拓链, 所以 $\{D_1, f_1(z)\}, \{D_4, f_4(z)\}$ 互为(间接)解析延拓. ■

7.20.3 证明级数可延拓到整个复平面

例 8.1.10 证明: 函数 $f(z) = \sum_{n=0}^{\infty} \left(\frac{2z-1}{z-2} \right)^n$ 在单位圆 $|z| < 1$ 内解析, 并且可解析延拓到除点 $z = -1$ 外的整个 z 平面.

证 和函数 $f(z)$ 的解析性区域为 $\left| \frac{2z-1}{z-2} \right| < 1$, 也就是单位圆 $|z| < 1$. 又

$$f(z) = \frac{1}{1 - \frac{2z-1}{z-2}} = \frac{2-z}{1+z} \quad (|z| < 1),$$

因函数 $F(z) = \frac{2-z}{1+z}$ 除 $z = -1$ 外, 在 z 平面上处处解析, 故由定义 8.1, $f(z)$ 可以解析延拓到除 $z = -1$ 外的整个 z 平面. ■

7.20.4 自然边界

例 8.1.11 设 θ 是一个无理数, $a = e^{i\theta\pi}$, 证明: 单位圆周是 $f(z) = \sum_{n=0}^{\infty} a^{n^2} z^n$ 的一个自然边界.

分析 由

$$\begin{aligned} f(z) &= 1 + \sum_{n=1}^{\infty} a^{n^2} z^n = 1 + az \sum_{n=1}^{\infty} a^{n^2-1} z^{n-1} \\ &= 1 + az \sum_{n-1=0}^{\infty} a^{(n-1)(n+1)} z^{n-1} \\ &= 1 + az \sum_{n=0}^{\infty} a^{n^2} (a^2 z)^n = 1 + az f(a^2 z) \end{aligned}$$

知函数 $f(z)$ 满足方程

$$f(z) - 1 = az f(a^2 z). \quad (1)$$

证 由柯西-阿达玛公式, 级数 $\sum_{n=0}^{\infty} a^{n^2} z^n$ 的收敛半径 $R=1$, 故在单位圆 $|z|<1$ 内 $f(z)$ 表示一个解析函数. 以下证明, $f(z)$ 的奇点稠密于单位圆周 $|z|=1$ 上.

首先, 因为 $f(z)$ 至少有一个奇点 z_0 在单位圆周 $|z|=1$ 上, 由(1)式知 $a^2 z_0$ 也是一个奇点; 重复这个过程, 对 $m=0, 1, 2, \dots$, $a^{2m} z_0$ 都是奇点, 即 $e^{i(m\theta)\pi} z_0$ 都是奇点. 因为 θ 是无理数, 这些奇点在单位圆周上是处处稠密的, 所以单位圆周是 $f(z)$ 的一个自然边界. ■

例 8.4 试证在单位圆 $|z|<1$ 内的解析函数

$$f(z) = \sum_{n=1}^{\infty} z^{2^n}$$

不能延拓到单位圆周 $|z|=1$ 的外部.

证 级数的收敛半径 $R=1$, 故在单位圆 $|z|<1$ 内 $f(z)$ 表示一解析函数. 以下证明, $f(z)$ 的奇点稠密于单位圆周 $|z|=1$ 上.

首先证明: 当 z 沿半径而趋于 1 时, $f(z) \rightarrow \infty$. 令

$z=x$, $0 < x < 1$,

则

$$f(x) = x^2 + x^4 + \cdots + x^{2^n} + \cdots > x^2 + x^4 + \cdots + x^{2^n},$$

故

$$\lim_{x \rightarrow 1} f(x) \geq \lim_{x \rightarrow 1} (x^2 + x^{2^2} + \cdots + x^{2^n}) = n,$$

由 n 的任意性, 故

$$\lim_{x \rightarrow 1} f(x) = +\infty,$$

显然, $z=1$ 是 $f(z)$ 的一个奇点.

其次, 由于

$$\begin{aligned} f(z) &= z^2 + z^4 + \cdots + z^{2^n} + (z^{2^{n+1}} + z^{2^{n+2}} + \cdots) \\ &= z^2 + z^4 + \cdots + z^{2^n} + f(z^{2^n}), \end{aligned}$$

故能使

$$z^{2^n} = 1$$

的点 z 均为 $f(z)$ 的奇点, 即单位圆周上的点

$$z_k = e^{\frac{2k\pi i}{2^n}} \quad (k=0, 1, \dots, 2^n - 1; n=1, 2, \dots)$$

均为 $f(z)$ 的奇点, 这种点稠密于圆周 $|z|=1$ 上. 故 $f(z)$ 不能再向单位圆周 $|z|=1$ 外延拓.

7.20.5 黎曼面的说明

下面我们举几个简单的例子来讨论黎曼面. 一般对于多值解析函数, 要用若干叶片来适当地“粘合”成一个黎曼面的工作是相当复杂的, 并且需要相当的技巧.

例 8.11 函数 $w=z^{\frac{1}{2}}$ 的黎曼面.

解 由于对 z 平面上每一个异于零的点, 此函数有两个值和它相对应; 如对不同函数值的相同的点 z 能加以区别, 就能满足我们的要求.

令 $z=re^{\theta}$, 于是相同的点 z 可由不同的 θ 来决定, 从而不同的函数值可用不同的 θ 来规定. 因当 z 绕原点一周, w 的值由 \sqrt{z} 变为 $-\sqrt{z}$; 当 z 再绕原点一周, $-\sqrt{z}$ 又变为 \sqrt{z} . 所以如 \sqrt{z} 相当于 $0 < \theta < 2\pi$, 则 $-\sqrt{z}$ 相当于 $2\pi < \theta < 4\pi$. 现在设想两个 z 平面相重叠, 原点的位置与实轴的方向都相同. 在上的平面用 M_0 表示, 相当于 $0 < \theta < 2\pi$; 在下的平面用 M_1 表示, 相当于 $2\pi < \theta < 4\pi$. 由于 $z=0$ 及 $z=\infty$ 是 $w=z^{\frac{1}{2}}$ 的两个支点, 我们现在可以选正实轴为支割线, 将两个平面各沿正实轴割开, 使 z 分别在 M_0 及 M_1 上不能越过支割线在同一平面上变动. 再沿支割线使 M_0 的下岸 ($\theta=2\pi$) 与 M_1 的上岸 ($\theta=2\pi$) 粘合, 并使 M_1 的下岸 ($\theta=4\pi$) 与 M_0 的上岸 ($\theta=0$) 粘合. 这样的模型就是 $w=z^{\frac{1}{2}}$ 的黎曼面(如图 8.13(a)), 两叶在截口处互相交叉, 其在支割线处的垂直纵截面如图 8.13(b).

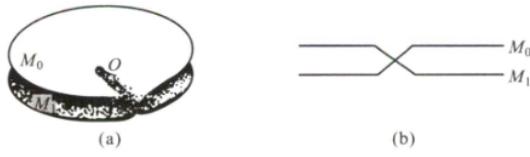


图 8.13

当点 z 在黎曼面的 M_0 叶上, 从正实轴上一点出发, 围绕支点 $z=0$ 以逆时针方向连续变动两圈时, 辐角 θ 先由 0 增至 2π , 并由叶片 M_0 进入叶片 M_1 , 在后一叶片 M_1 中, θ 再由 2π 增至 4π . 当点 z 再继续转动时, 它仍回到叶片 M_0 上(如图 8.14). 在 M_0 上, θ 的值可以认为是由 4π 增到 6π , 或由 0 增到 2π , 对于函数值没有影响. 余可类推. 这样一来, 函数 $w=z^{\frac{1}{2}}$ 就是黎曼面上的单值函数了. 只是在支割线处, 两个叶片的点需要设法去判别.

应该注意的是: 要想由以上的描述, 把黎曼面按通常办法具体地制作出来是办不到的. 因为当先把叶片 M_0 的截口的下岸与 M_1 截口的上岸粘合以后, 再把 M_1 的下岸与 M_0 的上岸粘合就无法进行, 因为中间已有一个粘合好的平面存在着. 尽管如此, 我们仍然要想象它们是粘合起来了.

这黎曼面的叶片 M_0 的像是上半 w 平面, 而叶片 M_1 的像是下半 w 平面. 因为

$$w_k = \sqrt{r} e^{i\frac{\theta}{2}} = \sqrt{r} e^{i\frac{\theta_0 + 2k\pi}{2}}, \quad 0 \leq \theta_0 < 2\pi, k=0,1,$$

在 M_0 上

$$0 < \frac{\theta}{2} < \pi,$$

在 M_1 上

$$\pi < \frac{\theta}{2} < 2\pi.$$

每个叶片上所确定的函数都是单值解析的, 并可连续到粘合的边界, 在粘合的边界上它们是等值的. 于是, 它们的一个就是另一支穿过支割线的解析延拓(由本章的潘勒韦连续延拓原理可见). 因此, 黎曼面上点的单值函数 $w=z^{\frac{1}{2}}$ 在除去支点 $z=0$ 及 $z=\infty$ 外, 到处都是解析的.

例 8.12 函数 $w=\sqrt[n]{z}$ (n 是正整数, $n>2$) 的黎曼面.

类似上例的讨论, 它的黎曼面是由 n 个沿着支割线正实轴割开的 z 平面粘合而成的, 仍只以 $z=0$ 及 $z=\infty$ 为支点.

令 $z=re^{i\vartheta}$, 则

$$w_k = \sqrt[n]{r} e^{i\frac{\theta_0 + 2k\pi}{n}}, \quad 0 \leq \theta_0 < 2\pi, \quad k=0,1,\dots,n-1,$$

图 8.15 是 $n=4$ 的情形.

例 8.13 函数 $w=\ln z$ 的黎曼面.

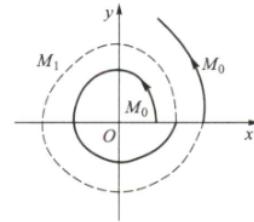


图 8.14

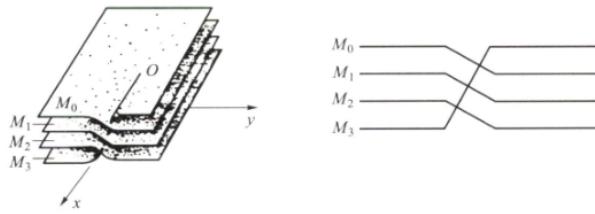


图 8.15

解 令 $z=re^{i\theta}$, 则

$$w_k = (\ln z)_k = \ln r + i(\theta_0 + 2k\pi), \quad 0 \leq \theta_0 < 2\pi, \quad k=0, \pm 1, \pm 2, \dots$$

仍只以 $z=0$ 及 $z=\infty$ 为支点, 取正实轴为支割线, 其黎曼面含有无穷多叶(图 8.16(a)). 其在支割线处的垂直纵截面如图 8.16(b).

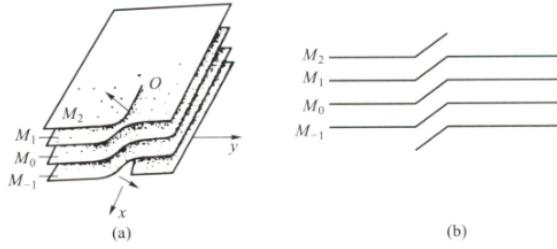


图 8.16

上面三个例题是应用透弧解析延拓制作 $w=\sqrt[n]{z}$ 及 $w=\ln z$ 的黎曼面, 当然也可应用相交区域的直接解析延拓制作它们的黎曼面.

$\sqrt[3]{(z-a)(z-b)}$ 的黎曼面

* 例 8.14 函数 $\sqrt[3]{(z-a)(z-b)}$ ($a \neq b$) 的黎曼面.

解 函数 $\sqrt[3]{(z-a)(z-b)}$ 以 $z=a, z=b$ 及 $z=\infty$ 为支点. 从 a 到 b 连接直线段, 从 b 到 ∞ 连接任一射线(不通过 a)作为支割线. 这样割开 z 平面后, 可以分出三个单值分支

$$f_k(z) = \omega^k f_0(z), \quad k=0, 1, 2,$$

其中 $\omega = e^{\frac{2\pi i}{3}}$, $f_0(z)$ 为 $\sqrt[3]{(z-a)(z-b)}$ 的一个确定支.

现将三张平面 M_k ($k=0, 1, 2$) 均按前述方法割破, 各分支 $f_k(z)$ 分别在 M_k 上是单值的, 但在割线两岸之值不同. 在 \vec{ab} 左岸(顺着 a 到 b 的方向)取值为 $f_0(z), f_1(z), f_2(z)$ 时连续变动到右岸, 由于绕 a 转了一周的缘故, 增加了一个因子 $\omega = e^{\frac{2\pi i}{3}}$, 就对应取值为 $f_1(z), f_2(z), f_0(z)$; 在 $\vec{b\infty}$ 左岸取值为 $f_0(z), f_1(z), f_2(z)$ 时连续变动到右岸, 由于绕 a, b 两点同时转了一周的缘故, 增加了一个因子 $\omega^2 = e^{\frac{4\pi i}{3}}$, 就对应取值为 $f_2(z), f_0(z), f_1(z)$. 即前者各分支构成以下置换:

$$\begin{bmatrix} f_0(z) & f_1(z) & f_2(z) \\ f_1(z) & f_2(z) & f_0(z) \end{bmatrix},$$

后者各分支构成以下置换:

$$\begin{bmatrix} f_0(z) & f_1(z) & f_2(z) \\ f_2(z) & f_0(z) & f_1(z) \end{bmatrix}.$$

将三个叶片叠起来,并粘合那些取值相同的各岸,即得所求的黎曼面.图 8.17 表示在割线处的粘合法.

黎曼面的最大优点是不仅使 w 平面上的点和 z 平面上的点建立一一对应,而且可使 z 平面上的连续曲线和 w 平面上的连续曲线成对应.利用黎曼面,所有关于单值函数的性质都可以推广到多值函数.

设 $f(z)$ 是定义于黎曼面上的函数,如 $z=a$ 是 $f(z)$ 的解析点,则在 a 的一个邻域内, $f(z)$ 可以展开为泰勒级数.其收敛圆可能有一部分在黎曼面的某一叶上,而另一部分则在与之粘合的另一叶上,但支点不能在收敛圆之内.

最后,我们还必须指出:一个多值解析函数与代表多个函数的一个表达式是有区别的.例如,函数 $w=\sqrt{z^2}$ 相当于 $w=+z$ 和 $w=-z$,而这两个单值解析函数不是互为解析延拓的.所以 $w=\sqrt{z^2}$ 仅是表示两个单值解析函数 $w=+z$ 与 $w=-z$ 的一个表达式.

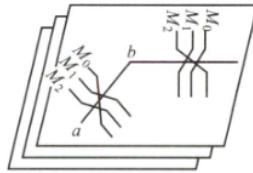


图 8.17

7.20.6 多角形区域共形映射

1. 了解克里斯托费尔(Christoffel)-施瓦茨公式及其几何意义.

定理 8.7 设(1) P_n 为有界 n 角形, 其顶点为 A_1, A_2, \dots, A_n , 其顶角为 $\alpha_1\pi, \alpha_2\pi, \dots, \alpha_n\pi$ ($0 < \alpha_j < 2, j = 1, 2, \dots, n$);

(2) 函数 $w = f(z)$ 将上半平面 $\operatorname{Im} z > 0$ 共形映射成 P_n ;

(3) z 平面实轴上对应于 w 平面多角形 P_n 的顶点 A_j 的那些点 a_j :

$$-\infty < a_1 < a_2 < \dots < a_j < \dots < a_n < +\infty$$

都是已知的, 则

$$f(z) = C \int_{z_0}^z (z - a_1)^{\alpha_1-1} (z - a_2)^{\alpha_2-1} \cdots (z - a_n)^{\alpha_n-1} dz + C_1, \quad (8.9)$$

其中 z_0, C 与 C_1 是三个复常数.

注 (1) 显然有

$$\sum_{j=1}^n \alpha_j = n - 2. \quad (8.10)$$

(2) 变换(8.9)的逆变换 $z = f^{-1}(w)$ 将 w 平面上的单连通区域——多角形 P_n ——共形映射成标准区域上半 z 平面.

2. 定理 8.7 的下列两个退化情形是很有用的:

(1) 若 n 角形 P_n 有一个顶点是 z 平面上无穷远点的像, 则在公式(8.9)中就丢掉那个关于这个顶点的因子.

在实际应用上, 可以利用这个事实来简化克里斯托费尔-施瓦茨积分.

(2) 对于一个或几个顶点在无穷远处的那些多角形(称为广义多角形)来说, 克里斯托费尔-施瓦茨公式仍然有效, 只需把顶点在无穷远处的那两条直线间的角度, 用这两条直线在有限点处的那个交角反号代替.

Remarque 25. 由此可以看出, 广义多角形实际上代表了扩充 w 平面上许多特殊形状的单连通区域(边界不止一点). 而变换(8.9)的逆变换 $z = f^{-1}(w)$ 就能把这许多特殊形状的单连通区域共形映射成(即“简化成”)标准区域——上半 z 平面.

广义多角形示例

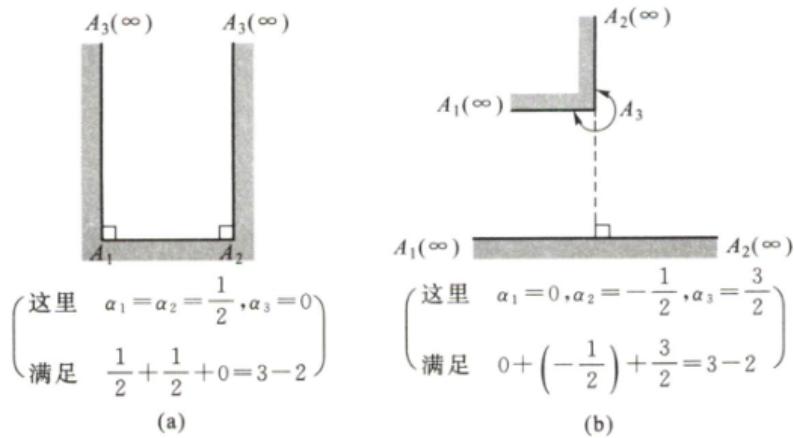
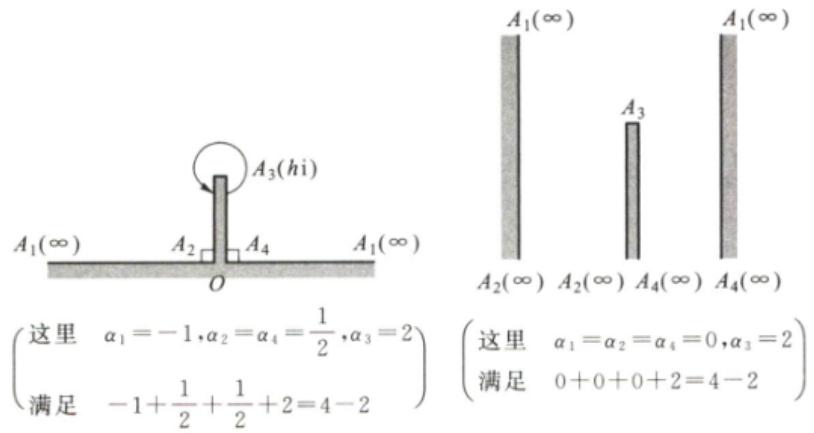


图 8.22



(a)

(b)



$$\left\{ \begin{array}{l} \text{这里 } a_1 = 0, a_2 = a_4 = \frac{3}{2}, a_3 = -1 \\ \text{满足 } 0 + \frac{3}{2} + \frac{3}{2} + (-1) = 4 - 2 \end{array} \right.$$

445 (c)

图 8.23

对于 (a) 我们有

A_j	α_j	a_j
$-\frac{\pi}{2}$	$\frac{1}{2}$	-1
$\frac{\pi}{2}$	$\frac{1}{2}$	1
∞	0	∞

于是根据 Schwarz-Christoffel 公式

$$\begin{aligned} w &= C \int_0^z (z+1)^{-\frac{1}{2}} (z-1)^{-\frac{1}{2}} dz + C_1 \\ &= C' \int_0^z \frac{1}{\sqrt{1-z^2}} dz + C_1 \\ &= C' \arcsin z + C_1 \end{aligned}$$

利用点 a_1, a_2 与 A_1, A_2 的对应关系, 我们得到

$$-\frac{\pi}{2} = -C' \cdot \frac{\pi}{2} + C_1, \quad \frac{\pi}{2} = C' \cdot \frac{\pi}{2} + C_1$$

因此 $C_1 = 0, C' = 1$. 故将上半 z 平面 $\operatorname{Im} z > 0$, 映射到半带形区域 $-\frac{\pi}{2} < \operatorname{Re} w < \frac{\pi}{2}, \operatorname{Im} w > 0$ 的函数为

$$w = \arcsin z$$

取 $z = 1$ 时, $w = \arcsin z = \frac{\pi}{2}$ 的那一支.

例 8.16 割去线段 $0 < v < h, u = 0$ ($w = u + iv$) 的上半平面 $v > 0$ 是广义四角形 (如图 8.23(a)), 它的已知量以及与顶点对应的点列于表 8.2 中(点 $a_j, j = 1, 2, 3, 4$ 中的三个是任意给定的, 第四个暂时用 ξ 来表示).

A_j	α_j	a_j
∞	-1	∞
0	$\frac{1}{2}$	-1
hi	2	0
0	$\frac{1}{2}$	ξ

在无穷远点的顶角等于 $-\pi$, 因为 $A_2 A_1$ 与 $A_4 A_1$ 一个是另一个的延长线, 即它们之间的角等于 π ; 而

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -1 + \frac{1}{2} + 2 + \frac{1}{2} = 4 - 2 = 2.$$

利用对称原理可以决定点 $a_4 = \xi$. 将由 z 平面的第二象限到 w 平面的第二象限, 且具有点的对应关系 $\infty \leftrightarrow \infty, h i \leftrightarrow 0$ 的共形映射解析延拓越过 y 轴的结果, 就得到所求的共形映射. 所以 a_4 应当关于虚轴与点 a_2 对称, 即 $a_4 = \xi = 1$.

克里斯托费尔-施瓦茨积分取形式

$$\begin{aligned} w &= C \int (z+1)^{-\frac{1}{2}} z(z-1)^{-\frac{1}{2}} dz \\ &= C \int \frac{z dz}{\sqrt{z^2-1}} = C' \sqrt{z^2-1} + C_1 \end{aligned}$$

(我们取不定积分, 而未取定积分, 是因为常数 C_1 是任意的). 为了确定常数 C' 与 C_1 , 我们利用点 a_2, A_2 与 a_3, A_3 的对应关系:

$$0 = 0 + C_1, \quad ih = iC' + C_1,$$

由此, $C_1 = 0, C' = h$, 因而所求的函数为

$$w = h \sqrt{z^2-1}$$

(取 $z=0$ 时, $\sqrt{z^2-1} = i$ 的那一支).

注 从上面关于广义多角形的举例可以看出, 广义多角形实际上代表了扩充 w 平面上许多特殊形状的单连通区域(边界不止一点), 而变换(8.9)的逆变换 $z = f^{-1}(w)$ 就能把这许多特殊形状的单连通区域共形映射成(即“简化成”)标准区域——上半 z 平面.

7.21 复变函数期末练习题

7.21.1 习题 1

1. 判断 $\sqrt{-2}\sqrt{-3} = \sqrt{6}$ 是否正确?

2. 构造一个共形映射将 $\mathbb{C} \setminus [-1, 1]$ 映射到单位圆的外部.

3. 计算

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}.$$

4. 设 $f(z)$ 为整函数且 $e^{f(z)}$ 有界. 证明: $f(z)$ 为常数.

5. 设 $f(z) = \sum_{n=0}^{\infty} a_n z^n$ 在 $|z| < 1$ 解析且满足 $|f(z)| < \frac{1}{1-|z|}$. 证明:

$$|a_n| \leq (n+1) \left(1 + \frac{1}{n}\right)^n < e(n+1).$$

6. 设 a_1, a_2, \dots, a_m 为非零复数, 定义

$$f(z) = \sum_{k=1}^m \frac{1}{1 - a_k z}.$$

(1) 求 $f(z)$ 在原点处幂级数展开式的收敛半径;

(2) 证明:

$$\overline{\lim}_{n \rightarrow \infty} \left| \sum_{k=1}^m a_k^n \right|^{1/n} = \max_{k=1, \dots, m} |a_k|.$$

7. 用 Rouché 定理来证明代数基本定理.

8. 求积分

$$\int_{|z|=\frac{1}{2}} \frac{dz}{z \sin^2(z)}.$$

9. 设 $f(z)$ 在 $\Delta = \{|z| < 1\}$ 内解析.

(1) 若 z_1, z_2, \dots, z_n 是 $f(z)$ 的零点. 证明:

$$|f(z)| \leq \prod_{k=1}^n |z_k|.$$

(2) 证明: 存在 $z_n \in \Delta$, $|z_n| \rightarrow 1$ 使得 $\{f(z_n)\}$ 是一个有界序列.

(1) 错, $\sqrt{-2} \cdot \sqrt{-3} = \sqrt{2} \cdot \sqrt{-1} \cdot \sqrt{3} \cdot \sqrt{-1} = -\sqrt{6}$.

(2) $w = \sqrt{z^2 - 1} + z$, 其中 $\sqrt{z^2 - 1}|_{z=i} = \sqrt{2}i$.

(3) $f(z) := (z + \frac{1}{z})^{2n} \frac{1}{z} = \dots + \binom{2n}{n} z^{-1} + \dots, \int_{|z|=1} f(z) dz = \text{res}_0 f = 2\pi i \binom{2n}{n}.$

(4) $e^{f(z)}$ 也是整函数 (整函数 e^z 复合整函数 f) 于是 $e^{f(z)} = C \Rightarrow f(z) = \log C.$

(5) $|a_n| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{1}{r^n(1-r)}, r$ 待定, 故可以取为 $\frac{n}{n+1}$, 则 $|a_n| \leq (n+1) \left(1 + \frac{1}{n}\right)^n < e(n+1).$

(6) $f(z) = \sum_{n=0}^{\infty} (\sum_{k=1}^m a_k^n) z^n$, 故 $R = \overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|\sum_{k=1}^m a_k^n|}$. 同时 $R = \min_k \left\{ \left| \frac{1}{|a_k|} \right| \right\}$ ², 故 $\overline{\lim}_{n \rightarrow \infty} |\sum_{k=1}^m a_k^n|^{1/n} = \max_k |a_k|.$

(7) 归纳即可

(8) $f(z) := \frac{1}{z \sin^2 z}$ 在 $|z| < \frac{1}{2}$ 内只有三阶极点 0, 洛朗展开得到 $f(z) = \frac{1}{z(z - \frac{z^3}{3} + o(z^4))^2} = \frac{1}{z^3} \cdot \frac{1}{(1 - \frac{z^2}{3} + o(z^3))^2} = z^{-3} \cdot \frac{1}{1 - \frac{2}{3}z^2 + o(z^3)} = z^{-3} + \frac{2}{3}z^{-1} + \dots$. 于是 $\text{res}_0 f = a_{-1} = \frac{2}{3}$. 故 $\int_{|z|<\frac{1}{2}} f(z) dz = 2\pi i \text{res}_0 f = \frac{4}{3}\pi i.$

(9) 题目有误: 应该补充条件 $|f(z)| < 1$, 且不等式改为 $|f(0)| \leq \prod_{k=1}^n |z_k|$.

(I) 构造函数 $g(z) = \frac{f(z)}{\prod_{k=1}^n \varphi_{z_k}(z)}$, 其中 $\varphi_{z_k}(z) := \frac{z-z_k}{1-\bar{z}_k z}$. 于是 $|z| \rightarrow 1$ 时, $|\varphi_{z_k}(z)| \rightarrow 1$, 故 $|g(z)| \rightarrow \lim_{|z| \rightarrow 1} |f(z)| \leq 1$. 由最大模定理³ $|g(z)| \leq 1, \forall z \in \Delta$. 注意到 $|\varphi_{z_k}(0)| = |z_k|$, 我们就有 $|f(0)| \leq \prod_{k=1}^n |\varphi_{z_k}(0)| = \prod_{k=1}^n |z_k|$.

(II) 反证而设, 对于任意 $\{|z_k|\} \rightarrow 1, |f(z_k)| \rightarrow \infty$. 接下来考虑 f 在 Δ 内的零点个数, 若为无穷个, 这些零点存在聚点, 则由非平凡解析函数的零点孤立性可知, 聚点都在 $\partial\Delta$ 上, 从而存在零点列 $\{|z_k|\} \rightarrow 1$, 其中 $|f(z_k)| = 0$, 故 $|f(z_k)| \not\rightarrow \infty$. 故零点个数有限, 按 (1) 中定义 g , 则 $\frac{1}{g} \rightarrow 0$, 在 $|z| \rightarrow 1$ 时. $g \in H(\Delta)$ 无零点, 故 $\frac{1}{g} \in H(\Delta)$, 故由最大模定理: $\frac{1}{g} \equiv 0$. g 必须是无穷大, 但这与 $g \in H(\Delta)$ 矛盾!

²一个函数在某点展开的幂级数, 其收敛半径是从该点到离它最近的奇点的距离。

³最大模定理可以推广到趋于边界上, 不需要在边界有定义, 详见 Rudin p.253

7.21.2 习题 2

1. 证明 $w = -\frac{1}{z}$ 是 z 在黎曼球面上的对径点.
2. 构造一个共形映射将区域 $\{z = x + iy : -1 < y < 1\} \setminus (-\infty, 0]$ 映射到上半平面.
3. 设 $f(z) = u(z) + iv(z)$ 在 $|z| \leq 1$ 解析且 $f(0) = 0$. 证明:

$$\iint_{|z| \leq 1} u^2 dx dy = \iint_{|z| \leq 1} v^2 dx dy.$$

4. 设 $f(z), g(z)$ 为整函数且当 $|z| \geq 2024^{2025}$ 时 $|f(z)| \leq |g(z)|$. 证明: $\frac{f(z)}{g(z)}$ 是一个有理函数.

5. 设 $f(z)$ 为整函数且当 $z \in \mathbb{R}$ 时, $|f(z)| = 1$. 证明: 存在整函数 $g(z)$ 使得

$$f(z) = e^{g(z)}.$$

6. 求积分

$$\int_0^\infty \frac{\cos x}{1+x^4} dx.$$

7. 设 $f(z)$ 在 $|z| < 1$ 内解析, 且 $f(z)$ 限制在 $\frac{1}{2} < |z| < 1$ 内为单射. 证明 $f(z)$ 在 $|z| < 1$ 内为单射.

8. 设 $f(z)$ 在 $|z| < 1$ 内解析且 $\operatorname{Re} f(z) > 0$. 证明:

$$|f'(0)| \leq 2 \operatorname{Re} f(0).$$

(1) 映射 $\mathbb{C} \rightarrow S^2$, $z = x + iy \mapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right)$. 而 $-\frac{1}{z} = \frac{-x-iy}{x^2+y^2} \mapsto \left(\frac{-2x}{x^2+y^2+1}, \frac{-2y}{x^2+y^2+1}, \frac{-(x^2+y^2-1)}{x^2+y^2+1} \right)$, 于是是 z 的对径点.

(2) $w = \sqrt{1 - e^{\pi z}}$.

(3) $\operatorname{Re} f^2(z) = u^2 - v^2$ 是调和函数, 由平均值公式 $\int_{|z| < 1} u^2 - v^2 dz = \pi \cdot 1^2 \cdot (u^2(0) - v^2(0)) = 0$.

(4) 记 $R = 2024^{2025}$, 在紧集 $|z| \leq R$ 内, 由零点孤立性, g 至多有有限个零点, $\frac{f}{g}$ 有有限个极点. 在 $|z| \geq R$ 上, $|f(z)| \leq |g(z)| \Rightarrow g$ 的零点都是 f 的零点, $\left| \frac{f}{g} \right| \leq 1$ 有界, 由黎曼可去奇点定理, $\frac{f}{g}$ 在该区域只有可去奇点. 且 $\frac{f}{g}$ 在无穷远点的邻域内有界, 则 ∞ 不是本性奇点, 只可能是可去奇点或极点. 因为任何在扩充复平面上亚纯的函数必然是有理函数, 所以 $\frac{f}{g}$ 是有理函数.

(5) 还不确定, 可以取 $g(z) = \int_{\gamma_0 \rightarrow z} \frac{f'(\zeta)}{f(\zeta)} d\zeta + i \arg f(0)$, 于是 $g' = \frac{f'}{f}$, 故验证这是整函数.

(6) $I = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos x}{1+x^4} dx$, 设 $f(z) = \frac{e^{iz}}{1+z^4}$. 考虑在上半平面圆周上的积分, $\left| \int_{\gamma_R^+} f(z) dz \right| \rightarrow 0$. 计算留数 $\operatorname{res}_{e^{i\pi/4}} f = \frac{1}{4} e^{-1/\sqrt{2}} e^{i(1/\sqrt{2}-3\pi/4)}$, $\operatorname{res}_{e^{3i\pi/4}} f = \frac{1}{4} e^{-1/\sqrt{2}} e^{i(-1/\sqrt{2}-\pi/4)}$. 于是 $I = \frac{1}{2} \int_{-\infty}^\infty f(z) dz = \pi i (\operatorname{res}_{e^{i\pi/4}} f + \operatorname{res}_{e^{3i\pi/4}} f) =$

$$\frac{\pi}{2}e^{-1/\sqrt{2}} \sin\left(\frac{1}{\sqrt{2}} + \frac{\pi}{4}\right).$$

- (7) 对于任意 $r \in (\frac{1}{2}, 1)$, 考虑 D_r 内 $f(z) = w_0$ 的解, w_0 是任意复数, 由于 f 在 C_r 上单射, 故 $f(C_r)$ 为简单闭曲线, 由幅角原理, $N(w_0) = \frac{1}{2\pi i} \oint_{C_r} \frac{f'(z)}{f(z)-w_0} dz$ 为 $f(C_r)$ 在 w_0 处的绕数, 根据若当曲线定理, 由于 $f(C_r)$ 是简单闭曲线, 那么对于任何不在 $f(C_r)$ 上的 w_0 , 绕数为 0 或 1. 于是对于 $f(C_r)$ 内部的 w_0 , $f(z) - w_0 = 0$ 只有一个解.
- (8) 定义 $g : \mathbb{D} \rightarrow \mathbb{D}$, $z \mapsto \frac{f(z)-f(0)}{f(z)+f(0)}$. 于是, $g(0) = 0$, 由 Schwarz 引理, $|g'(0)| \leq 1$, 即 $|f'(0)| \leq 2\operatorname{Re} f(0)$.

7.21.3 习题 3

1. 画出如下集合的图像:

$$\{z : \arg \frac{z+i}{z-i} < \frac{\pi}{4}\}.$$

2. 构造一个共形映射将区域 $\{|z| > 1\} \setminus (-\infty, -1)$ 映射成单位圆盘.

3. 设 $|a| < 1, |b| < 1$.

(1) 说明 $\sqrt{(z-a)(z-b)}$ 在 $|z| \geq 1$ 可取到单值解析分支.

(2) 取定 $\sqrt{(z-a)(z-b)}$ 在 $|z| \geq 1$ 的一个单值解析分支. 求积分

$$\int_{|z|=1} \frac{z}{\sqrt{(z-a)(z-b)}} dz.$$

4. 是否存在定义在 $|z| < 1$ 内的非常值解析函数 $f(z)$, 使得当 $|z| \rightarrow 1$ 时, $|f(z)| \rightarrow \infty$? 判断并给出证明.

5. 求积分

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx.$$

6. 设 $f(z)$ 在 $|z| < 1$ 内解析, $|f(z)| < 1$ 且 $f(0) \neq 0$. 证明: $f(z)$ 在圆盘 $\{|z| < |f(0)|\}$ 内部不存在零点.

7. 令 $\Delta = \{|z| < 1\}$. 设 $f(z) : \Delta \rightarrow \Delta$ 解析, 满足

$$f(0) = f\left(\frac{1}{2}\right) = f\left(-\frac{1}{2}\right) = 0.$$

证明:

$$\left|f\left(\frac{1}{4}\right)\right| \leq \frac{1}{21}.$$

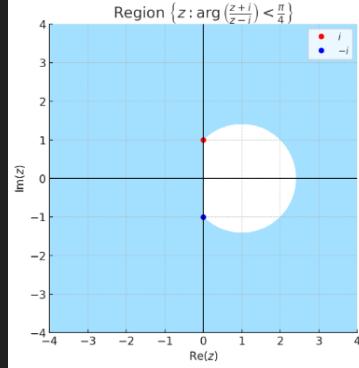
判断上界 $\frac{1}{21}$ 是否是最佳的.

8. 设 $f(z)$ 为整函数.

(1) 若存在常数 $C, n \in \mathbb{N}$ 使得 $|f(z)| \leq C|z|^n$, 证明: $f(z)$ 是一个多项式.

(2) 若将上述条件改成: 存在常数 $C, n \in \mathbb{N}$ 使得 $|\operatorname{Re} f(z)| \leq Cr^n$, 能否证明 $f(z)$ 是一个多项式? 给出证明或反例.

(1) 这题很细节.



上图展示了集合

$$\left\{ z \in \mathbb{C} : \arg \left(\frac{z+i}{z-i} \right) < \frac{\pi}{4} \right\}$$

- (2) $\phi_1 : \{|z| > 1\} \setminus (-\infty, -1) \rightarrow \{\operatorname{Im} z > 0; |z| > 1\}, z \mapsto \sqrt{-z}$. $\phi_2 : \{\operatorname{Im} z > 0; |z| > 1\} \rightarrow \{\operatorname{Im} z > 0\}, z \mapsto z + \frac{1}{z}$. $\phi_3 : \{\operatorname{Im} z > 0\} \rightarrow \{|z| < 1\}, z \mapsto \frac{z-i}{z+i}$. 于是 $\phi = \phi_3 \circ \phi_2 \circ \phi_1 : \{|z| > 1\} \setminus (-\infty, -1) \rightarrow \{|z| < 1\}, z \mapsto \frac{z-\sqrt{z}-1}{z+\sqrt{z}-1}$.
- (3) (1) f 在 $\{|z| \geq 1\}$ 内没有支点, 所以可以取单值分支. (2) 计算在 a, b 处的留数 (a_{-1}) 得到 $I = \pi i(a+b)$.

- (4) 不存在. 若存在, 则由非常数的解析函数的零点孤立性, f 只可能在 \mathbb{D} 内有有限多个零点 z_1, \dots, z_n . 考虑 $g(z) := \frac{f(z)}{\prod_{k=1}^n \varphi_{z_k}(z)} \in H(\mathbb{D})$, 且在 \mathbb{D} 内没有零点, 于是 $\frac{1}{g} \in H(\mathbb{D})$. 令 $|z| \rightarrow 1$, 我们有 $|\varphi_{z_k}(z)| \rightarrow 1, |f(z)| \rightarrow \infty$, 于是 $\left| \frac{1}{g} \right| \rightarrow 0$. 由最大模原理⁴, $\max_{|z|<1} \left| \frac{1}{g(z)} \right| \leq 0$. 于是 $\frac{1}{g} = 0$, g 在 \mathbb{D} 内处处取到无穷, 这与 $g \in H(\mathbb{D})$ 矛盾!

- (5) 记 $f(z) = \frac{z^2}{1+z^4}$, 考虑在上半圆周上积分, 在圆弧上积分趋于 0, 计算围道内留数: $\operatorname{res}_{e^{\pi i/4}} f = \frac{1-i}{4\sqrt{2}}$, $\operatorname{res}_{e^{3\pi i/4}} f = \frac{-1-i}{4\sqrt{2}}$. 于是 $I = 2\pi i(\operatorname{res}_{e^{\pi i/4}} f + \operatorname{res}_{e^{3\pi i/4}} f) = \frac{\pi}{\sqrt{2}}$.

- (6) 若存在 $|\alpha| < |f(0)|$ 使得 $f(\alpha) = 0$, 利用 Schwarz-pick lemma, $\left| \frac{f(b)-f(a)}{1-\bar{f}(a)f(b)} \right| \leq \left| \frac{b-a}{1-\bar{a}b} \right|, \forall a, b \in \mathbb{D}$, 取 $a = \alpha, b = 0$, 就有 $|f(0)| \leq |\alpha|$, 矛盾!

- 证明 Schwarz-pick lemma: 令 $\varphi_a(z) := \frac{a-z}{1-\bar{a}z}$, 对于 $f : \mathbb{D} \rightarrow \mathbb{D}$, 考虑 $\varphi_{f(a)} \circ f \circ \varphi_a : \mathbb{D} \rightarrow \mathbb{D}$, 它将 0 映射到 0, 于是由 Schwarz 引理, $|\varphi_{f(a)}(f(\varphi_a(z)))| \leq |z|$, 取 $z = \varphi_a(b)$, 则 $|\varphi_{f(a)}(f(b))| \leq |\varphi_a(b)|$, 即

⁴见 Rudin

$$\left| \frac{f(a)-f(b)}{1-f(a)f(b)} \right| \leq \left| \frac{a-b}{1-\bar{a}b} \right|.$$

(7) 记 $\varphi_a(z) := \frac{a-z}{1-\bar{a}z}$, 考虑 $g(z) := \frac{f(z)}{\varphi_0(z)\cdot\varphi_{1/2}(z)\cdot\varphi_{-1/2}(z)}$, 由于 $f \in H(\Delta)$ 在 $0, \frac{1}{2}, -\frac{1}{2}$ 有零点, 则 $g \in H(\Delta)$. 因为 $|f(z)| < 1$, 所以 $|z| \rightarrow 1$ 时, $|g(z)| \rightarrow \lim_{|z| \rightarrow 1} |f(z)| \leq 1$. 于是由最大模原理⁵, $\max_{z \in \Delta} |g(z)| \leq 1$. 令 $z = \frac{1}{4}$, 就有 $|g(\frac{1}{4})| = \frac{|f(\frac{1}{4})|}{|(-\frac{1}{4}) \cdot (\frac{2}{7}) \cdot (-\frac{2}{3})|} = 21 |f(\frac{1}{4})| \leq 1$, 即 $|f(\frac{1}{4})| \leq \frac{1}{21}$. 这是最佳上界, 因为等号成立当且仅当 $g \equiv 1$, 也就是 $f(z) = \varphi_0(z)\varphi_{1/2}(z)\varphi_{-1/2}(z)$.

(8) (1) $f \in H(\mathbb{C})$ 可写作 $f(z) = \sum_{k=0}^{\infty} a_k z^k$, 由 Cauchy 积分公式, $a_k = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{k+1}} dz$, 于是 $|a_k| \leq \frac{1}{2\pi} \int_{|z|=R} \frac{|f(z)|}{|z|^{k+1}} |dz| \leq \frac{1}{2\pi} \int_{|z|=R} \frac{C}{|z|^{k+1-n}} |dz| \leq \frac{C}{R^{k-n}}$. 当 $k > n$ 时, 由 R 的任意性, 令 $R \rightarrow \infty$, 则 $a_k = 0$. 于是 f 是一个至多 n 次多项式. (2) 利用 Borel-Carathéodory Theorem, 对于 $|z| \leq r$, $|f(z)| \leq \frac{2r}{R-r} \max_{|\zeta|=R} (\operatorname{Re} f(\zeta)) + \frac{R+r}{R-r} |f(0)|$, 其中 $0 < r < R$. 取 $R = 2r$. 于是 $|f(z)| \leq C' |z|^n + D$, 其中 C', D 是与 z 无关的常数. 类似 (1) 的讨论可知 f 是多项式.

• Borel-Carathéodory Theorem 的证明:

定义 $A = \sup_{|z| \leq R} \operatorname{Re} f(z)$.

首先设 $f(0) = 0$. 由于 $\operatorname{Re} f$ 是调和的, 可以取 $A > 0$. f 映到直线 $x = A$ 左边的半平面 P . 我们想把这个半平面映到圆盘上, 再用施瓦茨引理, 得到所要的不等式.

$w \mapsto w/A - 1$ 把 P 变成标准左半平面. $w \mapsto R(w+1)/(w-1)$ 把左半平面变成圆心在原点且半径为 R 的圆. 它们的复合映射把 0 映成 0, 就是所需要的映射:

$$w \mapsto \frac{Rw}{w-2A}$$

对上面这个映射与 f 的复合使用施瓦茨引理, 得到

$$\frac{|Rf(z)|}{|f(z) - 2A|} \leq |z|$$

取 $|z| < r$, 上式变为

$$R|f(z)| \leq r|f(z) - 2A| \leq r|f(z)| + 2Ar$$

所以

$$|f(z)| \leq \frac{2Ar}{R-r}$$

对于一般的情况, 考虑 $f(z) - f(0)$

$$\begin{aligned} |f(z)| - |f(0)| &\leq |f(z) - f(0)| \\ &\leq \frac{2r}{R-r} \sup_{|w| \leq R} \operatorname{Re} (f(w) - f(0)) \\ &\leq \frac{2r}{R-r} \left(\sup_{|w| \leq R} \operatorname{Re} f(w) + |f(0)| \right) \end{aligned}$$

整理后即得所要证明的不等式.

⁵ 只需要趋于边界时的最大值即可, 见 Rudin p.253

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1. 判断以下陈述是否正确

假设 $f : \mathbb{C} \rightarrow \mathbb{C}$ 是一个非常值解析函数. 则 $f(\mathbb{C})$ 是 \mathbb{C} 的一个稠密开子集.

存在解析函数 $f : \mathbb{D} \rightarrow \mathbb{C}$ 满足 $\lim_{|z| \rightarrow 1} |f(z)| = \infty$.

$z = 0$ 是 $\sin\left(\frac{1}{z}\right)$ 的一个极点.

假设 f 和 g 是定义在区域 Ω 中的解析函数, 满足 $|f|^2 + |g|^2 = 1$. 则 f 和 g 都是常数.

假设 u 是定义在复平面上的一个调和函数, $u > 0$. 则 u 一是常数.

- (1) 对. 反证而设 $w \notin \overline{f(\mathbb{C})}^{\mathbb{C}}$, 则存在 $r > 0$, 使得 $|f(z) - w| \geq r, \forall z \in \mathbb{C}$. 定义 $g = \frac{1}{f-w}$, 于是 $|g| \leq \frac{1}{r}$ 为有界整函数, 故 $g \equiv c$, $f \equiv c'$. 矛盾!
- (2) 错, 显然 f 只可能在 \mathbb{D} 内有有限零点, 除掉 Blaschke 乘积, 得到 \mathbb{D} 上无零点解析函数 $g := f/B$, 补充 $g|_{\partial\mathbb{D}} = 0$, 故 g 在 $\overline{\mathbb{D}}$ 连续, 在 \mathbb{D} 解析. 利用最大模原理 $\left|\frac{1}{g}\right| \leq \frac{1}{\infty} = 0$, 故 g 无定义. 矛盾!
- (3) 错. 考虑洛朗展开 $\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3} \frac{1}{z^3} + \dots$ 故 0 为本性奇点.
- (4) 对. 考虑 $f = u + iv, g = p + iq$. 于是 $|f|^2 + |g|^2 = 1 \Rightarrow u^2 + v^2 + p^2 + q^2 = 1$ 于是 $\Delta(u^2 + v^2 + p^2 + q^2) = 0$, 由于 $\Delta(\phi^2) = 2|\nabla\phi|^2 + 2\phi\nabla\phi$, 由于 $u = \operatorname{Re} f, v = \operatorname{Re}(-if)$ 为解析函数的实部, 故 $\Delta u^2 = 2|\nabla u|^2 \geq 0$, 等号必然成立, 故 $\nabla u = 0$, 故 $u = c$. 类似地, v, p, q 也为常数, 故 f, g 为常数.
- (5) 对. $u = \operatorname{Re} f$ for some $f \in H(\mathbb{C})$. 则 $f : \mathbb{C} \rightarrow \{\operatorname{Re} z > 0\}$, 像集不在 \mathbb{C} 中稠密, 故为常值. 或者考虑 $g : \{\operatorname{Re} z > 0\} \rightarrow \mathbb{D}, z \mapsto \frac{z-1}{z+1}$, $h = g \circ f : \mathbb{C} \rightarrow \mathbb{D}, z \mapsto \frac{f(z)-1}{f(z)+1}$. 于是 h 为有界整函数, 故 $h \equiv c$, 于是 $f \equiv c'$.

1. 判断以下陈述是否正确

如果 u 是一个调和函数, 则 $\frac{\partial u}{\partial z}$ 是一个解析函数.

假设 $f : \mathbb{D} \rightarrow \mathbb{C}$ 解析, $f = u + iv$. 如果 u 有界, 则 v 也有界.

假设 $f : \mathbb{D} \rightarrow \mathbb{C}$ 解析, $f = u + iv$. 则

$$\iint_{\mathbb{D}} u^2(z) dx dy = \iint_{\mathbb{D}} v^2(z) dx dy.$$

对任意的非零多项式 $P(z)$ 以及任意的 $\lambda \neq 0$, 方程 $P(z) = e^{\lambda z}$ 有无穷多个解.

不存在一个解析函数 $f : \mathbb{D} \rightarrow \mathbb{D}$ 使得 $z = 0, \frac{1}{2}, \dots, 1 - \frac{1}{n+1}, \dots$ 是 f 的零点.

- (1) 对. $\partial_{\bar{z}}(\partial_z u) = \Delta u = 0 \Rightarrow \partial_z u \in H$.
- (2) 错. 考虑 $f(z) := \log(1 - z)$.
- (3) 错. $u^2 - v^2 = \operatorname{Re}(f^2)$. 由调和函数平均值公式得到, $LHS - RHS = \pi(\operatorname{Re} f^2(0))$.
- (4) 对. 考虑 $f(z) := P(z)e^{-\lambda z}$ 这是超越整函数, 以 ∞ 为本性奇点, 利用 Picard 大定理, 在 ∞ 的任意邻域 $\{|z| > R\}$ 内, 除去至多一个点, f 都可以取到任意值无穷多次. 由于 P 为多项式, 考虑不包含 P 零点的 $\{|z| \in R\}$, f 取不到 0, 故除了 0 以外的任意值 (比如 1) 都可以取到无穷次.
- (5) 对. 由于 $f \in N$, 根据 Rudin theorem 15.23, $\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty$, 即 $\sum_{n=1}^{\infty} \frac{1}{n} < \infty$, 矛盾!

7.22 期末练习题

1. 写出方程 $az^2 + bz + c = 0$ 的两个复数解.
2. 用以下方法求解三次多项式方程 $z^3 + az + b = 0$. 选择适当的 α , 作变换 $z = \zeta + \frac{\alpha}{\zeta}$, 使得

$$z^3 + az + b = \zeta^6 + c_3\zeta^3 + c_0.$$

3. 证明: z_1, z_2, z_3 为等边三角形的顶点当且仅当

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

4. 证明: 复数 α, β, γ 共线的充要条件为

$$\begin{vmatrix} \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \\ \gamma & \bar{\gamma} & 1 \end{vmatrix} = 0.$$

5. 画出以下图形:

$$\{z \in \mathbb{C} : |z^2 - 1| = 1\}.$$

6. 设 $\mu \in \mathbb{C}, |\mu| < 1$. 画出以下图形:

$$\{z \in \mathbb{C} : |z + \mu\bar{z}| = 1\}.$$

(1) 略

(2) 略

(3) 等式成立当且仅当 $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$, 故可以不妨平移设 $z_1 = 0$, 不妨伸缩旋转设 $z_2 = 1$, 故显然.

(4) 不妨设 a, b, c 非零, 不妨设 $\alpha = 1$, 则必要性显然, 充分性也显然.

(5) 考虑卡西尼卵

(6) $\mu := a + bi$, then $x^2(1 + a^2 + b^2 + 2a) + y^2(1 + a^2 + b^2 - 2a) + 4bxy = 1$.

好的，卡西尼卵形线 (Cassini Oval) 是一个有趣的几何图形，它有许多独特的性质。以下是其主要性质的列举：

Définition 7

卡西尼卵形线是平面上所有满足到两个固定点（称为焦点 F_1 和 F_2 ）的距离之乘积为常数 k^2 的点的轨迹。如果点的坐标是 (x, y) ，焦点是 $(-c, 0)$ 和 $(c, 0)$ ，则其定义方程为：

$$\sqrt{(x+c)^2+y^2} \cdot \sqrt{(x-c)^2+y^2} = k^2$$

展开后可得到笛卡尔坐标方程：

$$(x^2+y^2)^2 - 2c^2(x^2-y^2) + c^4 = k^4$$

或极坐标方程：

$$r^4 - 2c^2r^2\cos(2\theta) + c^4 = k^4$$

卡西尼卵形线的形状非常多样，取决于常数 k 与焦距半距离 c 的关系：

- (1) $k > c$ (单瓣卵形)：曲线是一个单个的闭合回路，形状像一个压扁的椭圆，包含两个焦点。随着 k 的增大，它会变得越来越圆，当 $k \rightarrow \infty$ 时，趋近于一个以原点为圆心的大圆。
- (2) $k = c$ (伯努利双纽线 / Lemniscate of Bernoulli)：曲线是一个形如“8”字或无限符号 (∞) 的图形，在原点处自相交。这是卡西尼卵形线的一个特殊且著名的形式，它与抛物线的反演有关。
- (3) $k < c$ (双瓣卵形 / 两个分离的卵形)：曲线由两个分离的闭合回路组成，每个回路围绕一个焦点。随着 k 减小，这两个卵形会变得越来越小，并趋近于焦点本身。

卡西尼卵形线是关于 x 轴和 y 轴对称的。它也是关于原点（两个焦点的中点）中心对称的。

特殊点：

- 交点：卵形线可能与坐标轴有交点。例如，当 $y = 0$ 时，方程变为 $(x^2 - c^2)^2 = k^4$ ，解得 $x^2 = c^2 \pm k^2$ ，因此 $x = \pm\sqrt{c^2 \pm k^2}$ 。
- 焦点：当 $k = c$ 时，曲线通过原点（两焦点的中点）。

与椭圆和双曲线的关系：

- 椭圆定义为到两焦点的距离之和为常数。
- 双曲线定义为到两焦点的距离之差的绝对值为常数。
- 卡西尼卵形线定义为到两焦点的距离之乘积为常数。

它们都是圆锥曲线的推广，尽管卡西尼卵形线本身不是圆锥曲线。

反演性质：

伯努利双纽线 ($k = c$) 可以通过对一个双曲线进行反演（反转）来生成。或者说，对一个圆进行反演，可以得到一个双纽线。

应用：

- 在物理学中，卡西尼卵形线出现在某些力场（如电荷或质量分布）的等势线中。
- 在光学中，它们可以用来描述某些透镜的表面。
- 在数学研究中，它们是代数几何和复分析中的一个经典例子。

极值点：

卡西尼卵形线的曲率、最大/最小宽度等性质可以通过分析其方程的导数来确定。

总之，卡西尼卵形线是一个因其定义方式（距离乘积）而产生的丰富多样的曲线族，其形状随参数变化而呈现出显著的差异。

7. 画出

$$\Omega = \{z : |z^2 + az + b| < R^2\}$$

的图像. 当 R 取什么值时, Ω 是连通的?

8. 画出以下区域:

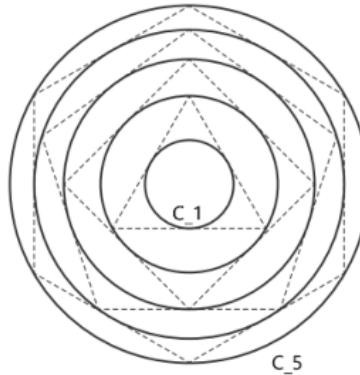
$$|z| < 1, \frac{|z - 1|}{1 - |z|} \leq M.$$

9. 定义

$$a_n = \prod_{k=1}^n \left(1 + \frac{i}{k}\right).$$

证明: $\{a_n\}$ 的极限点集是一个圆周.

10. 如图, 有一列同心圆 C_n , 是它一个正 $(n+2)$ 边形的内切圆, 同时包含一个正则 $(n+1)$ 边形的顶点. 当 $n \rightarrow \infty$ 时, C_n 的半径是否趋于 ∞ ? 给出证明.



- (1) When $|z_1 - z_2| > 2R$, i.e. $|\sqrt{a^2 - 4b}| > R$.
- (2) 是一个开圆盘, 在 $M = 1$ 时退化为 $[0, 1)$, 此外圆心在 $(-\infty, 0) \cup (0, 1)$.
- (3) $|a_n| \leq \prod_{k=1}^{\infty} \sqrt{1 + \frac{1}{k^2}} \leq \exp\left\{\frac{1}{2} \sum_{k=1}^{\infty} \log(1 + k^{-2})\right\} \leq \exp\left\{\frac{1}{2} \sum_{k=1}^{\infty} k^{-2}\right\} < \infty$. And $\arg a_n = \sum_{k=1}^n \arctan \frac{1}{k} \sim \sum k^{-1} \rightarrow \infty$, where $\arctan \frac{1}{k} \rightarrow 0$. More explicitly, we have $\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$, then $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{k^2}} = \sqrt{\frac{\sin i\pi}{\pi i}} = \sqrt{\frac{\sinh(\pi)}{\pi}}$.

(4) $r_{n+1} \cos \frac{\pi}{n+2} = r_n$. Then

$$\begin{aligned} r_n &= r_1 \prod_{k=2}^n \frac{1}{\cos \frac{\pi}{k+1}} = r_1 \cdot \exp \left\{ - \sum_{k=2}^n \log \cos \frac{\pi}{k+1} \right\} \\ &= r_1 \cdot \exp \left\{ - \sum_{k=2}^n \log \left(1 - \frac{1}{2} \left(\frac{\pi}{k+1} \right)^2 + o(k^{-3}) \right) \right\} \\ &= r_1 \cdot \exp \left\{ - \sum_{k=2}^n \left(-\frac{1}{2} \right) \left(\frac{\pi}{k+1} \right)^2 + o(1) \right\} \\ &< \infty \end{aligned}$$

11. 设 $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$. 令

$$R = \sqrt{1 + |a_{n-1}|^2 + \cdots + |a_1|^2 + |a_0|^2}.$$

证明: $P(z)$ 的零点都落在 $\{|z| < R\}$ 之内.

12. 将任意旋转变换 $A \in \text{SO}(3)$ 表示为分式线性变换.

13. 构造一个在 $\Delta = \{z : |z| < 1\}$ 上无界的解析函数, 使得对任意 $\zeta \in \partial\Delta$, 除了 $\zeta = 1$ 之外, $\lim_{z \rightarrow \zeta} |f(z)| = 1$.

14. 构造共形映射将区域 $\{\text{Im } z > 0\} \cap \{0 < \text{Re } z < \pi\}$ 映成单位圆盘 Δ .

15. 构造共形映射 f 将 $\{|z+i| \leq 1\} \cup \{|z-i| \leq 1\}$ 的外部映成单位圆盘的外部, 使得 $f(\infty) = \infty$.

16. 构造共形映射 f 将 $\{|z| > 1\} \setminus (-\infty, -1)$ 映成单位圆盘.

17. 构造共形映射将区域

$$\{\text{Im } z > 0\} \cap \{0 < \text{Re } z < \pi\}$$

映成单位圆盘 Δ .

18. 将任意旋转变换 $A \in SO(3)$ 表示为分式线性变换.

19. 设 $f(z) = \cos z - 1$. 证明: 在 $z = 0$ 附近, 存在解析函数 $h(z) = \sum a_n z^n$ 使得 $f(z) = h(z)^2$. 计算出 a_0, a_1, a_2 . 是否可以将 $h(z)$ 全纯延拓到 \mathbb{C} ?

(5) 这个不需要 Rouche 引理. 考虑 $P(z) = 0$, 有 $|z|^{2n} \leq (|a_{n-1}| |z|^{n-1} + \cdots + |a_0|)^2$, 由 Cauchy-Schwarz, $RHS \leq (|a_{n-1}|^2 + \cdots + |a_0|^2) (|z|^{2n-2} + \cdots + 1) = (|a_{n-1}|^2 + \cdots + |a_0|^2) \frac{|z|^{2n}-1}{|z|^2-1} < (R^2-1) \frac{|z|^{2n}}{|z|^2-1}$ 于是 $|z| < R$.

(6) 任意一个旋转变换 $A \in \text{SO}(3)$ 都可以通过以下步骤表示为一个分式线性

变换：1. 找到与旋转 A 对应的 $SU(2)$ 矩阵 $U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ 。2. 这个矩阵 U 定义了在黎曼球面上的一个分式线性变换 $f_A(w) = \frac{\alpha w + \beta}{-\bar{\beta}w + \bar{\alpha}}$ 。其中 w 是旋转前点在黎曼球面上的坐标， w' 是旋转后点在黎曼球面上的坐标。这个表示是李群论和复变函数论之间的一个重要联系。

- (7) 考虑 $w = \frac{1+z}{1-z}$, 在 $z \neq 1$ 的圆周上, 这是个纯虚数 $w = -i \cot(\arg z/2)$.
- (8) $z_1 : \Omega \rightarrow \{|z| < 1, \operatorname{Im} z > 0\}, z \mapsto e^{iz}$, $z_2 : \{|z| < 1, \operatorname{Im} z > 0\} \rightarrow \{\operatorname{Im} z > 0\}, z \mapsto -\frac{1}{2}(z + \frac{1}{z})$, $z_3 : \{\operatorname{Im} z > 0\} \rightarrow \{|z| < 1\}, z \mapsto \frac{z-i}{z+i}$.
- (9) $z_1 : \Omega \rightarrow \{|z| > 1, \operatorname{Re} z > 0\}, z \mapsto -\sqrt{z}$, $z_2 : \{|z| > 1, \operatorname{Re} z > 0\} \rightarrow \{\operatorname{Re} z > 0\}, z \mapsto \frac{1}{2}(z + \frac{1}{z})$, $z_3 : \{\operatorname{Re} z > 0\} \rightarrow \{|z| > 1\}, z \mapsto \frac{z+i}{z-i}$. 这个映射错了, 没有满足 $f(\infty) = \infty$.
- (10) 重复
- (11) 重复
- (12) 重复
- (13) $\left[\frac{f(z)}{z^2} \right]_{z=0} = -\frac{1}{2}$. 且 $\frac{f(z)}{z^2}$ 在 0 附近解析, 故在 0 附近存在解析函数 $g(z)$ 使得 $\frac{f(z)}{z^2} = e^{g(z)}$. 于是 $ze^{g(z)/2} = h(z)$. 我们知道 $h(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$, $f(z) = h^2(z) = a_0^2 + 2a_0 a_1 z + (a_1^2 + 2a_0 a_2)z^2 + (2a_0 a_3 + 2a_1 a_2)z^3 + \dots$, 且 $f(z)$ 自己展开得到 $-\frac{1}{2}z^2 + \frac{1}{24}z^4$. 故 $a_0 = 0$, $a_1^2 = -\frac{1}{2}$, $a_1 a_2 = 0$, 故 $a_1 = \pm \frac{1}{\sqrt{2}}i$, $a_2 = 0$.

20. 给定一列正实数 $r_i \rightarrow \infty$. 设 $N(r) = |\{i : r_i < r\}|$, $\rho = \inf\{\alpha > 0 : \sum r_i^{-\alpha} < \infty\}$. 证明:

$$\rho = \limsup_{r \rightarrow \infty} \frac{\log N(r)}{\log r}.$$

21. 证明 **Abel 极限定理**: 设级数 $\sum_n a_n$ 收敛, 则当 $z \in \mathbb{D}$, $z \rightarrow 1$ 且 $\frac{|1-z|}{1-|z|}$ 有界时, $\sum_n a_n z^n \rightarrow \sum_n a_n$.

22. 设 $P(z), Q(z)$ 在 $z = 0$ 附近复解析. 求方程 $u''(z) + P(z)u'(z) + Q(z)u(z)$ 在 $z = 0$ 附近的复解析解.

23. 设 g 在 $[0, 1]$ 上连续. 定义

$$f_n(z) = \int_0^1 \frac{g(t)}{(t-z)^n} dt.$$

证明: $f_n(z)$ 在 $\mathbb{C} \setminus [0, 1]$ 上全纯且 $f'_n(z) = n f_{n+1}(z)$.

24. 考虑多项式 $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$.

1) 证明: 存在 $z_0 \in \mathbb{C}, |z_0| = 1$ 使得 $|P(z_0)| \geq |a_0|$.

2) 假设当 $|z| \leq 1$ 时, $|P(z)| \leq 1$. 证明: $|a_k| \leq 1$, $k = 0, 1, \dots, n$.

3) 设 $f(z) = \sum_{n=0}^{\infty} a_n z^n$ 在 $|z| < 1$ 收敛, $|f(z)| < 1$. 证明: $|a_n| \leq 1$.

25. 设 $(a_0, a_1, a_2, a_3, \dots) = (1, 1, 2, 3, 5, 8, \dots)$ 为 Fibonacci 数列, $a_n = a_{n-1} + a_{n-2}$. 令 $f(z) = \sum_{n=0}^{\infty} a_n z^n$. 证明:

1) $f(z)$ 的收敛半径为 $\frac{1}{\gamma}$, $\gamma = \frac{1+\sqrt{5}}{2}$.

2) 存在 $\alpha > 0$ 使得 $\sum_{n=0}^{\infty} (a_n - \alpha \gamma^n) z^n$ 的收敛半径大于 1.

38. 设 $f : \mathbb{D} \rightarrow \mathbb{C}$ 解析. 定义

$$\|f\|_2 = \left(\int_{\mathbb{D}} |f(z)|^2 dx dy \right)^{1/2}.$$

证明:

$$|f(z)| \leq \frac{1}{\sqrt{\pi}(1-|z|)} \|f\|_2.$$

由平均值公式, 对于 $z \in \mathbb{D}$, $B_r(z) \subset \mathbb{D}$, 有

$$\begin{aligned} f(z) &= \frac{1}{\pi r^2} \iint_{B_r(z)} f(\zeta) dS \\ \iint_{B_r(z)} |f(\zeta)| dS &\leq \sqrt{\iint_{B_r(z)} 1 dS \cdot \iint_{B_r(z)} |f^2(\zeta)| dS} \\ &= \sqrt{\pi}r \cdot \|f\|_2 \end{aligned}$$

于是

$$|f(z)| \leq \frac{1}{\sqrt{\pi}r} \|f\|_2$$

令 $r \rightarrow 1 - |z|$, 就有

$$|f(z)| \leq \frac{1}{\sqrt{\pi}(1-|z|)} \|f\|_2$$

44. 令 $f(z) = \prod_{n=0}^{\infty} (1 + z^{2^n})$. 证明: (1) f 在单位圆内解析;

(2) $f(z) = (1+z)f(z^2)$; (3) $f(z) = \frac{1}{1-z}$.

(1) 在 $|z| \leq r < 1$ 内, $|f(z)| \leq \exp \left\{ \sum_{n=0}^{\infty} \log(1 + |z|^{2^n}) \right\} \leq \exp \{1/(1-r)\} < \infty$, 一致收敛, 故解析. (思考 why)

(2) 显然

(3) $f(z) = (1+z)f(z^2) = \cdots = (1+z) \cdots (1+z^{2^{N-1}})f(z^{2^N}) = \frac{1-z^{2^N}}{1-z} \cdot f(z^{2^N}) \rightarrow \frac{1}{1-z} f(0) = \frac{1}{1-z}$.

45. 证明: 存在解析函数 $f : \{z \in \mathbb{C} : |z| > 4\} \rightarrow \mathbb{C}$, 满足

$$f'(z) = \frac{z}{(z-1)(z-2)(z-3)}.$$

记 $g = \frac{z}{(z-1)(z-2)(z-3)}$, 于是计算得到 $\text{res}_\infty g = -\text{res}_1 g - \text{res}_2 g - \text{res}_3 g = 0$.
于是 g 在 $\{|z| > 4\}$ 内解析, 可以良好定义 $f(z) = \int_{z_0 \rightarrow z} g(z) dz$, 其中 $|z_0| > 4$.

47. 设 $f(z) = \sum_{n=0}^{\infty} a_n z^n$ 在 $|z| < 1$ 解析, $|f| \leq M$. 证明:

$$M|a_1| \leq M^2 - |a_0|^2.$$

不妨设 $M = 1$ 否则用 $\frac{f}{M}$ 代替 f .

利用 Schwarz-Pick 引理, $\varphi_a(z) := \frac{z-a}{1-\bar{a}z}$. $g(z) := \varphi_{f(\alpha)} \circ f \circ \varphi_{-\alpha}(z)$, 于是

$$\begin{aligned} |g'(0)| &= |\varphi'_{f(\alpha)}(f(\varphi_{-\alpha}(0)))f'(\varphi_{-\alpha}(0))\varphi'_{-\alpha}(0)| \\ &= |\varphi'_{f(\alpha)}(f(\alpha)) \cdot f'(\alpha) \cdot \varphi'_{-\alpha}(0)| \\ &= |1 - |f(\alpha)|^2|^{-1} \cdot |f'(\alpha)| \cdot |1 - |\alpha|^2| \\ &\leq 1 \end{aligned}$$

从而

$$|f'(\alpha)| \leq \frac{1 - |f(\alpha)|^2}{1 - |\alpha|^2}$$

令 $\alpha = 0$ 就有

$$|f'(0)| \leq 1 - |f(0)|^2$$

48. 设 $f : \mathbb{D} \rightarrow \mathbb{D}$ 解析, $f^n(z)$ 表示 n 次迭代 $f \circ f \circ \cdots \circ f(z)$.

证明: 要么存在 $z_0 \in \mathbb{D}$ s.t. $f(z_0) = z_0$; 要么对任意 $z \in \mathbb{D}$,
 $|f^n(z)| \rightarrow 1$.

This is Denjoy-Wolff theorem.

Exercice 35

5504

Suppose f is analytic in $U = \{|z| < 1\}$ with $f(0) = 0$ and $|f(z)| < 1$ for all $z \in U$. If the sequence $\{f_n\}$ is defined by composition

$$f_n(z) = \underbrace{f(f(\cdots f(z)) \cdots)}_n$$

and

$$f_n(z) \rightarrow g(z)$$

for all $z \in U$, prove that either $g(z) = 0$ or $g(z) = z$.

(Indiana-Purdue)

Solution.

By Schwarz's lemma, it follows from $f(0) = 0$ and $|f(z)| < 1$ that $|f(z)| \leq |z|$ for all $z \in U$, and if $|f(z)| = |z|$ for some $z \neq 0$, then $f(z) = e^{i\alpha}z$ where α is a real number.

In the case when $f(z) = e^{i\alpha}z$, $f_n(z) = e^{in\alpha}z$. Since $f_n(z)$ is convergent, we obtain $\alpha = 0$, which implies that $f(z) = z$ and $g(z) = z$.

In other cases, we have

$$\left| \frac{f(z)}{z} \right| < 1$$

for all $z \in U$. Let $0 < r < 1$. Then

$$\max_{|z| \leq r} \left| \frac{f(z)}{z} \right| = \lambda < 1.$$

For all $z \in \{|z| \leq r\}$, we have

$$\begin{aligned} |f(z)| &\leq \lambda|z|, \\ |f_2(z)| &= |f(f(z))| \leq \lambda|f(z)| \leq \lambda^2|z|, \\ &\dots \\ |f_n(z)| &= |f(f_{n-1}(z))| \leq \lambda|f_{n-1}(z)| \leq \lambda^n|z|, \\ &\dots \end{aligned}$$

Hence $f_n(z)$ converges to zero uniformly in $\{|z| \leq r\}$. Since $0 < r < 1$ is arbitrarily chosen, we obtain $g(z) = 0$ for all $z \in U$.

71. 设 f 在 \mathbb{C} 上解析, 且当 $z \rightarrow \infty$ 时 $f(z) \rightarrow \infty$. 证明: f 是一个多项式.

72. 设 f 是整函数. 证明: f 是多项式当且仅当

$$\limsup_{r \rightarrow \infty} \frac{\log M(r)}{\log r} < \infty,$$

其中 $M(r) = \sup_{|z|=r} |f(z)|$.

73. 设 $P(z)$ 是一个多项式, $\deg P = n$. 假设当 $|z| < 1$ 时, $|P(z)| \leq M$. 证明:

$$|P(z)| \leq M|z|^n, \forall |z| > 1.$$

- (1) $g(z) := f\left(\frac{1}{z}\right)$, 在 0 处有奇点, 若为本性奇点, 则由 Picard 大定理, g 在 0 去心邻域内, 除一个值以外, 取到任意值无穷次, 但 $\lim_{z \rightarrow 0} g(z) = \infty$. 故 0 为极点, $g(z) = c_{-n}z^{-n} + \dots + c_0 + c_1z + \dots$, 于是 $f(z) = \dots + c_1z^{-1} + c_0 + c_{-1}z + \dots + c_{-n}z^n$, 由于 $f \in H(\mathbb{C})$, f 不可能在 0 处存在奇点, 所以 $f(z) = c_0 + c_{-1}z + \dots + c_{-n}z^n$ 是多项式.
- (1) 使用 Cauchy 积分公式放缩
- (1) 利用最大模原理, 考虑解析函数 $Q(z) := z^n P\left(\frac{1}{z}\right)$, 对于 $|z| = 1$, 有 $|Q(z)| = |z^n P\left(\frac{1}{z}\right)| = |P\left(\frac{1}{z}\right)| \leq M$, 这利用了 $\left|\frac{1}{z}\right| = 1$, 以及 P 的连续性. 由最大模原理, 对于 $|z| > 1$, $\left|\frac{1}{z^n} P(z)\right| = |Q\left(\frac{1}{z}\right)| \leq \max_{\left|\frac{1}{z}\right|=1} |Q\left(\frac{1}{z}\right)| = M$.

4. 考虑解析函数族 $f_t(z) = 3z^5 + z^2 + t$.

(1) 证明: 对任意 $|t| < 1$, $f_t(z)$ 在单位圆 $\{z : |z| < 1\}$ 中有 5 个零点 (计重数).

(2) 定义

$$g(t) = \sum_{|z|<1, f_t(z)=0} z^{2022}.$$

证明: $g(t)$ 关于 $|t| < 1$ 解析.

Use the same idea as the Principle theorem.

$\{z_j\}$ are the 5 distinct zero's of $f_t(z)$ in \mathbb{D} , $f_t(z) := (z - z_j)h_t(z)$, then

$$\frac{f'_t(z)}{f_t(z)} = \frac{1}{z - z_j} + \frac{h'_t(z)}{h_t(z)}$$

So the residue of $\frac{f'_t(z)}{f_t(z)}$ at z_j is 1. Thus for holomorphic $\phi(z)$,

$$\operatorname{res}\left(\phi(z)\frac{f'_t(z)}{f_t(z)}; z_j\right) = \phi(z_j)$$

Then by residue theorem

$$\int_{|z|=1} \phi(z) \frac{f'_t(z)}{f_t(z)} dz = 2\pi i \sum_j \operatorname{res}\left(\phi(z)\frac{f'_t(z)}{f_t(z)}; z_j\right) = 2\pi i \sum_j \phi(z_j)$$

Then

$$g(t) = \sum \phi(z_j) = \frac{1}{2\pi i} \int_{|z|=1} \phi(z) \frac{f'_t(z)}{f_t(z)} dz$$

6. 设 f 在 \mathbb{D} 上解析, 不是单射. 证明: 存在 $|z| = |w| < 1$ 使得

$$f(z) = f(w).$$

辐角原理

假如对于任意满足 $f(z) = f(w)$ 的 z, w 都有 $|z| < |w| = r$, 由于 $z \in \mathbb{D}_{|z|}$, 所以 $f(z) \in f(\mathbb{D}_{|z|}) \subset f(\mathbb{D}_r)$, 同时 $f(z) = f(w) \in f(\partial\mathbb{D}_r) = \partial f(\mathbb{D}_r)$, 由开映射定理 $f(\mathbb{D}_r)$ 开, 故矛盾!

7.23 Rudin Chapter 11

7.23.1 Poisson Integral

The Poisson kernel is

$$P_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{int} \quad 0 \leq r < 1, t \text{ real}$$

If $z = re^{i\theta}$,

$$P_r(\theta - t) = \operatorname{Re} \left[\frac{e^{it} + z}{e^{it} - z} \right] = \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} \quad (7.7)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) dt = 1$$

From eq. (7.7), we have $P_r(t) > 0$, $P_r(t) = P_r(-t)$, $P_r(t) < P_r(\delta)$, for $0 < \delta < |t| \leq \pi$. Then

$$\lim_{r \rightarrow 1} P_r(\delta) = 0 \quad 0 < \delta \leq \pi$$

Now denote $B_1(0)$ by U , $\partial B_1(0)$ by T . Identify the spaces $L^p(T)$ and $C(T)$ with the corresponding spaces of 2π -period functions in \mathbb{R}^1 .

Regard $P_r(\theta - t)$ as $P(z = re^{i\theta}, e^{it})$, then

$$P(z, e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2} \quad z \in U, e^{it} \in T$$

The Poisson integral $P[f]$ is

$$F(re^{i\theta}) = (P_r * f)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)f(t) dt \quad f \in L^1(T)$$

If f is real,

$$P[f] = \operatorname{Re} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} f(t) dt \right] = \operatorname{Re} \underbrace{\left[\int_X \frac{d\mu(t)}{\varphi(t) - z} \right]}_{\in H(D) \text{ by Theorem 10.7} \atop \text{since } \varphi(X) \cap D = \emptyset} \quad \text{is harmonic.}$$

$f = u + iv$, then $P[f] = P[u] + iP[v]$ is harmonic.

Poisson integrals of continuous function behave well near the boundary of U .

11.8 Theorem If $f \in C(T)$ and if Hf is defined on the closed unit disc \bar{U} by

$$(Hf)(re^{i\theta}) = \begin{cases} f(e^{i\theta}) & \text{if } r = 1, \\ P[f](re^{i\theta}) & \text{if } 0 \leq r < 1, \end{cases} \quad (1)$$

then $Hf \in C(\bar{U})$.

Note. This theorem, using Poisson integral, solves the boundary value problem⁶: given $f \in C(T)$, find harmonic F in U .

The harmonic F is also unique.

11.9 Theorem Suppose u is a continuous real function on the closed unit disc \bar{U} , and suppose u is harmonic in U . Then (in U) u is the Poisson integral of its restriction to T , and u is the real part of the holomorphic function

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u(e^{it}) dt \quad (z \in U). \quad (1)$$

⁶the Dirichlet problem

证明. By theorem 10.7, $f \in H(D)$. Let $u_1 = \operatorname{Re} f$, u_1 is harmonic. NTS: $h := u - u_1 \equiv 0$ on U . Assume not, WLOG, $\exists z_0 \in U$, s.t. $h(z_0) > 0$; pick $0 < \epsilon < h(z_0)$, $g := h + \epsilon|z|^2$, then $g|_T = \epsilon < h(z_0) < g(z_0)$. As $\Delta g = 4\epsilon > 0$, g is subharmonic, by maximum principle, this is a contradiction. \square

For harmonic⁷ $u \in H(\overline{U})$,

$$\begin{aligned} u(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\operatorname{Re} \left[\frac{e^{it} + z}{e^{it} - z} \right]}_{=P_r(\theta-t)=\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}} u(e^{it}) dt \quad z \in U \end{aligned}$$

For $B_R(a)$, by eq. (7.7),

$$u(a + re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - t) + r^2} u(a + Re^{it}) dt \quad (7.8)$$

Let u harmonic in open Ω . For $\overline{B_R(a)} \subset \Omega$, u satisfies eq. (7.8)⁸, and $u = \operatorname{Re} f$ for some $f \in H(B_R(a))$, where f is unique up to $+i \cdot \text{const}$.

The Poisson integral also yields information about sequences of harmonic functions.

11.11 Harnack's Theorem *Let $\{u_n\}$ be a sequence of harmonic functions in a region Ω .*

- (a) *If $u_n \rightarrow u$ uniformly on compact subsets of Ω , then u is harmonic in Ω .*
- (b) *If $u_1 \leq u_2 \leq u_3 \leq \dots$, then either $\{u_n\}$ converges uniformly on compact subsets of Ω , or $u_n(z) \rightarrow \infty$ for every $z \in \Omega$.*

证明. (a) As $u_n \Rightarrow u$ on $\overline{B_R(a)} \subset \Omega$, u_n satisfies eq. (7.8), we have u satisfies eq. (7.8), thus u is harmonic.

(b) Assume that $u_1 \geq 0$, $u := \sup u_n$, $A := \{x : u(x) < \infty\}$, $B := \Omega - A$.

For any $B_r(a) \subset A$,

$$\begin{aligned} \frac{R-r}{R+r} &\leq \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - t) + r^2} \leq \frac{R+r}{R-r} \\ u_n(a+re^{i\theta}) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - t) + r^2} u_n(a+Re^{it}) dt \leq \frac{R+r}{R-r} \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} u_n(a+Re^{it}) dt}_{=u_n(a+0 \cdot e^{i\theta})=u(a)} \end{aligned}$$

⁷in U .

⁸eq. (7.8) is the criterion of harmonic function, due to the above two theorems.

Thus

$$\frac{R-r}{R+r}u_n(a) \leq u_n(a+re^{i\theta}) \leq \frac{R+r}{R-r}u_n(a) \quad (7.9)$$

Let $n \rightarrow \infty$, then eq. (7.9) also holds for u . Thus A, B are open. As Ω is connected, $A = \emptyset$ or Ω . We are done! \square

7.23.2 Schwarz reflection principle

11.14 Theorem (The Schwarz reflection principle) Suppose L is a segment of the real axis, Ω^+ is a region in Π^+ , and every $t \in L$ is the center of an open disc D_t such that $\Pi^+ \cap D_t$ lies in Ω^+ . Let Ω^- be the reflection of Ω^+ :

$$\Omega^- = \{z: \bar{z} \in \Omega^+\}. \quad (1)$$

Suppose $f = u + iv$ is holomorphic in Ω^+ , and

$$\lim_{n \rightarrow \infty} v(z_n) = 0 \quad (2)$$

for every sequence $\{z_n\}$ in Ω^+ which converges to a point of L .

Then there is a function F , holomorphic in $\Omega^+ \cup L \cup \Omega^-$, such that $F(z) = f(z)$ in Ω^+ ; this F satisfies the relation

$$F(\bar{z}) = \overline{F(z)} \quad (z \in \Omega^+ \cup L \cup \Omega^-). \quad (3)$$

The theorem asserts $f \in H(\Omega^+)$ can be extended to $F \in H(\Omega)$, where $\Omega := \Omega^+ \cup L \cup \Omega^-$. $f = u + iv$, we extend v to Ω by letting $v|_L = 0$, and $v|_{\Omega^-}(z) = \overline{v(\bar{z})}$.

7.23.3 Boundary Behavior of Poisson Integrals

Let $u_r(e^{i\theta}) := u(re^{i\theta})$, $0 \leq r < 1$. Theorem 11.8 can be restated as: for $f \in C(T)$, $F = P[f]$, then $F_r \rightrightarrows f$ on T as $r \rightarrow 1$. i.e. $\lim_{r \rightarrow 1} \|F_r - f\|_\infty = 0$.

We replace L^∞ by L^p :

11.16 Theorem If $1 \leq p \leq \infty$, $f \in L^p(T)$, and $u = P[f]$, then

$$\|u_r\|_p \leq \|f\|_p \quad (0 \leq r < 1). \quad (1)$$

If $1 \leq p < \infty$, then

$$\lim_{r \rightarrow 1} \|u_r - f\|_p = 0. \quad (2)$$

证明.

$$u_r(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) P_r(\theta - t) dx =: \int_X f(t) d\mu$$

Then

$$|u_r(e^{i\theta})|^p = \left| \int_X f(t) d\mu \right|^p \leq \left(\int_X |f(t)| d\mu \right)^p \leq \int_X |f(t)|^p dx \cdot \underbrace{\int_X 1^p d\mu}_{=1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p P_r(\theta - t) dt$$

Integrate θ over $[-\pi, \pi]$, then we obtain (1).

As $C(T)$ is dense in $L^p(T)$ for $1 \leq p < \infty$ ⁹, pick $g \in C(T)$ s.t. $\|f - g\|_p < \epsilon$, $v := P[g]$, then

$$\|u_r - f\|_p \leq \underbrace{\|u_r - v_r\|_p}_{=\|(u-v)_r\|_p \leq \|g-f\|_p < \epsilon} + \underbrace{\|v_r - g\|_p}_{\leq \|v_r - g\|_\infty} + \underbrace{\|g - f\|_p}_{< \epsilon} \leq 2\epsilon + \|v_r - g\|_\infty$$

By theorem 11.8, $\|v_r - g\|_\infty \rightarrow 0$, therefore $\|u_r - f\|_p \rightarrow 0$ as $r \rightarrow 1$. \square

7.23.4 The representation theorem

11.30 Theorem Suppose u is harmonic in U , $1 \leq p \leq \infty$, and

$$\sup_{0 < r < 1} \|u_r\|_p = M < \infty. \quad (1)$$

- (a) If $p = 1$, it follows that there is a unique complex Borel measure μ on T so that $u = P[d\mu]$.
- (b) If $p > 1$, it follows that there is a unique $f \in L^p(T)$ so that $u = P[f]$.
- (c) Every positive harmonic function in U is the Poisson integral of a unique positive Borel measure on T .

H^∞ space

11.31 Since holomorphic functions are harmonic, all of the preceding results (of which Theorems 11.16, 11.24, 11.25, 11.30 are the most significant) apply to holomorphic functions in U . This leads to the study of the H^p -spaces, a topic that will be taken up in Chap. 17.

At present we shall only give one application, to functions in the space H^∞ . This, by definition, is the space of all bounded holomorphic functions in U ; the norm

$$\|f\|_\infty = \sup \{ |f(z)| : z \in U \}$$

turns H^∞ into a Banach space.

⁹ $\overline{C(T)}^{L^\infty(T)} \subsetneq L^\infty(T)$

11.32 Theorem To every $f \in H^\infty$ corresponds a function $f^* \in L^\infty(T)$, defined almost everywhere by

$$f^*(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta}). \quad (1)$$

The equality $\|f\|_\infty = \|f^*\|_\infty$ holds.

If $f^*(e^{i\theta}) = 0$ for almost all $e^{i\theta}$ on some arc $I \subset T$, then $f(z) = 0$ for every $z \in U$.

A considerably stronger uniqueness theorem will be obtained later.

7.24 Rudin Chapter 12

Théorème 15: Maximum Modulus Principle

If K is the closure of a bounded region Ω , if f is continuous on K and holomorphic in Ω , then

$$|f(z)| \leq \|f\|_{\partial\Omega}$$

for every $z \in \Omega$. If equality holds at one point $z \in \Omega$, then f is constant.

The equality $\|f\|_\infty = \|f^*\|_\infty$, which is part of Theorem 11.32, implies that

$$|f(z)| \leq \|f^*\|_\infty \quad z \in U, f \in H^\infty(U)$$

This time boundedness on U is enough; we do not need continuity on \overline{U} .

This chapter contains further generalizations of the maximum modulus theorem, as well as some rather striking applications of it, and it concludes with a theorem which shows that the maximum property "almost" characterizes the class of holomorphic functions.

7.24.1 The Schwarz lemma

Théorème 16: Schwarz lemma

Suppose $f \in H^\infty$, $\|f\|_\infty \leq 1$, and $f(0) = 0$. Then

$$|f(z)| \leq |z| \quad (z \in U),$$

$$|f'(0)| \leq 1;$$

if equality holds in (1) for one $z \in U - \{0\}$, or if equality holds in (2), then
 $f(z) = \lambda z$, where λ is a constant, $|\lambda| = 1$.

证明. Apply Théorème 15 to $g(z) = \frac{f(z)}{z}$ which has a removable singularity at 0. \square

For any $\alpha \in U$, define

$$\varphi_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}$$

Then φ_α is bijection, $\varphi_\alpha(T) = T$, $\varphi_\alpha(U) = U$, $\varphi_\alpha(\alpha) = 0$. The inverse is $\varphi_{-\alpha}$. We have

$$\varphi'_\alpha(0) = 1 - |\alpha|^2 \quad \varphi'_\alpha(\alpha) = \frac{1}{1 - |\alpha|^2} \quad (7.10)$$

Suppose α and β are complex numbers, $|\alpha| < 1$, and $|\beta| < 1$. How large can $|f'(\alpha)|$ be if f is subject to the conditions $f \in H^\infty$, $\|f\|_\infty \leq 1$, and $f(\alpha) = \beta$?

To solve this, put $g = \varphi_\beta \circ f \circ \varphi_{-\alpha}$. Clearly, $g \in H^\infty$, $\|g\|_\infty \leq 1$, $g(0) = 0$. By Théorème 16, $|g'(0)| \leq 1$. The chain rule gives

$$g'(0) = \varphi'_\beta(\beta)f'(\alpha)\varphi'_{-\alpha}(0)$$

Apply eq. (7.10), we have

$$|f'(\alpha)| \leq \frac{1 - |\beta|^2}{1 - |\alpha|^2} \quad (7.11)$$

The equality holds iff $|g'(0)| = 1$, in which case g is a rotation, thus

$$f(z) = \varphi_{-\beta}(\lambda \varphi_\alpha(z)) \quad z \in U \quad (7.12)$$

for some constant λ with $|\lambda| = 1$.

Note. We imposed no smooth condition on the behavior of f near ∂U .

12.6 Theorem Suppose $f \in H(U)$, f is one-to-one, $f(U) = U$, $\alpha \in U$, and $f(\alpha) = 0$. Then there is a constant λ , $|\lambda| = 1$, such that

$$f(z) = \lambda \varphi_\alpha(z) \quad (z \in U). \quad (1)$$

In other words, we obtain f by composing the mapping φ_α with a rotation.

证明. Let g be the inverse of f , $f(g(z)) = z$. As f is injective, $f'(z) \neq 0$, $g'(z) = 1/f'(z)$ exists for any $z \in U$, thus $g \in H(U)$. By the chain rule, $g'(0)f'(\alpha) = 1$. By eq. (7.11),

$$|f'(\alpha)| \leq \frac{1 - |f(\alpha)|^2}{1 - |\alpha|^2} = \frac{1}{1 - |\alpha|^2} \quad |g'(0)| \leq \frac{1 - |g(0)|^2}{1 - 0^2} = 1 - |\alpha|^2$$

thus the equality holds. By eq. (7.12), with $\beta = 0$, $f(z) = \lambda\varphi_\alpha(z)$ for some $|\lambda| = 1$. \square

7.24.2 The Phragmen-Lindelof Method

For unbounded regions, $\|f\|_\Omega = \|f\|_{\partial\Omega}$ for $f \in H(\Omega)$ is no longer true.

Exemple 22. Let $\Omega = \{z = x + iy : -\frac{\pi}{2} < y < \frac{\pi}{2}\}$, the boundary $\partial\Omega$ is $y = \pm\frac{\pi}{2}$. Put $f(z) = e^{e^z}$, for real x ,

$$f\left(x \pm \frac{\pi i}{2}\right) = \exp(\pm ie^x)$$

$\|f\|_{\partial\Omega} = 1$, but $\|f\|_\Omega = \infty$ since $\lim_{x \rightarrow \infty} f(x) = \infty$.

Théorème 17

Suppose

$$\Omega = \{x + iy : a < x < b\}, \quad \overline{\Omega} = \{x + iy : a \leq x \leq b\},$$

f is continuous on $\overline{\Omega}$, $f \in H(\Omega)$, and suppose that $|f(z)| < B$ for all $z \in \Omega$ and some fixed $B < \infty$. If

$$M(x) = \sup\{|f(x + iy)| : -\infty < y < \infty\} \quad (a \leq x \leq b)$$

then we actually have

$$M(x)^{b-a} \leq M(a)^{b-x} M(b)^{x-a} \quad (a < x < b).$$

i.e. $\log M$ is a convex function on (a, b) .

证明. WLOG, assume that $M(a) = M(b) = 1$. Otherwise, replace f by $\frac{f(x+iy)}{M(a)^{b-x} M(b)^{x-a}}$, which also satisfies the conditions, where $M(a)^{b-x} := \exp\{(b-x)\log M(a)\}$.

We need to show $|f(z)| \leq 1$ for all $z \in \Omega$. For each $\epsilon > 0$, define an auxiliary function

$$h_\epsilon(z) = \frac{1}{1 + \epsilon(z - a)} \quad z \in \overline{\Omega}$$

Since $\operatorname{Re}(1 + \epsilon(z - a)) = 1 + \epsilon(\operatorname{Re} z - a) \geq 1$, $|h_\epsilon| \leq 1$ in $\overline{\Omega}$, thus $|f(z)h_\epsilon(z)| \leq 1, \forall z \in \partial\Omega$. Also, $|1 + \epsilon(z - a)| \geq \epsilon|y|$, so that $|f(z)h_\epsilon(z)| \leq \frac{B}{\epsilon|y|}, \forall z = x + iy \in \overline{\Omega}$.

Let R_ϵ be the rectangle cut off from $\bar{\Omega}$ by $y = \pm \frac{B}{\epsilon}$, then $|fh_\epsilon| \leq 1$ on ∂R_ϵ ; hence on R_ϵ by the maximum modulus theorem.

For fixed $z \in \Omega$, $z \in R_\epsilon$ for any sufficiently small ϵ , then $|f(z)h_\epsilon(z)| \leq 1$; let $\epsilon \rightarrow 0$, then $h_\epsilon \rightarrow 1$, $|f(z)| \leq 1$. We are done! \square

12.9 Theorem Suppose

$$\Omega = \left\{ x + iy : |y| < \frac{\pi}{2} \right\}, \quad \bar{\Omega} = \left\{ x + iy : |y| \leq \frac{\pi}{2} \right\}. \quad (1)$$

Suppose f is continuous on $\bar{\Omega}$, $f \in H(\Omega)$, there are constants $\alpha < 1$, $A < \infty$, such that

$$|f(z)| < \exp \{A \exp(\alpha|x|)\} \quad (z = x + iy \in \Omega), \quad (2)$$

and

$$\left| f\left(x \pm \frac{\pi i}{2}\right) \right| \leq 1 \quad (-\infty < x < \infty). \quad (3)$$

Then $|f(z)| \leq 1$ for all $z \in \Omega$.

Note that the conclusion does not follow if $\alpha = 1$, as is shown by the function $\exp(\exp z)$.

7.24.3 A Converse of the Maximum Modulus Theorem

The letter j will denote the identity function: $j(z) = z$. The function which assigns the number 1 to each $z \in U$ will be denoted by 1.

12.13 Theorem Suppose M is a vector space of continuous complex functions on the closed unit disc \bar{U} , with the following properties:

- (a) $1 \in M$.
- (b) If $f \in M$, then also $jf \in M$.
- (c) If $f \in M$, then $\|f\|_U = \|f\|_T$.

Then every $f \in M$ is holomorphic in U .

Note. Note that (c) is a rather weak form of the maximum modulus principle; (c) asserts only that the overall maximum of $|f|$ on \bar{U} is attained at some point of the boundary T , but (c) does not a priori exclude the existence of local maxima of $|f|$ in U .

证明. By (a) and (b), M contains all polynomials. In conjunction with (c), this shows that M satisfies the hypotheses of Theorem 5.25. Thus every $f \in M$ is

harmonic in U . We shall use (b) to show that every $f \in M$ actually satisfies the Cauchy-Riemann equation.

Let ∂ and $\bar{\partial}$ be the differential operators introduced in Sec.11.1. The product rule for differentiation gives

$$(\partial\bar{\partial})(fg) = f \cdot (\partial\bar{\partial}g) + (\partial f) \cdot (\bar{\partial}g) + (\bar{\partial}f) \cdot (\partial g) + (\partial\bar{\partial}f) \cdot g.$$

Fix $f \in M$, and take $g = j$. Then $fj \in M$. Hence f and fj are harmonic, so $\partial\bar{\partial}f = 0$ and $(\partial\bar{\partial})(f) = 0$. Also, $\bar{\partial}j = 0$ and $\partial j = 1$. The above identity therefore reduces to $\bar{\partial}f = 0$. Thus $f \in H(U)$. \square

5.25 Theorem Suppose A is a vector space of continuous complex functions on the closed unit disc \bar{U} . If A contains all polynomials, and if

$$\sup_{z \in U} |f(z)| = \sup_{z \in T} |f(z)| \quad (1)$$

for every $f \in A$ (where T is the unit circle, the boundary of U), then the Poisson integral representation

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} f(e^{it}) dt \quad (z = re^{i\theta}) \quad (2)$$

is valid for every $f \in A$ and every $z \in U$.

12.14 Radó's Theorem Assume $f \in C(\bar{U})$, Ω is the set of all $z \in U$ at which $f(z) \neq 0$, and f is holomorphic in Ω . Then f is holomorphic in U .

Note. In particular, the theorem asserts that $U - \Omega$ is at most countable, unless $\Omega = \emptyset$.

7.24.4 Applications of Maximum Modulus Principle

Exercice 36: Indiana

Let $Q = [0, 1] \times [0, 1] \subset \mathbb{C}$ be the unit square, and let f be holomorphic in

a neighborhood of Q . Suppose that

$$f(z+1) - f(z) \text{ is real and } \geq 0 \quad \text{for } z \in [0, i]$$

$$f(z+i) - f(z) \text{ is real and } \geq 0 \quad \text{for } z \in [0, 1]$$

Show that f is constant.

证明. Because f is holomorphic on the closed unit square Q , by Cauchy integral theorem, we have

$$\begin{aligned} \int_{\partial Q} f(z) dz &= \int_0^1 f(x) dx + \int_0^1 f(1+yi) idy - \int_0^1 f(x+i) dx - \int_0^1 f(yi) idy \\ &= \int_0^1 (f(x) - f(x+i)) dx + i \int_0^1 (f(1+yi) - f(yi)) dy \\ &= 0 \end{aligned}$$

As

$$f(x) - f(x+i) \leq 0$$

for $0 \leq x \leq 1$ and

$$f(1+yi) - f(yi) \geq 0$$

for $0 \leq y \leq 1$, by comparing the real and imaginary parts in the above identity, we obtain that $f(x+i) = f(x)$ for $0 \leq x \leq 1$ and $f(1+yi) = f(yi)$ for $0 \leq y \leq 1$. Hence $f(z)$ can be analytically extended to a double-periodic function by

$$f(z) = f(z+1) = f(z+i)$$

which is holomorphic in \mathbb{C} and satisfies

$$|f(z)| \leq \max_{z \in Q} \{|f(z)|\} < +\infty$$

This shows that $f(z)$ must be a constant. □

Exercice 37

设 $f(z)$ 在区域 $0 < |z| < 1$ 内解析, $|f(z)| < \log \frac{1}{|z|}$, 证明 $f(z) \equiv 0$.

设 $g(z) = zf(z)$, 由于 $f(z)$ 在 $0 < |z| < 1$ 内解析, $g(z)$ 在 $0 < |z| < 1$ 内也解析.

$$|g(z)| = |z| \cdot |f(z)| < -|z| \cdot \ln |z|$$

函数 $h(r) = -r \ln r$ 在 $(0, 1)$ 上的最大值为 $\frac{1}{e}$, 并且当 $r \rightarrow 0^+$ 或 $r \rightarrow 1^-$ 时, $h(r) \rightarrow 0$. 这说明 $g(z)$ 在该区域内有界的.

由于 $|g(z)| \rightarrow 0$ 当 $|z| \rightarrow 0^+, |z| \rightarrow 1^-$. 我们可以将 $g(z)$ 在 $z = 0$ 处补定义 $g(0) = 0$, 从而 $g(z)$ 在单位圆盘 $|z| < 1$ 内解析, 并且 $|g(z)| < \frac{1}{e}$. 由于 $|g(z)| \rightarrow 0$ 当 $|z| \rightarrow 1^-$, 根据最大模原理. $g(z) \equiv 0, 0 < |z| < 1$, 所以 $f(z) \equiv 0$

7.25 Rudin Chapter 15

See also Stein II, Chapter 5.

7.25.1 Jensen's formula

Lemme 2

$$\frac{1}{2\pi} \int_0^{2\pi} \log |1 - e^{i\theta}| d\theta = 0$$

PROOF Let $\Omega = \{z: \operatorname{Re} z < 1\}$. Since $1 - z \neq 0$ in Ω and Ω is simply connected, there exists an $h \in H(\Omega)$ such that

$$\exp \{h(z)\} = 1 - z$$

in Ω , and this h is uniquely determined if we require that $h(0) = 0$. Since $\operatorname{Re}(1 - z) > 0$ in Ω , we then have

$$\operatorname{Re} h(z) = \log |1 - z|, \quad |\operatorname{Im} h(z)| < \frac{\pi}{2} \quad (z \in \Omega). \quad (1)$$

For small $\delta > 0$, let Γ be the path

$$\Gamma(t) = e^{it} \quad (\delta \leq t \leq 2\pi - \delta), \quad (2)$$

and let γ be the circular arc whose center is at 1 and which passes from $e^{i\delta}$ to $e^{-i\delta}$ within U . Then

$$\frac{1}{2\pi} \int_{\delta}^{2\pi-\delta} \log |1 - e^{i\theta}| d\theta = \operatorname{Re} \left[\frac{1}{2\pi i} \int_{\Gamma} h(z) \frac{dz}{z} \right] = \operatorname{Re} \left[\frac{1}{2\pi i} \int_{\gamma} h(z) \frac{dz}{z} \right]. \quad (3)$$

The last equality depended on Cauchy's theorem; note that $h(0) = 0$.

The length of γ is less than $\pi\delta$, so (1) shows that the absolute value of the last integral in (3) is less than $C\delta \log(1/\delta)$, where C is a constant. This gives the result if $\delta \rightarrow 0$ in (3). ////

证明. □

15.18 Theorem Suppose $\Omega = D(0; R)$, $f \in H(\Omega)$, $f(0) \neq 0$, $0 < r < R$, and $\alpha_1, \dots, \alpha_N$ are the zeros of f in $\bar{D}(0; r)$, listed according to their multiplicities. Then

$$|f(0)| \prod_{n=1}^N \frac{r}{|\alpha_n|} = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| d\theta \right\}. \quad (1)$$

Note. This is known as Jensen's formula. The hypothesis $f(0) \neq 0$ causes no harm in applications, for if f has a zero of order k at 0, the formula can be applied to $f(z)/z^k$.

PROOF Order the points α_j so that $\alpha_1, \dots, \alpha_m$ are in $D(0; r)$ and $|\alpha_{m+1}| = \dots = |\alpha_N| = r$. (Of course, we may have $m = N$ or $m = 0$.) Put

$$g(z) = f(z) \prod_{n=1}^m \frac{r^2 - \bar{\alpha}_n z}{r(\alpha_n - z)} \prod_{n=m+1}^N \frac{\alpha_n}{\alpha_n - z}. \quad (2)$$

Then $g \in H(D)$, where $D = D(0; r + \epsilon)$ for some $\epsilon > 0$, g has no zero in D , hence $\log |g|$ is harmonic in D (Theorem 13.12), and so

$$\log |g(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |g(re^{i\theta})| d\theta. \quad (3)$$

By (2),

$$|g(0)| = |f(0)| \prod_{n=1}^m \frac{r}{|\alpha_n|}. \quad (4)$$

For $1 \leq n \leq m$, the factors in (2) have absolute value 1 if $|z| = r$. If $\alpha_n = re^{i\theta_n}$ for $m < n \leq N$, it follows that

$$\log |g(re^{i\theta})| = \log |f(re^{i\theta})| - \sum_{n=m+1}^N \log |1 - e^{i(\theta - \theta_n)}|. \quad (5)$$

Lemma 15.17 shows therefore that the integral in (3) is unchanged if g is replaced by f . Comparison with (4) now gives (1). ////

证明.

□

Applications of Jensen's formula: zeros of entire function

15.20 Zeros of Entire Functions Suppose f is an entire function,

$$M(r) = \sup_{\theta} |f(re^{i\theta})| \quad (0 < r < \infty), \quad (1)$$

and $n(r)$ is the number of zeros of f in $\bar{D}(0; r)$. Assume $f(0) = 1$, for simplicity. Jensen's formula gives

$$M(2r) \geq \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(2re^{i\theta})| d\theta \right\} = \prod_{n=1}^{n(2r)} \frac{2r}{|\alpha_n|} \geq \prod_{n=1}^{n(r)} \frac{2r}{|\alpha_n|} \geq 2^{n(r)},$$

if $\{\alpha_n\}$ is the sequence of zeros of f , arranged so that $|\alpha_1| \leq |\alpha_2| \leq \dots$. Hence

$$n(r) \log 2 \leq \log M(2r). \quad (2)$$

Thus the rapidity with which $n(r)$ can increase (i.e., the density of the zeros of f) is controlled by the rate of growth of $M(r)$. Suppose, to look at a more specific situation, that for large r

$$M(r) < \exp \{Ar^k\} \quad (3)$$

where A and k are given positive numbers. Then (2) leads to

$$\limsup_{r \rightarrow \infty} \frac{\log n(r)}{\log r} \leq k. \quad (4)$$

For example, if k is a positive integer and

$$f(z) = 1 - e^{z^k}, \quad (5)$$

then $n(r)$ is about $\pi^{-1}kr^k$, so that

$$\lim_{r \rightarrow \infty} \frac{\log n(r)}{\log r} = k. \quad (6)$$

This shows that the estimate (4) cannot be improved.

7.26 Rudin Chapter 13

7.26.1 Simply Connected Regions

13.11 Theorem For a plane region Ω , each of the following nine conditions implies all the others.

- (a) Ω is homeomorphic to the open unit disc U .
- (b) Ω is simply connected.
- (c) $\text{Ind}_\gamma(\alpha) = 0$ for every closed path γ in Ω and for every $\alpha \in S^2 - \Omega$.
- (d) $S^2 - \Omega$ is connected.
- (e) Every $f \in H(\Omega)$ can be approximated by polynomials, uniformly on compact subsets of Ω .
- (f) For every $f \in H(\Omega)$ and every closed path γ in Ω ,

$$\int_\gamma f(z) dz = 0.$$

- (g) To every $f \in H(\Omega)$ corresponds an $F \in H(\Omega)$ such that $F' = f$.
- (h) If $f \in H(\Omega)$ and $1/f \in H(\Omega)$, there exists a $g \in H(\Omega)$ such that $f = \exp(g)$.
- (j) If $f \in H(\Omega)$ and $1/f \in H(\Omega)$, there exists a $\varphi \in H(\Omega)$ such that $f = \varphi^2$.

13.12 Theorem If $f \in H(\Omega)$, where Ω is any open set in the plane, and if f has no zero in Ω , then $\log |f|$ is harmonic in Ω .

PROOF To every disc $D \subset \Omega$ there corresponds a function $g \in H(D)$ such that $f = e^g$ in D . If $u = \operatorname{Re} g$, then u is harmonic in D , and $|f| = e^u$. Thus $\log |f|$ is harmonic in every disc in Ω , and this gives the desired conclusion. \square

7.27 Rudin Chapter 14

The direction of z is $A[z] := \frac{\bar{z}}{|z|}$.

Définition 8: preserves angles

Suppose f is a mapping of a region Ω into the plane, $z_0 \in \Omega$, and z_0 has a deleted neighborhood $D'(z_0; r) \subset \Omega$ in which $f(z) \neq f(z_0)$. We say that f preserves angles at z_0 if

$$\lim_{r \rightarrow 0} e^{-i\theta} A[f(z_0 + re^{i\theta}) - f(z_0)] \quad (r > 0) \tag{2}$$

exists and is independent of θ .

In less precise language, the requirement is that for any two rays L and L' ,

starting at z_0 , the angle which their images $f(L)$ and $f(L')$ make at $f(z_0)$ is the same as that made by L and L' , in size as well as in orientation.

The property of preserving angles at each point of a region is characteristic of holomorphic functions whose derivative has no zero in that region.

Définition 9

Suppose $\mathcal{F} \subset H(\Omega)$, for some region Ω . We call \mathcal{F} a **normal family** if every sequence of members of \mathcal{F} contains a subsequence which converges^a uniformly on compact subsets of Ω . The limit function is not required to belong to \mathcal{F} .

^amay converges to ∞

Théorème 18

Suppose $\mathcal{F} \subset H(\Omega)$ and \mathcal{F} is uniformly bounded on each compact subset of the region Ω . Then \mathcal{F} is a normal family.

7.27.1 The Riemann Mapping Theorem

14.7 Conformal Equivalence We call two regions Ω_1 and Ω_2 *conformally equivalent* if there exists a $\varphi \in H(\Omega_1)$ such that φ is one-to-one in Ω_1 and such that $\varphi(\Omega_1) = \Omega_2$, i.e., if there exists a conformal one-to-one mapping of Ω_1 onto Ω_2 .

The Riemann mapping theorem reduces the study of $H(\Omega)$ to $H(U)$, for any simply connected proper subregion of the plane.

14.8 Theorem *Every simply connected region Ω in the plane (other than the plane itself) is conformally equivalent to the open unit disc U .*

7.27.2 The class \mathcal{S}

Définition 10

\mathcal{S} is the class of all $f \in H(U)$ which are injective in U and which satisfy

$$f(0) = 0, \quad f'(0) = 1.$$

Thus every $f \in \mathcal{G}$ has a power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U).$$

The class \mathcal{S} is not closed under addition or multiplication, but has many other interesting properties. We shall develop only a few of these in this section.

Theorem 14.15 will be used in the proof of Mergelyan's theorem, in Chap. 20.

14.12 Theorem (a) If $f \in \mathcal{S}$, $|\alpha| = 1$, and $g(z) = \bar{\alpha}f(\alpha z)$, then $g \in \mathcal{S}$. (b) If $f \in \mathcal{S}$ there exists a $g \in \mathcal{S}$ such that

$$g^2(z) = f(z^2) \quad (z \in U). \quad (1)$$

证明. (a) is clear. To prove (b), write $f(z) = z\varphi(z)$. Then $\varphi \in H(U)$, $\varphi(0) = 1$, and φ has no zero in U since f has no zero in $U \setminus \{0\}$. Hence there exists $h \in H(U)$, with $h(0) = 1$, $h^2(z) = \varphi(z)$. Put

$$g(z) = zh(z^2) \quad z \in U$$

Then $g^2(z) = z^2 h^2(z^2) = z^2 \varphi(z^2) = f(z^2)$. NTS: $g \in \mathcal{S}$. Clearly, $g(0) = 0$, $g'(0) = 1$. For $z, w \in U$, $g(z) = g(w)$, then, $z^2 = w^2$; as f is injective, $z = \pm w$. If $z = -w$, then $g(z) = zh(z^2) = -wh(w^2) = -g(w)$; it follows that $g(z) = g(w) = 0$, and since g has no zero in $U \setminus \{0\}$, we have $z = w = 0$. \square

Théorème 19: area theorem (Theorem 14.13)

If $F \in H(U - \{0\})$, F is one-to-one in U , and

$$F(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \alpha_n z^n \quad (z \in U),$$

then

$$\sum_{n=1}^{\infty} n |\alpha_n|^2 \leq 1. \quad (7.13)$$

证明. WLOG, assume $\alpha_0 = 0$, α_1 is real. Put $U_r := \{|z| < r\}$, $C_r := \{|z| = r\}$, $V_r := \{r < |z| < 1\}$. Then $F(U_r)$ is a neighborhood of ∞ ; the sets $F(U_r)$, $F(C_r)$ and $F(V_r)$ are disjoint, as f is injective. $F(V_r)$ is in the interior of $F(C_r)$. Write

$$F(z) = \frac{1}{z} + \alpha_1 z + \varphi(z) \quad z \in U$$

$F = u + iv$ and

$$A = \frac{1}{r} + \alpha_1 r \quad B = \frac{1}{r} - \alpha_1 r$$

For $z = re^{i\theta}$, we then obtain

$$u = A \cos \theta + \operatorname{Re} \varphi \quad \text{and} \quad v = -B \sin \theta + \operatorname{Im} \varphi$$

Then

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1 + \underbrace{\frac{2 \cos \theta}{A}}_{=O(r)} \underbrace{\operatorname{Re} \varphi}_{=O(r^2)} + \underbrace{\left(\frac{\operatorname{Re} \varphi}{A} \right)^2}_{=O(r^6)} - \underbrace{\frac{2 \sin \theta}{B} \operatorname{Im} \varphi}_{=O(r^3)} + \underbrace{\left(\frac{\operatorname{Im} \varphi}{B} \right)^2}_{=O(r^6)}$$

There exists $\eta > 0$, if sufficiently small r , we have

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} < 1 + \eta r^3 \quad z = re^{i\theta}$$

This says $F(C_r)$ lies in the interior of ellipse E_r , whose semiaxes are $A\sqrt{1+\eta r^3}$ and $B\sqrt{1+\eta r^3}$, and which therefore bounds an area

$$\pi AB(1+\eta r^3) = \pi \left(\frac{1}{r} + \alpha_1 r \right) \left(\frac{1}{r} - \alpha_1 r \right) (1+\eta r^3) \leq \frac{\pi}{r^2} (1+\eta r^3) \quad (7.14)$$

Since $F(V_r)$ is in the interior of $F(C_r)$, the area of $F(V_r)$ is no larger than eq. (7.14). The Cauchy-Riemann equations show that the Jacobian of $(x, y) \rightarrow (u, v)$ is $|F'|^2$, then

$$\begin{aligned} \frac{\pi}{r^2} (1+\eta r^2) &\geq \iint_{V_r} |F'|^2 \\ &= \int_r^1 t dt \int_0^{2\pi} \left| -t^{-2} e^{-2i\theta} + \sum_{n=1}^{\infty} n \alpha_n t^{n-1} e^{i(n-1)\theta} \right| d\theta \\ &= 2\pi \int_r^1 \left(t^{-3} + \sum_{n=1}^{\infty} n^2 |\alpha_n|^2 t^{2n-1} \right) dt \\ &= \pi \left[r^{-2} - 1 + \sum_{n=1}^{\infty} n |\alpha_n|^2 (1 - r^{2n}) \right] \end{aligned}$$

Thus

$$\sum_{n=1}^N n |\alpha_n|^2 (1 - r^{2n}) \leq 1 + \eta r \quad \forall N > 0$$

Let $r \rightarrow 0$, then let $N \rightarrow \infty$. We are done! \square

Corollaire 5

Under the same hypothesis in eq. (7.13), $|\alpha_1| \leq 1$.

14.14 Theorem If $f \in \mathcal{S}$, and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

then (a) $|a_2| \leq 2$, and (b) $f(U) \supset D(0; \frac{1}{4})$.

The second assertion is that $f(U)$ contains all w with $|w| < \frac{1}{4}$.

Note. The second assertion is that $f(U)$ contains all w with $|w| < \frac{1}{4}$.

证明. (a) By theorem 14.12, there exists $f \in \mathcal{S}$ s.t. $g^2(z) = f(z^2)$. Denote $G := 1/g$, then eq. (7.13) applies to G and this yields (a). Since

$$f(z^2) = z^2 + \sum_{n=2}^{\infty} a_n z^{2n}$$

we have

$$g(z) = z \left(1 + \frac{1}{2} a_2 z^2 + \dots \right)$$

and hence

$$G(z) = \frac{1}{z} \left(1 - \frac{1}{2} a_2 z^2 + \dots \right)$$

By Corollaire 5, $|a_2| \leq 2$.

To prove (b), suppose $w \notin f(U)$. Define

$$h(z) = \frac{f(z)}{1 - f(z)/w}$$

Then $h \in H(U)$, h is injective in U , and

$$h(z) = (z + a_2 z^2 + \dots) \left(1 + \frac{z}{w} + \dots \right) = z + \left(a_2 + \frac{1}{w} \right) z^2 + \dots$$

so that $h \in \mathcal{S}$. Apply (a) to h : we have $|a_2 + \frac{1}{w}| < 2$ and since $|a_2| \leq 2$, we finally obtain $|1/w| \leq 4$. So $|w| \geq \frac{1}{4}$ for every $w \notin f(U)$. This complete the proof. \square

Moreover, given any $\alpha \neq 0$, one can find entire f with $f(0) = 0$, $f'(0) = 1$, that omit the value α . For example,

$$f(z) = \alpha(1 - e^{-z/\alpha})$$

Of course, by Theorem 14.14, that f is injective in U and $|\alpha| < \frac{1}{4}$ cannot happen at the same time.

14.15 Theorem Suppose $F \in H(U - \{0\})$, F is one-to-one in U , F has a pole of order 1 at $z = 0$, with residue 1, and neither w_1 nor w_2 are in $F(U)$.

Then $|w_1 - w_2| \leq 4$.

证明. Let $f := 1/(F - w_1)$, then $f \in \mathcal{S}$, hence $f(U) \supset B_{\frac{1}{4}}(0)$, so the image of U under $F - w_1$ contains all w with $|w| > 4$. Since $w_2 - w_1$ is not in this image, we have $|w_2 - w_1| \leq 4$. \square

7.28 Rudin Chapter 17

This chapter studies certain subspaces of $H(U)$ which are defined by certain growth conditions. These so-called H^p -spaces have a large number of interesting properties concerning factorizations, boundary values, and Cauchy-type representations in terms of measures on the boundary of U .

A convenient approach to the subject is via subharmonic functions, and we begin with a brief outline of their properties.

7.28.1 Subharmonic Functions

Définition 11

A function u defined in an open set Ω in the plane is said to be **subharmonic** if it has the following four properties.

- (1) $-\infty \leq u(z) < \infty$ for all $z \in \Omega$.
- (2) u is upper semicontinuous in Ω .
- (3) Whenever $D(a; r) \subset \Omega$, then

$$u(a) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a + re^{i\theta}) d\theta.$$

- (4) None of the integrals in (c) is $-\infty$.

Note that the integrals in (c) always exist and are not $+\infty$, since (a) and (b)

imply that u is bounded above on every compact $K \subset \Omega$.¹⁰ Hence (d) says that the integrands in (c) belong to $L^1(T)$.

Théorème 20: Theorem 17.2

If u is subharmonic in Ω , and if φ is a monotonically increasing convex function on \mathbb{R}^1 , then $\varphi \circ u$ is subharmonic.

Théorème 21: Theorem 17.3

If Ω is a region, $f \in H(\Omega)$, and f is not identically 0, then $\log|f|$ is subharmonic in Ω , and so are $\log^+|f|$ and $|f|^p(0 < p < \infty)$.^a

^aif f has no zero in Ω , $\log|f|$ is harmonic.

Théorème 22: Theorem 17.4

Suppose u is a continuous subharmonic function in Ω , K is a compact subset of Ω , h is a continuous real function on K which is harmonic in the interior V of K , and $u(z) \leq h(z)$ at all boundary points of K . Then $u(z) \leq h(z)$ for all $z \in K$.

Note. Clearly by maximum modulus principle. But we gives a proof without it.

证明. Put $u_1 = u - h$. Assume that $u_1(z) > 0$ for some $z \in V$. Since $u_1 \in C(K)$, u_1 attains its maximum m on K ; and since $u_1 \leq 0$ on the boundary of K , the set $E = \{z \in K : u_1(z) = m\}$ is a nonempty compact subset of V . Let z_0 be a boundary point of E . Then for some $r > 0$, we have $\overline{B_r(z_0)} \subset V$, but some subarc of the boundary of $\overline{B_r(z_0)}$ lies in the complement of E . Hence

$$u_1(z_0) = m > \frac{1}{2\pi} \int_{-\pi}^{\pi} u_1(z_0 + re^{i\theta}) d\theta$$

and this means u_1 is not subharmonic in V , which is a contradiction. \square

¹⁰Proof: If K_n is the set of all $z \in K$ at which $u(z) \geq n$, then $K \supset K_1 \supset K_2 \dots$, so either $K_n = \emptyset$ for some n , or $\bigcap K_n \neq \emptyset$, in which case $u(z) = \infty$ for some $z \in K$.

Théorème 23: Theorem 17.5

Suppose u is a continuous subharmonic function in U , and

$$m(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta}) d\theta \quad (0 \leq r < 1).$$

If $r_1 < r_2$, then $m(r_1) \leq m(r_2)$.

证明. Let h be the continuous function on $\overline{B_{r_2}(0)}$ which coincides with u on the boundary of $\overline{B_{r_2}(0)}$ and which is harmonic in $B_{r_2}(0)$. By Théorème 22, $u \leq h$ in $B_{r_2}(0)$. Hence

$$m(r_1) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} h(r_1 e^{i\theta}) d\theta = h(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(r_2 e^{i\theta}) d\theta = m(r_2)$$

□

7.28.2 The spaces H^p and N

Let σ denote Lebesgue measure on T , so normalized that $\sigma(T) = 1$.

$$\begin{aligned} \|f_r\|_p &= \left(\int_T |f_r|^p d\sigma \right)^{1/p} \quad 0 < p < \infty \\ \|f_r\|_\infty &= \sup_\theta |f(re^{i\theta})| \end{aligned}$$

and we also introduce

$$\|f_r\|_0 = \exp \left\{ \int_T \log^+ |f_r| d\sigma \right\}$$

Définition 12

If $f \in H(U)$ and $0 \leq p \leq \infty$, we put

$$\|f\|_p = \sup \{ \|f_r\|_p : 0 \leq r < 1 \}$$

If $0 < p \leq \infty$, H^p is defined to be the class of all $f \in H(U)$ for which $\|f\|_p < \infty$. (Note that this coincides with our previously introduced terminology in the case $p = \infty$.)

The class N consists of all $f \in H(U)$ for which $\|f\|_0 < \infty$.

It is clear that $H^\infty \subset H^p \subset H^s \subset N$ if $0 < s < p < \infty$.

7.29 hw1

1 p13 1(2)(3) 4 7 9(2) 11 14

Exercice 38

1. 计算:

$$(1) (1+i)\pm(1-2i) \text{ (并作图);}$$

$$(2) \frac{i}{(i-1)(i-2)(i-3)};$$

$$(3) \sqrt{2}(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta),$$

其中

$$0 < \alpha, \beta < \frac{\pi}{2}, \alpha = \arctan 2, \beta = \arctan 3.$$

$$\frac{i}{(i-1)(i-2)(i-3)} = \frac{i(i+1)(i+2)(i+3)}{(-2)(-5)(-10)} = \frac{1}{10}$$

$$\sqrt{2}(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) = \sqrt{2} \left(\frac{1}{\sqrt{5}} + i \frac{2}{\sqrt{5}} \right) \left(\frac{1}{\sqrt{10}} + i \frac{3}{\sqrt{10}} \right) = -1 + i$$

Exercice 39

4. 求复数 $\frac{z-1}{z+1}$ 的实部及虚部.

$z := a + ib$, then

$$\frac{z-1}{z+1} = \frac{a-1+ib}{a+1+ib} = \frac{(a-1+ib)(a+1-ib)}{(a+1)^2 - b^2} = \frac{a^2 - (1-ib)^2}{a^2 - b^2 + 2a + 1} = \frac{a^2 + b^2 + 2ib - 1}{a^2 - b^2 + 2a + 1}$$

Hence

$$\operatorname{Re} \left(\frac{z-1}{z+1} \right) = \frac{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2 - 1}{\operatorname{Re}(z)^2 - \operatorname{Im}(z)^2 + 2\operatorname{Re}(z) + 1}$$

$$\operatorname{Im} \left(\frac{z-1}{z+1} \right) = \frac{2\operatorname{Im}(z)}{\operatorname{Re}(z)^2 - \operatorname{Im}(z)^2 + 2\operatorname{Re}(z) + 1}$$

Exercice 40

7. 试证: 分别以 z_1, z_2, z_3 及 w_1, w_2, w_3 为顶点的两个三角形相似的必要与充分条件是

$$\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0.$$

[提示] 所给条件即 $\frac{z_2 - z_1}{z_3 - z_1} = \frac{w_2 - w_1}{w_3 - w_1}$.

Let $Z(\operatorname{Re}(z), \operatorname{Im}(z))$, then

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{w_2 - w_1}{w_3 - w_1} \iff \frac{|Z_1 Z_2|}{|Z_1 Z_3|} = \frac{|W_1 W_2|}{|W_1 W_3|} \text{ and } \angle Z_2 Z_1 Z_3 = \angle W_2 W_1 W_3 \iff \Delta Z_1 Z_2 Z_3 \sim \Delta W_1 W_2 W_3$$

Exercice 41

9. 应用棣莫弗公式, 证明:

$$(1) (1 + \cos \theta + i \sin \theta)^n = 2^n \cos^n \frac{\theta}{2} \left(\cos \frac{n\theta}{2} + i \sin \frac{n\theta}{2} \right);$$

$$(2) \cos 3\theta = 4\cos^3 \theta - 3\cos \theta \text{ 及 } \sin 3\theta = 3\sin \theta - 4\sin^3 \theta;$$

Définition 13: 棣莫弗公式

棣莫弗公式 (De Moivre's formula) 是复数和三角函数之间的一个重要关系, 它的数学表达式为:

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta),$$

$$\cos(3\theta) + i \sin(3\theta) = e^{3i\theta} = (\cos \theta + i \sin \theta)^3 = (4\cos^3 \theta - 3\cos \theta) + i(3\sin \theta - 4\sin^3 \theta)$$

Exercice 42

11. 求 $\frac{1}{2}(\sqrt{2}+i\sqrt{2})$ 的三次方根.

Solve the equation:

$$z^3 = \frac{1}{2}(\sqrt{2} + i\sqrt{2}) = e^{i\pi/4} = e^{i \cdot 9\pi/4} = e^{i \cdot 17\pi/4}$$

Then

$$z_1 = e^{i\pi/12}, z_2 = e^{i \cdot 3\pi/4}, z_3 = e^{i \cdot 17\pi/12}$$

Exercice 43

14. 设 $|z_0| < 1$. 证明:

如果 $|z| = 1$, 那么

$$\left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| = 1.$$

如果 $|z| < 1$, 那么

$$(1) \quad \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| < 1;$$

$$(2) \quad 1 - \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right|^2 = \frac{(1 - |z_0|^2)(1 - |z|^2)}{|1 - \bar{z}_0 z|^2};$$

$$(3) \quad \frac{|z| - |z_0|}{1 - |z_0||z|} \leq \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| \leq \frac{|z| + |z_0|}{1 + |z_0||z|};$$

$$(4) \quad \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| \leq |z| + |z_0|.$$

[提示] 利用第 5 题(1).

5. 设 z_1 及 z_2 是两复数. 求证:

$$(1) \quad |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2 \operatorname{Re}(z_1 \overline{z_2});$$

Note.

If $|z| = 1$, then

$$|1 - \bar{z}_0 z| = |1 - \operatorname{Re}(\bar{z}_0 z) - i \cdot \operatorname{Im}(\bar{z}_0 z)| = \sqrt{1 + |\bar{z}_0 z|^2 - 2 \operatorname{Re}(\bar{z}_0 z)} = \sqrt{1 + |z_0|^2 - 2 \operatorname{Re}(\bar{z}_0 z)}$$

On the other hand

$$|z - \bar{z}_0| = \sqrt{1 + |z_0|^2 - 2\operatorname{Re}(\bar{z}_0 z)}$$

Therefore

$$\left| \frac{z - \bar{z}_0}{1 - \bar{z}_0 z} \right| = 1$$

If $|z| < 1$, we have $(|z|^2 - 1)(|z_0|^2 - 1) > 0 \Rightarrow 1 + |z|^2|z_0|^2 > |z|^2 + |z_0|^2$ then

$$\left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| = \frac{\sqrt{|z|^2 + |z_0|^2 - 2\operatorname{Re}(\bar{z}_0 z)}}{\sqrt{1 + |z_0|^2|z|^2 - 2\operatorname{Re}(\bar{z}_0 z)}} < 1$$

And

$$1 - \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right|^2 = \frac{1 + |z_0|^2|z|^2 - |z|^2 - |z_0|^2}{1 + |z_0|^2|z|^2 - 2\operatorname{Re}(\bar{z}_0 z)} = \frac{(1 - |z_0|^2)(1 - |z|^2)}{|1 - \bar{z}_0 z|^2}$$

We have

$$\left(\frac{|z| - |z_0|}{1 - |z_0||z|} \right)^2 = \frac{|z|^2 + |z_0|^2 - 2|\bar{z}_0||z|}{1 + |z_0|^2|z|^2 - 2|z_0||z|}, \quad \left(\frac{|z| + |z_0|}{1 + |z_0||z|} \right)^2 = \frac{|z|^2 + |z_0|^2 + 2|\bar{z}_0||z|}{1 + |z_0|^2|z|^2 + 2|z_0||z|}$$

Consider the function

$$f(x) = \frac{|z|^2 + |z_0|^2 - x}{1 + |z_0|^2|z|^2 - x} = 1 - \frac{(1 - |z|^2)(1 - |z_0|^2)}{1 + |z_0|^2|z|^2 - x}$$

which is decreasing in $(-\infty, 1 + |z_0|^2|z|^2)$. Since $-2|\bar{z}_0||z| \leq 2\operatorname{Re}(\bar{z}_0 z) \leq 2|\bar{z}_0||z| < 1 + |z_0|^2|z|^2$, we have $f(-2|\bar{z}_0||z|) \geq f(2\operatorname{Re}(\bar{z}_0 z)) \geq f(2|\bar{z}_0||z|)$, that is

$$\frac{|z| - |z_0|}{1 - |z_0||z|} \leq \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| \leq \frac{|z| + |z_0|}{1 + |z_0||z|}$$

Therefore,

$$\left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| \leq \frac{|z| + |z_0|}{1 + |z_0||z|} \leq |z| + |z_0|$$

7.30 hw2

1 [p13 1(2)(3) 4 7 9(2) 11 14]

Exercice 44

1. 计算：

$$(1) (1+i) \pm (1-2i) \text{ (并作图)};$$

$$(2) \frac{i}{(i-1)(i-2)(i-3)};$$

$$(3) \sqrt{2}(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta),$$

其中

$$0 < \alpha, \beta < \frac{\pi}{2}, \alpha = \arctan 2, \beta = \arctan 3.$$

$$\frac{i}{(i-1)(i-2)(i-3)} = \frac{i(i+1)(i+2)(i+3)}{(-2)(-5)(-10)} = \frac{1}{10}$$

$$\sqrt{2}(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) = \sqrt{2} \left(\frac{1}{\sqrt{5}} + i \frac{2}{\sqrt{5}} \right) \left(\frac{1}{\sqrt{10}} + i \frac{3}{\sqrt{10}} \right) = -1 + i$$

Exercice 45

4. 求复数 $\frac{z-1}{z+1}$ 的实部及虚部.

$z := a + ib$, then

$$\frac{z-1}{z+1} = \frac{a-1+ib}{a+1+ib} = \frac{(a-1+ib)(a+1-ib)}{(a+1)^2 - b^2} = \frac{a^2 - (1-ib)^2}{a^2 - b^2 + 2a + 1} = \frac{a^2 + b^2 + 2ib - 1}{a^2 - b^2 + 2a + 1}$$

Hence

$$\operatorname{Re} \left(\frac{z-1}{z+1} \right) = \frac{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2 - 1}{\operatorname{Re}(z)^2 - \operatorname{Im}(z)^2 + 2\operatorname{Re}(a) + 1}$$

$$\operatorname{Im} \left(\frac{z-1}{z+1} \right) = \frac{2\operatorname{Im}(z)}{\operatorname{Re}(z)^2 - \operatorname{Im}(z)^2 + 2\operatorname{Re}(a) + 1}$$

Exercice 46

7. 试证：分别以 z_1, z_2, z_3 及 w_1, w_2, w_3 为顶点的两个三角形相似的必要与充分条件是

$$\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0.$$

[提示] 所给条件即 $\frac{z_2 - z_1}{z_3 - z_1} = \frac{w_2 - w_1}{w_3 - w_1}$.

Let $Z(\operatorname{Re}(z), \operatorname{Im}(z))$, then

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{w_2 - w_1}{w_3 - w_1} \iff \frac{|Z_1 Z_2|}{|Z_1 Z_3|} = \frac{|W_1 W_2|}{|W_1 W_3|} \text{ and } \angle Z_2 Z_1 Z_3 = \angle W_2 W_1 W_3 \iff \Delta Z_1 Z_2 Z_3 \sim \Delta W_1 W_2 W_3$$

Exercice 47

9. 应用棣莫弗公式，证明：

$$(1) (1 + \cos \theta + i \sin \theta)^n = 2^n \cos^n \frac{\theta}{2} \left(\cos \frac{n\theta}{2} + i \sin \frac{n\theta}{2} \right);$$

$$(2) \cos 3\theta = 4\cos^3 \theta - 3\cos \theta \text{ 及 } \sin 3\theta = 3\sin \theta - 4\sin^3 \theta;$$

Définition 14: 棣莫弗公式

棣莫弗公式 (De Moivre's formula) 是复数和三角函数之间的一个重要关系，它的数学表达式为：

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta),$$

$$\cos(3\theta) + i \sin(3\theta) = e^{3i\theta} = (\cos \theta + i \sin \theta)^3 = (4 \cos^3 \theta - 3\cos \theta) + i(3\sin \theta - 4\sin^3 \theta)$$

Exercice 48

11. 求 $\frac{1}{2}(\sqrt{2} + i\sqrt{2})$ 的三次方根。

Solve the equation:

$$z^3 = \frac{1}{2}(\sqrt{2} + i\sqrt{2}) = e^{i\pi/4} = e^{i \cdot 9\pi/4} = e^{i \cdot 17\pi/4}$$

Then

$$z_1 = e^{i\pi/12}, z_2 = e^{i \cdot 3\pi/4}, z_3 = e^{i \cdot 17\pi/12}$$

Exercice 49

14. 设 $|z_0| < 1$. 证明:

如果 $|z| = 1$, 那么

$$\left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| = 1.$$

如果 $|z| < 1$, 那么

$$(1) \quad \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| < 1;$$

$$(2) \quad 1 - \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right|^2 = \frac{(1 - |z_0|^2)(1 - |z|^2)}{|1 - \bar{z}_0 z|^2};$$

$$(3) \quad \frac{|z| - |z_0|}{1 - |z_0||z|} \leq \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| \leq \frac{|z| + |z_0|}{1 + |z_0||z|};$$

$$(4) \quad \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| \leq |z| + |z_0|.$$

[提示] 利用第 5 题(1).

5. 设 z_1 及 z_2 是两复数. 求证:

$$(1) \quad |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1 \bar{z}_2);$$

Note.

If $|z| = 1$, then

$$|1 - \bar{z}_0 z| = |1 - \operatorname{Re}(\bar{z}_0 z) - i \cdot \operatorname{Im}(\bar{z}_0 z)| = \sqrt{1 + |\bar{z}_0 z|^2 - 2\operatorname{Re}(\bar{z}_0 z)} = \sqrt{1 + |z_0|^2 - 2\operatorname{Re}(\bar{z}_0 z)}$$

On the other hand

$$|z - \bar{z}_0| = \sqrt{1 + |z_0|^2 - 2\operatorname{Re}(\bar{z}_0 z)}$$

Therefore

$$\left| \frac{z - \bar{z}_0}{1 - \bar{z}_0 z} \right| = 1$$

If $|z| < 1$, we have $(|z|^2 - 1)(|z_0|^2 - 1) > 0 \Rightarrow 1 + |z|^2|z_0|^2 > |z|^2 + |z_0|^2$ then

$$\left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| = \frac{\sqrt{|z|^2 + |z_0|^2 - 2\operatorname{Re}(\bar{z}_0 z)}}{\sqrt{1 + |z_0|^2|z|^2 - 2\operatorname{Re}(\bar{z}_0 z)}} < 1$$

And

$$1 - \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right|^2 = \frac{1 + |z_0|^2|z|^2 - |z|^2 - |z_0|^2}{1 + |z_0|^2|z|^2 - 2\operatorname{Re}(\bar{z}_0 z)} = \frac{(1 - |z_0|^2)(1 - |z|^2)}{|1 - \bar{z}_0 z|^2}$$

We have

$$\left(\frac{|z| - |z_0|}{1 - |z_0||z|} \right)^2 = \frac{|z|^2 + |z_0|^2 - 2|\bar{z}_0||z|}{1 + |z_0|^2|z|^2 - 2|z_0||z|}, \quad \left(\frac{|z| + |z_0|}{1 + |z_0||z|} \right)^2 = \frac{|z|^2 + |z_0|^2 + 2|\bar{z}_0||z|}{1 + |z_0|^2|z|^2 + 2|z_0||z|}$$

Consider the function

$$f(x) = \frac{|z|^2 + |z_0|^2 - x}{1 + |z_0|^2|z|^2 - x} = 1 - \frac{(1 - |z|^2)(1 - |z_0|^2)}{1 + |z_0|^2|z|^2 - x}$$

which is decreasing in $(-\infty, 1 + |z_0|^2|z|^2)$. Since $-2|\bar{z}_0||z| \leq 2\operatorname{Re}(\bar{z}_0 z) \leq 2|\bar{z}_0||z| < 1 + |z_0|^2|z|^2$, we have $f(-2|\bar{z}_0||z|) \geq f(2\operatorname{Re}(\bar{z}_0 z)) \geq f(2|\bar{z}_0||z|)$, that is

$$\frac{|z| - |z_0|}{1 - |z_0||z|} \leq \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| \leq \frac{|z| + |z_0|}{1 + |z_0||z|}$$

Therefore,

$$\left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| \leq \frac{|z| + |z_0|}{1 + |z_0||z|} \leq |z| + |z_0|$$

7.31 hw3

1	p15 16 17(3) (4) (8) (9) (10)
2	p38 3(3) (5) 4 5 6 8 10

16. 指出下列点集中哪些是开集,哪些是闭集,哪些是紧集:

(1) 全体整数集; (2) 有限个复数的集;

(3) $\{z \in \mathbf{C} : \operatorname{Im} z > 0\} \setminus \bigcup_{k=-\infty}^{+\infty} \{z \in \mathbf{C} : z = k + iy, y \in [0, 1]\}$;

(4) \mathbf{C}, \mathbf{C}_* 及空集 \emptyset .

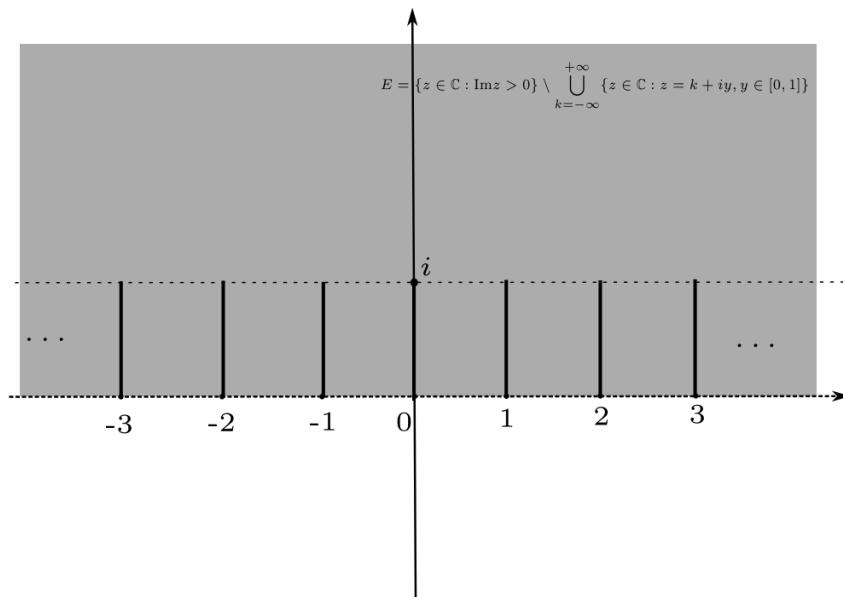
在复平面拓扑下:

(1) \mathbb{Z} 不含内点, 故不是开集, \mathbb{Z} 没有聚点, 故为闭集。由于 \mathbb{C} 同构于 \mathbb{R}^2 , 故 $E \subset \mathbb{C}$ 是紧集当且仅当 E 是有界闭集, 但 \mathbb{Z} 并非有界, 故 \mathbb{Z} 不是紧集。

(2) 同理于 (1), 该集合不是开集, 是闭集, 是紧集。

(3) 令

$$E = \{z \in \mathbb{C} : \operatorname{Im} z > 0\} \setminus \bigcup_{k=-\infty}^{+\infty} \{z \in \mathbb{C} : z = k + iy, y \in [0, 1]\}$$



所有点都是内点, 所以 E 是开集。而任意 $z = k + iy, k \in \mathbb{Z}, y \in [0, 1]$ 作为极限点, 却不在 E 中, 故 E 不是闭集, 进而不是紧集。

(4) 显然 \mathbb{C} 是开集, 不是闭集, 不是紧集。 \mathbb{C}_∞ 是闭集, 是开集, 是紧集。 \emptyset 是开集, 是闭集, 是紧集。

Remarque 26. \mathbb{C}_∞ 的定义是黎曼球极投影得到的, 它是依照 \mathbb{R}^3 中的 S^2 表面拓扑而定的。它是 \mathbb{C} 的紧致化。

17. 满足下列条件的点 z 所组成的点集是什么图形？如果是区域，是单连通区域还是多连通区域？

$$(1) \operatorname{Im} z = 3; \quad (2) \operatorname{Re} z > \frac{1}{2};$$

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第一章 复数及复平面

$$(3) |z-i| \leq |2+i|; \quad (4) |z-2| + |z+2| = 5;$$

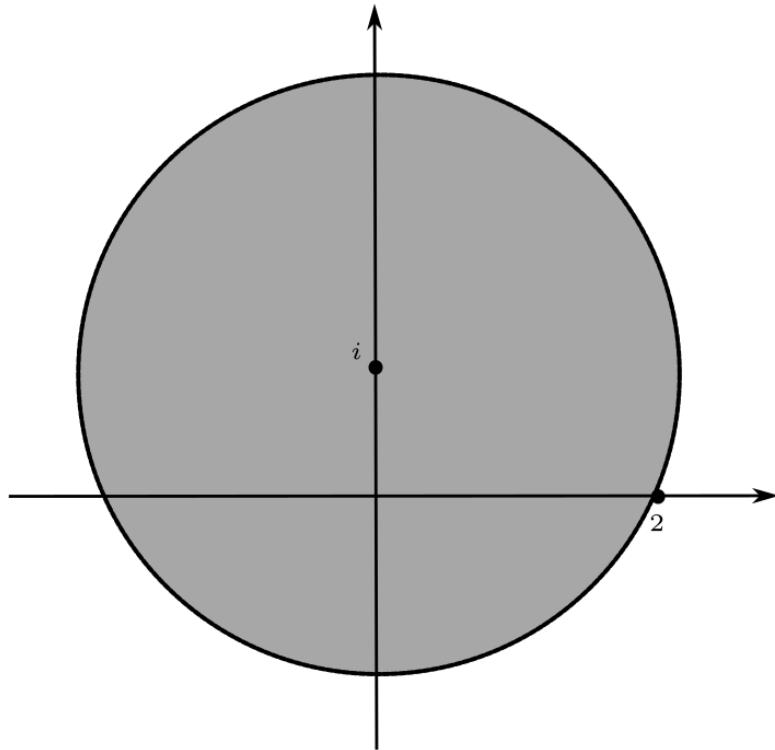
$$(5) \arg(z-i) = \frac{\pi}{4}; \quad (6) |z| < 1, \operatorname{Re} z \leq \frac{1}{2};$$

$$(7) 0 < |z+1+i| < 2; \quad (8) \left| \frac{z-1}{z+1} \right| \leq 2;$$

$$(9) 0 < \arg(z-1) < \frac{\pi}{4}, 2 < \operatorname{Re} z < 3;$$

$$(10) 0 < \arg \frac{z-i}{z+i} < \frac{\pi}{4}.$$

(3)



单连通的闭圆盘

(4)

$$|z - 2| + |z + 2| = 5$$

设 $z = x + iy$, 那么

$$\sqrt{(x - 2)^2 + y^2} + \sqrt{(x + 2)^2 + y^2} = 5$$

平方得到

$$(x - 2)^2 + y^2 = (5 - \sqrt{(x + 2)^2 + y^2})^2 = 25 + (x + 2)^2 + y^2 - 10\sqrt{(x + 2)^2 + y^2}$$

化简得到

$$10\sqrt{(x + 2)^2 + y^2} = 25 + 8x$$

平方得到

$$100(x + 2)^2 + 100y^2 = 625 + 400x + 64x^2$$

化简得到

$$36x^2 + 100y^2 = 625$$

是一个椭圆, 显然单连通。

(8)

$$\left| \frac{z - 1}{z + 1} \right| \leq 2$$

设 $z = x + iy$, 则

$$\sqrt{(x + 1)^2 + y^2} \neq 0, \quad \sqrt{(x - 1)^2 + y^2} \leq 2\sqrt{(x + 1)^2 + y^2}$$

于是 $(x, y) \neq (-1, 0)$. 平方得到

$$x^2 - 2x + 1 + y^2 \leq 4x^2 + 8x + 4 + 4y^2$$

化简得到

$$3x^2 + 10x + 3y^2 + 3 \geq 0$$

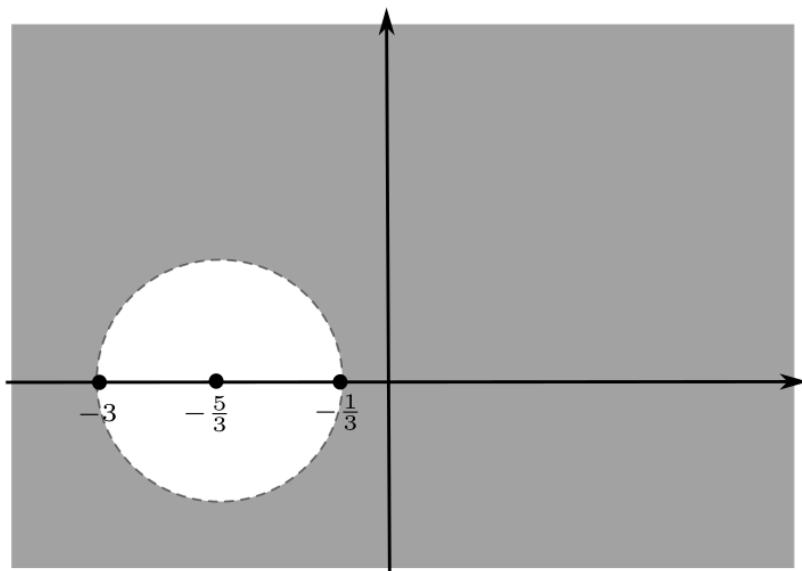
也就是说

$$\left(x^2 + \frac{10}{3}x + \frac{25}{9} \right) + y^2 \geq \frac{16}{9}$$

即

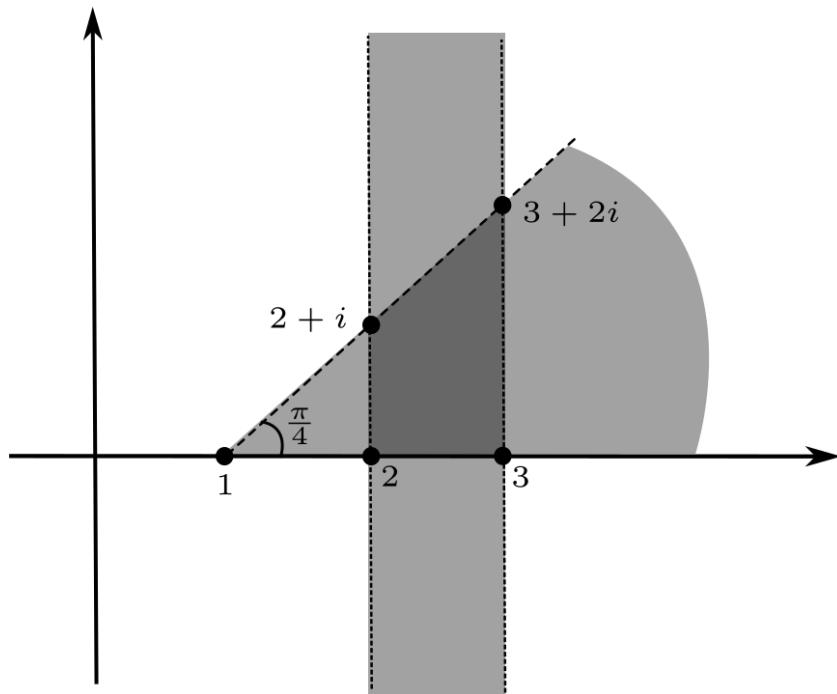
$$\left(x + \frac{5}{3} \right)^2 + y^2 \geq \frac{16}{9} \text{ 且 } (x, y) \neq (-1, 0)$$

这是复平面挖掉一个圆盘, 显然单连通。



(9)

$$0 < \arg(z - 1) < \frac{\pi}{4}, 2 < \operatorname{Re} z < 3$$



这是单连通区域

(10)

$$0 < \arg \frac{z-i}{z+i} < \frac{\pi}{4}$$

设 $z = x + iy$, 则

$$\frac{z-i}{z+i} = \frac{x+i(y-1)}{x+i(y+1)} = \frac{(x+i(y-1))(x-i(y+1))}{x^2+(y+1)^2} = \frac{x^2+y^2-1-2xi}{x^2+(y+1)^2}$$

于是, 对于 $x = 0$, $\arg \frac{z-i}{z+i} = 0$, 对于 $x \neq 0$, 则

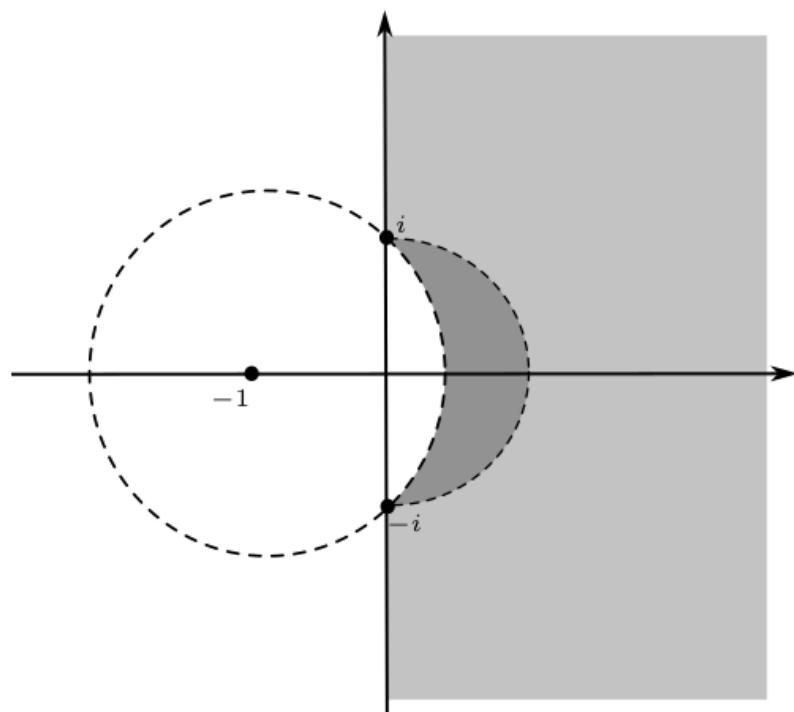
$$\arg \frac{z-i}{z+i} = \arctan \frac{x^2+y^2-1}{-2x} \in \left(0, \frac{\pi}{4}\right) \Rightarrow \frac{x^2+y^2-1}{-2x} \in (0, 1)$$

若 $x > 0$, 则

$$-2x < x^2 + y^2 - 1 < 0$$

也就是说

$$x^2 + y^2 < 1 \text{ 且 } (x+1)^2 + y^2 > 2$$

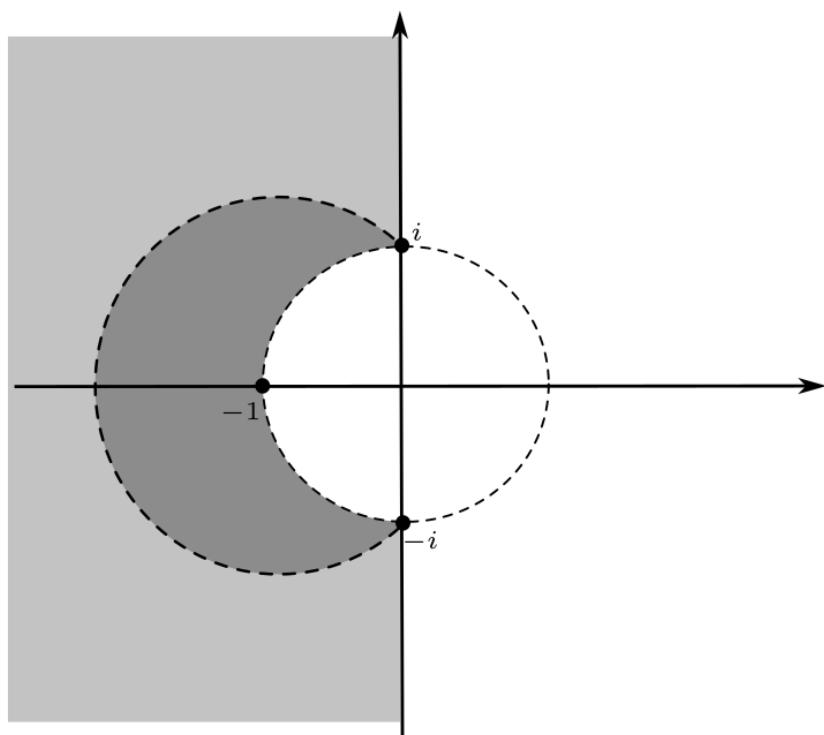


若 $x < 0$, 则

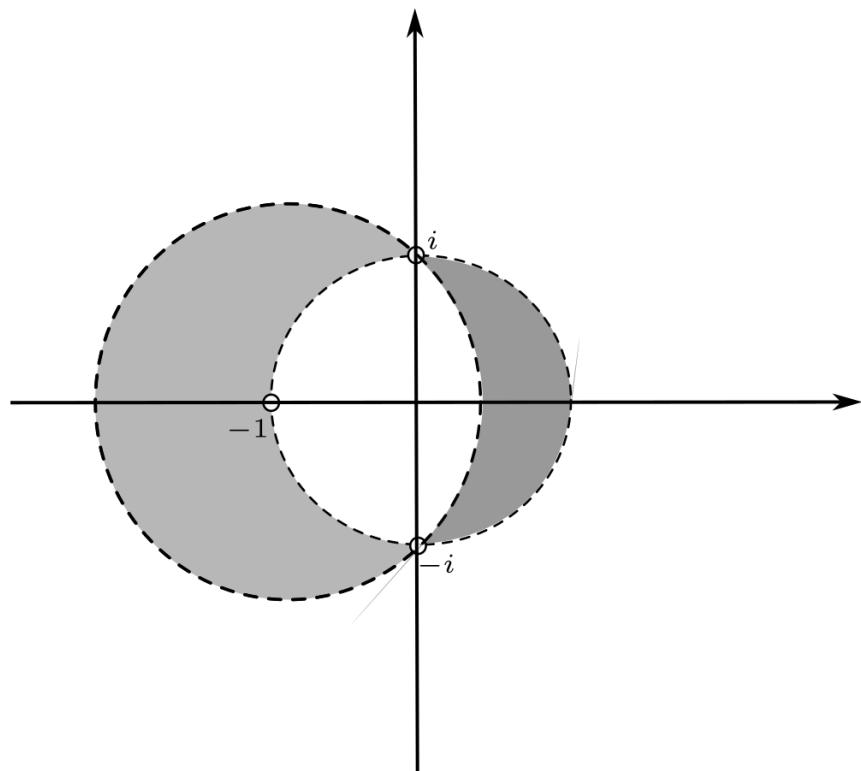
$$0 < x^2 + y^2 - 1 < -2x$$

也就是说

$$x^2 + y^2 > 1 \text{ 且 } (x+1)^2 + y^2 < 2$$



总而言之，该区域为



是多连通的开区域

3. 下列函数在何处可微? 在何处解析?

- | | | |
|----------------------------|---|---------------------|
| (1) $f(z) = x^2 - iy;$ | (2) $f(z) = 2x^3 + iy^3;$ | (3) $f(z) = z ^2;$ |
| (4) $f(z) = xy^2 + ix^2y;$ | (5) $e^{x^2-y^2} \cos 2xy + ie^{x^2-y^2} \sin 2xy.$ | |

Définition 15: 解析函数

定义 如果函数 $f(z)$ 在区域 D 内每一点可微, 那么 $f(z)$ 称为在区域 D 内解析. 如果 $f(z)$ 在 z_0 的一个邻域内解析, 那么我们说 $f(z)$ 在 z_0 解析. 如果 $f(z)$ 在区域 G 内解析, 而闭区域 \bar{D} 上每一点都属于 G , 那么我们说 $f(z)$ 在 \bar{D} 上解析. 如果 $f(z)$ 在区域 D 内某些点以外, 到处解析, 那么这些例外点称为 $f(z)$ 的奇点; $f(z)$ 不在奇点的任一邻域内解析.

Théorème 24

定理 3.1 设函数 $f(z) = u(x, y) + iv(x, y)$ 在区域 D 内确定, 那么 $f(z)$ 在点

§ 1. 解析函数

· 23 ·

$z=x+iy \in D$ 可微的必要与充分条件是: 在点 $z=x+iy$, $u(x, y)$ 及 $v(x, y)$ 可微, 并且

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad (3.1)$$

Théorème 25

定理 3.2 设函数 $f(z) = u(x, y) + iv(x, y)$ 在区域 D 内确定. $f(z)$ 在区域 D 内解析的必要与充分条件是: $u(x, y)$ 及 $v(x, y)$ 在 D 内可微, 而且在 D 内 (3.1) 成立.

(3) $f(z) = x^2 + y^2 =: u(x, y) + iv(x, y)$ so $u_x = 2x, u_y = 2y, v_x = v_y = 0$. If $f(z) = |z|^2$ is differentiable at $z_0 = x_0 + iy_0$ then $u_x = v_y, u_y = -v_x$ implies that $x_0 = y_0 = 0$ thus $z_0 = 0$. Hence 0 is the only point at which f is differentiable. For any neighborhood U of 0, f is not differentiable in $U \setminus \{0\}$ thus f is not holomorphic over U . Therefore f is holomorphic at no point.
(5) $f(z) = e^{x^2-y^2} \cos(2xy) + ie^{x^2-y^2} \sin(2xy) =: u(x, y) + iv(x, y)$ so $u_x = 2x \cos(2xy)e^{x^2-y^2} - 2y \sin(2xy)e^{x^2-y^2} = v_y$, and $u_y = -2y \cos(2xy)e^{x^2-y^2} - 2x \sin(2xy)e^{x^2-y^2} = -v_x$.

4. 设函数 $f(z)$ 在区域 D 内解析. 证明: 如果对每一点 $z \in D$, 有

$$f'(z) = 0,$$

那么 $f(z)$ 在 D 内为常数.

$f(z) := u(x, y) + iv(x, y)$ for $z = x+iy$, since $f \in H(D)$ then $\bar{\partial}f(z) = 0, z \in D$ where $\bar{\partial} := \frac{1}{2}(\partial_x + i\partial_y)$, and $\partial f(z) = f'(z) = 0, z \in D$ where $\partial := \frac{1}{2}(\partial_x - i\partial_y)$, we have $\partial_x f = 0, \partial_y f = 0$, i.e. $u_x = v_x = 0, u_y = v_y = 0$. (the differentiability is derived from $f \in H(D)$ and theorem 3.2) Therefore $u \equiv c_1, v \equiv c_2$ constant, and $f \equiv u + iv = c_1 + c_2$ is constant in D .

5. 设函数 $f(z)$ 在区域 D 内解析. 证明: 如果 $f(z)$ 满足下列条件之一, 那么它在 D 内为常数:

- (1) $\operatorname{Re} f(z)$ 或 $\operatorname{Im} f(z)$ 在 D 内为常数;
- (2) $|f(z)|$ 在 D 内为常数.

(1) $f(z) := u(x, y) + iv(x, y)$ for $z = x + iy$. $f \in H(D)$ means $u, v \in H(D)$ and $\bar{\partial}f(z) = 0, z \in D$, i.e. $\frac{1}{2}(\partial_x f(z) + i\partial_y f(z)) = \frac{1}{2}(\partial_x u + i\partial_x v + i\partial_y u - \partial_y v) = 0$. If $u = \operatorname{Re} f$ is constant in D then $\partial_x u = \partial_y u = 0$, combined with the fact that $\partial_x u - \partial_y v = 0$ and $\partial_x v + \partial_y u = 0$, we have $\partial_x u = \partial_y u = \partial_x v = \partial_y v = 0$. Therefore f is constant in D . There is a similar proof for $\operatorname{Im} f \equiv \text{Const.}$ in D .

(2) If $|f| \equiv c$ is a constant in D then $\bar{f} = \frac{c^2}{f}$ is holomorphic in D . However, if both $u + iv$ and $u - iv$ satisfies the C-R equation, it is easy to get $u_x = u_y = v_x = v_y = 0$ and thus f is constant.

6. 证明: 若函数 $f(z)$ 在上半平面解析, 那么函数 $\overline{f(\bar{z})}$ 在下半平面解析.

$f(z) := u(x, y) + iv(x, y)$ for $z = x + iy$. $g(z) := \overline{f(\bar{z})} = u(x, -y) - iv(x, -y)$. By theorem 3.2 it suffices to show that $u(x, -y)$ and $v(x, -y)$ are differentiable in \mathbb{R}_- and $\bar{\partial}g(z) = 0$ for $(x, y) \in \mathbb{R}_-$. By theorem 3.2 we have $u(x, y), v(x, y)$ are differentiable in \mathbb{R}_+^2 and $\bar{\partial}f(z) = 0$, i.e. $\partial_x u = \partial_y v, \partial_x v = -\partial_y u$.

$$\begin{aligned}\bar{\partial}g(z) &= \frac{1}{2}(\partial_x + i\partial_y)(u(x, -y) + iv(x, -y)) \\ &= \frac{1}{2}\partial_x u(x, -y) - \frac{1}{2}i\partial_x v(x, -y) + \frac{1}{2}i\partial_y u(x, -y) - \frac{1}{2}\partial_y u(x, -y) \\ &= \frac{1}{2}(u_x + v_y)(x, -y) + \frac{1}{2}i(u_y - v_x)(x, -y) = 0 \quad \text{where } (x, -y) \in \mathbb{R}_+^2\end{aligned}$$

Therefore $\overline{f(\bar{z})}$ is holomorphic in the lower half plane.

8. 证明在极坐标下函数 $f(z) = u(x, y) + iv(x, y)$ 的柯西-黎曼条件是:

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}.$$

$x = r \cos \theta, y = r \sin \theta$. So we have

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta \\ \frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial v}{\partial x} \cos \theta + \frac{\partial v}{\partial y} \sin \theta \\ \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r \sin \theta) + \frac{\partial u}{\partial y} r \cos \theta \\ \frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x}(-r \sin \theta) + \frac{\partial v}{\partial y} r \cos \theta\end{aligned}$$

Then

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \iff u_x r \cos \theta + u_y r \sin \theta = -v_x r \sin \theta + v_y r \cos \theta \iff (u_x - v_y)x = -(v_x + u_y)y$$

And

$$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r} \iff -u_x r \sin \theta + u_y r \cos \theta = -v_x r \cos \theta - v_y r \sin \theta \iff (u_x - v_y)y = (u_y + v_x)x$$

The Cauchy-Riemann condition is $u_x = v_y, u_y + v_x = 0$, which means

$$u_r \cdot \frac{\partial r}{\partial x} + u_\theta \cdot \frac{\partial \theta}{\partial x} = v_r \cdot \frac{\partial r}{\partial y} + v_\theta \cdot \frac{\partial \theta}{\partial y} \text{ and } u_r \cdot \frac{\partial r}{\partial y} + u_\theta \cdot \frac{\partial \theta}{\partial y} + v_r \cdot \frac{\partial r}{\partial x} + v_\theta \cdot \frac{\partial \theta}{\partial x} = 0$$

where $r = \sqrt{x^2 + y^2}, \theta = \arctan \frac{y}{x}$ then

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r}, \frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{y}{r^2}, \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{x}{r^2}$$

Therefore

$$u_r \frac{x}{r} - u_\theta \frac{y}{r^2} = v_r \frac{y}{r} + v_\theta \frac{x}{r^2} \text{ and } u_r \frac{y}{r} + u_\theta \frac{x}{r^2} + v_r \frac{x}{r} - v_\theta \frac{y}{r^2} = 0$$

Hence

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$$

10. 设两个实变数的函数 $u(x, y)$ 有偏导数. 这一函数可以写成 $z=x+iy$ 及 \bar{z} 的函数

$$u = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

证明:

$$\frac{\partial u}{\partial z} = \frac{1}{2} \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right), \quad \frac{\partial u}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right).$$

设复变函数 $f(z)$ 的实部及虚部分别是 $u(x, y)$ 及 $v(x, y)$, 并且它们都有偏导数, 求证: 对于 $f(z)$, 柯西-黎曼条件可以写成

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i \frac{\partial v}{\partial \bar{z}} = 0.$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial z} \left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i} \right) = \frac{1}{2} \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right), \quad \frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} \left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i} \right) = \frac{1}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i \frac{\partial v}{\partial \bar{z}} = \frac{1}{2}(u_x + iu_y) + \frac{1}{2}i(v_x + iv_y) = \frac{1}{2}(u_x - v_y) + \frac{1}{2}i(u_y + v_x) = 0 \iff u_x = v_y, u_y = -v_x$$

which is Cauchy-Riemann equation.

7.32 hw4

1 p40 11 14 15 16

Exercice 50

11. 试求出 e^{2+i} , $\ln(1+i)$, i^i , $1^{\sqrt{2}}$, $(-2)^{\sqrt{2}}$ 的值.

$$e^{2+i} = e^2 \cdot e^i = e^2 \cos 1 + ie^2 \sin 1$$

$$\ln(1+i) = \ln(\sqrt{2}e^{i\pi/4}) = \ln(\sqrt{2}) + i\frac{\pi}{4} + 2k\pi i, \quad k \in \mathbb{Z}$$

$$i^i = (e^{i\pi/2})^i = e^{-\pi/2}$$

$$1^{\sqrt{2}} = e^{\sqrt{2}\ln(1)} = e^{2\sqrt{2}k\pi i} = \cos(2\sqrt{2}k\pi) + i \sin(2\sqrt{2}k\pi), \quad k \in \mathbb{Z}$$

$$(-2)^{\sqrt{2}} = e^{\sqrt{2}\ln(-2)} = e^{\sqrt{2}(2+\pi i+2k\pi i)} = e^{2\sqrt{2}} \cos((2k+1)\sqrt{2}\pi) + ie^{2\sqrt{2}} \sin((2k+1)\sqrt{2}\pi), \quad k \in \mathbb{Z}$$

Exercice 51

14. 设函数 $f(\frac{1}{z})$ 在 $z=0$ 解析, 那么我们说 $f(z)$ 在 $z=\infty$ 解析. 下列函数中, 哪些在无穷远点解析?

$$e^z, \ln\left(\frac{z+1}{z-1}\right), \frac{a_0 + a_1 z + \cdots + a_m z^m}{b_0 + b_1 z + \cdots + b_n z^n}, \frac{\sqrt{z}}{1 + \sqrt{z}}$$

$e^{1/z}$ 在 $z=0$ 无定义, 故不解析, 故 e^z 在 $z=\infty$ 不解析。

$\ln\left(\frac{1/z+1}{1/z-1}\right) = \ln\left(-\frac{z+1}{z-1}\right)$ 在 $z=0$ 解析, 故 $\ln\left(\frac{z+1}{z-1}\right)$ 在 $z=\infty$ 解析。

$\frac{a_0 + a_1 z^{-1} + \cdots + a_m z^{-m}}{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}} = \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^n + b_1 z^{n-1} + \cdots + b_n} \cdot z^{n-m}$, 当 $n \geq m$ 且 $b_n \neq 0$ 时,

在 $z=0$ 解析, 故 $\frac{a_0 + a_1 z + \cdots + a_m z^m}{b_0 + b_1 z + \cdots + b_n z^n}$ 在 $z=\infty$ 解析, 否则在 $z=\infty$ 不解析。

$\frac{\sqrt{z}}{1 + \sqrt{z}} = \frac{\sqrt{1/z}}{1 + \sqrt{1/z}} = \frac{1}{\sqrt{z} + 1}$ 是多值函数, 在 $z=0$ 的每个分支内解析, 故 $\frac{\sqrt{z}}{1 + \sqrt{z}}$ 是多值解析函数。

Exercice 52

15. 在复平面上取上半虚轴作割线. 试在所得区域内分别取定函数 \sqrt{z} 与 $\ln z$ 在正实轴取正实值的一个解析分支, 并求它们在上半虚轴左沿的点及右沿的点 $z = i$ 处的值.

对于 \sqrt{z} , 它有两个解析分支

$$\sqrt{z} = |z| \cdot e^{\frac{i}{2}(\arg z)} \quad \text{和} \quad \sqrt{z} = |z| \cdot e^{\frac{i}{2}(\arg z + 2\pi)} = |z| \cdot e^{\frac{i}{2}(\arg z) + i\pi}$$

令 z 在正实轴取正实值, 于是 \sqrt{z} 在 $\sqrt{z} = |z| \cdot e^{\frac{i}{2}(\arg z)}$ 的解析分支. 在上半虚轴右沿, 令 $z = i$, 那么 $\arg z = \pi/2$, $\sqrt{z} = 1 \cdot e^{i\pi/4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$.

$$\text{在左沿 } \arg z = -3\pi/2, \sqrt{z} = 1 \cdot e^{-3\pi i/4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$$

对于 $\ln z$, 它有无穷多个解析分支

$$\ln z = \ln|z| + i \arg z + 2k\pi i, \quad k \in \mathbb{Z}$$

令 z 在正实轴取正实值, 于是 $\ln z$ 在 $\ln z = \ln|z| + i \arg z$ 的解析分支. 在上半虚轴右沿, 令 $z = i$, 那么 $\arg z = \pi/2$, $\ln z = \ln 1 + i\pi/2 = i\pi/2$.

$$\text{在左沿, } \arg z = -3\pi/2, \ln z = \ln 1 - 3\pi i/2 = -3\pi i/2.$$

Exercice 53

16. 在复平面上取正实轴作割线. 试在所得的区域内:(1) 取定函数 z^α ($-1 < \alpha < 0$) 在正实轴上沿取正实值的一个解析分支, 并求这一分支在

$$z = -1$$

处的值; 在正实轴下沿的值. (2) 取定函数 $\ln z$ 在正实轴上沿取实值的一个解析分支, 并求这一分支在 $z = -1$ 处的值; 在正实轴下沿的值.

(1)

$$z^\alpha = e^{\alpha \ln z} = e^{\alpha \ln|z| + \alpha \cdot i \arg z + 2k\pi \alpha i}, \quad k \in \mathbb{Z}$$

在正实轴上沿取正实值的解析分支为 $z^\alpha = e^{\alpha \ln|z| + \alpha \cdot i \arg z}$. 令 $z = -1$ 则

$$z^\alpha = e^{\alpha \cdot i \arg(-1)} = e^{\alpha \cdot i\pi}$$

在正实轴下沿

$$z^\alpha = |z|^\alpha \cdot e^{2\pi\alpha i} = |z|^\alpha (\cos(2\pi\alpha) + i \sin(2\pi\alpha))$$

(2)

$$\operatorname{Ln} z = \ln|z| + i \arg z + 2k\pi i, \quad k \in \mathbb{Z}$$

在正实轴上沿取正值的解析分支为 $\operatorname{Ln} z = \ln|z| + i \arg z$, 令 $z = -1$ 则

$$\operatorname{Ln} z = i\pi$$

在正实轴下沿

$$\operatorname{Ln} z = \ln|z| + 2\pi i$$

7.33 hw5

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Exercice 54

1. 分别计算沿着(1)直线段;(2)单位圆($|z|=1$)的左半圆;(3)单位圆的右半圆的下列积分:

$$I = \int_{-1}^1 |z| dz.$$

记

$$\gamma_1 : [0, 1] \rightarrow l_{-i \rightarrow i} \quad t \mapsto -i + 2ti$$

$$\gamma_2 : [0, 1] \rightarrow C_{-i \rightarrow i}^- \quad t \mapsto e^{-\frac{\pi}{2}i - t\pi i}$$

$$\gamma_3 : [0, 1] \rightarrow C_{-i \rightarrow i}^+ \quad t \mapsto e^{-\frac{\pi}{2}i + t\pi i}$$

于是

(1)

$$I_1 = \int_{\gamma_1} |z| dz = \int_0^1 |-i + 2ti| 2i dt = i$$

(2)

$$I_2 = \int_{\gamma_2} |z| dz = \int_0^1 |e^{-\frac{\pi}{2}i - t\pi i}| e^{-\frac{\pi}{2}i - t\pi i} \cdot (-\pi i) dt = 2i$$

(3)

$$I_3 = \int_{\gamma_3} |z| dz = \int_0^1 |e^{-\frac{\pi}{2}i + t\pi i}| \pi i \cdot e^{-\frac{\pi}{2}i + t\pi i} dt = 2i$$

Exercice 55

2. 计算积分

$$I = \int_L \operatorname{Re} z dz,$$

在这里 L 分别表示:(1) 单位圆(按反时针方向从 1 到 1 取积分);(2) 从 z_1 沿直线段到 z_2 .

记

$$\gamma_1 : [0, 1] \rightarrow C_{1 \rightarrow 1}^+ \quad t \mapsto e^{2\pi i t}$$

$$\gamma_2 : [0, 1] \rightarrow l_{z_1 \rightarrow z_2} \quad t \mapsto z_1 + t(z_2 - z_1)$$

(1)

$$I_1 = \int_{\gamma_1} \operatorname{Re} z dz = \int_0^1 \cos(2\pi i t) e^{2\pi i t} 2\pi i dt = \frac{1}{2}(-1 + i \sinh(2\pi) + \cosh(2\pi))$$

(2)

$$I_2 = \int_{\gamma_2} \operatorname{Re} z dz = \int_0^1 \operatorname{Re}[z_1 + t(z_2 - z_1)] \cdot (z_2 - z_1) dt = \frac{1}{2}(\operatorname{Re} z_2 - \operatorname{Re} z_1)(z_2 - z_1)$$

Exercice 56

3. 设函数 $f(z)$ 当 $|z - z_0| > r_0$ ($0 < r_0 < r$) 时是连续的. 令 $M(r)$ 表示 $|f(z)|$ 在 $|z - z_0| = r > r_0$ 上的最大值, 并且假定

$$\lim_{r \rightarrow +\infty} r M(r) = 0.$$

试证明

$$\lim_{r \rightarrow +\infty} \int_{K_r} f(z) dz = 0.$$

在这里 K_r 是圆 $|z - z_0| = r$.

证明. 当 $r > r_0$ 时,

$$\left| \int_{K_r} f(z) dz \right| \leq \int_{K_r} M(r) dz = 2\pi r \cdot M(r)$$

令 $r \rightarrow \infty$ 得到

$$\lim_{r \rightarrow \infty} \int_{K_r} f(z) dz = 0$$

□

Exercice 57

7. 计算积分：

$$(1) I = \int_C \frac{dz}{\sqrt{z}}; \quad (2) I = \int_C \ln z dz,$$

在这里用 C 表示单位圆(按反时针方向从 1 到 1 取积分), 而被积函数分别取为按下列各值决定的解析分支: (1) $\sqrt{1} = 1$ 或 $\sqrt{1} = -1$; (2) $\ln 1 = 0$ 或 $\ln 1 = 2\pi i$.

$$(1) \sqrt{z} = \sqrt{|z|} \cdot e^{\frac{i}{2} \arg z},$$

$$I = \int_C \frac{1}{\sqrt{|z|}} e^{-\frac{i}{2} \arg z} dz = \int_0^{2\pi} e^{-\frac{i}{2}\theta} \cdot ie^{i\theta} d\theta = \int_0^{2\pi} i \cdot e^{\frac{i}{2}\theta} d\theta = -4$$

$$\sqrt{z} = \sqrt{|z|} e^{\frac{i}{2} \arg z},$$

$$I = \int_C \frac{1}{\sqrt{|z|}} e^{-\frac{i}{2} \arg z} dz = 4$$

$$(2) \ln z = \ln|z| + i \arg z,$$

$$\begin{aligned} I &= \int_0^{2\pi} (\ln|e^{i\theta}| + i\theta) ie^{i\theta} d\theta = \int_0^{2\pi} -\theta e^{i\theta} d\theta \\ &= \int_0^{2\pi} i\theta de^{i\theta} = (i\theta e^{i\theta}) \Big|_0^{2\pi} - \int_0^{2\pi} e^{i\theta} d(i\theta) \\ &= 2\pi i \end{aligned}$$

$$\ln z = \ln|z| + i \arg z + 2\pi i,$$

$$\begin{aligned} I &= \int_0^{2\pi} (\ln|e^{i\theta}| + i\theta + 2\pi i) \cdot ie^{i\theta} d\theta \\ &= \int_0^{2\pi} -(2\pi + \theta) \cdot e^{i\theta} d\theta \\ &= - \int_0^{2\pi} \theta e^{i\theta} d\theta \\ &= 2\pi i \end{aligned}$$

Exercice 58

8. 如果积分路线不经过点 $\pm i$, 那么

$$\int_0^1 \frac{dz}{1+z^2} = \frac{\pi}{4} + k\pi \quad (k = 0, \pm 1, \pm 2, \dots).$$

Let

$$\gamma_0 : [0, 1] \rightarrow [0, 1] \quad t \mapsto t$$

Then

$$\int_{\gamma_0} \frac{1}{1+z^2} dz = \int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4}$$

$f(z) := \frac{1}{1+z^2}$ is holomorphic in $\mathbb{C} \setminus \{\pm i\}$, then for any

$$\gamma : [0, 1] \rightarrow C_{0 \rightarrow 1}$$

with $\text{Ind}_i(\gamma) = n, \text{Ind}_{-i}(\gamma) = m$. We have

$$\int_{\gamma} f(z) dz + \int_{\gamma_0^-} f(z) dz + n \int_{C_i(\epsilon)} f(z) dz + m \int_{C_{-i}(\epsilon)} f(z) dz = 0$$

where

$$\begin{aligned} \int_{C_i(\epsilon)} f(z) dz &= \int_{C_i(\epsilon)} \frac{1}{1+z^2} dz = \frac{1}{2i} \int_{C_i(\epsilon)} \left(\frac{1}{z-i} - \frac{1}{z+i} \right) dz \\ &= \underbrace{\frac{1}{2i} \int_{C_i(\epsilon)} \frac{1}{z-i} dz}_{=2\pi i} - \underbrace{\frac{1}{2i} \int_{C_i(\epsilon)} \frac{1}{z+i} dz}_{=0} \\ &= \pi \end{aligned}$$

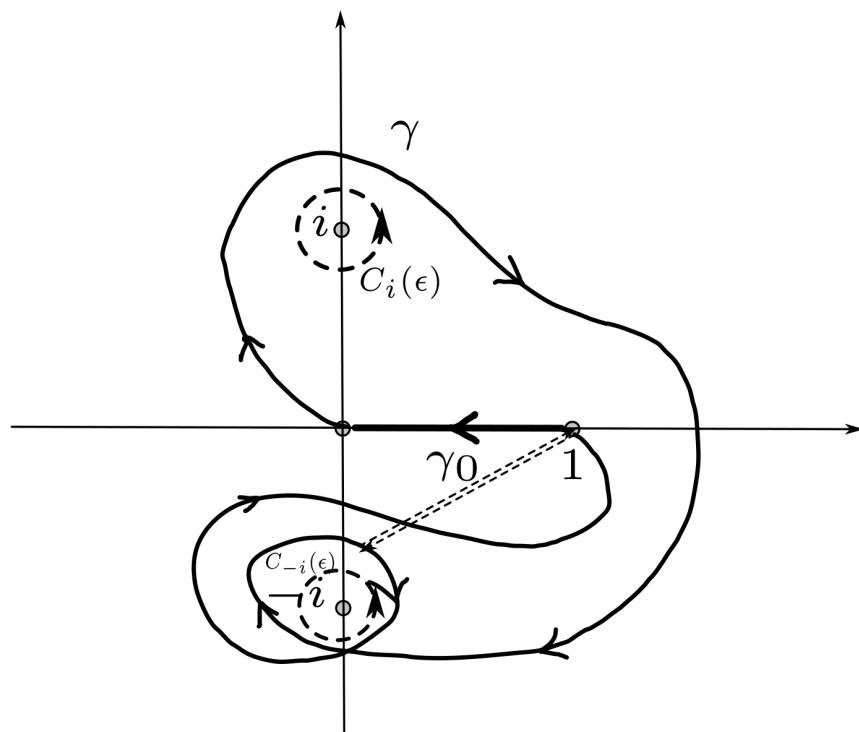
Similarly,

$$\int_{C_{-i}(\epsilon)} f(z) dz = -\pi$$

Therefore

$$\begin{aligned} \int_{\gamma} f(z) dz - \frac{\pi}{4} + (n-m)\pi &= 0 \\ \int_{\gamma} \frac{1}{1+z^2} dz &= \frac{\pi}{4} + k\pi \quad k = 0, \pm 1, \pm 2, \dots \end{aligned}$$

where $k = m - n = \text{Ind}_i(\gamma) - \text{Ind}_{-i}(\gamma)$.



Exercice 59

9. 证明：

- (1) $\left| \int_C (x^2 + iy^2) dz \right| \leq 2$, C 为联 $-i$ 到 i 的线段；
- (2) $\left| \int_C (x^2 + iy^2) dy \right| \leq \pi$, C 为右半单位圆 $|z| = 1$, $\operatorname{Re} z \geq 0$ ；
- (3) $\left| \int_C \frac{dz}{z^2} \right| \leq 2$, C 为联 i 到 $i+1$ 的线段.

(1)

$$C : [0, 1] \rightarrow l_{-i \rightarrow i} \quad t \mapsto -i + 2ti$$

$$\left| \int_C (x^2 + iy^2) dz \right| = \left| \int_0^1 i(2t-1)^2(-1)(2i) dt \right| = \left| \int_0^1 8t^2 - 8t + 2 dt \right| = \frac{2}{3} \leq 2$$

(2)

$$\begin{aligned} |I| &\leq \int_{|z|=1, \operatorname{Re} z \geq 0} |x^2 + iy^2| dy = \int_{|z|=1, \operatorname{Re} z \geq 0} \sqrt{x^4 + y^4} dy \\ &\leq \int_{|z|=1, \operatorname{Re} z \geq 0} \underbrace{(x^2 + y^2)^2}_{=1} dy \leq \int_{|z|=1, \operatorname{Re} z \geq 0} dz = \pi \end{aligned}$$

(3)

$$\begin{aligned} |I| &\leq \int_{l_{i \rightarrow i+1}} \underbrace{\frac{1}{|z|^2}}_{\leq 1/(1/\sqrt{2})^2} dz \leq \int_{l_{i \rightarrow i+1}} 2 dz = 2 \end{aligned}$$

Exercice 60

10. 设 $f(z)$ 在原点的邻域内连续, 那么

$$\lim_{r \rightarrow 0} \int_0^{2\pi} f(re^{i\theta}) d\theta = 2\pi f(0).$$

Since f is continuous in the neighborhood of 0, then for any $\epsilon > 0$, there exists $\delta > 0$, s.t.

$$|f(z) - f(0)| < \epsilon, \quad \forall |z| < \delta$$

Let $r < \delta$, then

$$\left| \int_0^{2\pi} f(re^{i\theta}) - f(0) d\theta \right| \leq \int_0^{2\pi} \underbrace{|f(re^{i\theta}) - f(0)|}_{\leq \epsilon} d\theta \leq 2\pi\epsilon$$

Since r can be arbitrarily small, thus we have

$$\lim_{r \rightarrow 0} \int_0^{2\pi} f(re^{i\theta}) d\theta = 2\pi f(0)$$

7.34 hw6

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11. 计算积分：

$$(1) \int_{|z|=1} \frac{e^z}{z} dz; \quad (2) \int_{|z|=2} \frac{dz}{z^2 + 2};$$

$$(3) \int_{|z|=1} \frac{dz}{z^2 + 2}; \quad (4) \int_{|z|=1} \frac{z dz}{(2z+1)(z-2)}.$$

$$(1) \int_{|z|=1} \frac{e^z}{z} dz \stackrel{\text{Cauchy}}{=} 2\pi i (e^z)|_{z=0} = 2\pi i$$

$$(2) \int_{|z|=2} \frac{1}{z^2 + 2} dz = \frac{1}{2\sqrt{2}i} \int_{|z|=2} \left(\frac{1}{z - \sqrt{2}i} - \frac{1}{z + \sqrt{2}i} \right) dz = 0$$

(3) Since $\frac{1}{z^2+2}$ is holomorphic in the unit disc and its boundary,

$$\int_{|z|=1} \frac{1}{z^2 + 2} dz = 0$$

$$(4) \int_{|z|=1} \frac{z}{(2z+1)(z-2)} dz = 2\pi i \left(\frac{z}{2(z-2)} \right)_{z=-1/2} = \frac{\pi i}{5}$$

12. 证明

$$\left(\frac{z^n}{n!} \right)^2 = \frac{1}{2\pi i} \int_C \frac{z^n e^{z\zeta}}{n!} \frac{d\zeta}{\zeta^n},$$

在这里 C 是围绕原点的一条简单闭曲线.

Apply Cauchy's integral formula to $f(\eta) = e^{z\eta}$ at $\eta = 0$, then

$$z^n = f^{(n)}(0) = \frac{n!}{2\pi i} \int_C \frac{e^{z\zeta}}{\zeta^n} d\zeta$$

Therefore

$$\left(\frac{z^n}{n!} \right)^2 = \frac{1}{2\pi i} \int_C \frac{z^n}{n!} \cdot \frac{e^{z\zeta}}{\zeta^n} d\zeta$$

14. 通过计算

$$\int_{|z|=1} \left(z + \frac{1}{z} \right)^{2n} \cdot \frac{dz}{z} \quad (n = 1, 2, \dots),$$

证明

$$\int_0^{2\pi} \cos^{2n} \theta d\theta = 2\pi \cdot \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n}.$$

$$\int_{|z|=1} \left(z + \frac{1}{z} \right)^{2n} \cdot \frac{1}{z} dz = \int_{|z|=1} \frac{(z^2 + 1)^{2n} \cdot z^{-n}}{z^{n+1}} dz$$

Let

$$f(z) := (z^2 + 1)^{2n} \cdot z^{-n} = \sum_{k=0}^{2n} C_{2n}^k z^{2k-n}$$

Then

$$f^{(n)}(z) = \sum_{k=0}^{2n} C_{2n}^k \cdot (2k-n) \dots (2k-2n+1) \cdot z^{2k-2n}$$

Let $z = 0$ then

$$f^{(n)}(0) = C_{2n}^n \cdot n!$$

Thus

$$\int_{|z|=1} \left(z + \frac{1}{z} \right)^{2n} \cdot \frac{1}{z} dz = \int_{|z|=1} \frac{f(z)}{z^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(0) = 2\pi i \cdot \frac{(2n)!}{n! \cdot n!}$$

On the other hand

$$\int_{|z|=1} \left(z + \frac{1}{z} \right)^{2n} \cdot \frac{1}{z} dz = \int_0^{2\pi} (e^{i\theta} + e^{-i\theta})^{2n} \cdot i d\theta = i \cdot \int_0^{2\pi} 2^{2n} \cdot \cos^{2n} \theta d\theta$$

Therefore

$$\int_0^{2\pi} \cos^{2n} \theta d\theta = 2\pi \cdot \frac{(2n-1)!!}{(2n)!!}$$

15. 如果在 $|z| < 1$ 内, $f(z)$ 解析, 并且

$$|f(z)| \leq \frac{1}{1 - |z|},$$

证明

$$|f^{(n)}(0)| \leq (n+1)! \left(1 + \frac{1}{n} \right)^n < e(n+1)! \quad (n = 1, 2, \dots).$$

[提示] 考虑 $\frac{f(z)}{z^{n+1}}$ 在 $|z| = \frac{n}{n+1}$ 上的积分.

For $0 < r < 1$, apply the Cauchy's integral formula

$$|f^{(n)}(0)| = \left| \frac{n!}{2\pi i} \int_{|z|=r} \frac{|f(\zeta)|}{\zeta^{n+1}} d\zeta \right| \leq \frac{n!}{2\pi} \cdot \frac{2\pi r}{(1-r)r^{n+1}} = \frac{n!}{(1-r)r^n}$$

Clearly, $g(r) = (1-r)r^n$ reaches its maximum at $r = \frac{n}{n+1}$. Let $r = \frac{n}{n+1} < 1$ then

$$|f^{(n)}(0)| \leq (n+1)! \cdot \left(1 + \frac{1}{n}\right)^n < e \cdot (n+1)!$$

17. 如果函数 $f(z)$ 在简单闭曲线 C 的外区域 D 内及 C 上每一点解析. 并且

$$\lim_{z \rightarrow \infty} f(z) = \alpha,$$

那么

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \begin{cases} -f(z) + \alpha & (\text{当 } z \in D \text{ 时}), \\ \alpha & (\text{当 } z \in C \text{ 的内区域时}), \end{cases}$$

这里沿 C 的积分是按反时针方向取的.

本题是关于含无穷远点的区域的柯西公式.

[提示] 应用柯西公式证明.

When $z \in C^\circ$,

Let R large enough such that C is contained in C_R , where C_R is the circle centered at z with radius R , counterclockwise. Then

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$\forall \epsilon > 0, \exists A > 0$ s.t. $\forall |z| > A$, we have

$$|f(z) - \alpha| < \epsilon$$

Then for $R > A$

$$\left| \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta - \alpha \right| = \left| \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta) - \alpha}{\zeta - z} d\zeta \right| \leq \frac{\epsilon}{2\pi} \int_{C_R} \underbrace{\frac{1}{|\zeta - z|}}_{=R} d\zeta = \epsilon$$

Thus

$$\left| \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta - \alpha \right| \leq \epsilon$$

Since ϵ is arbitrary,

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \alpha$$

When $z \in D$, let C_ϵ be the circle centered at z with radius ϵ , clockwise. ϵ is small enough such that C_ϵ is contained in D . Then

$$\frac{1}{2\pi i} \left(- \int_C + \int_{C_R} + \int_{C_\epsilon} \right) \frac{f(\zeta)}{\zeta - z} d\zeta = 0$$

By Cauchy's integral formula

$$\frac{1}{2\pi i} \int_{C_\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta = -2\pi i f(z)$$

Hence

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = -f(z) + \alpha$$

18. 设 $f(z)$ 在单连通区域 D 内解析，并且不等于零。那么

- (1) $\exists g(z)$ 在 D 内解析，使得 $e^{g(z)} = f(z)$ ；
 - (2) 对于整数 $q \geq 2$, $\exists h(z)$ 在 D 内解析，使得 $[h(z)]^q = f(z)$ 。
- [提示] 设 $g(z)$ 是 $f'(z)/f(z)$ 在 D 内的原函数，考虑 $e^{-g(z)} f(z)$.

(1) Fix a point $z_0 \in D$ and define

$$g(z) = \int_\gamma \frac{f'(w)}{f(w)} dw + c_0$$

where γ is any path in D connecting z_0 to z and c_0 is a complex number so that $e^{c_0} = f(z_0)$. This definition is independent of the path γ since D is simple connected. We find that g is holomorphic with

$$g'(z) = \frac{f'(z)}{f(z)}$$

and a simple calculation gives

$$\frac{d}{dz} (f(z)e^{-g(z)}) = 0$$

so that $f(z)e^{-g(z)}$ is constant. Evaluating this expression at z_0 we find $f(z_0)e^{-g(z_0)} = 1$ so that $f(z) = e^{g(z)}$ for all $z \in D$, and the proof is complete.

(2)

(1) implies (2). Put $h(z) = e^{\frac{1}{q}g(z)}$, then $h^q(z) = e^{g(z)} = f(z)$.

7.35 hw7

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Exercice 61

5. 试求下列幂级数的收敛半径:

$$(1) \sum_{n=0}^{+\infty} q^{\frac{n^2}{n}} z^n, \text{ 其中 } |q| < 1;$$

$$(2) \sum_{n=1}^{+\infty} z^{n!};$$

$$(3) \sum_{n=0}^{+\infty} n^p z^n, \text{ 其中 } p \text{ 是一正整数};$$

$$(4) \sum_{n=0}^{+\infty} [3 + (-1)^n]^n z^n;$$

$$(5) \sum_{n=1}^{+\infty} \frac{n!}{n^n} z^n;$$

$$(6) 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{2! c(c+1)} z^2 + \dots + \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n! c(c+1)\cdots(c+n-1)} z^n + \dots,$$

其中 a, b, c 是复数, 但 c 不是零或负整数.

$$(1) \limsup_{n \rightarrow \infty} \sqrt[n]{q^{n^2}} = \limsup_{n \rightarrow \infty} q^n = 0 \text{ then } R = \infty.$$

$$(2) \sum_{n=1}^{\infty} z^{n!} = \sum_{n=1}^{\infty} a_n z^n, \text{ where}$$

$$a_n = \begin{cases} 1 & \text{if } n = k! \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Then } \limsup_{n \rightarrow \infty} \sqrt[n]{a_n} = 1, R = \frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{a_n}} = 1.$$

$$(3) \limsup_{n \rightarrow \infty} \sqrt[n]{n^p} = \limsup_{n \rightarrow \infty} \exp\left(\frac{p \ln n}{n}\right) = 1 \text{ then } R = \frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{n^p}} =$$

1.

$$(4) \limsup_{n \rightarrow \infty} \sqrt[n]{[3 + (-1)^n]^n} = 4 \text{ then } R = \frac{1}{4}.$$

$$(5) \limsup_{n \rightarrow \infty} \sqrt[n]{\frac{n!}{n^n}} = \limsup_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n^n} \cdot \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + o(1))} = e^{-1}.$$

Then $R = e$.

(6)

$$a_n = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n) \cdot n!}$$

Lemme 3

$$\Gamma(\alpha + 1) = \alpha^\alpha e^{-\alpha} \sqrt{2\pi\alpha} \left(1 + \frac{1}{12\alpha} + O(\alpha^{-2}) \right), |\alpha| \rightarrow +\infty$$

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \limsup_{n \rightarrow \infty} \sqrt[n]{\left| \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n) \cdot n!} \right|} \\ &= \limsup_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(a+n-1)^{a+n-1}(b+n-1)^{b+n-1}e^{-(a+b+2n-2)}2\pi\sqrt{(a+n-1)(b+n-1)}\Gamma(c)}{\Gamma(a)\Gamma(b)(c+n-1)^{c+n-1}e^{-(c+n-1)}\sqrt{2\pi(c+n-1)}\sqrt{2\pi nn^n}e^{-n}} \right|} \\ &= \limsup_{n \rightarrow \infty} \sqrt[n]{\frac{n^{a+n-1} \cdot n^{b+n-1}e^{-(a+b-2)}e^{-2n}2\pi \cdot n \cdot \Gamma(c)}{\Gamma(a)\Gamma(b)n^{c+n-1}n \cdot 2\pi \cdot n^n \cdot e^{-n}e^{-c+1-n}}} \\ &= \limsup_{n \rightarrow \infty} \sqrt[n]{\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}} e^{c-a-b+1} n^{a+b-c-1} \\ &= 1 \end{aligned}$$

Thus $R = 1$.

Exercice 62

6. 设在 $|z| < R$ 内解析的函数 $f(z)$ 有泰勒展式

$$f(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_n z^n + \cdots,$$

试证:(1) 令 $M(r) = \max_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|$. 我们有

$$|\alpha_n| \leq \frac{M(r)}{r^n} \quad (\text{柯西不等式}),$$

在这里 $n=0,1,2,\cdots; 0 < r < R$.

(2) 由(1)证明刘维尔定理.

(3) 当 $0 \leq r < R$ 时,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{+\infty} |\alpha_n|^2 r^{2n}.$$

证明. (1)

$$\alpha_n = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^{n+1}} dz$$

Then

$$|\alpha_n| \leq \frac{1}{2\pi} \oint_{|z|=r} \frac{|f(z)|}{|z|^{n+1}} dz \leq \frac{1}{2\pi} 2\pi r \frac{M(r)}{r^{n+1}} = \frac{M(r)}{r^n}$$

(2) If f is entire, since $M(r)$ is bounded, then for each $n \geq 1$, let $r \rightarrow \infty$,

$$|\alpha_n| \leq \frac{M(r)}{r^n} \rightarrow 0$$

Therefore $f \equiv a_0$ is constant.

(3)

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta &= \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) \overline{f(re^{i\theta})} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=0}^{\infty} \alpha_n r^n e^{in\theta} \sum_{m=0}^{\infty} \overline{\alpha_m} r^m e^{-im\theta} d\theta \\ &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_0^{2\pi} \alpha_n \overline{\alpha_m} r^{n+m} e^{i\theta(n-m)} d\theta \\ &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_0^{2\pi} \alpha_n \overline{\alpha_m} r^{n+m} e^{i\theta(n-m)} \delta_{mn} d\theta \\ &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \int_0^{2\pi} \alpha_n \overline{\alpha_n} r^{2n} d\theta \\ &= \sum_{n=0}^{\infty} |\alpha_n|^2 r^{2n} \end{aligned}$$

□

Exercice 63

7. 证明: 如果在 $|z| < r$ 上及 $|z| < \rho$ 内, 我们分别有

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n \text{ 及 } g(z) = \sum_{n=0}^{+\infty} b_n z^n,$$

其中 $0 < r, \rho < +\infty$, 而且 $f(z)$ 在 $|z| \leq r$ 上连续, 那么在 $|z| < \rho r$ 内,

$$\sum_{n=0}^{+\infty} a_n b_n z^n = \frac{1}{2\pi i} \int_{|\zeta|=r} f(\zeta) g\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta}.$$

证明.

$$\begin{aligned} &\frac{1}{2\pi i} \oint_{|\zeta|=r} \sum_{n=0}^{\infty} a_n \zeta^n \sum_{m=0}^{\infty} b_m \frac{z^m}{\zeta^m} \frac{1}{\zeta} d\zeta \\ &= \sum_{m=0}^{\infty} z^m \left(\sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{|\zeta|=r} \frac{a_n b_m}{\zeta^{m-n+1}} d\zeta \right) \\ &= \sum_{m=0}^{\infty} z^m \left(\sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{|\zeta|=r} \frac{a_n b_m}{\zeta^{m-n+1}} \delta_{mn} d\zeta \right) \\ &= \sum_{m=0}^{\infty} a_m b_m z^m \end{aligned}$$

□

Exercice 64

8. 设 z 是任一复数, 证明 $|e^z - 1| \leq e^{|z|} - 1 \leq |z|e^{|z|}$.

证明.

$$|e^z - 1| = \left| \sum_{n=1}^{\infty} \frac{z^n}{n!} \right| \leq \sum_{n=1}^{\infty} \frac{|z|^n}{n!} = e^{|z|} - 1$$

Let $f(x) = xe^x - e^x + 1$, $x \in \mathbb{R}^+$, then

$$f'(x) = xe^x \geq 0$$

$f(x) \geq f(0) = 0$. Thus

$$f(|z|) \geq 0 \implies |z|e^{|z|} \geq e^{|z|} - 1$$

□

Exercice 65

9. 求下列解析函数或多值函数的解析分支在 $z=0$ 的泰勒展式:

$$(1) \sin^2 z; (2) e^z \cos z; (3) \frac{1}{2} \left(\ln \frac{1}{1-z} \right)^2;$$

$$(4) (2-z)^{\frac{3}{4}}; (5) \tan z (\text{计算到 } z^5 \text{ 的系数}).$$

(1)

$$\sin^2 z = \frac{1}{2}(1 - \cos 2z) = \frac{1}{2} \left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^{2n}}{2 \cdot (2n)!}$$

(2)

$$\begin{aligned} e^z \cos z &= \operatorname{Re}(e^z \cdot e^{iz}) = \operatorname{Re}(e^{(1+i)z}) = \operatorname{Re} \left(\sum_{n=0}^{\infty} \frac{(1+i)^n z^n}{n!} \right) \\ &= \operatorname{Re} \left(\sum_{n=0}^{\infty} \frac{2^{\frac{n}{2}} e^{i\pi \frac{n}{4}} z^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \frac{2^{\frac{n}{2}} \cos \left(\frac{n\pi}{4} \right) z^n}{n!} \end{aligned}$$

(3)

$$\begin{aligned}
 \frac{1}{2} \left(\ln \frac{1}{1-z} \right)^2 &= \frac{1}{2} (2k\pi - \ln(1-z))^2 = \frac{1}{2} \left(2k\pi + \sum_{n=1}^{\infty} \frac{z^n}{n} \right)^2 \\
 &= \frac{1}{2} \left(4k^2\pi^2 + 4k\pi \sum_{n=1}^{\infty} \frac{z^n}{n} + \sum_{n=2}^{\infty} \left(\frac{2}{m} \sum_{m=1}^{n-1} \frac{1}{m} \right) z^n \right) \\
 &= 2k^2\pi^2 + 2k\pi \sum_{n=1}^{\infty} \frac{z^n}{n} + \sum_{n=2}^{\infty} \left(\frac{1}{n} \sum_{m=1}^{n-1} \frac{1}{m} \right) z^n
 \end{aligned}$$

(4) 主值支:

$$(2-z)^{\frac{3}{4}} = 2^{\frac{3}{4}} \left(1 - \frac{z}{2}\right)^{\frac{3}{4}} = 2^{\frac{3}{4}} \cdot \sum_{n=0}^{\infty} \left(-\frac{z}{2}\right)^n \cdot \frac{\Gamma(\frac{3}{4}+n)}{\Gamma(\frac{3}{4})} = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{3}{4}+n)}{\Gamma(\frac{3}{4}) 2^{n-\frac{3}{4}}} z^n$$

(5)

$$\tan z = \frac{\sin z}{\cos z} = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \dots \quad \left(|z| < \frac{\pi}{2}\right)$$

7.36 hw8

1	10	11
2	12	(3)(5)(6) 13

Exercice 66

10. 设 $f(z)$ 是一整函数, 并且假定存在一个正整数 n , 以及两个正数 R 及 M , 使得当 $|z| \geq R$ 时,

$$|f(z)| \leq M |z|^n.$$

证明 $f(z)$ 是一个至多 n 次的多项式或一常数.

证明. 由于 f 是整函数, 考虑它的幂级数表示形式

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots$$

由 Cauchy 定理,

$$a_n = \frac{1}{2\pi i} \oint_{C_r} \frac{f(z)}{z^{n+1}} dz$$

其中 C_r 是以原点为中心, 半径为 r 的逆时针方向的圆. 对于任意 $m \geq n+1$, 有

$$|a_m| \leq \frac{1}{2\pi} \oint_{C_r} \frac{|f(z)|}{|z|^{m+1}} dz \leq \frac{1}{2\pi} \oint_{C_r} \frac{M}{|z|^{m-n+1}} dz = \frac{M}{2\pi} \oint_{C_r} \frac{1}{r^{m-n+1}} dz = \frac{M}{r^{m-n}}$$

由 r 的任意性, 令 $r \rightarrow \infty$, 就有 $a_m = 0, \forall m \geq n+1$. 于是 $f(z)$ 是一个至多 n 次的多项式或者常数. \square

Exercice 67

11. 证明: $\forall z \in \mathbb{C}, \lim_{n \rightarrow +\infty} \left(1 + \frac{z}{n}\right)^n = e^z$.

[提示] 考虑 $n \ln\left(1 + \frac{z}{n}\right)$ 及对数函数的泰勒展式.

证明. 只需要证明

$$\lim_{n \rightarrow \infty} \ln\left(1 + \frac{z}{n}\right)^n = z + 2k\pi i \quad \text{for some } k \in \mathbb{Z}$$

其中

$$\begin{aligned} \ln\left(1 + \frac{z}{n}\right)^n &= n \ln\left(1 + \frac{z}{n}\right) \\ &= n \left(2m\pi i + \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{z}{n}\right)^k \frac{1}{k}\right) \\ &= 2nm\pi i + \sum_{k=1}^{\infty} (-1)^{k+1} z^k \cdot n^{-k+1} \cdot k^{-1} \end{aligned}$$

当 $n > 2$ 时, 有

$$\begin{aligned} \left| \sum_{k=1}^{\infty} (-1)^{k+1} z^k \cdot n^{-k+1} \cdot k^{-1} - z \right| &\leq \sum_{k=2}^{\infty} \frac{|z|^k}{n^{k-1} \cdot k} \\ &\leq \frac{|z|}{2} \sum_{k=1}^{\infty} \left(\frac{|z|}{n}\right)^k \\ &= \frac{|z|}{2} \cdot \frac{n^{-1}|z|}{1 - n^{-1}|z|} \\ &= \frac{|z|^2}{2(n - |z|)} \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

故得证! \square

Exercice 68

12. 求下列解析函数或多值函数的解析分支在指定区域内的洛朗展式：

$$(1) \frac{e^z}{z(z^2+1)} \text{ 在 } 0 < |z| < 1 \text{ 内};$$

$$(2) \frac{1}{(z^5-1)(z-3)} \text{ 在 } 1 < |z| < 3 \text{ 内};$$

$$(3) \sin \frac{z}{z-1} \text{ 在 } 0 < |z-1| < 1 \text{ 内};$$

$$(4) e^{\frac{z}{z+2}} \text{ 在 } 2 < |z| < +\infty \text{ 内};$$

$$(5) \frac{1}{z^a(1+z)} \text{ 在 } 0 < |z+1| < 1 \text{ 内, 其中 } 0 < a < 1, z^a (1^a = 1);$$

$$(6) \frac{\ln z}{z^2-1} \text{ 在 } 0 < |z-1| < 1 \text{ 及 } 0 < |z+1| < 1 \text{ 内, } \ln z (\ln 1 = 0).$$

(3)

$$\begin{aligned} \sin \frac{z}{z-1} &= \sin \left(1 + \frac{1}{z-1} \right) \\ &= \sin 1 \cos \frac{1}{z-1} + \cos 1 \sin \frac{1}{z-1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \sin 1}{(2n)! \cdot (z-1)^{2n}} + \sum_{n=0}^{\infty} \frac{(-1)^n \cos 1}{(2n+1)! \cdot (z-1)^{2n+1}} \end{aligned}$$

(5)

$$\begin{aligned} \frac{1}{z^a(z+1)} &= (z+1)^{-1} (-1)^a (1-(z+1))^a \\ &= e^{i\pi a} (z+1)^{-1} \left[1 + \sum_{n=1}^{\infty} \frac{a(a-1)\dots(a-n+1)}{n!} (z+1)^n \right] \\ &= e^{ia\pi} (z+1)^{-1} + \sum_{n=0}^{\infty} \frac{a(a-1)\dots(a-n)}{(n+1)!} (z+1)^n \end{aligned}$$

(6) 在 $0 < |z-1| < 1$

$$\begin{aligned} \frac{\ln z}{z^2-1} &= \frac{\ln(1+z-1)}{(z-1)(z-1+2)} \\ &= (z-1)^{-1} \left(\sum_{n=1}^{\infty} \frac{(z-1)^n (-1)^{n-1}}{n} \right) \cdot \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{z-1}{2} \right)^n \\ &= \sum_{k=0}^{\infty} \left(\sum_{n=0}^k \frac{(-1)^k}{n+1} \cdot \left(\frac{1}{2} \right)^{k-n+1} \right) \cdot (z-1)^k \end{aligned}$$

在 $0 < |z + 1| < 1$

$$\begin{aligned}\frac{\ln z}{z^2 - 1} &= \sum_{k=0}^{\infty} \left(\sum_{n=1}^{k+1} \frac{1}{n} \cdot \frac{1}{2^{k+2-n}} \right) (z+1)^k - i\pi \sum_{k=-1}^{\infty} \frac{(z+1)^k}{2^{k+2}} \\ &= -\frac{i\pi}{2}(z+1)^{-1} + \sum_{k=0}^{\infty} \left(\sum_{n=1}^{k+1} \frac{2^{n-k-2}}{n} - \frac{i\pi}{2^{k+2}} \right) (z+1)^k\end{aligned}$$

Exercice 69

13. 问下列各函数有哪些孤立奇点？各属于哪一种类型？

- (1) $\frac{z-1}{z(z^2+4)^2}$;
- (2) $\cot z$;
- (3) $\frac{1}{\sin z - \sin \alpha}$, 其中 α 是一常数;
- (4) $\frac{e^{z-1}}{e^z - 1}$;
- (5) $\sin \frac{1}{1-z}$;
- (6) $\frac{\tan(z-1)}{z-1}$.

(1)

$$\frac{z-1}{z(z^2+4)^2} = \frac{z-1}{z(z-2i)^2(z+2i)^2}$$

有孤立奇点 $0, 2i, -2i, \infty$, 其中 0 为 1 阶极点, $2i$ 和 $-2i$ 为 2 阶极点. ∞ 为可去奇点.

(2)

$$\cot z = \frac{\cos z}{\sin z}$$

z 为 $\cot z$ 奇点当且仅当 $\sin z = 0$, 即 $e^{iz} = e^{-iz}$. 也就是 $e^{2iz} = 1$, 即 $z = k\pi$ 其中 $k \in \mathbb{Z}$. 于是 $z = k\pi$, $k \in \mathbb{Z}$ 为 $\cot z$ 的孤立奇点, 都是 1 阶奇点.

(3)

$$\frac{1}{\sin z - \sin \alpha} = \frac{1}{2 \sin \frac{z-\alpha}{2} \cos \frac{z+\alpha}{2}}$$

于是孤立奇点为 $z = \alpha + 2k\pi, z = \pi - \alpha + 2k\pi$ 其中 $k \in \mathbb{Z}$, 若 $\alpha = \pi - \alpha + 2n\pi$, $n \in \mathbb{Z}$, 那么 $z = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$ 是所有孤立奇点, 都是 2 阶极点. 否则 $z = \alpha + 2k\pi, z = \pi - \alpha + 2k\pi$, $k \in \mathbb{Z}$ 是所有孤立奇点, 都是 1 阶极点.

(4)

$$\frac{e^{\frac{1}{z-1}}}{e^z - 1}$$

孤立奇点为 $z = k\pi, k \in \mathbb{Z}$ 和 $z = 1$, 其中 $z = k\pi$ 为 1 阶极点, $z = 1$ 是本性奇点.

(5)

$$\sin \frac{1}{1-z} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} (1-z)^{-(2n+1)}$$

孤立奇点为 $1, \infty$, 其中 $z = 1$ 是本性奇点, $z = \infty$ 是可去奇点.

(6)

$$\frac{\tan(z-1)}{z-1} = \frac{1}{\cos(z-1)} \frac{\sin(z-1)}{z-1}$$

$\cos(z-1) = 0 \iff z-1 = \frac{\pi}{2} + n\pi i$, 即 $z = 1 + \frac{\pi}{2} + n\pi i, \forall n \in \mathbb{Z}$. $z = 1$ 也是奇点, 但 $\lim_{z \rightarrow 1} \frac{\tan(z-1)}{z-1} = 1$, 故 $z = 1$ 是可去奇点. 综上: $z = 1, 1 + \frac{\pi}{2} + n\pi i, n \in \mathbb{Z}$ 是所有孤立奇点, 其中 $z = 1$ 是可去奇点, $z = 1 + \frac{\pi}{2} + n\pi i$ 是 1 阶极点

7.37 hw9

1	第4章: 14 17 18 19
2	第5章: 1 2 3 7

Exercice 70

14. 证明: 在扩充复平面上只有一个一阶极点的解析函数 $f(z)$ 必有下面的形式:

$$f(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \alpha\delta - \beta\gamma \neq 0.$$

证明.

$$f(z) = c_{-1}(z - z_0)^{-1} + c_0 + \sum_{n=1}^{\infty} c_n(z - z_0)^n$$

Since $f \in H(\overline{\mathbb{C}})$,

$$f\left(\frac{1}{z}\right) = c_{-1} \frac{z}{1 - z_0 z} + c_0 + \sum_{n=1}^{\infty} c_n(z^{-1} - z_0)^n$$

is holomorphic at $z = 0$. Thus $c_n = 0, \forall n \geq 1$.

$$f(z) = \frac{c_{-1}}{z - z_0} + c_0 = \frac{c_0 z - c_0 z_0 + c_{-1}}{z - z_0}$$

where

$$\alpha = c_0, \quad \beta = -c_0 z_0 + c_{-1}, \quad \gamma = 1, \quad \delta = -z_0$$

And $\alpha\delta - \beta\gamma \neq 0$. □

Exercice 71

17. 设函数 $f(z)$ 在区域 D 内解析. 证明: 如果对某一点 $z_0 \in D$, 有

$$f^{(n)}(z_0) = 0, n = 1, 2, \dots,$$

那么, $f(z)$ 在 D 内为常数.

证明. For $z \in D$, we have

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

where $c_n = \frac{f^{(n)}(z_0)}{n!}$. Since $f^{(n)}(z_0) = 0$ for $n = 1, 2, \dots$. Then $c_n = 0$ for $n = 1, 2, \dots$. Therefore

$$f(z) \equiv c_0$$

is constant in D .

□

Exercice 72

18. 问是否存在满足下列条件, 并且在原点解析的函数 $f(z)$?

$$(1) f\left(\frac{1}{2n-1}\right) = 0, \quad f\left(\frac{1}{2n}\right) = \frac{1}{2n};$$

$$(2) f\left(\frac{1}{n}\right) = \frac{1}{n+1};$$

$$(3) f\left(\frac{1}{2n-1}\right) = f\left(\frac{1}{2n}\right) = \frac{1}{2n},$$

在这里 $n = 1, 2, 3, \dots$.

证明. (1) 不存在, 由于非平凡解析函数零点孤立, 而 $\left\{\frac{1}{2n-1}\right\} \rightarrow 0$, f 在 0 的某个邻域内恒为 0. 当 n 充分大的时候 $\frac{1}{2n}$ 落在这个邻域内, 但 $f\left(\frac{1}{2n}\right) = \frac{1}{2n} \neq 0$.

(2) 可做函数

$$f(z) = \frac{z}{z+1}$$

(3) 不存在, 事实上点列 $\left\{\frac{1}{2n}\right\} \rightarrow 0$, 故在 0 的某个邻域内, $f(z) - z \equiv 0$, 这与题设 $f\left(\frac{1}{2n-1}\right) = \frac{1}{2n}$ 矛盾.

□

Exercice 73

19. 函数 $\sin \frac{1}{1-z}$ 的零点 $1 - \frac{1}{n\pi}$ ($n = \pm 1, \pm 2, \pm 3, \dots$) 所成的集有聚点 1, 但这函数不恒等于零. 问这与解析函数的唯一性是否相矛盾?

证明. 并不矛盾, 因为 1 是函数的奇点, 函数不在 1 处解析.

□

Exercice 74

1. 试求下列各解析函数或多值函数的解析分支在指定各点的留数:

- | | |
|---|--|
| (1) $\frac{z^2}{(z^2+1)^2}$, 在 $z=\pm i$; | (2) $\frac{1}{1-e^z}$, 在 $z=2n\pi i$, n 为整数; |
| (3) $\frac{\sqrt{z}}{1-z}$, 在 $z=1$; | (4) $\sin \frac{1}{z-1}$, 在 $z=1$. |

(1)

$$\begin{aligned} \text{res}_i \frac{z^2}{(z^2+1)^2} &= \frac{1}{1!} \cdot \lim_{z \rightarrow i} \frac{d}{dz} \left[(z-i)^2 \frac{z^2}{(z-i)^2(z+i)^2} \right] \\ &= \lim_{z \rightarrow i} \frac{z^2}{(z+i)^2} \\ &= \frac{1}{4} \\ \text{res}_{-i} \frac{z^2}{(z^2+1)^2} &= \frac{1}{1!} \cdot \lim_{z \rightarrow -i} \frac{d}{dz} \left[(z+i)^2 \frac{z^2}{(z-i)^2(z+i)^2} \right] \\ &= \lim_{z \rightarrow -i} \frac{z^2}{(z-i)^2} \\ &= \frac{1}{4} \end{aligned}$$

(2)

$$\text{res}_{2n\pi i} \left(\frac{1}{1-e^z} \right) = \lim_{z \rightarrow 2n\pi i} \frac{z-2n\pi i}{1-e^z} = -1$$

(3)

$$\text{res}_1 \left(\frac{\sqrt{z}}{1-z} \right) = 1$$

(4)

$$\text{res}_1 \left(\sin \frac{1}{z-1} \right) = \lim_{z \rightarrow 1} (z-1) \sin \frac{1}{z-1} = 1$$

Exercice 75

2. 函数 $\frac{\ln z}{z^2 - 1}$ 的各解析分支在 $z = \pm 1$ 各有怎样的孤立奇点？求它们在这些点的留数。

$$\frac{\ln z}{z^2 - 1} = \frac{\ln|z| + i \arg z + 2n\pi i}{(z-1)(z+1)} =: f(z)$$

$n = 0$ 时， $f(z)$ 在 $z = 1$ 处有可去奇点，在 $z = -1$ 处有一阶极点。

$$\text{res}_1 f(z) = 0$$

$$\text{res}_{-1} f(z) = \lim_{z \rightarrow -1} (z+1)f(z) = -\frac{i\pi}{2}$$

$n \neq 0$ 时， $f(z)$ 在 $z = 1$ 处有一阶极点，在 $z = -1$ 处有一阶极点。

$$\text{res}_1 f(z) = \lim_{z \rightarrow 1} (z-1)f(z) = n\pi i$$

$$\text{res}_{-1} f(z) = \lim_{z \rightarrow -1} (z+1)f(z) = -n\pi i$$

Exercice 76

3. 计算下列积分：

$$(1) \int_C \frac{z dz}{(z-1)(z-2)^2}, \text{ 其中 } C \text{ 是 } |z-2| = \frac{1}{2};$$

$$(2) \int_C \frac{e^z dz}{z^2(z^2-9)}, \text{ 其中 } C \text{ 是 } |z| = 1;$$

$$(3) \int_C \tan \pi z dz, \text{ 其中 } C \text{ 是 } |z| = n (n = 1, 2, 3, \dots).$$

(1)

$$\begin{aligned} \int_C \frac{z}{(z-1)(z-2)} dz &= 2\pi i \cdot \text{res}_2 \left(\frac{z}{(z-1)(z-2)} \right) \\ &= 2\pi i \cdot \lim_{z \rightarrow 2} (z-2) \cdot \frac{z}{(z-1)(z-2)} \\ &= 4\pi i \end{aligned}$$

(2)

$$\begin{aligned} \int_C \frac{e^z}{z^2(z^2-9)} dz &= 2\pi i \cdot \text{res}_0 \left(\frac{e^z}{z^2(z^2-9)} \right) \\ &= 2\pi i \cdot \lim_{z \rightarrow 0} \frac{d}{dz} \left[z^2 \cdot \frac{e^z}{z^2(z^2-9)} \right] \\ &= 2\pi i \cdot \left(-\frac{1}{9} \right) \\ &= -\frac{2}{9}\pi i \end{aligned}$$

(3)

$$\begin{aligned} \int_C \tan \pi z dz &= 2\pi i \cdot \sum_{j=-n+1}^n \text{res}_{j-\frac{1}{2}}(\tan \pi z) \\ &= 2\pi i \cdot \sum_{j=-n+1}^n \lim_{z \rightarrow j-\frac{1}{2}} \left(z - j + \frac{1}{2} \right) \cdot \frac{\sin \pi z}{\cos \pi z} \\ &= 2\pi i \cdot \sum_{j=-n+1}^n -\frac{1}{\pi} \\ &= -4ni \end{aligned}$$

Exercice 77

7. 证明: 如果 $f(z)$ 在扩充复平面上除了有限个奇点外, 在每一点解析, 那么这函数在所有奇点上的留数(包括在无穷远点的留数)之和是零.

用此结果计算积分

$$\frac{1}{2\pi i} \int_{|z|=2} \frac{dz}{(z^5-1)(z-3)}.$$

记奇点为 z_1, z_2, \dots, z_n , 我们知道

$$2\pi i \sum_{j=1}^n \text{res}_{z_j} f(z) = -2\pi i \cdot \text{res}_\infty f(z)$$

由于 f 在 $z=\infty$ 处解析, 所以 $\text{res}_\infty f(z) = 0$. 从而

$$\sum_{j=1}^n \text{res}_{z_j} f(z) = 0$$

记

$$f(z) = \frac{1}{(z^5-1)(z-3)}$$

在扩充复平面上解析, 因为洛朗展开没有正项, 所以

$$\sum_{n=1}^5 \text{res}_{e^{i \cdot 2n\pi/5}} f(z) = -\text{res}_3 f(z) = -\lim_{z \rightarrow 3} (z-3)f(z) = -\frac{1}{242}$$

因此

$$\frac{1}{2\pi i} \int_{|z|=2} f(z) dz = -\frac{1}{242}$$

7.38 hw10

1 p110 8(1)(2)(4)

Exercice 78

8. 求下列各积分：

$$(1) \int_0^{+\infty} \frac{x^2 dx}{(x^2 + 1)^2};$$

$$(2) \int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}, \text{ 其中 } 0 < a < 1;$$

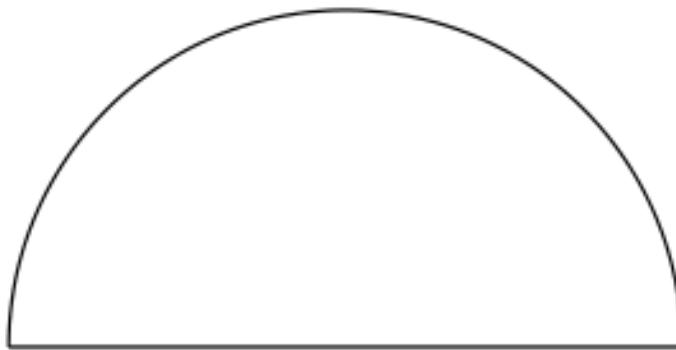
$$(3) \int_0^{\frac{\pi}{2}} \frac{dx}{a + \sin^2 x}, \text{ 其中 } a > 0;$$

$$(4) \int_0^{+\infty} \frac{x \sin x}{x^2 + 1} dx;$$

(1) Let

$$f(z) = \frac{z^2}{(z^2 + 1)^2} = \frac{z^2}{(z+i)^2(z-i)^2}$$

Consider the toy contour



Semicircle

We have

$$\begin{aligned}
\int_{-R}^R + \int_{\gamma_R^+} f(z) dz &= 2\pi i \cdot \text{res}_i f = 2\pi i \cdot \lim_{z \rightarrow i} \frac{d}{dz} \frac{1}{1!} (z-i)^2 f(z) \\
&= 2\pi i \cdot \lim_{z \rightarrow i} \frac{d}{dz} \left(\frac{z^2}{(z+i)^2} \right) \\
&= 2\pi i \cdot \lim_{z \rightarrow i} \left(\frac{2z}{(z+i)^2} - \frac{2z^2}{(z+i)^3} \right) \\
&= 2\pi i \cdot \lim_{z \rightarrow i} \left(-\frac{i}{4} \right) \\
&= \frac{\pi}{2}
\end{aligned} \tag{7.15}$$

On γ_R^+ ,

$$\left| \int_{\gamma_R^+} f(z) dz \right| \leq \int_{\gamma_R^+} |f(z)| dz = \int_{\gamma_R^+} \frac{|z^2|}{|z^2 + 1|^2} dz \leq \frac{R^2}{(R^2 - 1)^2} \cdot \pi R \rightarrow 0 \quad \text{as } R \rightarrow \infty$$

Let $R \rightarrow \infty$ in eq. (7.15), then

$$\int_{-\infty}^{\infty} f(z) dz = \frac{\pi}{2}$$

Hence

$$\int_0^{\infty} f(x) dx = \frac{1}{2} \int_{-\infty}^{\infty} f(z) dz = \frac{\pi}{4}$$

(2)

$$I = \oint_{|z|=1} \frac{1}{1 - 2a \left(\frac{z+z^{-1}}{2} \right) + a^2} \frac{dz}{iz} = \frac{i}{a} \oint_{|z|=1} \underbrace{\frac{1}{z^2 - (\frac{1+a^2}{a}) z + 1}}_{=: f(z)} dz$$

The singularities are $\frac{1}{a}$ and a . Since $a \in (0, 1)$, then

$$I = \frac{i}{a} \cdot 2\pi i \cdot \text{res}_a f = \frac{i}{a} \cdot 2\pi i \cdot \lim_{z \rightarrow a} (z-a)f(z) = \frac{i}{a} \cdot 2\pi i \cdot \frac{1}{a - \frac{1}{a}} = \frac{2\pi}{1 - a^2}$$

(4)

$$\int_0^{+\infty} \frac{x \sin x}{x^2 + 1} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 1} dx = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + 1} dx$$

Let

$$f(z) = \frac{ze^{iz}}{z^2 + 1} = \frac{ze^{iz}}{(z-i)(z+i)}$$

Consider the toy contour, an indented semicircle in the upper plane.

$$\int_{-\infty}^{\infty} + \int_{\gamma_R^+} f(z) dz = 2\pi i \cdot \text{res}_i f = 2\pi i \cdot \lim_{z \rightarrow i} (z-i)f(z) = 2\pi i \cdot \frac{ie^{-1}}{2i} = \frac{i\pi}{e} \tag{7.16}$$

On γ_R^+ ,

$$\left| \int_{\gamma_R^+} f(z) dz \right| \leq \int_{\gamma_R^+} |f(z)| |dz| = \int_0^\pi \frac{R \cdot e^{-R \sin \theta}}{|Re^{i\theta} - i| \cdot |Re^{i\theta} + i|} R d\theta \leq C \cdot e^{-R \sin \theta} \rightarrow 0 \quad \text{as } R \rightarrow \infty$$

where $\theta \in (0, \pi)$, thus $\sin \theta > 0$.

Let $R \rightarrow \infty$ in eq. (7.16), then

$$\int_{-\infty}^{\infty} f(z) dz = \frac{i\pi}{e}$$

Thus

$$\int_0^{+\infty} \frac{x \sin x}{x^2 + 1} dx = \frac{1}{2i} \int_{-\infty}^{\infty} f(z) dz = \frac{\pi}{2e}$$

7.39 hw11

1	p110 8 (6)(7)(8)(10)(12)(13)(14)
2	p111 11 12 14 15

Exercice 79

$$(6) \int_0^{+\infty} \frac{\ln x}{(x^2 + 1)^2} dx;$$

[提示] 取 $\frac{(\ln z)^2}{(z^2 + 1)^2}$ 的适当解析分支沿图 24 中的闭曲线积分.

$$(7) \int_0^{+\infty} \frac{x^{1-a}}{1+x^2} dx, \text{ 其中 } 0 < a < 2;$$

$$(8) \int_0^{+\infty} \frac{e^{ax} - e^{-ax}}{e^{\pi x} - e^{-\pi x}} dx, \text{ 其中 } -\pi < a < \pi;$$

[提示] 从顶点为 $-X_1, X_2, X_2 + 2\pi i$ 以及 $-X_1 + 2\pi i$ ($X_1, X_2 > 0$) 的矩形中挖去以 0 及 i 为心的小半圆盘及小圆盘. 考虑 $\frac{e^{az}}{e^{\pi z} - e^{-\pi z}}$ 沿这区域边界的积分.

$$(6) -\frac{\pi}{4}$$

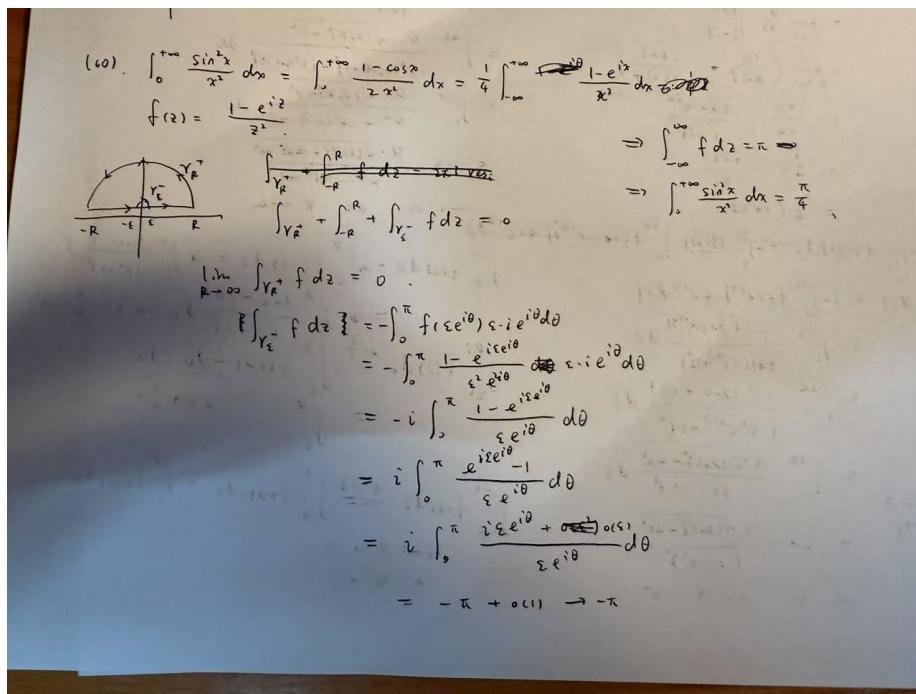
$$(7) \frac{1}{2}\pi \csc\left(\frac{a\pi}{2}\right)$$

$$(8) \frac{1}{2} \tan\left(\frac{a}{2}\right)$$

Exercice 80

$$(10) \int_0^{+\infty} \frac{\sin^2 x}{x^2} dx;$$

[提示] 这一积分可化为 $\int_0^{+\infty} \frac{1 - \cos x}{x^2} dx$.



Exercice 81

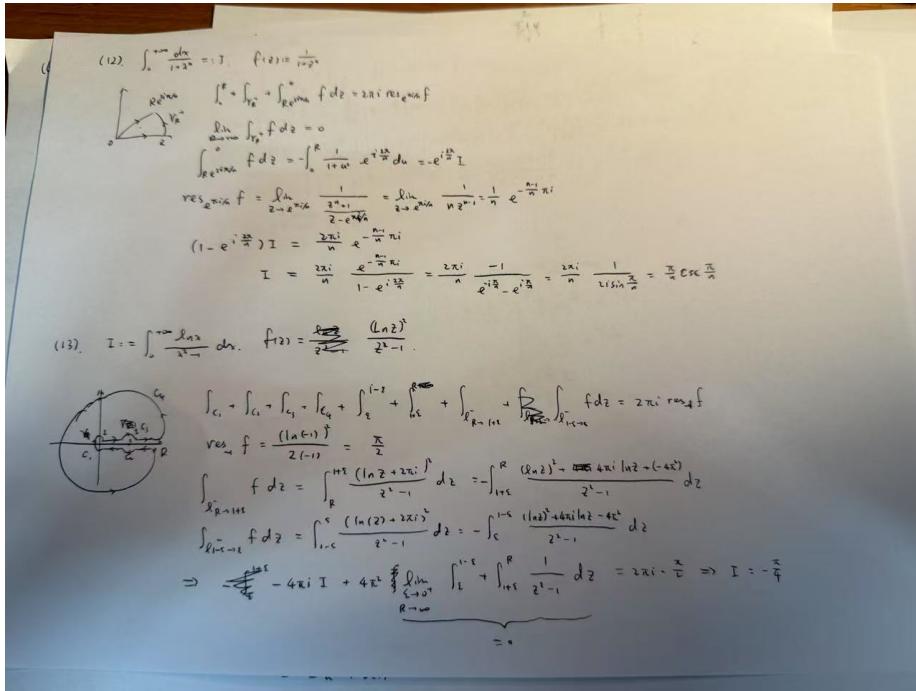
$$(12) \int_0^{+\infty} \frac{dx}{1+x^n} \text{ 其中整数 } n \geq 2;$$

[提示] 考虑第一象限内以半射线 $\arg z=0$ 及 $\arg z=\frac{2\pi}{n}$ 为边界的扇形.

$$(13) \int_0^{+\infty} \frac{\ln x}{x^2 - 1} dx;$$

$$(12) \frac{\pi}{n} \csc \frac{\pi}{n}.$$

$$(13) \frac{\pi^2}{4}.$$



Exercice 82

$$(14) \int_{-1}^1 \frac{dx}{\sqrt[3]{(1-x)(1+x)^2}};$$

$$\frac{8\pi}{9\sqrt{3}}$$

Exercice 83

11. 应用鲁歇定理,求下列方程在 $|z|<1$ 内根的个数:

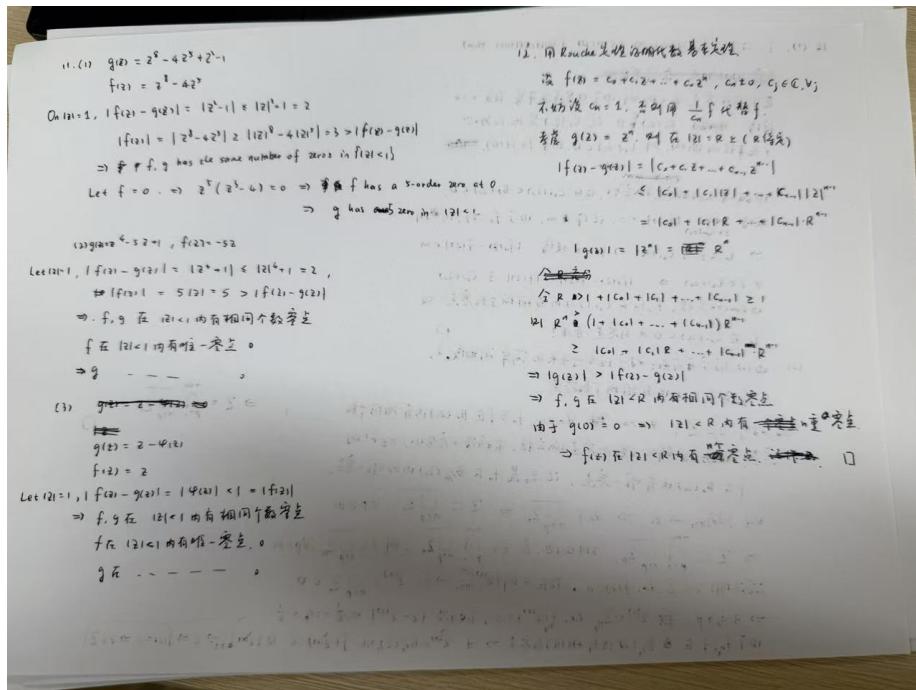
$$(1) z^8 - 4z^5 + z^2 - 1 = 0;$$

[提示] 令 $f(z) = z^8 - 4z^5$.

$$(2) z^4 - 5z + 1 = 0;$$

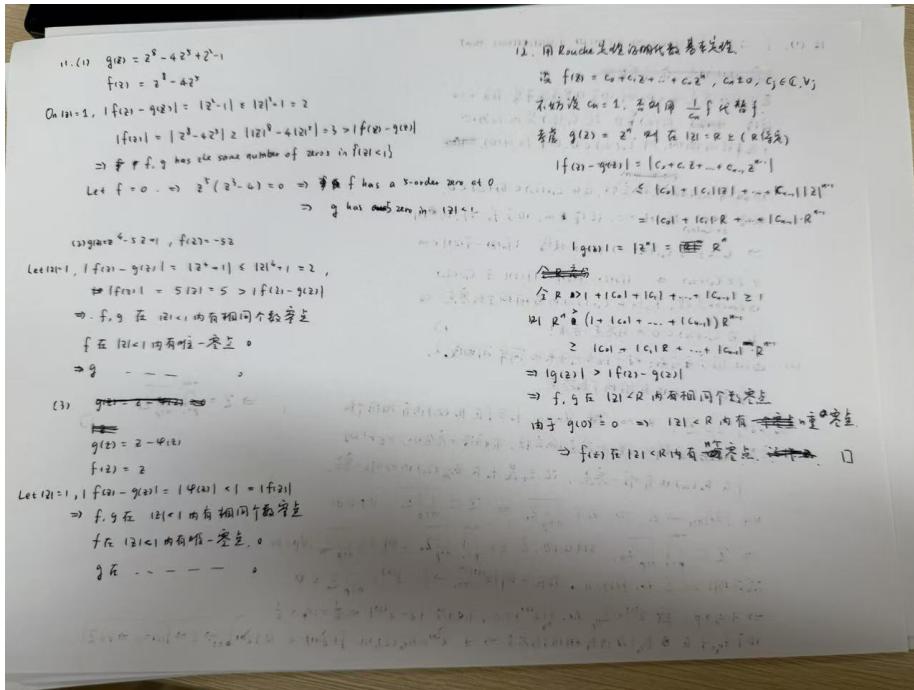
[提示] 令 $f(z) = -5z$.

(3) $z = \varphi(z)$, 其中 $\varphi(z)$ 在 $|z| \leq 1$ 上解析, 并且 $|\varphi(z)| < 1$.



Exercice 84

12. 试用鲁歇定理证明代数基本定理.



Exercice 85

* 14. 设解析函数序列 $|f_n(z)|$ 在区域 D 内内闭一致收敛于不恒等于零的函数 $f(z)$. 应用鲁歇定理, 证明:

- (1) 如果 $f_n(z)$ ($n=1, 2, \dots$) 在 D 内没有零点, 那么 $f(z)$ 在 D 内也没有零点;
- (2) 用 Z_n 及 Z 分别表示 $f_n(z)$ 及 $f(z)$ 在 D 内的零点集, 那么对于任何正整数 p ,

$$Z \subset \overline{\bigcup_{n \geq p} Z_n}, \text{ 而且 } Z = \bigcap_{p=1}^{\infty} \overline{\bigcup_{n=p}^{\infty} Z_n},$$

其中 $\overline{\bigcup_{n \geq p} Z_n}$ 表示 $\bigcup_{n \geq p} Z_n$ 的闭包, 即 $\bigcup_{n \geq p} Z_n$ 与其所有聚点组成的集的并集.

14. (1). $\int_n f_n \in H(D) \Rightarrow f_n \in H(D)$ (Weierstrass thm)

若 f 在 D 内有零点, 则可取长半径 $r > 0$

使得 $B_r(0) \subset D$, 且 $C_r(0)$ 是以 z_0 为心

r 为半径的圆周, 则 $C_r(0) \subset D$. 由于 $f \in H(D)$, 故

由解析函数的零点孤立性, 存在 $C_{r/2}(0) \subset B_r(0) \subset D$,

使得 $\min_{z \in C_{r/2}(0)} |f(z)| > 0$. 记作 m . 由于 $f_n \rightarrow f$ 在 D 上一致收敛,

$\Rightarrow f_n \equiv f$ 在 D 上一致收敛. $|f_n(z) - f(z)| < m$

$\forall z \in C_{r/2}(0) \Rightarrow |f_n(z) - f(z)| < m \leq C_{r/2}(0)$

由 Cauchy 定理, f, f_n 在 $C_{r/2}(0)$ 内有相同个数零点,

f_n 在 $C_{r/2}(0) \subset D$ 内有零点, 故而 f 在 D 内有零点.

(2) 由 (1) 推广至多变量: 对于任意可带长的简单闭曲线 γ ,

f 在 γ 内有相同个数零点.

故对于 $z \in \gamma$, $\forall r > 0$, f_n 与 f 在 $B_r(z)$ 内有相同个数零点.

由解析函数的零点孤立性, 不妨设 r 充分小, $r \leq r'$ 时

f 在 $B_{r'}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r'}(z)$ 内的唯一零点.

又由于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r''$, $r \leq r'$ 时

f 在 $B_{r''}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r''}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r'''$, $r \leq r''$ 时

f 在 $B_{r'''}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r'''}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(1)}$, $r \leq r'''$ 时

f 在 $B_{r^{(1)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(1)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(2)}$, $r \leq r^{(1)}$ 时

f 在 $B_{r^{(2)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(2)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(3)}$, $r \leq r^{(2)}$ 时

f 在 $B_{r^{(3)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(3)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(4)}$, $r \leq r^{(3)}$ 时

f 在 $B_{r^{(4)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(4)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(5)}$, $r \leq r^{(4)}$ 时

f 在 $B_{r^{(5)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(5)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(6)}$, $r \leq r^{(5)}$ 时

f 在 $B_{r^{(6)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(6)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(7)}$, $r \leq r^{(6)}$ 时

f 在 $B_{r^{(7)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(7)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(8)}$, $r \leq r^{(7)}$ 时

f 在 $B_{r^{(8)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(8)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(9)}$, $r \leq r^{(8)}$ 时

f 在 $B_{r^{(9)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(9)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(10)}$, $r \leq r^{(9)}$ 时

f 在 $B_{r^{(10)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(10)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(11)}$, $r \leq r^{(10)}$ 时

f 在 $B_{r^{(11)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(11)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(12)}$, $r \leq r^{(11)}$ 时

f 在 $B_{r^{(12)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(12)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(13)}$, $r \leq r^{(12)}$ 时

f 在 $B_{r^{(13)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(13)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(14)}$, $r \leq r^{(13)}$ 时

f 在 $B_{r^{(14)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(14)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(15)}$, $r \leq r^{(14)}$ 时

f 在 $B_{r^{(15)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(15)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(16)}$, $r \leq r^{(15)}$ 时

f 在 $B_{r^{(16)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(16)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(17)}$, $r \leq r^{(16)}$ 时

f 在 $B_{r^{(17)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(17)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(18)}$, $r \leq r^{(17)}$ 时

f 在 $B_{r^{(18)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(18)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(19)}$, $r \leq r^{(18)}$ 时

f 在 $B_{r^{(19)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(19)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(20)}$, $r \leq r^{(19)}$ 时

f 在 $B_{r^{(20)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(20)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(21)}$, $r \leq r^{(20)}$ 时

f 在 $B_{r^{(21)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(21)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(22)}$, $r \leq r^{(21)}$ 时

f 在 $B_{r^{(22)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(22)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(23)}$, $r \leq r^{(22)}$ 时

f 在 $B_{r^{(23)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(23)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(24)}$, $r \leq r^{(23)}$ 时

f 在 $B_{r^{(24)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(24)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(25)}$, $r \leq r^{(24)}$ 时

f 在 $B_{r^{(25)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(25)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(26)}$, $r \leq r^{(25)}$ 时

f 在 $B_{r^{(26)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(26)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(27)}$, $r \leq r^{(26)}$ 时

f 在 $B_{r^{(27)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(27)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(28)}$, $r \leq r^{(27)}$ 时

f 在 $B_{r^{(28)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(28)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(29)}$, $r \leq r^{(28)}$ 时

f 在 $B_{r^{(29)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(29)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(30)}$, $r \leq r^{(29)}$ 时

f 在 $B_{r^{(30)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(30)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(31)}$, $r \leq r^{(30)}$ 时

f 在 $B_{r^{(31)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(31)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(32)}$, $r \leq r^{(31)}$ 时

f 在 $B_{r^{(32)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(32)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

由解析函数的零点孤立性, 不妨设 $r \leq r^{(33)}$, $r \leq r^{(32)}$ 时

f 在 $B_{r^{(33)}}(z)$ 内有唯一零点, 记 z_0 是 f 在 $B_{r^{(33)}}(z)$ 内的唯一零点.

故对于 $z \in \gamma$, $\forall n \geq 1$, $z_0 \in \overline{\bigcup_{n \geq 1} Z_n}$.

(除了有限个极点外), 并且在 C 上没有零点和极点。那么, 当 z 沿着曲线 C 逆时针方向绕行一周时, $f(z)$ 的辐角变化量等于 2π 乘以函数 $f(z)$ 在曲线 C 内部零点的个数 (计重数) 减去极点的个数 (计重数)。用公式表示:

$$\Delta_C \arg f(z) = 2\pi(N - P)$$

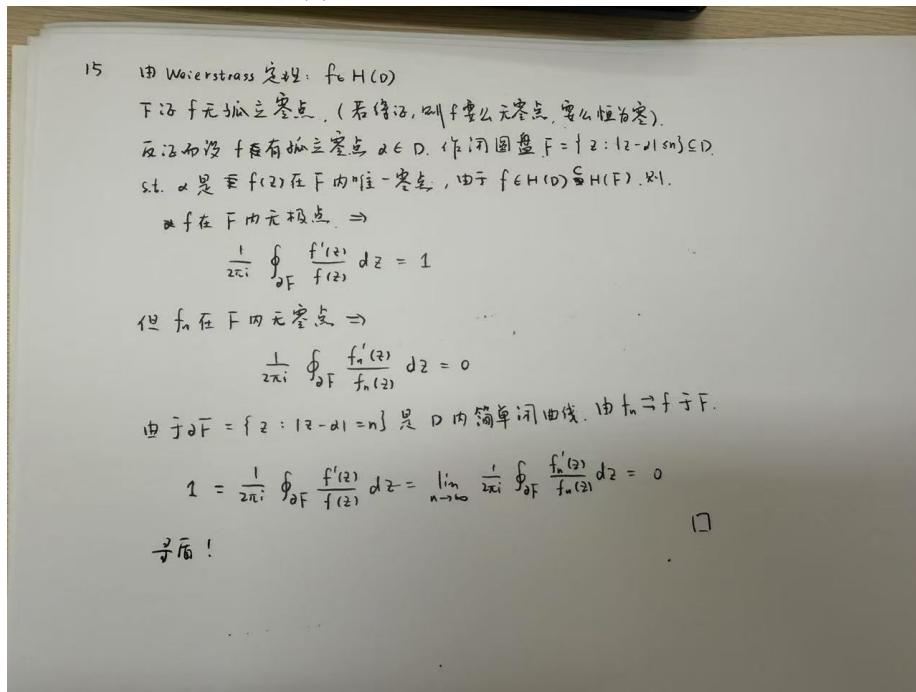
其中:

- $\Delta_C \arg f(z)$ 表示当 z 沿曲线 C 逆时针方向绕行一周时, $f(z)$ 的辐角变化量。
- N 表示函数 $f(z)$ 在曲线 C 内部零点的个数 (如果一个零点是 k 重的, 则计为 k 个)。
- P 表示函数 $f(z)$ 在曲线 C 内部极点的个数 (如果一个极点是 k 重的, 则计为 k 个)。

另一种等价的表述形式是利用对数导数:

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N - P$$

这个积分实际上计算了 $f(z)$ 的像在原点周围缠绕的圈数。



7.40 hw12

1 p134 1 3 4 5 6 7

Exercice 87

1. 如果单叶解析函数 $w=f(z)$ 把 z 平面上可求面积的区域 D 映射成 w 平面上的区域 D^* , 把 D 中分段光滑曲线 l 映射成 D^* 中的曲线 l^* . 证明 l^* 的长度是

$$\int_l |f'(z)| |dz|,$$

D^* 的面积是

$$\iint_D |f'(z)|^2 dx dy.$$

Consider the parameterization of l ,

$$\gamma : [0, 1] \rightarrow l \subset D$$

Then the parameterization of l^* is

$$f \circ \gamma : [0, 1] \rightarrow l \rightarrow l^* \subset D^*$$

Since $w = f(z)$ is injective, the path $f \circ \gamma$ is well-defined simple closed curve.

Then

$$\begin{aligned} \text{the length of } l^* &= \ell(f \circ \gamma) \\ &= \int_0^1 |(f \circ \gamma)'(t)| dt \\ &= \int_0^1 |f'(\gamma(t))| \cdot |\gamma'(t)| dt \end{aligned}$$

On the other hand,

$$\int_l |f'(z)| |dz| = \int_{\gamma} |f'(z)| |dz| = \int_0^1 |f'(\gamma(t))| \cdot |\gamma'(t)| dt = \int_0^1 |f'(\gamma(t))| \cdot |\gamma'(t)| dt$$

Therefore

$$\text{the length of } l^* = \int_l |f'(z)| |dz|$$

Denote $w = f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$.

The area $A(D^*)$ of the region D^* in the w -plane can be obtained by variable substitution using the Jacobian determinant:

$$A(D^*) = \iint_{D^*} du dv$$

According to the variable substitution formula for double integrals, we have:

$$du \ dv = \left| \frac{\partial(u, v)}{\partial(x, y)} \right| dx \ dy$$

where $\frac{\partial(u, v)}{\partial(x, y)}$ is the Jacobian determinant:

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}$$

Since $f(z)$ is an analytic function, its real part $u(x, y)$ and imaginary part $v(x, y)$ satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Substituting the Cauchy-Riemann equations into the Jacobian determinant:

$$\frac{\partial(u, v)}{\partial(x, y)} = \left(\frac{\partial u}{\partial x} \right)^2 - \left(-\frac{\partial v}{\partial x} \right) \frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2$$

We know that the complex derivative $f'(z)$ can be expressed as:

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Therefore, the square of the modulus of the complex derivative is:

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2$$

So, the value of the Jacobian determinant is equal to $|f'(z)|^2$:

$$\left| \frac{\partial(u, v)}{\partial(x, y)} \right| = |f'(z)|^2$$

Now we substitute this result back into the area integral formula:

$$A(D^*) = \iint_D \left| \frac{\partial(u, v)}{\partial(x, y)} \right| dx dy = \iint_D |f'(z)|^2 dx dy$$

This proves the second conclusion.

Exercice 88

3. (1) 证明 $w = z + \frac{1}{n}z^n$ 在 $|z| < 1$ 内单叶;

(2) 证明 $w = \frac{z}{(1-z)^3}$ 在 $|z| < \frac{1}{2}$ 内单叶;

(3) 证明 $w = z + z^2$ 在 $|z| < \frac{1}{2}$ 内单叶.

(1) For $z_1, z_2 \in \{z : |z| < 1\}$, if

$$z_1 + \frac{1}{n}z_1^n = z_2 + \frac{1}{n}z_2^n$$

Then

$$\begin{aligned} z_1 - z_2 &= \frac{1}{n}(z_2^n - z_1^n) \\ |z_1 - z_2| &= \frac{1}{n}|z_2^n - z_1^n| = \frac{1}{n}|(z_2 - z_1)(z_2^{n-1} + z_2^{n-2}z_1 + \dots + z_1^{n-1})| \\ &\leq \frac{1}{n}|z_2 - z_1|(|z_2|^{n-1} + |z_2|^{n-2}|z_1| + \dots + |z_1|^{n-1}) \\ &< \frac{1}{n}|z_2 - z_1| \cdot n \\ &= |z_2 - z_1| \end{aligned}$$

Thus $z_1 = z_2$. w is injective.

(2) For $z_1, z_2 \in \{z : |z| < \frac{1}{2}\}$, if

$$\frac{z_1}{(1-z)^3} = \frac{z_2}{(1-z_2)^3}$$

Then

$$\begin{aligned} z_1 - 3z_1z_2 + 3z_1z_2^2 - z_1z_2^3 &= z_2 - 3z_2z_1 + 3z_2z_1^2 - z_2z_1^3 \\ |z_1 - z_2| &= |3z_1z_2(z_1 - z_2) + z_1z_2(z_2 - z_1)(z_1 + z_2)| \\ &\leq 3|z_1z_2||z_1 - z_2| + |z_1z_2||z_1 + z_2||z_2 - z_1| \\ &< \frac{3}{4}|z_1 - z_2| + \frac{1}{4}|z_1 - z_2| = |z_1 - z_2| \end{aligned}$$

Thus $z_1 = z_2$. w is injective.

(3) For $z_1, z_2 \in \{z : |z| < \frac{1}{2}\}$, if

$$z_1 + z_1^2 = z_2 + z_2^2$$

Then

$$|z_1 - z_2| = |z_1^2 - z_2^2| = |z_1 - z_2||z_1 + z_2| \leq (|z_1| + |z_2|)|z_1 - z_2| < |z_1 - z_2|$$

Thus $z_1 = z_2$. w is injective.

Exercice 89

4. 设 $f(z) = \frac{1}{2}z^2 + \sum_{n=3}^{+\infty} \frac{a_n}{n} z^n$ 在 $|z| < R$ 内解析，并且 $|f'(z)| \leq M$ ，其中 $R, M \in (0, +\infty)$ 。证明： $\forall z_0 \in \{z : 0 < |z_0| < \frac{R^2}{M+R}\}$ ， $f'(z_0) \neq 0$ ，从而 $f(z)$ 在 z_0 的一个邻域内单叶。

[提示] 应用柯西不等式。

Denote

$$g(z) := f'(z) = z + \sum_{n=2}^{\infty} a_{n+1} z^n$$

Then

$$|g(z)| \leq M, \forall |z| < R$$

By Cauchy's formula,

$$a_{n+1} = \frac{1}{2\pi i} \oint_{|z|=r} \frac{g(\xi)}{\xi^{n+1}} d\xi$$

where $r \in (0, R)$. Then

$$|a_{n+1}| \leq \frac{1}{2\pi} \oint_{|z|=r} \left| \frac{g(\xi)}{\xi^{n+1}} \right| d\xi \leq \frac{1}{2\pi} \cdot 2\pi r \cdot \frac{M}{r^{n+1}} = \frac{M}{r^n}$$

Let $r \rightarrow R$, then

$$|a_{n+1}| \leq \frac{M}{R^n}$$

Thus for any $z \in \left\{ z : |z| < \frac{R^2}{M+R} \right\}$,

$$\begin{aligned} \left| \sum_{n=2}^{\infty} a_{n+1} z^n \right| &\leq \sum_{n=2}^{\infty} \underbrace{|a_{n+1}|}_{\leq \frac{M}{R^n}} |z|^n \\ &\leq \sum_{n=2}^{\infty} M \left| \frac{z}{R} \right|^n \\ &= |z| \cdot \frac{M}{R} \sum_{n=2}^{\infty} \left| \frac{z}{R} \right|^{n-1} \\ &< |z| \cdot \frac{M}{R} \sum_{n=2}^{\infty} \left| \frac{R}{M+R} \right|^{n-1} \\ &= |z| \cdot \frac{M}{R} \cdot \frac{\frac{R}{M+R}}{1 - \frac{R}{M+R}} \\ &= |z| \end{aligned}$$

By Rouché's theorem, z and $g(z)$ has the same numbers of zero's in $\{z : |z| < \frac{R^2}{M+R}\}$, i.e. only 1 zero. As $g(0) = 0$, $g(z)$ has no zero in $\{z : 0 < |z| < \frac{R^2}{M+R}\}$.

Assume that f is not injective in $\{z : 0 < |z| < \frac{R^2}{M+R}\}$. There exists z_1, z_2 s.t.

$$f(z_1) = f(z_2)$$

By the MVT, there exists $\theta \in [0, 1]$, s.t.

$$f'(\theta z_1 + (1 - \theta)z_2) = 0$$

By the convexity of $\{z : 0 < |z| < \frac{R^2}{M+R}\}$,

$$\theta z_1 + (1 - \theta)z_2 \in \left\{ z : 0 < |z| < \frac{R^2}{M+R} \right\}$$

is a zero of $g(z)$, which is a contradiction. Hence, f is injective in $\{z : 0 < |z| < \frac{R^2}{M+R}\}$; for any $z_0 \in \{z : 0 < |z| < \frac{R^2}{M+R}\}$, it is an interior point of $\{z : 0 < |z| < \frac{R^2}{M+R}\}$, thus has a neighborhood where f is injective.

Exercice 90

5. 设 $w = \frac{az+b}{cz+d}$ 是一分式线性函数. 如果采用矩阵记号, 令

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$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

那么已知函数的反函数是

$$z = \frac{\alpha w + \beta}{\gamma w + \delta}.$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

Then

$$\begin{aligned}
 \frac{a\frac{\alpha w+\beta}{\gamma w+\delta}+b}{c\frac{\alpha w+\beta}{\gamma w+\delta}+d} &= \frac{a\alpha w + a\beta + b\gamma w + b\delta}{c\alpha w + c\beta + d\gamma w + d\delta} \\
 &= \frac{a\frac{d}{ad-bc}w + a\left(-\frac{b}{ad-bc}\right) + b\left(-\frac{c}{ad-bc}\right)w + b\frac{a}{ad-bc}}{c\frac{d}{ad-bc}w + c\left(-\frac{b}{ad-bc}\right) + d\left(-\frac{c}{ad-bc}\right)w + d\frac{a}{ad-bc}} \\
 &= \frac{adw - ab - bcw + ab}{cdw - bc - cdw + ad} \\
 &= w \\
 \frac{\alpha\frac{az+b}{cz+d}+\beta}{\gamma\frac{az+b}{cz+d}+\delta} &= \frac{\frac{d}{ad-bc}\frac{az+b}{cz+d} - \frac{b}{ad-bc}}{-\frac{c}{ad-bc}\frac{az+b}{cz+d} + \frac{a}{ad-bc}} \\
 &= \frac{adz + bd - bcz - bd}{-acz - bc + acz + ad} \\
 &= z
 \end{aligned}$$

Thus the inverse of $w = \frac{az+b}{cz+d}$ is

$$z = \frac{\alpha w + \beta}{\gamma w + \delta}$$

Exercice 91

6. 设有两分式线性函数 $w = \frac{\alpha_1 w_1 + \beta_1}{\gamma_1 w_1 + \delta_1}$ 及 $w_1 = \frac{\alpha_2 z + \beta_2}{\gamma_2 z + \delta_2}$,

如果采用矩阵记号, 令

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix},$$

那么 $w = \frac{az + b}{cz + d}$.

$$\begin{aligned}
 \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 + \beta_1\gamma_2 & \alpha_1\beta_2 + \beta_1\delta_2 \\ \alpha_2\gamma_1 + \gamma_2\delta_1 & \beta_2\gamma_1 + \delta_1\delta_2 \end{pmatrix} \\
 w &= \frac{\alpha_1 w_1 + \beta_1}{\gamma_1 w_1 + \delta_1} \quad w_1 = \frac{\alpha_2 z + \beta_2}{\gamma_2 z + \delta_2}
 \end{aligned}$$

Then

$$w = \frac{\alpha_1 \frac{\alpha_2 z + \beta_2}{\gamma_2 z + \delta_2} + \beta_1}{\gamma_1 \frac{\alpha_2 z + \beta_2}{\gamma_2 z + \delta_2} + \delta_1} = \frac{\alpha_1\beta_2 + \beta_1\delta_2 + \alpha_1\alpha_2 z + \beta_1\gamma_2 z}{\beta_2\gamma_1 + \delta_1\delta_2 + \alpha_2\gamma_1 z + \gamma_2\delta_1 z} = \frac{az + b}{cz + d}$$

Exercice 92

7. 试用交比,求满足下列条件的分式线性函数:

- (1) 把 $-1, i, 1+i$ 分别映射成 $0, 2i, 1-i$;
- (2) 把 $-1, \infty, i$ 分别映射成 $\infty, i, 1$.

Theorem 7.4 Cross-Ratios and Linear Fractional Transformations

If $w = T(z)$ is a linear fractional transformation that maps the distinct points z_1, z_2 , and z_3 onto the distinct points w_1, w_2 , and w_3 , respectively, then

$$\frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1} = \frac{w - w_1}{w - w_3} \frac{w_2 - w_3}{w_2 - w_1} \quad (12)$$

for all z .

(1)

$$\frac{z - (-1)}{z - (1+i)} \cdot \frac{i - (1+i)}{i - (-1)} = \frac{w - 0}{w - (1-i)} \cdot \frac{2i - (1-i)}{2i - 0}$$

Then

$$w = \frac{(1-i)z + (1-i)}{(2+2i)z + (2-3i)}$$

(2)

$$\frac{z - (-1)}{z - i} \cdot \frac{\infty - i}{\infty - (-1)} = \frac{w - \infty}{w - 1} \cdot \frac{i - 1}{i - \infty}$$

i.e.

$$\frac{z + 1}{z - i} = \frac{i - 1}{w - 1}$$

Then

$$w = \frac{iz + (2+i)}{z + 1}$$

7.41 hw13

1 [p135 8 10 11]

Exercice 93

8. 如果 $f(z)$ 在区域 D 内解析, 不为常数, 且没有零点, 证明 $|f(z)|$ 不可能在 D 内达到最小值.

$f \in H(D)$ then $1/f \in H(D)$, by maximum module principle, $|1/f|$ cannot attain its maximum inside D .

Exercice 94

10. 设在 $|z| < 1$ 内, $f(z)$ 解析, 并且 $|f(z)| < 1$; 又设 $|\alpha| < 1$. 证明: 在 $|z| < 1$ 内, 有不等式

$$\left| \frac{f(z) - f(\alpha)}{1 - \overline{f(\alpha)}f(z)} \right| \leq \left| \frac{z - \alpha}{1 - \bar{\alpha}z} \right|.$$

Consider $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, for $|a| < 1$; let $g = \varphi_b \circ f \circ \varphi_a^{-1}$, where $b = f(a)$. Then

$$|g(0)| = |\varphi_b(f(\varphi_a^{-1}(0)))| = |\varphi_b(f(a))| = 0$$

For $z \in \mathbb{D}$, $|\varphi_a^{-1}(z)| = |\varphi_{-a}(z)| < 1$,

$$|g(z)| = |\varphi_b(f(\underbrace{\varphi_a^{-1}(z)}_{\in \mathbb{D}}))| < 1$$

$$\underbrace{\varphi_b}_{\in \mathbb{D}} \circ \underbrace{f}_{\in \mathbb{D}} \circ \underbrace{\varphi_a^{-1}}_{\in \mathbb{D}}(z)$$

By Schwarz lemma, for any $z \in \mathbb{D}$,

$$|g(z)| \leq |z|$$

where

$$|g(z)| = |\varphi_b(f(\varphi_{-a}(z)))| = \left| \frac{f(a) - f(\varphi_{-a}(z))}{1 - \overline{f(a)}f(\varphi_{-a}(z))} \right| \leq |z|$$

Replace z by $\varphi_a(z)$ then

$$\left| \frac{f(a) - f(z)}{1 - \overline{f(a)}f(z)} \right| \leq \left| \frac{a - z}{1 - \bar{a}z} \right|$$

Exercice 95

11. 应用施瓦茨引理, 证明: 把 $|z|<1$ 变为 $|w|<1$, 且把 α 变为 0 的保形双射一定有下列形状:

$$w = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha}z},$$

这里 θ 是实常数, α 是满足 $|\alpha|<1$ 的复常数.

Clearly, $\varphi_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}$ maps α to 0, and is automorphism in \mathbb{D} . For any $f \in \text{Aut}(\mathbb{D})$ with $f(\alpha) = 0$, consider $g = f \circ \varphi_\alpha^{-1} = f \circ \varphi_{-\alpha}$; $g(0) = 0$, $|g(z)| < 1$, by Schwarz lemma, $|g(z)| \leq |z|$. But the equality holds at α , thus $g(z) = e^{i\theta}z$ for some real θ . i.e. $f(z) = e^{i\theta}\varphi_\alpha(z)$.

7.42 hw14

¹ chapter 6: 12 17 19 20

² chapter 7: 2 3

Exercice 96

12. 试作保形映射:

- (1) 把带形区域 $\pi < y < 2\pi$ 映射成上半平面;
- (2) 把去掉上半虚轴的复平面映射成上半平面.

(1) $w = -e^z$.

(2) $w = (-iz)^{1/2}$.

Exercice 97

17. 试作一单叶解析函数 $w=f(z)$, 把 $|z|<1$ 映射成 $|w|<1$, 并且使 $f(0)=\frac{1}{2}, f'(0)>0$.

$$f(z) = \frac{\frac{1}{2} + z}{1 + \frac{z}{2}}$$

Then for any $|z| < 1$,

$$|f(z)|^2 = \frac{\frac{1}{2} + z}{1 + \frac{z}{2}} \cdot \frac{\frac{1}{2} + \bar{z}}{1 + \frac{\bar{z}}{2}} = \frac{\frac{1}{4} + |z|^2 + \frac{1}{2}(z + \bar{z})}{1 + \frac{1}{4}|z|^2 + \frac{1}{2}(z + \bar{z})} = 1 + \underbrace{\frac{\frac{3}{4}(|z|^2 - 1)}{1 + \frac{1}{4}|z|^2 + \frac{1}{2}(z + \bar{z})}}_{=|1+\frac{z}{2}|^2>0} < 1$$

Then $|f(z)| < 1$. For $|z| = 1$, we have $|f(z)| = 1$. Observe that

$$f(0) = \frac{1}{2} \quad f\left(-\frac{1}{2}\right) = 0$$

Then $g := -f(-f(z))$, maps \mathbb{D} into \mathbb{D} , and

$$g(0) = -f(-f(0)) = -f\left(-\frac{1}{2}\right) = 0 \quad g\left(-\frac{1}{2}\right) = -f\left(-f\left(-\frac{1}{2}\right)\right) = -f(0) = -\frac{1}{2}$$

By the Schwarz lemma, $g : z \mapsto ze^{i\theta}$, and $\theta = 0$. Thus g is identity map of \mathbb{D} , thus f is automorphism of \mathbb{D} . Also

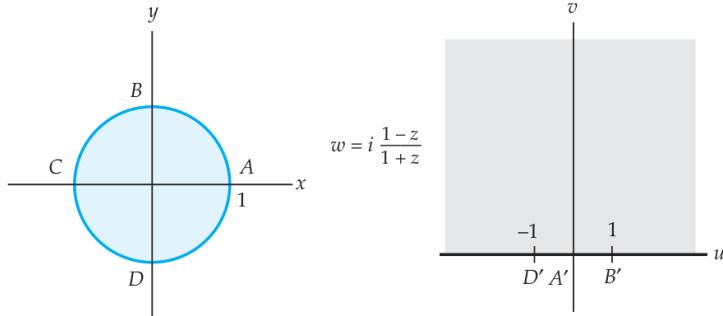
$$f'(z) = \frac{1}{1 + \frac{z}{2}} - \frac{\frac{1}{2}\left(\frac{1}{2} + z\right)}{\left(1 + \frac{z}{2}\right)^2}$$

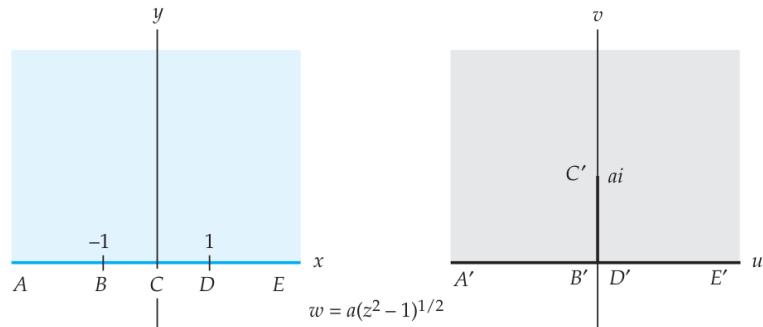
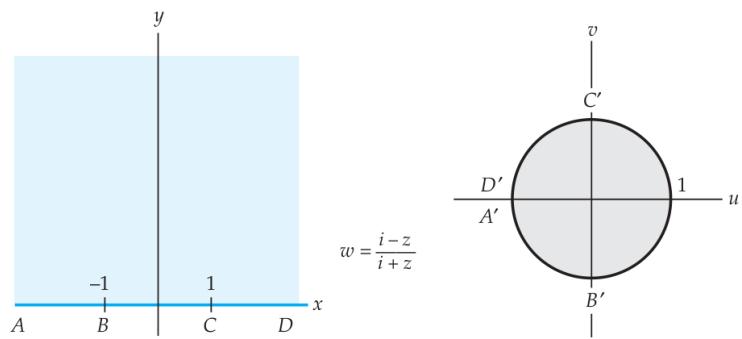
Then $f'(0) = 1 - \frac{1}{4} = \frac{3}{4} > 0$. We are done!

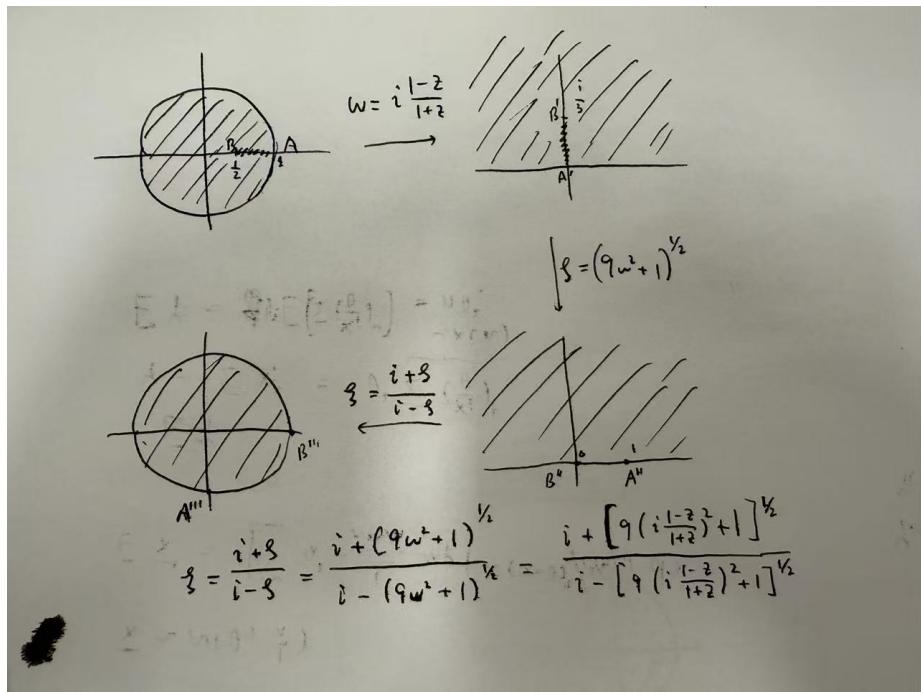
Exercice 98

19. 在圆盘 $|z| < 1$ 中除去实轴上的半闭区间 $\left[\frac{1}{2}, 1\right)$, 得一区域. 试把这一区域保形映射成圆盘 $|w| < 1$.

H-1



M-4**C-4**



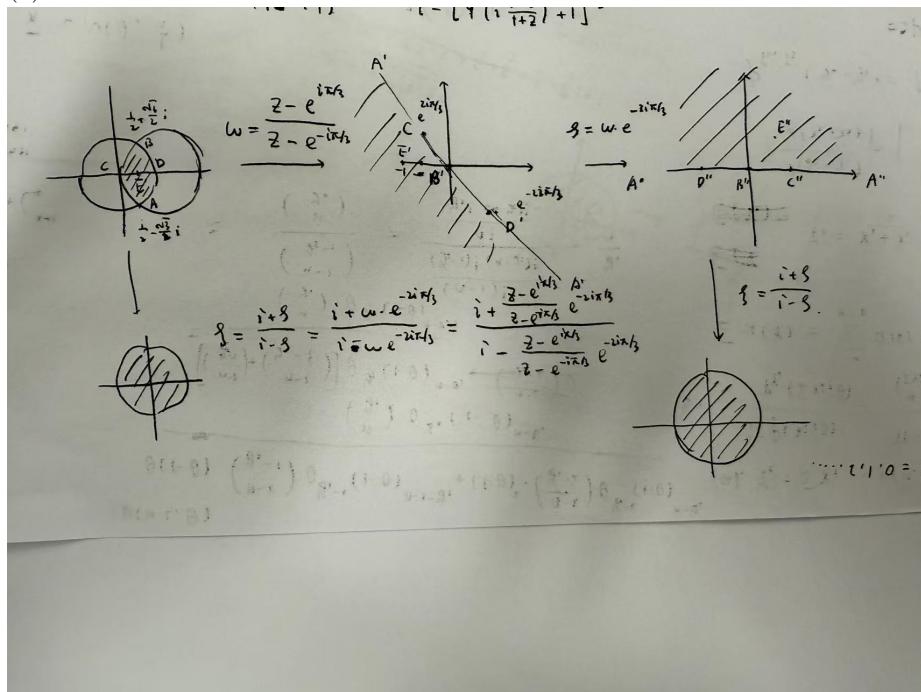
$$\xi = \frac{i + \left[9 \left(i \frac{1-z}{1+z} \right)^2 + 1 \right]^{1/2}}{i - \left[9 \left(i \frac{1-z}{1+z} \right)^2 + 1 \right]^{1/2}}$$

Exercice 99

20. 试作保形映射:

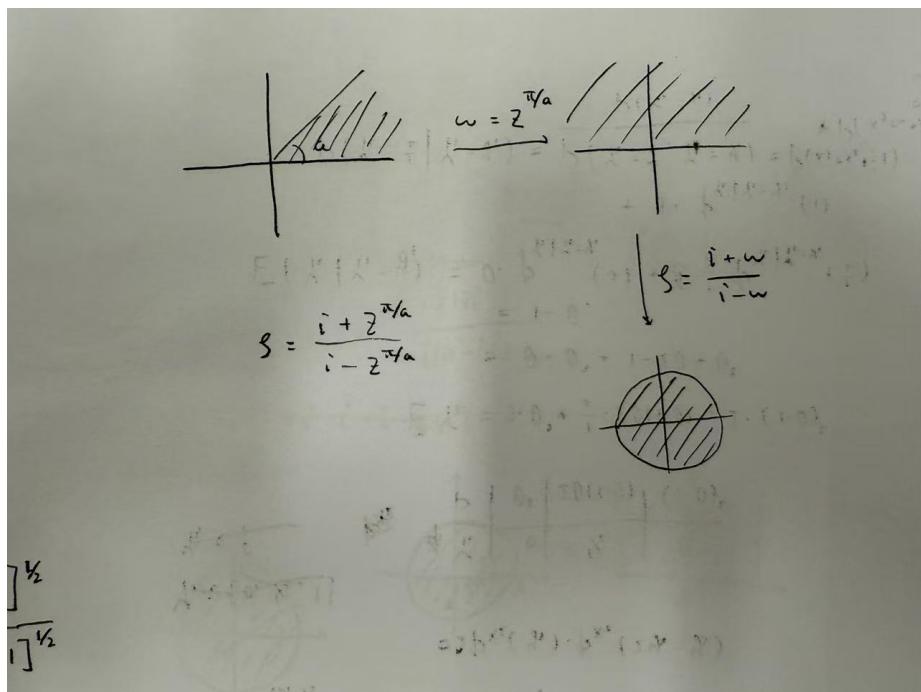
- (1) 把 $|z|<1$ 及 $|z-1|<1$ 的公共部分映射成 $|w|<1$;
- (2) 把扇形 $0 < \arg z < a (< 2\pi)$, $|z|<1$ 映射成 $|w|<1$;
- (3) 把圆 $|z|=2$ 及 $|z-1|=1$ 所夹的区域映射成 $|w|<1$;
- (4) 把圆 $|z|<1$ 映射成带形 $0 < v < 1$, 并把 $-1, 1, i$ 映射成 ∞, ∞, i .

(1)



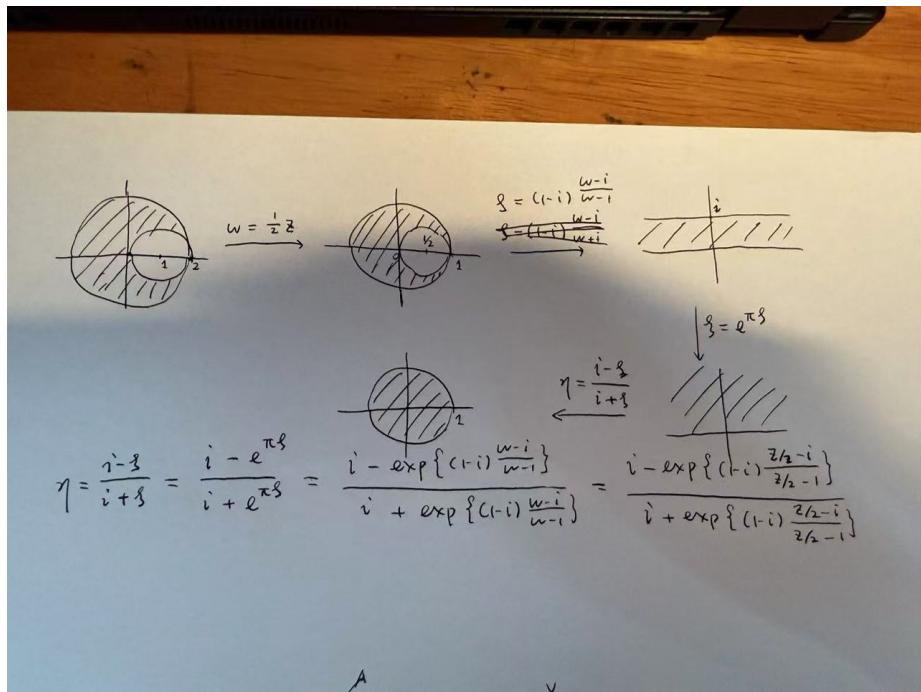
(2)

$$\xi = \frac{i + \frac{z - e^{i\pi/3}}{z - e^{i\pi/3}} e^{-2i\pi/3}}{i - \frac{z - e^{i\pi/3}}{z - e^{i\pi/3}} e^{-2i\pi/3}}$$

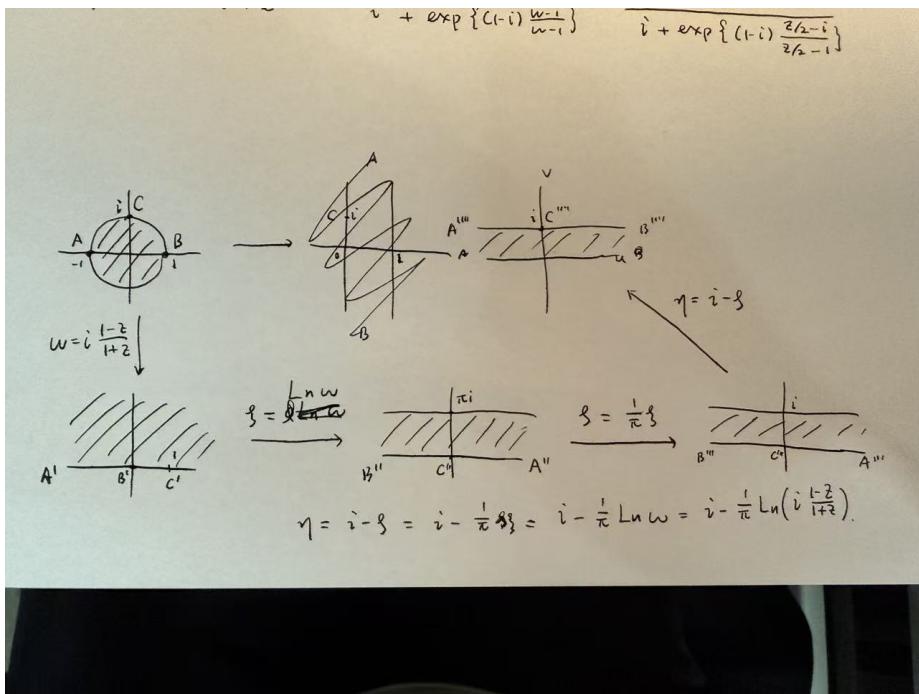


$$\zeta = \frac{i + z^{\pi/a}}{i - z^{\pi/a}}$$

(3)



(4)



$$\eta = i - \frac{1}{\pi} \operatorname{Ln} \left(i \frac{1-z}{1+z} \right)$$

Exercice 100

2. 设函数 $w=f(z)$ 在 $\operatorname{Im} z \geq 0$ 上单叶解析，并且把 $\operatorname{Im} z > 0$ 保形映射成 $|w| < 1$ ，把 $\operatorname{Im} z = 0$ 映射成 $|w| = 1$. 证明 $f(z)$ 一定是分式线性函数.

Proposition 5.2.2 Any conformal map of $D = \{z \text{ such that } |z| < 1\}$ onto itself is a fractional linear transformation of the form

$$T(z) = e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}$$

for some fixed $z_0 \in D$ and $\theta \in [0, 2\pi[$; moreover, any T of this form is a conformal map of D onto D .

考虑共形映射

$$g : \overline{\mathbb{D}} \rightarrow \{\operatorname{Im} z \geq 0\} \quad z \mapsto i \frac{1-z}{1+z}$$

$$h : \{\operatorname{Im} z \geq 0\} \rightarrow \overline{\mathbb{D}} \quad z \mapsto \frac{i-z}{i+z}$$

他们互为逆映射，由于 $f|_{\{\operatorname{Im} z > 0\}}$ 的值域为 \mathbb{D} , $g|_{\mathbb{D}}$ 的值域为 $\{\operatorname{Im} z > 0\}$ ，于是 $f \circ g$ 是 \mathbb{D} 的自同构，且为共形映射。从而

$$f(g(z)) = e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}, \quad \forall z \in \mathbb{D}$$

于是

$$f|_{\{\operatorname{Im} z > 0\}}(z) = e^{i\theta} \frac{h(z) - z_0}{1 - \bar{z}_0 h(z)}$$

是分式线性映射。由唯一性原理可知， f 是分式线性映射。

单叶解析函数和共形映射是复变函数理论中的两个重要概念，它们之间有密切的联系，但含义不同。

单叶解析函数

一个函数 $f(z)$ 如果在区域 D 内是解析的，并且对于 D 中任意不同的两点 $z_1 \neq z_2$ ，都有 $f(z_1) \neq f(z_2)$ ，则称 $f(z)$ 在 D 内是单叶的（或单价的、单射的）。

简单来说，单叶就是指函数在定义域内是一对一的（injective）。这是函数的一个整体性质。

共形映射

一个函数 $f(z)$ 如果在区域 D 内是解析的，并且对于 D 内的每一点 z_0 ，它都保持经过 z_0 的任意两条光滑曲线的夹角大小和定向不变，则称 $f(z)$ 在 D 内是共形的。

解析函数保持角度和定向的充要条件是其在该点的导数不为零。所以，一个解析函数 $f(z)$ 在 D 内是共形的，等价于 $f(z)$ 在 D 内是解析的且对于所有的 $z \in D$ ，都有 $f'(z) \neq 0$ 。

共形性是函数的一个局部性质（在每一点 z_0 附近保持角度）。

区别与联系

(1) 定义不同：单叶强调的是函数的整体一对一致性；共形强调的是函数在每一点的局部保角和保向性（对于解析函数，等价于导数不为零）。

(2) 性质包含：

- 如果一个解析函数在区域 D 内是单叶的，那么它在该区域内一定是共形的。这是因为，如果 $f(z)$ 是单叶解析的，可以证明其导数在 D 内不可能为零（如果 $f'(z_0) = 0$ 且 z_0 是零点，解析函数的零点是孤立的，局部上函数行为类似 $c(z - z_0)^m$ ($m \geq 2$)，这会破坏单叶性）。而解析且导数不为零正是共形的定义。
- 然而，如果一个解析函数在区域 D 内是共形的（即解析且 $f'(z) \neq 0$ ），它不一定是单叶的。例如，函数 $f(z) = z^2$ 在区域 $D = \{z : |z| < 2\}$

内是解析的，且对于 $z \neq 0$, 有 $f'(z) = 2z \neq 0$ 。因此, $f(z) = z^2$ 在 $D \setminus \{0\}$ 内是共形的。但在 D 内, 它不是单叶的, 例如 $f(1) = 1$, $f(-1) = 1$ 。

总结:

对于解析函数而言, 单叶性是比共形性更强的条件。单叶解析函数一定是共形映射, 但共形映射不一定是单叶解析函数。单叶性关注的是映射的整体结构 (一对一), 而共形性 (对解析函数而言) 关注的是映射在局部的几何性质 (保角保向, 导数不为零)。

Exercice 101

3. 证明: 如果整函数 $f(z) = \sum_{n=0}^{\infty} a_n z^n$ 在实轴上取实值, 在虚轴上取虚值, 那么 $f(z)$ 是奇函数.

Théorème 26: 唯一性原理

设 f 和 g 是区域 D 上的解析函数。如果存在 D 中的一个非空开集 U , 使得对于所有 $z \in U$ 都有 $f(z) = g(z)$, 那么对于所有 $z \in D$ 都有 $f(z) = g(z)$ 。

Théorème 27: 对称原理

设 $G \subset \mathbb{C}$ 是一个区域, 并且 $G \cap \mathbb{R} \neq \emptyset$. 记 $G^+ = \{z \in G : \operatorname{Im} z > 0\}$, 且假设 $f : G^+ \rightarrow \mathbb{C}$ 是解析的, 并且对于每个 $x \in G \cap \mathbb{R}$, 极限 $\lim_{z \rightarrow x} f(z)$ 存在, 且为实数. 那么存在一个解析函数 $\tilde{f} : G \cup G^* \cup (G \cap \mathbb{R}) \rightarrow \mathbb{C}$, 使得 $\tilde{f}(z) = f(z)$, 对所有 $z \in G^+$ 成立. 这里 $G^* = \{z : \bar{z} \in G\}$, 且对 $z \in G^-$ 有 $\tilde{f}(z) = \overline{f(\bar{z})}$.

由唯一性原理, 我们可以把下半平面上的 $f(z)$ 看作上半平面的 $f(z)$ 由对称原理解析延拓而来, 于是在下半平面上应该有 $f(z) = \overline{f(\bar{z})}$.

$$\sum_{n=0}^{\infty} a_n z^n = \overline{\sum_{n=0}^{\infty} a_n \bar{z}^n} = \sum_{n=0}^{\infty} \overline{a_n} z^n$$

于是 a_n 都是实数. 由于 $f(z)$ 在虚轴上取虚值, 对于任意实数 x , 我们有

$$f(xi) = -\overline{f(xi)}$$

也就是

$$\sum_{n=0}^{\infty} a_n (xi)^n = - \sum_{n=0}^{\infty} a_n (-xi)^n$$

由 x 的任意性可知, $a_n = (-1)^{n-1} a_n$, 于是对于偶数 n , 我们有 $a_n = 0$. 从而

$$f(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}$$

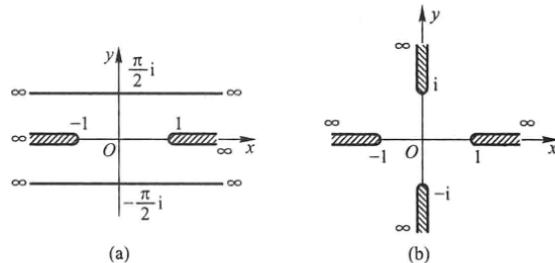
从而 $f(z) = f(-z)$, f 是奇函数.

7.43 hw15

1 p159 4 5 9

Exercice 102

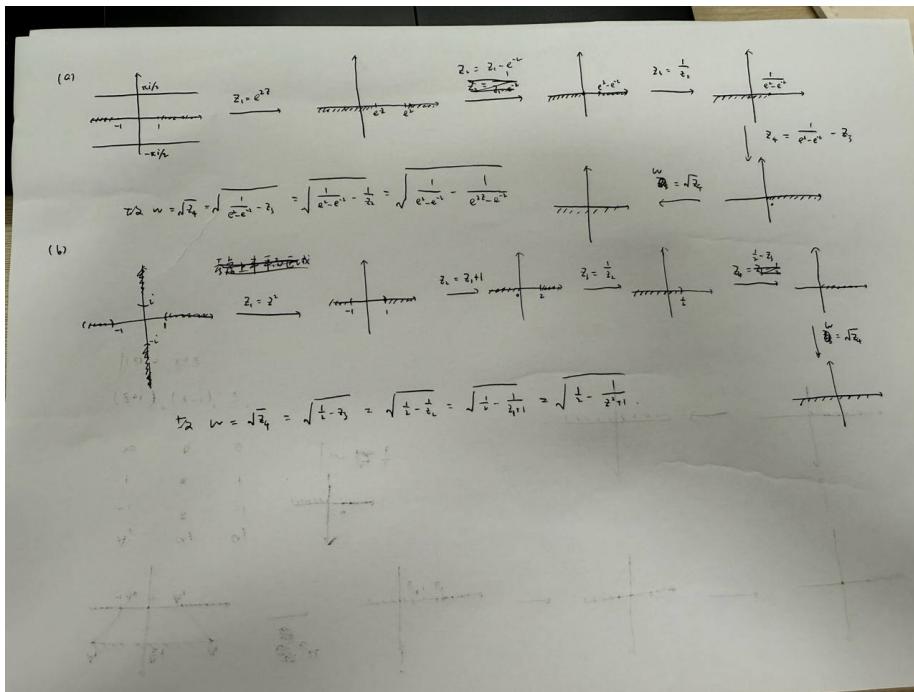
4. 试用对称原理把下列图形所示的区域分别保形映射为上半平面:



第 4 题图

Théorème 28: 对称原理 (或反射原理)

设区域 D 在实轴的上方, 其边界 ∂D 包含实轴上的一个开区间 I . 设函数 $f(z)$ 在 D 内解析, 并且在 I 上连续且取实数值. 则 $f(z)$ 可以延拓到区域 $D \cup I \cup D^*$ (其中 $D^* = \{z^*: z \in D\}$, z^* 表示 z 的共轭复数) 上成为一个解析函数 $\hat{f}(z)$, 并且对于任意 $z \in D \cup I \cup D^*$, 有 $\hat{f}(z^*) = (\hat{f}(z))^*$. 特别地, 当 $z \in D$, 我们有 $f(z^*) = (f(z))^*$.



Exercice 103

5. 设有圆环 $D = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$ 及 $G = \{w \in \mathbb{C} : R_1 < |w| < R_2\}$, 其中 r_1, r_2, R_1 及 R_2 是有限数. 求证: 存在单叶函数 $w=f(z)$, 使得 $G=f(D)$ 的必要与充分条件是 $\frac{r_1}{r_2} = \frac{R_1}{R_2}$.

若 $\frac{r_1}{r_2} = \frac{R_1}{R_2}$, 那么可以取单叶函数 $f(z) = z \cdot \frac{r_1}{R_1}$. 考虑微分同胚 $a : z \mapsto z/r_2$ 和 $b : z \mapsto z/R_2$, 我们有 $a(D) = \{z \in \mathbb{C} : r_1/r_2 < |z| < 1\}$ 和 $b(D) = \{z \in \mathbb{C} : R_1/R_2 < |z| < 1\}$. 只需要构造单叶满射函数 $w^* : a(D) \rightarrow b(D)$, 利用第 8 章第 1 段例题 4 可知, $r_1/r_2 = R_1/R_2$.

或者说, 若存在 f , 设 f 将 C_{r_1} 映射到 C_{R_1} , 利用推广的对称原理 (关于圆周的), 我们可以把 $f : D \rightarrow G$ 解析延拓到 $D_{r_1^2/r_2, r_1} = \{z \in \mathbb{C} : r_1^2/r_2 < |z| \leq r_1\}$ 上的单叶解析函数. 同样可以把逆函数 $f^{-1} : G \rightarrow D$ 解析延拓到 $G_{R_1^2/R_2, R_1} = \{w \in \mathbb{C} : R_1^2/R_2 < |w| \leq R_1\}$ 上的单叶解析函数. 以此类推, f 可以延拓到 $\{z \in \mathbb{C} : 0 < |z| < r_1\} \rightarrow \{w \in \mathbb{C} : 0 < |w| < R_1\}$. 由于 $\lim_{z \rightarrow 0} f(z) = \lim_{w \rightarrow 0} f^{-1}(w) = 0$, 补充在 0 处的定义. 再利用推广的 Schwarz 引理, 就有 $f(z) = \lambda z$ 其中 $|\lambda| = \frac{R_2}{r_2}$. 从而 $R_1 = |f(C_{r_1})| = \frac{R_2}{r_2} r_1$.

Note. There is another proof in Rudin.

For $0 < r < R$, let

$$A(r, R) = \{z : r < |z| < R\} \quad (1)$$

14.22 Theorem $A(r_1, R_1)$ and $A(r_2, R_2)$ are conformally equivalent if and only if $R_1/r_1 = R_2/r_2$.

PROOF Assume $r_1 = r_2 = 1$, without loss of generality. Put

$$A_1 = A(1, R_1), \quad A_2 = A(1, R_2), \quad (1)$$

and assume there exists $f \in H(A_1)$ such that f is one-to-one and $f(A_1) = A_2$. Let K be the circle with center at 0 and radius $r = \sqrt{R_2}$. Since $f^{-1}: A_2 \rightarrow A_1$ is also holomorphic, $f^{-1}(K)$ is compact. Hence

$$A(1, 1 + \epsilon) \cap f^{-1}(K) = \emptyset \quad (2)$$

for some $\epsilon > 0$. Then $V = f(A(1, 1 + \epsilon))$ is a connected subset of A_2 which does not intersect K , so that $V \subset A(1, r)$ or $V \subset A(r, R_2)$. In the latter case, replace f by R_2/f . So we can assume that $V \subset A(1, r)$. If $1 < |z_n| < 1 + \epsilon$ and $|z_n| \rightarrow 1$, then $f(z_n) \in V$ and $\{f(z_n)\}$ has no limit point in A_2 (since f^{-1} is continuous); thus $|f(z_n)| \rightarrow 1$. In the same manner we see that $|f(z_n)| \rightarrow R_2$ if $|z_n| \rightarrow R_1$.

Now define

$$\alpha = \frac{\log R_2}{\log R_1} \quad (3)$$

and

$$u(z) = 2 \log |f(z)| - 2\alpha \log |z| \quad (z \in A_1). \quad (4)$$

Let ∂ be one of the Cauchy-Riemann operators. Since $\partial \bar{f} = 0$ and $\partial f = f'$, the chain rule gives

$$\partial(2 \log |f|) = \partial(\log (\bar{f}f)) = f'/f, \quad (5)$$

so that

$$(\partial u)(z) = \frac{f'(z)}{f(z)} - \frac{\alpha}{z} \quad (z \in A_1). \quad (6)$$

Thus u is a harmonic function in A_1 which, by the first paragraph of this proof, extends to a continuous function on \bar{A}_1 which is 0 on the boundary of A_1 . Since nonconstant harmonic functions have no local maxima or minima, we conclude that $u = 0$. Thus

$$\frac{f'(z)}{f(z)} = \frac{\alpha}{z} \quad (z \in A_1). \quad (7)$$

Put $\gamma(t) = \sqrt{R_1} e^{it}$ ($-\pi \leq t \leq \pi$); put $\Gamma = f \circ \gamma$. As in the proof of Theorem 10.43, (7) gives

$$\alpha = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \text{Ind}_{\Gamma}(0). \quad (8)$$

Thus α is an integer. By (3), $\alpha > 0$. By (7), the derivative of $z^{-\alpha} f(z)$ is 0 in A_1 . Thus $f(z) = cz^{\alpha}$. Since f is one-to-one in A_1 , $\alpha = 1$. Hence $R_2 = R_1$. ////

Exercice 104

9. 试问下列实变数实值函数能否解析开拓到复平面上:

(1) $f(x) = |x|$;

(2) $f(x) = \begin{cases} e^{-1/x^2} & (\text{当 } x \neq 0 \text{ 时}), \\ 0 & (\text{当 } x = 0 \text{ 时}); \end{cases}$

(3) $f(x)$ 在 $[a, b]$ 上任一点可展开成实幂级数.

9. (1) $f'(0)$ 不存在, 因而 $f(x)$ 不能解析开拓到复平面.

(2) $\forall n \in \mathbb{N}, f^{(n)}(0) = 0$. 因此 $f(x)$ 在原点不能展开成实幂级数, 从而不能解析开拓到复平面.

(3) 在 $[a, b]$ 上任一点, $f(x)$ 可以展开成实幂级数, 因而可以解析开拓到以该点为心的收敛圆盘内. 从与 $[a, b]$ 上所有点相应的这种收敛圆盘中, 选取有限个覆盖 $[a, b]$, $f(x)$ 可以解析开拓到这有限个圆盘的并集中.

7.44 hw16

1 p169 1 2 5 6

Exercice 105

1. 设函数 $f(z)$ 在区域 D 内解析, 而且不等于零. 直接计算证明: 在 D 内, $\Delta \ln |f(z)| = 0, \Delta |f(z)| > 0$.

显然 $f \in H(D)$, $\frac{1}{f} \in H(D)$, D 单连通, 故由 Rudin Theorem 13.11 可知, 存在 $g \in H(D)$, 使得 $f = \exp(g)$, 于是

$$\ln |f(z)| = \ln |e^{g(z)}| = \ln |e^{\operatorname{Re} g(z) + i \operatorname{Im} g(z)}| = \ln e^{\operatorname{Re} g(z)} = \operatorname{Re} g(z)$$

是调和函数. 记 $u := \operatorname{Re} g(z)$, 于是 $\Delta u = 0$, 所以

$$\Delta |f(z)| = (\partial_x^2 + \partial_y^2)(e^u) = e^u(\partial_x^2 u + |\partial_x u|^2 + \partial_y^2 u + |\partial_y u|^2) = e^u(|\partial_x u|^2 + |\partial_y u|^2) \geq 0$$

记 $g(z) = \alpha(x, y) + i\beta(x, y)$, 于是由 C-R 公式: $\alpha_x = \beta_y, \alpha_y = -\beta_x$. 于是 $|\partial_x \alpha|^2 + |\partial_y \alpha|^2 = \beta_y^2 + \beta_x^2 = |g'(z)|^2$. 取等当且仅当 $g' \equiv 0$, 即 $g \equiv \text{const}$. 也就是 $f = e^g \equiv \text{const}$.

Note. 不知道能不能证出来严格 > 0 .

Exercice 106

2. 求一解析函数, 使其实部为 $e^x(x \cos y - y \sin y)$.

对于解析函数 $f(z) = u(x, y) + iv(x, y)$, 由 Cauchy-Riemann 方程: $u_x = v_y, u_y = -v_x$. 也就是

$$\begin{cases} v_y &= e^x(x+1) \cos y - e^x y \sin y \\ v_x &= e^x(x+1) \sin y + e^x y \cos y \end{cases}$$

于是 $v(x, y) = \int v_y dy + \phi(x) = e^x(x \sin(y) + y \cos(y)) + \phi(x)$, 对 x 求导得到 $v_x = e^x(x \sin y + y \cos y + \sin y) + \phi'(x)$. 于是 $\phi'(x) = 0$, 故 $\phi(x) = \text{const}$. 我们可以取该解析函数为

$$f(z) = e^x(x \cos y - y \sin y) + ie^x(x \sin y + y \cos y)$$

Exercice 107

5. 试用调和函数的中值公式, 证明

$$\int_0^\pi \ln(1 - 2r \cos \theta + r^2) d\theta = 0,$$

其中 $-1 < r < 1$.

由对称性, 只需要证明

$$\frac{1}{2\pi} \int_0^{2\pi} \ln(1 - 2r \cos \theta + r^2) d\theta = 0$$

下面说明 $f(z) = \ln(1 - 2\operatorname{Re} z + |z|^2) = \ln(1 - 2x + x^2 + y^2) = \ln((x-1)^2 + y^2)$ 是调和函数 (由于 $|r| < 1$, 该函数良好定义), 直接求导得到

$$\partial_x^2 f(z) = \frac{2y^2 - 2(x-1)^2}{[(x-1)^2 + y^2]^2}, \quad \partial_y^2 f(z) = \frac{2(x-1)^2 - 2y^2}{[(x-1)^2 + y^2]^2}$$

于是 $\Delta f(z) = 0$, 故由调和函数的平均值公式

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta$$

其中 $f(0) = \ln 1 = 0$, $\frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \ln(1 - 2r \cos \theta + r^2) d\theta$. 于是

$$\int_0^\pi \ln(1 - 2r \cos \theta + r^2) d\theta = 0$$

Exercice 108

6. 证明: 如果在整个 z 平面调和的函数 $u(z)$ 是有界的, 那么 $u(z)$ 恒等于常数.

证明. 利用平均值公式可知: $|\partial_x u(z)| \leq \frac{n}{r} \max_{\bar{B}_r(z)} |u|$, $|\partial_y u(z)| \leq \frac{n}{r} \max_{\bar{B}_r(z)} |u|$. 令 $r \rightarrow \infty$, 就有 $|\partial_x u| = |\partial_y u| = 0$, 故 $u(z)$ 恒为常数. \square

THEOREM 2.7. Suppose that $u \in C^2(\Omega)$ is harmonic in the open set Ω and $B_r(x) \Subset \Omega$. Then for any $1 \leq i \leq n$,

$$|\partial_i u(x)| \leq \frac{n}{r} \max_{\bar{B}_r(x)} |u|.$$

PROOF. Since u is smooth, differentiation of Laplace's equation with respect to x_i shows that $\partial_i u$ is harmonic, so by the mean value property for balls and the divergence theorem

$$\partial_i u = \oint_{B_r(x)} \partial_i u dx = \frac{1}{\alpha_n r^n} \int_{\partial B_r(x)} u \nu_i dS.$$

Taking the absolute value of this equation and using the estimate

$$\left| \int_{\partial B_r(x)} u \nu_i dS \right| \leq n \alpha_n r^{n-1} \max_{\bar{B}_r(x)} |u|$$

we get the result. \square

COROLLARY 2.8. If $u \in C^2(\mathbb{R}^n)$ is bounded and harmonic in \mathbb{R}^n , then u is constant.

PROOF. If $|u| \leq M$ on \mathbb{R}^n , then Theorem 2.7 implies that

$$|\partial_i u(x)| \leq \frac{Mn}{r}$$

for any $r > 0$. Taking the limit as $r \rightarrow \infty$, we conclude that $Du = 0$, so u is constant. \square

更强的, 我们证明命题在 u 仅为上调和函数 ($\Delta u \geq 0$) 的情况下依然成立.
这就是 2025 丘赛分析赛道第 3 题.

定义 $M(r) = \max_{|z|=r} u(z)$, 任意给定 $0 < r_1 < r_2 < \infty$, 在区域 $\{z \in \mathbb{C} : r_1 < |z| < r_2\}$ 内, 断言 $\ln M(r)$ 是下凸函数. 定义

$$h(z) = \frac{M(r_1)(\ln r_2 - \ln|z|) + M(r_2)(\ln|z| - \ln r_1)}{\ln r_2 - \ln r_1}, \quad \forall r_1 \leq |z| \leq r_2$$

h 是调和函数, 因为 $\Delta \ln|z| = (\partial_x^2 + \partial_y^2) \left(\frac{1}{2} \ln(x^2 + y^2) \right) = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0$.
而且

$$u(z) \leq |u(z)| \leq M(r_1) = h(r_1) \quad \forall |z| = r_1$$

$$u(z) \leq |u(z)| \leq M(r_2) = h(r_2) \quad \forall |z| = r_2$$

利用 Rudin Theorem 17.4 可知

$$u(z) \leq h(z) \quad \forall r_1 \leq |z| \leq r_2$$

于是对于任意 $r \in [r_1, r_2]$, 我们有

$$M(r) \leq \frac{M(r_1)(\ln r_2 - \ln r) + M(r_2)(\ln r - \ln r_1)}{\ln r_2 - \ln r_1} \tag{7.17}$$

由于 u 有上界, 所以我们可以在 eq. (7.17) 中令 $r_2 \rightarrow \infty$, 得到

$$M(r) \leq M(r_1) \quad \forall r \geq r_1$$

也可以在 eq. (7.17) 中令 $r_1 \rightarrow 0^+$, 得到

$$M(r) \leq M(r_2) \quad \forall r \leq r_2$$

由 r_1, r_2 的任意性 (这里 r_1, r_2 是独立的, 不再有大小关系), 我们知道 $M(r)$ 为常数. 考虑任意 \mathbb{C} 内开区域 G , 利用上调和函数的强极值原理, 我们知道 u 在 G 内部取到极大值, 于是 u 为常数.

接下来给出 Rudin Theorem 17.4 和上调和函数的强极值原理的证明

17.4 Theorem Suppose u is a continuous subharmonic function in Ω , K is a compact subset of Ω , h is a continuous real function on K which is harmonic in the interior V of K , and $u(z) \leq h(z)$ at all boundary points of K . Then $u(z) \leq h(z)$ for all $z \in K$.

This theorem accounts for the term “subharmonic.” Continuity of u is not necessary here, but we shall not need the general case and leave it as an exercise.

PROOF Put $u_1 = u - h$, and assume, to get a contradiction, that $u_1(z) > 0$ for some $z \in V$. Since u_1 is continuous on K , u_1 attains its maximum m on K ; and since $u_1 \leq 0$ on the boundary of K , the set $E = \{z \in K : u_1(z) = m\}$ is a nonempty compact subset of V . Let z_0 be a boundary point of E . Then for

some $r > 0$ we have $\bar{D}(z_0; r) \subset V$, but some subarc of the boundary of $\bar{D}(z_0; r)$ lies in the complement of E . Hence

$$u_1(z_0) = m > \frac{1}{2\pi} \int_{-\pi}^{\pi} u_1(z_0 + re^{i\theta}) d\theta,$$

and this means that u_1 is not subharmonic in V . But if u is subharmonic, so is $u - h$, by the mean value property of harmonic functions, and we have our contradiction. ////

上调和函数的强极值原理:

先叙述上调和函数的平均值公式:

THEOREM 2.5. Suppose that Ω is an open set, $B_r(x) \Subset \Omega$, and $u \in C^2(\Omega)$. If u is subharmonic in Ω , then

$$(2.5) \quad u(x) \leq \int_{B_r(x)} u \, dx, \quad u(x) \leq \int_{\partial B_r(x)} u \, dS.$$

THEOREM 2.13. Suppose that Ω is a connected open set and $u \in C^2(\Omega)$. If u is subharmonic and attains a global maximum value in Ω , then u is constant in Ω .

PROOF. By assumption, u is bounded from above and attains its maximum in Ω . Let

$$M = \max_{\Omega} u,$$

and consider

$$F = u^{-1}(\{M\}) = \{x \in \Omega : u(x) = M\}.$$

Then F is nonempty and relatively closed in Ω since u is continuous. (A subset F is relatively closed in Ω if $F = \tilde{F} \cap \Omega$ where \tilde{F} is closed in \mathbb{R}^n .) If $x \in F$ and $B_r(x) \Subset \Omega$, then the mean value inequality (2.5) for subharmonic functions implies that

$$\int_{B_r(x)} [u(y) - u(x)] dy = \int_{B_r(x)} u(y) dy - u(x) \geq 0.$$

Since u attains its maximum at x , we have $u(y) - u(x) \leq 0$ for all $y \in \Omega$, and it follows that $u(y) = u(x)$ in $B_r(x)$. Therefore F is open as well as closed. Since Ω is connected, and F is nonempty, we must have $F = \Omega$, so u is constant in Ω . \square

7.45 期末练习题

7.46 复变函数期末练习题

7.46.1 习题 1

1. 判断 $\sqrt{-2}\sqrt{-3} = \sqrt{6}$ 是否正确?
2. 构造一个共形映射将 $\mathbb{C} \setminus [-1, 1]$ 映射到单位圆的外部.

3. 计算

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}.$$

4. 设 $f(z)$ 为整函数且 $e^{f(z)}$ 有界. 证明: $f(z)$ 为常数.
5. 设 $f(z) = \sum_{n=0}^{\infty} a_n z^n$ 在 $|z| < 1$ 解析且满足 $|f(z)| < \frac{1}{1-|z|}$. 证明:

$$|a_n| \leq (n+1) \left(1 + \frac{1}{n}\right)^n < e(n+1).$$

6. 设 a_1, a_2, \dots, a_m 为非零复数, 定义

$$f(z) = \sum_{k=1}^m \frac{1}{1 - a_k z}.$$

(1) 求 $f(z)$ 在原点处幂级数展开式的收敛半径;

(2) 证明:

$$\limsup_{n \rightarrow \infty} \left| \sum_{k=1}^m a_k^n \right|^{1/n} = \max_{k=1, \dots, m} |a_k|.$$

7. 用 Rouché 定理来证明代数基本定理.

8. 求积分

$$\int_{|z|=\frac{1}{2}} \frac{dz}{z \sin^2(z)}.$$

9. 设 $f(z)$ 在 $\Delta = \{|z| < 1\}$ 内解析.

(1) 若 z_1, z_2, \dots, z_n 是 $f(z)$ 的零点. 证明:

$$|f(z)| \leq \prod_{k=1}^n |z_k|.$$

(2) 证明: 存在 $z_n \in \Delta$, $|z_n| \rightarrow 1$ 使得 $\{f(z_n)\}$ 是一个有界序列.

- (1) 错, $\sqrt{-2} \cdot \sqrt{-3} = \sqrt{2} \cdot \sqrt{-1} \cdot \sqrt{3} \cdot \sqrt{-1} = -\sqrt{6}$.
- (2) $w = \sqrt{z^2 - 1} + z$, 其中 $\sqrt{z^2 - 1}|_{z=i} = \sqrt{2}i$.
- (3) $f(z) := (z + \frac{1}{z})^{2n} \frac{1}{z} = \dots + \binom{2n}{n} z^{-1} + \dots$, $\int_{|z|=1} f(z) dz = \text{res}_0 f = 2\pi i \binom{2n}{n}$.
- (4) $e^{f(z)}$ 也是整函数 (整函数 e^z 复合整函数 f) 于是 $e^{f(z)} = C \Rightarrow f(z) = \log C$.
- (5) $|a_n| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{1}{r^n(1-r)}$, r 待定, 故可以取为 $\frac{n}{n+1}$, 则 $|a_n| \leq (n+1) \left(1 + \frac{1}{n}\right)^n < e(n+1)$.
- (6) $f(z) = \sum_{n=0}^{\infty} (\sum_{k=1}^m a_k^n) z^n$, 故 $R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|\sum_{k=1}^m a_k^n|}}$. 同时 $R = \min_k \left\{ \left| \frac{1}{|a_k|} \right| \right\}$ ¹¹, 故 $\lim_{n \rightarrow \infty} |\sum_{k=1}^m a_k^n|^{1/n} = \max_k |a_k|$.
- (7) 归纳即可
- (8) $f(z) := \frac{1}{z \sin^2 z}$ 在 $|z| < \frac{1}{2}$ 内只有三阶极点 0, 洛朗展开得到 $f(z) = \frac{1}{z(z - \frac{z^3}{3} + o(z^4))^2} = \frac{1}{z^3} \cdot \frac{1}{(1 - \frac{z^2}{3} + o(z^3))^2} = z^{-3} \cdot \frac{1}{1 - \frac{2}{3}z^2 + o(z^3)} = z^{-3} + \frac{2}{3}z^{-1} + \dots$. 于是 $\text{res}_0 f = a_{-1} = \frac{2}{3}$. 故 $\int_{|z|<\frac{1}{2}} f(z) dz = 2\pi i \text{res}_0 f = \frac{4}{3}\pi i$.
- (9) 题目有误: 应该补充条件 $|f(z)| < 1$, 且不等式改为 $|f(0)| \leq \prod_{k=1}^n |z_k|$.
- (I) 构造函数 $g(z) = \frac{f(z)}{\prod_{k=1}^n \varphi_{z_k}(z)}$, 其中 $\varphi_{z_k}(z) := \frac{z-z_k}{1-\bar{z}_k z}$. 于是 $|z| \rightarrow 1$ 时, $|\varphi_{z_k}(z)| \rightarrow 1$, 故 $|g(z)| \rightarrow \lim_{|z| \rightarrow 1} |f(z)| \leq 1$. 由最大模定理¹² $|g(z)| \leq 1, \forall z \in \Delta$. 注意到 $|\varphi_{z_k}(0)| = |z_k|$, 我们就有 $|f(0)| \leq \prod_{k=1}^n |\varphi_{z_k}(0)| = \prod_{k=1}^n |z_k|$.
- (II) 反证而设, 对于任意 $\{|z_k|\} \rightarrow 1, |f(z_k)| \rightarrow \infty$. 接下来考虑 f 在 Δ 内的零点个数, 若为无穷个, 这些零点存在聚点, 则由非平凡解析函数的零点孤立性可知, 聚点都在 $\partial\Delta$ 上, 从而存在零点列 $\{|z_k|\} \rightarrow 1$, 其中 $|f(z_k)| = 0$, 故 $|f(z_k)| \not\rightarrow \infty$. 故零点个数有限, 按 (1) 中定义 g , 则 $\frac{1}{g} \rightarrow 0$, 在 $|z| \rightarrow 1$ 时. $g \in H(\Delta)$ 无零点, 故 $\frac{1}{g} \in H(\Delta)$, 故由最大模定理: $\frac{1}{g} \equiv 0$. g 必须是无穷大, 但这与 $g \in H(\Delta)$ 矛盾!

¹¹一个函数在某点展开的幂级数, 其收敛半径是从该点到离它最近的奇点的距离。¹²最大模定理可以推广到趋于边界上, 不需要在边界有定义, 详见 Rudin p.253

7.46.2 习题 2

1. 证明 $w = -\frac{1}{z}$ 是 z 在黎曼球面上的对径点.
2. 构造一个共形映射将区域 $\{z = x + iy : -1 < y < 1\} \setminus (-\infty, 0]$ 映射到上半平面.
3. 设 $f(z) = u(z) + iv(z)$ 在 $|z| \leq 1$ 解析且 $f(0) = 0$. 证明:

$$\iint_{|z| \leq 1} u^2 dx dy = \iint_{|z| \leq 1} v^2 dx dy.$$

4. 设 $f(z), g(z)$ 为整函数且当 $|z| \geq 2024^{2025}$ 时 $|f(z)| \leq |g(z)|$. 证明: $\frac{f(z)}{g(z)}$ 是一个有理函数.

5. 设 $f(z)$ 为整函数且当 $z \in \mathbb{R}$ 时, $|f(z)| = 1$. 证明: 存在整函数 $g(z)$ 使得

$$f(z) = e^{g(z)}.$$

6. 求积分

$$\int_0^\infty \frac{\cos x}{1+x^4} dx.$$

7. 设 $f(z)$ 在 $|z| < 1$ 内解析, 且 $f(z)$ 限制在 $\frac{1}{2} < |z| < 1$ 内为单射. 证明 $f(z)$ 在 $|z| < 1$ 内为单射.

8. 设 $f(z)$ 在 $|z| < 1$ 内解析且 $\operatorname{Re} f(z) > 0$. 证明:

$$|f'(0)| \leq 2 \operatorname{Re} f(0).$$

(1) 映射 $\mathbb{C} \rightarrow S^2$, $z = x + iy \mapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right)$. 而 $-\frac{1}{z} = \frac{-x-iy}{x^2+y^2} \mapsto \left(\frac{-2x}{x^2+y^2+1}, \frac{-2y}{x^2+y^2+1}, \frac{-(x^2+y^2-1)}{x^2+y^2+1} \right)$, 于是是 z 的对径点.

(2) $w = \sqrt{1 - e^{\pi z}}$.

(3) $\operatorname{Re} f^2(z) = u^2 - v^2$ 是调和函数, 由平均值公式 $\int_{|z| < 1} u^2 - v^2 dz = \pi \cdot 1^2 \cdot (u^2(0) - v^2(0)) = 0$.

(4) 记 $R = 2024^{2025}$, 在紧集 $|z| \leq R$ 内, 由零点孤立性, g 至多有有限个零点, $\frac{f}{g}$ 有有限个极点. 在 $|z| \geq R$ 上, $|f(z)| \leq |g(z)| \Rightarrow g$ 的零点都是 f 的零点, $\left| \frac{f}{g} \right| \leq 1$ 有界, 由黎曼可去奇点定理, $\frac{f}{g}$ 在该区域只有可去奇点. 且 $\frac{f}{g}$ 在无穷远点的邻域内有界, 则 ∞ 不是本性奇点, 只可能是可去奇点或极点. 因为任何在扩充复平面上亚纯的函数必然是有理函数, 所以 $\frac{f}{g}$ 是有理函数.

(5) 还不确定, 可以取 $g(z) = \int_{\gamma_0 \rightarrow z} \frac{f'(\zeta)}{f(\zeta)} d\zeta + i \arg f(0)$, 于是 $g' = \frac{f'}{f}$, 故验证这是整函数.

(6) $I = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos x}{1+x^4} dx$, 设 $f(z) = \frac{e^{iz}}{1+z^4}$. 考虑在上半平面圆周上的积分, $\left| \int_{\gamma_R^+} f(z) dz \right| \rightarrow 0$. 计算留数 $\operatorname{res}_{e^{i\pi/4}} f = \frac{1}{4} e^{-1/\sqrt{2}} e^{i(1/\sqrt{2}-3\pi/4)}$, $\operatorname{res}_{e^{3i\pi/4}} f = \frac{1}{4} e^{-1/\sqrt{2}} e^{i(-1/\sqrt{2}-\pi/4)}$. 于是 $I = \frac{1}{2} \int_{-\infty}^\infty f(z) dz = \pi i (\operatorname{res}_{e^{i\pi/4}} f + \operatorname{res}_{e^{3i\pi/4}} f) =$

$$\frac{\pi}{2}e^{-1/\sqrt{2}} \sin\left(\frac{1}{\sqrt{2}} + \frac{\pi}{4}\right).$$

- (7) 对于任意 $r \in (\frac{1}{2}, 1)$, 考虑 D_r 内 $f(z) = w_0$ 的解, w_0 是任意复数, 由于 f 在 C_r 上单射, 故 $f(C_r)$ 为简单闭曲线, 由幅角原理, $N(w_0) = \frac{1}{2\pi i} \oint_{C_r} \frac{f'(z)}{f(z)-w_0} dz$ 为 $f(C_r)$ 在 w_0 处的绕数, 根据若当曲线定理, 由于 $f(C_r)$ 是简单闭曲线, 那么对于任何不在 $f(C_r)$ 上的 w_0 , 绕数为 0 或 1. 于是对于 $f(C_r)$ 内部的 w_0 , $f(z) - w_0 = 0$ 只有一个解.
- (8) 定义 $g : \mathbb{D} \rightarrow \mathbb{D}$, $z \mapsto \frac{f(z)-f(0)}{f(z)+f(0)}$. 于是, $g(0) = 0$, 由 Schwarz 引理, $|g'(0)| \leq 1$, 即 $|f'(0)| \leq 2\operatorname{Re} f(0)$.

7.46.3 习题 3

1. 画出如下集合的图像:

$$\{z : \arg \frac{z+i}{z-i} < \frac{\pi}{4}\}.$$

2. 构造一个共形映射将区域 $\{|z| > 1\} \setminus (-\infty, -1)$ 映射成单位圆盘.

3. 设 $|a| < 1, |b| < 1$.

(1) 说明 $\sqrt{(z-a)(z-b)}$ 在 $|z| \geq 1$ 可取到单值解析分支.

(2) 取定 $\sqrt{(z-a)(z-b)}$ 在 $|z| \geq 1$ 的一个单值解析分支. 求积分

$$\int_{|z|=1} \frac{z}{\sqrt{(z-a)(z-b)}} dz.$$

4. 是否存在定义在 $|z| < 1$ 内的非常值解析函数 $f(z)$, 使得当 $|z| \rightarrow 1$ 时, $|f(z)| \rightarrow \infty$? 判断并给出证明.

5. 求积分

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx.$$

6. 设 $f(z)$ 在 $|z| < 1$ 内解析, $|f(z)| < 1$ 且 $f(0) \neq 0$. 证明: $f(z)$ 在圆盘 $\{|z| < |f(0)|\}$ 内部不存在零点.

7. 令 $\Delta = \{|z| < 1\}$. 设 $f(z) : \Delta \rightarrow \Delta$ 解析, 满足

$$f(0) = f\left(\frac{1}{2}\right) = f\left(-\frac{1}{2}\right) = 0.$$

证明:

$$\left|f\left(\frac{1}{4}\right)\right| \leq \frac{1}{21}.$$

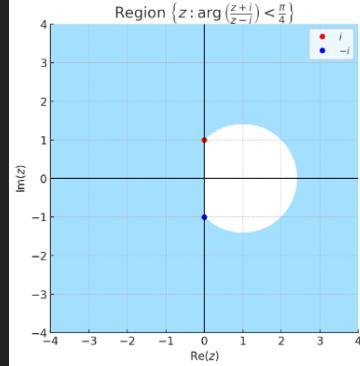
判断上界 $\frac{1}{21}$ 是否是最佳的.

8. 设 $f(z)$ 为整函数.

(1) 若存在常数 $C, n \in \mathbb{N}$ 使得 $|f(z)| \leq C|z|^n$, 证明: $f(z)$ 是一个多项式.

(2) 若将上述条件改成: 存在常数 $C, n \in \mathbb{N}$ 使得 $|\operatorname{Re} f(z)| \leq Cr^n$, 能否证明 $f(z)$ 是一个多项式? 给出证明或反例.

(1) 这题很细节.



上图展示了集合

$$\left\{ z \in \mathbb{C} : \arg \left(\frac{z+i}{z-i} \right) < \frac{\pi}{4} \right\}$$

- (2) $\phi_1 : \{|z| > 1\} \setminus (-\infty, -1) \rightarrow \{\operatorname{Im} z > 0; |z| > 1\}, z \mapsto \sqrt{-z}$. $\phi_2 : \{\operatorname{Im} z > 0; |z| > 1\} \rightarrow \{\operatorname{Im} z > 0\}, z \mapsto z + \frac{1}{z}$. $\phi_3 : \{\operatorname{Im} z > 0\} \rightarrow \{|z| < 1\}, z \mapsto \frac{z-i}{z+i}$. 于是 $\phi = \phi_3 \circ \phi_2 \circ \phi_1 : \{|z| > 1\} \setminus (-\infty, -1) \rightarrow \{|z| < 1\}, z \mapsto \frac{z-\sqrt{z}-1}{z+\sqrt{z}-1}$.
- (3) (1) f 在 $\{|z| \geq 1\}$ 内没有支点, 所以可以取单值分支. (2) 计算在 a, b 处的留数 (a_{-1}) 得到 $I = \pi i(a + b)$.
- (4) 不存在. 若存在, 则由非常数的解析函数的零点孤立性, f 只可能在 \mathbb{D} 内有有限多个零点 z_1, \dots, z_n . 考虑 $g(z) := \frac{f(z)}{\prod_{k=1}^n \varphi_{z_k}(z)} \in H(\mathbb{D})$, 且在 \mathbb{D} 内没有零点, 于是 $\frac{1}{g} \in H(\mathbb{D})$. 令 $|z| \rightarrow 1$, 我们有 $|\varphi_{z_k}(z)| \rightarrow 1, |f(z)| \rightarrow \infty$, 于是 $\left| \frac{1}{g} \right| \rightarrow 0$. 由最大模原理¹³, $\max_{|z|<1} \left| \frac{1}{g(z)} \right| \leq 0$. 于是 $\frac{1}{g} = 0$, g 在 \mathbb{D} 内处处取到无穷, 这与 $g \in H(\mathbb{D})$ 矛盾!
- (5) 记 $f(z) = \frac{z^2}{1+z^4}$, 考虑在上半圆周上积分, 在圆弧上积分趋于 0, 计算围道内留数: $\operatorname{res}_{e^{\pi i/4}} f = \frac{1-i}{4\sqrt{2}}$, $\operatorname{res}_{e^{3\pi i/4}} f = \frac{-1-i}{4\sqrt{2}}$. 于是 $I = 2\pi i(\operatorname{res}_{e^{\pi i/4}} f + \operatorname{res}_{e^{3\pi i/4}} f) = \frac{\pi}{\sqrt{2}}$.
- (6) 若存在 $|\alpha| < |f(0)|$ 使得 $f(\alpha) = 0$, 利用 Schwarz-pick lemma, $\left| \frac{f(b)-f(a)}{1-\overline{f(a)}f(b)} \right| \leq \left| \frac{b-a}{1-\bar{a}b} \right|, \forall a, b \in \mathbb{D}$, 取 $a = \alpha, b = 0$, 就有 $|f(0)| \leq |\alpha|$, 矛盾!
- 证明 Schwarz-pick lemma: 令 $\varphi_a(z) := \frac{a-z}{1-\bar{a}z}$, 对于 $f : \mathbb{D} \rightarrow \mathbb{D}$, 考虑 $\varphi_{f(a)} \circ f \circ \varphi_a : \mathbb{D} \rightarrow \mathbb{D}$, 它将 0 映射到 0, 于是由 Schwarz 引理, $|\varphi_{f(a)}(f(\varphi_a(z)))| \leq |z|$, 取 $z = \varphi_a(b)$, 则 $|\varphi_{f(a)}(f(b))| \leq |\varphi_a(b)|$, 即

¹³ 见 Rudin

$$\left| \frac{f(a)-f(b)}{1-f(a)f(b)} \right| \leq \left| \frac{a-b}{1-\bar{a}b} \right|.$$

(7) 记 $\varphi_a(z) := \frac{a-z}{1-\bar{a}z}$, 考虑 $g(z) := \frac{f(z)}{\varphi_0(z) \cdot \varphi_{1/2}(z) \cdot \varphi_{-1/2}(z)}$, 由于 $f \in H(\Delta)$ 在 $0, \frac{1}{2}, -\frac{1}{2}$ 有零点, 则 $g \in H(\Delta)$. 因为 $|f(z)| < 1$, 所以 $|z| \rightarrow 1$ 时, $|g(z)| \rightarrow \lim_{|z| \rightarrow 1} |f(z)| \leq 1$. 于是由最大模原理¹⁴, $\max_{z \in \Delta} |g(z)| \leq 1$. 令 $z = \frac{1}{4}$, 就有 $|g\left(\frac{1}{4}\right)| = \frac{|f\left(\frac{1}{4}\right)|}{\left|(-\frac{1}{4}) \cdot (\frac{2}{7}) \cdot (-\frac{2}{3})\right|} = 21 |f\left(\frac{1}{4}\right)| \leq 1$, 即 $|f\left(\frac{1}{4}\right)| \leq \frac{1}{21}$. 这是最佳上界, 因为等号成立当且仅当 $g \equiv 1$, 也就是 $f(z) = \varphi_0(z)\varphi_{1/2}(z)\varphi_{-1/2}(z)$.

(8) (1) $f \in H(\mathbb{C})$ 可写作 $f(z) = \sum_{k=0}^{\infty} a_k z^k$, 由 Cauchy 积分公式, $a_k = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{k+1}} dz$, 于是 $|a_k| \leq \frac{1}{2\pi} \int_{|z|=R} \frac{|f(z)|}{|z|^{k+1}} |dz| \leq \frac{1}{2\pi} \int_{|z|=R} \frac{C}{|z|^{k+1-n}} |dz| \leq \frac{C}{R^{k-n}}$. 当 $k > n$ 时, 由 R 的任意性, 令 $R \rightarrow \infty$, 则 $a_k = 0$. 于是 f 是一个至多 n 次多项式. (2) 利用 Borel-Carathéodory Theorem, 对于 $|z| \leq r$, $|f(z)| \leq \frac{2r}{R-r} \max_{|\zeta|=R} (\operatorname{Re} f(\zeta)) + \frac{R+r}{R-r} |f(0)|$, 其中 $0 < r < R$. 取 $R = 2r$. 于是 $|f(z)| \leq C'|z|^n + D$, 其中 C', D 是与 z 无关的常数. 类似 (1) 的讨论可知 f 是多项式.

• Borel-Carathéodory Theorem 的证明:

定义 $A = \sup_{|z| \leq R} \operatorname{Re} f(z)$.

首先设 $f(0) = 0$. 由于 $\operatorname{Re} f$ 是调和的, 可以取 $A > 0$. f 映到直线 $x = A$ 左边的半平面 P . 我们想把这个半平面映到圆盘上, 再用施瓦茨引理, 得到所要的不等式.

$w \mapsto w/A - 1$ 把 P 变成标准左半平面. $w \mapsto R(w+1)/(w-1)$ 把左半平面变成圆心在原点且半径为 R 的圆. 它们的复合映射把 0 映成 0, 就是所需要的映射:

$$w \mapsto \frac{Rw}{w-2A}$$

对上面这个映射与 f 的复合使用施瓦茨引理, 得到

$$\frac{|Rf(z)|}{|f(z) - 2A|} \leq |z|$$

取 $|z| < r$, 上式变为

$$R|f(z)| \leq r|f(z) - 2A| \leq r|f(z)| + 2Ar$$

所以

$$|f(z)| \leq \frac{2Ar}{R-r}$$

对于一般的情况, 考虑 $f(z) - f(0)$

$$\begin{aligned} |f(z)| - |f(0)| &\leq |f(z) - f(0)| \\ &\leq \frac{2r}{R-r} \sup_{|w| \leq R} \operatorname{Re} (f(w) - f(0)) \\ &\leq \frac{2r}{R-r} \left(\sup_{|w| \leq R} \operatorname{Re} f(w) + |f(0)| \right) \end{aligned}$$

整理后即得所要证明的不等式.

¹⁴ 只需要趋于边界时的最大值即可, 见 Rudin p.253

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第八章 近世代数

8.1 Broad classifications

Let G be a group. The letter p , q , and r are used for primes. Moreover, they are assumed to be distinct. n is a positive integer.

8.1.1 Order p

G is simple.

8.1.2 Order p^n

$n \geq 2$. G has nontrivial center due to the class equation. Assume $Z(G) = \{1\}$ then

$$\underbrace{|G|}_{=p^n} = \underbrace{|Z(G)|}_{=1} + \sum_i \underbrace{[G : C_G(x_i)]}_{p \text{ power}}$$

which is a contradiction. Then $Z(G)$ is a nontrivial normal subgroup of G .

8.1.3 Order mp^n

$1 < m < p$. G is not simple since $n_p \equiv 1 \pmod{p}$, $n_p | m \implies n_p = 1$ by Sylow III theorem.

8.1.4 Order p^2q

Claim G is not simple.

Case 1: $p > q$

By Sylow III, we know $n_p|q, n_p \equiv 1 \pmod{p}$. Then $n_p = 1$. The Sylow p -subgroups is normal, thus G is not simple.

Case 2: $p < q$

By Sylow III, $n_q \in \{1, p, p^2\}$. If $n_q = 1$ then we are done. As $p < q$, $n_q \equiv 1 \pmod{p}$, this cannot happen for $q > 3$. Now we consider the case $n_q = p^2$.

Since each Sylow- q subgroup has order q , the intersection of two distinct Sylow- q subgroups must be trivial. Thus there are $(q-1)p^2$ elements of order q . There left p^2 elements. Since $n_p \geq 1$ by Sylow I, the remaining elements must form the unique Sylow- p subgroup and we get $n_p = 1$. Thus G is not simple.

8.1.5 Order pqr

G is not simple.

Assume G is simple, then $n_p > 1, n_q > 1, n_r > 1$. Since $n_p|qr, n_q|pr, n_r|pq$, and $r > q > p$, then $n_r = pq, n_q \geq p, n_p \geq q$. Then there exists $pq(r-1)$ elements of order $r, p(q-1)$ of order q and $q(p-1)$ of order p .

$$n \geq pq(r-1) + p(q-1) + q(p-1) + \underbrace{1}_{\{1\}} = pqr + \underbrace{(p-1)(q-1)}_{>0} > n$$

which is a contradiction.

8.1.6 Order $2^n \cdot 3$

$$n > 1.$$

Claim G is not simple.

$n_2 \equiv 1 \pmod{2}, n_p|3 \implies n_2 \in \{1, 3\}$. If $n_2 = 1$ then we are done.

If $n_2 = 3$ then consider the action

$$\phi : G \rightarrow \text{Aut}(\text{Syl}_2(G)) = \text{Aut}\{P_1, P_2, P_3\} \cong S_3 \quad g \mapsto \text{Ad}_g$$

$$\text{Ad}_g : G \rightarrow G \quad x \mapsto gxg^{-1}$$

Then $[G : \ker \phi] | |S_3| = 6$. Since the action is transitive, $\ker \phi \neq G$. Since $n > 1$ then $\ker \phi \neq \{1\}$, otherwise $12 \leq [G : \ker \phi] = |G| \leq 6$. Therefore $\ker \phi$ is a proper nontrivial normal¹ subgroup of G .

8.1.7 Order $3^n \cdot 4$

$$n > 1.$$

Claim G is not simple.

$n_3 \in \{1, 4\}$. If $n_3 = 1$ then we are done.

If $n_3 = 4$, consider the action

$$\phi : G \rightarrow \text{Aut}(\text{Syl}_3(G)) = \text{Aut}\{P_1, P_2, P_3, P_4\} \cong S_4 \quad g \mapsto \text{Ad}_g$$

$$\text{Ad}_g : G \rightarrow G \quad x \mapsto gxg^{-1}$$

$[G : \ker \phi] | |S_4| = 24$. $\ker \phi \neq G$ as the action is transitive. $\ker \phi \neq 1$ as $|G| \geq 36 > 24$. Thus $\ker \phi$ is proper nontrivial normal subgroup of G .

8.2 Simplicity of A_n

[Ansimplle.pdf](#)

Some examples of groups:

- A finite group of prime order is simple.
- A finite abelian group G not of prime order is not simple.
 - Let prime $p \mid |G|$, by Cauchy's theorem, G contains a subgroup of order p , which is abelian² thus normal in G .
- When $n \geq 3$, S_n is not simple since $A_n \triangleleft S_n$ with index 2.

A meaningful result is as follows.

¹The kernel is always normal.

²As G is abelian

Théorème 1

For $n \geq 5$, A_n is simple.

证明. See [Ansimplle.pdf](#), where 5 proofs are listed. \square

Next we show 3 useful lemmas and a theorem.

Lemme 1

For $n \geq 3$, A_n is generated by 3-cycles. For $n \geq 5$, A_n is generated by permutations of type $(2, 2)$.

The 3-cyclics in S_n are all conjugate in S_n , since permutations of cycle type in S_n are conjugate. Are 3-cycles conjugate in A_n ?

Lemme 2

For $n \geq 5$, all 3-cycles in A_n are conjugate in A_n .

Lemme 3

For $n \geq 5$, the only nontrivial proper normal subgroup of S_n is A_n . In particular, the only subgroup of S_n with index 2 is A_n .

Corollaire 1

By Lemme 3, for $n \geq 5$ each homomorphic image of S_n not isomorphic to S_n has order 1 or 2. So there is no surjective homomorphism $S_n \rightarrow \mathbb{Z}/(m)$ for $m \geq 3$.

8.2.1 The index of subgroup of S_n

Théorème 2

For $n \geq 5$, no subgroup of S_n has index strictly between 2 and n . Moreover, each subgroup of index n in S_n is isomorphic to S_{n-1} .

证明. Let H be a proper subgroup of S_n and let $m := [S_n : H]$, so $m \geq 2$. If $m = 2$ then $H = A_n$ by Lemma 2.4. If $m < n$ then we will show $m = 2$. The left multiplication action of S_n on S_n/H gives a group homomorphism

$$\varphi : S_n \rightarrow \text{Sym}(S_n/H) \cong S_m$$

By hypothesis $m < n$, so φ is not injective. Let K be the kernel of φ , so $K \subset H$ and K is nontrivial. Since $K \triangleleft S_n$, Lemma 2.4 says $K = A_n$ or S_n . Since $K \subset H$, we get $H = A_n$ or S_n , which implies $m = 2$. Therefore we can't have $2 < m < n$.

Now let H be a subgroup of S_n with index n . Consider the left multiplication action of S_n on S_n/H . This is a homomorphism $\ell : S_n \rightarrow \text{Sym}(S_n/H)$. Since S_n/H has order n , $\text{Sym}(S_n/H)$ is isomorphic to S_n . The kernel of ℓ is a normal subgroup of S_n that lies in H (why?). Therefore the kernel has index at least n in S_n . Since the only normal subgroups of S_n are 1, A_n , and S_n , the kernel of ℓ is trivial, so ℓ is an isomorphism. What is the image $\ell(H)$ in $\text{Sym}(S_n/H)$? Since $gH = H$ if and only if $g \in H$, $\ell(H)$ is the group of permutations of S_n/H that fixes the "point" H in S_n/H . The subgroup fixing a point in a symmetric group isomorphic to S_n is isomorphic to S_{n-1} . Therefore $H \cong \ell(H) \cong S_{n-1}$. \square

Corollaire 2

Let F be a field. If $f \in F[X_1, \dots, X_n]$ and $n \geq 5$, the number of different polynomials we get from f by permuting its variables is either 1, 2, or at least n .

证明. Letting S_n act on $F[X_1, \dots, X_n]$ by permutations of the variables, the polynomials we get by permuting the variables of f is the S_n -orbit of f . The size

of this orbit is $[S_n : H]$, where $H = \text{Stab}_f = \{\sigma \in S_n : \sigma f = f\}$. By Theorem 2.7, this index is either 1, 2, or at least n . \square

8.2.2 The index of subgroup of A_n

Corollaire 3

For $n \geq 5$, each proper subgroup of A_n has index at least n .

Note. The proof is similar to Théorème 2

证明. Let H be a proper subgroup of A_n , with index $m > 1$. Consider the left multiplication action of A_n on A_n/H . This gives a group homomorphism

$$\varphi : A_n \rightarrow \text{Sym}(A_n/H) \cong S_m$$

Let K be the kernel of φ , so $K \subset H$ (why?) and $K \triangleleft A_n$. By simplicity of A_n , K is trivial. Therefore A_n injects into S_m , so $(n!/2) \mid m!$, which implies $n \leq m$. \square

8.3 Decomposition of Finite Abelian Groups

[finite-abelian.pdf](#)

Définition 1: indecomposable

Let A be a nontrivial finite abelian group. Call A **indecomposable** if we can't write $A = B \oplus C$ for some nontrivial subgroups B and C . Call A **decomposable** if we can write $A = B \oplus C$ for two nontrivial subgroups B and C .

- A group of prime order is abelian (it's cyclic) and is indecomposable.
- **A cyclic group of prime-power order is indecomposable** because if it could be decomposed into $B \oplus C$ for some nontrivial subgroups B and C , both B and C would contain a subgroup of order p . This would imply that the original group A has more than one subgroup of order p , contradicting the property that a **cyclic group has at most one subgroup of each size**.

Théorème 3

Let G be a cyclic group. Then, for every divisor d of $|G|$, there is a **unique** subgroup of G of order d .

证明. Let $G = \langle a \rangle$ and $|G| = n$. Suppose that $d \mid n$, and let $n = dk$. Then $|a^k| = d$, and so $\langle a^k \rangle$ is a subgroup of G of order d .

Now suppose that H is any subgroup of G of order d . Then $H = \langle a^m \rangle$ for some integer m . Also, $a^{md} = e$, and so $n \mid md$. Thus, $md = nt$ for some integer t , and so $md = dkt$. Therefore, $m = kt$, and so $H = \langle a^m \rangle = \langle a^{kt} \rangle = \langle a^k \rangle$. \square

Remarque 27. The groups $\mathbf{Z}/(4)$ and $\mathbf{Z}/(2) \oplus \mathbf{Z}/(2)$ are not isomorphic since $\mathbf{Z}/(4)$ is indecomposable. Or since $\mathbf{Z}/(4)$ has an element of order 4 and $\mathbf{Z}/(2) \oplus \mathbf{Z}/(2)$ does not.

Théorème 4: Theorem 2.5

A nontrivial finite abelian group is a direct sum of indecomposable subgroups.

Note. The proof is trivial.

证明. This argument will be the same as the standard proof of the existence of prime factorization in the positive integers. We argue by induction on the order n of the group.

For the base case $n = 2$, abelian groups of order 2 are indecomposable since 2 is prime. Suppose $n > 2$ and each nontrivial abelian group of order less than n is a direct sum of indecomposable subgroups. Let A be abelian of order n .

Case 1: A is indecomposable. We are done, since A is a direct sum of itself (one term).

Case 2: A is decomposable. We have $A = B \oplus C$ for nontrivial subgroups B and C . Then $n = |B||C|$ with $|B|$ and $|C|$ being greater than 1, so they are less than n . By induction,

$$B = P_1 \oplus \cdots \oplus P_r, \quad C = Q_1 \oplus \cdots \oplus Q_s$$

for indecomposable P_i and Q_j . Then $A = P_1 \oplus \cdots \oplus P_r \oplus Q_1 \oplus \cdots \oplus Q_s$. \square

8.3.1 Classification of indecomposable finite abelian groups

Lemme 4

If A is an abelian group and $|A| = mn$ where $(m, n) = 1$ then $A = A_m \oplus A_n$ for the subgroups $A_m = \{a \in A : ma = 0\}$ and $A_n = \{a \in A : na = 0\}$.

证明. The subsets A_m and A_n are subgroups because A is abelian, e.g., if $ma = 0$ and $ma' = 0$ then $m(a + a') = ma + ma' = 0 + 0 = 0$. Using multiplicative notation for a moment, we have $(gh)^m = g^m h^m$ when $gh = hg$, but it might not be true if $gh \neq hg$.

To show $A = A_m + A_n$, write $1 = mx + ny$ for $x, y \in \mathbf{Z}$ since $(m, n) = 1$. For all $a \in A$,

$$a = 1 \cdot a = (mx + ny)a = (mx)a + (ny)a.$$

We have $(ny)a \in A_m$ since $m((ny)a) = (mn)(ya) = |A|ya = 0$, and similarly $(mx)a \in A_n$. Thus $A = A_m + A_n$.

To show $A_m \cap A_n = \{0\}$, if $a \in A_m \cap A_n$ then $ma = 0$ and $na = 0$, so $a = (mx + ny)a = x(ma) + y(na) = 0 + 0 = 0$. Alternatively, the order of a divides m and n , so the order divides $(m, n) = 1$ and thus $a = 0$.

We have shown $A = A_m + A_n$ and $A_m \cap A_n = \{0\}$, so $A = A_m \oplus A_n$. \square

Remarque 28. Lemme 4 has a uniqueness aspect: $|A_m| = m$, $|A_n| = n$, and these are the unique subgroups of A with orders m and n . We will not need this.

Théorème 5

An indecomposable finite abelian group has prime-power order.

证明. Let A be a nontrivial abelian group. We will prove the contrapositive of the theorem for A : if $|A|$ is not a prime power then A is decomposable. By Lemme 4, we are done. \square

Lemme 5: 3.4

A nontrivial finite abelian p -group with a unique subgroup of order p is cyclic.

证明. Let A be a finite abelian p -group with a unique subgroup of order p and let p^m be the largest order of the elements of A . Then $m \geq 1$ and each element of A has order p^j where $j \leq m$ (A is a p -group and p^m is the maximal order), so all elements of A have order dividing $p^m : p^m A = \{0\}$.

Let $a \in A$ have order p^m . Since $p^{m-1}a$ has order p , $\langle p^{m-1}a \rangle$ is a subgroup of order p , so it is the only one by assumption. To prove $A = \langle a \rangle$, we'll assume $A \neq \langle a \rangle$ and get a contradiction.

The quotient group $A/\langle a \rangle$ (this makes sense since A is abelian) is nontrivial, abelian, and of p -power order. By Cauchy's theorem, $A/\langle a \rangle$ has an element of order p , say b . That means $b \notin \langle a \rangle$ and $pb \in \langle a \rangle$. So we can write

$$pb = ja$$

for some $j \in \mathbf{Z}$. Since $p^m b = 0$ (all elements of A have order dividing p^m) and $m \geq 1$,

$$0 = p^m b = p^{m-1}(pb) = p^{m-1}(ja) = (p^{m-1}j)a.$$

Since a has order p^m , $p^m \mid p^{m-1}j$, so $p \mid j$. Thus $j = pn$ for some $n \in \mathbf{Z}$, so $pb = (pn)a$. Rewrite that as $p(b - na) = 0$. The only subgroup of order p is in $\langle a \rangle$, so $b \in na + \langle a \rangle \subset \langle a \rangle$. This contradicts $b \notin \langle a \rangle$, so $A = \langle a \rangle$. \square

Remarque 29. Lemme 5 is true without assuming A is abelian when $p > 2$, but the quaternion group Q_8 has a unique subgroup of order 2 and is not cyclic.

Théorème 6

Let G be a finite abelian group and let $g \in G$ have maximal order in G . There is a subgroup $H \subset G$ such that $G \cong H \times \langle g \rangle$.

Note. See r.f. [charthy.pdf](#) section 5.

4.4. **Finitely generated abelian groups.** Recall that a group G is finitely generated if there exist a finite subset A of G such that $G = \langle A \rangle$.

Theorem 4.4.1 (Fundamental theorem of finitely generated abelian groups). *Let G be a finitely generated abelian group. Then*

$$G \cong \mathbb{Z}^r \times Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_s}$$

for some integers $r \geq 0$, $2 \leq n_1 \leq n_2 \leq \cdots \leq n_s$ satisfying $n_i | n_{i+1}$. Moreover these integers r, n_1, \dots, n_s are unique.

The integer r is called the **rank** of the abelian group G .

We will explain later in the semester that abelian groups = \mathbb{Z} -modules. So this theorem will follow from the classification of modules over a PID (Theorem 13.3.4).

The goal in this lecture is to see how to characterize finitely generated abelian groups.

Lemma 4.4.2. *If $m, n \in \mathbb{N}_{\geq 2}$ satisfying $\gcd(m, n) = 1$, then*

$$\mathbf{Z}_{mn} \cong \mathbf{Z}_m \times \mathbf{Z}_n.$$

Proof. Consider the group homomorphism

$$\begin{aligned} \mathbf{Z}_{mn} &\xrightarrow{\phi} \mathbf{Z}_m \times \mathbf{Z}_n \\ a &\longmapsto (a \bmod m, a \bmod n). \end{aligned}$$

We compute the kernel of ϕ :

$$\ker \phi = \left\{ a \bmod mn \mid \begin{array}{l} a \equiv 0 \bmod m \\ a \equiv 0 \bmod n \end{array} \right\} = \{0 \bmod mn\}.$$

So ϕ is injective. But $|\mathbf{Z}_{mn}| = |\mathbf{Z}_m| \cdot |\mathbf{Z}_n|$. So ϕ must be a bijection and hence an isomorphism. \square

Corollary 4.4.3. *Every finitely generated abelian group is of the form*

$$G = \mathbb{Z}^r \times (\mathbf{Z}_{p_1^{r_{11}}} \times \cdots \mathbf{Z}_{p_1^{r_{1s_1}}}) \times (\mathbf{Z}_{p_2^{r_{21}}} \times \cdots \mathbf{Z}_{p_2^{r_{2s_2}}}) \times \cdots$$

Here, $r, p_1, p_2, \dots, r_{11}, \dots, r_{1s_1}, r_{21}, \dots$ are unique up to permutation.

8.4 Galois-theory-at-work

8.4.1 CMC 高

Exercice 1

证明:

- (1) 存在恰有 q 个元素的域 \mathbb{F}_q ;
- (2) 如果 K 是一个 q 元域, 则 $K \cong \mathbb{F}_q$;
- (3) 设 $L \supset \mathbb{F}_q$ 是有限域扩张, 则映射 $\eta_q : L \rightarrow L, x \mapsto x^q$ 是域自同构, 且满足

$$\eta_q(a) = a, \quad \forall a \in \mathbb{F}_q;$$

- (1) 域扩张 $L \supset \mathbb{F}_q$ 的 Galois 群 $\text{Gal}(L/\mathbb{F}_q)$ 是由 η_q 生成的循环群;
- (2) $L^* = L \setminus \{0\}$ 关于 L 中乘法成为一个循环群. 特别地, 存在 $\alpha \in L$ 使

$$L = \mathbb{F}_q[\alpha].$$

证明. (1) Let \mathbb{F}_q be the splitting field of the polynomial $f(x) = x^q - x \in \mathbb{F}_p[x]$.

Since $f'(x) = -1$, $f(x) = x^q - x$ has exactly q roots in the splitting field \mathbb{F}_q , denoted as $R_f = \{\alpha_1, \dots, \alpha_q\} \subset \mathbb{F}_q$. Directly verifying that R_f is a subfield, we conclude that $R_f = \mathbb{F}_q$.

- (2) If $\text{Char}(K) = p_1$, $[K : \mathbb{F}_{p_1}] = m$, then $|K| = p_1^m = q$, which implies $p_1 = p$, $m = n$. Thus, K is also an n -th degree extension of \mathbb{F}_p , similar to \mathbb{F}_q . Since K^* forms a group of order $q - 1$ under multiplication, we have $u^{q-1} = 1$ for all $u \in K^*$. This means that the field K consists of the q roots of $f(x) = x^q - x \in \mathbb{F}_p[x]$ in K . Therefore, K and \mathbb{F}_q are both splitting fields of the same polynomial, implying $K \cong \mathbb{F}_q$.
- (3) From the proof of conclusion (1), we know that elements in \mathbb{F}_q satisfy $x^q = x$, i.e., $\eta_q(a) = a (\forall a \in \mathbb{F}_q)$. Because $q\alpha = 0 (\forall \alpha \in L)$, we have $(\alpha + \beta)^q = \alpha^q + \beta^q (\forall \alpha, \beta \in L)$. It's easy to prove that η_q is a field isomorphism.
- (4) Let $L \supset \mathbb{F}_q$ be an m -th degree extension; then L is a field with q^m elements. From the proof of conclusion (2), we know that L is a splitting field of the separable polynomial $f(x) = x^{q^m} - x \in \mathbb{F}_q[x]$, so $|\text{Gal}(L/\mathbb{F}_q)| = [L : \mathbb{F}_q] = m$. On the other hand, for any $0 < k < m$, $L \xrightarrow{\eta_q^k} L$ is not an identity map (otherwise, the polynomial $x^{q^k} - x$ would have q^m roots in L , which contradicts $q^k < q^m$), so $\text{Gal}(L/\mathbb{F}_q)$ is a cyclic group generated by η_q .
- (5) Let $\alpha \in L^*$ be an element of maximal order N in the finite abelian group L^* . We claim that

$$\beta^N = 1, \quad \forall \beta \in L^*.$$

Indeed, suppose the order of β is N' . If $\beta^N \neq 1$, then $N' \nmid N$. Let $d = (N, N')$ be the **greatest common divisor** of N and N' .

$$d = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$$

is the irreducible factorization, $d_1 = \prod_{p_i \nmid \frac{N'}{d}} p_i^{k_i}$, $d_2 = \frac{d}{d_1}$. Then $\left(\frac{N'}{d_1}, \frac{N}{d_2}\right) = 1$, the order of β^{d_1} is $\frac{N'}{d_1}$, and the order of α^{d_2} is $\frac{N}{d_2}$. It is easy to prove that the order of $\alpha^{d_2} \beta^{d_1}$ is $\frac{N}{d_2} \cdot \frac{N'}{d_1} = N \cdot \frac{N'}{d} > N$, which contradicts the choice

of N . Therefore, all elements in L^* are roots of the equation $x^N - 1 = 0$, so $|L^*| = q^m - 1 \leq N$, which means L^* is a cyclic group generated by $\alpha \in L^*$.

□

8.4.2 $\mathbb{Q}(\sqrt[4]{2}, \zeta_8)$

[galoisapp.pdf](#)

$K = \mathbb{Q}(\sqrt[4]{2}, \zeta_8)$ is a Galois extension of \mathbb{Q} because:

- (1) $\zeta_8 = e^{2\pi i/8} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, and $\sqrt{2} = \zeta_8 + \zeta_8^{-1}$, $i = \zeta_8^2$.
- (2) $\sqrt[4]{2}$ is a root of $x^4 - 2$, and all roots of $x^4 - 2$ are in K because $i \in K$.
- (3) ζ_8 is a root of $x^4 + 1$, and all roots of $x^4 + 1$ are in K .
- (4) K is a splitting field of $(x^4 - 2)(x^4 + 1)$ over \mathbb{Q} , making it a normal extension.
- (5) Since \mathbb{Q} has characteristic 0, K is a separable extension of \mathbb{Q} .
- (6) Therefore, K is a Galois extension of \mathbb{Q} .

Définition 2: normal extension

An algebraic field extension E/F is a **normal extension** if every irreducible polynomial in $F[x]$ that has a root in E factors completely into linear factors in $E[x]$.

Définition 3: separable extension

An algebraic extension E/F is a **separable extension** if every element of E is separable over F , i.e., is a root of a separable polynomial in $F[x]$.

Définition 4: separable polynomial

A polynomial is called **separable** if all its roots are distinct.

Task: why splitting field of $(x^4 - 2)(x^4 + 1)$ over \mathbb{Q} is separable.

Output: The splitting field of $(x^4 - 2)(x^4 + 1)$ over \mathbb{Q} is separable because we are in characteristic 0. More generally, if K is a field of characteristic 0 and

$f(x) \in K[x]$, then the splitting field of $f(x)$ over K is separable. This is because every irreducible polynomial over a field of characteristic 0 is separable.

In this case, the splitting field L is an extension of \mathbb{Q} . Since \mathbb{Q} has characteristic 0, L also has characteristic 0. The polynomials $x^4 - 2$ and $x^4 + 1$ have distinct roots. The roots of $x^4 - 2$ are $\sqrt[4]{2}, i\sqrt[4]{2}, -\sqrt[4]{2}, -i\sqrt[4]{2}$. The roots of $x^4 + 1$ are $e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$. Since these are all distinct, the extension is separable.

Théorème 7

Every irreducible polynomial over a field of characteristic 0 is **separable**.

Counterexample:

Let $K = \mathbb{F}_p(t)$ be the field of rational functions in one variable over \mathbb{F}_p . Then $f(x) = x^p - t \in K[x]$ is irreducible by Eisenstein's criterion. However, $f'(x) = px^{p-1} = 0$, so $\gcd(f(x), f'(x)) = f(x)$. Thus, $f(x)$ is not separable because a polynomial is separable if and only if it has no repeated roots in its splitting field, which is equivalent to the polynomial and its derivative being coprime (i.e., their greatest common divisor is 1). Since $\gcd(f(x), f'(x)) = f(x) \neq 1$, $f(x)$ has repeated roots and is therefore not separable.

The field extension $[\mathbb{Q}(\sqrt[4]{2}, \zeta_8) : \mathbb{Q}]$ has degree 8, not 16, because $\mathbb{Q}(\sqrt[4]{2}, \zeta_8) = \mathbb{Q}(\sqrt[4]{2}, i)$. The Galois group $\text{Gal}(\mathbb{Q}(\sqrt[4]{2}, \zeta_8)/\mathbb{Q})$ has at most 16 automorphisms, determined by $\sigma(\zeta_8) = \zeta_8^a$ and $\sigma(\sqrt[4]{2}) = i^b \sqrt[4]{2}$, but the choices of a and b are not independent. Since $\zeta_8 + \zeta_8^{-1} = \sqrt{2} = \sqrt[4]{2}^2$, $\mathbb{Q}(\sqrt{2})$ is a common subfield of both $\mathbb{Q}(\zeta_8)$ and $\mathbb{Q}(\sqrt[4]{2})$.

The effect of $\sigma \in \text{Gal}(\mathbb{Q}(\sqrt[4]{2}, \zeta_8)/\mathbb{Q})$ on $\sqrt[4]{2}$ partially determines it on ζ_8 , and conversely: $(\sigma(\sqrt[4]{2}))^2 = \sigma(\zeta_8) + \sigma(\zeta_8)^{-1}$, which gives the relation

$$(-1)^b = \frac{\zeta_8^a + \zeta_8^{-a}}{\sqrt{2}} \quad (8.1)$$

This tells us that if $a \equiv 1, 7 \pmod{8}$ then $(-1)^b = 1$, so $b \equiv 0, 2 \pmod{4}$, while if $a \equiv 3, 5 \pmod{8}$ then $(-1)^b = -1$, so $b \equiv 1, 3 \pmod{4}$. For example, σ can't both fix $\sqrt[4]{2}(b=0)$ and send ζ_8 to $\zeta_8^3(a=3)$ because eq. (8.1) would not hold.

Théorème 8

Let L_1 and L_2 be Galois over K . There is an injective homomorphism

$$\text{Gal}(L_1 L_2 / K) \hookrightarrow \text{Gal}(L_1 / K) \times \text{Gal}(L_2 / K)$$

given by $\sigma \mapsto (\sigma|_{L_1}, \sigma|_{L_2})$. In particular, if L_1/K and L_2/K are finite abelian extensions then $L_1 L_2$ is a finite abelian extension of K .

证明. We omit this proof □

Proposition 1

Consider a finite extension L/K . L/K is normal if and only if any of the following equivalent conditions hold:

- (1) Every irreducible polynomial in $K[x]$ that has at least one root in L splits completely into linear factors over L .
- (2) L is the splitting field of some polynomial in $K[x]$.
- (3) Every K -embedding of L into an algebraic closure \bar{K} of K has its image contained in L . That is, if $\sigma : L \hookrightarrow \bar{K}$ is a field homomorphism such that $\sigma(k) = k$ for all $k \in K$, then $\sigma(L) \subseteq L$.

Task: The Galois group of $(X^2 - 2)(X^2 - 3)$ over \mathbf{Q} is $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Its Galois group over \mathbf{R} is trivial. Why trivial?

Answer:

The Galois group of a polynomial over a field F is the group of automorphisms of the splitting field of the polynomial that fix F .

In this case, we are considering the polynomial $p(x) = (x^2 - 2)(x^2 - 3)$ over \mathbf{R} . The roots of $x^2 - 2$ are $\pm\sqrt{2}$, and the roots of $x^2 - 3$ are $\pm\sqrt{3}$. Since $\sqrt{2}$ and $\sqrt{3}$ are both real numbers, all four roots $\sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}$ are in \mathbf{R} . Thus, the splitting field of $p(x)$ over \mathbf{R} is \mathbf{R} itself.

The Galois group consists of automorphisms of the splitting field that fix the base field. In this case, we seek automorphisms of \mathbf{R} that fix \mathbf{R} . The only such automorphism is the identity map. Therefore, the Galois group is trivial.

Example 4.4. Consider the Galois group of $X^4 - 2$ over \mathbf{Q} . The polynomial has 4 roots: $\sqrt[4]{2}, i\sqrt[4]{2}, -\sqrt[4]{2}, -i\sqrt[4]{2}$. Take as generators of $\text{Gal}(\mathbf{Q}(\sqrt[4]{2}, i)/\mathbf{Q})$ the automorphisms r and s from Example 1.2. The effect of the Galois group on $\sqrt[4]{2}$ and i is in Table 3.

Automorphism	1	r	r^2	r^3	s	rs	r^2s	r^3s
Value on $\sqrt[4]{2}$	$\sqrt[4]{2}$	$i\sqrt[4]{2}$	$-\sqrt[4]{2}$	$-i\sqrt[4]{2}$	$\sqrt[4]{2}$	$i\sqrt[4]{2}$	$-\sqrt[4]{2}$	$-i\sqrt[4]{2}$
Value on i	i	i	i	i	$-i$	$-i$	$-i$	$-i$

TABLE 3

Passing from $\sqrt[4]{2}$ and i to the four roots of $X^4 - 2$, the effect of r on the roots of $X^4 - 2$ is

$$r(\sqrt[4]{2}) = i\sqrt[4]{2}, \quad r(i\sqrt[4]{2}) = -\sqrt[4]{2}, \quad r(-\sqrt[4]{2}) = -i\sqrt[4]{2}, \quad r(-i\sqrt[4]{2}) = \sqrt[4]{2},$$

which is a 4-cycle, while the effect of s on the roots of $X^4 - 2$ is

$$s(\sqrt[4]{2}) = \sqrt[4]{2}, \quad s(i\sqrt[4]{2}) = -i\sqrt[4]{2}, \quad s(-i\sqrt[4]{2}) = i\sqrt[4]{2}, \quad s(-\sqrt[4]{2}) = -\sqrt[4]{2},$$

which swaps $i\sqrt[4]{2}$ and $-i\sqrt[4]{2}$ while fixing $\sqrt[4]{2}$ and $-\sqrt[4]{2}$. So s is a 2-cycle on the roots.

Indexing the roots of $X^4 - 2$ as

$$(4.1) \quad \alpha_1 = \sqrt[4]{2}, \quad \alpha_2 = i\sqrt[4]{2}, \quad \alpha_3 = -\sqrt[4]{2}, \quad \alpha_4 = -i\sqrt[4]{2},$$

the automorphism r acts on the roots like (1234) and the automorphism s acts on the roots like (24). With this indexing of the roots, the Galois group of $X^4 - 2$ over \mathbf{Q} becomes the group of permutations in S_4 in Table 4.

Automorphism	1	r	r^2	r^3	s	rs	r^2s	r^3s
Permutation	(1)	(1234)	(13)(24)	(1432)	(24)	(12)(34)	(13)	(14)(23)

Théorème 9: Theorem 4.10

Let $f(X) \in K[X]$ be a separable polynomial of degree n . (a) If $f(X)$ is irreducible in $K[X]$ then its Galois group over K has order divisible by n .
(b) The polynomial $f(X)$ is irreducible in $K[X]$ if and only if its Galois group over K is a transitive subgroup of S_n .

证明. (a) For a root r of $f(X)$ in K , $[K(r) : K] = n$ is a factor of the degree of the splitting field over K , which is the size of the Galois group over K .

(b) First suppose $f(X)$ is irreducible. For two roots r_i and r_j of $f(X)$, we can write $r_j = \sigma(r_i)$ for some σ in the Galois group of $f(X)$ over K . Therefore the Galois group, as a subgroup of S_n , sends i to j , so it is a transitive subgroup. Now suppose $f(X)$ is reducible (so $n \geq 2$). It is a product of distinct irreducibles since it is separable. Let r_i and r_j be roots of different irreducible factors of $f(X)$. These irreducible factors are the minimal polynomials of r_i and r_j over K . For any σ in the Galois group of $f(X)$ over K , $\sigma(r_i)$ has the same minimal polynomial over K as r_i , so we can't have $\sigma(r_i) = r_j$. Therefore, as a subgroup

of S_n , the Galois group of $f(X)$ does not send i to j , so it is not a transitive subgroup of S_n . \square

Définition 5: transitive subgroup

A subgroup G of S_n is called **transitive** if for every $i, j \in \{1, 2, \dots, n\}$, there exists a $g \in G$ such that $g(i) = j$.

Théorème 10

Let $f(X) \in \mathbf{Q}[X]$ be an irreducible polynomial of prime degree p with all but two roots in \mathbf{R} . The Galois group of $f(X)$ over \mathbf{Q} is isomorphic to S_p .

证明. Let $L = \mathbf{Q}(r_1, \dots, r_p)$ be the splitting field of $f(X)$ over \mathbf{Q} . The permutations of the r_i 's by $\text{Gal}(L/\mathbf{Q})$ provide an embedding $\text{Gal}(L/\mathbf{Q}) \hookrightarrow S_p$ and $\#\text{Gal}(L/\mathbf{Q})$ is divisible by p by Theorem 4.10, so $\text{Gal}(L/\mathbf{Q})$ contains an element of order p by Cauchy's theorem. In S_p , the only permutations of order p are p -cycles (why?). So the image of $\text{Gal}(L/\mathbf{Q})$ in S_p contains a p -cycle.

We may take L to be a subfield of \mathbf{C} , since \mathbf{C} is algebraically closed. Complex conjugation restricted to L is a member of $\text{Gal}(L/\mathbf{Q})$. Since $f(X)$ has only two non-real roots by hypothesis, **complex conjugation transposes two of the roots of $f(X)$ and fixes the others**. Therefore $\text{Gal}(L/\mathbf{Q})$ contains a transposition of the roots of $f(X)$. (This is the reason for the hypothesis about all but two roots being real.)

We now show the only subgroup of S_p containing a p -cycle and a transposition is S_p , so $\text{Gal}(L/\mathbf{Q}) \cong S_p$. By suitable labeling of the numbers from 1 to p , we may let 1 be a number moved by the transposition, so our subgroup contains a transposition $\tau = (1a)$. Let σ be a p -cycle in the subgroup. As a p -cycle, σ acts on $\{1, 2, \dots, p\}$ by a single orbit, so some σ^i with $1 \leq i \leq p-1$ sends 1 to a : $\sigma^i = (1a\dots)$. This is also a p -cycle, because σ^i has order p in S_p and all elements of order p in S_p are p -cycles, so writing σ^i as σ and suitably reordering the numbers $2, \dots, p$ (which replaces our subgroup by a conjugate subgroup), we may suppose our subgroup of S_p contains the particular transposition (12) and the particular p -cycle $(12\dots p)$. For $n \geq 2$, it

is a theorem in group theory that the particular transposition (12) and n -cycle $(12 \dots n)$ generate S_n , so our subgroup is S_p . \square

Task: Why? In S_p , the only permutations of order p are p -cycles

Output: The order of a permutation written in cycle notation is the **least common multiple** of the lengths of its cycles. To get order p , where p is prime, we need the least common multiple of the cycle lengths to be p . Since p is prime, the cycle lengths must be 1 or p . Since the lengths must add up to p , we must have one cycle of length p and all the other cycles of length 1. Thus, we have a p -cycle.

Task: why? in group theory that the particular transposition (12) and n -cycle $(12 \dots n)$ generate S_n .

Output: The symmetric group S_n is the group of all permutations of n elements. We want to show that the transposition (12) and the n -cycle $(12 \dots n)$ generate S_n .

Let $\tau = (12)$ and $\sigma = (12 \dots n)$. We want to show that any permutation in S_n can be written as a product of τ and σ .

First, we can generate any transposition of the form $(1i)$ for $i = 2, \dots, n$.

We have

$$\sigma^{i-2} \tau \sigma^{-(i-2)} = (1i)$$

For example,

- $i = 2 : \sigma^0 \tau \sigma^0 = (12)$
- $i = 3 : \sigma^1 \tau \sigma^{-1} = (23)(12)(1n) \dots (23) = (13)$
- $i = 4 : \sigma^2 \tau \sigma^{-2} = (34)(23)(12)(1n) \dots (34) = (14)$

So we can generate any transposition of the form $(1i)$.

Next, we can generate any transposition (ij) for $1 \leq i, j \leq n$. We have

$$(ij) = (1i)(1j)(1i)$$

Since we can generate any $(1i)$ and $(1j)$, we can generate any (ij) .

Finally, since any permutation can be written as a product of transpositions, and we can generate any transposition (ij) using τ and σ , we can generate any permutation in S_n using τ and σ .

Therefore, the transposition (12) and the n -cycle $(12 \dots n)$ generate S_n .

Exemple 23 (4.13). *The polynomial $X^3 - X - 1$ is irreducible in $\mathbf{Q}[X]$ since it is irreducible mod 2 or since it is a cubic without any rational roots. It has one real root (approximately 1.3247), and one root of a cubic is all but two roots, so its Galois group over \mathbf{Q} is isomorphic to S_3 .*

Exemple 24 (4.14). *The polynomials $X^3 - 3X - 1$ and $X^3 - 4X - 1$ are both irreducible in $\mathbf{Q}[X]$ since they are cubics without any rational roots. Each polynomial has three real roots (check!), so we can't use Theorem 4.11 to determine their Galois groups over \mathbf{Q} .*

Exemple 25 (4.15). *The quintic polynomial $X^5 - X - 1$ is irreducible in $\mathbf{Q}[X]$ since it is irreducible mod 3. It has one real root, so Theorem 4.11 does not tell us the Galois group.*

Exemple 26 (4.16). *The quintic polynomial $X^5 - 4X - 1$ is irreducible in $\mathbf{Q}[X]$ since it is irreducible mod 3. It has three real roots, which is all but two roots, so its Galois group over \mathbf{Q} is isomorphic to S_5 .*

8.4.3 The Galois Group of $\mathbb{Q}(\sqrt[n]{a}, \zeta_n)$ over \mathbb{Q}

In general, if $\theta = \sqrt[n]{a}$ with $n \in \mathbb{N}$ and $a \in \mathbb{Q}$ such that for any divisor m of n , $\sqrt[m]{a} \notin \mathbb{Q}$, then the normal closure of $\mathbb{Q}(\theta)$ is $K = \mathbb{Q}(\theta, \zeta_n)$. The Galois group $\text{Gal}(K/\mathbb{Q})$ is always a subgroup of semidirect product $\mathbf{Z}_n \rtimes \mathbf{Z}_n^\times$.

In "majority" cases, the Galois group $\text{Gal}(K/\mathbb{Q})$ is isomorphic to $\mathbf{Z}_n \rtimes \mathbf{Z}_n^\times$.

Note. \mathbf{Z}_n is a normal subgroup of $\mathbf{Z}_n \rtimes \mathbf{Z}_n^\times$, which corresponds to that $\mathbb{Q}(\zeta_n)$ is a Galois extension of \mathbb{Q} . On the other hand, \mathbf{Z}_n^\times is not a normal subgroup and thus $\mathbb{Q}(\sqrt[n]{a})$ is not normal over \mathbb{Q} .

Définition 6: transitive

A subgroup G of the symmetric group S_n is said to be **transitive** if for every $i, j \in \{1, 2, \dots, n\}$, there exists a $g \in G$ such that $g(i) = j$. In other words, for any two elements in the set $\{1, 2, \dots, n\}$, there is a permutation in the subgroup that maps the first element to the second.

8.4.4 Discriminant

Définition 7: discriminant

For a nonconstant $f(X) \in K[X]$ of degree n that factors over a splitting field as

$$f(X) = c(X - r_1) \cdots (X - r_n),$$

the **discriminant** of $f(X)$ is defined to be

$$\text{disc } f = \prod_{i < j} (r_j - r_i)^2.$$

The number $\text{disc } f$ is nonzero if $f(X)$ is separable and is 0 if $f(X)$ is not separable.

When f is separable, $\text{disc } f$ is a symmetric polynomial in the r_i 's, so it is fixed by $\text{Gal}(K(r_1, \dots, r_n)/K)$ and therefore $\text{disc } f \in K$ by Galois theory.

Théorème 11

Let K not have characteristic 2 and let $f(X)$ be a separable cubic in $K[X]$ with a root r and discriminant D . The splitting field of $f(X)$ over K is $K(r, \sqrt{D})$. Note we are not assuming $f(X)$ is irreducible here.

Théorème 12: Theorem 4.23

Let $f(X) \in K[X]$ be a **separable polynomial** of degree n . If K does not have characteristic 2, the embedding of the Galois group of $f(X)$ over K into S_n as permutations of the roots of $f(X)$ has image in A_n if and only if $\text{disc } f$ is a square in K .

证明. 设 $\delta = \prod_{i < j} (r_j - r_i) \neq 0$, 因此 $\delta \in K(r_1, \dots, r_n)$ 且 $\delta^2 = \text{disc } f \in K$ 。因此, $\text{disc } f$ 是 K 中的一个平方, 当且仅当 $\delta \in K$ 。

对于任何 $\sigma \in \text{Gal}(K(r_1, \dots, r_n)/K)$, 设 $\varepsilon_\sigma = \pm 1$ 是其作为 r_i 的排列的符号。根据排列符号的定义之一,

$$\sigma(\delta) = \prod_{i < j} (\sigma(r_j) - \sigma(r_i)) = \varepsilon_\sigma \prod_{i < j} (r_j - r_i) = \varepsilon_\sigma \delta$$

因此 $\sigma(\delta) = \pm \delta$ 。由于 $\delta \neq 0$ 且 K 没有特征 2, $\delta \neq -\delta$ 。当且仅当 $\varepsilon_\sigma = 1$ 时, 我

们有 $\sigma \in A_n$, 因此当且仅当 $\sigma(\delta) = \delta$ 时, $\sigma \in A_n$ 。因此, $f(X)$ 在 K 上的伽罗瓦群在 A_n 中当且仅当 δ 被伽罗瓦群固定, 这与 $\delta \in K$ 相同。 \square

Théorème 13: Theorem 4.25

Let K not have characteristic 2 and let $f(X)$ be a separable irreducible cubic in $K[X]$. (a) If $\text{disc } f$ is a square in K then the Galois group of $f(X)$ over K is isomorphic to A_3 . (b) If $\text{disc } f$ is not a square in K then the Galois group of $f(X)$ over K is isomorphic to S_3 .

Remarque 30. If the characteristic of the field is 2, then $1 = -1$, which means every permutation is even since $\text{sgn}(\sigma) = \pm 1 = 1$.

Discriminants in low-degree cases

In low-degree cases, explicit formulas for discriminants of some trinomials are

$$\begin{aligned}\text{disc}(X^2 + aX + b) &= a^2 - 4b, \\ \text{disc}(X^3 + aX + b) &= -4a^3 - 27b^2, \\ \text{disc}(X^4 + aX + b) &= -27a^4 + 256b^3, \\ \text{disc}(X^5 + aX + b) &= 256a^5 + 3125b^4.\end{aligned}$$

8.4.5 Determine the Galois group

Théorème 14: Dedekind

Let $f(X) \in \mathbf{Z}[X]$ be monic irreducible over \mathbf{Q} of degree n . For any prime p not dividing $\text{disc } f$, let the monic irreducible factorization of $f(X) \bmod p$ be

$$f(X) \equiv \pi_1(X) \cdots \pi_k(X) \bmod p$$

and set $d_i = \deg \pi_i(X)$, so $d_1 + \cdots + d_k = n$. The Galois group of $f(X)$ over \mathbf{Q} , viewed as a subgroup of S_n , contains a permutation of type (d_1, \dots, d_k) .

The nicest proof of Théorème 14 uses algebraic number theory and is beyond the scope of these notes.

Exemple 27. We compute the Galois group of $X^4 - X - 1$ over \mathbf{Q} using Théorème 14. This polynomial is irreducible mod 2, so it is irreducible over \mathbf{Q} . Let its roots be r_1, r_2, r_3, r_4 . The extension $\mathbf{Q}(r_1)/\mathbf{Q}$ has degree 4, so the Galois group of $X^4 - X - 1$ over \mathbf{Q} has order divisible by 4. Since the Galois group embeds into S_4 , its size is either 4, 8, 12, or 24. The discriminant of $X^4 - X - 1$ is -283, which is not a rational square, so the Galois group is not a subgroup of A_4 . This eliminates the possibility of the Galois group having order 12, because the only subgroup of S_4 with order 12 is A_4 . (Quite generally, the only subgroup of index 2 in S_n is A_n for $n \geq 2$.) There are subgroups of S_4 with orders 4, 8, and (of course) 24 outside of A_4 , so no other size but 12 is eliminated yet. Using Théorème 14 with $p = 7$,

$$X^4 - X - 1 \equiv (X + 4)(X^3 + 3X^2 + 2X + 5) \pmod{7}$$

This is an irreducible factorization, so Théorème 14 says that the Galois group of $X^4 - X - 1$ over \mathbf{Q} contains a permutation of the roots with cycle type (1, 3), which means the Galois group has order divisible by 3, and that proves the Galois group is S_4 .

Task: why implies irreducibility? We compute the Galois group of $X^4 - X - 1$ over \mathbf{Q} using Théorème 14. This polynomial is irreducible mod 2, so it is irreducible over \mathbf{Q} .

A polynomial $f(x)$ is **irreducible over \mathbf{Q}** if it cannot be factored into non-constant polynomials with rational coefficients. Reducing a polynomial modulo a prime p can be used to prove irreducibility over \mathbf{Q} because if $f(x)$ factors over \mathbf{Q} , then it must also factor modulo p for almost all primes p . Thus, if $f(x) \pmod{p}$ is irreducible for some prime p , then $f(x)$ must be irreducible over \mathbf{Q} .

In this specific case, since $X^4 - X - 1$ is irreducible modulo 2, it follows that $X^4 - X - 1$ is irreducible over \mathbf{Q} . This is a consequence of the fact that if $X^4 - X - 1$ were reducible over \mathbf{Q} , then it would also be reducible modulo 2, which contradicts the given information that it is irreducible modulo 2.

Example

Let's determine the Galois group of $X^5 - X - 1$ over \mathbf{Q} , which was left unresolved in Example 4.15. Its irreducible factorization mod 2 is

$$X^5 - X - 1 \equiv (X^2 + X + 1)(X^3 + X^2 + 1) \pmod{2}.$$

Because the polynomial is irreducible over \mathbf{Q} , 5 divides the size of the Galois group. From the mod 2 factorization, the Galois group contains a permutation of the roots with cycle type $(2, 3)$, which has order 6, so the Galois group has size divisible by $5 \cdot 6 = 30$. Since the Galois group is a subgroup of S_5 , its size is either 30, 60, or 120.

It turns out that there is no subgroup of S_5 with order 30 and the only subgroup of order 60 is A_5 . The discriminant of $f(x)$ is $2869 = 19 \cdot 151$, which is not a rational square, so the Galois group is not in A_5 by Theorem 4.23. Therefore the Galois group is S_5 .

Why divisible by 30 We want to understand why the size of the Galois group G of the polynomial $f(X) = X^5 - X - 1$ over the rational numbers \mathbf{Q} must be divisible by $5 \times 6 = 30$.

This conclusion is derived from two key facts about the polynomial and fundamental theorems in Galois theory and group theory.

(1) Divisibility by 5

- **Reason:** The polynomial $f(X) = X^5 - X - 1$ is **irreducible** over \mathbf{Q} .
- **Theorem:** For an **irreducible** polynomial of degree n over a field F , the order of the Galois group of its splitting field over F is divisible by n . Since $f(X)$ is irreducible and has degree $n = 5$, the size of its Galois group $G = \text{Gal}(K/\mathbf{Q})$ must be divisible by 5. Equivalently, the Galois group G acts **transitively** on the 5 roots, so $|G|$ is divisible by 5.

(2) Divisibility by 6

- **Reason:** The factorization of $f(X)$ modulo 2 is given as:

$$X^5 - X - 1 \equiv (X^2 + X + 1)(X^3 + X^2 + 1) \pmod{2}$$

Both factors, $X^2 + X + 1$ and $X^3 + X^2 + 1$, are irreducible over the field \mathbf{F}_2 .

- **Theorem (Dedekind's Theorem / Frobenius Elements):** If a polynomial with integer coefficients factors modulo a prime p into distinct irreducible factors of degrees n_1, n_2, \dots, n_k , then its Galois group G contains an element σ having the cycle structure (n_1, n_2, \dots, n_k) . Thus, the Galois group G , seen as a subgroup of S_5 , must contain a permutation σ with cycle structure $(2, 3)$.
- The order of permutation σ is $\text{lcm}(2, 3) = 6$. Since G contains an element σ of order 6, the order of the group $|G|$ must be divisible by 6.

Théorème 15: Lagrange's Theorem

In any finite group, the order of any element must divide the order of the group.

证明. By considering the partition of a finite group G into disjoint left cosets of a subgroup $H = \langle g \rangle$, we can deduce that the order of H divides the order of G , and consequently, the order of any element g in G must also divide the order of G . \square

Some Strong Corollaries

From Théorème 10 and Théorème 14, we have

Corollaire 4

Let $f(X) \in \mathbf{Z}[X]$ be monic irreducible over \mathbf{Q} of prime degree p . If there is a prime number ℓ not dividing disc f such that $f(X) \bmod \ell$ has all but two roots in \mathbf{F}_ℓ , then the Galois group of $f(X)$ over \mathbf{Q} is isomorphic to S_p .

证明. The proof of Théorème 10 can be used again except for the step explaining why the Galois group of $f(X)$ over \mathbf{Q} contains a transposition. In Théorème 10 this came from the use of complex conjugation to transpose two non-real roots, assuming there are only two non-real roots. We aren't assuming that anymore.

By hypothesis the factorization of $f(X) \bmod \ell$ has all linear factors except for one quadratic irreducible factor. Therefore Theorem 4.29 says the Galois group contains a permutation of the roots with cycle type $(1, 1, \dots, 1, 2)$, which is a transposition in S_p . \square

If we seek an analogue of Théorème 10 for a Galois group to be isomorphic to A_p , using 3-cycles in place of transpositions, there is no analogue since an irreducible polynomial over \mathbf{Q} can't have all but three roots in \mathbf{R} (the number of non-real roots is always even). But $f(X) \bmod \ell$ could have all but three roots in \mathbf{F}_ℓ for some ℓ . This suggests the next result.

Corollaire 5

Let $f(X) \in \mathbf{Z}[X]$ be monic irreducible over \mathbf{Q} of prime degree $p \geq 3$ with $\text{disc } f$ a perfect square. If there is a prime number ℓ not dividing $\text{disc } f$ such that $f(X) \bmod \ell$ has all but three roots in \mathbf{F}_ℓ , then the Galois group of $f(X)$ over \mathbf{Q} is isomorphic to A_p .

证明. Let G be the Galois group, so G is a subgroup of A_p since $\text{disc } f$ is a square. The Galois group has order divisible by p , so it contains a p -cycle. From the factorization of $f(X) \bmod \ell$ and Théorème 14, G contains a 3-cycle. It is a theorem of C. Jordan that for any prime $p \geq 3$, any p -cycle and any 3-cycle in S_p generate A_p , so $G \cong A_p$. \square

Exemple 28. The polynomial $X^5 + 20X + 16$ has discriminant $2^{16}5^6$. It is irreducible mod 3, so it's irreducible over \mathbf{Q} . Modulo 7, its irreducible factorization is

$$X^5 + 20X + 16 \equiv (X - 4)(X - 5)(X^3 + 2X^2 + 5X + 5) \bmod 7.$$

This has all but three roots in \mathbf{F}_7 , so the Galois group of $X^5 + 20X + 16$ over \mathbf{Q} is isomorphic to A_5 .

Remarque 31. It is a hard theorem of Chebotarev that the sufficient conditions for $f(X)$ to have Galois group S_p in Corollaire 4 and A_p in Corollaire 5 are also necessary in a strong sense: if $f(X) \in \mathbf{Z}[X]$ is monic irreducible of prime degree p with Galois group over \mathbf{Q} isomorphic to S_p (resp., A_p) then there are infinitely many primes ℓ not dividing $\text{disc } f$ such that

Decomposing $X^3 - X - 1 \pmod{5}$ To decompose $X^3 - X - 1 \pmod{5}$, we want to find its roots in $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. We can test each element:

- For $x = 0$, $0^3 - 0 - 1 = -1 \equiv 4 \pmod{5}$.
- For $x = 1$, $1^3 - 1 - 1 = -1 \equiv 4 \pmod{5}$.
- For $x = 2$, $2^3 - 2 - 1 = 8 - 2 - 1 = 5 \equiv 0 \pmod{5}$.
- For $x = 3$, $3^3 - 3 - 1 = 27 - 3 - 1 = 23 \equiv 3 \pmod{5}$.
- For $x = 4$, $4^3 - 4 - 1 = 64 - 4 - 1 = 59 \equiv 4 \pmod{5}$.

Since $x = 2$ is a root, $(X - 2)$ is a factor. Then we can perform polynomial division to find the other factor. Note that $-2 \equiv 3 \pmod{5}$, so we can write $(X - 2) = (X + 3)$.

$$X^3 - X - 1 = (X - 2)(X^2 + 2X + 3) \text{ in } \mathbb{Z}_5[X].$$

Now we consider the quadratic $X^2 + 2X + 3 \pmod{5}$. We want to see if it can be factored further. We can check its discriminant: $\Delta = b^2 - 4ac = 2^2 - 4(1)(3) = 4 - 12 = -8 \equiv 2 \pmod{5}$. Since 2 is not a quadratic residue modulo 5 (i.e., there is no x such that $x^2 \equiv 2 \pmod{5}$), the quadratic has no roots in \mathbb{Z}_5 . The quadratic residues modulo 5 are $0^2 \equiv 0$, $1^2 \equiv 1$, $2^2 \equiv 4$, $3^2 \equiv 9 \equiv 4$, $4^2 \equiv 16 \equiv 1 \pmod{5}$.

Thus, $X^2 + 2X + 3$ is irreducible over \mathbb{Z}_5 .

Therefore, the decomposition of $X^3 - X - 1 \pmod{5}$ is $(X - 2)(X^2 + 2X + 3)$.

8.4.6 Methods to Verify Polynomial Irreducibility in a Field

Verifying whether a polynomial is irreducible in a given field is a fundamental problem in algebra. Here are several methods and theorems that can be used:

Eisenstein's Criterion

Théorème 16: Eisenstein's Criterion

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial with integer coefficients. If there exists a prime number p such that:

- $p \mid a_i$ for all $0 \leq i < n$,
- $p \nmid a_n$,

- $p^2 \nmid a_0$,

then $f(x)$ is irreducible over \mathbb{Q} .

This criterion directly proves irreducibility over \mathbb{Q} by checking divisibility conditions of the polynomial's coefficients by a prime number.

Reduction Modulo p

- Choose a prime p such that the degree of the polynomial $f(x)$ with integer coefficients remains the same when reduced modulo p .
- If the reduced polynomial is irreducible over \mathbb{Z}_p , then $f(x)$ is irreducible over \mathbb{Q} .
- If the reduced polynomial is reducible, no conclusion can be made about the irreducibility of $f(x)$ over \mathbb{Q} .

This method involves reducing the polynomial modulo a prime p and checking for irreducibility in the finite field \mathbb{Z}_p .

Rational Root Theorem

- For a polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_0$ with integer coefficients, any rational root must have the form $\frac{p}{q}$, where p divides a_0 and q divides a_n .
- A polynomial of degree 2 or 3 with no rational roots is irreducible over \mathbb{Q} .

This theorem helps find potential rational roots, and if none exist for polynomials of degree 2 or 3, it confirms irreducibility.

Adjoining Roots

- If $f(x)$ is a polynomial over a field F , and α is a root of $f(x)$ in some extension field, then $f(x)$ is irreducible if and only if $F(\alpha) = F[x]/(f(x))$ has degree equal to the degree of $f(x)$.

This method checks irreducibility by examining the degree of the field extension formed by adjoining a root of the polynomial.

Polynomial Decomposition Algorithms

- Algorithms like Berlekamp's algorithm determine if a polynomial is irreducible over a finite field.

These algorithms provide computational methods to test for irreducibility, especially over finite fields.

8.4.7 Powerful Theorems

Gauss's Lemma

Lemme 6: Gauss's Lemma

If a polynomial with integer coefficients can be factored into two polynomials with rational coefficients, then it can be factored into two polynomials with integer coefficients.

Proof Sketch: Suppose that $f(x)$ is a polynomial with integer coefficients. Assume that it can be factored into two polynomials with rational coefficients, say $f(x) = A(x)B(x)$, where $A(x)$ and $B(x)$ have rational coefficients.

- (1) **Clear denominators:** Multiply $A(x)$ and $B(x)$ by suitable integers a and b respectively, such that $aA(x)$ and $bB(x)$ have integer coefficients. Thus, we have $abf(x) = aA(x) \cdot bB(x)$, where $aA(x)$ and $bB(x)$ are polynomials with integer coefficients.
- (2) **Remove common factors:** Divide $aA(x)$ and $bB(x)$ by the greatest common divisor of their coefficients. Let $A_1(x) = \frac{aA(x)}{c_1}$ and $B_1(x) = \frac{bB(x)}{c_2}$, where c_1 is the greatest common divisor of the coefficients of $aA(x)$ and c_2 is the greatest common divisor of the coefficients of $bB(x)$. Then $A_1(x)$ and $B_1(x)$ are primitive polynomials (polynomials with integer coefficients whose coefficients have a greatest common divisor of 1). We then have $abf(x) = c_1c_2A_1(x)B_1(x)$.
- (3) **Reduce to primitive polynomials:** Let $C = \frac{ab}{c_1c_2}$. Then $Cf(x) = A_1(x)B_1(x)$. Since $A_1(x)$ and $B_1(x)$ have integer coefficients, their product $A_1(x)B_1(x)$ also has integer coefficients. Furthermore, $f(x)$ has integer coefficients.

ger coefficients. Hence C must be a rational number such that $Cf(x)$ has integer coefficients.

- (4) **Show C is an integer:** Since $A_1(x)$ and $B_1(x)$ are primitive polynomials, their product $A_1(x)B_1(x)$ is also a primitive polynomial. This is a key step and is often proven separately. Now, since $Cf(x) = A_1(x)B_1(x)$ and $f(x)$ is a polynomial with integer coefficients, and $A_1(x)B_1(x)$ is primitive, it follows that C must be an integer.
- (5) **Conclude the factorization:** Since C is an integer and $Cf(x) = A_1(x)B_1(x)$, we have a factorization of $f(x)$ into two polynomials with integer coefficients: $f(x) = \frac{1}{C}A_1(x)B_1(x)$, where $\frac{1}{C}A_1(x)$ and $B_1(x)$ have integer coefficients after adjusting for the constant factor. We can then rewrite $f(x) = A_2(x)B_1(x)$ where $A_2(x) = \frac{1}{C}A_1(x)$.

Thus, $f(x)$ can be factored into two polynomials with integer coefficients.

Remarque 32. *This lemma relates irreducibility over \mathbb{Z} to irreducibility over \mathbb{Q} , ensuring that if a polynomial is irreducible over the integers, it is also irreducible over the rationals.*

Hilbert's Irreducibility Theorem

Théorème 17: Hilbert's Irreducibility Theorem

If $f(x_1, \dots, x_n, y)$ is an irreducible polynomial in $\mathbb{Q}[x_1, \dots, x_n, y]$, then there exist infinitely many tuples (a_1, \dots, a_n) of rational numbers such that $f(a_1, \dots, a_n, y)$ is irreducible in $\mathbb{Q}[y]$.

This theorem is used in advanced settings to prove the existence of irreducible polynomials by specializing variables.

Capelli's Lemma

Lemme 7: Capelli's Lemma

Let K be a field and $f(x)$ be an irreducible polynomial over K . Let α be a root of $f(x)$. If $g(x) \in K(\alpha)[x]$, then $g(x)$ is irreducible over $K(\alpha)$ if and only if $f(x)$ is irreducible over $K(\alpha)$.

Given an irreducible polynomial $f(x)$ over a field K and a root α of $f(x)$, this lemma relates the irreducibility of another polynomial $g(x)$ over $K(\alpha)$ to the irreducibility of $f(x)$ over $K(\alpha)$.

Irreducibility over Finite Fields**Théorème 18: Irreducibility over Finite Fields**

Let $f(x)$ be a polynomial of degree n over \mathbb{F}_q . Then $f(x)$ is irreducible if and only if $f(x)$ divides $x^{q^n} - x$ and $\gcd(f(x), x^{q^k} - x) = 1$ for all k with $1 \leq k < n$.

A polynomial $f(x)$ of degree n over a finite field \mathbb{F}_q is irreducible if and only if $f(x)$ divides $x^{q^n} - x$ and $\gcd(f(x), x^{q^k} - x) = 1$ for all k with $1 \leq k < n$.

8.4.8 Trace and Norm**Définition 8: trace, norm**

If L/K is a finite Galois extension with Galois group G , the characteristic polynomial of $\alpha \in L$ is $\chi_{\alpha, L/K}(X) = \prod_{\sigma \in G} (X - \sigma(\alpha))$. In particular,

$$\text{Tr}_{L/K}(\alpha) = \sum_{\sigma \in G} \sigma(\alpha), \quad \text{N}_{L/K}(\alpha) = \prod_{\sigma \in G} \sigma(\alpha).$$

Example 1

If $K = \mathbf{Q}$ and $L = \mathbf{Q}(\sqrt{d})$ for a nonsquare d in \mathbf{Q}^\times , the two elements of $\text{Gal}(L/K)$ are determined by $\sigma_1(\sqrt{d}) = \sqrt{d}$ and $\sigma_2(\sqrt{d}) = -\sqrt{d}$, so

$$\text{Tr}_{\mathbf{Q}(\sqrt{d})/\mathbf{Q}}(a + b\sqrt{d}) = (a + b\sqrt{d}) + (a - b\sqrt{d}) = 2a$$

and

$$\mathbf{N}_{\mathbf{Q}(\sqrt{d})/\mathbf{Q}}(a + b\sqrt{d}) = (a + b\sqrt{d})(a - b\sqrt{d}) = a^2 - db^2.$$

Also

$$\chi_{a+b\sqrt{d}, \mathbf{Q}(\sqrt{d})/\mathbf{Q}}(X) = (X - (a + b\sqrt{d}))(X - (a - b\sqrt{d})) = X^2 - 2aX - (a^2 - db^2).$$

Example 2

For $\alpha \in \mathbf{F}_{p^n}$,

$$\mathrm{Tr}_{\mathbf{F}_{p^n}/\mathbf{F}_p}(\alpha) = \alpha + \alpha^p + \cdots + \alpha^{p^{n-1}} \text{ and } \mathrm{N}_{\mathbf{F}_{p^n}/\mathbf{F}_p}(\alpha) = \alpha\alpha^p \cdots \alpha^{p^{n-1}}.$$

Galois Group of $\mathbf{F}_{p^n}/\mathbf{F}_p$ The Galois group of $\mathbf{F}_{p^n}/\mathbf{F}_p$ is a cyclic group of order n , generated by the **Frobenius automorphism** σ , where $\sigma(x) = x^p$.

$$\mathrm{Gal}(\mathbf{F}_{p^n}/\mathbf{F}_p) \cong \mathbb{Z}/n\mathbb{Z}$$

8.4.9 Relations among Galois Groups

Théorème 19: Theorem 6.1

Let L_1 and L_2 be Galois over K . a) The embedding

$$\mathrm{Gal}(L_1 L_2 / K) \hookrightarrow \mathrm{Gal}(L_1 / K) \times \mathrm{Gal}(L_2 / K)$$

given by $\sigma \mapsto (\sigma|_{L_1}, \sigma|_{L_2})$ is an isomorphism if and only if $L_1 \cap L_2 = K$.

In particular, $[L_1 L_2 : K] = [L_1 : K][L_2 : K]$ if and only if $L_1 \cap L_2 = K$. b)

The image of the embedding in part a is the set of compatible pairs of automorphisms: $\{(\tau_1, \tau_2) \in \mathrm{Gal}(L_1 / K) \times \mathrm{Gal}(L_2 / K) : \tau_1 = \tau_2 \text{ on } L_1 \cap L_2\}$.

证明. We omit the proof. □

Exemple 29. Let $L_1 = \mathbf{Q}(\sqrt{2}, \sqrt{3})$ and $L_2 = \mathbf{Q}(\sqrt[4]{2}, i)$. Both are Galois over \mathbf{Q} , we know their Galois groups, and $L_1 \cap L_2 = \mathbf{Q}(\sqrt{2})$. Define $\tau_1 \in \mathrm{Gal}(L_1 / \mathbf{Q})$ and $\tau_2 \in \mathrm{Gal}(L_2 / \mathbf{Q})$ by the conditions

$$\tau_1(\sqrt{2}) = -\sqrt{2}, \quad \tau_1(\sqrt{3}) = \sqrt{3}, \quad \tau_2(\sqrt[4]{2}) = i\sqrt[4]{2}, \quad \tau_2(i) = i$$

These agree on $\sqrt{2}$ since $\tau_2(\sqrt{2}) = \tau_2(\sqrt[4]{2})^2 = -\sqrt{2}$, so there is a unique $\sigma \in \mathrm{Gal}(L_1 L_2 / \mathbf{Q})$ that restricts to τ_1 on L_1 and τ_2 on L_2 .

Exemple 30. If $K = \mathbf{Q}$, $L_1 = \mathbf{Q}(\sqrt[3]{2})$, and $L_2 = \mathbf{Q}(\omega\sqrt[3]{2})$, then $L_1 \cap L_2 = K$ but $[L_1 L_2 : K] = 6 \neq [L_1 : K][L_2 : K]$. See the field diagram below on the left.

If one of L_1/K and L_2/K is Galois, Théorème 19 still holds.

Théorème 20: Theorem 6.6

Let L/K and F/K be finite extensions with L/K a Galois extension.

- The extension LF/F is finite Galois and $\text{Gal}(LF/F) \cong \text{Gal}(L/L \cap F)$ by restriction. In particular,

$$[LF : F] = [L : L \cap F] \quad \text{and} \quad [LF : K] = \frac{[L : K][F : K]}{[L \cap F : K]}$$

so $[LF : K] = [L : K][F : K]$ if and only if $L \cap F = K$.

- The sets of intermediate fields $\{M : F \subset M \subset LF\}$ and $\{M' : L \cap F \subset M' \subset L\}$ are in bijection by $M \mapsto L \cap M$, with inverse $M' \mapsto M'F$.

In particular, every field between F and LF has the form $F(\alpha)$ where $\alpha \in L$, and if M and M' correspond by the bijection then M/F is Galois if and only if $M'/L \cap F$ is Galois, in which case $\text{Gal}(M/F) \cong \text{Gal}(M'/L \cap F)$ by restriction.

The extension $\mathbf{Q}(i, \sqrt[4]{2})/\mathbf{Q}$ is Galois, with Galois group D_4 . If we translate this extension by $\mathbf{Q}(i, \sqrt[3]{2})$ (which is not Galois over \mathbf{Q}), we get the Galois extension $\mathbf{Q}(i, \sqrt[3]{2}, \sqrt[4]{2})/\mathbf{Q}(i, \sqrt[3]{2})$. What is its Galois group? By Théorème 20,

$$\text{Gal}(\mathbf{Q}(i, \sqrt[3]{2}, \sqrt[4]{2})/\mathbf{Q}(i, \sqrt[3]{2})) \cong \text{Gal}(\mathbf{Q}(i, \sqrt[4]{2})/F)$$

where $F = \mathbf{Q}(i, \sqrt[4]{2}) \cap \mathbf{Q}(i, \sqrt[3]{2})$. Obviously, $\mathbf{Q}(i) \subset F$, and since $[\mathbf{Q}(i, \sqrt[4]{2}) : \mathbf{Q}(i)] = 4$, $[\mathbf{Q}(i, \sqrt[3]{2}) : \mathbf{Q}(i)] = 3$, we have $F = \mathbf{Q}(i)$.

8.4.10 Cyclotomic Extension

Définition 9

For a positive integer $n \in \mathbb{N}$,

$$\mu_n := \{n\text{th roots of unity in } \mathbb{C}\} = \langle \zeta_n \rangle \cong \mathbf{Z}_n$$

where $\zeta_n = e^{2\pi i/n}$. Define $\mathbb{Q}(\mu_n) = \mathbb{Q}(\zeta_n) \subseteq \mathbb{C}$; it is a finite field extension of \mathbb{Q} , called the **n th cyclotomic extension** of \mathbb{Q} . A **primitive n th root of unity** is a generator of μ_n ; it is equal to ζ_n^a for some $a \in \mathbb{Z}_n^\times$. Define

$$\Phi_n(x) := \prod_{a \in \mathbb{Z}_n^\times} (x - \zeta_n^a)$$

it is called the **n th cyclotomic polynomial**.

Exemple 31. We have $\Phi_1(x) = x - 1$, $\Phi_2(x) = x + 1$, $\Phi_3(x) = x^2 + x + 1$.

Lemme 8

We have

$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$

Each $\Phi_n(x)$ is a polynomial of degree $\varphi(n)$ with coefficients in \mathbb{Z} .

证明. The first equality is easy:

$$x^n - 1 = \prod_{b \in \mathbb{Z}_n} (x - \zeta_n^b) = \prod_{d|n} \prod_{i \in \mathbb{Z}_d^\times} (x - \zeta_n^{di}) = \prod_{d|n} \Phi_d(x). \quad (8.2)$$

We will prove that $\Phi_n(x)$ has coefficients in \mathbb{Z} and its coefficients have gcd = 1. Assume that this has been proved for smaller n . Then eq. (8.2) and Gauss' lemma implies that $\Phi_n(x)$ has coefficients in \mathbb{Z} and has coefficients' gcd = 1. \square

8.4.11 Generating a Composite Field with a Sum

Théorème 21: Theorem 7.1

If K has characteristic 0 and $K(\alpha, \beta)/K$ is a finite extension such that $K(\alpha)/K$ and $K(\beta)/K$ are both Galois and $K(\alpha) \cap K(\beta) = K$, then $K(\alpha, \beta) = K(\alpha + \beta)$.

证明. Let $H = \text{Gal}(K(\alpha, \beta)/K(\alpha + \beta))$. We will show this group is trivial.

Pick $\sigma \in H$, so $\sigma(\alpha + \beta) = \alpha + \beta$. Therefore

$$\sigma(\alpha) - \alpha = \beta - \sigma(\beta)$$

Since $K(\alpha)$ and $K(\beta)$ are Galois over K , $\sigma(\alpha) \in K(\alpha)$ and $\sigma(\beta) \in K(\beta)$, so $\sigma(\alpha) - \alpha \in K(\alpha)$ and $\beta - \sigma(\beta) \in K(\beta)$. This common difference is therefore in $K(\alpha) \cap K(\beta) = K$. Write $\sigma(\alpha) - \alpha = t$, so

$$\sigma(\alpha) = \alpha + t, \quad \sigma(\beta) = \beta - t$$

Applying σ repeatedly, $\sigma^j(\alpha) = \alpha + jt$ for all integers j . Choose $j \geq 1$ such that σ^j is the identity (for instance, let $j = [K(\alpha, \beta) : K]$). Then $\alpha = \alpha + jt$, so $jt = 0$. Since we are in characteristic 0 and j is a positive integer, we must have $t = 0$, so $\sigma(\alpha) = \alpha$ and $\sigma(\beta) = \beta$. Therefore σ is the identity on $K(\alpha, \beta)$. \square

Théorème 22: Theorem 7.6

If K has characteristic 0 and $K(\alpha, \beta)/K$ is a finite extension such that $[K(\alpha, \beta) : K] = [K(\alpha) : K][K(\beta) : K]$ then $K(\alpha, \beta) = K(\alpha + \beta)$.

Remarque 33. The degree hypothesis in Théorème 22 is equivalent to $K(\alpha) \cap K(\beta) = K$ when one of $K(\alpha)$ or $K(\beta)$ is Galois over K , so Théorème 21 is also true when only one of $K(\alpha)$ or $K(\beta)$ is Galois over K .

Exemple 32. Théorème 22 implies $\mathbb{Q}(\sqrt[3]{2}, \omega) = \mathbb{Q}(\sqrt[3]{2} + \omega)$ and $\mathbb{Q}(\sqrt[4]{2}, i) = \mathbb{Q}(\sqrt[4]{2} + i)$.

Exemple 33. We know $\mathbb{Q}(\sqrt[3]{2}, \omega\sqrt[3]{2}) \neq \mathbb{Q}(\sqrt[3]{2} + \omega\sqrt[3]{2})$ and this example does not fit Théorème 22 since $[\mathbb{Q}(\sqrt[3]{2}, \omega\sqrt[3]{2}) : \mathbb{Q}] = 6$ while $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}][\mathbb{Q}(\omega\sqrt[3]{2}) : \mathbb{Q}] = 9$.

Exemple 34. Letting r and r' be two roots of $X^4 + 8X + 12$, we can't decide if $\mathbb{Q}(r, r')$ equals $\mathbb{Q}(r + r')$ from Théorème 22 since $[\mathbb{Q}(r, r') : \mathbb{Q}] = 12$ and $[\mathbb{Q}(r) : \mathbb{Q}][\mathbb{Q}(r') : \mathbb{Q}] = 16$. The two fields are not the same since $\mathbb{Q}(r + r')$ has degree 6 over \mathbb{Q} .

Exemple 35. Does $\mathbb{Q}(\sqrt[4]{2}, \zeta_8) = \mathbb{Q}(\sqrt[4]{2} + \zeta_8)$? Since $[\mathbb{Q}(\sqrt[4]{2}, \zeta_8) : \mathbb{Q}] = 8$ (by Example 1.4) and $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}][\mathbb{Q}(\zeta_8) : \mathbb{Q}] = 16$, we can't answer this with Théorème 22. Since $\mathbb{Q}(\sqrt[4]{2}, \zeta_8) = \mathbb{Q}(\sqrt[4]{2}, i)$, you can check the Galois orbit of $\sqrt[4]{2} + \zeta_8$ has size 8, so in fact $\mathbb{Q}(\sqrt[4]{2}, \zeta_8) = \mathbb{Q}(\sqrt[4]{2} + \zeta_8)$. (The minimal polynomial of $\sqrt[4]{2} + \zeta_8$ over \mathbb{Q} is $X^8 - 8X^5 - 2X^4 + 16X^3 + 32X^2 + 24X + 9$.) Thus the degree hypothesis of Théorème 22 is sufficient to imply $K(\alpha, \beta) = K(\alpha + \beta)$ in characteristic 0, but it is not necessary.

8.4.12 The Inverse Galois Problem

Théorème 23: Theorem 8.1

Every finite group is the Galois group of some finite Galois extension in any characteristic.

Théorème 24: Theorem 8.2

If every finite group can be realized as a Galois group over \mathbb{Q} then every finite group can be realized as a Galois group over any finite extension of \mathbb{Q} .

8.4.13 What Next?

There are two important aspects of field extensions that are missing by a study of Galois theory of finite extensions, and we briefly address them:

- (1) Galois theory for infinite extensions
- (2) transcendental extensions

An example of an infinite Galois extension of \mathbb{Q} is

$$\mathbb{Q}(\mu_{p^\infty}) = \bigcup_{n \geq 1} \mathbb{Q}(\mu_{p^n})$$

the union of all p -th power cyclotomic extensions of \mathbb{Q} , where p is a fixed prime.

8.5 Irreducibility of Polynomial

Task: For a general field F there is no simple way to determine if an arbitrary polynomial in $F[T]$ is irreducible. Here we will focus on the case $F = \mathbb{Q}$ and describe two useful irreducibility tests in $\mathbb{Q}[T]$ for monic polynomials in $\mathbb{Z}[T]$. Let

$$f(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_1T + a_0 \in \mathbb{Z}[T].$$

The two tests are

- (1) Reduction mod p : for a prime p , reducing coefficients of $f(T)$ modulo p leads to

$$\bar{f}(T) = T^n + \bar{a}_{n-1}T^{n-1} + \cdots + \bar{a}_0 \in (\mathbf{Z}/p\mathbf{Z})[T].$$

If $\bar{f}(T)$ is irreducible in $(\mathbf{Z}/p\mathbf{Z})[T]$ for some p , then f is irreducible in $\mathbf{Q}[T]$.

- (2) Eisenstein criterion: call $f(T)$ Eisenstein at p if $p \mid a_i$ for all i and $p^2 \nmid a_0$. If f is Eisenstein for some p , then f is irreducible in $\mathbf{Q}[T]$.

These tests each depend on a choice of a prime number, but they use the prime number in different ways.

8.6 Permutation-Groups

See 丘维声, 肖梁, dummit&foote.

Théorème 25

Any permutation can be decomposed to disjoint transpositions. (Unique without considering the order)

8.6.1 Decomposition of the permutation

$$(1 \ 2 \ 3) = (1 \ 3) \circ (1 \ 2)$$

And

$$(1 \ 2 \ 3) = (2 \ 3 \ 1) = (2 \ 1) \circ (2 \ 3)$$

Generally,

$$(1 \ 2 \ \dots \ n) = (1 \ n) \circ (1 \ n-1) \circ \cdots \circ (1 \ 2)$$

Since

$$(i \ j) = (1 \ i) \circ (1 \ j) \circ (1 \ i)$$

Then

$$S_n = \langle (12), (13), \dots, (1n) \rangle$$

8.6.2 Alternating group A_n

When $n = 4$,

$$S_4 = \langle e, (12), (13), (14) \rangle$$

$$A_4 = \left\langle e, (12)(13), (12)(14), \underbrace{(13)(14)}_{= ((12)(13))^{-1}} \right\rangle = \left\langle e, \underbrace{(12)(13)}_{=(132)}, \underbrace{(12)(14)}_{=(142)} \right\rangle$$

When $n \geq 3$,

$$A_n = \langle (123), (124), \dots, (12n) \rangle$$

Remarque 34. An element of odd order in a symmetric group is an even permutation, i.e. lies in A_n . That's because the odd permutation can be decomposed to product of even transformations.

8.7 Semidirect-product

See [semidirect-product.pdf](#)

8.7.1 When $G \cong H \times K$

Théorème 26: recognition theorem

Let G be a group with subgroups H and K where

- (1) $G = HK$; that is, every element of G has the form hk for some $h \in H$ and $k \in K$,
- (2) $H \cap K = \{1\}$ in G ,
- (3) $hk = kh$ for all $h \in H$ and $k \in K$.

Then the map $H \times K \rightarrow G$ by $(h, k) \mapsto hk$ is an isomorphism.

Example 1

In $G = \text{Aff}(\mathbf{R})$, let

$$H = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} : y \in \mathbf{R} \right\} \cong \mathbf{R}, \quad K = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} : x \in \mathbf{R}^\times \right\} \cong \mathbf{R}^\times.$$

Since

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}}_{\in H} \underbrace{\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}}_{\in K}$$

we have $G = HK$, and clearly $H \cap K$ is trivial, but matrices in H and in K often do not commute with each other. You can find your own such matrices (nearly any random choice will work), but also observe that if elements of H and of K always commute with one another then $G \cong H \times K$ by Théorème 26, but $G \not\cong H \times K$ since $H \times K$ is abelian (H and K are abelian) while G is nonabelian.

Example 2

In $G = S_4$, let H be a 2-Sylow subgroup and K be a 3-Sylow subgroup (so $H \cong D_4$ and K is cyclic of order 3). In the Sylow theorems for S_4 , $n_2 = 3$ and $n_3 = 4$, so H and K are not normal in S_4 . The set HK can be written as H -cosets Hk and as K -cosets hK , so $|HK|$ is divisible by $|H| = 8$ and by $|K| = 3$, so $|HK| = 24$. Therefore $S_4 = HK$. The subgroups H and K intersect trivially. We have $S_4 \not\cong H \times K \cong D_4 \times \mathbf{Z}/(3)$ since $D_4 \times \mathbf{Z}/(3)$ has a nontrivial center (**D_4 and $\mathbf{Z}/(3)$ both have nontrivial center**) while S_4 has a trivial center.

The center of D_{2n} depends on whether n is even or odd. Recall that $D_{2n} = \langle r, s \mid r^n = s^2 = 1, srs = r^{-1} \rangle$.

- If n is odd, then $Z(D_{2n}) = \{1\}$.

证明. Suppose that $x \in Z(D_{2n})$. Write $x = r^i$ or $x = r^i s$. Assume that $x = r^i$. Then $srs^{-1} = r^{-1}$. Hence $sr^i s^{-1} = r^{-i}$. But $x \in Z(D_{2n})$, so $sr^i s^{-1} = r^i$. Thus $r^i = r^{-i}$, which implies that $r^{2i} = 1$. Thus $n \mid 2i$, so $i = n/2$ or $i = 0$. If $i = 0$, then $x = 1$. If $i = n/2$, then n is even. So if n is odd, then $x = 1$. Now assume that $x = r^i s$. Then $r^{-1}xr = x$. Thus $r^{-1}r^i sr = r^i s$, so $r^{-1}r^i sr = r^i s$. Hence $r^{-1}r^i sr = r^i s$, so $r^{-1}r^i sr = r^i s$. Thus $r^{-1}r^i sr = r^i s$, so $r^{-1}r^i sr = r^i s$. But $r^{-1}sr = r^{-1}sr = sr^{-2}$. Thus $r^{i-2}s = r^i s$, so $r^{-2} = 1$. This implies that $n \mid 2$, so $n = 1$ or $n = 2$. But n is odd and $n > 1$, a contradiction. Therefore, $x = 1$. \square

- If n is even, then $Z(D_{2n}) = \{1, r^{n/2}\}$.

证明. Suppose that $x \in Z(D_{2n})$. Write $x = r^i$ or $x = r^i s$. Assume that $x = r^i$. Then $srs^{-1} = r^{-1}$. Hence $sr^i s^{-1} = r^{-i}$. But $x \in Z(D_{2n})$, so $sr^i s^{-1} = r^i$. Thus $r^i = r^{-i}$, which implies that $r^{2i} = 1$. Thus $n \mid 2i$, so $i = n/2$ or $i = 0$. If $i = 0$, then $x = 1$. If $i = n/2$, then $x = r^{n/2}$. Now assume that $x = r^i s$. Then $r^{-1}xr = x$. Thus $r^{-1}r^i sr = r^i s$, so $r^{-1}r^i sr = r^i s$. Hence $r^{-1}r^i sr = r^i s$, so $r^{-1}r^i sr = r^i s$. Thus $r^{-1}r^i sr = r^i s$, so $r^{-1}r^i sr = r^i s$. But $r^{-1}sr = r^{-1}sr = sr^{-2}$. Thus $r^{i-2}s = r^i s$, so $r^{-2} = 1$. This implies that $n \mid 2$, so $n = 1$ or $n = 2$. If $n = 1$, then $D_2 = \{1, s\}$, so $Z(D_2) = D_2$, a contradiction since $r^{n/2} = r^{1/2} \notin D_2$. If $n = 2$, then $r^i s \notin Z(D_4)$. Thus $Z(D_{2n}) = \{1, r^{n/2}\}$. \square

In summary:

$$Z(D_{2n}) = \begin{cases} \{1\} & \text{if } n \text{ is odd} \\ \{1, r^{n/2}\} & \text{if } n \text{ is even} \end{cases}$$

8.7.2 Motivation of Semidirect Product: $G \not\cong H \times K$

For subgroups H, K of G , HK might not be a subgroup. However, if H or K is normal in G , then HK is a subgroup. Take $H \triangleleft G$, e.g.

$$\begin{aligned} (hk)(h'k') &= (\underbrace{h \underbrace{kh'k^{-1}}_{\in H}}_{\in H})(\underbrace{kk'}_{\in K}) \stackrel{=Ad_k(h')}{\in} HK \\ (hk)^{-1} &= k^{-1}h^{-1} = (\underbrace{k^{-1}h^{-1}k}_{\in H})k^{-1} \stackrel{=Ad_{k^{-1}}(h^{-1})}{\in} HK \end{aligned}$$

Consider $\varphi_k = \text{Ad}_k \in \text{Aut}(H)$ induced by $k \in K$, then we have the natural definition:

Définition 10: semidirect product

For two groups H and K and an action $\varphi : K \rightarrow \text{Aut}(H)$ of K on H by automorphisms, the corresponding **semidirect product** $H \rtimes_{\varphi} K$ is defined as follows: as a set it is $H \times K = \{(h, k) : h \in H, k \in K\}$. The

group law on $H \rtimes_{\varphi} K$ is

$$(h, k)(h', k') = (h\varphi_k(h'), kk').$$

Note. Note that H, K might not be subgroups of the same group, e.g. $H = D_4, K = \mathbb{Z}_2$, thus hk fails to be defined, but $h\varphi_k(h')$ has definition.

Remarque 35. The notation $H \rtimes K$ means H is normal " \triangleleft ", and K has a "twisted" action on H .

Exemple 36. In $H \rtimes_{\varphi} K$, $(h, k)^2 = (h, k)(h, k) = (h\varphi_k(h), k^2)$, so $(h, k)^2 = (1, 1)$ if and only if $\varphi_k(h) = h^{-1}$ and $k^2 = 1$.

Exemple 37. Take $H = \mathbf{R}, K = \mathbf{R}^{\times}$, and $\varphi : \mathbf{R}^{\times} \rightarrow \text{Aut}(\mathbf{R})$ where $\varphi_x : \mathbf{R} \rightarrow \mathbf{R}$ by $\varphi_x(y) = xy$. Note φ_x is an automorphism of \mathbf{R} as an additive group and $\varphi_x \circ \varphi_{x'} = \varphi_{xx'}$ since $x(x'y) = (xx')y$ for all $y \in \mathbf{R}$. The group $\mathbf{R} \rtimes_{\varphi} \mathbf{R}^{\times}$ has the operation

$$(a, b)(a', b') = (a + \varphi_b(a'), bb') = (a + ba', bb').$$

This resembles the multiplication in $\text{Aff}(\mathbf{R})$, where $\begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b' & a' \\ 0 & 1 \end{pmatrix} =$

$\begin{pmatrix} bb' & ba' + a \\ 0 & 1 \end{pmatrix}$. In the affine matrices we multiply in the upper left, while in $\mathbf{R} \rtimes_{\varphi} \mathbf{R}^{\times}$, components multiply in the second coordinate. That suggests turning

$(a, b) \in \mathbf{R} \rtimes_{\varphi} \mathbf{R}^{\times}$ into $\begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix}$: $\mathbf{R} \rtimes_{\varphi} \mathbf{R}^{\times} \cong \text{Aff}(\mathbf{R})$ by $(a, b) \mapsto \begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix} =$

$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}$. In the group $\text{Aff}(\mathbf{R})$ you have to be careful about how you decompose a matrix:

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \quad \text{for } x \neq 1, y \neq 0.$$

The nice decomposition puts the matrix associated to y first, before that associated to x .

Exemple 38. In the previous example, replace \mathbb{R} with $\mathbb{Z}/(m)$ and \mathbb{R}^\times with $(\mathbb{Z}/(m))^\times$. We have $\text{Aut}(\mathbb{Z}/(m)) \cong (\mathbb{Z}/(m))^\times$ since automorphisms of the additive group $\mathbb{Z}/(m)$ are the mappings $\varphi_a : x \pmod{m} \mapsto ax \pmod{m}$ for $a \in (\mathbb{Z}/(m))^\times$. Let $\varphi : (\mathbb{Z}/(m))^\times \rightarrow \text{Aut}(\mathbb{Z}/(m))$ by making $\varphi_a : \mathbb{Z}/(m) \rightarrow \mathbb{Z}/(m)$ for each a be multiplication by a . The semidirect product $\mathbb{Z}/(m) \rtimes_\varphi (\mathbb{Z}/(m))^\times$ has operation

$$(a, b)(a', b') = (a + ba', b + b')$$

and is isomorphic to $\text{Aff}(\mathbb{Z}/(m))$ by $(a, b) \pmod{m} \mapsto \begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix} \pmod{m}$.

Exemple 39. Since ± 1 acts as additive automorphisms on \mathbb{Z} , we have a semidirect product $\mathbb{Z} \rtimes \{\pm 1\}$ where $(a, \varepsilon)(a', \varepsilon') = (a + \varepsilon a', \varepsilon \varepsilon')$. The homomorphism $\mathbb{Z} \rightarrow \{\pm 1\}$ given by $n \mapsto (-1)^n$ leads to a semidirect product $\mathbb{Z} \rtimes \mathbb{Z}$ by $(m, n)(m', n') = (m + (-1)^n m', n + n')$.

Théorème 27: Theorem 3.7

Inside $H \rtimes_\varphi K$, we have

$$H \cong \{(h, 1) : h \in H\} \text{ by } h \mapsto (h, 1), \quad K \cong \{(1, k) : k \in K\} \text{ by } k \mapsto (1, k),$$

and $(h, k) = (h, 1)(1, k) = (1, k)(\varphi_k^{-1}(h), 1)$. The copy of H in $H \rtimes_\varphi K$ is a normal subgroup with conjugation by k being described with φ_k :

$$(1, k)(h, 1)(1, k)^{-1} = (\varphi_k(h), 1).$$

In particular, $(1, k)$ commutes with each $(h, 1)$ if and only if $k \in \ker \varphi$, and every $(1, k)$ and $(h, 1)$ commute if and only if $\varphi : K \rightarrow \text{Aut}(H)$ is trivial on $K : \varphi_k = \text{id}_H$ for all $k \in K$.

证明. The proof is routine. □

Exemple 40. Let $H = \mathbf{R}$, $K = \mathbf{R}^\times$, and $\varphi : \mathbf{R}^\times \rightarrow \mathbf{R}$ by $\varphi_x(y) = xy$. We saw

in Example 3.4 that $\text{Aff}(\mathbf{R}) \cong \mathbf{R} \rtimes_{\varphi} \mathbf{R}^{\times}$ by $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mapsto (y, x)$.

Théorème 27 says the effect of φ_x on \mathbf{R} looks like conjugation in $\mathbf{R} \rtimes_{\varphi} \mathbf{R}^{\times}$, and this is related to the affine group conjugation formula:

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & xy \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \varphi_x(y) \\ 0 & 1 \end{pmatrix}.$$

Exemple 41. Example 3.10. Let H be an abelian group written additively, so negation $\text{neg}: h \mapsto -h$ is an automorphism of order 2. Let $\varphi: \mathbb{Z}/(2) \rightarrow \text{Aut}(H)$ by $\varphi_0 = \text{id}_H$ and $\varphi_1 = [h \mapsto -h]$. Then φ is a homomorphism (tip: when a group G contains an element g of order m , we always get a homomorphism $\mathbb{Z}/(m) \rightarrow G$ by $a \bmod m \mapsto g^a$, and here we're using the special case $G = \text{Aut}(H)$ and g is inversion on H). The group $H \rtimes_{\varphi} \mathbb{Z}/(2)$ has operation (3.6) $(h, a \bmod 2)(h', a' \bmod 2) = (h + \text{neg}^a(h'), a + a' \bmod 2) = (h + (-1)^a h', a + a' \bmod 2)$. Then

$$(h, 0)(0, 1) = (h, 1) \text{ and } (0, 1)(h, 0) = (0 + \text{neg}(h), 1 + 0) = (-h, 1).$$

Thus $(h, 0)$ and $(0, 1)$ commute in $H \rtimes_{\varphi} \mathbb{Z}/(2)$ if and only if $h = -h$. If all nonzero elements of H have order 2 (so negation on H is the identity) then $H \rtimes_{\varphi} \mathbb{Z}/(2) = H \times \mathbb{Z}/(2)$. If some nonzero element of H does not have order 2 then $H \rtimes_{\varphi} \mathbb{Z}/(2)$ is nonabelian. Consider the case $H = \mathbb{Z}/(n)$ where $n \geq 3$. The group $\mathbb{Z}/(n) \rtimes_{\varphi} \mathbb{Z}/(2)$ has order $2n$ and the group law (3.6) in this special case is

$$(j, k)(j', k') = (j + (-1)^k j', k + k').$$

This may look like a weird group of order $2n$, but in fact it is isomorphic to D_n . If we identify $(1, 0)$ with r and $(0, 1)$ with s in D_n then $\mathbb{Z}/(n) \rtimes_{\varphi} \mathbb{Z}/(2) \cong D_n$ by $(1, 0) \mapsto r$ and $(0, 1) \mapsto s$. For example, (3.7) says $(0, 1)(1, 0) = (-1, 1) = (-1, 0)(0, 1)$, which matches the familiar dihedral relation $sr = r^{-1}s$. (The general multiplication rule in D_n is $(r^j s^k)(r^{j'} s^{k'}) = r^{j+(-1)^k j'} s^{k+k'}$, where the exponents on r and s look like (3.7).)

8.7.3 Recognize Semidirect Products

Similar to Théorème 26, we have a recognition theorem for semidirect products.

Théorème 28: Theorem 4.1

Let G be a group with subgroups H and K such that

- (1) $G = HK$,
- (2) $H \cap K = \{1\}$,
- (3) $H \triangleleft G$.

Let $\varphi : K \rightarrow \text{Aut}(H)$ be conjugation: $\varphi_k(h) = khk^{-1}$. Then φ is a homomorphism and the map $f : H \rtimes_{\varphi} K \rightarrow G$ where $f(h, k) = hk$ is an isomorphism.

证明. That φ makes sense at all is due to (3). That it is a homomorphism means $\varphi_k \circ \varphi_{k'} = \varphi_{kk'}$, and this is left to the reader to check. The function

$$f : H \rtimes_{\varphi} K \rightarrow G$$

where $f(h, k) = hk$ is surjective by (1), and f is injective by (2) using the same argument for injectivity as in the proof of Theorem 2.1. To show f is a homomorphism, calculate

$$\begin{aligned} f((h, k)(h', k')) &= f(h\varphi_k(h'), kk') \\ &= h\varphi_k(h')kk' \\ &= hkh'k^{-1}kk' \\ &= hkh'k' \\ &= f(h, k)f(h', k'). \end{aligned}$$

Hence f is an isomorphism. \square

Exemple 42. We will show for odd $n > 1$ that the direct product $\text{SL}_n(\mathbf{R}) \times \mathbf{R}^{\times}$ is isomorphic to a nontrivial semidirect product $\text{SL}_n(\mathbf{R}) \rtimes \mathbf{R}^{\times}$.

We have met several examples of groups that are isomorphic to a semidirect product of two groups:

- (1) $\text{Aff}(\mathbf{R}) \cong \mathbf{R} \rtimes \mathbf{R}^\times$
- (2) $S_n \cong A_n \rtimes \mathbf{Z}/(2)$;
- (3) $D_n \cong \mathbf{Z}/(n) \rtimes \mathbf{Z}/(2)$;
- (4) $\text{GL}_2(\mathbf{R}) \cong \text{SL}_2(\mathbf{R}) \rtimes \mathbf{R}^\times$;

In these respective groups,

- (1) \mathbf{R}^\times acts on \mathbf{R} by multiplication maps $\varphi_x : y \mapsto xy$ for $x \in \mathbf{R}^\times$;
- (2) $\mathbf{Z}/(2)$ is identified with $\{1, \tau\}$ for any transposition τ in S_n ;
- (3) $\mathbf{Z}/(n)$ is identified with $\langle r \rangle$ and $\mathbf{Z}/(2)$ is identified with $\{1, s\}$ in D_n ;
- (4) \mathbf{R}^\times is identified with the group of matrices $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ (not with the matrices $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$);

8.7.4 Building Semidirect Product

The case when H and K are finite with $(|K|, |\text{Aut}(H)|) = 1$ is boring, since $\varphi : K \rightarrow \text{Aut}(H)$ has a trivial image and thus the only semidirect product $H \rtimes K$ is the direct product $H \times K$.

Exemple 43. A semidirect product $\mathbf{Z}/(5) \rtimes \mathbf{Z}/(3)$ has order 15 and it must be a direct product: $\text{Aut}(\mathbf{Z}/(5)) = (\mathbf{Z}/(5))^\times$ has order 4 and that is relatively prime to 3. In fact, all groups of order 15 are cyclic; groups of order pq will be studied below.

Exemple 44. Example 5.2. Is there a nontrivial $\mathbf{Z}/(25) \rtimes \mathbf{Z}/(15)$? Since $\text{Aut}(\mathbf{Z}/(25)) = (\mathbf{Z}/(25))^\times$ has size 20 and $(20, 15) \neq 1$, there might be nontrivial examples. We seek a nontrivial homomorphism

$$\varphi : \mathbf{Z}/(15) \rightarrow (\mathbf{Z}/(25))^\times.$$

It is necessary that $\varphi(1)$ goes to a g such that $g^{15} = 1$. If $g^{15} \equiv 1 \pmod{25}$, then $g^{15} \equiv 1 \pmod{5}$, so (by testing mod 5) $g \equiv 1 \pmod{5}$. Then

$$g \in \{1, 6, 11, 16, 21\}.$$

Choose $g = 6$. Check $6^{15} \equiv 1 \pmod{25}$ (in fact, if $a \equiv 1 \pmod{5}$ then $a^5 \equiv 1 \pmod{25}$, so $a^{15} \equiv 1 \pmod{25}$). Let $\varphi : \mathbb{Z}/(15) \rightarrow (\mathbb{Z}/(25))^\times$ by $\varphi(1) = 6$, so $\varphi(k \pmod{15}) = 6^k \pmod{25}$ (e.g., $\varphi(2) = \varphi(1+1) = \varphi(1)\varphi(1) = 6^2$). We get a nontrivial semidirect product $\mathbb{Z}/(25) \rtimes_\varphi \mathbb{Z}/(15)$ with the group law

$$(a, b)(c, d) = (a + \varphi_b(c), b + d) = (a + 6^b c, b + d).$$

This is a nonabelian group of order $15 \cdot 25 = 375$ built from two cyclic groups of order 15 and 25.

8.7.5 Groups of Order pq

We'll use the Sylow theorems and semidirect products to find all groups of order pq up to isomorphism.

Théorème 29: Theorem 6.1

If primes $p < q$ satisfy $q \not\equiv 1 \pmod{p}$ then all groups of order pq are cyclic.

A useful lemma:

Lemme 9: Lemma 6.2

A semidirect product $H \rtimes_\varphi K$ is **unchanged up to isomorphism** if the action $\varphi : K \rightarrow \text{Aut}(H)$ is composed with an automorphism of K : for automorphisms $f : K \rightarrow K$, $H \rtimes_{\varphi \circ f} K \cong H \rtimes_\varphi K$.

Moreover, if $H \cong H'$, $K \cong K'$, then

$$H \rtimes K \cong H' \rtimes K'$$

e.g. if $|H| = 7$, $|K| = 11$, then

$$H \rtimes K \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_{11}$$

Théorème 30: Theorem 6.3

If primes $p < q$ satisfy $q \equiv 1 \pmod{p}$ then there are two groups of order pq up to isomorphism: one is cyclic and one is nonabelian.

Explicitly, for $q \equiv 1 \pmod{p}$, a nonabelian matrix group of order pq is

$$\left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x \in (\mathbb{Z}/(q))^\times, y \in \mathbb{Z}/(q), x^p \equiv 1 \pmod{q} \right\} \subset \text{Aff}(\mathbb{Z}/(q)).$$

8.8 Groups of order pq

8.9 Complementary Subgroups

If a group G contains subgroups H and K such that $G = HK$ and $H \cap K = \{1\}$, then H and K are called complementary subgroups. For a normal subgroup $H \triangleleft G$, is there always a complementary subgroup $K \subset G$? If so, we'd then have $G \cong H \rtimes_\varphi K$

The answer is no!

Théorème 31: Schur-Zassenhaus

If $H \triangleleft G$ and $(|H|, |G/H|) = 1$, then H has a complementary subgroup in G and all complementary subgroups to H in G are conjugate.

8.10 Sylow 定理习题

Dummit&Foote 4.5

Exercice 2

Prove that if $P \in Syl_P(G)$ and H is a subgroup of G containing P , then $P \in Syl_P(H)$. Give an example to show that, in general, a Sylow p -subgroup of a subgroup of G need not be a Sylow p -subgroup of G .

证明. We are given $P \leq H \leq G$ with $|P| = p^\alpha$ and $|G| = p^\alpha m$ such that $p \nmid m$. As $|P||H|$, we get that $|H| = p^\alpha k$ for some $k \in \mathbb{N}$. Moreover, $|H||G|$ and thus, $k|m$. As $p \nmid m$, we get that $p \nmid k$. Thus, P is a Sylow p -subgroup of H .

For the second part, consider $G = \mathbb{Z}_4$ and $H = \{\bar{0}, \bar{2}\}$. Then $P = H$ is a Sylow 2-subgroup of H but not one of G . \square

Exercice 3

Prove that if H is a subgroup of G and $Q \in Syl_p(H)$ then $gQg^{-1} \in Syl_p(gHg^{-1})$ for all $g \in G$.

证明. We first note that gQg^{-1} is indeed a subgroup of gHg^{-1} . The assertion follows from the fact that $|gQg^{-1}| = |Q|$ and $|gHg^{-1}| = |H|$. \square

Exercice 4

Use Sylow's theorem to prove Cauchy's theorem.

证明. Recall that Cauchy's theorem says that if G is a finite group such that $p||G|$ for some prime p , then G has an element of order p .

To prove this, let G and p be as above and P be a Sylow p -subgroup of G . (Existence of P is given by the Sylow theorems.)

Let $G = p^\alpha m$ with usual meanings. Then, $|P| = p^\alpha$.

Consider $x \in P$ such that $x \neq 1$. Then, the order of x is p^β for some $1 \leq \beta \leq \alpha$. Consider $y = x^{p^{\beta-1}}$. Then, order of y is p . (How? It is clear that $y^p = 1$. Thus, the only other possibility for the order is 1 but that is not possible since $y \neq 1$ as $p^{\beta-1} < p^\beta$, the order of x .) \square

Exercice 5

Exhibit all Sylow 2-subgroups and Sylow 3-subgroups of D_{12} and $S_3 \times S_3$.

证明. D_{12} : Note that $|D_{12}| = 12 = 2^2 \cdot 3$. Thus, the Sylow 2-subgroup(s) will have order 4 and Sylow 3-subgroup(s) will have order 3.

Sylow 3: Note that a Sylow 3 -subgroup must necessarily be isomorphic to \mathbb{Z}_3 . Since all elements not of the form r^k have order 2, we just find powers of r which have order 3. These turn out to be r^2 and r^4 . As $\langle r^2 \rangle = \langle r^4 \rangle$, we have a unique Sylow 3-subgroup:

$$\langle r^2 \rangle$$

Sylow 2: Note that no element of D_{12} has order 4. (Any such element would have to be a power of r but 4 does not divide 6, the order of r .) Thus, any Sylow 2-subgroup must be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Every element of this group must have order 2. r^3 is the only power of r which has order 2. Thus, the other two elements must be of the form sr^k . Noting that the multiplication of two such elements is:

$$sr^k \cdot sr^{k'} = r^{k'-k}$$

we get that $|k - k'| = 3$. This gives us three such groups which are all the Sylow 2subgroups:

$$\underbrace{\langle s, sr^3 \rangle}_{\langle s \rangle \times \langle r^3 \rangle}, \underbrace{\langle sr, sr^4 \rangle}_{\langle sr \rangle \times \langle r^3 \rangle}, \underbrace{\langle sr^2, sr^5 \rangle}_{\langle sr^2 \rangle \times \langle r^3 \rangle}.$$

(It can be checked that all of these are distinct subgroups.) $S_3 \times S_3$: The order of the group is $36 = 2^2 \cdot 3^2$. Thus, the Sylow 2-subgroup(s) will have order 4 and Sylow 3-subgroup(s) will have order 9.

Sylow 2: Note that there are three subgroups of order 2 of S_3 : $\langle (12) \rangle, \langle (13) \rangle, \langle (23) \rangle$. Let these be H_1, H_2, H_3 . (Note that these are indeed distinct.) Then, the nine products:

$$H_i \times H_j \quad i, j \in \{1, 2, 3\}$$

are subgroups of $S_3 \times S_3$ have order 4. This gives us nine Sylow 2-subgroups. However, by the Sylow theorems, $n_2 \mid 9$ and thus, there can't be any more.

Sylow 3: Let P be a Sylow 3-subgroup. Note that no element of $S_3 \times S_3$ has order 9. Thus, every non-identity element of P must have order 3. Now, note that the order of an element $(\sigma, \tau) \in S_3 \times S_3$ is the lcm of the orders of σ and τ . This gives us that there are 8 elements of order 3. These elements together with the identity do form a subgroup and moreover, there can't be any other. Thus, there is a unique Sylow 3-subgroup which is:

$$\langle (123) \rangle \times \langle (123) \rangle.$$

□

Exercice 6

Show that a Sylow p -subgroup of D_{2n} is cyclic and normal for every odd prime p .

证明. Let $2n = p^\alpha m$ as usual. As p is odd and $2n$ is even, we must have that $m = 2m'$ for some $m' \in \mathbb{N}$. Thus, $n = p^\alpha m'$ with $p \nmid m'$.

Note that r has order n and hence, $r^{m'}$ has order p^α . Hence, $\langle r^{m'} \rangle$ is Sylow p -subgroup of D_{2n}

We now show that is normal. To do this, it is enough to work with the generators of D_{2n} . Clearly, $r(r^{m'})r^{-1} \in \langle r^{m'} \rangle$. Also, $s(r^{m'})s^{-1} = s^2 r^{-m'} = r^{-m'} \in \langle r^{m'} \rangle$. Thus, $\langle r^{m'} \rangle$ is normal.

As all Sylow p -subgroups are conjugates, we get that the above Sylow p -subgroup is the unique Sylow p -subgroup. As it is cyclic and normal, we are done. □

Exercice 7

Exhibit all Sylow 3-subgroups of A_4 and all Sylow 3-subgroups of S_4 .

证明. A_4 : Clearly, $n_3 \mid 4$. We exhibit 4 such Sylow 3 -subgroups now:

$$\langle(123)\rangle, \langle(124)\rangle, \langle(134)\rangle, \langle(234)\rangle.$$

(Note that these are indeed distinct subgroups.)

These were easy to find as a Sylow 3-subgroup must have order 3 and thus, must be isomorphic to \mathbb{Z}_3 . It was then a matter of finding elements of order 3 .

S_4 : Same as earlier. Note that there are 8 elements of order 3. If we consider all the 8 subgroups generated by them, we see that we get only four distinct ones, listed above. \square

Exercice 8

Exhibit all Sylow 2-subgroups of S_4 and find elements of S_4 which conjugate one of these into each of the others.

证明. A little bit of experimenting with elements of order 2 and 4 gives the following subgroups:

$$\langle(1234), (13)\rangle, \langle(1243), (14)\rangle, \langle(1324), (12)\rangle.$$

It can be verified that these are distinct. (For example, the 2 cycles listed in the generators don't appear in any of the other subgroups.) It can also be verified that all of these have order 8. Moreover, there can't be any more as $n_2 \mid 3$.

As for the conjugation question, (34) conjugates the first to the second and (24) the second to the third.³ \square

³Note that the conjugate of D_{2n} works by transforming the places of the elements.

Exercice 9

Exhibit two distinct Sylow 2-subgroups of S_5 and an element of S_5 that conjugates one into the other.

证明. Note that $|S_5| = 2^3 \cdot 15$ and thus, any two of the Sylow 2 -subgroups listed earlier will work again. (With the understanding that the elements are now elements of S_5) \square

Lemme 10

The order of $GL_2(\mathbb{F}_q)$ is $(q^2 - 1)(q^2 - q)$.

证明. Consider the choice of the first column. There are $q^2 - 1$ choices for the first column. The second column must be a vector independent of the first column. There are $q^2 - q$ choices for the second column. \square

Exercice 10

Exhibit all Sylow 3-subgroups of $SL_2(\mathbb{F}_3)$.

证明. Note that the order of $GL_2(\mathbb{F}_3)$ is $(3^2 - 1)(3^2 - 3) = 48$. It can be easily shown that the order of SL_2 is half of that. Thus, $n_3 = 1$ or 4 . Also, note that any Sylow 3 -subgroup will have order 3 and hence, is isomorphic to \mathbb{Z}_3 . It is easy to find two distinct subgroups of order 3 by observing the following matrices to have order 3:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Also, these matrices generate distinct subgroups. Thus, $n_3 > 1$ which forces $n_3 = 4$. **We may now conjugate the above matrices to get the other two.** In any case, we are left with the following:

$$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \right\rangle.$$

□

Exercice 11

Prove that the subgroup of $SL_2(\mathbb{F}_3)$ generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the unique Sylow 2-subgroup of $SL_2(\mathbb{F}_3)$.

证明. Hmm.

□

8.11 Sylow more[sylowmore.pdf](#)

It is natural to ask how the Sylow theorems can be extended to p -subgroups that are not p -Sylow subgroups.

Théorème 32: Theorem 4.1

If $p^d \mid |G|$ then there is a subgroup of G with size p^d .

Note. First we directly prove Théorème 32 .

证明. We induct on the size of G . The case when $|G| = 1$ or prime is trivial. Now suppose $|G| > 1$ and the theorem is proved for all groups of smaller size. That is, we assume each group G' with $|G'| < |G|$ has a subgroup of size equal to an arbitrary prime power dividing $|G'|$.

Choose a prime power p^d dividing $|G|$, with $p^d > 1$. We seek a subgroup of G with size p^d . If G has a proper subgroup H such that $p^d \mid |H|$, then we're done: H has a subgroup of size p^d by induction (since $|H| < |G|$) and this subgroup is in G too.

Now we suppose every proper subgroup $H \subset G$ has size not divisible by p^d . Since $|G| = |H|[G : H]$ is divisible by p^d , we see every proper subgroup of G has index divisible by p . Consider the class equation

$$|G| = |Z(G)| + \sum_{i=1}^r [G : Z(g_i)]$$

where g_1, \dots, g_r represent the conjugacy classes of size greater than 1. We have $p \mid |G|$ and $p \mid [G : Z(g_i)]$ for each i since each subgroup $Z(g_i)$ of G is proper (if $Z(g_i) = G$ then g_i would be in a conjugacy class of size 1, which isn't true). Therefore $p \mid |Z(G)|$. By Cauchy's theorem, $Z(G)$ has an element of order p , say z . As $z \in Z(G)$, $\langle z \rangle \triangleleft G$.

We now consider the quotient group $G/\langle z \rangle$, which is a group with size less than that of G . Since $p^{d-1} \mid |G/\langle z \rangle|$, by induction $G/\langle z \rangle$ has a subgroup with size p^{d-1} . Its inverse image under $G \rightarrow G/\langle z \rangle$ is a subgroup of G with size $p \cdot p^{d-1} = p^d$. \square

Théorème 33: Cauchy's Theorem

Let G be a finite group and let p be a prime. If p divides $|G|$, then G contains an element of order p .

Note. Here is a proof: [cauchypf.pdf](#).

Théorème 34

Let G be a finite group. If $p^d \mid |G|$ and $d > 0$ then each subgroup of G with size p^{d-1} has index p in a subgroup of G .

From Théorème 34 we can build a nested chain

$$\{e\} \subset G_1 \subset G_2 \subset \cdots \subset G_k \subset G$$

where $[G_i : G_{i-1}] = p$, so $|G_i| = p^i$. Théorème 32 is a special case of Théorème 34, so it suffices to prove Théorème 34.

证明. The case $d = 1$ says there is a subgroup of size p in G . This is Cauchy's theorem.

Now take $d > 1$. Let H be a subgroup of G with size p^{d-1} . We want to find a subgroup $K \subset G$ in which H has index p . Consider the left multiplication action of H (not $G!$) on G/H . Since H is a non-trivial p -group,

$$|G/H| \equiv |\{\text{fixed points}\}| \pmod{p} \quad (8.3)$$

The left side of the congruence is $[G : H]$, which is divisible by p . Which cosets in G/H are fixed points? They are

$$\begin{aligned} \{gH : hgH = gH \text{ for all } h \in H\} &= \{gH : g^{-1}hg \in H \text{ for all } h \in H\} \\ &= \{gH : g^{-1}Hg = H\} \\ &= \{gH : g \in N(H)\} \\ &= N(H)/H \end{aligned}$$

Therefore the set of fixed points of H acting on G/H is $N(H)/H$, which has the structure of a group since $H \triangleleft N(H)$. By (4.2), $p \mid |N(H)/H|$, so Cauchy tells us there is a subgroup $H' \subset N(H)/H$ of order p . Its inverse image under $N(H) \rightarrow N(H)/H$ is a subgroup of $N(H)$ with size $p \cdot p^{d-1} = p^d$, and it contains H with index p . \square

Théorème 35: Orbit-Stabilizer Theorem

Let G be a group acting on a set X . For any $x \in X$, we have

$$|G| = |\mathcal{O}_x| \cdot |\text{Stab}(x)|$$

where \mathcal{O}_x is the orbit of x and $\text{Stab}(x)$ is the stabilizer of x . In other words,

$$|\mathcal{O}_x| = [G : \text{Stab}(x)]$$

Explain the reason why eq. (8.3) holds.

Let's consider the orbits of the action of H on G/H . For any $gH \in G/H$, the orbit of gH under the action of H is given by $\text{Orb}(gH) = \{h \cdot (gH) \mid h \in H\} = \{(hg)H \mid h \in H\}$. The size of this orbit is given by the Orbit-Stabilizer Theorem: $|\text{Orb}(gH)| = |H| / |\text{Stab}_H(gH)|$,

If gH is not a fixed point, then there exists at least one $h \in H$ such that $h \cdot (gH) \neq gH$, which means $|\text{Stab}_H(gH)|$ is a proper subgroup of H . Since H is a

p -group of order $p^{d-1} > 1$, the size of any proper subgroup of H must be a power of p strictly less than p^{d-1} . Therefore, the index $|H| / |\text{Stab}_H(gH)| = |\text{Orb}(gH)|$ must be a multiple of p .

We can partition the set G/H into disjoint orbits under the action of H . Let F be the set of fixed points. Then,

$$|G/H| = |F| + \sum_{\text{orbits } \mathcal{O} \text{ not in } F} |\mathcal{O}|$$

We know that $|F| = |N_G(H)/H|$, and for every orbit \mathcal{O} not in F , its size $|\mathcal{O}|$ is a multiple of p . Therefore, when we take this equation modulo p , the sum on the right-hand side becomes 0 (mod p), and we are left with:

$$|G/H| \equiv |F| \pmod{p}$$

Corollaire 6

Let H be a p -subgroup of the finite group G . Then

$$[G : H] \equiv [N(H) : H] \pmod{p}.$$

In particular, if $p \mid [G : H]$ then $H \neq N(H)$.

证明. The congruence here is (4.2). When $p \mid [G : H], [N(H) : H] \neq 1$, so $H \neq N(H)$. \square

Théorème 36: Frobenius

If $p^r \mid |G|$, the number of subgroups of G with size p^r is $\equiv 1 \pmod{p}$.

8.12 Sylow 定理习题

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[sylowapp.pdf](#)

[Expository papers by K. Conrad](#)

Dummit&Foote 4.5

Note. It should be pointed out that the definition of cosets does not require normality, but the definition of quotient group requires. What's the difference? The former (as set) admits disjoint orbits, but $aH \cdot bH$ need not be $ab \cdot H$. The latter (as group) admits both.

8.12.1 Definitions

The **affine group**

$$\text{Aff}(\mathbf{Z}_5) := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbf{Z}_5^\times, b \in \mathbf{Z}_5 \right\}$$

It's defined by the transformation

$$x \mapsto ax + b \quad \pmod{5}$$

$$\begin{pmatrix} x' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}$$

The **Heisenberg group** over $\mathbb{Z}/p\mathbb{Z}$, denoted by $\text{Heis}(\mathbb{Z}/p\mathbb{Z})$, is the set of 3×3 upper triangular matrices of the form:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

where:

- $x, y, z \in \mathbb{Z}/p\mathbb{Z}$, meaning they are elements of the finite field $\mathbb{Z}/p\mathbb{Z}$ (the integers modulo p).
- The group operation is matrix multiplication.

8.12.2 CONSEQUENCES OF THE SYLOW THEOREMS

[sylowapp.pdf](#)

Theorem 2.4. For a prime p , each element of $\mathrm{GL}_2(\mathbf{Z}/(p))$ with order p is conjugate to a strictly upper-triangular matrix $(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix})$. The number of p -Sylow subgroups is $p+1$.

Proof. The size of $\mathrm{GL}_2(\mathbf{Z}/(p))$ is $(p^2-1)(p^2-p) = p(p-1)(p^2-1)$. Therefore a p -Sylow subgroup has size p . The matrix $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ has order p , so it generates a p -Sylow subgroup $P = \langle (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \rangle = \{(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix})\}$. Since all p -Sylow subgroups are conjugate, a matrix with order p is conjugate to some power of $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$.

The number of p -Sylow subgroups is $[\mathrm{GL}_2(\mathbf{Z}/(p)) : N(P)]$ by Sylow III*. We'll compute $N(P)$ and then find its index. For $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ to lie in $N(P)$ means it conjugates $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ to some power $(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix})$. Since

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right) \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)^{-1} = \left(\begin{array}{cc} 1 - ac/\Delta & a^2/\Delta \\ -c^2/\Delta & 1 + ac/\Delta \end{array}\right),$$

where $\Delta = ad - bc \neq 0$, $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in N(P)$ precisely when $c = 0$. Therefore $N(P) = \{(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix})\}$ in $\mathrm{GL}_2(\mathbf{Z}/(p))$. The size of $N(P)$ is $(p-1)^2 p$. Since $\mathrm{GL}_2(\mathbf{Z}/(p))$ has size $p(p-1)(p^2-1)$, the index of $N(P)$ is $n_p = p+1$. \square

Theorem 2.7. For prime p , $\mathrm{Heis}(\mathbf{Z}/(p))$ is the unique p -Sylow subgroup of the group of invertible upper-triangular matrices

$$(2.1) \quad \left(\begin{array}{ccc} d_1 & a & b \\ 0 & d_2 & c \\ 0 & 0 & d_3 \end{array}\right)$$

in $\mathrm{GL}_3(\mathbf{Z}/(p))$.

Proof. This matrix group, call it U , has size $(p-1)^3 p^3$, so $\mathrm{Heis}(\mathbf{Z}/(p))$ is a p -Sylow subgroup of U . To show it is the only p -Sylow subgroup, the relations in Sylow III are *not* adequate. They tell us $n_p \mid (p-1)^3$ and $n_p \equiv 1 \pmod{p}$, but it does not follow from this that n_p must be 1. For instance, $(p-1)^2$ satisfies these two conditions in place of n_p .

To show $n_p = 1$, we will prove $\mathrm{Heis}(\mathbf{Z}/(p)) \triangleleft U$. Projecting a matrix in U onto its 3 diagonal entries is a function from U to the 3-fold direct product $(\mathbf{Z}/(p))^{\times} \times (\mathbf{Z}/(p))^{\times} \times (\mathbf{Z}/(p))^{\times}$. This is a homomorphism with kernel $\mathrm{Heis}(\mathbf{Z}/(p))$, so $\mathrm{Heis}(\mathbf{Z}/(p)) \triangleleft U$. \square

Theorem 3.3. All the Sylow subgroups of a finite group are normal if and only if the group is isomorphic to the direct product of its Sylow subgroups.

Proof. If a group is isomorphic to the direct product of its Sylow subgroups then its Sylow subgroups are normal since a group that is one of the factors in a direct product is a normal subgroup of the direct product. Conversely, suppose G is finite and its Sylow subgroups are all normal. Write the nontrivial Sylow subgroups as P_1, \dots, P_m . Elements in P_i and P_j commute with each other for $i \neq j$, by Theorem 3.2, so the map $P_1 \times \dots \times P_m \rightarrow G$ given by

$$(x_1, \dots, x_m) \mapsto x_1 \cdots x_m$$

is a homomorphism. It is injective since the order of a product of commuting elements with relatively prime orders is equal to the product of their orders. Our map is between two groups of equal size, so from injectivity we get surjectivity, so we have an isomorphism. \square

Theorem 4.3. Every group of size 45 is abelian.

Proof. Let G have size 45. In G , a 3-Sylow subgroup has size 9 and a 5-Sylow subgroup has size 5. Using Sylow III,

$$n_3 \mid 5, \quad n_3 \equiv 1 \pmod{3} \implies n_3 = 1$$

and

$$n_5 \mid 9, \quad n_5 \equiv 1 \pmod{5} \implies n_5 = 1.$$

Therefore G has normal 3-Sylow and 5-Sylow subgroups. Denote them by P and Q respectively, so $|P| = 9$ and $|Q| = 5$. Then P is abelian and Q is cyclic (thus abelian).

The set $PQ = \{ab : a \in P, b \in Q\}$ is a subgroup of G by Lemma 4.1 since P and Q are normal subgroups (we really only need one of them to be normal to use that lemma). Since PQ contains P and Q as subgroups, Lagrange tells us $|PQ|$ is divisible by both 9 and 5. Therefore $45 \mid |PQ|$, so $PQ = G$. Since P and Q are both abelian, we will know G is abelian once we show each element of P commutes with each element of Q . This commutativity is Theorem 3.2. \square

Remark 4.4. The reader can check that the same argument shows every group of size p^2q with primes $p < q$ and $q \not\equiv 1 \pmod{p}$ is abelian. Examples include $99 = 3^2 \cdot 11$ and $175 = 5^2 \cdot 7$.

Theorem 5.4. If $|G| = 12$ then G has a normal 2-Sylow or 3-Sylow subgroup.

Proof. By Sylow III, $n_2 \mid 3$, so $n_2 = 1$ or 3. Also $n_3 \mid 4$ and $n_3 \equiv 1 \pmod{3}$, so $n_3 = 1$ or 4. We want to show $n_2 = 1$ or $n_3 = 1$.

Assume $n_3 \neq 1$, so $n_3 = 4$. Since the 3-Sylows have size 3, Lemma 5.3 says G has $n_3 \cdot 2 = 8$ elements of order 3. The number of remaining elements is $12 - 8 = 4$. A 2-Sylow subgroup has size 4, and thus fills up the remaining elements. Therefore $n_2 = 1$. \square

For example, A_4 has $n_2 = 1$ and $n_3 = 4$, while D_6 has $n_2 = 3$ and $n_3 = 1$.

Theorem 5.5. If $|G| = 24$ then G has a normal subgroup of size 4 or 8.

Proof. Let P be a 2-Sylow subgroup, so $|P| = 8$. Consider the left multiplication action $\ell: G \rightarrow \text{Sym}(G/P) \cong S_3$. Set K to be the kernel of ℓ . Then

- $K \subset P$, so $|K| \mid 8$,
- G/K embeds into S_3 , so $[G : K] \mid 6$. That is, $4 \mid |K|$.

This tells us $|K| = 4$ or 8. Since K is the kernel of ℓ , $K \triangleleft G$. \square

8.12.3 Exercises from Dummit&Foote

[Sylow exercises](#) | Aryaman Maithani

Exercice 12

Prove that if $P \in Syl_p(G)$ and H is a subgroup of G containing P , then $P \in Syl_p(H)$. Give an example to show that, in general, a Sylow p -subgroup of a subgroup of G need not be a Sylow p -subgroup of G .

证明. We are given $P \leq H \leq G$ with $|P| = p^\alpha$ and $|G| = p^\alpha m$ such that $p \nmid m$. As $|P||H|$, we get that $|H| = p^\alpha k$ for some $k \in \mathbb{N}$. Moreover, $|H||G|$ and thus, $k|m$. As $p \nmid m$, we get that $p \nmid k$. Thus, P is a Sylow p -subgroup of H .

For the second part, consider $G = \mathbb{Z}_4$ and $H = \{\bar{0}, \bar{2}\}$. Then $P = H$ is a Sylow 2-subgroup of H but not one of G . \square

Exercice 13

Prove that if H is a subgroup of G and $Q \in Syl_p(H)$ then $gQg^{-1} \in Syl_p(gHg^{-1})$ for all $g \in G$.

证明. We first note that gQg^{-1} is indeed a subgroup of gHg^{-1} . The assertion follows from the fact that $|gQg^{-1}| = |Q|$ and $|gHg^{-1}| = |H|$. \square

Exercice 14

Use Sylow's theorem to prove Cauchy's theorem.

证明. Recall that Cauchy's theorem says that if G is a finite group such that $p||G|$ for some prime p , then G has an element of order p .

To prove this, let G and p be as above and P be a Sylow p -subgroup of G . (Existence of P is given by the Sylow theorems.)

Let $G = p^\alpha m$ with usual meanings. Then, $|P| = p^\alpha$.

Consider $x \in P$ such that $x \neq 1$. Then, the order of x is p^β for some $1 \leq \beta \leq \alpha$. Consider $y = x^{p^{\beta-1}}$. Then, order of y is p . (How? It is clear that $y^p = 1$. Thus, the only other possibility for the order is 1 but that is not possible since $y \neq 1$ as $p^{\beta-1} < p^\beta$, the order of x .) \square

Exercice 15

Exhibit all Sylow 2-subgroups and Sylow 3-subgroups of D_{12} and $S_3 \times S_3$.

证明. D_{12} : Note that $|D_{12}| = 12 = 2^2 \cdot 3$. Thus, the Sylow 2-subgroup(s) will have order 4 and Sylow 3-subgroup(s) will have order 3.

Sylow 3: Note that a Sylow 3-subgroup must necessarily be isomorphic to \mathbb{Z}_3 . Since \underline{all elements not of the form r^k have order 2}, we just find powers of r which have order 3. These turn out to be r^2 and r^4 . As $\langle r^2 \rangle = \langle r^4 \rangle$, we have a unique Sylow 3-subgroup:

$$\langle r^2 \rangle$$

Sylow 2: Note that no element of D_{12} has order 4. (Any such element would have to be a power of r but 4 does not divide 6, the order of r .) Thus, any Sylow 2-subgroup must be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Every element of this group must have order 2. r^3 is the only power of r which has order 2. Thus, the other two elements must be of the form sr^k . Noting that the multiplication of two such elements is:

$$sr^k \cdot sr^{k'} = r^{k'-k}$$

we get that $|k - k'| = 3$. This gives us three such groups which are all the Sylow 2-subgroups:

$$\underbrace{\langle s, sr^3 \rangle}_{\langle s \rangle \times \langle r^3 \rangle}, \underbrace{\langle sr, sr^4 \rangle}_{\langle sr \rangle \times \langle r^3 \rangle}, \underbrace{\langle sr^2, sr^5 \rangle}_{\langle sr^2 \rangle \times \langle r^3 \rangle}.$$

(It can be checked that all of these are distinct subgroups.)

$S_3 \times S_3$: The order of the group is $36 = 2^2 \cdot 3^2$. Thus, the Sylow 2-subgroup(s) will have order 4 and Sylow 3-subgroup(s) will have order 9.

Sylow 2: Note that there are three subgroups of order 2 of S_3 : $\langle (12) \rangle, \langle (13) \rangle, \langle (23) \rangle$. Let these be H_1, H_2, H_3 . (Note that these are indeed distinct.) Then, the nine products:

$$H_i \times H_j \quad i, j \in \{1, 2, 3\}$$

are subgroups of $S_3 \times S_3$ have order 4. This gives us nine Sylow 2-subgroups. However, by the Sylow theorems, $n_2 \mid 9$ and thus, there can't be any more.

Sylow 3: Let P be a Sylow 3-subgroup. Note that no element of $S_3 \times S_3$ has order 9. Thus, every non-identity element of P must have order 3. Now, note that the order of an element $(\sigma, \tau) \in S_3 \times S_3$ is the lcm of the orders of σ and τ . This gives us that there are 8 elements of order 3. These elements together with the identity do form a subgroup and moreover, there can't be any other. Thus, there is a unique Sylow 3-subgroup which is:

$$\langle (123) \rangle \times \langle (123) \rangle.$$

□

Exercice 16

Show that a Sylow p -subgroup of D_{2n} is cyclic and normal for every odd prime p .

证明. Let $2n = p^\alpha m$ as usual. As p is odd and $2n$ is even, we must have that $m = 2m'$ for some $m' \in \mathbb{N}$. Thus, $n = p^\alpha m'$ with $p \nmid m'$.

Note that r has order n and hence, $r^{m'}$ has order p^α . Hence, $\langle r^{m'} \rangle$ is Sylow p -subgroup of D_{2n} .

We now show that it is normal. To do this, it is enough to work with the generators of D_{2n} . Clearly, $r(r^{m'})r^{-1} \in \langle r^{m'} \rangle$. Also, $s(r^{m'})s^{-1} = s^2r^{-m'} = r^{-m'} \in \langle r^{m'} \rangle$. Thus, $\langle r^{m'} \rangle$ is normal.

As all Sylow p -subgroups are conjugates, we get that the above Sylow p -subgroup is the unique Sylow p -subgroup. As it is cyclic and normal, we are done. □

Exercice 17

Exhibit all Sylow 3-subgroups of A_4 and all Sylow 3-subgroups of S_4 .

证明. A_4 : Clearly, $n_3 \mid 4$. We exhibit 4 such Sylow 3-subgroups now:

$$\langle (123) \rangle, \langle (124) \rangle, \langle (134) \rangle, \langle (234) \rangle.$$

(Note that these are indeed distinct subgroups.)

These were easy to find as a Sylow 3-subgroup must have order 3 and thus, must be isomorphic to \mathbb{Z}_3 . It was then a matter of finding elements of order 3.

S_4 : Same as earlier. Note that there are 8 elements of order 3. If we consider all the 8 subgroups generated by them, we see that we get \underline{only four distinct ones}, listed above. \square

Exercice 18

Exhibit all Sylow 2-subgroups of S_4 and find elements of S_4 which conjugate one of these into each of the others.

证明. A little bit of experimenting with elements of order 2 and 4 gives the following subgroups:

$$\langle(1234), (13)\rangle, \langle(1243), (14)\rangle, \langle(1324), (12)\rangle.$$

It can be verified that these are distinct. (For example, the 2 cycles listed in the generators don't appear in any of the other subgroups.) It can also be verified that all of these have order 8. Moreover, there can't be any more as $n_2 \mid 3$.

As for the conjugation question, (34) conjugates the first to the second and (24) the second to the third.\footnote{Note that the conjugate of D_{2n} works by transforming the places of the elements.} \square

Exercice 19

Exhibit two distinct Sylow 2-subgroups of S_5 and an element of S_5 that conjugates one into the other.

证明. Note that $|S_5| = 2^3 \cdot 15$ and thus, any two of the Sylow 2-subgroups listed earlier will work again. (With the understanding that the elements are now elements of S_5) \square

Lemme 11

The order of $GL_2(\mathbb{F}_q)$ is $(q^2 - 1)(q^2 - q)$.

证明. Consider the choice of the first column. There are $q^2 - 1$ choices for the first column. The second column must be a vector \underline{independent} of the first column. There are $q^2 - q$ choices for the second column. \square

Exercice 20

Exhibit all Sylow 3-subgroups of $SL_2(\mathbb{F}_3)$.

证明. Note that the order of $GL_2(\mathbb{F}_3)$ is $(3^2 - 1)(3^2 - 3) = 48$. It can be easily shown that the order of SL_2 is half of that. Thus, $n_3 = 1$ or 4 . Also, note that any Sylow 3-subgroup will have order 3 and hence, is isomorphic to \mathbb{Z}_3 . It is easy to find two distinct subgroups of order 3 by observing the following matrices to have order 3:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Also, these matrices generate distinct subgroups. Thus, $n_3 > 1$ which forces $n_3 = 4$. We may now conjugate the above matrices to get the other two. In any case, we are left with the following:

$$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \right\rangle.$$

\square

Exercice 21

Prove that the subgroup of $SL_2(\mathbb{F}_3)$ generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the unique Sylow 2-subgroup of $SL_2(\mathbb{F}_3)$.

证明. Hmm. □

Exercice 22

Prove that a group G of order 351 has a normal Sylow p -subgroup for some prime p dividing its order.

证明. Note that $312 = 3^3 \cdot 13$. By the Sylow theorems, $n_{13} \mid 27$ and $n_{13} \equiv 1 \pmod{13}$. This forces $n_{13} = 1, 27$. If $n_{13} = 1$, then we are done. Let us assume that $n_{13} \neq 1$. Thus, $n_{13} = 27$.

Thus, there are 27 Sylow 13-subgroups of G . **As 13 is a prime, this forces the intersection of two distinct Sylow 13-subgroups to be trivial.** Thus, the number of elements having order 13 equals $27(13 - 1) = 3^3(12)$.

This leaves us with 3^3 remaining elements which are not part of any Sylow 13-subgroup. Now, by Sylow Theorems, we know that $n_3 \geq 1$. However, no non-identity element can be part of a Sylow 3 as well as a Sylow 13-subgroup. Thus, **the remaining 3^3 elements form the unique Sylow 3-subgroup** which gives us that $n_3 = 1$ and thus, we are done. □

Exercice 23

Let $|G| = pqr$, where p, q , and r are primes with $p < q < r$. Prove that G has a normal Sylow subgroup for either p, q or r .

证明. Let p, q , and r be distinct prime numbers. WLOG, we may assume that $p < q < r$. Let G be a group with order pqr .

We shall be appealing to the Sylow theorems without mentioning it explicitly. If any of n_p, n_q , or n_r is equal to 1, then we know that G is not simple. For sake of contradiction assume that each of the above is strictly greater than

1. As $n_r \mid pq$ and $p, q < r$, we get that $n_r = pq$. (Since we have assumed that $n_r > 1$.) Thus, there are pq Sylow- r subgroups of G .

Now, note that each such Sylow- r subgroup has order r , a prime and thus, the intersection of two distinct Sylow- r subgroups must be trivial, id est, (1). Thus, the number of elements having order r equals $o_r = pq(r - 1)$.

Now, $n_q > 1$ and $n_q \mid pr$. Thus, $n_q \in \{p, r, pr\}$. However, $n_q \equiv 1 \pmod{q}$ and thus, $n_q \neq p$. This gives us that $n_q \geq r$. Thus, the number of elements having order q equals $o_q \geq r(q - 1)$.

Lastly, similar argument as earlier gives us that $o_p \geq q(p - 1)$. Note that o_r, o_q , and o_p are counting distinct non-identity elements and thus,

$$\begin{aligned} |G| &\geq o_r + o_q + o_p + 1 \\ &\geq pq(r - 1) + r(q - 1) + q(p - 1) \\ &\geq pqr + rq - r - q + 1 \\ &= pqr + \underbrace{(r - 1)(q - 1)}_{>0} \\ &> pqr \end{aligned}$$

Thus, we have a contradiction as $|G| = pqr$ and we are done! \square

Exercice 24

Prove that if $|G| = 105$, then G has a normal Sylow 5-subgroup and a normal Sylow 7 subgroup.

证明. Assume not. Then, $n_5 > 1$ and $n_7 > 1$.

Note that $105 = 3 \cdot 5 \cdot 7$. The Sylow theorems force $n_5 = 21$ and $n_7 = 15$.

Note that the intersection of any two of these 21+15 Sylow subgroups must be trivial (by Lagrange's Theorem). Thus, their union contains exactly

$$21(5 - 1) + 15(7 - 1) + 1 = 175 \text{ elements.}$$

This is clearly a contradiction. (As $175 > 105$). \square

Exercice 25

Prove that if $|G| = 6545$ then G is not simple.

证明. Note that $6545 = 5 \cdot 7 \cdot 11 \cdot 17$. For the sake of contradiction, let us assume that G is not simple. Thus, n_5, n_7, n_{11} , and n_{17} are all strictly greater than 1. Now, looking at the possibilities for each by considering the factors of the "complimentary part", we get that

$$n_5 = 11, n_7 = 85, n_{11} = 594, n_{17} = 35$$

Note that the intersection of any two of these Sylow subgroups must be trivial (by Lagrange's Theorem). Thus, their union contains exactly

$$11(5 - 1) + 85(7 - 1) + 594(11 - 1) + 35(17 - 1) + 1 = 7055 \text{ elements.}$$

This is clearly a contradiction. (As $7055 > 6545$.) \square

Exercice 26

Prove that if $|G| = 1365$ then G is not simple.

证明. Note that $1365 = 3 \cdot 5 \cdot 7 \cdot 13$. For the sake of contradiction, let us assume that G is not simple. Thus, n_3, n_7 , and n_{13} are all strictly greater than 1. Now, looking at the possibilities for each by considering the factors of the "complimentary part", we get that

$$n_3 \geq 7, n_7 \geq 15, n_{13} = 105$$

Note that the intersection of any two of these Sylow subgroups must be trivial (by Lagrange's Theorem). Thus, their union contains at least

$$7(3 - 1) + 15(7 - 1) + 105(13 - 1) > 1365 \text{ elements.}$$

This is clearly a contradiction. \square

Exercice 27

Prove that if $|G| = 132$ then G is not simple.

证明. Let G be a group with order 132 . We show that G is not simple.

Note that $|G| = 2^2 \cdot 3 \cdot 11$. Let us assume that G is simple and arrive at a contradiction. By simplicity, we know that $n_{11} > 1$, $n_3 > 1$, and $n_2 > 1$.

By Sylow Theorem (3), it is forced that $n_{11} = 12$. Also, $n_3 \geq 4$ and $n_2 \geq 3$.

Note the following:

- (1) Intersection of any two Sylow- 11 subgroups is trivial.
- (2) Intersection of any two Sylow-3 subgroups is trivial.
- (3) Intersection of any Sylow- p subgroup with any Sylow- q subgroup is trivial.

The above facts follow by considering the fact that the intersection would be a subgroup of the two bigger subgroups and would have to divide their orders.

Now, if we consider the union of all the Sylow-11 and Sylow- 3 subgroups, it contains at least $12(11-1) + 4(3-1) + 1 = 129$ elements. Thus, the remaining elements are at most 3 . However, we do need at least 3 more elements to form a Sylow- 2 subgroup.

These 3 elements, along with the identity must form the unique Sylow- 2 subgroup of order 4 . However, we have reached a contradiction as we get that $n_2 = 1$.

Thus, we are done!

□

Exercice 28

Prove that if $|G| = 462$ then G is not simple.

证明. Note that $462 = 11 \cdot 42$. Considering the factors of 42 , we see that n_{11} is forced to be 1 .

□

Exercice 29

Prove that if G is a group of order 231 then $Z(G)$ contains a Sylow 11-subgroup of G and a Sylow 7-subgroup is normal in G .

证明. Note that $231 = 3 \cdot 7 \cdot 11$. The restrictions from Sylow Theorems force that $n_7 = n_{11} = 1$. In particular, this shows that G has a (unique) Sylow 7-subgroup which is normal in it.

Now, we show that the Sylow 11-subgroup is contained in $Z(G)$. Let us denote this subgroup by P . Note that P is normal. Consider the action of G on P given by conjugation, that is, $(g, p) \mapsto gpg^{-1}$.

This induces a homomorphism $\Phi : G \rightarrow \text{Aut}(P)$. Note that $P = (\mathbb{Z}/11\mathbb{Z})$ is cyclic and thus, $\text{Aut}(P) = (\mathbb{Z}/11\mathbb{Z})^\times = (\mathbb{Z}/10\mathbb{Z})$. (The last equality may be manually verified by noting that $\bar{2}$ is a generator for $(\mathbb{Z}/11\mathbb{Z})^\times$ which has 10 elements.)

Thus, $G/\ker \Phi$ is isomorphic to a subgroup of \mathbb{Z}_{10} (First Isomorphism Theorem). In particular, $|G/\ker \Phi| \mid 10$. Looking at the divisors of $|G|$, we see that this forces $|G/\ker \Phi| = 1$ or $G = \ker \Phi$.

This means that given any $g \in G$, the map $\Phi(g)$ is the identity map. In other words, $\Phi(g)(p) = p$ for all $g \in G$ and $p \in P$. The above is equivalent to $gpg^{-1} = p$ or $gp = pg$ for all $g \in G$ and $p \in P$. Thus, $P \leq Z(G)$, by the definition of the center.

Remark. We didn't really require the fact that $\mathbb{Z}_{11}^\times \cong \mathbb{Z}_{10}$. Only $|\mathbb{Z}_{11}^\times| = 10$ would've been sufficient (which is more straightforward as well). \square

Exercice 30

Let G be a group of order 105. Prove that if a Sylow 3-subgroup of G is normal then G is abelian.

证明. Let N be the Sylow 3-subgroup.

Claim 1: $N \leq Z(G)$.

Note that $|G/C_G(N)|$ must divide both $|\text{Aut}(N)| = 2$ and $|G| = 105$. This forces $G = C_G(N)$ or $N \leq Z(G)$. (We have used the fact that N is normal in

the above by considering the Φ defined similarly in the previous exercise.)

Claim 2: G/N is cyclic.

Note that G/N has order $35 = 5 \cdot 7$. Noting that $5 \nmid 6$ proves the claim.

(See this.)

Claim 3: $G/Z(G)$ is cyclic.

By the third isomorphism theorem, we have that

$$G/Z(G) \cong (G/N)/(Z(G)/N)$$

(Note that Claim 1 (and the fact that $N \triangleleft G$) tells us that $Z(G)/N$ makes sense.) By the previous claim, G/N is cyclic. The claim follows as quotients of a cyclic group are cyclic.

Claim 4: G is abelian.

This is a consequence of the above claim.

□

Lemme 12

$\text{Aut}(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is isomorphic to $\text{GL}(2, \mathbb{F}_3)$.

证明. Consider $\mathbb{Z}_3 \times \mathbb{Z}_3$ as a 2-dimension vector in \mathbb{F}_3^2 . Then $\text{Aut}(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is a nondegenerate transform above \mathbb{F}_3^2 . □

Exercice 31

Let G be a group of order 315 which has a normal Sylow 3-subgroup.

Prove that $Z(G)$ contains a Sylow 3-subgroup of G and deduce that G is abelian.

证明. Let P be the normal Sylow 3-subgroup. (It must necessarily be unique.)

Consider the action of G on P given by conjugation. (This is an action as P is normal.) Consider the induced homomorphism $\Phi : G \rightarrow \text{Aut}(P)$.

Now, note that P is a group of order 9. Thus, it is isomorphic to either \mathbb{Z}_9 or $\mathbb{Z}_3 \times \mathbb{Z}_3$. Accordingly, Aut has order either 6 or 48.

As $G/\ker \Phi \cong \Phi(G)$ by the first isomorphism theorem, we get that $|G/\ker \Phi|$ divides 48. (6 divides 48 anyway.)

Also, $|G/\ker \Phi| | |G|$. Thus, $|G/\ker \Phi|$ is either 1 or 3.

If $|G/\ker \Phi| = 3$, then $|\ker \Phi| = 315/3 = 105$. However, note that $P \leq \ker \Phi = C_G(P)$ and hence $|P| | |C_G(P)|$ which is a contradiction as $9 \nmid 105$. Hence, we get that $G/\ker \Phi = (1)$ or $G = \ker \Phi$.

The above shows that the image of Φ is just the identity map. In other words, $gpg^{-1} = p$ for all $p \in P$ and $g \in G$. This is precisely what it means for $P \leq Z(G)$.

The rest of the solution is now the same as the earlier one. We note that G/P has order 35 and hence, is cyclic.

We also have

$$G/Z(G) \cong (G/P)/(Z(G)/P)$$

and thus, $G/Z(G)$ is cyclic. This proves that G is abelian. \square

Exercice 32

Let G be a group of order 1575. Prove that if a Sylow 3-subgroup of G is normal then a Sylow 5-subgroup and a Sylow 7-subgroup are normal. In this situation prove that G is abelian.

证明. By the same arguments as earlier, we get that G is abelian. Now, by the Sylow Theorems, there do exist Sylow 5 and 7-subgroups. As G is abelian, they must be normal. \square

Exercice 33

How many elements of order 7 must there be in a simple group of order 168?

证明. Note that $168 = 7 \cdot 24$. The Sylow Theorems force n_7 to be either 1 or 8. If $n_7 = 1$, then there is a unique Sylow 7-subgroup which must be normal. This contradicts the assumption that the group is simple.

Thus, $n_7 = 8$. This means that every Sylow 7-subgroup is isomorphic to \mathbb{Z}_7 . In particular, every non-identity has order 7.

Moreover, two distinct Sylow 7-subgroups can only intersect trivially, by Lagrange's theorem. Thus, the total number of order 7 elements contained in these Sylow subgroups is

$$8(7 - 1) = 48$$

Now, we show that there are no more. That is, any order 7 element is contained in some Sylow 7-subgroup.⁴

Let x be any order 7 element. Then, $\langle x \rangle$ is a 7-group and thus, must be contained in some Sylow 7-subgroup, as desired. (In fact, $\langle x \rangle$ is itself a Sylow 7-subgroup.) \square

Exercice 34

For $p = 2, 3$, and 5 , find $n_p(A_5)$ and $n_p(S_5)$.

证明. A_5 : Note that Sylow Theorems force $n_3 \in \{1, 10\}$ and $n_5 \in \{1, 6\}$.

As A_5 is simple, n_p cannot be 1 for any p and hence, $n_3 = 10$ and $n_5 = 6$.

Now, we show that $n_2 = 5$. Note that $\langle (12)(34), (13)(24) \rangle$ is a Sylow 2-subgroup. (The way I found this was by taking a Sylow 2-subgroup from Q_8 . And intersecting it with A_5 .) Note that its elements are

$$\{1, (12)(34), (13)(24), (14)(23)\}$$

That is, identity along with all possible elements of the cycle type 1-2-2 formed using the numbers 1, 2, 3, 4.

Recalling that all Sylow 2-subgroups are conjugates and also recalling how conjugates in S_n look, we see that any Sylow 2-subgroup is of this form:

$$\{1, (ab)(cd), (ac)(bd), (ad)(bc)\}$$

(Where a, b, c, d are distinct elements of $\{1, 2, 3, 4, 5\}$.)

This shows us that picking any four numbers from $\{1, \dots, 5\}$ uniquely defines Sylow 2 subgroup. (In the sense, that there's no over-counting or under-counting.) Thus, the number of such subgroups is $\binom{5}{4} = 5$.

To conclude, we have, for A_5 :

$$n_2 = 5, n_3 = 10, n_5 = 6$$

⁴This is important.

S_5 :

$$|S_5| = 120 = 2^3 \cdot 3 \cdot 5$$

For n_3 , we have the possibilities 1, 10, and 40. Note that it can't be 1 as any Sylow 3 subgroup of A_5 is again a Sylow 3-subgroup of S_5 and we already had 10 there.

Now, if n_3 were 40, then there would be a total of 80 elements of order 3. However, it can be verified that only elements of the cycle type (abc) have order 3 and there are only 20 of those. Thus, $n_3 = 10$.

For n_5 , the discussion is even simpler as there are no new possibilities and we have $n_5 = 6$ as before.

As before, we have $n_2 \leq 15$. We show that $n_2 = 15$ by demonstrating fifteen such subgroups. Let $a, b, c, d \in \{1, 2, 3, 4, 5\}$ be distinct. Assume that a is the smallest amongst the four.

Consider the subgroup:

$$\langle (abcd), (ac) \rangle$$

It can be verified that the above does have 8 elements. Moreover, to determine such a subgroup, we first need to pick 4 elements from $\{1, \dots, 5\}$ and then from those 4, pick an element to be paired with the smallest.

It can be verified that different such pickings will give different subgroups. (Simply by writing explicitly the elements of the above subgroup.)

This gives us $5 \times 3 = 15$ different subgroups and hence, we are done. To conclude, we have, for S_5 :

$$n_2 = 15, n_3 = 10, n_5 = 6$$

□

8.13 Discussion of the intersection of Sylow 3

subgroups, when $n_3 \neq 1$

[Order 90 | Aryaman Maithani](#)

[Order 144 | Aryaman Maithani](#)

8.13.1 Group with order 180 is not simple

Order 180 | Aryaman Maithani

Exercice 35

Let G be a group with order 180. Show that G is not simple.

证明. Assume that G is simple, note that $|G| = 2^2 \cdot 3^2 \cdot 5$; by Sylow III theorem, $n_3 \in \{4, 10\}$, $n_5 \in \{6, 36\}$.

Claim that $n_3 \neq 4$. Suppose not, $n_3 = 4$, then there is a homomorphism

$$\varphi : G \rightarrow \text{Aut}(\text{Syl}_3(G)) \cong S_4 \quad g \mapsto \text{Ad}_g$$

recall that $\text{Syl}_p(G) := \{\text{Sylow } p\text{-subgroups of } G\}$. By Sylow II theorem, φ is transitive, thus $\ker \varphi \neq G$. As G is simple, $\ker \varphi$ is trivial, otherwise the non-trivial $\ker \varphi \triangleleft G$; thus G is not simple. So $\ker \varphi = \{e\}$, then

$$G \cong G/\ker \varphi \cong \text{Im} \varphi \leq S_4 \Rightarrow 180 = |G| \mid |S_4| = 24 \Rightarrow 180 \leq 24$$

which is a contradiction.

Claim that $n_5 \neq 6$. Suppose not, $n_5 = 6$, then there is a homomorphism

$$\psi : G \rightarrow \text{Aut}(\text{Syl}_5(G)) \cong S_6 \quad g \mapsto \text{Ad}_g$$

Similarly, $\ker \psi = \{e\}$, then

$$G \cong G/\ker \psi \cong \text{Im} \psi \leq S_6$$

We have $[S_6 : G] = 4$, strictly between 2 and $n = 6$, which is impossible by the following lemma.

Lemme 13

For $n \geq 5$, no subgroup of S_n has index strictly between 2 and n . Moreover, each subgroup of index n in S_n is isomorphic to S_{n-1} .

Applying the idea in the proof of Lemme 13, we consider the left multiplication action of S_6 on S_6/G ,

$$\Phi : S_6 \rightarrow \text{Aut}(S_6/G) \cong S_4 \quad g \mapsto \ell_g$$

Then

$$S_6 / \ker \Phi \cong \text{Im} \Phi \leq S_4$$

Clearly, $\ker \Phi \triangleleft S_6$, then $\ker \Phi$ is trivial or $\ker \Phi = A_6$. Since $\ker \Phi \subset G$ (by the definition of Φ), we have $[S_6 : \ker \Phi] \geq [S_6 : G] = 4$, thus $\ker \Phi$ must be $\{e\}$. Then

$$180 = |S_6| = |S_6 / \ker \Phi| = |\text{Im} \Phi| \cdot |S_4| = 24$$

which is a contradiction. Hence $n_5 \neq 6$.

Note that the intersection of any two Sylow-5 subgroups is trivial, since $P_1 \cap P_2$ is subgroup of P_1 , then $|P_1 \cap P_2| \mid 5$; $P_1 \cap P_2 = \{e\}$.

Remarque 36. When $p^2 \mid |G|$, the intersection of Sylow p -subgroups may be nontrivial. A counterexample is $G = S_3 \times \mathbf{Z}_2$; consider $P_1 = \{(e, 0), (e, 1), ((12), 0), ((12), 1)\}$, $P_2 = \{(e, 0), (e, 1), ((13), 0), ((13), 1)\}$, which are Sylow 2-subgroups of G ; but $P_1 \cap P_2 = \{(e, 0), (e, 1)\}$.

Thus we now arrive the conclusion that $n_3 = 10$ and $n_5 = 36$. Then there are two cases,

- Intersection of any two Sylow-3 subgroups is trivial.
- Intersection of some two Sylow-3 subgroups is non-trivial.

In the first case,

- The number of elements of order 9 is $10 \times (9 - 1) = 80$.
- The number of elements of order 5 is $36 \times (5 - 1) = 144$.

$80 + 144 > 180$; this is a contradiction.

In the second case, let P_1 and P_2 have nontrivial intersection. Note that $|P_1| = |P_2| = 9$ and thus $P = P_1 \cap P_2$ must contain exactly 3 elements. As groups of p^2 order are abelian, we have $P_1 \cup P_2 \subset N_G(P) =: N$. By the definition of group, the set $P_1 P_2 := \{p_1 p_2 : p_1 \in P_1, p_2 \in P_2\}$ is contained in N , as $P_1 \subset N$, $P_2 \subset N$. This gives

$$|N| \geq |P_1 P_2| = \frac{|P_1| \cdot |P_2|}{|P_1 \cap P_2|} = 27$$

Thus the coset

$$|G/N| = \frac{|G|}{|N|} \leq \frac{180}{27} < 7 \implies n := |G/N| \leq 6$$

Let G act on G/N by left multiplication,

$$\phi : G \rightarrow \text{Aut}(G/N) \cong S_n \quad g \mapsto \ell_g$$

Since G is simple, $\ker \phi$ is trivial; since the action is transitive, $\ker \phi \neq G$; thus $\ker \phi = \{e\}$.

$$G \cong G/\ker \phi \cong \text{Im} \phi \leq S_n \implies 180 = |G| \mid |S_n| = n!$$

Therefore $n = 6$. But by Lemme 13, $[S_n : \text{Im} \phi] \neq 4$, as $\text{Im} \phi$ is a subgroup of S_n , $n = 6 \geq 5$.

□

8.14 Group with order 36 has either a normal 2-Sylow or 3-Sylow subgroup

Note. This topic involves extensive discussion and various techniques in group theory.

Let $|G| = 36$. Then $n_3 = 1$ or 4 . We will show that if $n_3 = 4$ then $n_2 = 1$. Assume $n_3 = 4$ and $n_2 > 1$. Since $n_2 > 1$, G has no subgroup of size 18 (it would have index 2 and therefore be normal, so a 3-Sylow subgroup of it would be normal in G by Lemma 5.8, which contradicts $n_3 > 1$). Since $n_2 > 1$, G is nonabelian. Our goal is to get a contradiction. We will try to count elements of different orders in G and find the total comes out to more than 36 elements. That will be our contradiction.

Let Q be a 3-Sylow in G , so $[G : Q] = 4$. Left multiplication of G on G/Q gives a homomorphism $G \rightarrow \text{Sym}(G/Q) \cong S_4$. Since $|G| > |S_4|$, the kernel K is nontrivial. Since $K \subset Q$, either $|K| = 3$ or $K = Q$. Since $Q \not\subset G$, Q does not equal K , so $|K| = 3$.

Since $K \triangleleft G$, we can make G act on K by conjugations. This is a homomorphism $G \rightarrow \text{Aut}(K) \cong \mathbb{Z}/(2)$. If this homomorphism is onto (that is, some element of G conjugates on K in a nontrivial way) then the kernel is a subgroup of G with size 18, which G does not have. So the conjugation action of G on K is trivial, which means every element of G commutes with the elements of K , so $K \subset Z(G)$. Then $3|Z(G)|$, so the size of $Z(G)$ is one of the numbers

in $\{3, 6, 9, 12, 18, 36\}$. Since G is nonabelian and a group is abelian when the quotient by its center is cyclic, $|Z(G)|$ can't be 12, 18, or 36. Since $n_3 > 1$ there is no normal subgroup of size 9, so $|Z(G)| \neq 9$. If $|Z(G)| = 6$ then the product set $Z(G)Q$ is a subgroup of size 18, a contradiction. So we must have $|Z(G)| = 3$, which means $Z(G) = K$.

Now we start counting elements with various orders. The center is a 3-subgroup of G , so by the conjugacy of 3-Sylow subgroups every 3-Sylow subgroup contains K . Each pair of different 3-Sylow subgroups have K as their intersection, so we can count the total number of elements of 3-power order: $|K| + n_3 \cdot (9 - 3) = 27$.

Let $g \in G$ have order 2. Then $K\langle g \rangle$ is a subgroup of order 6 by Lemma 4.1, and it is abelian and in fact cyclic since $K \subset Z(G)$. The cyclic group $K\langle g \rangle$ has a unique element of order 2, which must be g . Therefore when g and g' are different elements of order 2 in G , the groups $K\langle g \rangle$ and $K\langle g' \rangle$ have K as their intersection. So each element of order 2 in G provides us with 2 new elements of order 6. Let n be the number of elements of order 2 in G , so there are at least $2n$ elements of order 6, giving at least $3n$ elements in total with order 2 or 6. Since we already found 27 elements with 3-power order (including the identity), $3n \leq 36 - 27$, or $n \leq 3$. We can get an inequality on n in the other direction: $n \geq 2$. Indeed, no element of order 2 lies in $Z(G) = K$, so some conjugate of an element of order 2 is a second element of order 2. Thus $n \geq 2$.

Since $|\{g \in G : g^2 = e\}|$ is even (by McKay's proof of Cauchy's theorem) and this number is $1+n$, n is odd, so $n = 3$. Therefore G has 3 elements of order 2, so at least $3n = 9$ elements of order 2 or 6. Adding this to 27 from before gives $9 + 27 = 36 = |G|$, so each element of G has 3-power order or order 2 or 6. In particular, the 2-Sylow subgroup of G is isomorphic to $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ (no elements of order 4 in G). Then different 2-Sylow subgroups meet at most in a group of order 2, which gives us 5 elements of order 2 from both subgroups. We saw before that there are only 3 elements of order 2. This is a contradiction.

8.15 群论杂题

见《近世代数三百题》

Proposition 2

1. 9. 9. 设 n_1, \dots, n_r 为自然数, 则 (1) $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \cong \mathbb{Z}_{n_1 n_2}$ 当且仅当 $(n_1, n_2) = 1$. (2) 如果 n_1, \dots, n_r 两两互素, 则 $\mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_r} \cong \mathbb{Z}_{n_1 \dots n_r}$.

Proposition 3

1. 9. 10. 试证 7·11·13 阶群一定是循环群.

证明. 首先利用 Sylow 定理证明 $\mathbb{Z}_7 \triangleleft G, \mathbb{Z}_{11} \triangleleft G, \mathbb{Z}_{13} \triangleleft G$. 借助 Proposition 2 可知 $\mathbb{Z}_7 \times \mathbb{Z}_{11} \times \mathbb{Z}_{13} \cong \mathbb{Z}_{7 \cdot 11 \cdot 13}$ 是循环群. \square

Définition 11

Let H be a subgroup of G . Define

$$K := \bigcap_{g \in G} gHg^{-1}$$

to be the intersection of all conjugates of H .

Exercice 36

- (1) Show that K is a normal subgroup of G .
- (2) Show that if $[G : H]$ is finite, then $[G : K]$ is finite. (Hint: first show that the intersection above defining K is essentially a finite intersection.)

证明. (1) We check that for any $s \in G$,

$$sKs^{-1} := s \left(\bigcap_{g \in G} gHg^{-1} \right) s^{-1} = \bigcap_{g \in G} sgHg^{-1}s^{-1} = \bigcap_{g' \in G} g'Hg'^{-1} = K$$

with $g' = sg$ in the notation. So K is a normal subgroup of G .

(2) We start with a lemma: if H_1 and H_2 are subgroups of G of finite index. Then $H_1 \cap H_2$ is a subgroup of G of finite index. The easiest way to see this is to let H_1 act on the left cosets G/H_2 by left multiplication. Then the stabilizer

group at H_2 is precisely $H_1 \cap H_2$. We know that the index of $H_1 \cap H_2$ inside H_1 is precisely the number of elements in the orbit of the identity coset H_2 in G/H_2 under this action. In particular, $[H_1 : H_1 \cap H_2] \leq \#(G/H_2)$. It then follows that $[G : H_1 \cap H_2] \leq [G : H_1] \cdot [G : H_2]$.

Now, we come back to the proof of (2). As $[G : H]$ is assumed to be finite, we may choose a finite set of coset representatives g_1H, \dots, g_rH of G/H . Then for every element $g \in g_iH$ (writing $g = g_i h$), we have

$$gHg^{-1} = g_i h H h^{-1} g_i^{-1} = g_i H g_i^{-1}.$$

So K is the intersection

$$\bigcap_{i=1}^r g_i H g_i^{-1},$$

which is the intersection of finitely many finite index subgroups. By the lemma above, $[G : K]$ is finite as well. \square

8.16 环论

Théorème 37

设 $(R, +, \cdot)$ 为含 $1 \neq 0$ 的结合环, $a, b \in R$. 若 $a + b = ba$, 且关于 x 的方程

$$\begin{cases} x^2 - (ax^2 + x^2a) + ax^2a = 1 \\ x + a - (ax + xa) + axa = 1 \end{cases}$$

在 R 中有解. 证明: $ab = ba$.

证明. 1) 首先注意到

$$\begin{aligned} & \begin{cases} x^2 - (ax^2 + x^2a) + ax^2a = 1 \\ x + a - (ax + xa) + axa = 1 \end{cases} \\ \Leftrightarrow & \begin{cases} (1-a)x^2(1-a) = 1 \\ (1-a)x(1-a) = 1-a \end{cases} \end{aligned}$$

结果有：

$$\begin{aligned}
 (1-a)x &= (1-a)x\{(1-a)x^2(1-a)\} \\
 &= (1-a)x(1-a)x^2(1-a) = (1-a)x^2(1-a) = 1 \\
 x(1-a) &= (1-a)x^2(1-a)x(1-a) \\
 &= (1-a)x^2 \cdot (1-a)x(1-a) = (1-a)x^2(1-a) = 1
 \end{aligned}$$

因此有 $1-a$ 可逆且 $(1-a)^{-1} = x$

2) 现在考虑 $(1-b)(1-a)$ ，则有 $(1-b)(1-a) = 1 - a - b + ba = 1$ ，结合前面所证 $1-a$ 可逆，因此得 $(1-a)^{-1} = 1-b$ 。进而有

$$1 = (1-a)(1-b) = 1 - a - b + ab = 1 - ba + ab$$

亦即 $ab = ba$ 。

□

Exercice 37: Kaplansky

含幺环中某元若有多于一个右逆，则它必然有无限多个右逆。

证明。设 $a \in R$ 有多于一个右逆，则它是左零因子，设理想

$$I := \{x \in R : ax = 0\}$$

只需要证明： I 是无限集。这是因为若 b 是 a 的一个右逆，则 $a(b+I) = 0$ ， $b+x, \forall x \in I$ 也是 a 的右逆。

考虑反证，假设 I 是有限集 $\{x_0, x_1, \dots, x_n\}$ ，($x_0 = 0$) 于是 $ax_k = 0, k \in \{1, \dots, n\}$ 。考虑 $x_k a$ ，它们显然也在 I 内，但 $x_k a \neq x_l a$ ，两两不同，故 $\{x_1 a, \dots, x_n a\}$ 是 $\{x_1, \dots, x_n\}$ 的一个置换。

取 b 为 a 的一个右逆，因为 a 是零因子，故 $ba \neq 1$ （若 b 是 a 的左右逆，则 a 为 unit，不是零因子）从而 $ba - 1 \neq 0$ ，但是 $a(ba - 1) = a - a = 0$ ，故 $ba - 1 \in I$ 。于是 $ba - 1 = x_i \neq 0$ for some $i \neq 0$ 。同时 $x_i b = (ba - 1)b = 0$ ， $x_i = x_j a$ for some $j \neq 0$ 。于是

$$0 = x_i b = x_j a b = x_j \neq 0$$

矛盾！

□

Exercice 38: 注意环的乘法不包含可逆性

设 D 为整环, m 和 n 为互素的正整数, $a, b \in D$. 如果 $a^m = b^m$, $a^n = b^n$.

求证: $a = b$.

证明. 不妨设 $a \neq 0, b \neq 0$, 由于 $(m, n) = 1$, 故存在整数 s, t 使得 $sm + tn = 1$, 如果 $s \geq 0$ 则 $t \leq 0$ 故

$$a \cdot a^{sm} = a \cdot b^{sm} = a \cdot b^{1-tn} = ba \cdot b^{-tn} = ba \cdot a^{-tn} = ba^{1-tn} = ba^{sm}$$

由于 D 是整环, $(a - b)a^{sm} = 0$, $a \neq 0$ 不是零因子, 所以 $a - b$ 是零因子, 故为 0. 也就是 $a = b$. \square

8.17 Elementary Ring Theory

See 肖梁

In this course, all rings are assumed to be unital and $1 \neq 0$.

Proposition 10.3.3. Let R be a commutative ring. An ideal $\mathfrak{m} \subseteq R$ is maximal if and only if the quotient R/\mathfrak{m} is a field.

Proof. By lattice isomorphism theorem, $\mathfrak{m} \subseteq R$ if and only if $\bar{R} := R/\mathfrak{m}$ has only two ideals (0) and (1) . We claim that the latter statement is equivalent to that \bar{R} is a field.

If \bar{R} is a field, then clearly it has only two ideals (0) and (1) . Conversely, if \bar{R} has only two ideals (0) and (1) , then for any nonzero element $a \in \bar{R}$, the ideal $(a) \neq (0)$. Thus $(a) = (1)$, namely there exists $a' \in \bar{R}$ such that $aa' = 1$. This implies that $a \in \bar{R}^\times$. So \bar{R} is a field. \square

Note. Field only has zero ideal.

If I is an ideal such that $M \subset I \subset S$ then $(0) \subset I/M \subset S/M$, I/M is an ideal of the field S/M thus is zero. Therefore, M is maximal.

Exercice 39

Let R be a commutative ring. If all submodules of finitely generated free modules over R are free over R , then R is a PID.

Définition 12: Finitely generated free module over R

A module M over a ring R is called a **finitely generated free module** if it has a basis consisting of a finite number of elements. This means there exists a finite set of elements $\{b_1, \dots, b_n\} \subset M$ such that every element $m \in M$ can be written uniquely in the form

$$m = r_1 b_1 + r_2 b_2 + \cdots + r_n b_n$$

for some unique coefficients $r_1, \dots, r_n \in R$. Such a module is isomorphic to the direct sum of n copies of R , denoted R^n .

Note. The definition of M as R -module is similar to V as K -linear space.

Définition 13: free R -module

An R -module F is a **free R -module** if it has a basis, i.e., there exists a subset $B \subseteq F$ such that for every $x \in F$, there is a unique finite subset $\{b_1, \dots, b_n\} \subseteq B$ and unique scalars $\{r_1, \dots, r_n\} \subseteq R$ such that $x = r_1 b_1 + \cdots + r_n b_n$. Equivalently, F is isomorphic to a direct sum of copies of R , denoted $R^{(I)}$ for some index set I .

8.17.1 Problem 1**Théorème 38**

Let R be a commutative ring. If all submodules of finitely generated free modules over R are free over R , then R is a PID.

证明. First prove that R is an integral domain. Suppose that $a, b \in R \setminus \{0\}$ with $ab = 0$. Consider $aR \subseteq R$; it is a submodule of free module; so aR is a free R -module. However, we know that b kills all of aR because $ab = 0$; this contradicts that aR is a free R -module.

Let $I \subseteq R$ be an ideal; it is then a free R -submodule of R . We claim that I is a free module of rank 1 over R . Suppose not, then there exist $(\alpha_j)_{j \in J}$ forming an R -basis of I . But if $\#J \geq 2$, we know that $\alpha_j \cdot (\alpha_i) + \alpha_i \cdot (-\alpha_j) = 0$,

contradiction! This says that I is free of rank one, i.e. I is generated by one element $\alpha \in I$ so that $I = (\alpha)$ is a principal ideal. Thus R is a PID. \square

Explanation

Okay, let's break down this proof step by step.

The Statement to be Proven:

"Let R be a commutative ring. If all submodules of finitely generated free modules over R are free over R , then R is a PID."

Let's define the key terms:

- **Commutative Ring (R):** A set with addition and multiplication operations that satisfy certain properties (associativity, commutativity of both operations, distributivity, additive identity 0, additive inverses, multiplicative identity 1). "Commutative" means $ab = ba$ for all $a, b \in R$.
- **Module over R (or R -module):** Similar to a vector space over a field, but scalars come from a ring R .
- **Free R -module:** An R -module that has a basis, i.e., a linearly independent spanning set. A finitely generated free module has a finite basis. R itself is a free R -module of rank 1 (with basis $\{1\}$). R^n (tuples of n elements from R) is a free R -module of rank n .
- **Submodule:** A subset of a module that is itself a module under the same operations and with scalars from the same ring.
- **PID (Principal Ideal Domain):** An integral domain in which every ideal is principal.
 - **Integral Domain:** A commutative ring with no zero divisors (if $a, b \in R$ and $ab = 0$, then either $a = 0$ or $b = 0$).
 - **Ideal:** A submodule of R (when R is considered as a module over itself). More formally, a subset $I \subseteq R$ such that $(I, +)$ is a subgroup of $(R, +)$, and for any $r \in R$ and $x \in I$, $rx \in I$.
 - **Principal Ideal:** An ideal generated by a single element, i.e., an ideal of the form $I = (a) = \{ra \mid r \in R\}$ for some $a \in R$.

The Hypothesis: All submodules of finitely generated free modules over R are free over R .

The Goal: Prove that R is a PID. This requires proving two things:

- (1) R is an integral domain.
- (2) Every ideal in R is principal.

Proof Breakdown:

Part 1: Prove that R is an integral domain.

- **"Suppose that $a, b \in R \setminus \{0\}$ with $ab = 0$."** This is the standard way to start a proof that R is an integral domain by contradiction. We assume R is *not* an integral domain, which means there exist non-zero elements a and b whose product is zero. Our goal is to show this assumption leads to a contradiction, forcing us to conclude that such a and b cannot exist (meaning if $ab = 0$, then $a = 0$ or $b = 0$).
- **"Consider $aR \subseteq R$; it is a submodule of free module; so aR is a free R -module."**
 - R itself can be considered as a (finitely generated) free R -module of rank 1 (its basis is just the element $\{1\}$).
 - aR is the set $\{ar \mid r \in R\}$. This is an ideal in R , and every ideal is a submodule of R (when R is viewed as an R -module).
 - Since R is a finitely generated free module, and aR is a submodule of R , the hypothesis of the theorem applies: " aR is a free R -module."
- **"However, we know that b kills all of aR because $ab = 0$; this contradicts that aR is a free R -module."**
 - "Kills" means $b \cdot x = 0$ for all $x \in aR$. Let $x \in aR$. Then $x = ar_0$ for some $r_0 \in R$.
 - So, $b \cdot x = b(ar_0) = (ba)r_0$. Since we assumed $ab = 0$ (and R is commutative, so $ba = 0$), we have $(ba)r_0 = 0 \cdot r_0 = 0$.
 - So, $b \cdot x = 0$ for all $x \in aR$.
 - Now, why is this a contradiction to aR being a free R -module?
 - * If aR is a free R -module, it has a basis.
 - * Case 1: $aR = \{0\}$ (the zero module). This means $a \cdot 1 = a = 0$. But we started by assuming $a \in R \setminus \{0\}$ (i.e., $a \neq 0$). So this case is ruled out by our initial assumption about a .

- * Case 2: $aR \neq \{0\}$. Then aR is a non-zero free R -module. Let $\{e_k\}_{k \in K}$ be a basis for aR . Since $aR \neq \{0\}$, there is at least one basis element, say $e_1 \neq 0$.
- * Since $e_1 \in aR$, we know from above that $b \cdot e_1 = 0$.
- * If an element $b \neq 0$ annihilates a non-zero element e_1 in a free module, this means that the module cannot be free *unless* b forces all coefficients to be zero in some sense.
- * More precisely: A free module M over a ring R has the property that if $r \cdot m = 0$ for some $r \in R$ and $m \in M$, then either r is a zero divisor for elements related to m , or $m = 0$. If M is free with basis $\{e_k\}$, then any $m \in M$ can be written as $m = \sum c_k e_k$. If $r \cdot m = \sum r c_k e_k = 0$, then by linear independence, $r c_k = 0$ for all k .
- * Here, aR is free. If $aR \neq \{0\}$, it has a non-empty basis, say $\{\beta_j\}_{j \in J}$. Take any basis element β_j . We know $b \cdot \beta_j = 0$. Since $\beta_j \neq 0$ (it's a basis element of a non-zero module) and $b \neq 0$ (by assumption), this implies that b is a zero-divisor "acting on" β_j .
- * The crucial property of a free module M over a ring R is that if $r \in R$ annihilates *every* element of M (i.e., $r \cdot m = 0$ for all $m \in M$), and $M \neq \{0\}$, then r must be 0.
- * In our situation, $b \neq 0$ and $aR \neq \{0\}$ (since $a \neq 0$). We have $b \cdot x = 0$ for all $x \in aR$. This means b is a non-zero element that annihilates the entire non-zero module aR . This can only happen if aR is not actually free (unless R itself has issues). The only way a non-zero scalar can annihilate every element of a non-zero free module is if the module itself is the zero module (which we've ruled out because $a \neq 0$) or if the ring itself isn't "nice."
- * **The more direct contradiction:** A non-zero free module cannot have a non-zero annihilator. The set of elements $r \in R$ such that $r \cdot m = 0$ for all $m \in M$ is called the annihilator of M , denoted $\text{Ann}_R(M)$. If M is a free R -module and $M \neq \{0\}$, then $\text{Ann}_R(M) = \{0\}$. Here, $M = aR$. Since $a \neq 0$, $aR \neq \{0\}$ (as $a \cdot 1 = a \in aR$). We found that $b \in \text{Ann}_R(aR)$. But we assumed

$b \neq 0$. This means $\text{Ann}_R(aR) \neq \{0\}$, which contradicts the fact that aR is a non-zero free module.

- This contradiction shows that our initial assumption ($a, b \in R \setminus \{0\}$ with $ab = 0$) must be false. Therefore, R must be an integral domain.

Part 2: Let $I \subseteq R$ be an ideal; it is then a free R -submodule of R . We claim that I is a free module of rank 1 over R .

- "Let $I \subseteq R$ be an ideal; it is then a free R -submodule of R ."
 - An ideal I of R is, by definition, a submodule of R (when R is considered as an R -module).
 - As established before, R is a finitely generated free R -module (rank 1, basis $\{1\}$).
 - By the hypothesis of the theorem, any submodule of a finitely generated free module is free. Thus, I must be a free R -module.
- "We claim that I is a free module of rank 1 over R ." This means I should have a basis consisting of a single element. If $I = (\alpha)$ for some $\alpha \in I$, then $\{\alpha\}$ would be its basis (assuming $\alpha \neq 0$).
- "Suppose not, then there exist $(\alpha_j)_{j \in J}$ forming an R -basis of I . But if $\#J \geq 2$, we know that $\alpha_j \cdot (\alpha_i) + \alpha_i \cdot (-\alpha_j) = 0$, contradiction!"
 - Let's assume the rank of I (as a free R -module) is not 1.
 - * Could the rank be 0? If rank is 0, $I = \{0\}$, which is the principal ideal (0) . This fits the "rank 1 or 0" pattern, as (0) is generated by one element. So the argument is primarily against rank ≥ 2 .
 - * So, assume the rank is greater than or equal to 2. This means there are at least two distinct basis elements in any basis for I . Let α_i and α_j be two distinct elements from such a basis.
 - Consider the expression: $\alpha_j \cdot (\alpha_i) + \alpha_i \cdot (-\alpha_j)$.
 - * Here, α_i and α_j are elements of the ideal I , and also elements of the ring R .
 - * The coefficients are (α_j) and $(-\alpha_j)$, which are also elements of R .
 - * The expression is a linear combination of the basis elements α_i and α_j .

- * $\alpha_j \cdot \alpha_i + \alpha_i \cdot (-\alpha_j) = \alpha_j \alpha_i - \alpha_i \alpha_j.$
- * Since R is a commutative ring (this was stated at the beginning),
 $\alpha_j \alpha_i = \alpha_i \alpha_j.$
- * So, $\alpha_j \alpha_i - \alpha_i \alpha_j = 0.$
- This is a linear combination of basis elements that equals zero: $c_1 \alpha_i + c_2 \alpha_j = 0$, where $c_1 = \alpha_j$ and $c_2 = -\alpha_j$.
- For a set of elements to be a basis, they must be linearly independent. This means if a linear combination of them is zero, all coefficients must be zero.
- So, we must have $c_1 = \alpha_j = 0$ and $c_2 = -\alpha_j = 0$.
- But if $\alpha_j = 0$, then it cannot be a basis element of a free module (unless the module is the zero module, which would have an empty basis or rank 0, not ≥ 2). Basis elements are non-zero.
- **The contradiction:** We found a linear combination of distinct basis elements α_i, α_j which is $\alpha_j(\alpha_i) + (-\alpha_i)(\alpha_j) = 0$. For this to not violate linear independence, the coefficients must be zero. That is, $\alpha_j = 0$ and $-\alpha_i = 0$. But basis elements of a free module must be non-zero if the module itself is non-zero (and if I has rank ≥ 2 , it's non-zero).
- This specific relation $\alpha_j(\alpha_i) - \alpha_i(\alpha_j) = 0$ is always true in a commutative ring. If α_i, α_j are part of a basis for an ideal $I \subseteq R$, and these are considered as elements of the module I , the coefficients in the linear combination are $\alpha_j \in R$ and $-\alpha_i \in R$.
- If I is non-zero, and its basis has at least two elements α_1, α_2 . Then $\alpha_2 \cdot \alpha_1 + (-\alpha_1) \cdot \alpha_2 = 0$.
- For this to be a non-trivial relation, at least one of the coefficients α_2 or $-\alpha_1$ must be non-zero. Since α_1, α_2 are basis elements, they are non-zero (assuming $I \neq (0)$). So the coefficients α_2 and $-\alpha_1$ are non-zero (since R is now an integral domain, if $-\alpha_1 = 0$, then $\alpha_1 = 0$, which is not possible for a basis element).
- So we have a linear combination of basis elements equal to zero, where the coefficients are non-zero elements from R . This contradicts the definition of a basis for a free module over an integral domain R

(unless R was the zero ring, which it isn't as it has a 1 and is an integral domain).

- Therefore, the assumption that the rank of I is ≥ 2 must be false.
- ”This says that I is free of rank one, i.e. I is generated by one element $\alpha \in I$ so that $I = (\alpha)$ is a principal ideal.”
 - Since I is a free R -module and its rank cannot be ≥ 2 , its rank must be 0 or 1.
 - * If rank is 0, then $I = \{0\}$, which is the principal ideal (0) .
 - * If rank is 1, then I has a basis consisting of a single element, say $\{\alpha\}$, where $\alpha \in I$. This means every element $x \in I$ can be written uniquely as $x = r \cdot \alpha$ for some $r \in R$. This is precisely the definition of the principal ideal generated by α , i.e., $I = (\alpha)$.
 - So, every ideal I in R is a principal ideal.

Conclusion:

- We have shown that R is an integral domain (Part 1).
- We have shown that every ideal I in R is a principal ideal (Part 2).

By definition, an integral domain in which every ideal is principal is a **Principal Ideal Domain (PID)**. Thus, the proof is complete.

8.17.2 Problem 2

Exercice 40

Let k be a finite field with q elements.

- (1) How many monic irreducible polynomials are there in $k[x]$ of each degree $d = 2, 3, 4, 5, 6$?
- (2) What are the possible Galois groups of the splitting field of a (not necessarily irreducible) polynomial of degree 5 over k ? Why?

证明. (1) Each irreducible polynomial of degree 2 has exactly two zeros in $\mathbb{F}_{q^2} - \mathbb{F}_q$.

Note. Denote one zero of the irreducible polynomial $f(x)$ by α, β , then $\alpha \notin \mathbb{F}_q$, otherwise $\alpha \in \mathbb{F}_q \Rightarrow \beta \in \mathbb{F}_q$, thus $f(x) \in \mathbb{F}_q[x]$, which is a contradiction. Claim that $\alpha \in \mathbb{F}_{q^2}$. Since $m_{\alpha, \mathbb{F}_q}(x) | f(x)$, $\alpha \notin \mathbb{F}_q$, then $x - \alpha \notin \mathbb{F}_q[x]$, $\deg m_{\alpha, \mathbb{F}_q}(x) \geq 2$; but $\deg m_{\alpha, \mathbb{F}_q}(x) \leq \deg f(x) = 2$ and the minimal polynomial of α is unique (by definition); so $m_{\alpha, \mathbb{F}_q}(x) = f(x)$. $\mathbb{F}_q(\alpha)$ is the splitting field of $f(x)$ over \mathbb{F}_q , with degree 2. $[\mathbb{F}_q(\alpha), \mathbb{F}_q] = 2$, which means $\mathbb{F}_q(\alpha)$ is a 2-dimensional vector space over \mathbb{F}_q ; $\mathbb{F}_q(\alpha) = \mathbb{F}_q \oplus \mathbb{F}_q$, thus $|\mathbb{F}_q(\alpha)| = q^2$. Claim that $\mathbb{F}_q(\alpha) \cong \mathbb{F}_{q^2}$. This refers to the construction of \mathbb{F}_{q^2} , more generally, \mathbb{F}_{q^n} , which is the splitting field of $x^{q^n} - x$ over \mathbb{F}_q . It suffices to verify that every field with q^2 elements is a⁵ splitting field of $x^{q^2} - x$ over \mathbb{F}_q . Consider the multiplicative group, $\mathbb{F}_q(\alpha)^\times$; it has $q^2 - 1$ elements. For any $a \in \mathbb{F}_q(\alpha)^\times$, by Lagrange's theorem from group theory, $q^2 - 1 = |\mathbb{F}_q(\alpha)^\times| = [\mathbb{F}_q(\alpha)^\times : a] \cdot |\langle a \rangle|$ is divided by $|a|$, the order of a ; thus $a^{q^2-1} = 1$; a is the solution of $x^{q^2-1} - 1 = 0$. Including 0, then any element in $\mathbb{F}_q(\alpha)$ is the solution of $x^{q^2} - x = 0$, which has q^2 distinct zeros. Since $\mathbb{F}_q(\alpha)$ also has q^2 distinct elements, they correspond to q^2 different roots of $x^{q^2} - x = 0$. By the definition of \mathbb{F}_{q^2} , $\mathbb{F}_q(\alpha) \cong \mathbb{F}_{q^2}$; thus $\alpha \in \mathbb{F}_{q^2}$; similarly, $\beta \in \mathbb{F}_{q^2}$.

So there are $\frac{q^2-q}{2}$ irreducible polynomials of degree 2.

Note. This is because, every element in the extension of \mathbb{F}_q has a unique minimal polynomial $m_{\alpha, \mathbb{F}_q}(x)$, which is the definition of minimal polynomial. We know that $f(x)$ is the minimal polynomial of exactly two elements, α, β in $\mathbb{F}_{q^2} - \mathbb{F}_q$; each irreducible polynomial of degree 2 corresponds to 2 different elements in $\mathbb{F}_{q^2} - \mathbb{F}_q$. Thus there are $\frac{q^2-q}{2}$ irreducible polynomials of degree 2. We are done!

Similarly, there are $\frac{q^3-q}{3}$ irreducible polynomials of degree 3, and there are $\frac{q^5-q}{5}$ irreducible polynomials of degree 5.

Each irreducible polynomial of degree 4 corresponds to four elements of $\mathbb{F}_{q^4} - \mathbb{F}_{q^2}$; so there are $\frac{q^4-q^2}{4}$ irreducible polynomials of degree 4.

Each irreducible polynomial of degree 6 corresponds to six elements of $\mathbb{F}_{q^6} - (\mathbb{F}_{q^2} \cup \mathbb{F}_{q^3})$; there are $\frac{q^6-q^3-q^2+q}{6}$ such polynomials.

⁵"a" means "the", since the splitting field is unique up to isomorphism.

Note. We know that

$$|\mathbb{F}_{q^2} \cup \mathbb{F}_{q^3}| = |\mathbb{F}_{q^2}| + |\mathbb{F}_{q^3}| - \underbrace{|\mathbb{F}_{q^2} \cap \mathbb{F}_{q^3}|}_{=\mathbb{F}_q} = q^2 + q^3 - q$$

(2) The factorization of f into irreducibles corresponds to partitions of 5:

If f is irreducible, the splitting field is \mathbb{F}_{q^5} . The associated Galois group is Z_5 .

If f factors as the product of a degree 2 and a degree 3 polynomials, the splitting field is \mathbb{F}_{q^6} . The associated Galois group is Z_6 .

In other cases, we can get similarly $\mathbb{F}_{q^4}, \mathbb{F}_{q^3}, \mathbb{F}_{q^2}$, and \mathbb{F}_q . The associated Galois groups are $Z_4, Z_3, Z_2, \{1\}$, respectively. \square

8.17.3 A UFD property

Proposition 11.2.7. In a UFD, the gcd of nonzero element exists. Namely, for two nonzero element $a, b \in R$, there exists an element $d = \gcd(a, b) \in R$ such that if $d' \in R$ satisfies $d' | a$ and $d' | b$, then $d' | d$.

Explicitly, if a and b factor as

$$a = up_1^{c_1} \cdots p_r^{c_r} \quad \text{and} \quad b = vp_1^{d_1} \cdots p_r^{d_r}$$

with p_1, \dots, p_r irreducible and pairwise non-associate, and $u, v \in R^\times$, $c_i, d_i \in \mathbb{Z}_{\geq 0}$, then

$$d = p_1^{\min(c_1, d_1)} \cdots p_r^{\min(c_r, d_r)}$$

is a gcd of a and b .

Problem 3

Exercice 41

Suppose that R is a UFD for which every nonzero prime ideal is maximal.

Show that R is a PID.

证明. We first prove that for two prime elements p and q , either they are associates, or there exists $a, b \in R$ such that $ap + bq = 1$. Indeed, if p and q are not associates, the ideals (p) and (q) cannot have containment relations (otherwise, say $(p) \subseteq (q)$, we must have $q | p$; which would immediately forces p and q to be associates). Now as nonzero prime ideals (such as (p) and (q)) are maximal, the ideal (p, q) must be the unit ideal, i.e. there exists $a, b \in R$ such that $ap + bq = 1$.

Next, we show that if, in the factorization of two elements $c, d \in R$, no prime factors of c are associates of prime factors of d , then there exists $a, b \in R$ such that $ac + bd = 1$. By induction, it suffices to prove that: if $(p_1, q) = (p_2, q) = (1)$, then $(p_1 p_2, q) = (1)$. Indeed, write $\lambda_1 p_1 + \mu_1 q = 1$ and $\lambda_2 p_2 + \mu_2 q = 1$ for $\lambda_1, \lambda_2, \mu_1, \mu_2 \in R$, then

$$\lambda_1 \lambda_2 p_1 p_2 = (1 - \mu_1 q)(1 - \mu_2 q) = 1 - (\mu_1 + \mu_2 - \mu_1 \mu_2 q)q$$

This implies that $(p_1 p_2, q) = (1)$.

We finally prove that R is a PID. Let I be a nonzero ideal. Pick an element $x \in I$ with minimal number of prime factors. We show that $I = (x)$. If $y \in I \setminus (x)$, then write $d = \gcd(x, y)$ and $x = dx_d$ and $y = dy_d$ with $x_d, y_d \in R$, and x_d, y_d have distinct prime factors. By the discussion above, there exist $a, b \in R$ such that $ax_d + by_d = 1$. This implies that $d \in I$, contradicting with the minimality of prime factors of $x \in I$. Thus I is a principal ideal.

□

Proposition 4

PID is Bézout integral domain; UFD may not.

Définition 14: Bézout domain

A **Bézout domain** is an integral domain in which every finitely generated ideal is principal.

Note. Equivalently, an integral domain is called Bézout domain provided that for any $a, b \in R$, $\gcd(a, b) =: d$ exists, i.e. $(a, b) = (d)$, i.e. $\exists x, y \in R$ s.t.

$$ax + by = d$$

By the definition, PID is Bézout domain. But there exists non-Bézout UFD, e.g. $\mathbb{Z}[x]$. Consider $2, x$ in $\mathbb{Z}[x]$; UFD admits gcd; $\gcd(2, x) = 1$, but no $p(x), q(x)$ satisfy $2 \cdot p(x) + x \cdot q(x) = 1$, as letting $x = 0$, then $2 \cdot p(0) = 1$. (a contradiction!)

8.18 Elementary Field Theory

Exercice 42

Question 3. Let k be an imperfect field of characteristic $p > 0$. Let $a \in k \setminus k^p$.

- (1) Show that the polynomial $X^p - a \in k[X]$ is irreducible.
- (2) Let $A = k[X]/(X^{p^2} - aX^p)$. Compute A_{red} , the quotient of A by its nilpotent radical.

A **perfect field** K can be characterized in several equivalent ways:

- (1) Separable Extensions: Every algebraic extension of K is a separable extension. (An algebraic extension is separable if the minimal polynomial of every element in the extension over K has distinct roots.)
- (2) Separable Polynomials: Every irreducible polynomial in $K[x]$ (the ring of polynomials with coefficients in K) is separable (has distinct roots in its splitting field).
- (3) Characteristic Property:
 - If K has characteristic 0 (like $\mathbb{Q}, \mathbb{R}, \mathbb{C}$), it is always perfect.
 - If K has characteristic $p > 0$ (where p is a prime number), then K is perfect if and only if the Frobenius endomorphism $F : K \rightarrow K$, defined by $F(a) = a^p$ for all $a \in K$, is surjective. This means that every element in K is a p -th power of some element in K (i.e., $K^p = K$). Since the Frobenius endomorphism is always injective for fields, this condition means it must be an automorphism.

Examples of perfect fields include:

- All fields of characteristic 0 (e.g., the rational numbers \mathbb{Q} , the real numbers \mathbb{R} , the complex numbers \mathbb{C}).
- All finite fields (e.g., $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for a prime p , or \mathbb{F}_{p^n}). In finite fields, any injective map from the field to itself must also be surjective.
- All algebraically closed fields.

Now, an **imperfect field** is a field that does not satisfy these conditions.

Based on the above:

- An imperfect field must have characteristic $p > 0$ for some prime p .

- In an imperfect field K of characteristic $p > 0$, the Frobenius endomorphism $a \mapsto a^p$ is not surjective. This means there exists at least one element in K that cannot be written as the p -th power of any element in K (i.e., $K^p \neq K$).
- Imperfect fields have at least one irreducible polynomial that is inseparable (i.e., has multiple roots in an extension field).
- There exist algebraic extensions of an imperfect field that are not separable.

8.19 p -adic numbers

See [Zuniga.pdf](#)

Définition 15: p -adic norm

Let p be a fixed prime number, and let x be a nonzero rational number. Then, $x = p^k \frac{a}{b}$, with $p \nmid ab$, and $k \in \mathbb{Z}$. The **p -adic absolute value** (or **p -adic norm**) of x is defined as

$$|x|_p = \begin{cases} p^{-k} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Définition 16: non-Archimedean metric

Let (X, d) be a metric space. The metric d is called **non-Archimedean** if

$$d(x, y) \leq \max \{d(x, z), d(z, y)\} \text{ for any } x, y, z \in X.$$

Exercice 43

Let $(F, |\cdot|)$ be a valued field, where $|\cdot|$ is a non-Archimedean absolute value. Assume that F is complete with respect to $|\cdot|$. Then, the series $\sum_{k \geq 0} a_k, a_k \in F$ converges if and only if $\lim_{k \rightarrow \infty} |a_k| = 0$.

Théorème 39: Ostrowski

Any non trivial absolute value on \mathbb{Q} is equivalent to $|\cdot|_p$ or to the standard absolute value $|\cdot|_\infty$.

8.19.1 The field of p -adic numbers**Lemme 14**

Consider the set

$$\mathbb{Q}_p := \left\{ x = p^\gamma \sum_{i=0}^{\infty} x_i p^i : \gamma \in \mathbb{Z}, x_i \in \{0, 1, \dots, p-1\}, x_0 \neq 0 \right\} \cup \{0\}$$

endowed with the p -adic norm $|\cdot|_p$. Then, the following assertions hold.

- (1) $(\mathbb{Q}_p, |\cdot|_p)$ is a complete metric space;
- (2) \mathbb{Q} is dense in \mathbb{Q}_p ;
- (3) \mathbb{Q}_p is a field of characteristic zero;
- (4) the completion of $(\mathbb{Q}, |\cdot|_p)$ is $(\mathbb{Q}_p, |\cdot|_p)$.

(i) I don't understand.

(ii) We set

$$\mathbb{Z}_p := \left\{ x \in \mathbb{Q}_p : x = p^\gamma \sum_{i=0}^{\infty} x_i p^i; \gamma \in \mathbb{N}, x_0 \neq 0 \right\}.$$

Then any $x \in \mathbb{Q}_p \setminus \{0\}$ can be written as $x = p^\gamma \tilde{x}$, with $\tilde{x} \in \mathbb{Z}_p$ and $|\tilde{x}|_p = 1$.

Given p^{-L} , $L > \gamma$, we have to show the existence of $\frac{a}{b} \in \mathbb{Q}$ such that $|x - \frac{a}{b}|_p < p^{-L}$. We take $b^{-1} = p^\gamma$ and $a \in \mathbb{Z}$ satisfying $|a - \tilde{x}|_p < p^{-L+\gamma}$.

(iii) omitted

(iv) combine (i), (ii), (iii).

The field of p -adic numbers \mathbb{Q}_p is defined as the completion of \mathbb{Q} w.r.t. the distance induced by $|\cdot|_p$.

The unit ball

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\} = \left\{ x \in \mathbb{Q}_p : x = \sum_{i=i_0}^{\infty} x_i p^i, i_0 \geq 0 \right\}$$

is a PID. Any ideal of \mathbb{Z}_p has the form

$$p^m \mathbb{Z}_p = \left\{ x \in \mathbb{Z}_p : x = \sum_{i \geq m} x_i p^i \right\}, \quad m \in \mathbb{N}$$

Indeed, let $I \subseteq \mathbb{Z}_p$ be an ideal. Set $m_0 = \min_{x \in I} \text{ord}(x) \in \mathbb{N}$, and let $x_0 \in I$ such that $\text{ord}(x_0) = m_0$. Then $I = x_0 \mathbb{Z}_p$.

From a geometric point of view, the ideals of the form $p^m \mathbb{Z}_p$, $m \in \mathbb{Z}$, constitute a fundamental system of neighborhoods around the origin in \mathbb{Q}_p .

The *residue field* of \mathbb{Q}_p is $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$ (the finite field with p elements). The group of units of \mathbb{Z}_p is

$$\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p : |x|_p = 1\}.$$

Exercice 44

$x = x_0 + x_1 p + \dots \in \mathbb{Z}_p$ is a unit iff $x_0 \neq 0$. Moreover if $x \in \mathbb{Q}_p \setminus \{0\}$, then $x = p^m u$, $m \in \mathbb{Z}$, $u \in \mathbb{Z}_p^\times$.

8.19.2 Topology of \mathbb{Q}_p

Define

$$B_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^r\}, r \in \mathbb{Z}$$

as the ball with center a and radius p^r , and

$$S_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p = p^r\}, r \in \mathbb{Z}$$

as the sphere with center a and radius p^r .

The radius are always integer powers of p .

Remarque 37. Notice that $B_r(a) = a + p^{-r} \mathbb{Z}_p$ and $S_r(a) = a + p^{-r} \mathbb{Z}_p^\times$.

We declare that the $B_r(a)$, $r \in \mathbb{Z}$, $a \in \mathbb{Q}_p$, are open subsets. These sets form a basis for the topology of \mathbb{Q}_p .

Proposition 5

$S_r(a), B_r(a)$ are open and closed sets in the topology of \mathbb{Q}_p .

证明. We first show that $S_r(a)$ is open. Notice that

$$\begin{aligned} S_r(a) &= a + p^{-r} \underbrace{\mathbb{Z}_p^\times}_{\substack{x_0 \\ \neq 0}} + x_1 p + \cdots \in \underbrace{\mathbb{Z}_p^\times}_{\sum_{i=0}^{\infty} y_i p^i \in} \\ &= \bigsqcup_{x_0=1,2,\dots,p-1} a + x_0 p^{-r} + p^{-r+1} \underbrace{\mathbb{Z}_p^\times}_{\sum_{i=0}^{\infty} y_i p^i \in} \\ &= \bigsqcup_{i \in \{1,\dots,p-1\}} \underbrace{B_{(r-1)}(a + p^{-r} i)}_{\text{open}} \end{aligned}$$

Consequently, $S_r(a)$ is open.

In order to show that $S_r(a)$ is closed, we take a sequence $\{x_n\}_{n \in \mathbb{N}}$ of points of $S_r(a)$ converging to $\tilde{x}_0 \in \mathbb{Q}_p$. We must show that $\tilde{x}_0 \in S_r(a)$. Note that $x_n = a + p^{-r} u_n$, $u_n \in \mathbb{Z}_p^\times$. Since $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence, we have

$$|x_n - x_m|_p = p^r |u_n - u_m|_p \rightarrow 0, \quad n, m \rightarrow \infty,$$

thus $\{u_n\}_{n \in \mathbb{N}}$ is also Cauchy, and since \mathbb{Q}_p is complete $u_n \rightarrow \tilde{u}_0$. Then $x_n \rightarrow a + p^{-r} \tilde{u}_0$, so in order to conclude our proof we must verify that $\tilde{u}_0 \in \mathbb{Z}_p^\times$. Because u_m is arbitrarily close to \tilde{u}_0 , their p -adic expansions must agree up to a big power of p , hence $\tilde{u}_0 \in \mathbb{Z}_p^\times$.

A similar argument shows that $B_r(a)$ is closed. \square

\square

Lemme 15

If $b \in B_r(a)$ then $B_r(b) = B_r(a)$, i.e. any point of the ball $B_r(a)$ is its center.

证明. Let $x \in B_r(b)$, then

$$|x - a|_p = |x - b + b - a|_p \leq \max\{|x - b|_p, |b - a|_p\} \leq p^r,$$

i.e. $B_r(b) \subseteq B_r(a)$.

Since $a \in B_r(b)$ (i.e. $|b - a|_p = |a - b|_p \leq p^r$), we can repeat the previous argument to show that $B_r(a) \subseteq B_r(b)$. \square

Exercice 45

Any two balls in \mathbb{Q}_p are either disjoint or one is contained in another.

Exercice 46

The boundary of any ball is the empty set.

Théorème 40

A set $K \subset \mathbb{Q}_p$ is compact iff closed and bounded.

8.19.3 The n -dimensional p -adic space

We extend the p -adic norm to \mathbb{Q}_p^n by taking

$$\|x\|_p := \max_{1 \leq i \leq d} |x_i|_p, \text{ for } x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$$

We define $\text{ord}(x) = \min_{1 \leq i \leq n} \{\text{ord}(x_i)\}$, then $\|x\|_p = p^{-\text{ord}(x)}$.

$$B_r^n(a) = \{x \in \mathbb{Q}_p^n : \|x - a\|_p \leq p^r\} = B_r(a_1) \times \cdots \times B_r(a_n)$$

$$S_r^n(a) = \{x \in \mathbb{Q}_p^n : \|x - a\|_p = p^r\}$$

Note that $S_0^1 = \mathbb{Z}_p^\times$ but $(\mathbb{Z}_p^\times)^n \subsetneq S_0^n$.

As a topological space $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is totally disconnected, i.e. the only connected subsets of \mathbb{Q}_p^n are the empty set and the points.

8.19.4 Exercises**Exercice 47: yau-2020-problem-5**

Problem 5. Let $p \geq 3$ be a prime number and let \mathbb{Z}_p be the ring of p -adic integers.

- (a) Show that an element in $1 + p\mathbb{Z}_p$ is a p -th power in \mathbb{Z}_p if and only if it lives in $1 + p^2\mathbb{Z}_p$.
- (b) Let \mathbb{Z}_p^\times denote the group of units in \mathbb{Z}_p . Show that there exist $a, b, c \in \mathbb{Z}_p^\times$ such that $a^p + b^p = c^p$ if and only if

$$\sum_{i=1}^{p-1} i^{p-2} t^i \equiv 0 \pmod{p}$$

for some integer $t \in \{2, 3, \dots, p-1\}$. (In particular, this condition holds for $p=7$ by taking $t=3$. Therefore, Fermat's Last Theorem does not hold for \mathbb{Z}_7 .)

Solution: Part (a): If an element in $1+p\mathbb{Z}_p$ is a p -th power, it must have form $(1+p\alpha)^p$ for some $\alpha \in \mathbb{Z}_p$. A simple calculation yields

$$(1+p\alpha)^p = 1 + \binom{p}{1}p\alpha + \binom{p}{2}(p\alpha)^2 + \dots \in 1+p^2\mathbb{Z}_p.$$

To prove sufficiency, recall the two functions

$$\exp : p\mathbb{Z}_p \rightarrow 1+p\mathbb{Z}_p, \quad \log : 1+p\mathbb{Z}_p \rightarrow p\mathbb{Z}_p$$

which are inverses to each other. For any $a = 1+p^2x \in 1+p^2\mathbb{Z}_p$, consider

$$a^{\frac{1}{p}} := \exp\left(\frac{1}{p}\log(a)\right).$$

Notice that

$$\frac{1}{p}\log(a) = \frac{1}{p}\log(1+p^2x) = \frac{1}{p}\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} (p^2x)^i \in p\mathbb{Z}_p$$

and hence $a^{\frac{1}{p}}$ is well-defined. It is clear that $\left(a^{\frac{1}{p}}\right)^p = a$.

Part (b): As an immediate corollary from Part (a), if we write an element $a \in \mathbb{Z}_p^\times$ in terms of Witt coordinates $a = (a_0, a_1, \dots)$, then a is a p -th power in \mathbb{Z}_p if and only if $a_1 = 0$. In particular, whether an element in \mathbb{Z}_p^\times is a p -th power can be detected by its image under the projection $\mathbb{Z}_p = W(\mathbb{F}_p) \rightarrow W_2(\mathbb{F}_p)$.

Hence, there exist $a, b, c \in \mathbb{Z}_p^\times$ such that $a^p + b^p = c^p$ if and only if there exist $a_0, b_0, c_0 \in \mathbb{F}_p^\times$ such that $(a_0, 0) + (b_0, 0) = (c_0, 0)$ in $W_2(\mathbb{F}_p)$. Using the addition formula of Witt coordinates, the later equation translates to $a_0 + b_0 = c_0$ and

$$\frac{1}{p}(a_0^p + b_0^p - (a_0 + b_0)^p) = 0.$$

Direct calculation gives

$$\begin{aligned} \frac{1}{p}(a_0^p + b_0^p - (a_0 + b_0)^p) &= -\sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} a_0^i b_0^{p-i} \\ &= -\sum_{i=1}^{p-1} \frac{1}{i} \frac{(p-1)(p-2)\cdots(p-i+1)}{(i-1)\cdots 1} a_0^i b_0^{p-i} \\ &\equiv \sum_{i=1}^{p-1} \frac{1}{i} (-1)^i a_0^i b_0^{p-i} \equiv \sum_{i=1}^{p-1} i^{p-2} \left(-\frac{a_0}{b_0}\right)^i \pmod{p} \end{aligned}$$

Since $a_0 + b_0 = c_0 \neq 0$, we have $-\frac{a_0}{b_0} \neq 1$. Namely, there exists $t \in \{2, 3, \dots, p-1\}$ such that

$$\sum_{i=1}^{p-1} i^{p-2} t^i \equiv 0 \pmod{p}.$$

All steps above are clearly reversible and hence cover both the “if” and “only if” parts. This completes the proof.

8.20 Integral ring extension

See 肖梁

Today, all rings are commutative.

22.1. **Integral ring extension.** Recall from the field extensions, we have proved that for a field extension K/F ,

$$K/F \text{ is a finite extension} \iff K/F \text{ is finitely generated and algebraic.}$$

We develop the corresponding theory for rings.

Definition 22.1.1. Let $A \subseteq B$ be a subring. An element $x \in B$ is called **integral** over A if it satisfies a *monic* equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

for some $a_0, \dots, a_{n-1} \in A$.

We point out that, since the polynomial ring over a general ring is no longer a PID, we do not have the notion of “minimal polynomial” here.

Proposition 22.1.2. *The following are equivalent.*

- (1) $x \in B$ is integral over A ;
- (2) $A[x]$ (*= ring of all elements in B that can be expressed by a polynomial in x with coefficients in A*) is a finitely generated A -module;
- (3) $A[x]$ is contained in a subring C of B such that C is a finitely generated A -module.

Proof. (1) \Rightarrow (2). Assume that x satisfies $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ for $a_0, \dots, a_{n-1} \in A$. So each x^{n+r} for $r \in \mathbb{Z}_{\geq 0}$ may be replaced by $-a_{n-1}x^{n+r-1} - \cdots - a_0x^r$. From this, we see that $A[x]$ is generated by $1, x, \dots, x^{n-1}$ as an A -module.

(2) \Rightarrow (3) Take $C = A[x]$.

(3) \Rightarrow (1) Assume that C is generated by e_1, \dots, e_n as an A -module (not necessarily a basis; so there might be relations). We may write each xe_j (for $j = 1, \dots, n$) as an A -linear combination of this set of generators, i.e.

$$xe_j = a_{1j}e_1 + a_{2j}e_2 + \cdots + a_{nj}e_n \quad \text{for } a_{1j}, \dots, a_{nj} \in A.$$

(There might be more than one way to write xe_j ; we take any such expression.) Writing this collectively, we have

$$(e_1, \dots, e_n)x = (e_1, \dots, e_n) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$(e_1, \dots, e_n) \begin{pmatrix} x - a_{11} & -a_{12} & \cdots & -a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & x - a_{nn} \end{pmatrix} = 0.$$

Write S for the matrix on the right. By Cayley–Hamilton theorem, $\det(S)$ kills all elements e_1, \dots, e_n . But 1 is a linear combination of e_1, \dots, e_n . So $\det(S) = 0$; this shows that x is integral over A . \square

Corollary 22.1.3. Let x_1, \dots, x_n be elements of B , each integral over A . Then $A[x_1, \dots, x_n]$ is a finitely generated A -module.

Proof. For each i , assume that $x_i^{m_i} + a_{i,m_i-1}x_i^{m_i-1} + \dots + a_{i,0} = 0$ for some $m_i \in \mathbb{N}$ and $a_{i,j} \in A$. Then $A[x_1, \dots, x_m]$ is generated as an A -module by monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with each $\alpha_i \in \{0, \dots, m_i - 1\}$. In particular $A[x_1, \dots, x_n]$ is a finitely generated A -module. \square

Corollary 22.1.4. The set C of elements of B which are integral over A is a subring of B containing A .

Proof. Given $x, y \in C$, the previous corollary implies that $A[x, y]$ is a finitely generated A -module. By Proposition 22.1.2, $x \pm y, xy \in A[x, y]$ are integral over A ; so $x \pm y, xy \in C$. Thus, C is a subring of B . \square

Definition 22.1.5. This C in Corollary 22.1.4 is called the **integral closure** of A in B .

- (1) If $C = A$, we say that A is **integrally closed** in B .
- (2) If $C = B$, we say that B is **integral over** A .

Corollary 22.1.6. If $A \subseteq B \subseteq C$ are rings and if B is integral over A and C is integral over B , then C is integral over A .

Proof. (Compare with the proof of Theorem 14.4.14.) Let $x \in C$, the integrality implies that $x^n + b_{n-1}x^{n-1} + \dots + b_0 = 0$ with $b_0, \dots, b_{n-1} \in B$. Consider the subring $B' = A[b_0, \dots, b_{n-1}] \subseteq B$. See the following diagram.

$$\begin{array}{ccc} C & \supseteq & B'[x] \\ | & & | \\ B & \supseteq & B' = A[b_0, \dots, b_{n-1}] \\ | & \nearrow & | \\ A & & \end{array}$$

This B' is a finitely generated A -module as each b_0, \dots, b_{n-1} is integral over A . Then $B'[x]$ is a finitely generated A -module, and hence x is integral over A . \square

Corollary 22.1.7. Let $A \subseteq B$ be rings and let C be the integral closure of A in B . Then C is integrally closed in B .

Proof. If $x \in C$ is integral over B , then x is integral over A by the previous corollary. Thus $x \in C$. \square

8.20.1 Determinant Trick

Théorème 41: The Determinant Trick

Let R be a commutative ring. Let M be an R -module. Let $A = (a_{ij})$ be

an $n \times n$ matrix with entries $a_{ij} \in R$. Let $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ be a column vector

where each $v_i \in M$. Suppose that $A\mathbf{v} = \mathbf{0}$, i.e.,

$$\sum_{j=1}^n a_{ij}v_j = 0 \quad \text{for each } i = 1, \dots, n$$

Then, $\det(A) \cdot v_k = 0$ for all $k = 1, \dots, n$. In other words, the determinant of the matrix A annihilates each element v_k of the vector \mathbf{v} .

证明. (1) **Recall the Adjugate Matrix:** For any $n \times n$ matrix A with entries in a commutative ring R , its adjugate (or classical adjoint), denoted $\text{adj}(A)$, satisfies the property:

$$\text{adj}(A) \cdot A = \det(A) \cdot I_n$$

where I_n is the $n \times n$ identity matrix. The entries of $\text{adj}(A)$ are also in R .

(2) **Multiply the System by the Adjugate:** We are given the system of equations $A\mathbf{v} = \mathbf{0}$. Multiply both sides of this equation on the left by $\text{adj}(A)$:

$$\text{adj}(A)(A\mathbf{v}) = \text{adj}(A) \cdot \mathbf{0}$$

(3) **Simplify:** The right side is simply the zero vector:

$$\text{adj}(A)(A\mathbf{v}) = \mathbf{0}$$

Using the associative property of matrix multiplication and the property of the adjugate:

$$(\text{adj}(A)A)\mathbf{v} = \mathbf{0}$$

$$(\det(A)I_n)\mathbf{v} = \mathbf{0}$$

(4) **Conclusion:** The product $(\det(A)I_n)\mathbf{v}$ is the vector:

$$\begin{pmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det(A) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \det(A)v_1 \\ \det(A)v_2 \\ \vdots \\ \det(A)v_n \end{pmatrix}$$

Since this vector is equal to the zero vector $\mathbf{0}$, each of its components must be zero:

$$\det(A) \cdot v_k = 0 \quad \text{for all } k = 1, \dots, n$$

□

This trick is a cornerstone in various proofs in module theory, including versions of Nakayama's Lemma and the Cayley-Hamilton theorem for modules over commutative rings. It elegantly connects the "linear algebra" of matrices over a ring with the structure of modules over that ring.

Définition 17: Adjoint of A

Let A be a matrix with entries in a ring R . The **adjoint** of A , denoted $\text{adj}(A)$, is the transpose of the matrix of cofactors of A .

In ring theory, the definition of the cofactor of a matrix is analogous to its definition in linear algebra over fields. Let A be an $n \times n$ matrix with entries in a commutative ring R .

The (i, j) -cofactor of A , denoted as C_{ij} , is defined as $(-1)^{i+j}$ times the determinant of the $(n - 1) \times (n - 1)$ matrix formed by deleting the i -th row and j -th column of A .

Specifically:

- (1) Let A_{ij} be the $(n - 1) \times (n - 1)$ matrix obtained by removing the i -th row and the j -th column from A .
- (2) The (i, j) -cofactor is $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

The matrix of cofactors is the matrix whose (i, j) -entry is C_{ij} . The adjugate (or classical adjoint) of A , denoted as $\text{adj}(A)$, is the transpose of the cofactor matrix. That is, the (i, j) -entry of $\text{adj}(A)$ is C_{ji} .

第九章 实变函数

9.1 Cardinality

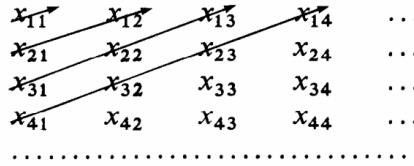
Théorème 1: theorem 2.12 in baby rudin

Let $\{E_n\}, n = 1, 2, \dots$ be a sequence of countable sets, and put

$$S = \bigcup_{n=1}^{\infty} E_n$$

Then S is countable.

Proof Let every set E_n be arranged in a sequence $\{x_{nk}\}, k = 1, 2, 3, \dots$, and consider the infinite array



in which the elements of E_n form the n th row. The array contains all elements of S . As indicated by the arrows, these elements can be arranged in a sequence

$$x_{11}; x_{21}, x_{12}; x_{31}, x_{22}, x_{13}; x_{41}, x_{32}, x_{23}, x_{14}; \dots$$

If any two of the sets E_n have elements in common, these will appear more than once in (17). Hence there is a subset T of the set of all positive integers such that $S \sim T$, which shows that S is at most countable (Theorem 2.8). Since $E_1 \subset S$, and E_1 is infinite, S is infinite, and thus countable.

Théorème 1 can be written as

$$\underbrace{\aleph_0 + \dots + \aleph_0}_{\aleph_0} = \aleph_0 \times \aleph_0 = \aleph_0$$

Théorème 2: theorem 2.13 in baby rudin

Let A be a countable set, and let B_n be the set of all n -tuples (a_1, \dots, a_n) , where $a_k \in A$ ($k = 1, \dots, n$) and the elements a_1, \dots, a_n need not be distinct. Then B_n is countable.

Théorème 2 is proved by induction from Théorème 1, which means

$$\underbrace{\aleph_0 \times \aleph_0 \times \dots \times \aleph_0}_n = \aleph_0$$

But

$$\underbrace{\aleph_0 \times \aleph_0 \times \cdots \times \aleph_0}_{\aleph_0} = \aleph_0^{\aleph_0} \sim \mathbb{R}$$

In particular, we have

Théorème 3: theorem 2.14 in baby rudin

Let A be the set of all sequences whose elements are the digits 0 and 1. This set A is uncountable. The elements of A are sequences like 1, 0, 0, 1, 0, 1, 1, 1, ...

The proof is very classic to show that A is uncountable.

Let E be a countable subset of A , and let E consist of the sequences s_1, s_2, s_3, \dots . We construct a sequence s as follows. If the n th digit in s_n is 1, we let the n th digit of s be 0, and vice versa. **Then the sequence s differs from every member of E in at least one place**; hence $s \notin E$. But clearly $s \in A$, so that E is a proper subset of A .

We have shown that every countable subset of A is a proper subset of A . It follows that A is uncountable (for otherwise A would be a proper subset of A , which is absurd).

Example

It should be noted that the mapping

$$\{0, 1\}^\infty \rightarrow [0, 1] \quad (n_1, n_2, \dots) \mapsto \sum_{k=1}^{\infty} \frac{n_k}{2^k}$$

is surjective but not injective. We denote the sets by

$$A = \{(\dots, 1, \underbrace{0}_i, 1, \dots), (\dots, 0, \underbrace{1}_i, 0, \dots) : i \in \mathbb{N}\}$$

$$B = \left\{ \frac{n}{2^m} : m \in \mathbb{N}, n \in \mathbb{N}, n \leq 2^m \right\}$$

It's easy to check that A, B are countable. Then

$$\{0, 1\}^\infty \setminus A \rightarrow [0, 1] \setminus B \quad (n_1, n_2, \dots) \mapsto \sum_{k=1}^{\infty} \frac{n_k}{2^k}$$

is injective, thus bijective. Therefore

$$\{0, 1\}^\infty \sim A \cup (\{0, 1\}^\infty \setminus A) \sim \{0, 1\}^\infty \setminus A \sim [0, 1] \setminus B \sim B \cup ([0, 1] \setminus B) \sim [0, 1] \sim \mathbb{R}$$

9.2 Outer measure: to construct Lebesgue measure

$\{I_k\}_{k=1}^{\infty}$ is a countable collection of nonempty open, bounded intervals that covers A . Define the **outer measure** of A to be

$$m^*(A) := \inf \left\{ \sum_{k=1}^{\infty} l(I_k) : A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}$$

The proof of countably subadditivity is very classic, i.e. check that

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k)$$

By the definition of outer measure, for any given $\epsilon > 0$, there is a collection $\{I_{k,i}\}_{i=1}^{\infty}$ for each k , such that

$$m^*(E_k) \leq \sum_{i=1}^{\infty} l(I_{k,i}) + \epsilon/2^k$$

Then $\{I_{k,i}\}_{k,i}$ is a open cover of $\bigcup_{k=1}^{\infty} E_k$, thus by the definition of $m^*(\bigcup_{k=1}^{\infty} E_k)$,

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{i,k} l(I_{k,i}) + \epsilon = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} l(I_{k,i}) + \epsilon/2^k \right) = \sum_{k=1}^{\infty} m^*(E_k)$$

9.2.1 Definition of measurable set

A set E is said to be **measurable** provided for any set A ,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C)$$

Clearly, LHS \leq RHS, it suffices to check that

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^C)$$

Clearly any set of outer measure 0 is measurable, since $m^*(A \cap E) \leq m^*(E) = 0$, $m^*(A \cap E^C) \leq m^*(A)$.

The proof of measurability preserving under countable union is classic. Let E be a countable union of measurable sets $\{E_k\}_{k=1}^{\infty}$. WLOG, assume that $\{E_k\}$ are disjoint, then the proof is routine.

The collection \mathcal{M} of measurable sets is a sigma-algebra containing all the Borel sets, i.e. each G_δ and F_σ sets.

9.2.2 Excision property

If A is a measurable set of finite outer measure contained in B , then

$$\underbrace{m^*(B \sim A)}_{=m^*(B \cap A^C)} = m^*(B) - \underbrace{m^*(A)}_{=m^*(B \cap A)}$$

9.2.3 Measurability: Approximation by G_δ (outer) and F_σ

(inner) sets

Theorem 11 Let E be any set of real numbers. Then each of the following four assertions is equivalent to the measurability of E .

(Outer Approximation by Open Sets and G_δ Sets)

- (i) For each $\epsilon > 0$, there is an open set \mathcal{O} containing E for which $m^*(\mathcal{O} \sim E) < \epsilon$.
- (ii) There is a G_δ set G containing E for which $m^*(G \sim E) = 0$.

(Inner Approximation by Closed Sets and F_σ Sets)

- (iii) For each $\epsilon > 0$, there is a closed set F contained in E for which $m^*(E \sim F) < \epsilon$.
- (iv) There is an F_σ set F contained in E for which $m^*(E \sim F) = 0$.

9.2.4 Definition of Lebesgue measure

If E is a measurable set then its Lebesgue measure is defined by

$$m(E) = m^*(E)$$

i.e.

$$m = m^*|_{\mathcal{M}}$$

9.2.5 Nonmeasurable sets

It's natural to be interested about the sets that is not measurable.

Théorème 4: Vitali

Any set E of real numbers with positive outer measure contains a subset that fails to be measurable.

Proof By the countable subadditivity of outer measure, we may suppose E is bounded. Let \mathcal{C}_E be any choice set for the rational equivalence relation on E . We claim that \mathcal{C}_E is not measurable. To verify this claim, we assume it is measurable and derive a contradiction.

Let Λ_0 be any bounded, countably infinite set of rational numbers. Since \mathcal{C}_E is measurable, and, by (16), the collection of translates of \mathcal{C}_E by members of Λ_0 is disjoint, it follows from Lemma 16 that $m(\mathcal{C}_E) = 0$. Hence, again using the translation invariance and the countable additivity of measure over countable disjoint unions of measurable sets,

$$m \left[\bigcup_{\lambda \in \Lambda_0} (\lambda + \mathcal{C}_E) \right] = \sum_{\lambda \in \Lambda_0} m(\lambda + \mathcal{C}_E) = 0.$$

To obtain a contradiction we make a special choice of Λ_0 . Because E is bounded it is contained in some interval $[-b, b]$. We choose

$$\Lambda_0 = [-2b, 2b] \cap \mathbb{Q}.$$

Then Λ_0 is bounded, and is countably infinite since the rationals are countable and dense.¹¹ We claim that

$$E \subseteq \bigcup_{\lambda \in [-2b, 2b] \cap \mathbb{Q}} (\lambda + \mathcal{C}_E). \quad (17)$$

Indeed, by the second characteristic property of \mathcal{C}_E , if x belongs to E , there is a number c in the choice set \mathcal{C}_E for which $x = c + q$ with q rational. But x and c belong to $[-b, b]$, so that q belongs to $[-2b, 2b]$. Thus the inclusion (17) holds. This is a contradiction because E , a set of positive outer measure, is not a subset of a set of measure zero. The assumption that \mathcal{C}_E is measurable has led to a contradiction and thus it must fail to be measurable. \square

Theorem 18 *There are disjoint sets of real numbers A and B for which*

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

Proof We prove this by contradiction. Assume $m^*(A \cup B) = m^*(A) + m^*(B)$ for every disjoint pair of sets A and B . Then, by the very definition of measurable set, every set must be measurable. This contradicts the preceding theorem. \square

9.3 General Measure Spaces

9.3.1 Definition of measurable space

Definition By a **measurable space** we mean a couple (X, \mathcal{M}) consisting of a set X and a σ -algebra \mathcal{M} of subsets of X . A subset E of X is called **measurable** (or measurable with respect to \mathcal{M}) provided E belongs to \mathcal{M} .

Definition By a **measure** μ on a measurable space (X, \mathcal{M}) we mean an extended real-valued nonnegative set function $\mu: \mathcal{M} \rightarrow [0, \infty]$ for which $\mu(\emptyset) = 0$ and which is **countably additive** in the sense that for any countable disjoint collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets,

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

By a **measure space** (X, \mathcal{M}, μ) we mean a measurable space (X, \mathcal{M}) together with a measure μ defined on \mathcal{M} .

9.3.2 Properties: Finite Additivity, Monotonicity, Excision, Countable Monotonicity, Continuity of Measure

Proposition 1 Let (X, \mathcal{M}, μ) be a measure space.

(Finite Additivity) For any finite disjoint collection $\{E_k\}_{k=1}^n$ of measurable sets,

$$\mu\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu(E_k).$$

(Monotonicity) If A and B are measurable sets and $A \subseteq B$, then

$$\mu(A) \leq \mu(B).$$

(Excision) If, moreover, $A \subseteq B$ and $\mu(A) < \infty$, then

$$\mu(B \sim A) = \mu(B) - \mu(A),$$

so that if $\mu(A) = 0$, then

$$\mu(B \sim A) = \mu(B).$$

(Countable Monotonicity) For any countable collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets that covers a measurable set E ,

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

A sequence of sets $\{E_k\}_{k=1}^{\infty}$ is called **ascending** provided for each k , $E_k \subseteq E_{k+1}$, and said to be **descending** provided for each k , $E_{k+1} \subseteq E_k$.

Proposition 2 (Continuity of Measure) Let (X, \mathcal{M}, μ) be a measure space.

(i) If $\{A_k\}_{k=1}^{\infty}$ is an ascending sequence of measurable sets, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \rightarrow \infty} \mu(A_k). \quad (1)$$

(ii) If $\{B_k\}_{k=1}^{\infty}$ is a descending sequence of measurable sets for which $\mu(B_1) < \infty$, then

$$\mu\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k \rightarrow \infty} \mu(B_k). \quad (2)$$

9.3.3 Borel-Cantelli Lemma

The Borel-Cantelli Lemma Let (X, \mathcal{M}, μ) be a measure space and $\{E_k\}_{k=1}^{\infty}$ a countable collection of measurable sets for which $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Then almost all x in X belong to at most a finite number of the E_k 's.

Proof For each n , by the countable monotonicity of μ , $\mu(\bigcup_{k=n}^{\infty} E_k) \leq \sum_{k=n}^{\infty} \mu(E_k)$. Hence, by the continuity of μ ,

$$\mu\left(\bigcap_{n=1}^{\infty} \left[\bigcup_{k=n}^{\infty} E_k\right]\right) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mu(E_k) = 0.$$

Observe that $\bigcap_{n=1}^{\infty} [\bigcup_{k=n}^{\infty} E_k]$ is the set of all points in X that belong to an infinite number of the E_k 's. \square

9.3.4 Definition of σ -finite, complete

Definition Let (X, \mathcal{M}, μ) be a measure space. The measure μ is called **finite** provided $\mu(X) < \infty$. It is called **σ -finite** provided X is the union of a countable collection of measurable sets, each of which has finite measure. A measurable set E is said to be of **finite measure** provided $\mu(E) < \infty$, and said to be **σ -finite** provided E is the union of a countable collection of measurable sets, each of which has finite measure.

For example, $(-\infty, \infty)$ is σ -finite under the Lebesgue measure on \mathbb{R}^1 .

Definition A measure space (X, \mathcal{M}, μ) is said to be **complete** provided \mathcal{M} contains all subsets of sets of measure zero, that is, if E belongs to \mathcal{M} and $\mu(E) = 0$, then every subset of E also belongs to \mathcal{M} .

The Lebesgue measure restricted to Borel sets on \mathbb{R}^1 is not complete, since a Borel sets with measure zero contains a subsets that is not Borel, see Royden

page 52.

9.3.5 Signed measure

See evans 偏微分方程笔记 Chap 3 Appendix Measures.

9.4 The Carathéodory measure induced by a outer measure

We now define the general concept of an outer measure and of measurability of a set with respect to an outer measure, and show that the Carathéodory strategy for the construction of Lebesgue measure on the real line is feasible in general.

9.4.1 Definition of outer measure, measurable sets

Definition A set function $\mu^*: 2^X \rightarrow [0, \infty]$ is called an **outer measure** provided $\mu^*(\emptyset) = 0$ and μ^* is countably monotone.

Definition For an outer measure $\mu^*: 2^X \rightarrow [0, \infty]$, we call a subset E of X **measurable** (with respect to μ^*) provided for every subset A of X ,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^C).$$

Since μ^* is finitely monotone, to show that $E \subseteq X$ is measurable it is only necessary to show that

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^C) \text{ for all } A \subseteq X \text{ such that } \mu^*(A) < \infty.$$

The union of a countable collection of measurable sets is measurable.

9.4.2 Construction by restriction

Theorem 8 Let μ^* be an outer measure on 2^X . Then the collection \mathcal{M} of sets that are measurable with respect to μ^* is a σ -algebra. If $\bar{\mu}$ is the restriction of μ^* to \mathcal{M} , then $(X, \mathcal{M}, \bar{\mu})$ is a complete measure space.

Proof We already observed that the complement in X of a measurable subset of X also is measurable. According to Proposition 7, the union of a countable collection of measurable sets is measurable. Therefore \mathcal{M} is a σ -algebra. By the definition of an outer measure, $\mu^*(\emptyset) = 0$ and therefore \emptyset is measurable and $\bar{\mu}(\emptyset) = 0$. To verify that $\bar{\mu}$ is a measure on \mathcal{M} , it remains to show it is countably additive. Since μ^* is countably monotone and μ^* is an extension of $\bar{\mu}$, the set function $\bar{\mu}$ is countably monotone. Therefore we only need show that if $\{E_k\}_{k=1}^\infty$ is a disjoint collection of measurable sets, then

$$\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} \mu^*(E_k). \quad (5)$$

However, μ^* is monotone and, by taking $A = X$ in Proposition 7, we see that μ^* is additive over finite disjoint unions of measurable sets. Therefore, for each n ,

$$\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \mu^*\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu^*(E_k).$$

The left-hand side of this inequality is independent of n and therefore (5) holds. \square

9.4.3 The Construction of Outer Measure

The definition of outer measure before in this section is by properties, while the following is by construction. We can construct μ^* by a set function (not measure) μ .

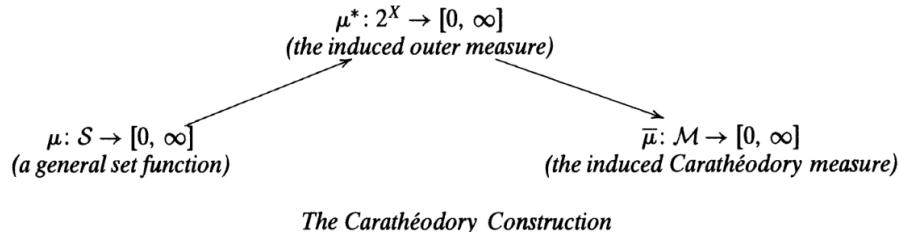
Theorem 9 Let \mathcal{S} be a collection of subsets of a set X and $\mu: \mathcal{S} \rightarrow [0, \infty]$ a set function. Define $\mu^*(\emptyset) = 0$ and for $E \subseteq X, E \neq \emptyset$, define

$$\mu^*(E) = \inf \sum_{k=1}^{\infty} \mu(E_k), \quad (6)$$

where the infimum is taken over all countable collections $\{E_k\}_{k=1}^\infty$ of sets in \mathcal{S} that cover E .¹ Then the set function $\mu^*: 2^X \rightarrow [0, \infty]$ is an outer measure called the **outer measure induced by μ** .

Carathéodory measure

Definition Let \mathcal{S} be a collection of subsets of X , $\mu: \mathcal{S} \rightarrow [0, \infty]$ a set function, and μ^* the outer measure induced by μ . The measure $\bar{\mu}$ that is the restriction of μ^* to the σ -algebra \mathcal{M} of μ^* -measurable sets is called the **Carathéodory measure induced by μ** .



9.4.4 Definition of premeasure, closed collection, semiring

Definition Let \mathcal{S} be a collection of subsets of a set X and $\mu: \mathcal{S} \rightarrow [0, \infty]$ a set function. Then μ is called a **premeasure** provided μ is both finitely additive and countably monotone and, if \emptyset belongs to \mathcal{S} , then $\mu(\emptyset) = 0$.

Definition A collection \mathcal{S} of subsets of X is said to be closed with respect to the formation of relative complements provided whenever A and B belong to \mathcal{S} , the relative complement $A \sim B$ belongs to \mathcal{S} . The collection \mathcal{S} is said to be closed with respect to the formation of finite intersections provided whenever A and B belong to \mathcal{S} , the intersection $A \cap B$ belongs to \mathcal{S} .

Definition A nonempty collection \mathcal{S} of subsets of a set X is called a **semiring** provided whenever A and B belong to \mathcal{S} , then $A \cap B$ also belongs to \mathcal{S} and there is a finite disjoint collection $\{C_k\}_{k=1}^n$ of sets in \mathcal{S} for which

$$A \sim B = \bigcup_{k=1}^n C_k.$$

9.4.5 Carathéodory extension of set function

Theorem 12 Let $\mu: \mathcal{S} \rightarrow [0, \infty]$ be a premeasure on a nonempty collection \mathcal{S} of subsets of X that is closed with respect to the formation of relative complements. Then the Carathéodory measure $\bar{\mu}: \mathcal{M} \rightarrow [0, \infty]$ induced by μ is an extension of μ : it is called the **Carathéodory extension** of μ .

Proof Let A belong to \mathcal{S} . To show that A is measurable with respect to the outer measure induced by μ it suffices to let E be any subset of X of finite outer measure, let $\epsilon > 0$ and verify that

$$\mu^*(E) + \epsilon \geq \mu^*(E \cap A) + \mu^*(E \cap A^C). \quad (8)$$

By the definition of outer measure, there is a collection $\{E_k\}_{k=1}^\infty$ of sets in \mathcal{S} that covers E and

$$\mu^*(E) + \epsilon \geq \sum_{k=1}^{\infty} \mu(E_k). \quad (9)$$

However, for each k , since \mathcal{S} is closed with respect to the formation of relative complements, $E_k \sim A$ belongs to \mathcal{S} and so does $E_k \cap A = E_k \sim [E_k \sim A]$. A premeasure is finitely additive. Therefore

$$\mu(E_k) = \mu(E_k \cap A) + \mu(E_k \cap A^C).$$

Sum these inequalities to conclude that

$$\sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^{\infty} \mu(E_k \cap A) + \sum_{k=1}^{\infty} \mu(E_k \cap A^C). \quad (10)$$

Observe that $\{E_k \cap A\}_{k=1}^\infty$ and $\{E_k \cap A^C\}_{k=1}^\infty$ are countable collections of sets in \mathcal{S} that cover $E \cap A$ and $E \cap A^C$, respectively. Therefore, by the very definition of outer measure,

$$\sum_{k=1}^{\infty} \mu(E_k \cap A) \geq \mu^*(E \cap A) \text{ and } \sum_{k=1}^{\infty} \mu(E_k \cap A^C) \geq \mu^*(E \cap A^C).$$

The desired inequality (8) follows from these two inequalities together with (9) and (10).

Clearly $\mu(E) = \mu^*(E)$ for each set $E \in \mathcal{S}$ if and only if μ is countable monotone. Hence for each $E \in \mathcal{S}$, $\mu(E) = \mu^*(E)$ and therefore, since each set $E \in \mathcal{S}$ is measurable, $\mu(E) = \bar{\mu}(E)$. \square

9.4.6 Unique extension to a premeasure

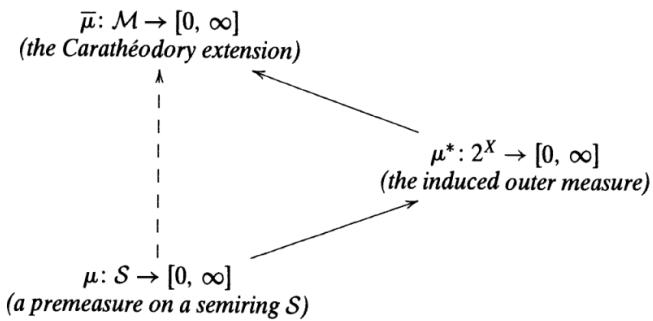
We show that a semiring S has the property that every premeasure on S has a unique extension to a premeasure on a collection of sets that is closed with respect to the formation of relative complements.

Proposition 13 Let S be a semiring of subsets of a set X . Define S' to be the collection of unions of finite disjoint collections of sets in S . Then S' is closed with respect to the formation of relative complements. Furthermore, any premeasure on S has a unique extension to a premeasure on S' .

The Carathéodory-Hahn Theorem

For \mathcal{S} a collection of subsets of X , a set function $\mu: \mathcal{S} \rightarrow [0, \infty]$ is said to be σ -finite provided $X = \bigcup_{k=1}^{\infty} S_k$ where for each k , $S_k \in \mathcal{S}$ and $\mu(S_k) < \infty$.

The Carathéodory-Hahn Theorem Let $\mu: \mathcal{S} \rightarrow [0, \infty]$ be a premeasure on a semiring \mathcal{S} of subsets of X . Then the Carathéodory measure $\bar{\mu}$ induced by μ is an extension of μ . Furthermore, if μ is σ -finite, then so is $\bar{\mu}$ and $\bar{\mu}$ is the unique measure on the σ -algebra of μ^* -measurable sets that extends μ .



The Carathéodory Construction Extends a Premeasure on a Semiring to a Measure

9.5 刷题：Lebesgue 测度-可测集-测度-Borel 集

参见《实变函数解题指南》· 周民强.

9.5.1 Lebesgue 测度

Définition 1: Lebesgue outer measure

Let $E \subset \mathbf{R}^n$. If $\{I_k\}$ is a countable collection of open rectangles in \mathbf{R}^n , and

$$E \subset \bigcup_{k \geq 1} I_k,$$

then $\{I_k\}$ is called an L -covering of E (obviously there are many such coverings, and each L -covering $\{I_k\}$ determines a non-negative generalized real value $\sum_{k \geq 1} |I_k|$ (can be $+\infty$, $|I_k|$ represents the volume of I_k), we call

$$m^*(E) = \inf \left\{ \sum_{k \geq 1} |I_k| : \{I_k\} \text{ is an } L - \text{covering of } E \right\}$$

the **Lebesgue outer measure** of the point set E .

Exercice 1

(5) 设 $E \subset [a, b]$, $m^*(E) > 0$, $0 < c < m^*(E)$, 则存在 E 的子集 A , 使得 $m^*(A) = c$.

证明. Let $f(x) = m^*([a, x] \cap E)$, $a \leq x \leq b$. Then $f(a) = 0$, $f(b) = m^*(E)$. Consider x and $x + \Delta x$. Without loss of generality, assume that $a \leq x < x + \Delta x \leq b$. Then, by

$$[a, x + \Delta x] \cap E = ([a, x] \cap E) \cup ([x, x + \Delta x] \cap E),$$

we have $f(x + \Delta x) \leq f(x) + \Delta x$, that is,

$$f(x + \Delta x) - f(x) \leq \Delta x.$$

Similarly, for $\Delta x < 0$, we can also prove similar inequalities. In summary, we have

$$|f(x + \Delta x) - f(x)| \leq |\Delta x|, \quad a \leq x \leq b.$$

This indicates that $f \in C([a, b])$. According to the intermediate value theorem for continuous functions, for $f(a) < c < f(b)$, there must exist $\xi \in (a, b)$ such that $f(\xi) = c$. Thus, taking $A = [a, \xi] \cap E$ yields the desired result. \square

Définition 2: 可数覆盖空间

设 X 是一个拓扑空间. 如果存在一个可数集 $A \subset X$, 使得 X 的任何开覆盖都有一个子覆盖, 其元素个数不超过 A 的元素个数, 那么 X 被称为可数覆盖空间.

一个拓扑空间 X 被称为可数覆盖空间, 如果存在一个可数子集 $A \subset X$, 使得 X 的任何开覆盖都有一个子覆盖, 其元素个数不超过 A 的元素个数. 换句话说, 存在一个可数集合 A , 使得对于 X 的任何开覆盖, 都可以从中选择一个可数子覆盖. 由于 \mathbb{R}^n 具有可数基, 因此 \mathbb{R}^n 是第二可数空间. 根据定义, 第二可数空间是 Lindelöf 空间, 这意味着 \mathbb{R}^n 的每个开覆盖都有一个可数子覆盖. 因此, \mathbb{R}^n 是一个可数覆盖空间.

Exercice 2

(5) 设 $E \subset \mathbf{R}^n$. 若对任意的 $x \in E$, 存在开球 $B(x, \delta_x)$, 使得 $m^*(E \cap B(x, \delta_x)) = 0$, 则 $m^*(E) = 0$.

证明. (5) By assumption, there exists a countable cover of E by balls $\{B_k \triangleq B(x_k, \delta_{x_k})\}$ such that $E \subset \bigcup_{k=1}^{\infty} B_k$, and $m^*(E \cap B_k) = 0$. It follows that

$$E = \bigcup_{k=1}^{\infty} (E \cap B_k), \quad m^*(E) \leq \sum_{k=1}^{\infty} m^*(E \cap B_k) = 0.$$

□

Proposition 1

设 $A, B \subset \mathbf{R}^n$, 且 $m^*(A), m^*(B) < \infty$, 则

$$|m^*(A) - m^*(B)| \leq m^*(A \Delta B);$$

Définition 3: Limit superior and Limit inferior of sets

Let A_1, A_2, \dots be a sequence of sets. The **limit superior** is defined by

$$\limsup_{n \rightarrow \infty} A_n \equiv \{x : x \in A_n \text{ for infinitely many } n\} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$$

The **limit inferior** is defined by

$$\liminf_{n \rightarrow \infty} A_n \equiv \{x : x \in A_n \text{ for all but finitely many } n\} = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i$$

Exercice 3

(2) 设 $E_k \subset \mathbf{R}^n (k \in \mathbf{N})$. 若 $\sum_{k=1}^{\infty} m^*(E_k) < +\infty$, 则 $m(\overline{\lim}_{k \rightarrow \infty} E_k) = 0$.
 . (3) 设定义在 \mathbf{R}^1 上的函数列 $\{f_n(x)\}$ 满足 ($\lambda_n > 0, n \in \mathbf{N}$)

$$\sum_{n=1}^{\infty} m^*(E_n) < +\infty \quad (E_n = \{x \in \mathbf{R}^1 : |f_n(x)| / \lambda_n > 1\}),$$

则存在 $Z \subset \mathbf{R}^1$ 且 $m(Z) = 0$, 使得 $\overline{\lim}_{n \rightarrow \infty} |f_n(x)| / \lambda_n \leq 1 (x \in \mathbf{R}^1 \setminus Z)$.

证明. (2) 注意 $\overline{\lim}_{k \rightarrow \infty} E_k = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k$, 且依题设知, 对任给 $\varepsilon > 0$, 存在 N , 使得 $\sum_{k=N}^{\infty} m^*(E_k) < \varepsilon$. 从而对任意 $j \in \mathbf{N}$, 有

$$m^*\left(\overline{\lim}_{k \rightarrow \infty} E_k\right) \leq m^*\left(\bigcup_{k=j}^{\infty} E_k\right) \leq \sum_{k=j}^{\infty} m^*(E_k).$$

由此知 $m^*(\overline{\lim}_{k \rightarrow \infty} E_k) \leq \sum_{k=N}^{\infty} m^*(E_k) < \varepsilon$. 证毕.

(3) 令 $Z = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n$, 则由题设知 $m(Z) = 0$. 因此当 $x \in \mathbf{R}^1 \setminus Z$ 时, 必存在 n_0 , 使得 $x \in E_n (n \geq n_0)$. 从而有 $\overline{\lim}_{n \rightarrow \infty} |f_n(x)| / \lambda_n \leq 1 (x \in \mathbf{R}^1 \setminus Z)$.

□

9.5.2 可测集与测度

Définition 4: Lebesgue 可测集, Carathéodory 条件

设 $E \subset \mathbf{R}^n$. 若对任意的点集 $T \subset \mathbf{R}^n$, 有

$$m^*(T) = m^*(T \cap E) + m^*(T \cap E^c),$$

则称 E 为 Lebesgue 可测集 (或 m^* -可测集), 简称为可测集, 其中 T 称为试验集 (这一定义可测集的等式也称为 Carathéodory 条件); 可测集的全体称为可测集类, 简记为 \mathcal{M} .

Proposition 2

设 $\{E_k\}$ 是可测集列, 则 $m(\underline{\lim}_{k \rightarrow \infty} E_k) \leq \overline{\lim}_{k \rightarrow \infty} m(E_k)$.

Exercice 4

设 $E \subset \mathbf{R}^n, E \in \mathcal{M}$ 的充要条件是: 对任给 $\epsilon > 0$, 存在可测集 $A, B \in \mathbf{R}^n$:
 $A \subset E \subset B$, 使得 $m(B \setminus A) < \epsilon$.

Exercice 5

(3) 设 $\{E_n\}$ 是 $[0, 1]$ 中的可测集列, 且满足 $\overline{\lim}_{n \rightarrow \infty} m(E_n) = 1$, 试证明对任意的 $\alpha : 0 < \alpha < 1$, 必存在 $\{E_{n_k}\}$, 使得 $m(\bigcap_{k=1}^{\infty} E_{n_k}) > \alpha$.

证明. (3) 由题设知, 对任意的 $k \in \mathbf{N}$, 存在 $\{n_k\}$, 使得

$$m(E_{n_k}) > 1 - (1 - \alpha)/2^k \quad (k \in \mathbf{N}).$$

由此知 $1 - m(E_{n_k}) < (1 - \alpha)/2^k (k \in \mathbf{N})$. 从而得到

$$\begin{aligned} [0, 1] \setminus \bigcap_{k=1}^{\infty} E_{n_k} &= \bigcup_{k=1}^{\infty} ([0, 1] \setminus E_{n_k}), \\ m\left([0, 1] \setminus \bigcap_{k=1}^{\infty} E_{n_k}\right) &\leq \sum_{k=1}^{\infty} m([0, 1] \setminus E_{n_k}) \\ &= \sum_{k=1}^{\infty} (1 - m(E_{n_k})) \leq \sum_{k=1}^{\infty} (1 - \alpha)/2^k = 1 - \alpha, \end{aligned}$$

故有 $m(\bigcap_{k=1}^{\infty} E_{n_k}) > \alpha$. □

Exercice 6

(5) 设有 \mathbf{R}^1 中可测集列 $\{E_k\}$, 且当 $k \geq k_0$ 时, $E_k \subset [a, b]$. 若存在 $\lim_{k \rightarrow \infty} E_k = E$, 试证明: $m(E) = \lim_{k \rightarrow \infty} m(E_k)$.

证明.

$$m(\lim_{k \rightarrow \infty} E_k) \geq \overline{\lim}_{k \rightarrow \infty} m(E_k) \geq \underline{\lim}_{k \rightarrow \infty} m(E_k) \geq m(\lim_{k \rightarrow \infty} E_k)$$

□

Exercice 7

构造零测的第二纲集.

证明. (2) 令 $[0, 1] \cap \mathbf{Q} = \{r_1, r_2, \dots, r_n, \dots\}$, 以及

$$I_{n,k} = \left(r_n - \frac{1}{2^{n+k}}, r_n + \frac{1}{2^{n+k}}\right) \quad (n, k \in \mathbf{N}),$$

则点集 $(-\infty, +\infty) \setminus \bigcup_{n,k=1}^{\infty} I_{n,k}$ 在 \mathbf{R}^1 中无处稠密. 我们有

$$m\left(\bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} I_{n,k}\right) = 0,$$

$\bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} I_{n,k}$ 是第二纲集. \square

Exercice 8

(3) 在 $[0, 1]$ 中作点集 $E = \{x \in [0, 1] : \text{在十进位小数表示式 } x = 0.a_1a_2\ldots \text{ 中的所有 } a_i \text{ 都不出现 } 10 \text{ 个数字中的某一个}\}$, 试证明 E 是不可数集, 且 $m(E) = 0$.

证明. Let E_k be a set defined as follows:

$$E_k = \{x \in [0, 1] : \text{in the decimal expansion } x = 0.a_1a_2\ldots, \text{ the digit } k \text{ does not appear for any } a_i\}$$

where $k \in \{0, 1, 2, \dots, 9\}$.

The set E_k is constructed by iteratively removing intervals from $[0, 1]$. In the first step, the interval $[0, 1]$ is divided into 10 equal subintervals, and the $(k+1)$ -th subinterval $[\frac{k}{10}, \frac{k+1}{10})$ is removed. This ensures that the remaining points have a decimal expansion where the first digit is not equal to k .

In the second step, each of the remaining 9 subintervals is divided into 10 equal subintervals, and the $(k+1)$ -th subinterval is removed from each. This process continues indefinitely.

The total length of the removed intervals can be calculated as the sum of a geometric series:

$$\frac{1}{10} + \frac{9}{10^2} + \frac{9^2}{10^3} + \cdots = \frac{1}{10} \sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n = \frac{1}{10} \cdot \frac{1}{1 - \frac{9}{10}} = \frac{1}{10} \cdot \frac{1}{\frac{1}{10}} = 1$$

Since the total length of the intervals removed is 1, the measure of the remaining set E_k is:

$$m(E_k) = 1 - 1 = 0$$

\square

Exercice 9

(5) 将 $[0, 1]$ 中的点用十进位小数展开, 令

$$E = \{x \in [0, 1] : x \text{ 的任一位小数是 } 2 \text{ 或 } 7\},$$

试问: (i) E 是闭集? (ii) E 是开集? (iii) E 是可数集? (iv) $\bar{E} = [0, 1]$? (v) E 是可测集? $m(E) = ?$

证明. (5) (i) Suppose $\{x_k\} \subset E : x_k \rightarrow x(k \rightarrow \infty)$, and let $x = \sum_{n=1}^{\infty} b_n / 10^n$. If $|x_k - x| < 1/10^p$, then $b_p = 2$ or 7 . Hence $x \in E$, i.e., E is a closed set.

(ii) Noticing (i) and $E \neq [0, 1]$, we know that E is not an open set.

(iii) $\#E = 2^{\aleph_0} = \aleph_1$.

(iv) Since E is a closed set and $E \neq [0, 1]$, we know that E is not dense in $[0, 1]$.

(v) E is a measurable set (see §2.3). $m(E) = 1 - 0.8 \times \sum_{m=0}^{\infty} (2/10)^m = 0$.

□

Exercice 10

(1) 设 $E \subset \mathbf{R}^1$, 且存在 $q : 0 < q < 1$, 使得对任一区间 (a, b) , 都有开区间列 $\{I_n\}$:

$$E \cap (a, b) \subset \bigcup_{n=1}^{\infty} I_n, \quad \sum_{n=1}^{\infty} m(I_n) < (b - a)q.$$

则 $m(E) = 0$.

证明. 证明 (1) 因为 $m^*(E) = m^*(E \cap \bigcup_{n=1}^{\infty} I_n) \leq \sum_{n=1}^{\infty} m^*(E \cap I_n)$, 所以只需指出对任意的 (a, b) , 有 $m^*(E \cap (a, b)) = 0$. 由题设知, 存在 $I_n = (a_n, b_n)$ ($n \in \mathbf{N}$), $\bigcup_{n=1}^{\infty} I_n \supset E \cap (a, b)$, 使得 $\sum_{n=1}^{\infty} (b_n - a_n) \leq q(b - a)$. 再对每个 (a_n, b_n) 作覆盖, 其总长度小于 $q(b_n - a_n)$. 依此程序继续作下去, 可得 (对任意 $k \in \mathbf{N}$)

$$m^*(E \cap (a, b)) \leq q \sum_{n=1}^{\infty} (b_n - a_n) \leq q^2(b - a) \cdots \leq q^k(b - a),$$

由此易知 $m^*(E \cap (a, b)) = 0$.

□

Exercice 11: yau-21-analysis-problem-3

- (1) For $f \in L^1(\mathbb{R}^n)$, $g \in L^\infty(\mathbb{R}^n)$, show that their convolution $f * g$ is a well-defined continuous function.
- (2) Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set with Lebesgue measure $m(E) > 0$. Prove that

$$E - E := \{x - y \mid x \in E, y \in E\}$$

contains an open neighborhood of $0 \in \mathbb{R}^n$.

证明. (a) This is standard: In fact, we have $\|f * g\|_{L^\infty} \leq \|f\|_{L^1} \|g\|_{L^\infty}$. Therefore, by the continuity argument, it suffices to prove the theorem for $f \in C_0^\infty(\mathbf{R}^n)$. In this case, we have

$$\begin{aligned} |f * g(x_0 + x) - f * g(x_0)| &= \left| \int_{\mathbf{R}^n} (f(x_0 + x - y) - f(x_0 - y))g(y) dy \right| \\ &\leq \|g\|_{L^\infty} \int_{\mathbf{R}^n} |f(x_0 + x - y) - f(x_0 - y)| dy \end{aligned}$$

Now let $x \rightarrow 0$, the integrand converges to 0 uniformly. This yields (a).

(b) It suffices to consider the case where $m(E) < \infty$. We take $f = \mathbf{1}_E$, $g = \mathbf{1}_{-E}$, thus $h(x) = f * g$ is a continuous function. In particular, $h(0) = m(E) > 0$. Therefore, there exists an open set U such that $0 \in U$ and $h|_U > \delta > 0$ for some $\delta > 0$. For $x \in U$, by definition,

$$h(z) = \int_{\mathbf{R}^n} \mathbf{1}_E(x - y)\mathbf{1}_{-E}(y) dy > 0.$$

Therefore, there must be some $y \in -E$, such that $x - y = x + (-y) \in E$. This implies $x \in E - (-y) \subset E - E$. Hence $U \subset E - E$. \square

9.6 Topological preliminaries

9.6.1 Definition of topology

1.2 Definition

- (a) A collection τ of subsets of a set X is said to be a *topology in X* if τ has the following three properties:
 - (i) $\emptyset \in \tau$ and $X \in \tau$.
 - (ii) If $V_i \in \tau$ for $i = 1, \dots, n$, then $V_1 \cap V_2 \cap \dots \cap V_n \in \tau$.
 - (iii) If $\{V_\alpha\}$ is an arbitrary collection of members of τ (finite, countable, or uncountable), then $\bigcup_\alpha V_\alpha \in \tau$.
- (b) If τ is a topology in X , then X is called a *topological space*, and the members of τ are called the *open sets* in X .
- (c) If X and Y are topological spaces and if f is a mapping of X into Y , then f is said to be *continuous* provided that $f^{-1}(V)$ is an open set in X for every open set V in Y .

9.6.2 General definitions of closed, closure, compact, neighborhood, Hausdorff space, locally compact, separated and connected sets

2.3 Definitions Let X be a topological space, as defined in Sec. 1.2.

- (a) A set $E \subset X$ is *closed* if its complement E^c is open. (Hence \emptyset and X are closed, finite unions of closed sets are closed, and arbitrary intersections of closed sets are closed.)
- (b) The *closure* \bar{E} of a set $E \subset X$ is the smallest closed set in X which contains E . (The following argument proves the existence of \bar{E} : The collection Ω of all closed subsets of X which contain E is not empty, since $X \in \Omega$; let \bar{E} be the intersection of all members of Ω .)
- (c) A set $K \subset X$ is *compact* if every open cover of K contains a finite sub-cover. More explicitly, the requirement is that if $\{V_\alpha\}$ is a collection of open sets whose union contains K , then the union of some finite sub-collection of $\{V_\alpha\}$ also contains K .
In particular, if X is itself compact, then X is called a *compact space*.
- (d) A *neighborhood* of a point $p \in X$ is any open subset of X which contains p . (The use of this term is not quite standardized; some use

- “neighborhood of p ” for any set which contains an open set containing p .)
- (e) X is a *Hausdorff space* if the following is true: If $p \in X$, $q \in X$, and $p \neq q$, then p has a neighborhood U and q has a neighborhood V such that $U \cap V = \emptyset$.
- (f) X is *locally compact* if every point of X has a neighborhood whose closure is compact.

Open sets remains to be open under arbitrary unions, while closed sets remains to be closed under arbitrary intersections. Imagine the special case of intervals to remember it.

The definition of compact is not technique to prove a set compact but a property.

2.45 Definition Two subsets A and B of a metric space X are said to be *separated* if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty, i.e., if no point of A lies in the closure of B and no point of B lies in the closure of A .

A set $E \subset X$ is said to be *connected* if E is not a union of two nonempty separated sets.

2.46 Remark Separated sets are of course disjoint, but disjoint sets need not be separated. For example, the interval $[0, 1]$ and the segment $(1, 2)$ are *not* separated, since 1 is a limit point of $(1, 2)$. However, the segments $(0, 1)$ and $(1, 2)$ are separated.

Another definition of closure:

Definition For a subset E of a topological space X , a point $x \in X$ is called a **point of closure** of E provided every neighborhood of x contains a point in E . The collection of points of closure of E is called the **closure** of E and denoted by \bar{E} .

9.6.3 Metric definition of neighborhood, limit point, closed, interior point, open, complement, perfect, bounded, dense

2.18 Definition Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X .

- (a) A neighborhood of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$, for some $r > 0$. The number r is called the radius of $N_r(p)$.
- (b) A point p is a limit point of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.
- (c) If $p \in E$ and p is not a limit point of E , then p is called an isolated point of E .
- (d) E is closed if every limit point of E is a point of E .
- (e) A point p is an interior point of E if there is a neighborhood N of p such that $N \subset E$.
- (f) E is open if every point of E is an interior point of E .
- (g) The complement of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.
- (h) E is perfect if E is closed and if every point of E is a limit point of E .
- (i) E is bounded if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.
- (j) E is dense in X if every point of X is a limit point of E , or a point of E (or both).

2.26 Definition If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X , then the closure of E is the set $\bar{E} = E \cup E'$.

Définition 5: relatively open

Suppose $E \subset Y \subset X$, where X is a metric space. Motivated by the idea that Y can also be a metric space, we say E is **open relative to Y** if to each $p \in E$ there is associated an $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$.

A set may be open relative to Y without being an open subset of X , e.g. $(a, b) \subset \mathbb{R} \subset \mathbb{R}^2$.

Théorème 5

Suppose $Y \subset X$. A subset E of Y is open relative to Y iff $E = Y \cap G$ for some open subset G of X .

Théorème 5 can be another definition of **relatively open**.

9.6.4 Some properties in metric spaces

- Every neighborhood is an open set.
- If p is a limit point of a set E , then every neighborhood of p contains infinitely many points of E .
- A finite point set has no limit points.
- Closed subsets of compact sets are compact.
- Perfect set in \mathbb{R}^k is uncountable.
- If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.
- If $\{K_n\}$ is a decreasing sequence of nonempty compact sets, then $\bigcap_{n=1}^{\infty} K_n$ is not empty.

9.6.5 Examples

2.21 Examples Let us consider the following subsets of R^2 :

- (a) The set of all complex z such that $|z| < 1$.
- (b) The set of all complex z such that $|z| \leq 1$.
- (c) A nonempty finite set.
- (d) The set of all integers.
- (e) The set consisting of the numbers $1/n$ ($n = 1, 2, 3, \dots$). Let us note that this set E has a limit point (namely, $z = 0$) but that no point of E is a limit point of E ; we wish to stress the difference between having a limit point and containing one.
- (f) The set of all complex numbers (that is, R^2).
- (g) The segment (a, b) .

Let us note that (d), (e), (g) can be regarded also as subsets of R^1 .

Some properties of these sets are tabulated below:

	<i>Closed</i>	<i>Open</i>	<i>Perfect</i>	<i>Bounded</i>
(a)	No	Yes	No	Yes
(b)	Yes	No	Yes	Yes
(c)	Yes	No	No	Yes
(d)	Yes	No	No	No
(e)	No	No	No	Yes
(f)	Yes	Yes	Yes	No
(g)	No		No	Yes

In (g), we left the second entry blank. The reason is that the segment (a, b) is not open if we regard it as a subset of R^2 , but it is an open subset of R^1 .

9.6.6 Heine-Borel theorem

Théorème 6: Heine-Borel theorem

The compact subsets of a Euclidean space \mathbb{R}^n are precisely those that are closed and bounded.

Moreover, the theorem is true for any locally compact Hausdorff space. Note that metric spaces are locally compact Hausdorff space.

Théorème 7: Closed subsets of compact sets are compact.

Suppose K is compact and F is closed, in a topological space X . If $F \subset K$ then F is compact.

证明. If $\{V_\alpha\}$ is an open cover of F and $W = F^c$ then $W \cup \bigcup_\alpha V_\alpha$ covers X ; hence there is a finite collection $\{V_{\alpha_i}\}$ such that

$$K \subset W \cup V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$$

Then $F \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. □

Corollaire 1

If $A \subset B$ and if B has compact closure, so does A .

Théorème 8

X Hausdorff, $K \subset X$, K compact, and $p \in K^c$. Then there are open sets U and W such that $p \in U$, $K \subset W$, and $U \cap W = \emptyset$.

证明. If $q \in K$, the Hausdorff separation axiom implies the existence of disjoint open sets U_q and V_q such that $p \in U_q$ and $q \in V_q$. Since K is compact, there are points $q_1, \dots, q_n \in K$ such that

$$K \subset V_{q_1} \cup \cdots \cup V_{q_n}$$

Our requirements are then satisfied by the sets

$$U = U_{q_1} \cap \cdots \cap U_{q_n} \quad W = V_{q_1} \cup \cdots \cup V_{q_n}$$

□

Corollaire 2

Compact subsets of Hausdorff spaces are closed. (the inverse is not true.)

Corollaire 3

If F closed, K compact in a Hausdorff space, then $F \cap K$ is compact.

2.6 Theorem If $\{K_\alpha\}$ is a collection of compact subsets of a Hausdorff space and if $\bigcap_\alpha K_\alpha = \emptyset$, then some finite subcollection of $\{K_\alpha\}$ also has empty intersection.

Use the definition of compact set.

2.7 Theorem Suppose U is open in a locally compact Hausdorff space X , $K \subset U$, and K is compact. Then there is an open set V with compact closure such that

$$K \subset V \subset \bar{V} \subset U.$$

Use the definition of compact set.

9.7 General properties

See Royden Chapter 11.

9.7.1 Bases and subbases

Definition For a topological space (X, T) and a point x in X , a collection of neighborhoods of x , \mathcal{B}_x , is called a **base for the topology** at x provided for any neighborhood \mathcal{U} of x , there is a set B in the collection \mathcal{B}_x for which $B \subseteq \mathcal{U}$. A collection of open sets \mathcal{B} is called a **base for the topology** T provided it contains a base for the topology at each point.

Proposition 2 For a nonempty set X , let \mathcal{B} be a collection of subsets of X . Then \mathcal{B} is a base for a topology for X if and only if

- (i) \mathcal{B} covers X , that is, $X = \bigcup_{B \in \mathcal{B}} B$.
- (ii) if B_1 and B_2 are in \mathcal{B} and $x \in B_1 \cap B_2$, then there is a set B in \mathcal{B} for which $x \in B \subseteq B_1 \cap B_2$.

The unique topology that has \mathcal{B} as its base consists of \emptyset and unions of subcollections of \mathcal{B} .

Definition For a topological space (X, T) , a subcollection \mathcal{S} of T that covers X is called a **subbase for the topology** T provided intersections of finite subcollections of \mathcal{S} are a base for T .

9.7.2 Separation properties

- **The Tychonoff Separation Property:**

- For each two points u and v in X , there is a neighborhood of u that does not contain v and a neighborhood of v that does not contain u .

- **The Hausdorff Separation Property:**

- Each two points in X can be separated by disjoint neighborhoods.

- **The Regular Separation Property:**

- The Tychonoff separation property holds and, moreover, each closed set and point not in the set can be separated by disjoint neighborhoods.

- **The Normal Separation Property:**

- The Tychonoff separation property holds and, moreover, each two disjoint closed sets can be separated by disjoint neighborhoods.

We have defined what we mean by a neighborhood of a point in a topological space. For a subset K of a topological space X , by a **neighborhood of K** we mean an open set that contains K . We say that two disjoint subsets A and B of X can be **separated by disjoint neighborhoods** provided there are neighborhoods of A and B , respectively, that are disjoint. For a topological space X , we consider the following four separation properties:

The Tychonoff Separation Property For each two points u and v in X , there is a neighborhood of u that does not contain v and a neighborhood of v that does not contain u .

The Hausdorff Separation Property Each two points in X can be separated by disjoint neighborhoods.

The Regular Separation Property The Tychonoff separation property holds and, moreover, each closed set and point not in the set can be separated by disjoint neighborhoods.

The Normal Separation Property The Tychonoff separation property holds and, moreover, each two disjoint closed sets can be separated by disjoint neighborhoods.

We naturally call a topological space Tychonoff, Hausdorff, regular, or normal, provided it satisfies the respective separation property.

Proposition 3

A topological space X is a Tychonoff space iff every set consisting of a single point is closed.

Proposition 4

Every metric space is normal.

Proposition 5

Let X be Tychonoff. Then X is normal iff whenever \mathcal{U} is a neighborhood of a closed subset F of X , there is another neighborhood of F whose closure is contained in \mathcal{U} , that is, there is an open set \mathcal{O} for which

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}$$

9.7.3 Countability and Separability

Definition A topological space X is said to be **first countable** provided there is a countable base at each point. The space X is said to be **second countable** provided there is a countable base for the topology.

It is clear that a second countable space is first countable.

Example Every metric space X is first countable since for $x \in X$, the countable collection of open balls $\{B(x, 1/n)\}_{n=1}^{\infty}$ is a base at x for the topology induced by the metric.

In a topological space that is not first countable, it is possible for a point to be a point of closure of a set and yet no sequence in the set converges to the point.

Definition A subset E of topological space X is said to be **dense in X** provided every open set in X contains a point of E . We call X **separable** provided it has a countable dense subset.

The Urysohn Metrization Theorem Let X be a second countable topological space. Then X is metrizable if and only if it is normal.

9.7.4 Strong, weak topology, induced topology, homeomorphism

Definition Given two topologies T_1 and T_2 for a set X , if $T_2 \subseteq T_1$, we say that T_2 is **weaker** than T_1 and that T_1 is **stronger** than T_2 .

Definition Let X be a nonempty set and consider a collection of mappings $\mathcal{F} = \{f_\alpha: X \rightarrow X_\alpha\}_{\alpha \in \Lambda}$, where each X_α is a topological space. The weakest topology for X that contains the collection of sets

$$\mathcal{F} = \left\{ f_\alpha^{-1}(\mathcal{O}_\alpha) \mid f_\alpha \in \mathcal{F}, \mathcal{O}_\alpha \text{ open in } X_\alpha \right\}$$

is called the **weak topology** for X induced by \mathcal{F} .

Definition A continuous mapping from a topological space X to a topological space Y is said to be a **homeomorphism** provided it is one-to-one, maps X onto Y , and has a continuous inverse f^{-1} from Y to X .

9.7.5 Compact and Sequentially Compact topological spaces

Definition A topological space X is said to be **compact** provided every open cover of X has a finite subcover. A subset K of X is called **compact** provided K , considered as a topological space with the subspace topology inherited from X , is compact.

In view of the definition of the subspace topology, a subset K of X is compact provided every covering of K by a collection of open subsets of X has a finite subcover.

Définition 6: finite intersection property

A collection of sets is said to be **finite intersection property** provided every finite subcollection has nonempty intersection.

Proposition 14 A topological space X is compact if and only if every collection of closed subsets of X that possesses the finite intersection property has nonempty intersection.

Proposition 15 A closed subset K of a compact topological space X is compact.

Definition A topological space X is said to be **sequentially compact** provided each sequence in X has a subsequence that converges to a point of X .

Proposition 17 *Let X be a second countable topological space. Then X is compact if and only if it is sequentially compact.*

Theorem 18 *A compact Hausdorff space is normal.*

Proposition 19 *A continuous one-to-one mapping f of a compact space X onto a Hausdorff space Y is a homeomorphism.*

Additionally, homeomorphism requires continuous inverse, so hypothesis upon space is necessary.

Proposition 20 *The continuous image of a compact topological space is compact.*

Regard a compact set as a compact topological space...

Corollary 21 *A continuous real-valued function on a compact topological space takes a maximum and minimum functional value.*

Définition 7: countably compact

A topological space is said to be **countably compact** provided every countable open cover has a finite subcover.

9.7.6 Separate, Connected, Intermediate value property

Définition 8: separate

Two nonempty open subsets of a topological spaces X are said to **separate** X if they are disjoint and their union is X .

Définition 9: connected

A topological space which cannot be separated by such a pair is said to be **connected**.

Connectness is preserved under continuous mapping.

Proposition 22 *Let f be a continuous mapping of a connected space X to a topological space Y . Then its image $f(Y)$ is connected.*

For a set C of real number, the following are equivalent:

- C is an interval.
- C is convex.
- C is connected.

Definition *A topological space X is said to have the intermediate value property provided the image of any continuous real-valued function on X is an interval.*

Proposition 23 *A topological space has the intermediate value property if and only if it is connected.*

9.8 Three Fundamental Theorems

参见汪林《实分析中的反例》

9.9 不同意义收敛的函数序列

设 $\{f_n\}$ 是可测集 E 上的 $p(1 \leq p < +\infty)$ 次幂 Lebesgue 可积函数序列，这时 $\{f_n\}$ 的收敛意义可以按多种意义解释。在这一章里，我们将考虑其中较常见的几种收敛意义，并指出它们之间的蕴涵关系，当没有蕴涵关系时就给出反例。我们要考虑的收敛意义是：

- (1) **一致收敛**: $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ 关于 $x \in E$ 一致地成立。
- (2) **近一致收敛**: 任给 $\delta > 0$ ，存在 E 的可测子集 E_δ ，使在 E_δ 上 $\{f_n\}$ 一致收敛于 f ，而 $m(E \setminus E_\delta) < \delta$ 。
- (3) **几乎处处收敛**: $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ 对于几乎所有的 $x \in E$ 成立。

(4) 测度收敛: 任给 $\varepsilon > 0$, $\lim_{n \rightarrow \infty} m\{x : |f_n(x) - f(x)| \geq \varepsilon\} = 0$.

(5) 平均收敛:

$$\lim_{n \rightarrow \infty} \int_E |f_n(x) - f(x)|^p dx = 0$$

(6) 弱收敛: 当 $1 < p < +\infty$ 时, 对每个 $g \in L^q(E)$, $1/p + 1/q = 1$, 有

$$\lim_{n \rightarrow \infty} \int_E f_n(x)g(x)dx = \int_E f(x)g(x)dx$$

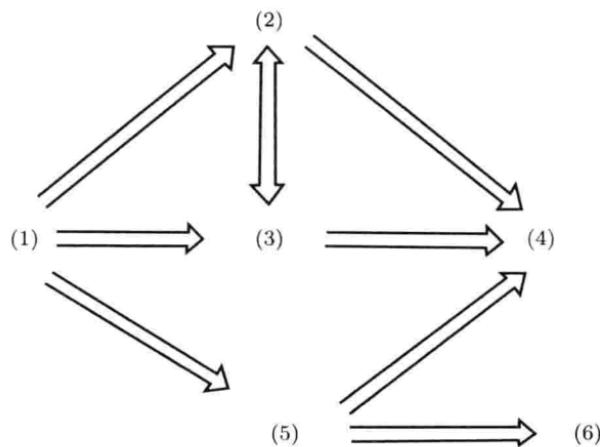
当 $p = 1$ 时, 对每个 $g \in L^\infty(E)$, 有

$$\lim_{n \rightarrow \infty} \int_E f_n(x)g(x)dx = \int_E f(x)g(x)dx$$

Remarque 38. 凡是下面没有出现的蕴含关系, 都存在反例!

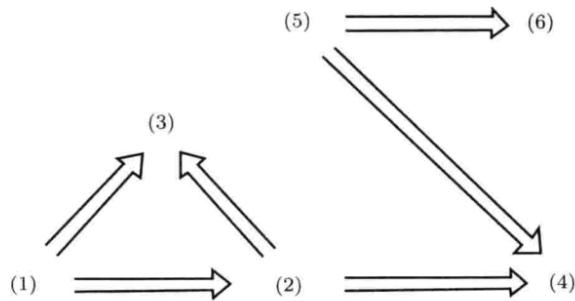
9.9.1 E 的测度有限时

若 $mE < +\infty$, 则有



9.9.2 E 的测度无限时

当 $mE = +\infty$ 时, 有



9.10 函数空间 L^p

参见《实变函数习题精选》徐森林

Remarque 39. 当 $1 \leq p < \infty$ 时, $\lim_{n \rightarrow \infty} f_n = f$ in L^p 与 $\lim_{n \rightarrow \infty} f_n(x) = f(x), x \in E$ 互不蕴含. 当 $p = \infty$ 时, 前者蕴含后者.

Théorème 9: L^{p_1}, L^{p_2} 空间包含关系

设 $m(E) < +\infty$, 且 $1 \leq p_1 < p_2 \leq +\infty$, 则 $L^{p_2}(E) \subset L^{p_1}(E)$, 且

$$\|f\|_{p_1} \leq [m(E)]^{p_1^{-1} - p_2^{-1}} \|f\|_{p_2} \quad (9.1)$$

Théorème 10: L^p 空间两边夹住

设 $f \in L^r(E) \cap L^s(E), 0 < \lambda < 1, \frac{1}{p} = \frac{\lambda}{r} + \frac{1-\lambda}{s}$, 则

$$\|f\|_p \leq \|f\|_r^\lambda \cdot \|f\|_s^{1-\lambda}$$

由此

$$\|f\|_p \leq \max\{\|f\|_r, \|f\|_s\} \quad (9.2)$$

Exercice 12: 逐点收敛加上什么条件能推出 L^p 收敛

设 $1 \leq p < +\infty, g \in L^p(E), f_n \in L^p(E), |f_n(x)| \leq g(x), n = 1, 2, \dots$ 在 E 上, $\lim_{n \rightarrow \infty} f_n(x) = f(x)$. 证明: $\lim_{n \rightarrow \infty} f_n = f$ in L^p .

证明. 由于 $\lim_{n \rightarrow \infty} f_n(x) = f(x)$. 于是逐点考虑

$$\lim_{n \rightarrow \infty} |f_n(x) - f(x)|^p = 0$$

又因为 $|f_n(x)| \leq g(x), n = 1, 2, \dots$, 故

$$|f(x)| = |\lim_{n \rightarrow \infty} f_n(x)| = \lim_{n \rightarrow \infty} |f_n(x)| \leq g(x)$$

$$|f_n(x) - f(x)|^p \leq [|f_n(x)| + |f(x)|]^p \leq 2^p [g(x)]^p$$

注意到 $g \in L^p(E), f_n \in L^p(E)$, 由 Lebesgue 控制收敛定理可知

$$\lim_{n \rightarrow \infty} \int_E |f_n(x) - f(x)|^p dx = \int_E \lim_{n \rightarrow \infty} |f_n(x) - f(x)|^p dx = \int_E 0 dx = 0$$

也就是

$$\lim_{n \rightarrow \infty} f_n = f \quad \text{in } L^p$$

□

Exercice 13: 有关 L^∞

设 $f \in \mathcal{L}^\infty(E), w(x) > 0$ 且 $\int_E w(x)dx = 1$. 证明:

$$\lim_{p \rightarrow +\infty} \left[\int_E |f(x)|^p w(x) dx \right]^{\frac{1}{p}} = \|f\|_\infty$$

证明. 因为 $f \in \mathcal{L}^\infty(E)$, 故 $\|f\|_\infty < +\infty$. 于是

$$\begin{aligned} \left[\int_E |f(x)|^p w(x) dx \right]^{\frac{1}{p}} &\leq \left[\int_E (\|f\|_\infty)^p w(x) dx \right]^{\frac{1}{p}} \\ &= \|f\|_\infty \left[\int_E w(x) dx \right]^{\frac{1}{p}} = \|f\|_\infty \cdot 1 = \|f\|_\infty \end{aligned}$$

另一方面, 对 $\forall \varepsilon > 0, \exists e \subset E, m(e) > 0$, s.t.

$$|f(x)| > \|f\|_\infty - \frac{\varepsilon}{2}, x \in e$$

从而可得

$$\begin{aligned} \left(\int_E |f(x)|^p w(x) dx \right)^{\frac{1}{p}} &> \left(\int_e |f(x)|^p w(x) dx \right)^{\frac{1}{p}} \\ &\geq \left(\|f\|_{\infty} - \frac{\varepsilon}{2} \right) \left(\int_e w(x) dx \right)^{\frac{1}{p}} \end{aligned}$$

又因 $\lim_{p \rightarrow +\infty} \left(\int_e w(x) dx \right)^{\frac{1}{p}} = 1$, 故 $\exists N \in \mathbb{N}$, 当 $p > N$ 时, 有

$$\left(\int_E |f(x)|^p w(x) dx \right)^{\frac{1}{p}} > \|f\|_{\infty} - \varepsilon$$

综上, 当 $p > N$ 时, 有

$$\|f\|_{\infty} - \varepsilon < \left(\int_E |f(x)|^p w(x) dx \right)^{\frac{1}{p}} \leq \|f\|_{\infty} < \|f\|_{\infty} + \varepsilon,$$

因此

$$\lim_{p \rightarrow +\infty} \left(\int_E |f(x)|^p w(x) dx \right)^{\frac{1}{p}} = \|f\|_{\infty}$$

□

Exercice 14: L^p, L^q 空间的包含关系

设 $0 < p_0 < q_0 < \infty$, 若 $L^{p_0}(E) \subset L^{q_0}(E)$ 证明对于 $0 < p < q$ 有

$$L^p(E) \subset L^q(E)$$

Note. 注意这里的包含关系和 eq. (9.1) 是相反的.

证明. 先证明 $m(E) < +\infty$. 反证而设 $m(E) = +\infty$, 取 E 中无交可测子集列 $\{E_n\}$, s.t. $m(E_n) = 1/n^3$. 作函数

$$f(x) = \sum_{n=1}^{\infty} n^{p_0^{-1} + q_0^{-1}} \chi_{E_n}(x)$$

易见

$$\begin{aligned} \|f\|_{p_0}^{p_0} &= \sum_{n=1}^{\infty} n^{(p_0^{-1} + q_0^{-1})p_0} \cdot m(E_n) \\ &= \sum_{n=1}^{\infty} n^{1+p_0/q_0} \cdot n^{-3} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{2-p_0/q_0}} < +\infty \end{aligned}$$

$$\begin{aligned}\|f\|_{q_0}^{q_0} &= \sum_{n=1}^{\infty} n^{(p_0^{-1} + q_0^{-1})q_0} \cdot m(E_n) \\ &= \sum_{n=1}^{\infty} n^{1+q_0/p_0} \cdot n^{-3} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{2-q_0/p_0}} = +\infty\end{aligned}$$

因此 $f \in L^{p_0}(E), f \notin L^{q_0}(E)$. 这与 $L^{p_0}(E) \subset L^{q_0}(E)$ 矛盾.

由于这个包含关系是反的，就会有一些神奇的性质，对于 $f \in L^p(E)$,

$$f^{p/p_0} \in L^{p_0}(E) \subset L^{q_0}(E) \Rightarrow f \in L^{p \cdot \frac{q_0}{p_0}}(E) \Rightarrow f \in L^{p \cdot \left(\frac{q_0}{p_0}\right)^n}(E), \forall n$$

于是可以取 n 使得 $p \cdot \left(\frac{p_0}{q_0}\right)^n > q$, 再用 eq. (9.2) 得到 $f \in L^q(E)$. 因此 $L^p(E) \subset L^q(E)$.

Note. 事实上，利用 eq. (9.1) 可知这里 $L^p(E) = L^q(E)$.

□

Exercice 15: 利用 Holder 不等式

设 $0 < m(E) < \infty$, 令

$$N_p(f) = \left[\frac{1}{m(E)} \int_E |f(x)|^p dx \right]^{1/p}$$

证明：当 $p_1 < p_2$ 时，有 $N_{p_1}(f) \leq N_{p_2}(f)$.

证明. 对于 $p_1 < p_2$, 记 $p = \frac{p_2}{p_1}, p' = \frac{p_2}{p_2 - p_1}$, 则 $\frac{1}{p} + \frac{1}{p'} = \frac{p_1}{p_2} + \frac{p_2 - p_1}{p_2} = 1$, 且

$$\begin{aligned}N_{p_1}(f) &= \left[\int_E |f(x)|^{p_1} \cdot \frac{1}{m(E)} dx \right]^{1/p_1} = \| |f|^{p_1} \cdot [m(E)]^{-1} \|_{L_1(E)}^{1/p_1} \\ &\leq [\| |f|^{p_1} \|_{L_p(E)} \cdot \| [m(E)]^{-1} \|_{L_{p'}(E)}]^{1/p_1} \\ &= \left[\left(\int_E |f(x)|^{p_2} dx \right)^{p_1/p_2} \cdot \left(\int_E [m(E)]^{-p'} dx \right)^{1/p'} \right]^{1/p_1} \\ &= \left[\left(\int_E |f(x)|^{p_2} dx \right)^{p_1/p_2} \cdot [m(E)]^{\overbrace{1/p' - 1}^{-p_1/p_2}} \right]^{1/p_1} \\ &= \left[\frac{1}{m(E)} \int_E |f(x)|^{p_2} dx \right]^{1/p_2} = N_{p_2}(f)\end{aligned}$$

□

9.10.1 L^2 空间 (Hilbert 空间)

$f, g \in L^2(E) \Rightarrow |fg| \leq \frac{1}{2}(f^2 + g^2) \Rightarrow fg \in L^1(E)$. 于是可以定义内积

$$\langle \cdot \rangle : L^2(E) \times L^2(E) \rightarrow \mathbb{R} \quad (f, g) \mapsto \langle f, g \rangle := \int_E f(x)g(x) dx$$

定义 f, g 的夹角 (进而有垂直的概念)

$$\cos \theta = \frac{\langle f, g \rangle}{\|f\|_2 \cdot \|g\|_2} \in [-1, 1]$$

Théorème 11: $L^2(E)$ 收敛蕴含弱收敛

若 $\|f_n - f\|_{L^2(E)} \rightarrow 0$ 则 $\langle f_n, g \rangle \rightarrow \langle f, g \rangle, \forall g \in L^2(E)$.

Remarque 40. 弱收敛不蕴含 $L^2(E)$ 收敛, 也不蕴含几乎处处收敛, 也不蕴含依测度收敛.

设 $\{\varphi_i : i \in \mathbb{N}\}$ 是 $L^2(E)$ 中的规范正交系¹² 对于 $f \in L^2(E)$ 定义 f 关于 $\{\varphi_i\}$ 的广义 Fourier 系数为

$$c_i = \langle f, \varphi_i \rangle := \int_E f(x)\varphi_i(x) dx \quad i = 1, 2, \dots$$

广义 Fourier 级数为

$$\sum_{i=1}^{\infty} c_i \varphi_i(x)$$

定理 4.2.4 设 $\{\varphi_i\}$ 为 $L^2(E)$ 中的规范正交系, $f \in L^2(E)$, 取定 k , 作

$$f_k(x) = \sum_{i=1}^k a_i \varphi_i(x),$$

其中 $a_i (i=1, 2, \dots, k)$ 为实数, 则当 $a_i = c_i = \langle f, \varphi_i \rangle (i=1, 2, \dots, k)$ 时, $\|f - f_k\|_2$ 达到最小值.

定理 4.2.5 (Bessel 不等式) 设 $\{\varphi_i\}$ 为 $L^2(E)$ 中的规范正交系, 且 $f \in L^2(E)$, 则 f 的广义 Fourier 系数 $\{c_i\}$ 满足

$$\sum_{i=1}^{\infty} c_i^2 \leq \|f\|_2^2,$$

并称它为 **Bessel 不等式**.

定义 4.2.4 设 $\{\varphi_i\}$ 为 $L^2(E)$ 中的规范正交系, $f \in L^2(E)$, $c_i = \langle f, \varphi_i \rangle$, 则称

$$\|f\|_2^2 = \sum_{i=1}^{\infty} c_i^2$$

为 f 的封闭公式, 也称为 **Parseval 等式**. 如果对 $\forall f \in L^2(E)$ 封闭公式成立, 则称 $\{\varphi_i\}$ 是封闭的.

¹可以证明该集合至多可数

²正交意味着任意两个元垂直, 规范意味着模为 1.

定义 4.2.5 设 $\{\varphi_i\} \subset \mathcal{L}^2(E)$, 如果 $\mathcal{L}^2(E)$ 中不存在非零元素 $[f]$ 与一切 $\varphi_i (\forall i)$ 都正交, 则称 $\{\varphi_i\}$ 为 $\mathcal{L}^2(E)$ 中的完全系. 换言之, 如果 $f \in \mathcal{L}^2(E)$, 且 $\langle f, \varphi_i \rangle = 0 (\forall i)$, 则必有 $f(x) \stackrel{m}{=} 0$.

定理 4.2.6 (Riesz-Fischer) 设 $\{\varphi_i\}$ 为 $\mathcal{L}^2(E)$ 中的规范正交系. 如果 $\{c_i\}$ 为满足

$$\sum_{i=1}^{\infty} c_i^2 < +\infty$$

的任一实数列, 则 $\exists f \in \mathcal{L}^2(E)$, s.t.

$$\langle f, \varphi_i \rangle = c_i, i = 1, 2, \dots,$$

即 $f \sim \sum_{i=1}^{\infty} c_i \varphi_i$.

令 $S_k = \sum_{i=1}^k c_i \varphi_i$, 则在 $\mathcal{L}^2(E)$ 中, S_k 收敛于 f , 即

$$(\mathcal{L}^2) \lim_{k \rightarrow +\infty} S_k = f$$

或

$$\left\| \sum_{i=1}^k c_i \varphi_i - f \right\|_2 = \| S_k - f \|_2 \rightarrow 0 (k \rightarrow +\infty),$$

且有封闭公式 $\| f \|_2^2 = \sum_{i=1}^{\infty} c_i^2$.

定理 4.2.7 设 $\{\varphi_i\}$ 为 $\mathcal{L}^2(E)$ 中的规范正交系.

(1) $\{\varphi_i\}$ 为 $\mathcal{L}^2(E)$ 中的完全系

\Leftrightarrow (2) 对 $\forall f \in \mathcal{L}^2(E)$, 有

$$\lim_{k \rightarrow +\infty} \left\| f - \sum_{i=1}^k c_i \varphi_i \right\|_2 = \lim_{k \rightarrow +\infty} \| f - S_k \|_2 = 0,$$

其中 $c_i = \langle f, \varphi_i \rangle, i = 1, 2, \dots$

\Leftrightarrow 对 $\forall f \in \mathcal{L}^2(E)$, 封闭公式 (Parseval 等式)

$$\| f \|_2^2 = \sum_{i=1}^{\infty} c_i^2$$

成立, 其中 $c_i = \langle f, \varphi_i \rangle$, 即 $\{\varphi_i\}$ 是封闭的.

定理 4.2.8 (推广的 Parseval 等式) 设 $f, g \in \mathcal{L}^2(E)$, $\{\varphi_i\}$ 为 $\mathcal{L}^2(E)$ 上的规范正交的完全系. 而 a_k, b_k 分别为 f, g 关于 $\{\varphi_i\}$ 的广义 Fourier 系数, 则

$$\int_E f(x) g(x) dx = \sum_{i=1}^{\infty} a_i b_i.$$

定理 4.2.9 (逐项积分) 设 $\{\varphi_i\}$ 为 $\mathcal{L}^2(E)$ 的规范正交系, $f \in \mathcal{L}^2(E)$ 的广义 Fourier 级数为

$$f(x) \sim \sum_{i=1}^{\infty} c_i \varphi_i(x),$$

则对任何 Lebesgue 可测集 $E_1 \subset E, m(E_1) < +\infty$, 有

$$\int_{E_1} f(x) dx = \sum_{i=1}^{\infty} c_i \int_{E_1} \varphi_i(x) dx.$$

定理 4.2.10 (惟一性定理) 设 $f, g \in \mathcal{L}^2(E)$, 且它们关于规范正交的完全系 $\{\varphi_i\}$ 有相同的广义 Fourier 级数(即有相同的广义 Fourier 系数), 则在 E 上, 有 $f \stackrel{m}{=} g$.

此处省略习题.... (太多了) 参见《实变函数习题精选》徐森林

9.11 测度空间、可测函数的收敛性、Lebesgue 可测函数的结构

Théorème 12: 用连续函数刻画 Lebesgue 可测函数

设 $E \subset \mathbb{R}^n$ 为 Lebesgue 可测集, f 为 E 上的 Lebesgue 可测函数, 则对于任意 $\delta > 0$, 存在 E 的闭子集 F_δ , 使得 $m(E - F_\delta) < \delta$, 且 f 为 F_δ 上的连续函数.

将 $f|_{F_\delta}$ 延拓到 \mathbb{R}^n 上, 我们有定理 Théorème 12 的另一种表述形式:

Théorème 13

设 $E \subset \mathbb{R}^n$ 为 Lebesgue 可测集, f 为 E 上的 Lebesgue 可测函数, 则对于任意 $\delta > 0$, 必有 \mathbb{R}^n 上的连续函数 h , 使得

$$m\{x \in E : f(x) \neq h(x)\} < \delta$$

如果 $|f| \leq M$ (或 $< M$) 则上述 h 可以同样 $|h| \leq M$ (或 $< M$).

Remarque 41. h 可以被选为具有紧³的支撑的函数.

一个推论是:

Corollaire 4

设 f 为 $E \subset \mathbb{R}^n$ 上几乎处处有限的 Lebesgue 可测函数, 则存在 \mathbb{R}^n 上的连续函数列 $\{f_k\}$ 使得在 E 上

$$\lim_{k \rightarrow \infty} f_k(x) = f(x)$$

连续函数 f 复合上 Lebesgue 可测函数 g 得到的 $f \circ g$ 依然可测, 这由 Lebesgue 可测性的“任意开集逆像为可测集”定义可得. 但 $g \circ f$ 不一定可测.

³ 在 \mathbb{R}^n 中, 由海涅定理, 等价于有界闭集.

Exercice 16: 逼近 · 几乎处处有限的可测函数

设 (X, \mathcal{R}, μ) 为测度空间, f 为 $E \in \mathcal{R}$ 上的几乎处处有限的可测函数, $\mu(E) < +\infty$. 证明: 对于任意 $\epsilon > 0$, 存在 E 上的有界可测函数 $g(x)$, 使得

$$\mu(\{x \in E : |f(x)g(x)| > 0\}) < \epsilon.$$

证明. 因为 f 为 $E \in \mathcal{R}$ 上的几乎处处有限的可测函数, 所以

$$F := \{x \in E : |f(x)| = +\infty\}$$

$$F_n := \{x \in E : |f(x)| > n\} = \{x \in E : |f(x)| < -n\} \cup \{x \in E : |f(x)| > n\}$$

都是可测集, 且 $\mu(F) = 0$, F_n 关于 n 单调递减, $F = \bigcap_{n=1}^{\infty} F_n$. 由于 $\mu(F_n) \leq \mu(E) < +\infty$, 所以有测度的下连续性

$$0 = \mu(F) = \mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \mu\left(\lim_{n \rightarrow \infty} F_n\right) = \lim_{n \rightarrow \infty} \mu(F_n)$$

因此对于任意 $\epsilon > 0$, 存在 $N > 0$, 使得当 $n > N$ 时, $\mu(F_n) < \epsilon$, 令

$$g(x) := \begin{cases} f(x) & x \in E - F_n \\ 0 & x \in F_n \end{cases}$$

即可. □

Remarque 42. 也可以反证.

证明. 假设存在 $\epsilon_0 > 0$, 使得对于任意 E 上的有界可测函数 $g(x)$ 有

$$\mu(\{x \in E : |f(x) - g(x)| > 0\}) \geq \epsilon_0$$

记 $E_n = E(|f| \leq n)$, 取

$$g_n(x) = \begin{cases} f(x) & x \in E_n \\ n & x \in E(f > n) \\ -n & x \in E(f < -n) \end{cases}$$

简单验证可知 g 有界可测. 于是

$$\underbrace{\mu(\{x \in E : |f(x)| > n\})}_{\rightarrow \mu(\{x \in E : |f(x)| = \infty\}) \text{ as } n \rightarrow \infty} = \mu(\{x \in E : |f(x) - g_n(x)| > 0\}) \geq \epsilon_0$$

这与 f 几乎处处有界矛盾! □

Définition 10: Baire 函数 (Borel 可测函数)

【180】 \mathcal{B} 是 \mathbb{R}^1 上的 Borel 集的全体, $(\mathbb{R}^1, \mathcal{B})$ 是 Borel 可测空间, 设 f 是定义在 $E \subset \mathbb{R}^1$ 上的有限实函数, 如果对 $\forall c \in \mathbb{R}$, 集合 $E(c \leq f)$ 都是 Borel 集, 则称 f 是 E 上的 **Borel 可测函数**, 也称作 **Baire 函数**.

设 f 在 E 上是 Borel 可测的, 则 f 必是 E 上 Lebesgue 可测函数.

证明 因为 f 在 E 上是 Borel 可测的, 所以对 $\forall c \in \mathbb{R}$, $E(c \leq f) \in \mathcal{B} \subset \mathcal{L}$, 因而 f 在 E 上是 Lebesgue 可测的. \square

Exercice 17: Baire 函数很接近有限的 Lebesgue 可测函数

设 f 是 $E \in \mathcal{R}$ 上有限的 Lebesgue 可测函数, 则一定存在全直线 \mathbb{R}^1 上的 Borel 可测函数 h 使得

$$m(E(f \neq h)) = 0$$

证明 根据例 3.3.2, 存在 E 上一列函数 $\{f_n\}$, 每个 f_n 是 Lebesgue 可测集 (E 的子集) 的特征函数的线性组合, 即

$$f_n = \sum_{i=1}^{l_n} \alpha_i^n \chi_{E_i^n}, \text{ s. t. } \{f_n\} \text{ 在 } E \text{ 上处处收敛于 } f.$$

又根据定理 2.3.1(5)知, 对每个 E_i^n , 存在 F_n 集 (Borel 集) B_i^n , s. t. $E_i^n \supset B_i^n$, 而且, $m(E_i^n - B_i^n) = 0$.

作直线 \mathbb{R}^1 上的函数

$$h_n = \sum_{i=1}^{l_n} \alpha_i^n \chi_{B_i^n}, n = 1, 2, \dots.$$

显然, $h_n (n=1, 2, \dots)$ 都是 Borel 可测函数. 而且

$$E(f_n \neq h_n) \subset \bigcup_{i=1}^{l_n} (E_i^n - B_i^n).$$

因此

$$m(E(f_n \neq h_n)) = 0.$$

记 $E_0 = \bigcup_{n=1}^{\infty} E(f_n \neq h_n)$. 显然, $m(E_0) = 0$. 所以

$$\lim_{n \rightarrow +\infty} h_n(x) = f(x), x \in E - E_0.$$

再根据定理 2.3.1(6)知, 对 E_0 , 有 Borel 集 $B_0 \supset E_0$, 适合 $m(B_0) = 0$. 令 $B_1 = \mathbb{R}^1 - B_0$, B_1 是 Borel 集. 因此, 从上式得到

$$\lim_{n \rightarrow +\infty} \chi_{B_1}(x) h_n(x) = \chi_{B_1}(x) f(x), \quad x \in E \cap B_1.$$

当 $x \in E - B_1$ 时, 上述两边在这种点上的值都是零. 于是, 上式实际上在 E 上也成立.

对原来只定义在 E 上的函数 $\tilde{h}(x) = \chi_{B_1}(x) f(x)$ 补充定义它在 $\mathbb{R}^1 - E$ 上的值为零, 补充定义后所得的全直线 \mathbb{R}^1 上定义的函数记为 $h(x)$. 显然, 在全直线 \mathbb{R}^1 上, 有

$$h(x) = \lim_{n \rightarrow +\infty} \chi_{B_1}(x) h_n(x).$$

因为 $\{\chi_{B_1} h_n\}$ 为直线 \mathbb{R}^1 上的 Borel 可测函数列. 由定理 3.1.4, h 为直线 \mathbb{R}^1 上的 Borel 可测函数. 显然, $E(f \neq h) \subset B_0$, 因而 $m(E(f \neq h)) = 0$. \square

证明.

□

9.12 积分理论

Théorème 14: Levi 定理

设 (X, \mathcal{R}, μ) 为测度空间, $\{f_k\}$ 为 $E \in \mathcal{R}$ 上的非负广义可测递增函数列, 且 $\lim_{k \rightarrow \infty} f_k(x) = f(x), \forall x \in E$. 则

$$\lim_{k \rightarrow \infty} \int_E f_k d\mu = \int_E f d\mu = \int_E \lim_{k \rightarrow \infty} f_k d\mu$$

Exercice 18

f 为 Lebesgue 可测集 $E \subset \mathbb{R}^n$ 上的几乎处处大于零的 Lebesgue 可测函数, 且满足

$$\int_E f dm = 0$$

证明: $m(E) = 0$.

证明. 考虑反证, 我们知道

$$E = E(f \leq 0) \bigcup \left(\bigcup_{k=1}^{\infty} E \left(f \geq \frac{1}{k} \right) \right)$$

由题意 $m(E(f \leq 0)) = 0$, 若 $m(E) > 0$ 则存在 k_0 使得 $m(E(f \geq \frac{1}{k_0})) > 0$, 那么

$$0 = \int_E f dm \geq \int_{E(f \geq \frac{1}{k_0})} f dm \geq \frac{1}{k_0} m(E(f \geq \frac{1}{k_0})) > 0$$

矛盾!

□

Exercice 19

设 f 为 $E \subset \mathbb{R}^n$ 上非负 Lebesgue 可积函数

$$E_k := \{x \in E : f(x) \geq k\} \quad k = 1, 2, \dots$$

证明: $\sum_{k=1}^{\infty} m(E_k) < +\infty$.

证明. 记 $F_k = \{x \in E : k \leq f(x) < k+1\}, k = 0, 1, \dots$, 则 $E = \bigsqcup_{k=0}^{\infty} F_k$, 于是

$$\begin{aligned} +oo \int_E f dm &\stackrel{f \in L^1}{>} \sum_{k=0}^{\infty} \int_{F_k} f dm \geq \sum_{k=0}^{\infty} k \cdot m(F_k) \\ &= [m(F_1) + m(F_2) + m(F_3) + \dots] + [m(F_2) + m(F_3) + \dots] + \dots \\ &= \sum_{k=1}^{\infty} m(E_k) \end{aligned}$$

□

Exercice 20

设 f 为 $E \subset \mathbb{R}^n$ 上的 Lebesgue 可测函数, $m(E) < +\infty$. 证明: f^2 为 E 上的 Lebesgue 可积函数 $\Leftrightarrow \sum_{k=1}^{\infty} k \cdot m(\{x \in E | |f(x)| > k\}) < +\infty$. 如果 $m(E) = +\infty$, 举例说明充分性不成立.

证明. 令 $E_k = \{x \in E | |f(x)| > k\}$, 则

$$\begin{aligned} k^2 [m(E_k) - m(E_{k+1})] &\leq \int_{E_k - E_{k+1}} f^2 dm \leq (k+1)^2 [m(E_k) - m(E_{k+1})], \\ \sum_{k=0}^{\infty} k^2 [m(E_k) - m(E_{k+1})] &\leq \int_E f^2 dm \leq \sum_{k=0}^{\infty} (k+1)^2 [m(E_k) - m(E_{k+1})], \\ \sum_{k=1}^{\infty} km(E_k) &\leq \sum_{k=0}^{\infty} (2k+1)m(E_{k+1}) \leq \int_E f^2 dm \leq \sum_{k=0}^{\infty} (2k+1)m(E_k) \\ &\leq 3 \sum_{k=1}^{\infty} km(E_k) + m(E_0) \leq 3 \sum_{k=1}^{\infty} km(E_k) + m(E). \end{aligned}$$

由此推得 f^2 在 E 上 Lebesgue 可积 $\Leftrightarrow \sum_{k=1}^{\infty} km(E_k) = \sum_{k=1}^{\infty} k \cdot m(\{x \in E | |f(x)| > k\}) < +\infty$.

举出反例: $f(x) = 1, \forall x \in E = \mathbb{R}^n$. 于是 $\sum km(E_k) = 0$ 但是 $f \notin L^2(E)$.

□

Exercice 21

设

$$\int_0^{2\pi} |f(x)| \cdot \ln(1 + |f(x)|) dx < +\infty$$

证明: $f \in L([0, 2\pi])$.

证明. 设 $E_1 = \{x \in [0, 2\pi] : |f(x)| < e - 1\}$, $E_2 = \{x \in [0, 2\pi] : |f(x)| \geq e - 1\}$. 则当 $x \in E_2$ 时,

$$\ln(1 + |f(x)|) \geq \ln(1 + e - 1) = 1$$

于是

$$\begin{aligned} \int_0^{2\pi} |f(x)| dx &\leq \int_{E_1} |f(x)| dx + \int_{E_2} |f(x)| dx \\ &\leq (e - 1)m(E_1) + \int_{E_2} |f(x)| \cdot \ln(1 + |f(x)|) dx \\ &\leq (e - 1)m(E_1) + \int_0^{2\pi} |f(x)| \cdot \ln(1 + |f(x)|) dx < +\infty \end{aligned}$$

于是 $f \in L([0, 2\pi])$.

□

9.13 勒贝格微分定理, 勒贝格点

Théorème 15: 勒贝格微分定理

设 $f \in L^1_{loc}(\mathbb{R}^d)$, 则对于几乎处处的 $x \in \mathbb{R}^d$, 我们有

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) dy = f(x).$$

Définition 11: Lebesgue 点

设 $f \in L^1_{loc}(\mathbb{R}^n)$. 对 \mathbb{R}^n 中的 x , 如果

$$\lim_{r \rightarrow 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y) - f(x)| dy = 0,$$

那么称 x 为 f 的 Lebesgue 点.

若 $f \in L^1_{loc}(E)$, 那么 E 几乎处处为 f 的 Lebesgue 点.

第十章 数理统计

10.1 Bootstrap

The bootstrap method estimates standard errors and computes confidence intervals for a statistic $T_n = g(X_1, \dots, X_n)$. It aims to estimate $\mathbb{V}_F(T_n)$, the variance of T_n , which depends on the unknown distribution function F . The method involves two steps:

- Step 1: Estimate $\mathbb{V}_F(T_n)$ with $\mathbb{V}_{\widehat{F}_n}(T_n)$.
- Step 2: Approximate $\mathbb{V}_{\widehat{F}_n}(T_n)$ using simulation.

For $T_n = \bar{X}_n$, Step 1 gives $\mathbb{V}_{\widehat{F}_n}(T_n) = \widehat{\sigma}^2/n$. In more complex cases, Step 2 is needed to approximate $\mathbb{V}_{\widehat{F}_n}(T_n)$ because a simple formula cannot be derived.

10.1.1 Simulation

Théorème 1: Law of Large Numbers

Let X_1, X_2, \dots be i.i.d. with $\mathbb{E}(X_i) = \mu$ and $\mathbb{V}(X_i) = \sigma^2 < \infty$. Let $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$. Then, for every $\epsilon > 0$,

$$\mathbb{P}(|\bar{X}_n - \mu| > \epsilon) \rightarrow 0$$

as $n \rightarrow \infty$. We say that \bar{X}_n converges in probability to μ and write $\bar{X}_n \xrightarrow{P} \mu$.

Let Y_1, \dots, Y_n i.i.d. then by law of large numbers, $\overline{h(Y_i)}_n \rightarrow \mathbb{E}(h(Y))$. Let $h(y) = (y - \bar{Y})^2$, then

$$\frac{1}{n} \sum_{j=1}^n (Y_j - \bar{Y})^2 \xrightarrow{P} \mathbb{V}(Y)$$

Hence, we can use the sample variance of the simulated values to approximate $\mathbb{V}(Y)$.

10.1.2 Bootstrap Variance Estimation

We can approximate $\mathbb{V}_{\widehat{F}_n}(T_n)$ by simulation. To simulate from the distribution of T_n when the data are assumed to have distribution \widehat{F}_n , we simulate X_1^*, \dots, X_n^* from \widehat{F}_n and then compute $T_n^* = g(X_1^*, \dots, X_n^*)$. Drawing an observation from \widehat{F}_n is equivalent to drawing one point at random from the original data set. Thus, to simulate $X_1^*, \dots, X_n^* \sim \widehat{F}_n$ it suffices to draw n observations with replacement from X_1, \dots, X_n .

Bootstrap Variance Estimation

- (1) Draw $X_1^*, \dots, X_n^* \sim \widehat{F}_n$.
- (2) Compute $T_n^* = g(X_1^*, \dots, X_n^*)$.
- (3) Repeat steps 1 and 2, B times, to get $T_{n,1}^*, \dots, T_{n,B}^*$.
- (4) Let

$$v_{\text{boot}} = \frac{1}{B} \sum_{b=1}^B \left(T_{n,b}^* - \frac{1}{B} \sum_{r=1}^B T_{n,r}^* \right)^2$$

10.1.3 Bootstrap Confidence Intervals

There are three methods to construct bootstrap confidence intervals.

See All of Statistic p 111

Note. 主要通过程序计算。

10.2 Probability and Distributions

Section 3 introduces the fundamental definitions of probability functions on sets endowed with a σ -algebra structure. Key concepts discussed include the interchange of $\lim_{n \rightarrow \infty}$ and $P(\cdot)$, properties of disjoint unions (\sqcup), and DeMorgan's laws.

Section 4 elaborates on conditional probability and independence, presenting them as generalizations of probability theory with broad applicability. The conditional probability is defined computationally as $P(B|A) = P(AB)/P(A)$,

while independence is defined as $P(B|A) = P(B)$, implying $P(AB) = P(A)P(B)$. The section concludes with a discussion of simulation methods for approximating probabilities in complex real-world scenarios, along with an introduction to error estimation, which is further explored in Chapter 4 in the context of **confidence intervals**.

Section 5 introduces the concept of random variables as functions mapping from a sample space (equipped with a sigma algebra) to \mathbb{R} . It distinguishes between the probability mass function (pmf) for discrete distributions and the probability density function (pdf) for continuous distributions, denoted by $p_X(x)$. The cumulative distribution function (cdf) is defined as $F_X(x) = P_X((-\infty, x]) = P(\{c \in \mathcal{C} : X(c) \leq x\})$, where \mathcal{C} represents the sample space. The notion of **equality in distribution** is defined as $X \stackrel{D}{=} Y$ if and only if $F_X = F_Y$. However, it is noted that $X \stackrel{D}{=} Y$ does not necessarily imply $X = Y$ (e.g., $X \sim U(0, 1), Y = 1 - X$). The section also outlines the properties of cdfs, including being nondecreasing, having a lower limit of 0, an upper limit of 1, and being right continuous. Additionally, it establishes that $P_X((a, b]) = F_X(b) - F_X(a)$.

Section 6 focuses on discrete random variables, whose sample space is at most countable, and explores transformations between random variables, i.e., $Y = g(X)$. In the case where g is one-to-one, the pmf of Y is given by $p_Y = p_X(g^{-1}(y))$ (X and Y are discrete). When g is piecewise one-to-one, the definition of p_Y requires a case-by-case analysis.

Section 7 introduces continuous random variables, where the cdf is given by $F_X(x) = \int_{-\infty}^x f_X(t) dt$. The quantile of order p of the distribution of X is defined as a value ξ_p such that $P(X < \xi_p) \leq p$ and $P(X \leq \xi_p) \geq p$, also known as the $(100p)$ th **percentile** of X . It is noted that the quantile may not be unique and that $\xi_p \in F_X^{-1}(p)$. For transformations, the pdf of Y is given by $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$ for y in $\mathcal{S}_Y = g(\mathcal{S}_X)$, the support of Y . For continuous random vectors, $\frac{dx}{dy}$ refers to the Jacobian of the transformation.

Section 8 discusses the **expectation** $E(X) = \int_{-\infty}^{\infty} xf(x) dx$, and $E(g(X)) = \int_{-\infty}^{\infty} g(x)f_X(x) dx$.

Section 9 covers specific expectations, including the mean $\mu = E(X)$, the variance $\sigma^2 = E((X - E(X))^2)$, and the **moment generating function** (mgf)

$M(t) := E(e^{tX})$ (if it exists for $t \in (-h, h)$). The derivatives of $M(t)$ are explored, noting that $M'(0) = \mu$, $M''(0) = E(X^2)$, and $M^{(m)}(0) = E(X^m)$ is called the m th moment of X .

Section 10 presents important inequalities: the existence of $E[|X|^m]$ implies the existence of $E[|X|^k]$ for $k \leq m$. Markov's Inequality is stated as $P(u(X) \geq c) \leq \frac{E(u(X))}{c}$ for nonnegative u and positive c . Chebyshev's Inequality is given as: for X with $\sigma^2 < \infty$, for $k > 0$ we have $P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$. Jensen's Inequality states that for a convex function ϕ ($\phi'' \geq 0$), we have $\phi(E(X)) \leq E(\phi(X))$.

Chapter 2 delves into multivariate distributions. Section 1 introduces the basic definitions and calculation techniques for two random variables. Section 2 presents transformations and the **moment generating function (mgf) technique**, which is effective for linear functions of random variables. If $Y = g(X_1, X_2)$ then $E(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_1, x_2) f_{X_1, X_2}(x_1, x_2) dx_1 dx_2$.

Section 3 introduces conditional distributions and expectations. The conditional pmf is given by $p_{X_1|X_2}(x_1|x_2) = \frac{p_{X_1, X_2}(x_1, x_2)}{p_{X_2}(x_2)}$, $x_1 \in \mathcal{S}_{X_1}$, and the conditional pdf is $f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1, X_2}(x_1, x_2)}{f_{X_2}(x_2)}$, $f_{X_2}(x_2) > 0$. The probability $P(X_2 \in (a, b)|X_1 = x_1) = \int_a^b f_{2|1}(x_2|x_1) dx_2$ and $P(X_1 \in (c, d)|X_2 = x_2) = \int_c^d f_{1|2}(x_1|x_2) dx_1$. The **conditional expectation** of $u(X_2)$ (u is a function of X_2) is $E[u(X_2)|x_1] = \int_{-\infty}^{\infty} u(x_2) f_{2|1}(x_2|x_1) dx_2$. The **conditional variance** is $\text{Var}(X_2|x_1) = E(X_2^2|x_1) - [E(X_2|x_1)]^2$. Conditional expectation satisfies $E[E(X_2|X_1)] = E(X_2)$, $\text{Var}[E(X_2|X_1)] \leq \text{Var}(X_2)$ which can serve as an alternative definition of conditional expectation.

Section 4 introduces the concept of independent random variables. If X_1, X_2 have the joint pdf $f(x_1, x_2)$, then X_1, X_2 are independent if and only if $f(x_1, x_2)$ can be written as $f(x_1, x_2) = g(x_1)h(x_2)$, where g and h are nonnegative functions. For independent X_1, X_2 , we have $E[u(X_1)v(X_2)] = E[u(X_1)]E[v(X_2)]$. If $M(t_1, t_2) = E(e^{t_1 X_1 + t_2 X_2})$ is the joint mgf, then X_1, X_2 are independent if and only if $M(t_1, t_2) = M(t_1, 0)M(0, t_2)$.

Section 5 introduces the correlation coefficient $\rho = \frac{\text{cov}(X, Y)}{\sigma_1 \sigma_2}$, where the covariance $\text{cov}(X, Y) = E[(X - \mu_1)(Y - \mu_2)]$. Independence implies that random variables are uncorrelated.

Section 6 extends the discussion to several random variables. X_1, X_2, \dots, X_n

are said to be mutually independent if and only if $f(x_1, \dots, x_n) = f_1(x_1) \dots f_n(x_n)$.

Consequently, $E[u_1(X_1) \dots u_n(X_n)] = E[u_1(X_1)] \dots E[u_n(X_n)]$. Let $\mathbf{X} = (X_1, \dots, X_n)'$ be a n -dimensional random vector, $E(\mathbf{X}) = (E(X_1), \dots, E(X_n))'$. If \mathbf{W} is an $m \times n$ matrix of random variables then the **mean** of \mathbf{W} is $E[\mathbf{W}] = [E(W_{ij})]$, the **variance-covariance matrix** is $Cov(\mathbf{X}) = E[(\mathbf{X} - \mu)(\mathbf{X} - \mu)'] = [\sigma_{ij}]$ ($\sigma_{ij} = Cov(X_i, X_j)$), and $Cov(\mathbf{X}) = E[\mathbf{X}\mathbf{X}'] - \mu\mu'$ and $Cov(\mathbf{A}\mathbf{X}) = \mathbf{A}Cov(\mathbf{X})\mathbf{A}'$. All variance-covariance matrices are positive semi-definite, because $\mathbf{a}'Cov(\mathbf{X})\mathbf{a} = Var(\mathbf{a}'\mathbf{X}) = Var(Y) \geq 0$ where $Y = \mathbf{a}'\mathbf{X}$ is a random variable.

Section 7 discusses transformations for several random variables. Section 8 addresses the linear combinations of random variables. If X_1, \dots, X_n are iid, then these random variables constitute a **random sample** of size n from that common distribution. The **sample variance** is defined by $S^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X})^2 = (n-1)^{-1} \left(\sum_{i=1}^n X_i^2 - n\bar{X}^2 \right)$.

10.3 Binomial and Related Distributions

$X \sim b(1, p)$, has the **Bernouli distribution**, with pmf:

$$p(x) = p^x(1-p)^{1-x}, \quad x = 0, 1$$

$$\mu = p, \sigma^2 = p(1-p).$$

For $X \sim b(n, p)$, the pmf is

$$p(x) = \binom{n}{x} p^x(1-p)^{n-x}, \quad x = 0, 1, \dots, n$$

The mgf of a binomial distribution is

$$M(t) = \sum_x e^{tx} p(x) = \sum_{x=0}^n \binom{n}{x} (pe^t)^x (1-p)^{n-x} = [(1-p) + pe^t]^n$$

Then $E(X) = M'(0) = np, \sigma^2 = M''(0) - \mu^2 = np(1-p)$.

10.4 Negative Binomial and Geometric Distributions

$$p_Y(y) = \binom{y+r-1}{r-1} p^r (1-p)^y \quad y = 0, 1, 2, \dots$$

A distribution with a pmf of the form $p_Y(y)$ is called a **negative binomial distribution**, which means the probability that exactly $r - 1$ successes in the first $y + r - 1$ trials and a success on the $(y + r)$ th trial. Its mgf is $M(t) = p^r[1 - (1 - p)e^t]^{-r}$, for $t < -\log(1 - p)$.

If $r = 1$, then $p_Y(y) = p(1 - p)^y$, $y = 0, 1, 2, \dots$ and $M(t) = p[1 - (1 - p)e^t]^{-1}$. We say that Y has a **geometric distribution**.

10.5 Multinomial Distribution

$X_1 + \dots + X_k = n$, $p_1 + \dots + p_k = 1$, we say $(X_1, X_2, \dots, X_{k-1})$ has a **multinomial distribution** with parameters n and p_1, \dots, p_{k-1} if

$$P(X_1 = x_1, \dots, X_{k-1} = x_{k-1}) = \frac{n!}{x_1! \dots x_{k-1}! x_k!} p_1^{x_1} \dots p_{k-1}^{x_{k-1}} p_k^{x_k}$$

The conditional pmf is interesting.

$$p_{2|1}(x_2|x_1) = \frac{p_{12}(x_1, x_2)}{p_1(x_1)} = \binom{n-x_1}{x_2} \left(\frac{p_2}{1-p_1}\right)^{x_2} \left(1 - \frac{p_2}{1-p_1}\right)^{n-x_1-x_2}$$

10.6 Hypergeometric Distribution

$$p(x) = \frac{\binom{N-D}{n-x} \binom{D}{x}}{\binom{N}{n}} \quad x = 0, 1, \dots, n$$

We say that X has a **hypergeometric distribution** with parameters (N, D, n) .

The mean of X is $E(X) = n \frac{D}{N}$ and $Var(X) = n \frac{D}{N} \frac{N-D}{N} \frac{N-n}{N-1}$.

10.7 The Poisson Distribution

$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$, $x = 0, 1, 2, \dots$ if $X_i \sim \text{Poi}(\lambda_i)$ then $X_1 + \dots + X_n \sim \text{Poi}(\sum_{i=1}^n \lambda_i)$. (check by mgf)

$$\varphi(t) = E(e^{iXt}) = \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \cdot e^{int} = e^{\lambda(e^{it}-1)}$$

$$E(X) = \frac{1}{i} \varphi'(0) = \frac{1}{i} (\lambda i e^{it} e^{\lambda(e^{it}-1)})_{t=0} = \lambda$$

$$E(X^2) = -\varphi''(0) = -(-\lambda e^{it} e^{\lambda(e^{it}-1)} - \lambda^2 e^{2it} e^{\lambda(e^{it}-1)})_{t=0} = \lambda^2 + \lambda$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \lambda$$

10.8 The Γ Distribution

The definition of **Gamma function** is

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

$X \sim \Gamma(\alpha, \beta)$ has the pmf

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \quad 0 < x < \infty$$

$\Gamma(1, \beta)$ is also called the **exponential distribution** with parameter $1/\beta$. $f(x) = \beta^{-1} e^{-x/\beta}$ for $x > 0$.

The characteristic function of $\Gamma(\alpha, \beta)$ is

$$\begin{aligned} \varphi(t) &= E[e^{iXt}] = \int_0^\infty e^{ixt} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{(it-1/\beta)x} dx \\ &= (1 - i\beta t)^{-\alpha} \end{aligned}$$

Note. If $X_i \sim \Gamma(\alpha_i, \beta)$ then $X_1 + \dots + X_n \sim \Gamma(\sum_{i=1}^n \alpha_i, \beta)$.

$$E(X) = \frac{1}{i} \varphi'(0) = \frac{1}{i} [(-\alpha)(1 - i\beta t)^{-\alpha-1} (-i\beta)]_{t=0} = \alpha\beta$$

$$E(X^2) = -\varphi''(0) = -[i(-\alpha)(-\beta)(-\alpha-1)(-i\beta)(1 - i\beta t)^{-\alpha-2}]_{t=0} = (\alpha^2 + \alpha)\beta^2$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \alpha\beta^2$$

When $\alpha = 1$, then the exponential distribution $\text{Exp}(\beta)$ has mean β and variance β^2 .

Remarque 43. Some texts define the PDF of $\text{Exp}(\lambda)$ as $p(x) = \lambda e^{-\lambda x}$, while others use $p(x) = \frac{1}{\lambda} e^{-x/\lambda}$.

10.9 The χ^2 -Distribution

Consider the special case $\Gamma(\alpha = r/2, \beta = 2)$ for given r . X has the pdf

$$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2} \quad 0 < x < \infty$$

and the mgf

$$M(t) = (1 - 2t)^{-r/2} \quad t < \frac{1}{2}$$

Because χ^2 -distribution is a subfamily of Γ -distribution, we have $\sum_{i=1}^n X_i \sim \chi^2(\sum_{i=1}^n r_i)$.

The mean and variance of $\chi^2(r)$ are:

$$\mathbb{E}(X) = r, \quad \text{Var}(X) = 2r$$

Théorème 2

Let X have a $\chi^2(r)$ distribution. If $k > -r/2$, then $\mathbb{E}(X^k)$ exists and it is given by

$$\mathbb{E}(X^k) = \frac{2^k \Gamma\left(\frac{r}{2} + k\right)}{\Gamma\left(\frac{r}{2}\right) 2^{r/2}}, \quad \text{if } k > -r/2. \quad (10.1)$$

证明. Note that

$$\mathbb{E}(X^k) = \int_0^\infty \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{r/2}} x^{(r/2)+k-1} e^{-x/2} dx.$$

Make the change of variable $u = x/2$ in the above integral. This results in

$$\mathbb{E}(X^k) = \int_0^\infty \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{r/2}} 2^{(r/2)+k} u^{(r/2)+k-1} e^{-u} du.$$

This simplifies to the desired result provided that $k > -(r/2)$. \square

10.10 The β -Distribution

Let $X_1 \sim \Gamma(\alpha, 1)$, $X_2 \sim \Gamma(\beta, 1)$, then $Y := X_1/(X_1 + X_2)$ has the **beta distribution**, with pdf:

$$f_Y(y) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1} (1-y)^{\beta-1} \quad 0 < y < 1$$

And

$$\mu = \frac{\alpha}{\alpha + \beta}, \quad \sigma^2 = \frac{\alpha\beta}{(\alpha + \beta + 1)(\alpha + \beta)^2}$$

10.11 The Normal Distribution

Let Z have the standard normal distribution then $P(Z \leq z) = \Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$. Φ is the CDF of $N(0, 1)$.

Définition 1: Normal Distribution

We say a random variable X has a **normal distribution** $N(\mu, \sigma^2)$ if its pdf is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \quad \text{for } -\infty < x < \infty$$

The parameters μ and σ^2 are the mean and variance of X , respectively.

Théorème 3

If $X \sim N(\mu, \sigma^2)$ then $V = \frac{(X-\mu)^2}{\sigma^2} \sim \chi^2(1)$. Particularly,

$$X \sim N(0, 1) \Rightarrow X^2 \sim \chi^2(1) \quad (10.2)$$

Note. Calculate the CDF of X^2 .

证明. Because $V = W^2$, where $W = (X - \mu)/\sigma$ is $N(0, 1)$, the cdf $G(v)$ for V is, for $v \geq 0$,

$$G(v) = P(W^2 \leq v) = P(-\sqrt{v} \leq W \leq \sqrt{v})$$

That is,

$$G(v) = 2 \int_0^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw, \quad 0 \leq v$$

And

$$G(v) = 0, \quad v < 0$$

If we change the variable of integration by writing $w = \sqrt{y}$, then

$$G(v) = \int_0^v \frac{1}{\sqrt{2\pi}\sqrt{y}} e^{-y/2} dy, \quad 0 \leq v$$

Hence the pdf $g(v) = G'(v)$ of the continuous-type random variable V is

$$g(v) = \begin{cases} \frac{1}{\sqrt{\pi}\sqrt{2}} v^{1/2-1} e^{-v/2} & 0 < v < \infty \\ 0 & \text{elsewhere} \end{cases}$$

Since $g(v)$ is a pdf

$$\int_0^\infty g(v) dv = 1$$

hence, it must be that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and thus V is $\chi^2(1)$. \square

Théorème 4

$X_i \sim N(\mu_i, \sigma_i^2)$ are iid, then $Y = \sum_{i=1}^n a_i X_i \sim N\left(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2\right)$.

Note. Prove by the characteristic function.

10.12 Multivariate Normal Distribution

10.12.1 When $n = 2$

We say that (X, Y) follows a bivariate normal distribution if its pdf is given by

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-q/2}, \quad -\infty < x < \infty, \quad -\infty < y < \infty$$

Where

$$q = \frac{1}{1-\rho^2} \left[\left(\frac{x-\mu_1}{\sigma_1} \right)^2 - 2\rho \left(\frac{x-\mu_1}{\sigma_1} \right) \left(\frac{y-\mu_2}{\sigma_2} \right) + \left(\frac{y-\mu_2}{\sigma_2} \right)^2 \right]$$

and $-\infty < \mu_i < \infty, \sigma_i > 0$, for $i = 1, 2$, and ρ satisfies $\rho^2 < 1$. Clearly, this function is positive everywhere in R^2 . As we show in Section 3.5.2, it is a pdf with the mgf given by:

$$M_{(X,Y)}(t_1, t_2) = \exp \left\{ t_1\mu_1 + t_2\mu_2 + \frac{1}{2} (t_1^2\sigma_1^2 + 2t_1t_2\rho\sigma_1\sigma_2 + t_2^2\sigma_2^2) \right\}$$

Then

$$E(XY) = \frac{\partial^2 M_{(X,Y)}}{\partial t_1 \partial t_2}(0, 0) = \rho\sigma_1\sigma_2 + \mu_1\mu_2$$

10.12.2 General Case

Consider the random vector $\mathbf{Z} = (Z_1, \dots, Z_n)'$ where Z_1, \dots, Z_n are iid $N(0, 1)$ random variables. Then the density of \mathbf{Z} is

$$f_{\mathbf{Z}}(\mathbf{z}) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} z_i^2 \right\} = \left(\frac{1}{2\pi} \right)^{n/2} \exp \left\{ -\frac{1}{2} \sum_{i=1}^n z_i^2 \right\} = \left(\frac{1}{2\pi} \right)^{n/2} \exp \left\{ -\frac{1}{2} \mathbf{z}' \mathbf{z} \right\} \quad \mathbf{z} \in \mathbf{R}^n$$

$$M_{\mathbf{Z}}(\mathbf{t}) = E[\exp\{\mathbf{t}' \mathbf{Z}\}] = E \left[\prod_{i=1}^n \exp\{t_i Z_i\} \right] = \exp \left\{ \frac{1}{2} \sum_{i=1}^n t_i^2 \right\} = \exp \left\{ \frac{1}{2} \mathbf{t}' \mathbf{t} \right\} \quad \mathbf{t} \in \mathbf{R}^n$$

We abbreviate this by saying that $\mathbf{Z} \sim N_n(\mathbf{0}, \mathbf{I}_n)$.

For the general case, suppose Σ is an $n \times n$, symmetric and positive semi-definite matrix. The **spectral decomposition** is $\Sigma = \Gamma' \Lambda \Gamma$, where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ are the eigenvalues of Σ and the columns of Γ' , $\mathbf{v}_1, \dots, \mathbf{v}_n$ are the corresponding eigenvectors. The matrix Γ is orthogonal, i.e., $\Gamma^{-1} = \Gamma'$. $\Sigma = \sum_{i=1}^n \lambda_i \mathbf{v}_i \mathbf{v}_i'$.

Définition 2: Multivariate Normal

Say an n -dimensional random vector \mathbf{X} has a **multivariate normal distribution** if its mgf is

$$M_{\mathbf{X}}(\mathbf{t}) = \exp \left\{ \mathbf{t}' \boldsymbol{\mu} + \frac{1}{2} \mathbf{t}' \boldsymbol{\Sigma} \mathbf{t} \right\} \quad \forall \mathbf{t} \in \mathbb{R}^n$$

where Σ is a symmetric, positive semi-definite matrix and $\boldsymbol{\mu} \in \mathbb{R}^n$. We abbreviate this by saying that \mathbf{X} has a $N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution.

If $\mathbf{Z} \sim N_n(0, \mathbf{I}_n)$ then $\mathbf{X} = \boldsymbol{\Sigma}^{1/2} \mathbf{Z} + \boldsymbol{\mu} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}^{1/2})$. And we have

$$E[\mathbf{X}] = \boldsymbol{\mu}, \quad \text{Cov}[\mathbf{X}] = \boldsymbol{\Sigma}^{1/2} \boldsymbol{\Sigma}^{1/2} = \boldsymbol{\Sigma}$$

Further, the mgf of \mathbf{X} is given by

$$\begin{aligned} M_{\mathbf{X}}(\mathbf{t}) &= E[\exp\{\mathbf{t}' \mathbf{X}\}] = E\left[\exp\left\{\mathbf{t}' \boldsymbol{\Sigma}^{1/2} \mathbf{Z} + \mathbf{t}' \boldsymbol{\mu}\right\}\right] \\ &= \exp\{\mathbf{t}' \boldsymbol{\mu}\} E\left[\exp\left\{\left(\boldsymbol{\Sigma}^{1/2} \mathbf{t}\right)' \mathbf{Z}\right\}\right] \\ &= \exp\{\mathbf{t}' \boldsymbol{\mu}\} \exp\left\{(1/2) \left(\boldsymbol{\Sigma}^{1/2} \mathbf{t}\right)' \boldsymbol{\Sigma}^{1/2} \mathbf{t}\right\} \\ &= \exp\{\mathbf{t}' \boldsymbol{\mu}\} \exp\{(1/2)\mathbf{t}' \boldsymbol{\Sigma} \mathbf{t}\} \end{aligned}$$

Théorème 5

If $\mathbf{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\boldsymbol{\Sigma}$ is positive definite then $Y = (\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{X} - \boldsymbol{\mu}) \sim \chi^2(n)$.

证明. It's a consequence of eq. (10.2). □

Théorème 6: Linearity

Suppose $\mathbf{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$ where \mathbf{A} is $m \times n$ and $\mathbf{b} \in \mathbb{R}^m$. Then $\mathbf{Y} \sim N_m(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$.

Théorème 7

Suppose $\mathbf{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$ where \mathbf{A} is $m \times n$ and $\mathbf{b} \in \mathbb{R}^m$. Then $\mathbf{Y} \sim N_m(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$.

证明. Calculate the mgf of \mathbf{Y} . □

Particularly, suppose

$$\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \sim N_n\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}\right)$$

Then $\mathbf{X}_1 \sim N_m(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$.

Théorème 8: decomposite to independent subvector

$\mathbf{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)'$, then \mathbf{X}_1 and \mathbf{X}_2 are independent iff $\boldsymbol{\Sigma}_{12} = \mathbf{0}$, i.e.

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

证明. The joint mgf is

$$M_{\mathbf{X}_1, \mathbf{X}_2}(\mathbf{t}_1, \mathbf{t}_2) = \exp \left\{ \mathbf{t}'_1 \boldsymbol{\mu}_1 + \mathbf{t}'_2 \boldsymbol{\mu}_2 + \frac{1}{2} (\mathbf{t}'_1 \boldsymbol{\Sigma}_{11} \mathbf{t}_1 + \mathbf{t}'_2 \boldsymbol{\Sigma}_{22} \mathbf{t}_2 + \mathbf{t}'_2 \boldsymbol{\Sigma}_{21} \mathbf{t}_1 + \mathbf{t}'_1 \boldsymbol{\Sigma}_{12} \mathbf{t}_2) \right\}$$

The product of their marginal mgfs is

$$M_{\mathbf{X}_1}(\mathbf{t}_1) M_{\mathbf{X}_2}(\mathbf{t}_2) = \exp \left\{ \mathbf{t}'_1 \boldsymbol{\mu}_1 + \mathbf{t}'_2 \boldsymbol{\mu}_2 + \frac{1}{2} (\mathbf{t}'_1 \boldsymbol{\Sigma}_{11} \mathbf{t}_1 + \mathbf{t}'_2 \boldsymbol{\Sigma}_{22} \mathbf{t}_2) \right\}.$$

Then $M_{\mathbf{X}_1, \mathbf{X}_2}(\mathbf{t}_1, \mathbf{t}_2) = M_{\mathbf{X}_1}(\mathbf{t}_1) M_{\mathbf{X}_2}(\mathbf{t}_2)$ iff $\boldsymbol{\Sigma}_{12} = \mathbf{0}$. □

Théorème 9: conditional distribution

$\mathbf{X}_1, \mathbf{X}_2$ are defined as above, then the conditional distribution of $\mathbf{X}_1 | \mathbf{X}_2$ is

$$N_m (\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} (\mathbf{X}_2 - \boldsymbol{\mu}_2), \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\Sigma}_{21})$$

证明. See proof from Hogg p.204. \square

For $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ with $\rho = \text{corr}(X_1, X_2)$. Then

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N \left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \right)$$

We decompose the symmetric matrix.

$$\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 & 1 & 0 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} (1-\rho^2)\sigma_1^2 & 0 & 1 & -\frac{\rho\sigma_1}{\sigma_2} \\ 0 & \sigma_2^2 & 0 & 1 \end{bmatrix}$$

Then

$$\begin{bmatrix} X_1 - \frac{\rho\sigma_1}{\sigma_2} X_2 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{\rho\sigma_1}{\sigma_2} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N \left(\begin{bmatrix} \mu_1 - \frac{\rho\sigma_1}{\sigma_2} \mu_2 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} (1-\rho^2)\sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \right)$$

Thus $X_1 - \frac{\rho\sigma_1}{\sigma_2} X_2$ and X_2 are independent. The conditional expectation is

$$\begin{aligned} \mathbb{E}[X_1 | X_2] &= \mathbb{E}\left[X_1 - \frac{\rho\sigma_1}{\sigma_2} X_2 | X_2\right] + \mathbb{E}\left[\frac{\rho\sigma_1}{\sigma_2} X_2 | X_2\right] \\ &= \mathbb{E}\left[X_1 - \frac{\rho\sigma_1}{\sigma_2} X_2\right] + \frac{\rho\sigma_1}{\sigma_2} X_2 \\ &= \mu_1 - \frac{\rho\sigma_1}{\sigma_2} \mu_2 + \frac{\rho\sigma_1}{\sigma_2} X_2 \\ &= \mathbb{E}[X_1] + \frac{\text{Cov}(X_1, X_2)}{\text{Var}(X_2)} (X_2 - \mu_2) \end{aligned}$$

$$p((1-p)((1-p)p^2 + p((1-p)p + p)) + p((1-p)((1-p)p + p) + p)) + (1-p)((1-p)p^3 + p((1-p)p^2 + p((1-p)p + p)))$$

10.13 The t -distribution

$W \sim N(0, 1)$, $V \sim \chi^2(r)$ independent; $T := \frac{W}{\sqrt{V/r}}$; $T \sim t(r)$ has the pdf

$$f(t) = \frac{\Gamma[(r+1)/2]}{\sqrt{\pi r} \Gamma(r/2)} \frac{1}{(1+t^2/r)^{(r+1)/2}} \quad -\infty < t < \infty$$

$$\mathbb{E}(T) = 0$$

$$\begin{aligned}\text{Var}(T) &= \mathbb{E}(T^2) \\ &= \mathbb{E} \left[W^2 \left(\frac{V}{r} \right)^{-1} \right] \\ &= \underbrace{\mathbb{E} \left(\underbrace{W^2}_{\sim \chi^2(1)} \right)}_{=1} \cdot \mathbb{E} \left[\left(\frac{V}{r} \right)^{-1} \right]\end{aligned}$$

By eq. (10.1), for $r > 2$,

$$\mathbb{E}(V^{-1}) = \frac{2^{-1}\Gamma(\frac{r}{2}-1)}{\Gamma(\frac{r}{2})} = \frac{1}{r-2}$$

Thus

$$\text{Var}(Y) = \frac{r}{r-2}$$

10.13.1 Computation

Note. We will show the technique to compute the pdf of $T = T(W, V)$.

Let independent $W \sim N(0, 1)$, $V \sim \chi^2(r)$. Then the joint pdf of W, V , say $h(w, v)$ is

$$h(w, v) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2} \frac{1}{\Gamma(r/2) 2^{r/2}} v^{r/2-1} e^{-v/2} \quad -\infty < w < \infty, 0 < v < \infty$$

Define a new random variable T by

$$T = \frac{W}{\sqrt{V/r}}$$

The transformation technique i used to obtain the pdf $g_1(t)$ of T . The equations

$$t = \frac{w}{\sqrt{v/r}}, \quad u = v$$

define a transformation from $\mathcal{S} = \{(w, v) : -\infty < w < \infty, 0 < v < \infty\}$ to $\mathcal{T} = \{(t, u) : -\infty < t < \infty, 0 < u < \infty\}$. Calculate the Jacobian of the transformation

$$|J| = \frac{\partial(w, v)}{\partial(t, u)} = \left[\frac{\partial(t, u)}{\partial(w, v)} \right]^{-1} = \begin{vmatrix} \frac{1}{\sqrt{v/r}} & 0 \\ 0 & 1 \end{vmatrix}^{-1} = \frac{\sqrt{u}}{\sqrt{r}}$$

Accordingly, the joint pdf of T and $U = V$ is given by

$$g(t, u) = h(w, v)|J| = h(t\sqrt{u}/\sqrt{r}, u)|J| = \frac{1}{\sqrt{2\pi}\Gamma(r/2)2^{r/2}}u^{r/2-1}\exp\left[-\frac{u}{2}\left(1+\frac{t^2}{r}\right)\right]\frac{\sqrt{u}}{\sqrt{r}}$$

where $|t| < \infty, 0 < u < \infty$.

The marginal pdf of T is then

$$g_1(t) = \int_{-\infty}^{\infty} g(t, u) du = \int_0^{\infty} \frac{1}{\sqrt{2\pi r}\Gamma(r/2)2^{r/2}}u^{(r+1)/2-1}\exp\left[-\frac{u}{2}\left(1+\frac{t^2}{r}\right)\right] du$$

In this integral let $z = y[1 + (t^2/r)]/2$ and it is seen that

$$g_1(t) = \frac{\Gamma[(r+1)/2]}{\sqrt{\pi r}\Gamma(r/2)} \frac{1}{(1+t^2/r)^{(r+1)/2}} \quad -\infty < t < \infty$$

The distribution of the random variable T is usually called a **t -distribution**.

$g_1(t) = g_1(-t)$, with max at 0. As $t \rightarrow \infty$, the t -distribution converges to the $N(0, 1)$.

10.14 The F -distribution

Let independent $U \sim \chi^2(r_1)$, $V \sim \chi^2(r_2)$. The joint pdf $h(u, v)$ of U and V is then

$$h(u, v) = \frac{1}{\Gamma(r_1/2)\Gamma(r_2/2)2^{(r_1+r_2)/2}}u^{r_1/2-1}v^{r_2/2-1}e^{-(u+v)/2} \quad 0 < u, v < \infty$$

We define the new random variable

$$W = \frac{U/r_1}{V/r_2}$$

and we propose finding the pdf $g_1(w)$ of W . The equations

$$w = \frac{u/r_1}{v/r_2}, \quad z = v$$

By calculations, we have

$$g_1(w) = \frac{\Gamma[(r_1+r_2)/2](r_1/r_2)^{r_1/2}}{\Gamma(r_1/2)\Gamma(r_2/2)} \frac{w^{r_1/2-1}}{(1+r_1w/r_2)^{(r_1+r_2)/2}} \quad 0 < w < \infty$$

The distribution of this random variable is usually called an F -distribution,
 $W \sim F(r_1, r_2)$.

By eq. (10.1), for $k > -\frac{r_1}{2}$, $k > -\frac{r_2}{2}$,

$$\begin{aligned}\mathbb{E}(W^k) &= \left(\frac{r_2}{r_1}\right)^k \mathbb{E}(U^k) \cdot E(V^{-k}) \\ &= \left(\frac{r_2}{r_1}\right)^k \cdot \frac{2^k \Gamma\left(\frac{r_1}{2} + k\right)}{\Gamma\left(\frac{r_1}{2}\right)} \cdot \frac{2^{-k} \Gamma\left(\frac{r_2}{2} - k\right)}{\Gamma\left(\frac{r_2}{2}\right)} \\ &= \left(\frac{r_2}{r_1}\right)^k \cdot \frac{\Gamma\left(\frac{r_1}{2} + k\right) \cdot \Gamma\left(\frac{r_2}{2} - k\right)}{\Gamma\left(\frac{r_1}{2}\right) \cdot \Gamma\left(\frac{r_2}{2}\right)}\end{aligned}$$

In particular,

$$\mathbb{E}(W) = \frac{r_2}{r_1} \cdot \underbrace{\frac{\Gamma\left(\frac{r_1}{2} + 1\right)}{\Gamma\left(\frac{r_1}{2}\right)}}_{=\frac{r_1}{2}} \cdot \underbrace{\frac{\Gamma\left(\frac{r_2}{2} - 1\right)}{\Gamma\left(\frac{r_2}{2}\right)}}_{=\left(\frac{r_2}{2} - 1\right)^{-1}} = \frac{r_2}{r_2 - 2}$$

10.15 Student's Theorem

Our final note concerns an important result for the later chapters on inference for normal random variables. It is a corollary to the t -distribution derived above and is often referred to as Student's Theorem.

Théorème 10: Student's theorem

Let X_1, \dots, X_n be iid random variables each having a normal distribution with mean μ and variance σ^2 . Define the random variables

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \text{ and } S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

Then

- (1) \bar{X} has a $N\left(\mu, \frac{\sigma^2}{n}\right)$ distribution.
- (2) \bar{X} and S^2 are independent.
- (3) $(n-1)S^2/\sigma^2$ has a $\chi^2(n-1)$ distribution.
- (4) The random variable

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has a Student t -distribution with $n-1$ degrees of freedom.

Remarque 44. In (b) we have proved that \bar{X} is independent of $(X_1 - \bar{X}, \dots, X_n - \bar{X})$. 这是因为协方差为 0 的正态分布是独立的.

证明. (1) is clear. (2) it suffices to show that \bar{X} and $X_i - \bar{X}$ are uncorrelated for any i , thus independent due to the normality.

$$\begin{aligned}\text{Cov}(\bar{X}, X_i - \bar{X}) &= \mathbb{E}[\bar{X}(X_i - \bar{X})] - \mathbb{E}[\bar{X}] \cdot \underbrace{\mathbb{E}[X_i - \bar{X}]}_{=\mu-\mu=0} \\ &= \mathbb{E}(\bar{X}X_i) - \mathbb{E}(\bar{X}^2) \\ &= \frac{1}{n} \left[\sum_{j \neq i} \mathbb{E}(X_j)\mathbb{E}(X_i) + \mathbb{E}(X_i^2) \right] - \left(\frac{\sigma^2}{n} + \mu^2 \right) \\ &= \frac{1}{n}[(n-1)\mu^2 + \sigma^2 + \mu^2] - \frac{\sigma^2}{n} - \mu^2 \\ &= 0\end{aligned}$$

As S^2 is a Borel function of $(X_1 - \bar{X}, \dots, X_n - \bar{X})$, \bar{X} and S^2 are independent.

(3) Consider the random variable

$$V = \sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma} \right)^2$$

Each term in this sum is the square of a $N(0, 1)$ random variable and, hence, has a $\chi^2(1)$ distribution (Theorem 3.4.1). Because the summands are independent, it follows from Corollary 3.3 .1 that V is a $\chi^2(n)$ random variable. Note the following identity:

$$\begin{aligned}V &= \sum_{i=1}^n \left(\frac{(X_i - \bar{X}) + (\bar{X} - \mu)}{\sigma} \right)^2 \\ &= \sum_{i=1}^n \left(\frac{X_i - \bar{X}}{\sigma} \right)^2 + \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right)^2 \\ &= \frac{(n-1)S^2}{\sigma^2} + \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right)^2\end{aligned}$$

By part (b), the two terms on the right side of the last equation are independent. Further, the second term is the square of a standard normal random variable and, hence, has a $\chi^2(1)$ distribution. Taking mgfs of both sides, we have

$$(1-2t)^{-n/2} = E[\exp\{t(n-1)S^2/\sigma^2\}] (1-2t)^{-1/2}$$

Solving for the mgf of $(n-1)S^2/\sigma^2$ on the right side we obtain part (c). Finally, part (d) follows immediately from parts (a)-(c) upon writing T , (3.6.9), as

$$T = \frac{(\bar{X} - \mu)/(\sigma/\sqrt{n})}{\sqrt{(n-1)S^2/(\sigma^2(n-1))}}$$

□

10.16 Mixture Distributions

For k distributions with pdfs $f_1(x), \dots, f_k(x)$, supports $\mathcal{S}_1, \dots, \mathcal{S}_k$, means μ_1, \dots, μ_k and variances $\sigma_1^2, \dots, \sigma_k^2$, with positive mixing probabilities p_1, \dots, p_k where $p_1 + \dots + p_k = 1$. Let $\mathcal{S} = \bigcup_{i=1}^k \mathcal{S}_i$ and consider the function

$$f(x) = \sum_{i=1}^k p_i f_i(x) \quad x \in \mathcal{S}$$

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(t) dt = \int_{-\infty}^x \sum_{i=1}^k p_i f_i(t) dt = \sum_{i=1}^k p_i \int_{-\infty}^x f_i(t) dt = \sum_{i=1}^k p_i F_i(x) \\ E(X) &= \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^{\infty} x \sum_{i=1}^k p_i f_i(x) dx = \sum_{i=1}^k p_i \int_{-\infty}^{\infty} x f_i(x) dx = \sum_{i=1}^k p_i \mu_i = \bar{\mu} \\ \text{Var}(X) &= \int_{-\infty}^{\infty} (x - \bar{\mu})^2 f(x) dx = \sum_{i=1}^k p_i \int_{-\infty}^{\infty} (x - \bar{\mu})^2 f_i(x) dx \\ &= \sum_{i=1}^k p_i \int_{-\infty}^{\infty} [(x - \mu_i) + (\mu_i - \bar{\mu})]^2 f_i(x) dx \\ &= \sum_{i=1}^k p_i \left[\int_{-\infty}^{\infty} (x - \mu_i)^2 f_i(x) dx + 2 \underbrace{\int_{-\infty}^{\infty} (x - \mu_i)(\mu_i - \bar{\mu}) f_i(x) dx}_{=0} + \int_{-\infty}^{\infty} (\mu_i - \bar{\mu})^2 f_i(x) dx \right] \\ &= \sum_{i=1}^k p_i \sigma_i^2 + \sum_{i=1}^k p_i (\mu_i - \bar{\mu})^2 \end{aligned}$$

10.17 Maximum Likelihood Estimator (mle)

We continue the concepts of samples and statistics in chapter 2 while introducing the main tools of inference: confidence intervals and tests of hypotheses. In a typical statistic problem, we have X but don't know its pdf $f(x)$ or pmf $p(x)$. There are two cases: completely unknown and known down to a parameter θ which may be a vector. Now we consider the second case, e.g. $X \sim Exp(\theta)$ with unknown θ , $X \sim N(\mu, \sigma^2)$ with unknown μ, σ^2 . X_1, X_2, \dots, X_n iid is called a **random sample** of size n , and a function of the sample $T = T(X_1, \dots, X_n)$ is called a **statistic**. Once the sample is drawn, then t is called the **realization** of T , where $t = T(x_1, \dots, x_n)$ and x_1, \dots, x_n is the realization of the sample.

Using the above terminology, the problem we discuss in this chapter is phrased as: Let X_1, X_2, \dots, X_n denote a random sample on a random variable X with a density or mass function of the form $f(x; \theta)$ or $p(x; \theta)$, where $\theta \in \Omega$ for a specified set Ω . In this situation, it makes sense to consider a statistic T , which is an **estimator** of θ . More formally, T is called a **point estimator** of θ . While we call T an estimator of θ , we call its realization t an **estimate** of θ .

Definition 4.1.3 (Unbiasedness). *Let X_1, X_2, \dots, X_n denote a sample on a random variable X with pdf $f(x; \theta)$, $\theta \in \Omega$. Let $T = T(X_1, X_2, \dots, X_n)$ be a statistic. We say that T is an **unbiased estimator** of θ if $E(T) = \theta$.*

In chapter 6 and 7 we discuss several theories of estimation in general. We briefly discuss the **maximum likelihood estimator (mle)** and then use it to obtain point estimators. Our discussion is for the continuous case. The information is involved in the **likelihood function** $L(\theta) = L(\theta; x_1, \dots, x_n) = \prod_{i=1}^n f(x_i; \theta)$. An often-used estimate is the value of θ that provides a maximum of $L(\theta)$. If unique, this is called the **maximum likelihood estimator (mle)** and we denote it as $\hat{\theta} = \text{Argmax } L(\theta)$. In practice, it's easier to work with $l(\theta) = \log L(\theta)$, and $\hat{\theta}$ solves $\frac{\partial l(\theta)}{\partial \theta} = 0$.

10.17.1 Examples

Exponential Distribution

Suppose $X_1, \dots, X_n \sim \Gamma(1, \theta)$ with density $f(x) = \theta^{-1} \exp\{-x/\theta\}$, $0 < x < \infty$. The log of the likelihood function is given by

$$l(\theta) = \log \prod_{i=1}^n \frac{1}{\theta} e^{-x_i/\theta} = -n \log \theta - \theta^{-1} \sum_{i=1}^n x_i$$

The first partial of the log-likelihood with respect to θ is

$$\frac{\partial l(\theta)}{\partial \theta} = -n\theta^{-1} + \theta^{-2} \sum_{i=1}^n x_i$$

Setting this partial to 0 then we obtain the solution \bar{x} , thus the statistic $\hat{\theta} = \bar{X}$ is the mle of θ . Because $E(X) = \theta$ we have $E(\bar{X}) = \theta$ hence $\hat{\theta}$ is an unbiased estimator of θ .

Binomial Distribution

Let X be one of zero. Let θ , $0 < \theta < 1$, denote the probability of success. Then the pmf of X is

$$p(x; \theta) = \theta^x (1 - \theta)^{1-x}, \quad x = 0 \text{ or } 1$$

If X_1, \dots, X_n is a random sample on X , then

$$L(\theta) = \prod_{i=1}^n p(x_i; \theta) = \theta^{\sum_{i=1}^n x_i} (1 - \theta)^{n - \sum_{i=1}^n x_i}, \quad x_i = 0 \text{ or } 1$$

Taking logs, we have

$$l(\theta) = \sum_{i=1}^n x_i \log \theta + \left(n - \sum_{i=1}^n x_i \right) \log(1 - \theta), \quad x_i = 0 \text{ or } 1$$

The partial derivative of $l(\theta)$ is

$$\frac{\partial l(\theta)}{\partial \theta} = \frac{\sum_{i=1}^n x_i}{\theta} - \frac{n - \sum_{i=1}^n x_i}{1 - \theta}$$

Thus $\hat{\theta} = \bar{X}$, $E(\hat{\theta}) = E(\bar{X}) = \theta$ then $\hat{\theta}$ is an unbiased estimator of θ .

Normal Distribution

Let $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ then $\boldsymbol{\theta} = (\mu, \sigma)$.

$$l(\mu, \sigma) = -\frac{n}{2} \log 2\pi - n \log \sigma - \frac{1}{2} \sum_{i=1}^n \left(\frac{x_i - \mu}{\sigma} \right)^2$$

The two partial derivatives simplify to

$$\begin{aligned} \frac{\partial l(\mu, \sigma)}{\partial \mu} &= -\sum_{i=1}^n \left(\frac{x_i - \mu}{\sigma} \right) \left(-\frac{1}{\sigma} \right) \\ \frac{\partial l(\mu, \sigma)}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^n (x_i - \mu)^2 \end{aligned}$$

Setting these to 0 and solving simultaneously, we see that the mles are

$$\hat{\mu} = \bar{X}, \quad \hat{\sigma}^2 = n^{-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

We know that \bar{X} is unbiased estimator for μ and $S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$ is an unbiased estimator of σ^2 . Thus for the mle of σ^2 , $E(\hat{\sigma}^2) = [n/(n-1)]\sigma^2$, which is a biased estimator of σ^2 . Note that the bias of $\hat{\sigma}^2$ is $E(\hat{\sigma}^2 - \sigma^2) = -\sigma^2/n$ which converges to 0 as $n \rightarrow \infty$. However S^2 is the preferred estimator of σ^2 .

Uniform Distribution

Let X_1, \dots, X_n be iid with the uniform $(0, \theta)$ density; i.e.

$$f(x; \theta) = \frac{1}{\theta} \mathbb{1}_{(0, \theta)}(x)$$

Because θ is in the support, **differentiation is not helpful** here. The likelihood function can be written as

$$L(\theta) = \prod_{i=1}^n f(x_i; \theta) = \theta^{-n} \prod_{i=1}^n \mathbb{1}_{(0, \theta)}(x_i) = \theta^{-n} \mathbb{1}_{(0, \theta)}(\max\{x_i\})$$

The function $L(\theta)$ is a decreasing function of θ for all $\theta \geq \max\{x_i\}$ and is 0 otherwise. So the maximum occurs at the smallest value that θ can assume; i.e. the mle is $\hat{\theta} = \max\{X_i\}$.

10.18 Histogram Estimates of pmfs and pdfs

In this section we briefly discuss a histogram of the sample, which is an estimate of the pmf, $p(x)$ or the pdf, $f(x)$, of X .

Discrete Distribution

Assume X discrete with pmf $p(x)$. Let X_1, \dots, X_n be a random sample on X . Suppose the space of X is finite, i.e. $\mathcal{D} = \{a_1, \dots, a_m\}$. An intuitive estimate of $p(a_j)$ is the relative frequency of a_j in the sample.

$$\hat{p}(a_j) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{a_j\}}(X_i)$$

These estimators $\{\hat{p}(a_1), \dots, \hat{p}(a_m)\}$ constitute the nonparametric estimate of the pmf $p(x)$. Because

$$E[\hat{p}(a_j)] = \frac{1}{n} \sum_{i=1}^n E[\mathbb{1}_{a_j}(X_i)] = \frac{1}{n} \sum_{i=1}^n p(a_j) = p(a_j)$$

$\hat{p}(a_j)$ is an unbiased estimator of $p(a_j)$.

Next suppose X infinite, i.e. $\mathcal{D} = \{a_1, a_2, \dots\}$. In practice, we select a value a_m and make the groupings

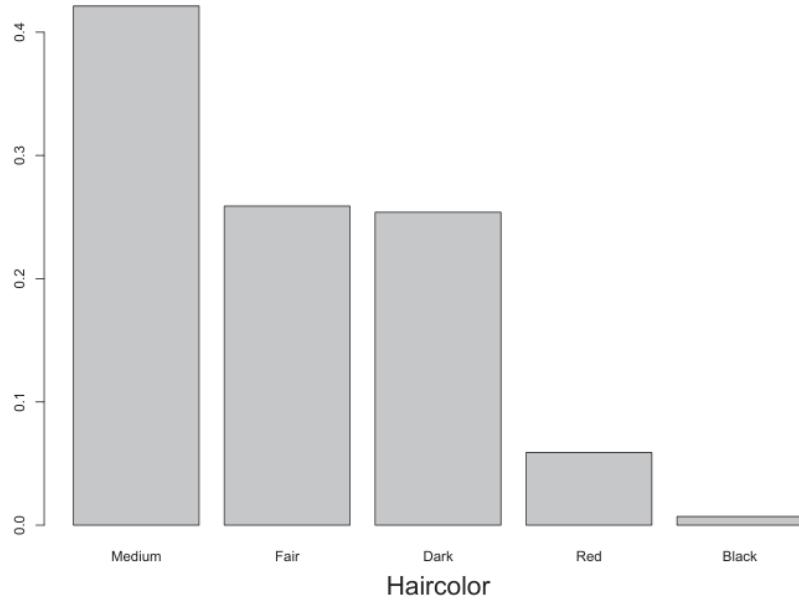
$$\{a_1\}, \{a_2\}, \dots, \{a_m\}, \tilde{a}_{m+1} = \{a_{m+1}, a_{m+2}, \dots\}$$

Let $\hat{p}(\tilde{a}_{m+1})$ be the proportion of sample items that are greater than or equal to a_{m+1} . Then the estimates $\{\hat{p}(a_1), \dots, \hat{p}(a_m), \hat{p}(\tilde{a}_{m+1})\}$ form our estimate of $p(x)$.

A histogram is a **barplot** of $\hat{p}(a_j)$ versus a_j . There are two cases to consider. For the first case, suppose the values a_j represent qualitative categories, e.g. hair colors of a population of people. Such histograms are usually called **bar charts**. An example is helpful here.

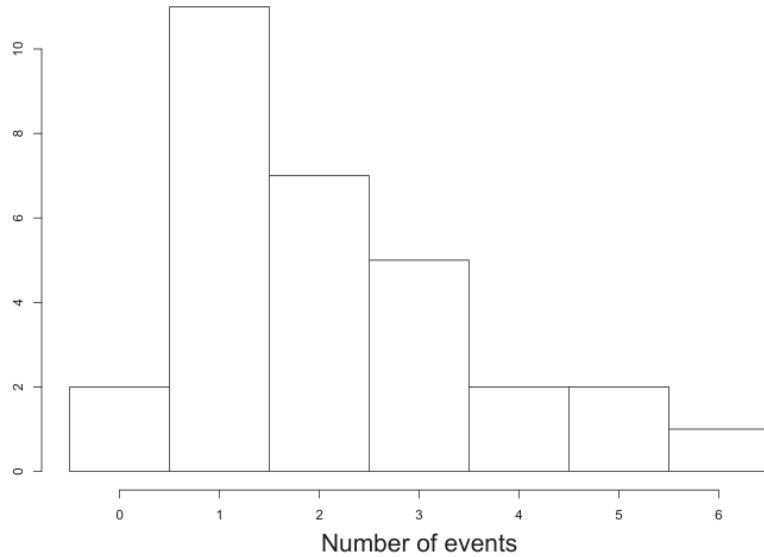
	Fair	Red	Medium	Dark	Black
Count	5789	1319	9418	5678	157
$\hat{p}(a_j)$	0.259	0.059	0.421	0.254	0.007

Bar Chart of Haircolor of Scottish Schoolchildren



For the second case, assume that the values in the space \mathcal{D} are **ordinal** in nature; i.e. the natural ordering of the a_j 's is numerically meaningful. In this case, the usual histogram is an abutting bar chart with heights $\hat{p}(a_j)$ that are plotted in the natural order of the a_j 's, as in the following example.

j	0	1	2	3	4	5	≥ 6
$\hat{p}(j)$	0.067	0.367	0.233	0.167	0.067	0.067	0.033

Histogram of Poisson Variates

Continuous Distribution

For this section, assume that the random sample X_1, \dots, X_n is from a continuous random variable X with continuous pdf $f(t)$. For an arbitrary but fixed point x and a given $h > 0$, consider the interval $(x - h, x + h)$. By the mean value theorem for integrals, we have for some ξ , $|x - \xi| < h$, that

$$P(x - h < X < x + h) = \int_{x-h}^{x+h} f(t) dt = f(\xi)2h \approx f(x)2h$$

Let the sample items fall in $(x - h, x + h)$, which suggests the following nonparametric estimate of $f(x)$ at a given x :

$$\hat{f}(x) = \frac{\#\{x - h < X_i < x + h\}}{2hn}$$

More formally, a nonparametric estimator of $f(x)$ is

$$\hat{f}(x) = \frac{1}{2hn} \sum_{i=1}^n \mathbb{1}_{(x-h,x+h)}(X_i)$$

Then

$$E[\hat{f}(x)] = \frac{1}{2hn} \sum_{i=1}^n E[\mathbb{1}_{(x-h,x+h)}(X_i)] = \frac{1}{2hn} \sum_{i=1}^n P(X_i \in (x-h, x+h)) = \frac{1}{2hn} n f(\xi) 2h = f(\xi) \rightarrow f(x) \quad \text{as } h \rightarrow 0$$

Hence $\hat{f}(x)$ is approximately an unbiased estimator of the density $f(x)$.

The histogram provides a somewhat crude but often used estimator of the pdf, so a few remarks on it are pertinent. Let x_1, \dots, x_n be the realized values of the random sample on a continuous random variable X with pdf $f(x)$. Our histogram estimate of $f(x)$ is obtained as follows. While for the discrete case, there are natural classes for the histogram, for the continuous case these classes must be chosen. One way of doing this is to select a positive integer m , an $h > 0$, and a value a such that $a < \min x_i$, so that the m intervals

$$(a-h, a+h], (a+h, a+3h], (a+3h, a+5h], \dots, (a+(2m-3)h, a+(2m-1)h] \quad (4.1.15)$$

cover the range of the sample $[\min x_i, \max x_i]$. These intervals form our classes. Let $A_j = (a + (2j - 3)h, a + (2j - 1)h]$ for $j = 1, \dots, m$.

Let $\hat{f}_h(x)$ denote our histogram estimate. If $x \leq a - h$ or $x > a + (2m - 1)h$ then define $\hat{f}_h(x) = 0$. For $a - h < x \leq a + (2m - 1)h$, x is in one, and only one, A_j . For $x \in A_j$, define $\hat{f}_h(x)$ to be:

$$\hat{f}_h(x) = \frac{\#\{x_i \in A_j\}}{2hn} \quad (4.1.16)$$

Note that $\hat{f}_h(x) \geq 0$ and that

$$\begin{aligned} \int_{-\infty}^{\infty} \hat{f}_h(x) dx &= \int_{a-h}^{a+(2m-1)h} \hat{f}_h(x) dx = \sum_{j=1}^m \int_{A_j} \frac{\#\{x_i \in A_j\}}{2hn} dx \\ &= \frac{1}{2hn} \sum_{j=1}^m \#\{x_i \in A_j\}[h(2j-1-2j+3)] = \frac{2h}{2hn} n = 1; \end{aligned}$$

so, $\hat{f}_h(x)$ satisfies the properties of a pdf.

For the discrete case, except when classes are merged, the histogram is unique. For the continuous case, though, the histogram depends on the classes chosen. The resulting picture can be quite different if the classes are changed. Unless there is a compelling reason for the class selection, we recommend using the default classes selected by the computational algorithm. The histogram algorithms in most statistical packages such as R are current on recent research for selection of classes. The histogram in the following example is based on default classes.

10.19 Confidence Intervals

Recall that the random variable of interest X has density $f(x; \theta), \theta \in \Omega$, where θ is unknown. In Section 4.1, we discussed estimating θ by a statistic $\hat{\theta} = \hat{\theta}(X_1, \dots, X_n)$, where X_1, \dots, X_n is a sample from the distribution of X . But how much did $\hat{\theta}$ miss θ ? In this section, we embody this estimate of error in terms of a confidence interval, which we now formally define:

Definition 4.2.1 (Confidence Interval). Let X_1, X_2, \dots, X_n be a sample on a random variable X , where X has pdf $f(x; \theta)$, $\theta \in \Omega$. Let $0 < \alpha < 1$ be specified. Let $L = L(X_1, X_2, \dots, X_n)$ and $U = U(X_1, X_2, \dots, X_n)$ be two statistics. We say that the interval (L, U) is a $(1 - \alpha)100\%$ confidence interval for θ if

$$1 - \alpha = P_\theta[\theta \in (L, U)]. \quad (4.2.1)$$

That is, the probability that the interval includes θ is $1 - \alpha$, which is called the confidence coefficient or the confidence level of the interval.

Example 4.2.1 (Confidence Interval for μ Under Normality). Suppose the random variables X_1, \dots, X_n are a random sample from a $N(\mu, \sigma^2)$ distribution. Let \bar{X} and S^2 denote the sample mean and sample variance, respectively. Recall from the last section that \bar{X} is the mle of μ and $[(n-1)/n]S^2$ is the mle of σ^2 . By part (d) of Theorem 3.6.1, the random variable $T = (\bar{X} - \mu)/(S/\sqrt{n})$ has a t -distribution with $n-1$ degrees of freedom. The random variable T is our pivot variable.

For $0 < \alpha < 1$, define $t_{\alpha/2, n-1}$ to be the upper $\alpha/2$ critical point of a t -distribution with $n-1$ degrees of freedom; i.e., $\alpha/2 = P(T > t_{\alpha/2, n-1})$. Using a simple algebraic derivation, we obtain

$$\begin{aligned} 1 - \alpha &= P(-t_{\alpha/2, n-1} < T < t_{\alpha/2, n-1}) \\ &= P_\mu \left(-t_{\alpha/2, n-1} < \frac{\bar{X} - \mu}{S/\sqrt{n}} < t_{\alpha/2, n-1} \right) \\ &= P_\mu \left(-t_{\alpha/2, n-1} \frac{S}{\sqrt{n}} < \bar{X} - \mu < t_{\alpha/2, n-1} \frac{S}{\sqrt{n}} \right) \\ &= P_\mu \left(\bar{X} - t_{\alpha/2, n-1} \frac{S}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2, n-1} \frac{S}{\sqrt{n}} \right). \end{aligned} \quad (4.2.2)$$

Once the sample is drawn, let \bar{x} and s denote the realized values of the statistics \bar{X} and S , respectively. Then a $(1 - \alpha)100\%$ confidence interval for μ is given by

$$(\bar{x} - t_{\alpha/2, n-1} s / \sqrt{n}, \bar{x} + t_{\alpha/2, n-1} s / \sqrt{n}). \quad (4.2.3)$$

This interval is often referred to as the $(1 - \alpha)100\%$ ***t*-interval** for μ . The estimate of the standard deviation of \bar{X} , s/\sqrt{n} , is referred to as the **standard error** of \bar{X} .

For $0 < \alpha < 1$, define $\alpha/2 = P(Z > z_{\alpha/2})$ for $Z \sim N(0, 1)$.

Motivated by the CLT, we have

Example 4.2.2 (Large Sample Confidence Interval for the Mean μ). Suppose X_1, X_2, \dots, X_n is a random sample on a random variable X with mean μ and variance σ^2 , but, unlike the last example, the distribution of X is not normal. However, from the above discussion we know that the distribution of Z_n , (4.2.4), is approximately $N(0, 1)$. Hence

$$1 - \alpha \approx P_\mu \left(-z_{\alpha/2} < \frac{\bar{X} - \mu}{S/\sqrt{n}} < z_{\alpha/2} \right).$$

Using the same algebraic derivation as in the last example, we obtain

$$1 - \alpha \approx P_\mu \left(\bar{X} - z_{\alpha/2} \frac{S}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2} \frac{S}{\sqrt{n}} \right). \quad (4.2.5)$$

Again, letting \bar{x} and s denote the realized values of the statistics \bar{X} and S , respectively, after the sample is drawn, an approximate $(1 - \alpha)100\%$ confidence interval for μ is given by

$$(\bar{x} - z_{\alpha/2}s/\sqrt{n}, \bar{x} + z_{\alpha/2}s/\sqrt{n}). \quad (4.2.6)$$

This is called a **large sample** confidence interval for μ . ■

Example 4.2.3 (Large Sample Confidence Interval for p). Let X be a Bernoulli random variable with probability of success p , where X is 1 or 0 if the outcome is success or failure, respectively. Suppose X_1, \dots, X_n is a random sample from the distribution of X . Let $\hat{p} = \bar{X}$ be the sample proportion of successes. Note that $\hat{p} = n^{-1} \sum_{i=1}^n X_i$ is a sample average and that $\text{Var}(\hat{p}) = p(1-p)/n$. It follows immediately from the CLT that the distribution of $Z = (\hat{p} - p)/\sqrt{p(1-p)/n}$ is approximately $N(0, 1)$. Referring to Example 5.1.1 of Chapter 5, we replace $p(1-p)$ with its estimate $\hat{p}(1-\hat{p})$. Then proceeding as in the last example, an approximate $(1 - \alpha)100\%$ confidence interval for p is given by

$$(\hat{p} - z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})/n}, \hat{p} + z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})/n}), \quad (4.2.7)$$

where $\sqrt{\hat{p}(1-\hat{p})/n}$ is called the standard error of \hat{p} .

In Example 4.1.2 we discussed a data set involving hip replacements, some of which were squeaky. The outcomes of a hip replacement were squeaky and non-squeaky which we labeled as success or failure, respectively. In the sample there were 28 successes out of 143 replacements. Using R, the 99% confidence interval for p , the probability of a squeaky hip replacement, is computed by `prop.test(28, 143, conf.level=.99)`, which results in the interval (0.122, 0.298). So with 99% confidence, we estimate the probability of a squeaky hip replacement to be between 0.122 and 0.298. ■

10.19.1 Confidence Intervals for Difference in Means

A practical problem of interest is the comparison of two distributions, X and Y . In this section, we compare the means of X and Y , denoted by μ_1 and μ_2 . In particular, we obtain confidence intervals for the difference $\Delta = \mu_1 - \mu_2$. Let X_1, \dots, X_{n_1} be a random sample from the distribution of X and Y_1, \dots, Y_{n_2} from Y , and all independent of one another. Let $\bar{X} = n_1^{-1} \sum_{i=1}^{n_1} X_i$

and $\bar{Y} = n_2^{-1} \sum_{i=1}^{n_2} Y_i$ and $\hat{\Delta} = \bar{X} - \bar{Y}$, which is an unbiased estimator of Δ . This difference $\hat{\Delta} - \Delta$ is the numerator of the pivot random variable. By independence of the samples,

$$\text{Var}(\hat{\Delta}) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Let $S_1^2 = (n_1 - 1)^{-1} \sum_{i=1}^{n_1} (X_i - \bar{X})^2$ and $S_2^2 = (n_2 - 1)^{-1} \sum_{i=1}^{n_2} (Y_i - \bar{Y})^2$ be the sample variances. Then estimating the variances by the sample variances, consider the random variable

$$Z = \frac{\hat{\Delta} - \Delta}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

By CLT, this pivot variable has an approximate $N(0, 1)$ distribution. This leads to the approximate $(1 - \alpha)100\%$ confidence interval for $\Delta = \mu_1 - \mu_2$ given by

$$\left((\bar{x} - \bar{y}) - z_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}, (\bar{x} - \bar{y}) + z_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \right)$$

where $\sqrt{(s_1^2/n_1) + (s_2^2/n_2)}$ is the standard error of $\bar{X} - \bar{Y}$. This is a large sample $(1 - \alpha)100\%$ confidence interval for $\mu_1 - \mu_2$.

The above confidence interval is approximate. In this situation we can obtain exact confidence intervals if we assume that the distributions of X and Y are normal with the same variance; i.e., $\sigma_1^2 = \sigma_2^2$. Thus the distributions can differ only in location, i.e., a **location model**. Assume then that X is distributed $N(\mu_1, \sigma^2)$ and Y is distributed $N(\mu_2, \sigma^2)$, where σ^2 is the common variance of X and Y . As above, let X_1, \dots, X_{n_1} be a random sample from the distribution of X , let Y_1, \dots, Y_{n_2} be a random sample from the distribution of Y , and assume that the samples are independent of one another. Let $n = n_1 + n_2$ be the total sample size. Our estimator of Δ is $\bar{X} - \bar{Y}$. Our goal is to show that a pivot random variable, defined below, has a t -distribution, which is defined in Section 3.6.

Because \bar{X} is distributed $N(\mu_1, \sigma^2/n_1)$, \bar{Y} is distributed $N(\mu_2, \sigma^2/n_2)$, and \bar{X} and \bar{Y} are independent, we have the result

$$\frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sigma \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \text{ has a } N(0, 1) \text{ distribution.} \quad (4.2.10)$$

This serves as the numerator of our T -statistic.

Let

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}. \quad (4.2.11)$$

Note that S_p^2 is a weighted average of S_1^2 and S_2^2 . It is easy to see that S_p^2 is an unbiased estimator of σ^2 . It is called the **pooled estimator** of σ^2 . Also, because $(n_1 - 1)S_1^2/\sigma^2$ has a $\chi^2(n_1 - 1)$ distribution, $(n_2 - 1)S_2^2/\sigma^2$ has a $\chi^2(n_2 - 1)$ distribution, and S_1^2 and S_2^2 are independent, we have that $(n - 2)S_p^2/\sigma^2$ has a $\chi^2(n - 2)$ distribution; see Corollary 3.3.1. Finally, because S_1^2 is independent of \bar{X} and S_2^2 is independent of \bar{Y} , and the random samples are independent of each other, it follows that S_p^2 is independent of expression (4.2.10). Therefore, from the

result of Section 3.6.1 concerning Student's t -distribution, we have that

$$\begin{aligned} T &= \frac{[(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)] / \sigma \sqrt{n_1^{-1} + n_2^{-1}}}{\sqrt{(n - 2)S_p^2 / (n - 2)\sigma^2}} \\ &= \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \end{aligned} \quad (4.2.12)$$

has a t -distribution with $n - 2$ degrees of freedom. From this last result, it is easy to see that the following interval is an exact $(1 - \alpha)100\%$ confidence interval for $\Delta = \mu_1 - \mu_2$:

$$\left((\bar{x} - \bar{y}) - t_{(\alpha/2, n-2)} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}, (\bar{x} - \bar{y}) + t_{(\alpha/2, n-2)} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right). \quad (4.2.13)$$

A consideration of the difficulty encountered when the unknown variances of the two normal distributions are not equal is assigned to one of the exercises.

10.19.2 Confidence Interval for Difference in Proportions

Omitted...

10.19.3 Confidence Intervals for Parameters of Discrete Distributions

Omitted...

10.20 Order Statistics

Définition 3: order statistic

Let X_1, X_2, \dots, X_n denote a random sample from a distribution of the *continuous type* having a pdf $f(x)$ that has support $\mathcal{S} = (a, b)$, where $-\infty \leq a < b \leq \infty$. Let Y_1 be the smallest of these X_i , Y_2 the next X_i in order of magnitude, \dots , and Y_n the largest of X_i . That is, $Y_1 < Y_2 < \dots < Y_n$ represent X_1, X_2, \dots, X_n when the latter are arranged in ascending order of magnitude. We call Y_i , $i = 1, 2, \dots, n$, the *i*th **order statistic** of the random sample X_1, X_2, \dots, X_n . Then the joint pdf of Y_1, Y_2, \dots, Y_n is given in the following theorem.

Theorem 4.4.1. *Using the above notation, let $Y_1 < Y_2 < \dots < Y_n$ denote the n order statistics based on the random sample X_1, X_2, \dots, X_n from a continuous distribution with pdf $f(x)$ and support (a, b) . Then the joint pdf of Y_1, Y_2, \dots, Y_n is given by*

$$g(y_1, y_2, \dots, y_n) = \begin{cases} n! f(y_1) f(y_2) \cdots f(y_n) & a < y_1 < y_2 < \dots < y_n < b \\ 0 & \text{elsewhere.} \end{cases} \quad (4.4.1)$$

Note that the support of X_1, \dots, X_n can be partitioned into $n!$ mutually disjoint sets that map onto the support of Y_1, \dots, Y_n namely, $\{(y_1, \dots, y_n) : a < y_1 < \dots < y_n < b\}$. One of these $n!$ sets is $a < x_1 < \dots < x_n < b$, and the others can be found by permuting the n x s in all possible way, thus the Jacobian equal to 1.

$$g(y_1, \dots, y_n) = \sum_{i=1}^{n!} |J_i| f(y_1) \dots f(y_n) = \begin{cases} n! f(y_1) \dots f(y_n) & a < y_1 < \dots < y_n < b \\ 0 & \text{elsewhere} \end{cases}$$

Example 4.4.1. Let X denote a random variable of the continuous type with a pdf $f(x)$ that is positive and continuous, with support $\mathcal{S} = (a, b)$, $-\infty \leq a < b \leq \infty$. The distribution function $F(x)$ of X may be written

$$F(x) = \int_a^x f(w) dw, \quad a < x < b.$$

If $x \leq a$, $F(x) = 0$; and if $b \leq x$, $F(x) = 1$. Thus there is a unique median m of the distribution with $F(m) = \frac{1}{2}$. Let X_1, X_2, X_3 denote a random sample from this distribution and let $Y_1 < Y_2 < Y_3$ denote the order statistics of the sample. Note that Y_2 is the sample median. We compute the probability that $Y_2 \leq m$. The joint pdf of the three order statistics is

$$g(y_1, y_2, y_3) = \begin{cases} 6f(y_1)f(y_2)f(y_3) & a < y_1 < y_2 < y_3 < b \\ 0 & \text{elsewhere.} \end{cases}$$

The pdf of Y_2 is then

$$\begin{aligned} h(y_2) &= 6f(y_2) \int_{y_2}^b \int_a^{y_2} f(y_1)f(y_3) dy_1 dy_3 \\ &= \begin{cases} 6f(y_2)F(y_2)[1 - F(y_2)] & a < y_2 < b \\ 0 & \text{elsewhere.} \end{cases} \end{aligned}$$

Accordingly,

$$\begin{aligned} P(Y_2 \leq m) &= 6 \int_a^m \{F(y_2)f(y_2) - [F(y_2)]^2 f(y_2)\} dy_2 \\ &= 6 \left\{ \frac{[F(y_2)]^2}{2} - \frac{[F(y_2)]^3}{3} \right\}_a^m = \frac{1}{2}. \end{aligned}$$

Hence, for this situation, the median of the sample median Y_2 is the population median m . ■

Once it is observed that

$$\int_a^x [F(w)]^{\alpha-1} f(w) dw = \frac{[F(x)]^\alpha}{\alpha}, \quad \alpha > 0,$$

and that

$$\int_y^b [1 - F(w)]^{\beta-1} f(w) dw = \frac{[1 - F(y)]^\beta}{\beta}, \quad \beta > 0,$$

it is easy to express the marginal pdf of any order statistic, say Y_k , in terms of $F(x)$ and $f(x)$. This is done by evaluating the integral

$$g_k(y_k) = \int_a^{y_k} \cdots \int_a^{y_2} \int_{y_k}^b \cdots \int_{y_{n-1}}^b n! f(y_1)f(y_2) \cdots f(y_n) dy_n \cdots dy_{k+1} dy_1 \cdots dy_{k-1}.$$

The result is

$$g_k(y_k) = \begin{cases} \frac{n!}{(k-1)!(n-k)!} [F(y_k)]^{k-1} [1 - F(y_k)]^{n-k} f(y_k) & a < y_k < b \\ 0 & \text{elsewhere.} \end{cases} \quad (4.4.2)$$

Omitted...

10.20.1 Quantiles

10.20.2 Confidence Intervals for Quantiles

X continuous rv with cdf $F(x)$. For $0 < p < 1$, define the $100p$ th distribution percentile to be ξ_p , where $F(\xi_p) = p$. Let $Y_1 < \dots < Y_n$ be the order statistics. Then

$$P(Y_i < \xi_p < Y_j) = \sum_{w=i}^{j-1} \binom{n}{w} p^w (1-p)^{n-w} =: \gamma$$

Then (y_i, y_j) serves as a $100\gamma\%$ confidence interval for ξ_p , the quantile of order p .

Confidence Interval for the Median

Example 4.4.7 (Confidence Interval for the Median). Let X be a continuous random variable with cdf $F(x)$. Let $\xi_{1/2}$ denote the median of $F(x)$; i.e., $\xi_{1/2}$ solves $F(\xi_{1/2}) = 1/2$. Suppose X_1, X_2, \dots, X_n is a random sample from the distribution of X with corresponding order statistics $Y_1 < Y_2 < \dots < Y_n$. As before, let Q_2 denote the sample median, which is a point estimator of $\xi_{1/2}$. Select α , so that $0 < \alpha < 1$. Take $c_{\alpha/2}$ to be the $\alpha/2$ th quantile of a binomial $b(n, 1/2)$ distribution;

that is, $P[S \leq c_{\alpha/2}] = \alpha/2$, where S is distributed $b(n, 1/2)$. Then note also that $P[S \geq n - c_{\alpha/2}] = \alpha/2$. (Because of the discreteness of the binomial distribution, either take a value of α for which these probabilities are correct or change the equalities to approximations.) Thus it follows from expression (4.4.8) that

$$P[Y_{c_{\alpha/2}+1} < \xi_{1/2} < Y_{n-c_{\alpha/2}}] = 1 - \alpha. \quad (4.4.9)$$

Hence, when the sample is drawn, if $y_{c_{\alpha/2}+1}$ and $y_{n-c_{\alpha/2}}$ are the realized values of the order statistics $Y_{c_{\alpha/2}+1}$ and $Y_{n-c_{\alpha/2}}$, then the interval

$$(y_{c_{\alpha/2}+1}, y_{n-c_{\alpha/2}}) \quad (4.4.10)$$

is a $(1 - \alpha)100\%$ confidence interval for $\xi_{1/2}$.

To illustrate this confidence interval, consider the data of Example 4.4.4. Suppose we want an 88% confidence interval for $\xi_{1/2}$. Then $\alpha/2 = 0.060$. Then $c_{\alpha/2} = 4$ because $P[S \leq 4] = \text{pbinom}(4, 15, .5) = 0.059$, where the distribution of S is binomial with $n = 15$ and $p = 0.5$. Therefore, an 88% confidence interval for $\xi_{1/2}$ is $(y_5, y_{11}) = (96, 106)$.

The R function `onesampsx(x)` computes a confidence interval for the median. For the data in Example 4.4.4, the code `onesampsx(x, alpha=.12)` computes the confidence interval $(96, 106)$ for the median. ■

10.21 Introduction to Hypothesis Testing

See All of statistics Chapter 10.

10.21.1 Definitions of critical (rejection) region, power function, size (significance level), level, type I (II) error, two-side test, one-side test

We partition the parameter space Θ into two disjoint sets Θ_0 and Θ_1 and we wish to test

$$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_1$$

We call H_0 the **null hypothesis** and H_1 the **alternative hypothesis**. Let X be a r.v. and \mathcal{X} be the range of X . To test a hypothesis, we aim to find the **rejection region**(critical region) $R \subset \mathcal{X}$. If $X \in R$ we reject H_0 , otherwise retain H_0 .

Définition 4: critical region (rejection region)

A test of H_0 versus H_1 is based on a subset C of \mathcal{D} . This set C is called the **critical region (rejection region)** and its corresponding decision rule (test) is

$$\begin{aligned} \text{Reject } H_0 \text{ (Accept } H_1) & \quad \text{if } (X_1, \dots, X_n) \in C \\ \text{Retain } H_0 \text{ (Reject } H_1) & \quad \text{if } (X_1, \dots, X_n) \in C^c. \end{aligned}$$

Usually, the **rejection region** R (critical region C) is of the form

$$R = \{x : T(x) > c\}$$

where T is a **test statistic** and c is called a **critical value**.

Définition 5: power function, size (significance level), level

The **power function** of a test with rejection region R is defined by

$$\beta(\theta) = \mathbb{P}_\theta(X \in R)$$

The **size** of a test is defined to be

$$\alpha = \sup_{\theta \in \Theta_0} \beta(\theta)$$

A test is said to have **level^a α** if its size is less than or equal to α .

^aThe definition is useless.

Définition 6: type I error, type II error

Rejecting H_0 when H_0 is true is called a **type I error**. Retaining H_0 when H_1 is true is called a **type II error**.

Définition 7: two-side test, one-side test

A test of the form

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0$$

is called a **two-side test**. A test of the form

$$H_0 : \theta \underset{\text{or } \geq}{\leq} \theta_0 \quad \text{versus} \quad H_1 : \theta \underset{\text{or } <}{>} \theta_0$$

is called a **one-side test**. The most common tests are two-sided.

Example

10.2 Example. Let $X_1, \dots, X_n \sim N(\mu, \sigma)$ where σ is known. We want to test $H_0 : \mu \leq 0$ versus $H_1 : \mu > 0$. Hence, $\Theta_0 = (-\infty, 0]$ and $\Theta_1 = (0, \infty)$. Consider the test:

$$\text{reject } H_0 \text{ if } T > c$$

where $T = \bar{X}$. The rejection region is

$$R = \left\{ (x_1, \dots, x_n) : T(x_1, \dots, x_n) > c \right\}.$$

Let Z denote a standard Normal random variable. The power function is

$$\begin{aligned} \beta(\mu) &= \mathbb{P}_\mu (\bar{X} > c) \\ &= \mathbb{P}_\mu \left(\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} > \frac{\sqrt{n}(c - \mu)}{\sigma} \right) \\ &= \mathbb{P} \left(Z > \frac{\sqrt{n}(c - \mu)}{\sigma} \right) \\ &= 1 - \Phi \left(\frac{\sqrt{n}(c - \mu)}{\sigma} \right). \end{aligned}$$

This function is increasing in μ . See Figure 10.1. Hence

$$\text{size} = \sup_{\mu \leq 0} \beta(\mu) = \beta(0) = 1 - \Phi \left(\frac{\sqrt{n}c}{\sigma} \right).$$

For a size α test, we set this equal to α and solve for c to get

$$c = \frac{\sigma \Phi^{-1}(1 - \alpha)}{\sqrt{n}}.$$

We reject when $\bar{X} > \sigma \Phi^{-1}(1 - \alpha)/\sqrt{n}$. Equivalently, we reject when

$$\frac{\sqrt{n}(\bar{X} - 0)}{\sigma} > z_\alpha,$$

where $z_\alpha = \Phi^{-1}(1 - \alpha)$. ■

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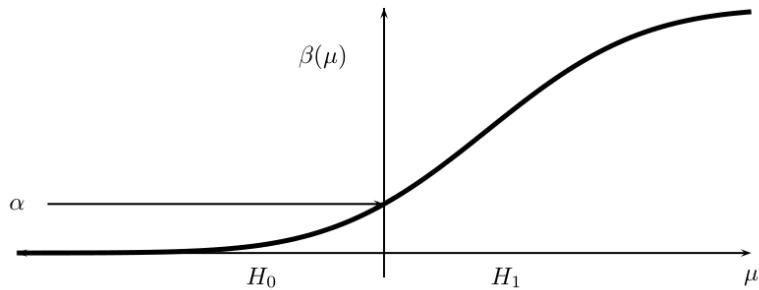


FIGURE 10.1. The power function for Example 10.2. The size of the test is the largest probability of rejecting H_0 when H_0 is true. This occurs at $\mu = 0$ hence the size is $\beta(0)$. We choose the critical value c so that $\beta(c) = \alpha$.

Next we consider four widely used tests: the Wald test, the χ^2 test, the permutation test, and the likelihood ratio test.

10.21.2 The Wald test

Let θ be a scalar parameter, let $\hat{\theta}$ be an estimate of θ and let $\hat{s}\hat{e}$ be the estimated standard error of $\hat{\theta}$.

Définition 8: The Wald Test

Consider testing $H_0 : \theta = \theta_0, H_1 : \theta \neq \theta_0$. Assume that the estimator $\hat{\theta}$ is asymptotically Normal, i.e.

$$W := \frac{\hat{\theta} - \theta_0}{\hat{s}\hat{e}} \xrightarrow{D} N(0, 1)$$

The size α **Wald test** is: reject H_0 when $|W| > z_{\alpha/2}$.

z_α means that for $Z \sim N(0, 1)$,

$$\mathbb{P}(Z > z_\alpha) = \alpha$$

Thus

$$z_\alpha := \Phi^{-1}(1 - \alpha)$$

where $\Phi(x) = \mathbb{P}(Z \leq x) = F_Z(x)$.

Remarque 45. An alternative version of the Wald test statistic is $W = (\hat{\theta} - \theta_0)/\widehat{se}_0$ where \widehat{se}_0 is the standard error computed at $\theta = \theta_0$. Both versions of the test is valid.

Let us consider the power of the Wald test when the null hypothesis is false.

10.6 Theorem. Suppose the true value of θ is $\theta_* \neq \theta_0$. The power $\beta(\theta_*)$ — the probability of correctly rejecting the null hypothesis — is given (approximately) by

$$1 - \Phi\left(\frac{\theta_0 - \theta_*}{\widehat{se}} + z_{\alpha/2}\right) + \Phi\left(\frac{\theta_0 - \theta_*}{\widehat{se}} - z_{\alpha/2}\right). \quad (10.6)$$

Théorème 11: The rejection region of size α Wald test

The size α Wald test rejects $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ iff $\theta_0 \notin C$ where

$$C = (\hat{\theta} - \widehat{se} \cdot z_{\alpha/2}, \hat{\theta} + \widehat{se} \cdot z_{\alpha/2})$$

Thus, testing the hypothesis is equivalent to checking whether the null value is in the confidence interval.

Remarque 46. When we reject H_0 we often say that the result is statistically significant.

Example 1

10.7 Example (Comparing Two Prediction Algorithms). We test a prediction algorithm on a test set of size m and we test a second prediction algorithm on a second test set of size n . Let X be the number of incorrect predictions for algorithm 1 and let Y be the number of incorrect predictions for algorithm 2. Then $X \sim \text{Binomial}(m, p_1)$ and $Y \sim \text{Binomial}(n, p_2)$. To test the null hypothesis that $p_1 = p_2$ write

$$H_0 : \delta = 0 \quad \text{versus} \quad H_1 : \delta \neq 0$$

where $\delta = p_1 - p_2$. The MLE is $\hat{\delta} = \hat{p}_1 - \hat{p}_2$ with estimated standard error

$$\widehat{\text{se}} = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{m} + \frac{\hat{p}_2(1-\hat{p}_2)}{n}}.$$

The size α Wald test is to reject H_0 when $|W| > z_{\alpha/2}$ where

$$W = \frac{\hat{\delta} - 0}{\widehat{\text{se}}} = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{m} + \frac{\hat{p}_2(1-\hat{p}_2)}{n}}}.$$

The power of this test will be largest when p_1 is far from p_2 and when the sample sizes are large.

Example 2

10.8 Example (Comparing Two Means). Let X_1, \dots, X_m and Y_1, \dots, Y_n be two independent samples from populations with means μ_1 and μ_2 , respectively. Let's test the null hypothesis that $\mu_1 = \mu_2$. Write this as $H_0 : \delta = 0$ versus $H_1 : \delta \neq 0$ where $\delta = \mu_1 - \mu_2$. Recall that the nonparametric plug-in estimate of δ is $\hat{\delta} = \bar{X} - \bar{Y}$ with estimated standard error

$$\widehat{\text{se}} = \sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}$$

where s_1^2 and s_2^2 are the sample variances. The size α Wald test rejects H_0 when $|W| > z_{\alpha/2}$ where

$$W = \frac{\hat{\delta} - 0}{\widehat{\text{se}}} = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}}. \blacksquare$$

10.21.3 p -values

For every $\alpha \in (0, 1)$ we have a size α test with rejection region R_α . Then

$$\text{p-value} = \inf\{\alpha : T(X) \in R_\alpha\}$$

p -value is the smallest level at which we can reject H_0 .

Remarque 47. Do not confuse the p -value with $\mathbb{P}(H_0|Data)$. The p -value is not the probability that the null hypothesis is true.

Suppose that the size α test is of the form

$$\text{reject } H_0 \quad \text{iff} \quad T(X) \geq c_\alpha$$

Then

$$\text{p-value} = \sup_{\theta \in \Theta_0} \mathbb{P}_\theta(T(X) > T(x))$$

where x is the observed value of X .

Note. The p -value is the probability (under H_0) of observing a value of the test statistic the same as or more extreme than what was actually observed.

Example

10.15 Example. Recall the cholesterol data from Example 7.15. To test if the means are different we compute

$$W = \frac{\widehat{\delta} - 0}{\widehat{\text{se}}} = \frac{\overline{X} - \overline{Y}}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} = \frac{216.2 - 195.3}{\sqrt{5^2 + 2.4^2}} = 3.78.$$

To compute the p-value, let $Z \sim N(0, 1)$ denote a standard Normal random variable. Then,

$$\text{p-value} = \mathbb{P}(|Z| > 3.78) = 2\mathbb{P}(Z < -3.78) = .0002$$

which is very strong evidence against the null hypothesis. To test if the medians are different, let $\widehat{\nu}_1$ and $\widehat{\nu}_2$ denote the sample medians. Then,

$$W = \frac{\widehat{\nu}_1 - \widehat{\nu}_2}{\widehat{\text{se}}} = \frac{212.5 - 194}{7.7} = 2.4$$

where the standard error 7.7 was found using the bootstrap. The p-value is

$$\text{p-value} = \mathbb{P}(|Z| > 2.4) = 2\mathbb{P}(Z < -2.4) = .02$$

which is strong evidence against the null hypothesis. ■

10.21.4 χ^2 test

Definition of χ^2 distribution

Before proceeding we need to discuss the χ^2 distribution. Let Z_1, \dots, Z_k be independent, standard Normals. Let $V = \sum_{i=1}^k Z_i^2$. Then we say that V has a χ^2 distribution with k degrees of freedom, written $V \sim \chi_k^2$. The probability density of V is

$$f(v) = \frac{v^{(k/2)-1} e^{-v/2}}{2^{k/2} \Gamma(k/2)}$$

for $v > 0$. It can be shown that $\mathbb{E}(V) = k$ and $\mathbb{V}(V) = 2k$. We define the upper α quantile $\chi_{k,\alpha}^2 = F^{-1}(1-\alpha)$ where F is the CDF. That is, $\mathbb{P}(\chi_k^2 > \chi_{k,\alpha}^2) = \alpha$.

Pearson's χ^2 test for multinomial data

The Pearson's χ^2 test is used for multinomial data. If $X = (X_1, \dots, X_k)$ has a multinomial (n, p) distribution, then the MLE of p is $\hat{p} = (\hat{p}_1, \dots, \hat{p}_k)$

Note. X_1, \dots, X_k need not to be independent!

Définition 9: Pearson's χ^2 statistic

Pearson's χ^2 statistic is

$$T = \sum_{j=1}^k \frac{(X_j - np_{0j})^2}{np_{0j}} = \sum_{j=1}^k \frac{(X_j - E_j)^2}{E_j}$$

where $E_j = \mathbb{E}(X_j) = np_{0j}$ is the expected value of X_j under H_0 .

Under H_0 , $T \rightsquigarrow \chi_{k-1}^2$. Hence the test: reject H_0 if $T > \chi_{k-1,\alpha}^2$ has asymptotic level α . The p-value is $\mathbb{P}(\chi_{k-1}^2 > t)$ where t is the observed value of the test statistic.

10.17 Theorem. Under H_0 , $T \rightsquigarrow \chi_{k-1}^2$. Hence the test: reject H_0 if $T > \chi_{k-1,\alpha}^2$ has asymptotic level α . The p-value is $\mathbb{P}(\chi_{k-1}^2 > t)$ where t is the observed value of the test statistic.

Theorem 10.17 is illustrated in Figure 10.5.

10.18 Example (Mendel's peas). Mendel bred peas with round yellow seeds and wrinkled green seeds. There are four types of progeny: round yellow, wrinkled yellow, round green, and wrinkled green. The number of each type is multinomial with probability $p = (p_1, p_2, p_3, p_4)$. His theory of inheritance predicts that p is equal to

$$p_0 \equiv \left(\frac{9}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{16} \right).$$

In $n = 556$ trials he observed $X = (315, 101, 108, 32)$. We will test $H_0 : p = p_0$ versus $H_1 : p \neq p_0$. Since, $np_{01} = 312.75$, $np_{02} = np_{03} = 104.25$, and $np_{04} = 34.75$, the test statistic is

$$\begin{aligned} \chi^2 &= \frac{(315 - 312.75)^2}{312.75} + \frac{(101 - 104.25)^2}{104.25} \\ &\quad + \frac{(108 - 104.25)^2}{104.25} + \frac{(32 - 34.75)^2}{34.75} = 0.47. \end{aligned}$$

The $\alpha = .05$ value for a χ_3^2 is 7.815. Since 0.47 is not larger than 7.815 we do not reject the null. The p-value is

$$\text{p-value} = \mathbb{P}(\chi_3^2 > .47) = .93$$

which is not evidence against H_0 . Hence, the data do not contradict Mendel's theory.³ ■

Example**Goodness-of-fit test ($H_0 : Y$ satisfies some distribution)**

Thus far we have used the chi-square test when the hypothesis H_0 is a simple hypothesis. More often we encounter hypotheses H_0 in which the multinomial probabilities p_1, p_2, \dots, p_k are not completely specified by the hypothesis H_0 . That is, under H_0 , these probabilities are functions of unknown parameters. For an illustration, suppose that a certain random variable Y can take on any real value. Let us partition the space $\{y : -\infty < y < \infty\}$ into k mutually disjoint sets A_1, A_2, \dots, A_k so that the events A_1, A_2, \dots, A_k are mutually exclusive and exhaustive. Let H_0 be the hypothesis that Y is $N(\mu, \sigma^2)$ with μ and σ^2 unspecified. Then each

$$p_i = \int_{A_i} \frac{1}{\sqrt{2\pi}\sigma} \exp[-(y-\mu)^2/2\sigma^2] dy, \quad i = 1, 2, \dots, k,$$

is a function of the unknown parameters μ and σ^2 . Suppose that we take a random sample Y_1, \dots, Y_n of size n from this distribution. If we let X_i denote the frequency of A_i , $i = 1, 2, \dots, k$, so that $X_1 + X_2 + \dots + X_k = n$, the random variable

$$Q_{k-1} = \sum_{i=1}^k \frac{(X_i - np_i)^2}{np_i}$$

cannot be computed once X_1, \dots, X_k have been observed, since each p_i , and hence Q_{k-1} , is a function of μ and σ^2 . Accordingly, choose the values of μ and σ^2 that minimize Q_{k-1} . These values depend upon the observed $X_1 = x_1, \dots, X_k = x_k$ and are called **minimum chi-square estimates** of μ and σ^2 . These point estimates of μ and σ^2 enable us to compute numerically the estimates of each p_i . Accordingly, if these values are used, Q_{k-1} can be computed once Y_1, Y_2, \dots, Y_n , and hence X_1, X_2, \dots, X_k , are observed. However, a very important aspect of the fact, which we accept without proof, is that now Q_{k-1} is approximately $\chi^2(k-3)$. That is, the number of degrees of freedom of the approximate chi-square distribution of Q_{k-1} is reduced by one for each parameter estimated by the observed data. This statement applies not only to the problem at hand but also to more general situations. Two examples are now be given. The first of these examples deals with the test of the hypothesis that two multinomial distributions are the same.

10.22 Consistency and Limiting Distributions

10.22.1 Convergence in Probability

In this section, we formalize the notion of a sequence of random variables $\{X_n\}$ getting "close" to another random variable X as $n \rightarrow \infty$. We say X_n **converges in probability** to X , denoted as $X_n \xrightarrow{P} X$, if $P(|X_n - X| \geq \epsilon) \rightarrow 0$ for all $\epsilon > 0$.

One way to demonstrate convergence in probability is to use Chebyshev's Theorem: for a random variable X with variance $\sigma^2 < \infty$, and for any $k > 0$, we have $P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$.

Let $\{X_n\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with mean μ and variance $\sigma^2 < \infty$. Let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$. Then $\bar{X}_n \xrightarrow{P} \mu$.

证明. The mean and variance of \bar{X}_n are μ and $\frac{\sigma^2}{n}$, respectively. By Chebyshev's Theorem, for all $\epsilon > 0$, we have

$$P(|\bar{X}_n - \mu| \geq \epsilon) = P\left(|\bar{X}_n - \mu| \geq \frac{\epsilon\sqrt{n}}{\sigma} \cdot \frac{\sigma}{\sqrt{n}}\right) \leq \frac{\sigma^2}{n\epsilon^2} \rightarrow 0.$$

□

The Weak Law of Large Numbers (WLLN) implies that all the mass of the distribution of \bar{X}_n converges to μ . In a sense, for large n , \bar{X}_n is close to μ . But how close is it? For example, if we were to estimate μ by \bar{X}_n , what can we say about the error of estimation? Actually, a Strong Law of Large Numbers (SLLN) can be proved. Moreover, we can weaken the hypothesis of the WLLN to require only that the X_i are i.i.d. with finite mean μ . Thus, the SLLN is a first moment theorem, while the WLLN requires the existence of the second moment.

Next, we list several theorems concerning convergence in probability, which is closed under linearity.

Théorème 12

Suppose $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$. Then $X_n + Y_n \xrightarrow{P} X + Y$.

证明. Using the triangle inequality, we have

$$P(|(X_n + Y_n) - (X + Y)| \geq \epsilon) \leq P(|X_n - X| + |Y_n - Y| \geq \epsilon) \leq P\left(|X_n - X| \geq \frac{\epsilon}{2}\right) + P\left(|Y_n - Y| \geq \frac{\epsilon}{2}\right).$$

□

Théorème 13

For any constant a , if $X_n \xrightarrow{P} X$, then $aX_n \xrightarrow{P} aX$.

Théorème 14

If $X_n \xrightarrow{P} a$ and the real function g is continuous at a , then $g(X_n) \xrightarrow{P} g(a)$.

证明. Since g is continuous at a , for all $\epsilon > 0$, there exists $\delta > 0$ such that $|g(x) - g(a)| < \epsilon$ for $|x - a| < \delta$. Thus, $|g(x) - g(a)| \geq \epsilon$ implies $|x - a| \geq \delta$. Substituting X_n for x , we obtain

$$P(|g(X_n) - g(a)| \geq \epsilon) \leq P(|X_n - a| \geq \delta) \rightarrow 0.$$

□

Exercice 1

If $X_n \xrightarrow{P} X$, then $g(X_n) \xrightarrow{P} g(X)$ for continuous g over \mathbb{R} .

证明. For r.v. X , since $F_X(x) \rightarrow 0$ as $x \rightarrow -\infty$ and $F_Y(x) \rightarrow 1$ as $x \rightarrow +\infty$, X is \mathbb{P} -bounded. Given $\epsilon > 0$, since $X_n \xrightarrow{P} X$, $\exists N > 0$, s.t. $\mathbb{P}(|X_n - X| \geq 1) < \epsilon$ for $n \geq N$ and $\exists M > 0$, s.t. $\mathbb{P}(|X| \geq M) < \epsilon$, thus $\mathbb{P}(|X_n| \geq M + 1) \leq \mathbb{P}(|X_n - X| \geq 1) + \mathbb{P}(|X| \geq M) < 2\epsilon$ for $n \geq N$.

As g is continuous on $[-M - 1, M + 1]$, thus uniformly continuous, then $\forall \eta > 0, \exists \delta > 0$, s.t. for $x, y \in [-M - 1, M + 1]$ $|x - y| < \delta \Rightarrow |g(x) - g(y)| < \eta$. Also $\exists N' > 0$, s.t. $\mathbb{P}(|X_n - X| \geq \delta) < \epsilon$ for $n \geq N'$. Then for $n \geq \max\{N, N'\}$,

$$\{\omega : |g(X_n) - g(X)| \geq \epsilon\} \subseteq \{\omega : |X_n| \geq M + 1\} \cup \{\omega : |X| \geq M + 1\} \cup \{\omega : |X_n - X| \geq \delta\}$$

thus

$$\mathbb{P}(|g(X_n) - g(X)| \geq \eta) \leq \underbrace{\mathbb{P}(|X_n| \geq M + 1)}_{\leq 2\epsilon} + \underbrace{\mathbb{P}(|X| \geq M + 1)}_{\leq \epsilon} + \underbrace{\mathbb{P}(|X_n - X| \geq \delta)}_{\leq \epsilon} \leq 4\epsilon$$

Since ϵ is arbitrary, we have $g(X_n) \xrightarrow{P} g(X)$. □

Théorème 15

Suppose $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$. Then $X_n Y_n \xrightarrow{P} XY$.

证明. We can write

$$X_n Y_n = \frac{1}{2}(X_n + Y_n)^2 - \frac{1}{2}(X_n - Y_n)^2.$$

Since $X_n + Y_n \xrightarrow{P} X + Y$ and $X_n - Y_n \xrightarrow{P} X - Y$, we have $(X_n + Y_n)^2 \xrightarrow{P} (X + Y)^2$ and $(X_n - Y_n)^2 \xrightarrow{P} (X - Y)^2$. Therefore,

$$X_n Y_n \xrightarrow{P} \frac{1}{2}(X + Y)^2 - \frac{1}{2}(X - Y)^2 = XY.$$

Alternatively, $X_n Y_n = \frac{1}{2}X_n^2 + \frac{1}{2}Y_n^2 - \frac{1}{2}(X_n - Y_n)^2 \xrightarrow{P} \frac{1}{2}X^2 + \frac{1}{2}Y^2 - \frac{1}{2}(X - Y)^2 = XY$. \square

Sampling and Statistics

Consider the situation where we have a random variable X whose pdf (or pmf) is written as $f(x; \theta)$ for an unknown parameter $\theta \in \Omega$. For example, the distribution of X is normal with unknown mean μ and variance σ^2 . Then $\theta = (\mu, \sigma^2)$ and $\Omega = \{\theta = (\mu, \sigma^2) : -\infty < \mu < \infty, \sigma > 0\}$. As another example, the distribution of X is $\Gamma(1, \beta)$, where $\beta > 0$ is unknown. Our information consists of a **random sample** X_1, X_2, \dots, X_n on X ; i.e., X_1, X_2, \dots, X_n are independent and identically distributed (iid) random variables with the common pdf $f(x; \theta)$, $\theta \in \Omega$. We say that T is a **statistic** if T is a function of the sample; i.e., $T = T(X_1, X_2, \dots, X_n)$. Here, we want to consider T as a **point estimator** of θ . For example, if μ is the unknown mean of X , then we may use as our point estimator the sample mean $\bar{X} = n^{-1} \sum_{i=1}^n X_i$. When the sample is drawn let x_1, x_2, \dots, x_n denote the observed values of X_1, X_2, \dots, X_n . We call these values the **realized** values of the sample and call the realized statistic $t = t(x_1, x_2, \dots, x_n)$ a **point estimate** of θ .

In Chapters 6 and 7, we discuss properties of point estimators in formal settings. For now, we consider two properties: **unbiasedness** and **consistency**. We say that the point estimator T for θ is **unbiased** if $E(T) = \theta$. Recall in Section 2.8, we showed that the sample mean \bar{X} and the sample variance S^2 are unbiased estimators of μ and σ^2 respectively; see equations (2.8.6) and (2.8.8). We next consider consistency of a point estimator.

Definition 5.1.2 (Consistency). Let X be a random variable with cdf $F(x, \theta)$, $\theta \in \Omega$. Let X_1, \dots, X_n be a sample from the distribution of X and let T_n denote a statistic. We say T_n is a **consistent** estimator of θ if

$$T_n \xrightarrow{P} \theta.$$

10.22.2 Convergence in Distribution

In many situations we can show statistic convergence without the distribution function of the statistic. But how close is the statistic to the estimator?

Definition 5.2.1 (Convergence in Distribution). *Let $\{X_n\}$ be a sequence of random variables and let X be a random variable. Let F_{X_n} and F_X be, respectively, the cdfs of X_n and X . Let $C(F_X)$ denote the set of all points where F_X is continuous. We say that X_n converges in distribution to X if*

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x), \quad \text{for all } x \in C(F_X).$$

We denote this convergence by

$$X_n \xrightarrow{D} X.$$

Example 5.2.3. Let T_n have a t -distribution with n degrees of freedom, $n = 1, 2, 3, \dots$. Thus its cdf is

$$F_n(t) = \int_{-\infty}^t \frac{\Gamma[(n+1)/2]}{\sqrt{\pi n}} \frac{1}{\Gamma(n/2)} \frac{1}{(1+y^2/n)^{(n+1)/2}} dy,$$

where the integrand is the pdf $f_n(y)$ of T_n . Accordingly,

$$\lim_{n \rightarrow \infty} F_n(t) = \lim_{n \rightarrow \infty} \int_{-\infty}^t f_n(y) dy = \int_{-\infty}^t \lim_{n \rightarrow \infty} f_n(y) dy,$$

by a result in analysis (the Lebesgue Dominated Convergence Theorem) that allows us to interchange the order of the limit and integration, provided that $|f_n(y)|$ is dominated by a function that is integrable. This is true because

$$|f_n(y)| \leq 10f_1(y)$$

and

$$\int_{-\infty}^t 10f_1(y) dy = \frac{10}{\pi} \arctan t < \infty,$$

for all real t . Hence we can find the limiting distribution by finding the limit of the pdf of T_n . It is

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n(y) &= \lim_{n \rightarrow \infty} \left\{ \frac{\Gamma[(n+1)/2]}{\sqrt{n/2} \Gamma(n/2)} \right\} \lim_{n \rightarrow \infty} \left\{ \frac{1}{(1+y^2/n)^{1/2}} \right\} \\ &\quad \times \lim_{n \rightarrow \infty} \left\{ \frac{1}{\sqrt{2\pi}} \left[\left(1 + \frac{y^2}{n} \right) \right]^{-n/2} \right\}. \end{aligned}$$

Using the fact from elementary calculus that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{y^2}{n} \right)^n = e^{y^2},$$

the limit associated with the third factor is clearly the pdf of the standard normal distribution. The second limit obviously equals 1. By Remark 5.2.2, the first limit also equals 1. Thus, we have

$$\lim_{n \rightarrow \infty} F_n(t) = \int_{-\infty}^t \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy,$$

and hence T_n has a limiting standard normal distribution. ■

Théorème 16: Stirling's formula

$$\Gamma(k+1) \sim \sqrt{2\pi} k^{k+1/2} e^{-k}$$

Example 5.2.4 (Maximum of a Sample from a Uniform Distribution, Continued). Recall Example 5.1.2, where X_1, \dots, X_n is a random sample from a uniform($0, \theta$) distribution. Again, let $Y_n = \max\{X_1, \dots, X_n\}$, but now consider the random variable $Z_n = n(\theta - Y_n)$. Let $t \in (0, n\theta)$. Then, using the cdf of Y_n , (5.1.1), the cdf of Z_n is

$$\begin{aligned} P[Z_n \leq t] &= P[Y_n \geq \theta - (t/n)] \\ &= 1 - \left(\frac{\theta - (t/n)}{\theta} \right)^n \\ &= 1 - \left(1 - \frac{t/\theta}{n} \right)^n \\ &\rightarrow 1 - e^{-t/\theta}. \end{aligned}$$

Note that the last quantity is the cdf of an exponential random variable with mean θ , (3.3.6), i.e., $\Gamma(1, \theta)$. So we say that $Z_n \xrightarrow{D} Z$, where Z is distributed $\Gamma(1, \theta)$. ■

Convergence in distribution is weaker than convergence in probability. Thus convergence in distribution is often called weak convergence.

Théorème 17

If X_n converges to X in probability, then X_n converges to X in distribution.

证明. Let x be a point of continuity of $F_X(x)$. For every $\epsilon > 0$,

$$\begin{aligned} F_{X_n}(x) &= P[X_n \leq x] \\ &= P[\{X_n \leq x\} \cap \{|X_n - X| < \epsilon\}] + P[\{X_n \leq x\} \cap \{|X_n - X| \geq \epsilon\}] \\ &\leq P[X \leq x + \epsilon] + P[|X_n - X| \geq \epsilon] \end{aligned}$$

Basd on this inequality and the fact that $X_n \xrightarrow{P} X$ we see that

$$\limsup_{n \rightarrow \infty} F_{X_n}(x) \leq F_X(x + \epsilon)$$

To get a lower bound, we proceed similarly with the complement to show that

$$P[X_n > x] \leq P[X \geq x - \epsilon] + P[|X_n - X| \geq \epsilon]$$

Hence

$$\liminf_{n \rightarrow \infty} F_{X_n}(x) \geq F_X(x - \epsilon)$$

Using a relationship between \limsup and \liminf , it follows that

$$F_X(x - \epsilon) \leq \liminf_{n \rightarrow \infty} F_{X_n}(x) \leq \limsup_{n \rightarrow \infty} F_{X_n}(x) \leq F_X(x + \epsilon)$$

Letting $\epsilon \downarrow 0$ gives us the desired result. \square

Reconsider the sequence of random variables $\{X_n\}$ defined by expression (5.2.1). Here, $X_n \xrightarrow{D} X$ but $X_n \not\xrightarrow{P} X$. So, in general, the converse of the above theorem is not true. However, it is true if X is degenerate, as shown by the following theorem.

distributions. Define the sequence of random variables X_n as

$$X_n = \begin{cases} X & \text{if } n \text{ is odd} \\ -X & \text{if } n \text{ is even.} \end{cases} \quad (5.2.1)$$

Théorème 18

If $X_n \xrightarrow{D} b$ constant, then $X_n \xrightarrow{P} b$.

Let $\epsilon > 0$ be given. Then

$$\lim_{n \rightarrow \infty} P[|X_n - b| \leq \epsilon] = \lim_{n \rightarrow \infty} F_{X_n}(b + \epsilon) - \lim_{n \rightarrow \infty} F_{X_n}[(b - \epsilon) - 0] = 1 - 0 = 1$$

The converse is not true.

Théorème 19

$X_n \xrightarrow{D} X, Y_n \xrightarrow{P} 0$ then $X_n + Y_n \xrightarrow{D} X$.

The proof is similar to the above theorem.

We often use this result as follows. Suppose it is difficult to show that X_n converges to X in distribution, but it is easy to show that Y_n converges in distribution to X and that $X_n - Y_n$ converges to 0 in probability. Hence by this last theorem, $X_n = Y_n + (X_n - Y_n) \xrightarrow{D} X$ as desired.

The next two theorems state general results.

Théorème 20

$X_n \xrightarrow{D} X$ and g continuous on the support of X . Then $g(X_n) \xrightarrow{D} g(X)$.

An often-used application of this theorem occurs when we have a sequence of random variables Z_n which converges in distribution to a standard normal random variable Z . Because the distribution of Z^2 is $\chi^2(1)$, it follows by the above theorem that Z_n^2 converges in distribution to a $\chi^2(1)$ distribution.

Théorème 21: Slutsky's theorem

If $X_n \xrightarrow{D} X$, $A_n \xrightarrow{P} a$ and $B_n \xrightarrow{P} b$ then $A_n + B_n X_n \xrightarrow{D} a + bX$.

The proof is similar to Théorème 17

10.22.3 Bounded in Probability

Definition 5.2.2 (Bounded in Probability). *We say that the sequence of random variables $\{X_n\}$ is bounded in probability if, for all $\epsilon > 0$, there exist a constant $B_\epsilon > 0$ and an integer N_ϵ such that*

$$n \geq N_\epsilon \Rightarrow P[|X_n| \leq B_\epsilon] \geq 1 - \epsilon.$$

Théorème 22

Theorem 5.2.6. *Let $\{X_n\}$ be a sequence of random variables and let X be a random variable. If $X_n \rightarrow X$ in distribution, then $\{X_n\}$ is bounded in probability.*

One way of thinking of a sequence that is bounded in probability (or one that is converging to a random variable in distribution) is that the probability mass of $|X_n|$ is not escaping to ∞ . At times we can use boundedness in probability instead of convergence in distribution. A property we will need later is given in the following theorem:

Théorème 23

Let $\{X_n\}$ be a sequence of random variables bounded in probability and let $\{Y_n\}$ be a sequence of random variables that converges to 0 in probability. Then

$$X_n Y_n \xrightarrow{P} 0$$

Let $\epsilon > 0$ is given. Choose $B_\epsilon > 0$ and an integer N_ϵ such that

$$n \geq N_\epsilon \Rightarrow P[|X_n| \leq B_\epsilon] \geq 1 - \epsilon$$

Then

$$\overline{\lim}_{n \rightarrow \infty} P[|X_n Y_n| \geq \epsilon] \leq \overline{\lim}_{n \rightarrow \infty} P[|X_n Y_n| \geq \epsilon, |X_n| \leq B_\epsilon] + \overline{\lim}_{n \rightarrow \infty} P[|X_n Y_n| \geq \epsilon, |X_n| > B_\epsilon] \leq \overline{\lim}_{n \rightarrow \infty} P[|Y_n| \geq \epsilon/B_\epsilon] -$$

Remarque 48. It's similar to $X_n, Y_n \xrightarrow{P} X, Y \Rightarrow X_n Y_n \xrightarrow{P} XY$.

10.22.4 Δ -Method

The Δ -method is employed to determine the asymptotic distribution of a function of a random variable, given the distribution of the random variable itself. This is analogous to problems discussed in previous chapters, such as Théorème 20 and Théorème 21.

Little-o Notation

The notation $Y_n = o_p(X_n)$ signifies that Y_n converges to 0 in probability relative to X_n , formally:

$$Y_n = o_p(X_n) \text{ if and only if } \frac{Y_n}{X_n} \xrightarrow{P} 0, \text{ as } n \rightarrow \infty.$$

Big-O Notation

The notation $Y_n = O_p(X_n)$ indicates that $\frac{Y_n}{X_n}$ is bounded in probability as $n \rightarrow \infty$.

Théorème 24: Theorem 5.2.8.

If $\{Y_n\}$ is a sequence of random variables that is bounded in probability and $X_n = o_p(Y_n)$, then $X_n \xrightarrow{P} 0$ as $n \rightarrow \infty$.

Proof of Theorem 5.2.8

Let $\epsilon > 0$. Then, there exist N_ϵ and B_ϵ such that for $n \geq N_\epsilon$, $P[|Y_n| \leq B_\epsilon] \geq 1 - \epsilon$. Since $\frac{X_n}{Y_n} \xrightarrow{P} 0$, we have:

$$P[|X_n| \geq \epsilon] = P[|X_n| \geq \epsilon, |Y_n| \leq B_\epsilon] + P[|X_n| \geq \epsilon, |Y_n| > B_\epsilon] \leq P\left[\frac{|X_n|}{|Y_n|} \geq \frac{\epsilon}{B_\epsilon}\right] + P[|Y_n| > B_\epsilon] \rightarrow \epsilon$$

Théorème 25: Theorem 5.2.9 (Δ -Method)

Let $\{X_n\}$ be a sequence of random variables such that

$$\sqrt{n}(X_n - \theta) \xrightarrow{D} N(0, \sigma^2).$$

If $g(x)$ is differentiable at θ and $g'(\theta) \neq 0$, then

$$\sqrt{n}(g(X_n) - g(\theta)) \xrightarrow{D} N(0, \sigma^2(g'(\theta))^2).$$

Proof of Theorem 5.2.9

Since $g(X_n) = g(\theta) + g'(\theta)(X_n - \theta) + o_p(|X_n - \theta|)$, it follows that

$$\sqrt{n}(g(X_n) - g(\theta)) = g'(\theta)\sqrt{n}(X_n - \theta) + o_p(\sqrt{n}|X_n - \theta|)$$

Because $\sqrt{n}(X_n - \theta) \xrightarrow{D} N(0, \sigma^2)$, it implies that $\sqrt{n}|X_n - \theta|$ is bounded in probability. Therefore, by Théorème 24, $o_p(\sqrt{n}|X_n - \theta|) \rightarrow 0$ in probability. Hence, the result follows.

10.22.5 Moment Generating Function Technique

It's difficult to obtain $\lim_{n \rightarrow \infty} F_{X_n}(x)$, but quite easier from the mgf M_n that corresponds to the cdf $F_{X_n}(x)$.

Theorem 5.2.10. Let $\{X_n\}$ be a sequence of random variables with mgf $M_{X_n}(t)$ that exists for $-h < t < h$ for all n . Let X be a random variable with mgf $M(t)$, which exists for $|t| \leq h_1 \leq h$. If $\lim_{n \rightarrow \infty} M_{X_n}(t) = M(t)$ for $|t| \leq h_1$, then $X_n \xrightarrow{D} X$.

Example 5.2.7. Let Z_n be $\chi^2(n)$. Then the mgf of Z_n is $(1 - 2t)^{-n/2}$, $t < \frac{1}{2}$. The mean and the variance of Z_n are, respectively, n and $2n$. The limiting distribution of the random variable $Y_n = (Z_n - n)/\sqrt{2n}$ will be investigated. Now the mgf of Y_n is

$$\begin{aligned} M_{Y_n}(t) &= E \left\{ \exp \left[t \left(\frac{Z_n - n}{\sqrt{2n}} \right) \right] \right\} \\ &= e^{-tn/\sqrt{2n}} E(e^{tZ_n/\sqrt{2n}}) \\ &= \exp \left[- \left(t \sqrt{\frac{2}{n}} \right) \left(\frac{n}{2} \right) \right] \left(1 - 2 \frac{t}{\sqrt{2n}} \right)^{-n/2}, \quad t < \frac{\sqrt{2n}}{2}. \end{aligned}$$

This may be written in the form

$$M_{Y_n}(t) = \left(e^{t\sqrt{2/n}} - t \sqrt{\frac{2}{n}} e^{t\sqrt{2/n}} \right)^{-n/2}, \quad t < \sqrt{\frac{n}{2}}.$$

In accordance with Taylor's formula, there exists a number $\xi(n)$, between 0 and $t\sqrt{2/n}$, such that

$$e^{t\sqrt{2/n}} = 1 + t \sqrt{\frac{2}{n}} + \frac{1}{2} \left(t \sqrt{\frac{2}{n}} \right)^2 + \frac{e^{\xi(n)}}{6} \left(t \sqrt{\frac{2}{n}} \right)^3.$$

If this sum is substituted for $e^{t\sqrt{2/n}}$ in the last expression for $M_{Y_n}(t)$, it is seen that

$$M_{Y_n}(t) = \left(1 - \frac{t^2}{n} + \frac{\psi(n)}{n} \right)^{-n/2},$$

where

$$\psi(n) = \frac{\sqrt{2}t^3 e^{\xi(n)}}{3\sqrt{n}} - \frac{\sqrt{2}t^3}{\sqrt{n}} - \frac{2t^4 e^{\xi(n)}}{3n}.$$

Since $\xi(n) \rightarrow 0$ as $n \rightarrow \infty$, then $\lim \psi(n) = 0$ for every fixed value of t . In accordance with the limit proposition cited earlier in this section, we have

$$\lim_{n \rightarrow \infty} M_{Y_n}(t) = e^{t^2/2}$$

for all real values of t . That is, the random variable $Y_n = (Z_n - n)/\sqrt{2n}$ has a limiting standard normal distribution.

10.22.6 Central Limit Theorem

Theorem 5.3.1 (Central Limit Theorem). *Let X_1, X_2, \dots, X_n denote the observations of a random sample from a distribution that has mean μ and positive variance σ^2 . Then the random variable $Y_n = (\sum_{i=1}^n X_i - n\mu)/\sqrt{n}\sigma = \sqrt{n}(\bar{X}_n - \mu)/\sigma$ converges in distribution to a random variable that has a normal distribution with mean zero and variance 1.*

Prove by characteristic function $\varphi(t) = E(e^{itX})$.

The Central Limit Theorem is saying that when n is large, fixed positive integer, the random variable \bar{X} has an approximate normal distribution with mean μ and variance σ^2/n . We can equivalently state the conclusion of the Central Limit Theorem as

$$\sqrt{n}(\bar{X} - \mu) \xrightarrow{\mathcal{D}} N(0, \sigma^2)$$

This is often a convenient formulation to use.

Remarque 49. We know that \bar{X} and $\sum_{i=1}^n X_i$ have approximately normal distributions, provided that n is large enough. Later, we find that other statistics also have approximate normal distributions, and this is the reason that the normal distribution is so important to statisticians. That is, while not many underlying distributions are normal, the distributions of statistics calculated from random samples arising from these distributions are often close to being normal.

We can combine Δ -method with Central Limit Theorem. Assume that X_1, \dots, X_n is a random sample on X which has finite mean μ and variance σ^2 . Then by the Central Limit Theorem, we have

$$\sqrt{n}(\bar{X} - \mu) \xrightarrow{\mathcal{D}} N(0, \sigma^2)$$

Hence by the Δ -method, we have

$$\sqrt{n}[g(\bar{X}) - g(\mu)] \xrightarrow{\mathcal{D}} N(0, \sigma^2(g'(\mu))^2)$$

for a continuous transformation $g(x)$ such that $g'(\mu) \neq 0$.

10.22.7 Extensions to Multivariate Distributions

This section discusses asymptotic concepts for sequences of random vectors.

Définition 10: convergence in probability

Let $\{\mathbf{X}_n\}$ be a sequence of p -dimensional random vectors, and let \mathbf{X} be a random vector, all defined on the same sample space. We say that $\{\mathbf{X}_n\}$ converges in probability to \mathbf{X} if

$$\lim_{n \rightarrow \infty} P[\|\mathbf{X}_n - \mathbf{X}\| \geq \epsilon] = 0$$

for all $\epsilon > 0$. As in the univariate case, we write $\mathbf{X}_n \xrightarrow{P} \mathbf{X}$.

Théorème 26: Theorem 5.4.1

Let $\{\mathbf{X}_n\}$ be a sequence of p -dimensional random vectors, and let \mathbf{X} be a random vector, all defined on the same sample space. Then $\mathbf{X}_n \xrightarrow{P} \mathbf{X}$ if and only if $X_{nj} \xrightarrow{P} X_j$ for all $j = 1, \dots, p$.

Based on Théorème 26, many theorems involving convergence in probability can be extended to the multivariate setting.

Let $\{\mathbf{X}_n\}$ be a sequence of i.i.d. random vectors with common mean vector $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$. Denote the vector of means by $\bar{\mathbf{X}}_n = \frac{1}{n} \sum_{i=1}^n \mathbf{X}_i$. By the Weak Law of Large Numbers, $\bar{X}_j \rightarrow \mu_j$ in probability for each j . Hence, by Théorème 26, $\bar{\mathbf{X}}_n \rightarrow \boldsymbol{\mu}$ in probability.

Now consider the analog of the sample variances. Let $\mathbf{X}_i = (X_{i1}, \dots, X_{ip})'$. Define the sample variances and covariances by

$$S_{n,jj} = S_{n,j}^2 = \frac{1}{n-1} \sum_{i=1}^n (X_{ij} - \bar{X}_j)^2 \quad \text{for } j = 1, \dots, p$$

$$S_{n,jk} = \frac{1}{n-1} \sum_{i=1}^n (X_{ij} - \bar{X}_j)(X_{ik} - \bar{X}_k) \quad \text{for } j \neq k = 1, \dots, p$$

If we define the $p \times p$ matrix $\mathbf{S} = (S_{n,jk})$, then $\mathbf{S} \rightarrow \boldsymbol{\Sigma}$ in probability.

Définition 11: convergence in distribution

Let $\{\mathbf{X}_n\}$ be a sequence of random vectors with \mathbf{X}_n having distribution function $F_n(\mathbf{x})$, and let \mathbf{X} be a random vector with distribution function $F(\mathbf{x})$. Then $\{\mathbf{X}_n\}$ **converges in distribution** to \mathbf{X} if

$$\lim_{n \rightarrow \infty} F_n(\mathbf{x}) = F(\mathbf{x}),$$

for all points \mathbf{x} at which $F(\mathbf{x})$ is continuous. We write $\mathbf{X}_n \xrightarrow{D} \mathbf{X}$.

Théorème 27

Let $\{\mathbf{X}_n\}$ be a sequence of random vectors that converges in distribution to a random vector \mathbf{X} , and let $g(\mathbf{x})$ be a function that is continuous on the support of \mathbf{X} . Then $g(\mathbf{X}_n)$ converges in distribution to $g(\mathbf{X})$.

Théorème 28

Let $\{\mathbf{X}_n\}$ be a sequence of random vectors with \mathbf{X}_n having distribution function $F_n(\mathbf{x})$ and moment generating function $M_n(\mathbf{t})$. Let \mathbf{X} be a random vector with distribution function $F(\mathbf{x})$ and moment generating function $M(\mathbf{t})$. Then $\{\mathbf{X}_n\}$ converges in distribution to \mathbf{X} if and only if, for some $h > 0$,

$$\lim_{n \rightarrow \infty} M_n(\mathbf{t}) = M(\mathbf{t}),$$

for all \mathbf{t} such that $\|\mathbf{t}\| < h$.

Théorème 29: Multivariate Central Limit Theorem

Let $\{\mathbf{X}_n\}$ be a sequence of i.i.d. random vectors with common mean vector $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$ which is positive definite. Assume that the common moment generating function $M(\mathbf{t})$ exists in an open neighborhood of $\mathbf{0}$. Let

$$\mathbf{Y}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n (\mathbf{X}_i - \boldsymbol{\mu}) = \sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu})$$

Then \mathbf{Y}_n converges in distribution to a $N_p(\mathbf{0}, \boldsymbol{\Sigma})$ distribution.

\begin{proof} Let $\mathbf{t} \in \mathbf{R}^p$ be a vector in the stipulated neighborhood of $\mathbf{0}$. The moment generating function of \mathbf{Y}_n is

$$M_n(\mathbf{t}) = \mathbb{E} \left[\exp \left\{ \mathbf{t}' \frac{1}{\sqrt{n}} \sum_{i=1}^n (\mathbf{X}_i - \boldsymbol{\mu}) \right\} \right] = \mathbb{E} \left[\exp \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbf{t}' (\mathbf{X}_i - \boldsymbol{\mu}) \right\} \right] = \mathbb{E} \left[\exp \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^n W_i \right\} \right]$$

where $W_i = \mathbf{t}' (\mathbf{X}_i - \boldsymbol{\mu})$. Note that W_i are i.i.d. with mean 0 and variance $\text{Var}(W_i) = \mathbf{t}' \boldsymbol{\Sigma} \mathbf{t}$. Hence, by the standard Central Limit Theorem,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n W_i \xrightarrow{D} N(0, \mathbf{t}' \boldsymbol{\Sigma} \mathbf{t})$$

Then $M_n(\mathbf{t})$ is the MGF of $(1/\sqrt{n}) \sum_{i=1}^n W_i$ evaluated at 1. Therefore, we must have

$$M_n(\mathbf{t}) = \mathbb{E} \left[\exp \left\{ 1 \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^n W_i \right\} \right] \rightarrow e^{1^2 \mathbf{t}' \boldsymbol{\Sigma} \mathbf{t}/2} = e^{\mathbf{t}' \boldsymbol{\Sigma} \mathbf{t}/2}.$$

Because the last quantity is the moment generating function of a $N_p(\mathbf{0}, \boldsymbol{\Sigma})$ distribution, the result follows. \end{proof}

Théorème 30: Theorem 5.4.5

Let $\{\mathbf{X}_n\}$ be a sequence of p -dimensional random vectors. Suppose $\mathbf{X}_n \xrightarrow{D} N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let \mathbf{A} be an $m \times p$ matrix of constants, and let \mathbf{b} be an m -dimensional vector of constants. Then $\mathbf{AX}_n + \mathbf{b} \xrightarrow{D} N(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$.

Théorème 31: Theorem 5.4.6

Let $\{\mathbf{X}_n\}$ be a sequence of p -dimensional random vectors. Suppose

$$\sqrt{n}(\mathbf{X}_n - \boldsymbol{\mu}_0) \xrightarrow{D} N_p(\mathbf{0}, \boldsymbol{\Sigma}).$$

Let \mathbf{g} be a transformation $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \dots, g_k(\mathbf{x}))'$ such that $1 \leq k \leq p$ and the $k \times p$ matrix of partial derivatives,

$$\mathbf{B} = \left[\frac{\partial g_i}{\partial \mu_j} \right], \quad i = 1, \dots, k; j = 1, \dots, p$$

are continuous and do not vanish in a neighborhood of $\boldsymbol{\mu}_0$. Let $\mathbf{B}_0 = \mathbf{B}$ at $\boldsymbol{\mu}_0$. Then

$$\sqrt{n}(\mathbf{g}(\mathbf{X}_n) - \mathbf{g}(\boldsymbol{\mu}_0)) \xrightarrow{D} N_k(\mathbf{0}, \mathbf{B}_0 \boldsymbol{\Sigma} \mathbf{B}_0').$$

10.23 Maximum Likelihood Estimation

10.23.1 Rao-Cramer Lower Bound and Efficiency

We need some regularity conditions.

Regularity conditions

Définition 12: Assumptions 6.1.1 (Regularity Conditions)

Regularity conditions (R0)-(R5) are

- (1) **(R0)** The cdfs are distinct; i.e., $\theta \neq \theta' \implies F(x_i; \theta) \neq F(x_i; \theta')$.
- (2) **(R1)** The pdfs have common support for all θ .
- (3) **(R2)** The point θ_0 is an interior point in Ω .
- (4) **(R3)** The pdf $f(x; \theta)$ is twice differentiable as a function of θ .
- (5) **(R4)** The integral $\int f(x; \theta) dx$ can be differentiated twice under the integral sign as a function of θ .
- (6) **(R5)** The pdf $f(x; \theta)$ is three times differentiable as a function of θ .

Further, for all $\theta \in \Omega$, there exist a constant c and a function $M(x)$ such that

$$\left| \frac{\partial^3}{\partial \theta^3} \log f(x; \theta) \right| \leq M(x),$$

with $\mathbb{E}_{\theta_0}[M(X)] < \infty$, for all $\theta_0 - c < \theta < \theta_0 + c$ and all x in the support of X .

Fisher information

Since

$$1 = \int_{-\infty}^{\infty} f(x; \theta) dx \implies 0 = \int_{-\infty}^{\infty} \frac{\partial f(x; \theta)}{\partial \theta} dx = \int_{-\infty}^{\infty} \frac{\partial \log f(x; \theta)}{\partial \theta} f(x; \theta) dx$$

thus

$$\mathbb{E}_{\theta} \left[\frac{\partial \log f(X; \theta)}{\partial \theta} \right] = 0$$

Apply $\frac{\partial}{\partial \theta}$ again,

$$0 = \underbrace{\int_{-\infty}^{\infty} \frac{\partial^2 \log f(x; \theta)}{\partial \theta^2} f(x; \theta) dx}_{= -\mathbb{E}_{\theta} \left[\frac{\partial^2 \log f(X; \theta)}{\partial \theta^2} \right]} + \underbrace{\int_{-\infty}^{\infty} \frac{\partial \log f(x; \theta)}{\partial \theta} \frac{\partial \log f(x; \theta)}{\partial \theta} f(x; \theta) dx}_{= \mathbb{E}_{\theta} \left[\left(\frac{\partial \log f(X; \theta)}{\partial \theta} \right)^2 \right]}$$

We define the **Fisher information**:

$$I(\theta) = \mathbb{E}_{\theta} \left[\left(\frac{\partial \log f(X; \theta)}{\partial \theta} \right)^2 \right] = -\mathbb{E}_{\theta} \left[\frac{\partial^2 \log f(X; \theta)}{\partial \theta^2} \right] \stackrel{\mathbb{E}_{\theta} \left[\frac{\partial \log f(X; \theta)}{\partial \theta} \right] = 0}{=} \text{Var} \left(\frac{\partial \log f(X; \theta)}{\partial \theta} \right)$$

The function $\frac{\partial \log f(x; \theta)}{\partial \theta}$ is called the **score function**. Recall the calculation of MLE, the mle $\hat{\theta}$ solves

$$\sum_{i=1}^n \frac{\partial \log f(x_i; \theta)}{\partial \theta} = 0$$

Rao-Cramer Lower Bound, efficient estimator, efficiency

Théorème 32: Rao-Cramér Lower Bound

Let X_1, \dots, X_n be iid with common pdf $f(x; \theta)$ for $\theta \in \Omega$. Assume that the regularity conditions (R0)-(R4) hold. Let $Y = u(X_1, X_2, \dots, X_n)$ be a statistic with mean $\mathbb{E}(Y) = \mathbb{E}[u(X_1, X_2, \dots, X_n)] =: k(\theta)$. Then

$$\text{Var}(Y) \geq \frac{[k'(\theta)]^2}{nI(\theta)} \quad (10.3)$$

Corollaire 1

Under the assumptions of eq. (10.3), if $Y = u(X_1, \dots, X_n)$ is an unbiased estimator of θ , so that $k(\theta) = \theta$, then the Rao-Cramér inequality becomes

$$\text{Var}(Y) \geq \frac{1}{nI(\theta)} \quad (10.4)$$

Définition 13: Efficiency

The **efficiency** of that estimator is

$$e(\hat{\theta}) := \frac{(k'(\theta))^2}{nI(\theta) \cdot \text{Var}(\hat{\theta})}$$

Définition 14: Efficient Estimator

Let Y be an unbiased estimator of a parameter θ . The statistic Y is called an **efficient estimator** of θ iff

$$\text{Var}(Y) = \frac{1}{nI(\theta)} \quad \text{or} \quad e(\hat{\theta}) = 1$$

Théorème 33

Assume X_1, \dots, X_n are iid with pdf $f(x; \theta_0)$ for $\theta_0 \in \Omega$ such that the regularity conditions (R0)-(R5) are satisfied. Suppose further that the Fisher information satisfies $0 < I(\theta_0) < \infty$. Then any consistent sequence of solutions of the mle equations satisfies

$$\sqrt{n} (\hat{\theta} - \theta_0) \xrightarrow{D} N\left(0, \frac{1}{I(\theta_0)}\right).$$

Définition 15

Let X_1, \dots, X_n be independent and identically distributed with probability density function $f(x; \theta)$. Suppose $\hat{\theta}_{1n} = \hat{\theta}_{1n}(X_1, \dots, X_n)$ is an estimator of θ_0 such that $\sqrt{n} (\hat{\theta}_{1n} - \theta_0) \xrightarrow{D} N\left(0, \sigma_{\hat{\theta}_{1n}}^2\right)$. Then

- (1) The **asymptotic efficiency** of $\hat{\theta}_{1n}$ is defined to be

$$e(\hat{\theta}_{1n}) = \frac{1/I(\theta_0)}{\sigma_{\hat{\theta}_{1n}}^2}.$$

- (1) The estimator $\hat{\theta}_{1n}$ is said to be **asymptotically efficient** if the ratio in part (a) is 1 .
 (2) Let $\hat{\theta}_{2n}$ be another estimator such that $\sqrt{n} (\hat{\theta}_{2n} - \theta_0) \xrightarrow{D} N\left(0, \sigma_{\hat{\theta}_{2n}}^2\right)$. Then the **asymptotic relative efficiency (ARE)** of $\hat{\theta}_{1n}$ to $\hat{\theta}_{2n}$ is the reciprocal of the ratio of their respective asymptotic variances; i.e.,

$$e(\hat{\theta}_{1n}, \hat{\theta}_{2n}) = \frac{\sigma_{\hat{\theta}_{2n}}^2}{\sigma_{\hat{\theta}_{1n}}^2}.$$

10.24 The Likelihood Ratio Test

See All of statistic.

Définition 16: likelihood ratio test

Consider testing

$$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \notin \Theta_0.$$

The **likelihood ratio statistic** is

$$\lambda = 2 \log \left(\frac{\sup_{\theta \in \Theta} \mathcal{L}(\theta)}{\sup_{\theta \in \Theta_0} \mathcal{L}(\theta)} \right) = 2 \log \left(\frac{\mathcal{L}(\hat{\theta})}{\mathcal{L}(\hat{\theta}_0)} \right)$$

where $\hat{\theta}$ is the MLE and $\hat{\theta}_0$ is the MLE when θ is restricted to lie in Θ_0 .

When $\dim \Theta_0 = 1$, $\Lambda := \frac{\mathcal{L}(\theta_0)}{\mathcal{L}(\hat{\theta})} = \frac{\sup_{\theta \in \Theta_0} \mathcal{L}(\theta_0)}{\sup_{\theta \in \Theta} \mathcal{L}(\theta)}$.¹ We have

$$\lambda \xrightarrow{\mathcal{D}} \chi^2(1)$$

- The p -value for the test is $\mathbb{P}(\chi^2_1 > \lambda)$, where λ is the observed value. We reject H_0 at level α if $\alpha \geq p$ -value.
- We reject H_0 at level α when $\lambda \geq \chi^2_{1,\alpha}$, where $\alpha = \mathbb{P}(\chi^2(1) \geq \chi^2_{1,\alpha})$.

Consider another statistic

$$\chi^2_W = \{ \sqrt{nI(\hat{\theta})} (\hat{\theta} - \theta_0) \}^2 = nI(\hat{\theta}) \cdot (\hat{\theta} - \theta_0)^2$$

Under H_0 , $\chi^2_W \xrightarrow{\mathcal{D}} \chi^2(1)$, then we construct a test²

Reject H_0 in favor of H_1 if $\chi^2_W \geq \chi^2_{\alpha}(1)$.

Note that under H_0 , $\chi^2_W - \lambda \xrightarrow{P} 0$.

The third test is called **Rao's score test**. The **scores** are

$$\mathbf{S}(\theta) = \left(\frac{\partial \log f(X_1; \theta)}{\partial \theta}, \dots, \frac{\partial \log f(X_n; \theta)}{\partial \theta} \right)'$$

Define the statistic

$$\chi^2_R = \left(\frac{l'(\theta)}{\sqrt{nI(\theta_0)}} \right)^2$$

Recall that

$$l'(\theta) = \frac{\partial}{\partial \theta} \left(\sum_{i=1}^n \log f(X_i; \theta) \right) = \sum_{i=1}^n \frac{\partial \log f(X_i; \theta)}{\partial \theta}$$

¹Defined in Hogg

²This test is often referred to as a **Wald-type test**.

10.24.1 Multiparameter Case: Estimation

Assume that $\boldsymbol{\theta} \in \Theta \subset \mathbf{R}^p$. Then the **Fisher information** is given by

$$\mathbf{I}(\boldsymbol{\theta}) = \text{Cov}(\nabla \log f(X; \boldsymbol{\theta})) = \left[\text{Cov}\left(\frac{\partial}{\partial \theta_j} \log f(X; \boldsymbol{\theta}), \frac{\partial}{\partial \theta_k} \log f(X; \boldsymbol{\theta}) \right) \right]_{j,k=1}^p$$

Recall that

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)], \quad \text{Cov}(\mathbf{X}, \mathbf{Y}) = \mathbb{E}[(\mathbf{X} - \mathbb{E}\mathbf{X})(\mathbf{Y} - \mathbb{E}\mathbf{Y})]$$

We also have

$$\mathbb{E}\left[\frac{\partial}{\partial \theta_j} \log f(X; \boldsymbol{\theta}) \frac{\partial}{\partial \theta_k} \log f(X; \boldsymbol{\theta})\right] = -\mathbb{E}\left[\frac{\partial^2}{\partial \theta_j \partial \theta_k} \log f(X; \boldsymbol{\theta})\right].$$

The diagonal entries of $\mathbf{I}(\boldsymbol{\theta})$ are

$$I_{ii}(\boldsymbol{\theta}) = \text{Var}\left[\frac{\partial \log f(X; \boldsymbol{\theta})}{\partial \theta_i}\right] = -\mathbb{E}\left[\frac{\partial^2}{\partial \theta_i^2} \log f(X_i; \boldsymbol{\theta})\right]$$

Similar to eq. (10.4), if $Y_j = u_j(\mathbf{X})$ is an unbiased estimate of θ_j , then

$$\text{Var}(Y_j) \geq \frac{1}{n} [\mathbf{I}^{-1}(\boldsymbol{\theta})]_{jj}$$

We call it **efficient** if attaining the lower bound.

Note. $[\mathbf{I}^{-1}(\boldsymbol{\theta})]_{jj} = (\mathbf{I}(\boldsymbol{\theta}))_{jj}^{-1}$ if $\mathbf{I}(\boldsymbol{\theta})$ is diagonal.

Asymptotic behavior

Théorème 34

Let X_1, \dots, X_n be iid with pdf $f(x; \boldsymbol{\theta})$ for $\boldsymbol{\theta} \in \Omega$. Assume the regularity conditions hold. Then

- (1) The likelihood equation,

$$\frac{\partial}{\partial \boldsymbol{\theta}} l(\boldsymbol{\theta}) = \mathbf{0}$$

has a solution $\widehat{\boldsymbol{\theta}}_n$ such that $\widehat{\boldsymbol{\theta}}_n \xrightarrow{P} \boldsymbol{\theta}$. 2. For any sequence that satisfies (1),

$$\sqrt{n} (\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \xrightarrow{D} N_p(\mathbf{0}, \mathbf{I}^{-1}(\boldsymbol{\theta})).$$

For $\boldsymbol{\eta} = \mathbf{g}(\boldsymbol{\theta}) = (g_1(\boldsymbol{\theta}), \dots, g_k(\boldsymbol{\theta}))$, we have

$$\sqrt{n}(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}) \xrightarrow{\mathcal{D}} N_k(\mathbf{0}, \mathbf{B}\mathbf{I}^{-1}(\boldsymbol{\theta})\mathbf{B}')$$

where

$$\mathbf{B} = \left[\frac{\partial g_i}{\partial \theta_j} \right] \quad i = 1, \dots, k, \quad j = 1, \dots, p.$$

Example: MLE Under the Normal Model

Suppose X_1, \dots, X_n are i.i.d. $N(\mu, \sigma^2)$. In this case $\boldsymbol{\theta} = (\mu, \sigma^2)'$ and $\Omega = (-\infty, \infty) \times (0, \infty)$. Then

$$l(\mu, \sigma^2) = -\frac{n}{2} \log 2\pi - n \log \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^n (X_i - \mu)^2$$

Then

$$\frac{\partial}{\partial \boldsymbol{\theta}} l(\boldsymbol{\theta}) = \left(\frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \mu), -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^n (X_i - \mu)^2 \right)$$

Let $\frac{\partial}{\partial \boldsymbol{\theta}} l(\boldsymbol{\theta}) = 0$, then

$$\hat{\mu} = \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \quad \hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2}$$

Note that $\hat{\mu}$ is unbiased, $\hat{\sigma}^2$ is biased.

10.25 Chapter 7 Summary: Sufficiency

This chapter focuses on finding "optimal" point estimators, particularly Minimum Variance Unbiased Estimators (MVUEs), by leveraging the concept of sufficiency.

10.25.1 7.1 Measures of Quality of Estimators

- **Goal:** Find estimators that are close to the true parameter value.

- **Properties Reviewed:**

- Consistency: Estimator converges in probability to the parameter as sample size increases.
- Unbiasedness: Expected value of the estimator equals the parameter.

- **Minimum Variance Unbiased Estimator (MVUE):**

Définition 17

An estimator $Y = u(X_1, \dots, X_n)$ is an MVUE of θ if:

- (1) It is unbiased: $E(Y) = \theta$.
- (2) Its variance is less than or equal to the variance of *every* other unbiased estimator of θ .

- **Loss Functions and Risk:**

- **Decision function**³ δ : $\delta(y)$ is the function of the observed value of the statistic Y which is the point estimate of θ . One value of the decision function, say $\delta(y)$, is called a **decision**.
- **Loss Function** $\mathcal{L}[\theta, \delta(y)]$: Measures the "cost" of estimating θ with $\delta(y)$. Common example: Squared Error Loss $\mathcal{L}[\theta, \delta(y)] = (\theta - \delta(y))^2$.
- **Risk Function** $R(\theta, \delta)$: Expected value of the loss function, $R(\theta, \delta) = E\{\mathcal{L}[\theta, \delta(Y)]\}$.
- Minimizing risk uniformly is usually impossible.

- **Minimax Principle:** Choose the decision rule δ_0 that minimizes the maximum risk: $\max_\theta R[\theta, \delta_0(y)] \leq \max_\theta R[\theta, \delta(y)]$ for all other rules δ .

- **Likelihood Principle:** If two experiments yield likelihood functions $L_1(\theta)$ and $L_2(\theta)$ that are proportional ($L_1 \propto L_2$), they contain the same information about θ , and the inference should be the same.

10.25.2 7.2 A Sufficient Statistic for a Parameter

- **Idea:** A statistic $Y_1 = u_1(X_1, \dots, X_n)$ is sufficient if it captures all the information about θ contained in the sample. Knowing the value of Y_1 is equivalent to knowing the original sample for the purpose of inference about θ .

- **Formal Definition:**

³or decision rule

Définition 18

A statistic $Y_1 = u_1(X_1, \dots, X_n)$ with pdf/pmf $f_{Y_1}(y_1; \theta)$ is **sufficient** for θ if and only if the ratio

$$\frac{f(x_1; \theta) \cdots f(x_n; \theta)}{f_{Y_1}(u_1(x_1, \dots, x_n); \theta)} = H(x_1, \dots, x_n)$$

does not depend on θ . (This means the conditional distribution of the sample given $Y_1 = y_1$ does not depend on θ).

- **Neyman Factorization Theorem:** A much easier way to find sufficient statistics.

Théorème 35: Neyman

$Y_1 = u_1(X_1, \dots, X_n)$ is sufficient for θ if and only if the joint pdf/pmf can be factored as:

$$f(x_1; \theta) \cdots f(x_n; \theta) = k_1[u_1(x_1, \dots, x_n); \theta] \times k_2(x_1, \dots, x_n)$$

where k_2 does not depend on θ . The function k_1 depends on the data *only* through the statistic Y_1 .

- **Example (Normal Mean, Known Variance):** Let $X_1, \dots, X_n \sim N(\theta, \sigma^2)$ with σ^2 known. The joint pdf is:

$$\left(\frac{1}{\sigma \sqrt{2\pi}} \right)^n \exp \left[-\frac{\sum (x_i - \theta)^2}{2\sigma^2} \right] = \left\{ \exp \left[-\frac{n(\bar{x} - \theta)^2}{2\sigma^2} \right] \right\} \times \left\{ \frac{\exp \left[-\frac{\sum (x_i - \bar{x})^2}{2\sigma^2} \right]}{(\sigma \sqrt{2\pi})^n} \right\}$$

Here, $k_1[\bar{x}; \theta] = \exp \left[-\frac{n(\bar{x} - \theta)^2}{2\sigma^2} \right]$ and $k_2(x_1, \dots, x_n)$ is the second term, which does not involve θ . Thus, \bar{X} is a sufficient statistic for θ .

- **Example (Poisson):** Let $X_1, \dots, X_n \sim \text{Poisson}(\theta)$. Joint pmf:

$$\prod_{i=1}^n \frac{e^{-\theta} \theta^{x_i}}{x_i!} = \frac{e^{-n\theta} \theta^{\sum x_i}}{\prod x_i!} = \left(e^{-n\theta} \theta^{\sum x_i} \right) \times \left(\frac{1}{\prod x_i!} \right)$$

Here $k_1[\sum x_i; \theta] = e^{-n\theta} \theta^{\sum x_i}$ and $k_2(x_1, \dots, x_n) = 1 / \prod x_i!$. Thus, $Y_1 = \sum X_i$ is sufficient for θ .

10.25.3 7.3 Properties of a Sufficient Statistic

- **Rao-Blackwell Theorem:** Provides a method to improve any unbiased estimator by conditioning on a sufficient statistic.

Théorème 36: Rao-Blackwell

Let Y_1 be a sufficient statistic for θ , and let Y_2 be any unbiased estimator of θ . Define $\varphi(y_1) = E(Y_2|Y_1 = y_1)$. Then:

- (1) $\varphi(Y_1)$ is a statistic (does not depend on θ).
- (2) $E[\varphi(Y_1)] = \theta$ (it's unbiased).
- (3) $\text{Var}[\varphi(Y_1)] \leq \text{Var}(Y_2)$. Equality holds iff Y_2 is already a function of Y_1 .

- **Implication:** To find an MVUE, we only need to consider functions of a sufficient statistic.
- **Sufficiency and MLEs:**

Théorème 37

If a sufficient statistic Y_1 exists and a unique Maximum Likelihood Estimator (MLE) $\hat{\theta}$ exists, then $\hat{\theta}$ must be a function of Y_1 .

- Often, the MLE is a good starting point to find an MVUE. It might be biased, but we can adjust it (if necessary) to make it unbiased, and it will still be a function of the sufficient statistic.
- **Example (Exponential):** $X_1, \dots, X_n \sim \text{Exp}(\theta)$ (mean $1/\theta$). Pdf is $f(x; \theta) = \theta e^{-\theta x}$.
 - $Y_1 = \sum X_i$ is sufficient.
 - MLE is $\hat{\theta} = 1/\bar{X} = n/Y_1$.
 - $E(n/Y_1) = \theta \frac{n}{n-1}$ (biased).
 - MVUE is $\frac{n-1}{n}\hat{\theta} = \frac{n-1}{Y_1}$.

10.25.4 7.4 Completeness and Uniqueness

- **Completeness:** A property of the *family* of distributions of a statistic.

Définition 19

A family of pdfs/pdfs $\{h(z; \theta) : \theta \in \Omega\}$ for a statistic Z is **complete** if $E[u(Z)] = 0$ for all $\theta \in \Omega$ implies $u(z) = 0$ (almost everywhere w.r.t. the distributions in the family).

- **Intuition:** If a family is complete, the only unbiased estimator of 0 that is a function of Z is the function $u(z) = 0$.
- **Lehmann-Scheffé Theorem:** Connects sufficiency, completeness, and MVUEs.

Théorème 38: Lehmann-Scheffé

If Y_1 is a **complete sufficient statistic** for θ , and a function $\varphi(Y_1)$ is an unbiased estimator of θ , then $\varphi(Y_1)$ is the **unique MVUE** of θ . (Uniqueness means any other MVUE must be equal to $\varphi(Y_1)$ almost everywhere).

- **Complete Sufficient Statistic:** A statistic Y_1 that is both sufficient for θ and whose family of distributions $\{f_{Y_1}(y_1; \theta) : \theta \in \Omega\}$ is complete.
- **Example (Uniform):** $X_1, \dots, X_n \sim U(0, \theta)$.
 - $Y_n = \max(X_i)$ is sufficient.
 - The family of pdfs for Y_n is $g(y_n; \theta) = ny_n^{n-1}/\theta^n$ for $0 < y_n < \theta$. This family can be shown to be complete.
 - $E(Y_n) = \frac{n}{n+1}\theta$.
 - Therefore, $\frac{n+1}{n}Y_n$ is the unique MVUE of θ .

10.25.5 7.5 The Exponential Class of Distributions

- A large class of common distributions where finding complete sufficient statistics is straightforward.
- **Definition (Regular Exponential Class):** A family $\{f(x; \theta) : \theta \in \Omega\}$ is a regular exponential class if:
 - (1) $f(x; \theta) = \exp[p(\theta)K(x) + H(x) + q(\theta)]$ for $x \in \mathcal{S}$.
 - (2) The support \mathcal{S} does not depend on θ .

- (3) $p(\theta)$ is a nontrivial continuous function of θ .
- (4) $K'(x) \not\equiv 0$ and $H(x)$ are continuous (continuous case) or $K(x)$ is nontrivial (discrete case).

- **Key Result:** For a random sample X_1, \dots, X_n from a regular exponential class distribution:

Théorème 39

The statistic $Y_1 = \sum_{i=1}^n K(X_i)$ is a **complete sufficient statistic** for θ .

- **Implication:** If you can write the pdf/pmf in the exponential form, you can immediately identify the complete sufficient statistic. Then, if you find *any* function of that statistic which is unbiased for θ , it is automatically the unique MVUE by Lehmann-Scheffé.
- **Example (Normal Mean, Known Variance):** $N(\theta, \sigma^2)$, σ^2 known. $f(x; \theta) = \exp\left[\frac{\theta}{\sigma^2}x - \frac{x^2}{2\sigma^2} - \log\sqrt{2\pi\sigma^2} - \frac{\theta^2}{2\sigma^2}\right]$. Here $K(x) = x$. So $Y_1 = \sum X_i = n\bar{X}$ is complete sufficient. Since $E(\bar{X}) = E(Y_1/n) = \theta$, \bar{X} is the unique MVUE.

10.25.6 7.6 Functions of a Parameter

- Often interested in estimating $g(\theta)$, not just θ .
- If Y_1 is a complete sufficient statistic for θ , we seek a function $\psi(Y_1)$ such that $E[\psi(Y_1)] = g(\theta)$. If found, $\psi(Y_1)$ is the unique MVUE of $g(\theta)$.
- **Techniques:**
 - (1) Find an unbiased estimator T (perhaps not a function of Y_1) and compute $E(T|Y_1)$. (Rao-Blackwell).
 - (2) Guess a function $\psi(Y_1)$ (often related to the MLE of $g(\theta)$) and check if $E[\psi(Y_1)] = g(\theta)$. Adjust if necessary.
- **Example (Bernoulli Variance):** $X_1, \dots, X_n \sim b(1, \theta)$. Estimate $\delta = \theta(1 - \theta)$.
 - $Y = \sum X_i$ is complete sufficient. $\bar{X} = Y/n$ is MVUE of θ .
 - MLE of δ is $\tilde{\delta} = \bar{X}(1 - \bar{X}) = (Y/n)(1 - Y/n)$.
 - $E(\tilde{\delta}) = \frac{n-1}{n}\theta(1 - \theta) = \frac{n-1}{n}\delta$.

- MVUE of δ is $\hat{\delta} = \frac{n}{n-1}\tilde{\delta} = \frac{n}{n-1}\frac{Y}{n}(1 - \frac{Y}{n})$.

- **Bootstrap Standard Errors:** Can compute standard errors for MVUEs using bootstrap resampling (nonparametric or parametric).

10.25.7 7.7 The Case of Several Parameters

- Concepts extend to vector parameters $\boldsymbol{\theta} = (\theta_1, \dots, \theta_p)'$.
- **Joint Sufficiency:** A vector of statistics $\mathbf{Y} = (Y_1, \dots, Y_m)'$ is jointly sufficient if the definition or factorization theorem holds.

$$\prod f(x_i; \boldsymbol{\theta}) = k_1(\mathbf{y}; \boldsymbol{\theta})k_2(x_1, \dots, x_n)$$

- **Completeness:** $E[u(\mathbf{Y})] = 0$ for all $\boldsymbol{\theta} \in \Omega$ implies $u(\mathbf{y}) = 0$.
- **Rao-Blackwell & Lehmann-Scheffé:** Apply similarly. If \mathbf{Y} is jointly complete sufficient for $\boldsymbol{\theta}$, and $T(\mathbf{Y})$ is unbiased for $g(\boldsymbol{\theta})$, then $T(\mathbf{Y})$ is the unique MVUE of $g(\boldsymbol{\theta})$.
- **Exponential Class (Multiparameter):**

$$f(x; \boldsymbol{\theta}) = \exp \left[\sum_{j=1}^m p_j(\boldsymbol{\theta}) K_j(x) + H(x) + q(\boldsymbol{\theta}) \right]$$

If regular, then $Y_j = \sum_{i=1}^n K_j(X_i)$ for $j = 1, \dots, m$ are **jointly complete sufficient statistics** for $\boldsymbol{\theta}$.

- **Example (Normal):** $X_1, \dots, X_n \sim N(\theta_1, \theta_2)$. $f(x; \theta_1, \theta_2) = \exp \left[\frac{\theta_1}{\theta_2}x - \frac{1}{2\theta_2}x^2 - \frac{\theta_1^2}{2\theta_2} - \ln \sqrt{2\pi\theta_2} \right]$. $K_1(x) = x$, $K_2(x) = x^2$. $Y_1 = \sum X_i$, $Y_2 = \sum X_i^2$ are jointly complete sufficient. Equivalently, $\bar{X} = Y_1/n$ and $S^2 = (\sum X_i^2 - (\sum X_i)^2/n)/(n-1)$ are jointly complete sufficient. Since $E(\bar{X}) = \theta_1$ and $E(S^2) = \theta_2$, \bar{X} and S^2 are the unique MVUEs of θ_1 and θ_2 .

10.25.8 7.8 Minimal Sufficiency and Ancillary Statistics

- **Minimal Sufficient Statistic:** A sufficient statistic that is a function of every other sufficient statistic. It achieves the maximum possible data reduction while retaining sufficiency.
- **Lehmann-Scheffe criterion:** A statistic $T(\mathbf{X})$ is minimal sufficient if for any two data points \mathbf{x} and \mathbf{z} , the ratio $K(\mathbf{x}, \mathbf{z}; \boldsymbol{\theta}) = \frac{L(\boldsymbol{\theta}; \mathbf{x})}{L(\boldsymbol{\theta}; \mathbf{z})}$ is independent of $\boldsymbol{\theta}$ if and only if $T(\mathbf{x}) = T(\mathbf{z})$.

- Complete sufficient statistics are minimal. (See [Completeness, Ancillarity, and Basu's Theorem](#))
- Sufficient MLE is also minimal.
- **Ancillary Statistic:** A statistic Z whose distribution does *not* depend on the parameter θ .
 - **Location-Invariant:** If $X_i = \theta + W_i$, a statistic $Z = u(X_1, \dots, X_n)$ is location-invariant if $u(x_1 + d, \dots, x_n + d) = u(x_1, \dots, x_n)$. Such statistics are ancillary for θ . Ex: S^2 , Range.
 - **Scale-Invariant:** If $X_i = \theta W_i$, a statistic $Z = u(X_1, \dots, X_n)$ is scale-invariant if $u(cx_1, \dots, cx_n) = u(x_1, \dots, x_n)$ for $c > 0$. Such statistics are ancillary for θ . Ex: $X_1/(X_1 + X_2)$.
 - **Location- and Scale-Invariant:** Combination of the above. Ex: $(X_i - \bar{X})/S$.

10.25.9 7.9 Sufficiency, Completeness, and Independence

- Key Relationship (Basu's Theorem):

Théorème 40: Basu

If Y_1 is a **complete sufficient statistic** for θ , then Y_1 is independent of any **ancillary statistic** Z .

- The converse also holds: If Y_1 is sufficient and is independent of an ancillary statistic Z , then Y_1 must be complete.
- This provides a powerful tool for proving independence.

- **Example (Normal \bar{X} and S^2):**

- Let the model be $N(\mu, \sigma^2)$.
- If σ^2 is known, \bar{X} is complete sufficient for μ . S^2 is location-invariant, hence ancillary for μ . By Basu's Theorem, \bar{X} and S^2 are independent.
- If μ is known, $\sum(X_i - \mu)^2$ is complete sufficient for σ^2 . \bar{X} is not ancillary for σ^2 (its distribution depends on σ^2). We cannot use Basu's theorem here (though they are still independent).

- **If Sufficiency is Not Complete:** An ancillary statistic might still provide information about the precision of an estimator based on the sufficient statistic (e.g., Example 7.9.5).

10.26 Chapter 8 Summary: Optimal Tests of Hypotheses

This chapter develops the theory of optimal hypothesis testing, building from the most powerful tests for simple hypotheses to uniformly most powerful tests, likelihood ratio tests, sequential tests, and minimax decision procedures.

10.26.1 8.1 Most Powerful Tests

- **Goal:** Find the "best" test for simple hypothesis $H_0 : \theta = \theta'$ versus simple alternative $H_1 : \theta = \theta''$.
- **Critical Region:** A test is defined by its critical region C . The decision rule is:
 - Reject H_0 if $\mathbf{X} \in C$
 - Retain H_0 if $\mathbf{X} \in C^c$
- **Error Types:**
 - Type I Error: Reject H_0 when H_0 is true, probability = $\alpha = \max_{\theta \in \omega_0} P_\theta(\mathbf{X} \in C)$
 - Type II Error: Accept H_0 when H_1 is true, probability = β
 - Power Function: $\gamma_C(\theta) = P_\theta(\mathbf{X} \in C)$ for $\theta \in \omega_1$

Définition 20: Best Critical Region

A critical region C is a **best critical region** of size α for testing $H_0 : \theta = \theta'$ against $H_1 : \theta = \theta''$ if:

- (1) $P_{\theta'}[\mathbf{X} \in C] = \alpha$
- (2) For every other region A with $P_{\theta'}[\mathbf{X} \in A] = \alpha$: $P_{\theta''}[\mathbf{X} \in C] \geq P_{\theta''}[\mathbf{X} \in A]$

Théorème 41: Neyman-Pearson Theorem

Let C be a subset of the sample space such that:

- (1) $\frac{L(\theta'; \mathbf{x})}{L(\theta''; \mathbf{x})} \leq k$ for each $\mathbf{x} \in C$
- (2) $\frac{L(\theta'; \mathbf{x})}{L(\theta''; \mathbf{x})} \geq k$ for each $\mathbf{x} \in C^c$
- (3) $\alpha = P_{H_0}[\mathbf{X} \in C]$

Then C is a best critical region of size α for testing $H_0 : \theta = \theta'$ against $H_1 : \theta = \theta''$.

- **Significant level:** the probability of rejecting H_0 when H_0 is true, denoted by α .
- **Power:** the probability of rejecting H_0 , when H_0 is false, is the **power** of the test at H_1 , i.e. $\mathbb{P}_{H_1}(\mathbf{X} \in C)$, which forms the **power function** $\gamma(\theta)$.
- **Unbiased Tests:**

Définition 21: Unbiased Test

A test with critical region C and level α is **unbiased** if $P_\theta(\mathbf{X} \in C) \geq \alpha$ for all $\theta \in \omega_1$.

- **Example (Normal Mean):** $X_1, \dots, X_n \sim N(\theta, 1)$, test $H_0 : \theta = 0$ vs $H_1 : \theta = 1$.
 - Likelihood ratio: $\frac{L(0)}{L(1)} = \exp(-\sum x_i + n/2)$
 - Best test: Reject H_0 if $\sum X_i \geq c$ or equivalently $\bar{X} \geq c/n$
 - Critical value determined by $P_{H_0}(\bar{X} \geq c/n) = \alpha$
- **Example (Binomial):** $X \sim \text{Binomial}(5, \theta)$, test $H_0 : \theta = 1/2$ vs $H_1 : \theta = 3/4$.
 - For $\alpha = 1/32$, best critical region is $C = \{x : x = 5\}$
 - Power = $P_{H_1}(X = 5) = (3/4)^5 = 243/1024$

10.26.2 8.2 Uniformly Most Powerful Tests

- **Extension:** Testing simple H_0 against composite alternative H_1 .

Définition 22: Uniformly Most Powerful (UMP) Test

A critical region C is **uniformly most powerful** of size α for testing $H_0 : \theta = \theta'$ against $H_1 : \theta \in \omega_1$ if C is a best critical region of size α for testing H_0 against each simple hypothesis in H_1 .

- **Monotone Likelihood Ratio (MLR):**

Définition 23: Monotone Likelihood Ratio

The likelihood $L(\theta, \mathbf{x})$ has **monotone likelihood ratio** in statistic $Y = u(\mathbf{x})$ if, for $\theta_1 < \theta_2$, the ratio $\frac{L(\theta_1, \mathbf{x})}{L(\theta_2, \mathbf{x})}$ is a monotone function of Y .

Théorème 42: UMP Test for MLR Families

If the likelihood has monotone decreasing likelihood ratio in Y , then for testing $H_0 : \theta \leq \theta'$ vs $H_1 : \theta > \theta'$, the UMP level α test is: **Reject H_0 if $Y \geq c_Y$** where c_Y is determined by $\alpha = P_{\theta'}[Y \geq c_Y]$.

- **Exponential Family Result:** For regular exponential family $f(x; \theta) = \exp[p(\theta)K(x) + H(x) + q(\theta)]$ with $p(\theta)$ increasing:
 - Likelihood has MLR in $\sum K(X_i)$
 - UMP test for $H_0 : \theta \leq \theta'$ vs $H_1 : \theta > \theta'$: Reject H_0 if $\sum K(X_i) \geq c$

- **Example (Normal Variance):** $X_1, \dots, X_n \sim N(0, \theta)$, test $H_0 : \theta = \theta'$ vs $H_1 : \theta > \theta'$.
 - UMP test: Reject H_0 if $\sum X_i^2 \geq c$
 - Under H_0 : $\sum X_i^2 / \theta' \sim \chi^2(n)$

- **Example (Bernoulli):** $X_1, \dots, X_n \sim \text{Bernoulli}(p)$, test $H_0 : p \leq p_0$ vs $H_1 : p > p_0$.
 - UMP test: Reject H_0 if $\sum X_i \geq c$

- **No UMP for Two-Sided Tests:** For $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$, generally no UMP test exists.

10.26.3 8.3 Likelihood Ratio Tests

- **General Framework:** For testing $H_0 : \boldsymbol{\theta} \in \omega$ vs $H_1 : \boldsymbol{\theta} \in \Omega \cap \omega^c$.
- **Likelihood Ratio Statistic:**

$$\Lambda = \frac{L(\hat{\boldsymbol{\theta}})}{L(\hat{\Omega})} = \frac{\max_{\boldsymbol{\theta} \in \omega} L(\boldsymbol{\theta})}{\max_{\boldsymbol{\theta} \in \Omega} L(\boldsymbol{\theta})}$$

- **Decision Rule:** Reject H_0 if $\Lambda \leq \lambda_0$ (or equivalently if $-2 \log \Lambda \geq c$).

8.3.1 Tests for Normal Means

- **One-Sample t-test:** For $X_1, \dots, X_n \sim N(\mu, \sigma^2)$, testing $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$.
 - Test statistic: $T = \frac{\sqrt{n}(\bar{X} - \mu_0)}{S}$
 - Under H_0 : $T \sim t_{n-1}$
 - Reject H_0 if $|T| \geq t_{\alpha/2, n-1}$
- **Two-Sample t-test:** For independent samples from $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, testing $H_0 : \mu_1 = \mu_2$.
 - Test statistic: $T = \frac{\sqrt{nm/(n+m)}(\bar{X} - \bar{Y})}{S_p}$
 - Pooled variance: $S_p^2 = \frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2}$
 - Under H_0 : $T \sim t_{n+m-2}$
- **Power Analysis with Noncentral t-distribution:**

Définition 24: Noncentral t-distribution

If $W \sim N(\delta, 1)$, $V \sim \chi^2(r)$, and W, V independent, then $T = \frac{W}{\sqrt{V/r}}$ has a **noncentral t-distribution** with r degrees of freedom and noncentrality parameter δ .

- **One-sample test power:** Noncentrality parameter $\delta = \frac{\sqrt{n}(\mu_1 - \mu_0)}{\sigma}$
- **Two-sample test power:** Noncentrality parameter $\delta = \frac{\sqrt{nm/(n+m)}(\mu_1 - \mu_2)}{\sigma}$

8.3.2 Tests for Normal Variances

- **Two-Sample F-test:** For testing $H_0 : \sigma_1^2 = \sigma_2^2$ vs $H_1 : \sigma_1^2 \neq \sigma_2^2$.
 - Test statistic: $F = \frac{S_1^2}{S_2^2}$

- Under H_0 : $F \sim F_{n_1-1, n_2-1}$
- **Warning:** F-test is not robust to non-normality!

- **Robustness Considerations:**

- t-tests are asymptotically valid for non-normal distributions with finite variance
- F-tests for variances are not robust and can have severely inflated Type I error rates

10.26.4 8.4 Sequential Probability Ratio Test

- **Motivation:** Sample size is not fixed in advance; continue sampling until a decision can be made.

- **Procedure:** Given constants $k_0 < k_1$:

- Continue sampling while $k_0 < \frac{L(\theta', n)}{L(\theta'', n)} < k_1$
- Reject H_0 if $\frac{L(\theta', n)}{L(\theta'', n)} \leq k_0$
- Accept H_0 if $\frac{L(\theta', n)}{L(\theta'', n)} \geq k_1$

- **Error Control:** Choose $k_0 = \frac{\alpha_a}{1-\beta_a}$ and $k_1 = \frac{1-\alpha_a}{\beta_a}$ for desired error probabilities α_a, β_a .

- **Bounds on Actual Errors:**

$$\frac{\alpha}{1-\beta} \leq k_0, \quad k_1 \leq \frac{1-\alpha}{\beta}$$

- **Example (Bernoulli):** Test $H_0 : p = 1/3$ vs $H_1 : p = 2/3$.

- Continue while $\frac{n}{2} - \frac{1}{2} \log_2 k_1 < \sum X_i < \frac{n}{2} - \frac{1}{2} \log_2 k_0$

- **Example (Normal):** Test $H_0 : \mu = 75$ vs $H_1 : \mu = 78$ with $\sigma^2 = 100$.

- Continue while $\frac{153n}{2} - \frac{100}{3} \log 9 < \sum X_i < \frac{153n}{2} + \frac{100}{3} \log 9$

- **Advantages:** Expected sample size is typically smaller than fixed-sample tests with same error probabilities.

10.26.5 8.5 Minimax and Classification Procedures

8.5.1 Minimax Procedures

- **Decision Theory Framework:**

- Loss function $\mathcal{L}(\theta, \delta)$ measures cost of decision δ when true parameter is θ
- Risk function $R(\theta, \delta) = E[\mathcal{L}(\theta, \delta)]$
- **Minimax Solution:** Choose critical region C to minimize $\max[R(\theta', C), R(\theta'', C)]$.

Théorème 43: Minimax Test

The critical region $C = \{\mathbf{x} : \frac{L(\theta')}{L(\theta'')} \leq k\}$ provides a minimax solution if k is chosen so that $R(\theta', C) = R(\theta'', C)$.

8.5.2 Classification Problems

- **Setup:** Observe (X, Y) and classify into one of two populations with parameters θ' or θ'' .
- **Optimal Classification Rule:**

$$\frac{f(x, y; \theta')}{f(x, y; \theta'')} \leq k \Rightarrow \text{classify as } \theta''$$

- **Choice of k :**

- Equal prior probabilities: $k = 1$
- General case: $k = \frac{\pi''}{\pi'}$ (ratio of prior probabilities)

- **Example (Bivariate Normal):** For bivariate normal distributions with different means but same covariance structure, the classification boundary is linear: $ax + by \leq c$.
- **Fisher's Linear Discriminant:** When parameters are unknown, replace with sample estimates to get practical classification rule.

10.26.6 Problem-Solving Techniques

1. Identifying Test Type

- **Simple vs Simple** → Use Neyman-Pearson Theorem
- **Simple vs Composite (One-sided)** → Check for MLR, use UMP theory
- **Composite vs Composite** → Use Likelihood Ratio Test

2. Working with Exponential Families

- For exponential family, sufficient statistic $\sum K(X_i)$ often provides the test statistic
- MLR property is automatic if $p(\theta)$ is monotone

3. Normal Distribution Patterns

- Mean tests (known variance) → Z-tests
- Mean tests (unknown variance) → t-tests
- Variance tests → F-tests or χ^2 -tests

4. Power Calculations

- Use noncentral distributions when available
- One-sided tests generally more powerful than two-sided
- Power increases with: larger sample size, larger effect size, higher significance level

5. Sequential Testing Strategy

- Useful when early stopping is desired
- Average sample size typically smaller than fixed-sample tests
- Important in clinical trials and quality control

10.26.7 Key Formulas and Inequalities

Neyman-Pearson Inequality

$$\frac{L(\theta')}{L(\theta'')} \leq k \Leftrightarrow \text{Reject } H_0$$

Exponential Family Likelihood Ratio

$$\log \frac{L(\theta')}{L(\theta'')} = [p(\theta') - p(\theta'')] \sum K(X_i) + n[q(\theta') - q(\theta'')]$$

Noncentrality Parameters

- One-sample: $\delta = \frac{\sqrt{n}(\mu_1 - \mu_0)}{\sigma}$
- Two-sample: $\delta = \frac{\sqrt{nm/(n+m)}(\mu_1 - \mu_2)}{\sigma}$

Sequential Test Boundaries

$$k_0 = \frac{\alpha_a}{1 - \beta_a}, \quad k_1 = \frac{1 - \alpha_a}{\beta_a}$$

10.26.8 Practical Considerations

- (1) **Robustness:** t-tests are robust to non-normality; F-tests are not
- (2) **Sample Size Planning:** Use power analysis to determine required sample sizes
- (3) **Multiple Testing:** Adjust significance levels when performing multiple tests
- (4) **Sequential Methods:** Valuable in applications requiring early stopping
- (5) **Classification:** Prior information is crucial for optimal performance

This chapter establishes the theoretical foundation for hypothesis testing, providing the framework for understanding modern statistical inference procedures.

10.27 Chapter 9 Summary: Inferences About Normal Linear Models

This chapter covers analysis of variance (ANOVA), regression, and related inference procedures for normal linear models. These are among the most widely used statistical methods.

10.27.1 9.1 Introduction

- **Linear Models:** Models where the mean response is linear in parameters (even if nonlinear in variables)
- **Quadratic Forms:** Used extensively in ANOVA theory

- General form:

$$q(X_1, \dots, X_n) = \sum_{i=1}^n \sum_{j=1}^n X_i a_{ij} X_j$$

- Sample variance is a quadratic form:

$$(n-1)S^2 = \sum_{i=1}^n X_i^2 - \frac{1}{n} \left(\sum_{i=1}^n X_i \right)^2$$

10.27.2 9.2 One-Way ANOVA

Model

$$X_{ij} = \mu_j + e_{ij}, \quad i = 1, \dots, n_j, \quad j = 1, \dots, b$$

where $e_{ij} \sim \text{iid } N(0, \sigma^2)$.

Hypotheses

$$H_0 : \mu_1 = \mu_2 = \dots = \mu_b \quad \text{vs} \quad H_1 : \mu_j \neq \mu_{j'} \text{ for some } j \neq j'$$

Théorème 44: One-Way ANOVA F-Test

The test statistic is:

$$F = \frac{Q_4/(b-1)}{Q_3/(n-b)}$$

where:

- $Q_4 = \sum_{j=1}^b n_j (\bar{X}_{\cdot j} - \bar{X}_{\cdot \cdot})^2$ (between groups sum of squares)
- $Q_3 = \sum_{j=1}^b \sum_{i=1}^{n_j} (X_{ij} - \bar{X}_{\cdot j})^2$ (within groups sum of squares)

Under H_0 : $F \sim F(b-1, n-b)$

Problem-Solving Approach

- (1) **Check assumptions:** Normality, equal variances, independence
- (2) **Compute group means:** $\bar{X}_{\cdot j} = \frac{1}{n_j} \sum_{i=1}^{n_j} X_{ij}$
- (3) **Calculate sums of squares:**

- Total: $Q = \sum_{j=1}^b \sum_{i=1}^{n_j} (X_{ij} - \bar{X}_{\cdot \cdot})^2$

- Between: $Q_4 = \sum_{j=1}^b n_j (\bar{X}_{.j} - \bar{X}_{..})^2$
- Within: $Q_3 = Q - Q_4$

(4) **Identity:** $Q = Q_3 + Q_4$ (total = within + between)

10.27.3 9.3 Noncentral χ^2 and F-Distributions

Noncentral Chi-Square

Définition 25: Noncentral χ^2 Distribution

A random variable has $\chi^2(r, \theta)$ distribution if its mgf is:

$$M(t) = \frac{1}{(1-2t)^{r/2}} e^{t\theta/(1-2t)}, \quad t < \frac{1}{2}$$

where r = degrees of freedom, θ = noncentrality parameter.

- **Mean:** $E(Y) = r + \theta$
- **When $\theta = 0$:** Reduces to central $\chi^2(r)$
- **Noncentrality parameter:** $\theta = \frac{1}{\sigma^2} \sum_{i=1}^n \mu_i^2$ for $\sum X_i^2 / \sigma^2$ where $X_i \sim N(\mu_i, \sigma^2)$

Noncentral F-Distribution

If $U \sim \chi^2(r_1, \theta)$ and $V \sim \chi^2(r_2)$ independently, then:

$$F = \frac{r_2 U}{r_1 V} \sim F(r_1, r_2, \theta)$$

Power Analysis: Use noncentral distributions to compute power of ANOVA tests.

10.27.4 9.4 Multiple Comparisons

When making multiple pairwise comparisons, need to control overall error rate.

Bonferroni Procedure

Définition 26: Bonferroni Correction

For k comparisons at overall level α , use individual level α/k :

$$\bar{X}_{\cdot j} - \bar{X}_{\cdot j'} \pm t_{\alpha/(2k), n-b} \hat{\sigma}_\Omega \sqrt{\frac{1}{n_j} + \frac{1}{n_{j'}}}$$

Tukey's Procedure**Définition 27: Tukey-Kramer Method**

Uses studentized range distribution:

$$\bar{X}_{\cdot j} - \bar{X}_{\cdot j'} \pm \frac{q_{1-\alpha, b, n-b}}{\sqrt{2}} \hat{\sigma}_\Omega \sqrt{\frac{1}{n_j} + \frac{1}{n_{j'}}}$$

Fisher's PLSD

Two-stage procedure:

- (1) **Stage 1:** Overall F-test at level α
- (2) **Stage 2:** If reject, use standard t-intervals for all pairs

10.27.5 9.5 Two-Way ANOVA**Additive Model**

$$\mu_{ij} = \mu + \alpha_i + \beta_j$$

where $\sum_{i=1}^a \alpha_i = 0$, $\sum_{j=1}^b \beta_j = 0$.

Interaction Model

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$$

where additionally $\sum_{i=1}^a \gamma_{ij} = \sum_{j=1}^b \gamma_{ij} = 0$.

Hypotheses and Tests

(1) **Interaction:** $H_{0AB} : \gamma_{ij} = 0$ for all i, j

$$F_{AB} = \frac{MS_{AB}}{MS_E} \sim F((a-1)(b-1), ab(c-1))$$

(2) **Main Effects (if no interaction):**

- Factor A: $H_{0A} : \alpha_i = 0$ for all i
- Factor B: $H_{0B} : \beta_j = 0$ for all j

Diagnostic Tools

- **Mean Profile Plots:** Parallel lines indicate no interaction
- **Residual Analysis:** Check model assumptions

10.27.6 9.6 Regression Analysis

Simple Linear Model

$$Y_i = \alpha + \beta(x_i - \bar{x}) + e_i, \quad e_i \sim \text{iid } N(0, \sigma^2)$$

Maximum Likelihood/Least Squares Estimates

Théorème 45: Regression Estimates

$$\begin{aligned}\hat{\alpha} &= \bar{Y} \\ \hat{\beta} &= \frac{\sum_{i=1}^n (Y_i - \bar{Y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} \\ \hat{\sigma}^2 &= \frac{1}{n} \sum_{i=1}^n (Y_i - \hat{\alpha} - \hat{\beta}(x_i - \bar{x}))^2\end{aligned}$$

Distributions of Estimators

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \sim N_2 \left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \sigma^2 \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{\sum(x_i - \bar{x})^2} \end{pmatrix} \right)$$

Inference for Parameters

Théorème 46: t-Statistics for Regression

$$T_1 = \frac{\hat{\alpha} - \alpha}{\sqrt{\hat{\sigma}^2/(n-2)}} \sim t(n-2)$$

$$T_2 = \frac{\hat{\beta} - \beta}{\sqrt{n\hat{\sigma}^2/[(n-2)\sum(x_i - \bar{x})^2]}} \sim t(n-2)$$

Prediction and Confidence Intervals

- Confidence interval for mean at x_0 :

$$\hat{\eta}_0 \pm t_{\alpha/2, n-2} \sqrt{\hat{\sigma}^2 \left[\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\sum(x_i - \bar{x})^2} \right]}$$

- Prediction interval for new observation:

$$\hat{Y}_0 \pm t_{\alpha/2, n-2} \sqrt{\hat{\sigma}^2 \left[1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\sum(x_i - \bar{x})^2} \right]}$$

Residual Analysis

- Fitted values: $\hat{y}_i = \hat{\alpha} + \hat{\beta}(x_i - \bar{x})$
- Residuals: $\hat{e}_i = y_i - \hat{y}_i$
- Diagnostic plot: Plot residuals vs fitted values for model checking

10.27.7 9.7 Test of Independence (Correlation)

Sample Correlation Coefficient

$$R = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2 \sum_{i=1}^n (Y_i - \bar{Y})^2}}$$

Test for Independence

Théorème 47: Test for $\rho = 0$

Under $H_0 : \rho = 0$:

$$T = \frac{R\sqrt{n-2}}{\sqrt{1-R^2}} \sim t(n-2)$$

Reject H_0 if $|T| \geq t_{\alpha/2, n-2}$

Asymptotic Methods

For large n , use Fisher's z-transformation:

$$W = \frac{1}{2} \log \left(\frac{1+R}{1-R} \right) \approx N \left(\frac{1}{2} \log \left(\frac{1+\rho}{1-\rho} \right), \frac{1}{n-3} \right)$$

10.27.8 Problem-Solving Strategies

1. ANOVA Problems

- **Check assumptions:** Use residual plots, normality tests
- **Choose appropriate model:** One-way, two-way, interaction
- **Multiple comparisons:** Consider family-wise error rate
- **Effect sizes:** Practical vs statistical significance

2. Regression Problems

- **Model adequacy:** Residual plots are crucial
- **Outliers and influence:** Check for unusual observations
- **Prediction vs confidence intervals:** Different purposes
- **Assumptions:** Linearity, normality, equal variance, independence

3. General Linear Model Approach

- **Matrix formulation:** $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$
- **Least squares:** $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$
- **Geometry:** Projection onto column space of \mathbf{X}

10.27.9 9.8 The Distributions of Certain Quadratic Forms

Definition and Properties

Définition 28: Quadratic Form

Let $\mathbf{X} = (X_1, \dots, X_n)'$ be an n -dimensional random vector and \mathbf{A} be a real $n \times n$ symmetric matrix. Then:

$$Q = \mathbf{X}' \mathbf{A} \mathbf{X} = \sum_{i=1}^n \sum_{j=1}^n a_{ij} X_i X_j$$

Trace Operator

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^n a_{ii}$$

Properties:

- **Linearity:** $\text{tr}(a\mathbf{A} + b\mathbf{B}) = a\text{tr}(\mathbf{A}) + b\text{tr}(\mathbf{B})$
- **Cyclic property:** $\text{tr}(\mathbf{ABC}) = \text{tr}(\mathbf{BCA}) = \text{tr}(\mathbf{CAB})$

Mean of Quadratic Forms**Théorème 48: Theorem 9.8.1**

If \mathbf{X} has mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, then:

$$E(Q) = \text{tr}(\mathbf{A}\boldsymbol{\Sigma}) + \boldsymbol{\mu}' \mathbf{A} \boldsymbol{\mu}$$

MGF of Quadratic Forms in Normal Variables**Théorème 49: Theorem 9.8.2**

Let $\mathbf{X}' = (X_1, \dots, X_n)$ where $X_i \sim \text{iid } N(0, \sigma^2)$. For $Q = \sigma^{-2} \mathbf{X}' \mathbf{A} \mathbf{X}$ with \mathbf{A} symmetric of rank r :

$$M_Q(t) = \prod_{i=1}^r (1 - 2t\lambda_i)^{-1/2} = |\mathbf{I} - 2t\mathbf{A}|^{-1/2}$$

where $\lambda_1, \dots, \lambda_r$ are the nonzero eigenvalues of \mathbf{A} .

Idempotent Matrices

Définition 29: Idempotent Matrix

A symmetric matrix \mathbf{A} is idempotent if $\mathbf{A}^2 = \mathbf{A}$.

Key Properties:

- Eigenvalues are only 0 or 1
- $\text{rank}(\mathbf{A}) = \text{tr}(\mathbf{A})$ for idempotent matrices
- Example: $\mathbf{I} - \frac{1}{n}\mathbf{J}$ (centering matrix)

Chi-Square Distribution Characterization

Théorème 50: Theorem 9.8.4

Let $\mathbf{X}' = (X_1, \dots, X_n)$ where $X_i \sim \text{iid } N(0, \sigma^2)$. For $Q = \sigma^{-2}\mathbf{X}'\mathbf{A}\mathbf{X}$ with symmetric \mathbf{A} of rank r : $Q \sim \chi^2(r)$ if and only if \mathbf{A} is idempotent.

Applications

Sample Variance:

$$\frac{(n-1)S^2}{\sigma^2} = \sigma^{-2}\mathbf{X}'(\mathbf{I} - \frac{1}{n}\mathbf{J})\mathbf{X} \sim \chi^2(n-1)$$

Noncentral Case:

- When $\mathbf{X} \sim N_n(\boldsymbol{\mu}, \sigma^2\mathbf{I})$ and $\mathbf{A}^2 = \mathbf{A}$:
- $Q/\sigma^2 \sim \chi^2(r, \theta)$ where $\theta = \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}/\sigma^2$

10.27.10 9.9 Independence of Certain Quadratic Forms

Main Independence Theorem

Théorème 51: Theorem 9.9.1

Let X_1, \dots, X_n be iid $N(0, \sigma^2)$. Let $Q_1 = \mathbf{X}'\mathbf{A}\mathbf{X}$ and $Q_2 = \mathbf{X}'\mathbf{B}\mathbf{X}$ be quadratic forms with symmetric matrices \mathbf{A} and \mathbf{B} . Then Q_1 and Q_2 are independent if and only if $\mathbf{AB} = \mathbf{0}$.

Proof Strategy:

- Uses MGF method with spectral decomposition
- Shows that $\mathbf{AB} = \mathbf{0}$ implies factorization of joint MGF
- Geometric interpretation: orthogonal projections

Extensions of Independence**Théorème 52: Theorem 9.9.2 (Hogg and Craig)**

Define $Q = Q_1 + \dots + Q_k$ where all are quadratic forms in a sample from $N(0, \sigma^2)$. If:

- $Q/\sigma^2 \sim \chi^2(r)$
- $Q_i/\sigma^2 \sim \chi^2(r_i)$ for $i = 1, \dots, k-1$
- $Q_k \geq 0$

Then Q_1, \dots, Q_k are independent and $Q_k/\sigma^2 \sim \chi^2(r_k)$ where $r_k = r - r_1 - \dots - r_{k-1}$.

Cochran's Theorem**Théorème 53: Theorem 9.9.3 (Cochran)**

Let $X_1, \dots, X_n \sim$ iid $N(0, \sigma^2)$ and suppose:

$$\sum_{i=1}^n X_i^2 = Q_1 + Q_2 + \dots + Q_k$$

where each Q_j is a quadratic form with matrix \mathbf{A}_j of rank r_j . Then Q_1, \dots, Q_k are independent and $Q_j/\sigma^2 \sim \chi^2(r_j)$ if and only if $\sum_{j=1}^k r_j = n$.

Applications in Linear Models

ANOVA Decomposition: In one-way ANOVA:

$$\sum_{i,j} (X_{ij} - \bar{X}_{..})^2 = \sum_{i,j} (X_{ij} - \bar{X}_{.j})^2 + \sum_j n_j (\bar{X}_{.j} - \bar{X}_{..})^2$$

- Within-groups SS and between-groups SS are independent
- Each divided by σ^2 follows chi-square distributions
- Enables F-test construction

Regression Analysis:

$$\sum_{i=1}^n (Y_i - \bar{Y})^2 = \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 + \sum_{i=1}^n (\hat{Y}_i - \bar{Y})^2$$

- Residual SS and regression SS are independent
- Forms basis for regression F-tests and confidence intervals

Problem-Solving with Quadratic Forms

- (1) **Identify the matrices:** Express quadratic forms as $\mathbf{X}'\mathbf{A}\mathbf{X}$
- (2) **Check idempotency:** For chi-square, verify $\mathbf{A}^2 = \mathbf{A}$
- (3) **Verify independence:** Check if $\mathbf{A}\mathbf{B} = \mathbf{0}$ for independence
- (4) **Count degrees of freedom:** Use $\text{rank}(\mathbf{A}) = \text{tr}(\mathbf{A})$ for idempotent matrices
- (5) **Apply Cochran's theorem:** Verify rank condition $\sum r_j = n$

General Linear Model Context

For the model $\mathbf{Y} = \mathbf{X}\beta + \mathbf{e}$ where $\mathbf{e} \sim N_n(\mathbf{0}, \sigma^2 \mathbf{I})$:

Key Quadratic Forms:

- **Total SS:** $\mathbf{Y}'\mathbf{Y}$
- **Regression SS:** $\hat{\beta}'\mathbf{X}'\mathbf{X}\hat{\beta}$
- **Error SS:** $(\mathbf{Y} - \mathbf{X}\hat{\beta})'(\mathbf{Y} - \mathbf{X}\hat{\beta})$

Independence Results:

- $\hat{\beta}$ and error SS are independent
- Enables construction of t and F statistics
- Foundation for confidence intervals and hypothesis tests

10.27.11 Key Distributions and Tests

F-Tests

- One-way ANOVA: $F(b - 1, n - b)$
- Two-way main effects: $F(a - 1, ab(c - 1))$, $F(b - 1, ab(c - 1))$
- Two-way interaction: $F((a - 1)(b - 1), ab(c - 1))$
- Regression overall: $F(p - 1, n - p)$

t-Tests

- Regression coefficients: $t(n - 2)$ for simple regression
- Pairwise comparisons in ANOVA
- Correlation coefficient test

Chi-Square Tests

- Variance estimation: $\chi^2(n - 2)$ in regression
- Goodness of fit tests

10.27.12 Applications and Extensions

Practical Considerations

- (1) **Sample size planning:** Use power analysis
- (2) **Robust methods:** When assumptions violated
- (3) **Transformation:** Achieve normality and equal variance
- (4) **Model selection:** Balance complexity and interpretability

Advanced Topics

- **Nonparametric alternatives:** When normality fails
- **Mixed effects models:** Random factors
- **Multiple regression:** More than one predictor
- **Analysis of covariance (ANCOVA):** Combining categorical and continuous predictors

This chapter provides the foundation for most applied statistical analysis involving normal linear models, from basic ANOVA to regression analysis and

correlation studies. The theory of quadratic forms in normal variables underpins the distributional results that enable inference in these models.

10.28 Chapter 10 Summary: Nonparametric and Robust Statistics

This chapter introduces nonparametric procedures for location problems, emphasizing distribution-free tests and rank-based estimation. It covers asymptotic relative efficiencies (AREs) for comparing methods and aims for asymptotically efficient estimators.

10.28.1 10.1 Location Models

- **Functionals:** Parameters like mean ($\mu_X = \mathbb{E}(X)$) or median ($\xi = F_X^{-1}(1/2)$) are treated as functionals $T(F_X)$ of the cdf/pdf.
- **Natural Nonparametric Estimators:**
 - **Empirical Distribution Function (EDF):** $\hat{F}_n(x) = n^{-1}[\#\{x_i \leq x\}]$.
 - Induced estimator of $T(F)$ is $T(\hat{F}_n)$.
 - * e.g., Sample mean \bar{x} for $T(F_X) = \mathbb{E}(X)$.
 - * e.g., Sample median Q_2 for $T(F_X) = F_X^{-1}(1/2)$.
- **Location Functional:** A functional $T(F_X)$ is a **location functional** if:
 1. $Y = X + a \implies T(F_Y) = T(F_X) + a$; 2. $Y = aX \implies T(F_Y) = aT(F_X)$ for $a \neq 0$.
- Mean and median are location functionals. Only the median among percentiles is a location functional.
- **Location Model:** $X_i = \theta_X + \varepsilon_i$, where $\theta_X = T(F_X)$ is a chosen location functional, and ε_i are iid with $T(F_\varepsilon) = 0$. The pdf of X_i is $f_X(x) = f(x - \theta_X)$, where f is the pdf of ε_i .
- **Symmetry:** If X is symmetric about a , then any location functional $T(F_X) = a$.
- **Scale Functional:** A functional $T(F_X)$ is a **scale functional** if: 1. $T(F_{aX}) = aT(F_X)$ for $a > 0$; 2. $T(F_{X+b}) = T(F_X)$ for all b ; 3. $T(F_{-X}) =$

$T(F_X)$.

- Standard deviation and interquartile range are scale functionals.

10.28.2 10.2 Sample Median and the Sign Test

Considers inference for the median θ using the location model $X_i = \theta + \varepsilon_i$, where ε_i are iid with median 0.

- **Sign Statistic:** For $H_0 : \theta = \theta_0$ vs $H_1 : \theta > \theta_0$.
 - $S(\theta_0) = \#\{X_i > \theta_0\} = \sum I(X_i > \theta_0)$.
 - Null distribution: $S(\theta_0) \sim \text{Binomial}(n, 1/2)$ (distribution-free).
 - Test: Reject H_0 if $S(\theta_0) \geq c_\alpha$.
 - Large sample test: Standardized $S(\theta_0)$ is approx. $N(0, 1)$.
- **Two-sided test:** $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$. Reject if $S(\theta_0) \leq c_1$ or $S(\theta_0) \geq n - c_1$.
- **Properties:**

- The function $S(\theta) = \#\{X_i > \theta\}$ is a decreasing step function of θ .
- $P_\theta[S(0) \geq k] = P_0[S(-\theta) \geq k]$.
- Power function $\gamma(\theta)$ for $H_1 : \theta > \theta_0$ is nondecreasing. Test is unbiased. Allows $H_0 : \theta \leq \theta_0$.
- Under alternative θ_1 , $S(\theta_0) \sim \text{Binomial}(n, p_1)$ where $p_1 = P_{\theta_1}(X > \theta_0)$, not distribution-free.

10.2.1 Asymptotic Relative Efficiency (ARE)

- Sequence of local alternatives: $H_{1n} : \theta_n = \delta/\sqrt{n}$ (for $H_0 : \theta = 0$).
- Asymptotic power: $\lim_{n \rightarrow \infty} \gamma(\theta_n) = 1 - \Phi(z_\alpha - \delta\tau_S^{-1})$, where $\tau_S = 1/[2f(0)]$.
- **Efficacy c_S :** $c_S = 2f(0) = \tau_S^{-1}$. Asymptotic power is $1 - \Phi(z_\alpha - \delta c_S)$.
- **Efficacy of t -test c_t :** $c_t = 1/\sigma$ (assuming symmetric errors, θ = mean, variance σ^2).
- **ARE(Sign, t -test):** $\text{ARE}(S, t) = \sigma^2/\tau_S^2 = c_S^2/c_t^2$.
 - **Normal data:** $\text{ARE}(S, t) = 2/\pi \approx 0.637$.
 - **Laplace data:** $\text{ARE}(S, t) = 2$.
 - **Contaminated normals:** $\text{ARE}(S, t)$ increases with contamination (heavier tails).

10.2.2 Estimating Equations Based on the Sign Test

- Estimator $\hat{\theta}$ for median θ by minimizing L_1 norm: $\hat{\theta} = \text{Argmin} \sum |X_i - \theta|$.
- Leads to estimating equation $\sum \text{sgn}(X_i - \theta) = 0$, solved by sample median Q_2 .
- Also solves $S(\theta) \approx n/2$ (estimation by test inversion).
- Asymptotic distribution of sample median: $\sqrt{n}(Q_2 - \theta) \rightarrow N(0, \tau_S^2)$, where $\tau_S = (2f(0))^{-1}$.

10.2.3 Confidence Interval for the Median

- Invert sign test: $1 - \alpha = P_\theta[c_1 < S(\theta) < n - c_1]$.
- $c_1 < S(\theta) < n - c_1 \iff Y_{c_1+1} \leq \theta < Y_{n-c_1}$.
- **CI for θ :** $[Y_{c_1+1}, Y_{n-c_1}]$.
- **Large sample approximation for c_1 :** $c_1 \approx n/2 - z_{\alpha/2}\sqrt{n}/2 - 1/2$.

10.2.3 10.3 Signed-Rank Wilcoxon Test

Assumes Model (10.2.1) and symmetric pdf $f(x)$ for errors ε_i . θ is the center of symmetry. Aims for higher efficiency than sign test, especially for near-normal data.

- **Hypotheses (WLOG):** $H_0 : \theta = 0$ vs $H_1 : \theta > 0$.
- **Wilcoxon Signed-Rank Statistic T :** $T = \sum_{i=1}^n \text{sgn}(X_i)R(|X_i|)$, where $R(|X_i|)$ is rank of $|X_i|$ among absolute values.
- **Alternative form T^+ :** Sum of ranks of positive X_i . $T = 2T^+ - n(n + 1)/2$.
- **Null Distribution Properties:**
 - T (and T^+) is distribution-free under H_0 .
 - $\mathbb{E}_{H_0}(T) = 0$, $\text{Var}_{H_0}(T) = n(n + 1)(2n + 1)/6$.
 - $\mathbb{E}_{H_0}(T^+) = n(n + 1)/4$, $\text{Var}_{H_0}(T^+) = n(n + 1)(2n + 1)/24$.
 - Standardized T (or T^+) is asymptotically $N(0, 1)$. Distribution is symmetric.
- **Walsh Averages:** $T^+ = \#\{i \leq j \mid (X_i + X_j)/2 > 0\}$.
- **Process $T^+(\theta)$:** $T^+(\theta) = \#\{i \leq j \mid (X_j + X_i)/2 > \theta\}$. Decreasing step function of θ .

- Power function is nondecreasing; test is unbiased.

10.3.1 Asymptotic Relative Efficiency (Wilcoxon)

- For local alternatives $\theta_n = \delta/\sqrt{n}$, asymptotic power is $\lim_{n \rightarrow \infty} \gamma_{SR}(\theta_n) = 1 - \Phi(z_\alpha - \delta\tau_W^{-1})$, where $\tau_W = 1/[\sqrt{12} \int f^2(x)dx]$.
- Efficacy $c_{T^+} = \tau_W^{-1} = \sqrt{12} \int f^2(x)dx$.
- **ARE(Wilcoxon, t-test):** $\text{ARE}(W, t) = \sigma^2/\tau_W^2$.
 - **Normal data:** $\text{ARE}(W, t) = 3/\pi \approx 0.955$. Highly efficient.
 - **Contaminated normals:** $\text{ARE}(W, t)$ often > 1 , increases with contamination.

10.3.2 Estimating Equations (Wilcoxon)

- **Hodges-Lehmann Estimator $\hat{\theta}_W$:** Solves $T^+(\hat{\theta}_W) = n(n+1)/4$.
 - $\hat{\theta}_W = \text{median}\{(X_i + X_j)/2\}$ (median of Walsh averages).
- Asymptotic distribution of $\hat{\theta}_W$: $\sqrt{n}(\hat{\theta}_W - \theta) \rightarrow N(0, \tau_W^2)$.

10.3.3 Confidence Interval for Median (Wilcoxon)

- Invert $T^+(\theta)$ test: $P_\theta[W_{c_{W1}+1} \leq \theta < W_{m-c_{W1}}] = 1 - \alpha$.
- **CI for θ :** $[W_{c_{W1}+1}, W_{m-c_{W1}}]$, where W_k is k -th ordered Walsh average, $m = n(n+1)/2$.
- **Approx. for c_{W1} :** $c_{W1} \approx n(n+1)/4 - z_{\alpha/2}\sqrt{n(n+1)(2n+1)/24} - 1/2$.

10.3.4 Monte Carlo Investigation

- Method to estimate finite sample relative efficiency (RE) of estimators.
- $\widehat{\text{RE}}_n(\hat{\theta}_1, \hat{\theta}_2) = \text{MSE}_2/\text{MSE}_1$.
- **Example:** HL vs sample mean for contaminated normals shows simulation RE close to asymptotic ARE.

10.28.4 10.4 Mann-Whitney-Wilcoxon (MWW) Procedure

Two independent samples: $X_1, \dots, X_{n_1} \sim F(x)$ and $Y_1, \dots, Y_{n_2} \sim G(x)$.
 Focus on shift model: $G(x) = F(x - \Delta)$. $H_0 : \Delta = 0$.

- **MWW Statistic W :** $W = \sum_{j=1}^{n_2} R(Y_j)$, where $R(Y_j)$ is rank of Y_j in combined sample ($n = n_1 + n_2$).
- Test for $H_1 : \Delta > 0$: Reject H_0 if $W \geq c$.
- **Null Distribution Properties:**
 - W is distribution-free under H_0 .
 - $\mathbb{E}_{H_0}(W) = n_2(n+1)/2$.
 - $\text{Var}_{H_0}(W) = n_1 n_2 (n+1)/12$.
 - Standardized W is asymptotically $N(0, 1)$. Distribution is symmetric.
- **Equivalent Statistic U :** $U = \#\{Y_j > X_i\}$. Then $W = U + n_2(n_2 + 1)/2$.
 - $\mathbb{E}_{H_0}(U) = n_1 n_2 / 2$. $\text{Var}_{H_0}(U) = n_1 n_2 (n+1)/12$.
- **Process $U(\Delta)$:** $U(\Delta) = \#\{Y_j - X_i > \Delta\}$. Decreasing step function of Δ .
- Power function is nondecreasing; test is unbiased.

10.4.1 Asymptotic Relative Efficiency (MWW)

- Local alternatives $\Delta_n = \delta/\sqrt{n}$. Assume $n_1/n \rightarrow \lambda_1, n_2/n \rightarrow \lambda_2$.
- Efficacy $c_U = \sqrt{12\lambda_1\lambda_2} \int f^2(x)dx = \sqrt{\lambda_1\lambda_2}\tau_W^{-1}$.
- Asymptotic power for MWW: $\lim_{n \rightarrow \infty} \gamma_U(\Delta_n) = 1 - \Phi(z_\alpha - c_U\delta)$.
- **ARE(MWW, Two-Sample t -test):** $\text{ARE}(\text{MWW}, \text{LS}) = \sigma^2/\tau_W^2$.
 - Same as one-sample Wilcoxon vs one-sample t -test.
 - **Normal data:** ≈ 0.955 .

10.4.2 Estimating Equations (MWW)

- **Hodges-Lehmann type estimator $\hat{\Delta}_U$:** Solves $U(\hat{\Delta}_U) = n_1 n_2 / 2$.
 - $\hat{\Delta}_U = \text{median}_{i,j}\{Y_j - X_i\}$.
- Asymptotic distribution of $\hat{\Delta}_U$: Approx. $N(\Delta, \tau_W^2(1/n_1 + 1/n_2))$.

10.4.3 Confidence Interval for Shift Δ (MWW)

- Invert $U(\Delta)$ test. **CI:** $[D_{c+1}, D_{n_1 n_2 - c}]$, where D_k is k -th ordered difference $Y_j - X_i$.

- **Approx. for c :** $c \approx n_1 n_2 / 2 - z_{\alpha/2} \sqrt{n_1 n_2 (n+1)/12} - 1/2$.

10.4.4 Monte Carlo Investigation of Power

- Compares power of MWW and two-sample t -test for finite samples.
- **Example:** Contaminated normal errors, MWW much more powerful than t -test.

10.28.5 10.5 General Rank Scores

Addresses finding optimal distribution-free procedures. Focus on two-sample location problem.

- **Score function $\varphi(u)$:** Nondecreasing on $(0, 1)$, standardized ($\int \varphi du = 0, \int \varphi^2 du = 1$).
- Scores $a_\varphi(i) = \varphi[i/(n+1)]$.
- **Test Statistic W_φ :** $W_\varphi = \sum_{j=1}^{n_2} a_\varphi(R(Y_j))$. Reject $H_0 : \Delta = 0$ if $W_\varphi \geq c$.
 - MWW is special case with $\varphi(u) = \sqrt{12}(u - 1/2)$.
- **Null Distribution:** W_φ is distribution-free. $\mathbb{E}_{H_0}(W_\varphi) = 0$. $\text{Var}_{H_0}(W_\varphi) = \frac{n_1 n_2}{n(n-1)} s_a^2$, where $s_a^2 = \sum a_\varphi^2(i) \approx n$. Asymptotically normal.
- Process $W_\varphi(\Delta) = \sum a_\varphi(R(Y_j - \Delta))$ is decreasing step function. Test is unbiased.

10.5.1 Efficacy (General Scores)

- Efficacy $c_\varphi = \sqrt{\lambda_1 \lambda_2} \int \varphi'[F(y)] f^2(y) dy$.
- Asymptotic power: $\lim \gamma_\varphi(\Delta_n) = 1 - \Phi(z_\alpha - c_\varphi \delta)$.

10.5.2 Estimating Equations (General Scores)

- Estimator $\hat{\Delta}_\varphi$ solves $W_\varphi(\hat{\Delta}_\varphi) \approx 0$.
- Asymptotic distribution: Approx. $N(\Delta, \tau_\varphi^2(1/n_1 + 1/n_2))$, with $\tau_\varphi = [\int \varphi'[F(y)] f^2(y) dy]^{-1}$.
- $c_\varphi = \sqrt{\lambda_1 \lambda_2} \tau_\varphi^{-1}$.

10.5.3 Optimization: Best Estimates

- **Goal:** Choose φ to maximize c_φ .

- **Optimal score function** $\varphi_f(u)$: $\varphi_f(u) = -\kappa \frac{f'(F^{-1}(u))}{f(F^{-1}(u))}$.
- Max efficacy c_{φ_f} corresponds to $\tau_{\varphi_f}^{-2} = I(f)$ (Fisher Information for location).
- Estimator $\hat{\Delta}_{\varphi_f}$ is asymptotically efficient (ARE=1 with MLE).
- **Examples:**
 - **Normal Scores:** For normal f , $\varphi_N(u) = \Phi^{-1}(u)$. Fully efficient at normal. $\text{ARE}(\hat{\Delta}_N, \bar{Y} - \bar{X}) \geq 1$ for all symmetric dist.
 - **Wilcoxon Scores:** For logistic f , $\varphi_W(u) = \sqrt{12}(u-1/2)$. $\text{ARE}(\hat{\Delta}_W, \bar{Y} - \bar{X}) \geq 0.864$ for symmetric dist.
 - **Sign Scores:** For Laplace f , $\varphi_S(u) = \text{sgn}(u - 1/2)$. Estimator $\hat{\Delta}_S = \text{median}\{Y_j\} - \text{median}\{X_i\}$. (Related to Mood's Median Test).

10.28.6 10.6 Adaptive Procedures

Selects score function based on data, aiming to maintain α and improve power.

- **Method:** Selector statistic Q (function of order statistics, independent of test statistics W_i) chooses which W_i to use. Overall significance level remains α .
- **Hogg's Adaptive Procedure for two samples:**
 - Uses Q_1 (skewness) and Q_2 (tail weight) based on combined order statistics.
 - Selects one of four score functions: Wilcoxon, Sign, short-tailed, right-skewed.
 - R function `hogg.test` in package `npsm`.
- Adaptive estimation is more complex, often uses residuals from an initial estimate.

10.28.7 10.7 Simple Linear Model

Rank-based procedures for $Y_i = \alpha + \beta(x_i - \bar{x}) + \varepsilon_i$.

- $H_0 : \beta = 0$.
- **Test statistic** $T_\varphi = \sum(x_i - \bar{x})a_\varphi(R(Y_i))$.

- **Null Distribution:** $\mathbb{E}_{H_0}(T_\varphi) = 0$. $\text{Var}_{H_0}(T_\varphi) = \frac{s_a^2}{n-1} \sum (x_i - \bar{x})^2$. Asymptotically normal.
- **Estimator** $\hat{\beta}_\varphi$ solves $T_\varphi(\hat{\beta}_\varphi) \approx 0$, where $T_\varphi(\beta) = \sum (x_i - \bar{x}) a_\varphi(R(Y_i - x_i \beta))$.
 - $T_\varphi(\beta)$ is decreasing step function of β .
- **Intercept estimate:** $\hat{\alpha} = \text{median}\{Y_i - \hat{\beta}_\varphi(x_i - \bar{x})\}$.
- **Efficacy** $c_T = \sigma_x \int \varphi'(F(y)) f^2(y) dy$, where $\sigma_x^2 = \lim n^{-1} \sum (x_i - \bar{x})^2$.
- **Asymptotic distribution of $\hat{\beta}_\varphi$:** Approx. $N(\beta, \tau_\varphi^2 / \sum (x_i - \bar{x})^2)$, with $\tau_\varphi = (\int \varphi'(F(y)) f^2(y) dy)^{-1}$.
- $\text{ARE}(\hat{\beta}_\varphi, \hat{\beta}_{LS}) = \sigma^2 / \tau_\varphi^2$. Same as in location models.
- **Computation** via Rfit package.

10.29 Chapter 11 Summary: Bayesian

Statistics

This chapter introduces the Bayesian approach to statistical inference, where parameters are treated as random variables with prior distributions.

Bayesian methods incorporate prior knowledge and provide a coherent framework for sequential learning.

10.29.1 11.1 Bayesian Procedures

11.1.1 Prior and Posterior Distributions

The fundamental Bayesian model consists of:

Définition 30: Bayesian Model Components

$$X \mid \theta \sim f(x \mid \theta)$$

(likelihood)

$$\Theta \sim h(\theta)$$

(prior distribution) The **posterior distribution** is:

$$k(\theta \mid \mathbf{x}) = \frac{L(\mathbf{x} \mid \theta)h(\theta)}{g_1(\mathbf{x})}$$

where $g_1(\mathbf{x}) = \int L(\mathbf{x} | \theta)h(\theta)d\theta$ (marginal distribution)

Key Relationship

$$k(\theta | \mathbf{x}) \propto L(\mathbf{x} | \theta)h(\theta)$$

Interpretation:

- Prior: $h(\theta)$ represents belief about θ before observing data
- Posterior: $k(\theta | \mathbf{x})$ represents updated belief after observing data
- Likelihood: $L(\mathbf{x} | \theta)$ represents information from the data

Sufficient Statistics

If $Y = u(\mathbf{X})$ is sufficient for θ , then:

$$k(\theta | y) \propto g(y | \theta)h(\theta)$$

11.1.2 Bayesian Point Estimation

Définition 31: Bayes Estimator

A **Bayes estimator** minimizes the posterior expected loss:

$$\delta(\mathbf{x}) = \operatorname{argmin} \int \mathcal{L}[\theta, \delta(\mathbf{x})]k(\theta | \mathbf{x})d\theta$$

Common Loss Functions:

- (1) Squared-error loss: $\mathcal{L}[\theta, \delta] = [\theta - \delta]^2$
 - Bayes estimator = posterior mean: $E(\Theta | \mathbf{x})$
- (2) Absolute-error loss: $\mathcal{L}[\theta, \delta] = |\theta - \delta|$
 - Bayes estimator = posterior median

Key Examples

Example 1: Beta-Binomial Model

$$X_i | \theta \sim \text{iid Bernoulli}(\theta)$$

$$\Theta \sim \text{Beta}(\alpha, \beta)$$

Posterior: $\Theta | \mathbf{x} \sim \text{Beta}(\alpha + \sum x_i, \beta + n - \sum x_i)$

Bayes estimator (squared-error):

$$\delta(y) = \frac{\alpha + y}{\alpha + \beta + n} = \frac{n}{n + \alpha + \beta} \cdot \frac{y}{n} + \frac{\alpha + \beta}{n + \alpha + \beta} \cdot \frac{\alpha}{\alpha + \beta}$$

This is a **weighted average** of MLE and prior mean!

Example 2: Normal Model with Known Variance

$$X_i | \theta \sim \text{iid } N(\theta, \sigma^2)$$

$$\Theta \sim N(\theta_0, \sigma_0^2)$$

Posterior: $\Theta | \bar{x} \sim N\left(\frac{\bar{x}\sigma_0^2 + \theta_0(\sigma^2/n)}{\sigma_0^2 + \sigma^2/n}, \frac{(\sigma^2/n)\sigma_0^2}{\sigma_0^2 + \sigma^2/n}\right)$

Bayes estimator:

$$\delta(\bar{x}) = \frac{\sigma_0^2}{\sigma_0^2 + \sigma^2/n} \bar{x} + \frac{\sigma^2/n}{\sigma_0^2 + \sigma^2/n} \theta_0$$

11.1.3 Bayesian Interval Estimation

Définition 32: Credible Interval

An interval $(u(\mathbf{x}), v(\mathbf{x}))$ is a $(1 - \alpha)100\%$ **credible interval** if:

$$P[u(\mathbf{x}) < \Theta < v(\mathbf{x}) | \mathbf{X} = \mathbf{x}] = 1 - \alpha$$

Construction Methods:

- (1) Equal-tail intervals: $\alpha/2$ probability in each tail
- (2) Highest density region (HDR): Shortest interval with given probability

11.1.4 Bayesian Testing Procedures

For hypotheses $H_0 : \theta \in \omega_0$ vs $H_1 : \theta \in \omega_1$:

Decision Rule:

- Accept H_0 if $P(\Theta \in \omega_0 | \mathbf{x}) \geq P(\Theta \in \omega_1 | \mathbf{x})$
- Otherwise accept H_1

Advantages:

- Direct probability statements about hypotheses
- Coherent with other Bayesian procedures
- Natural for sequential analysis

11.1.5 Bayesian Sequential Procedures

Sequential Update: Previous posterior becomes new prior

- (1) Start with prior $h_0(\theta)$
- (2) Observe \mathbf{x}_1 , get posterior $k_1(\theta | \mathbf{x}_1)$
- (3) Use k_1 as prior for next observation
- (4) Continue indefinitely

This provides an elegant framework for sequential learning.

10.29.2 11.2 More Bayesian Terminology and Ideas

Conjugate Priors

Définition 33: Conjugate Family

A class of prior distributions is **conjugate** for a likelihood family if the posterior is in the same family as the prior.

Common Conjugate Pairs:

- Binomial + Beta \rightarrow Beta
- Poisson + Gamma \rightarrow Gamma
- Normal (known variance) + Normal \rightarrow Normal
- Normal (known mean) + Inverse-Gamma \rightarrow Inverse-Gamma

Advantages:

- Analytical tractability
- Easy computation
- Clear interpretation of hyperparameters

Improper and Noninformative Priors

Définition 34: Improper Prior

A prior $h(\theta) \geq 0$ that doesn't integrate to 1, but yields a proper posterior:

$$\int k(\theta | \mathbf{x}) d\theta = 1$$

Examples:

- Uniform on $(-\infty, \infty)$: $h(\theta) \propto 1$
- Scale-invariant: $h(\sigma) \propto 1/\sigma$

Définition 35: Jeffreys' Prior

$$h(\theta) \propto \sqrt{I(\theta)}$$

where $I(\theta)$ is Fisher information. This prior is **invariant** under reparameterization.

Problem-Solving Strategy for Priors

- (1) Identify conjugate family if available
- (2) Choose hyperparameters based on prior belief
- (3) Check sensitivity to prior specification
- (4) Use noninformative priors when little prior knowledge exists

10.29.3 11.3 Gibbs Sampler

Motivation

When analytical integration is difficult, use **Monte Carlo methods**.

Théorème 54: Generation Algorithm

To generate $X \sim f_X(x)$:

- (1) Generate $Y \sim f_Y(y)$
- (2) Generate $X \sim f_{X|Y}(x | Y)$

Then X has the desired distribution $f_X(x)$.

Gibbs Sampler Algorithm

Définition 36: Gibbs Sampler

For joint distribution (X, Y) , starting with X_0 : For $i = 1, 2, \dots, m$:

- (1) Generate $Y_i | X_{i-1} \sim f(y | x_{i-1})$
- (2) Generate $X_i | Y_i \sim f(x | y_i)$

As $i \rightarrow \infty$: $(X_i, Y_i) \xrightarrow{D} (X, Y)$

Key Properties:

- Markov chain: Future depends only on present state
- Convergence: Reaches equilibrium distribution
- Estimation: $\bar{W} = m^{-1} \sum_{i=1}^m W(X_i) \xrightarrow{P} E[W(X)]$

Implementation Guidelines

- (1) Burn-in period: Discard first m iterations
- (2) Multiple chains: Check convergence
- (3) Diagnostic plots: Monitor chain behavior
- (4) Thinning: Reduce autocorrelation if needed

Example: Gamma-Poisson Model

$$f(x, y) \propto \frac{y^{\alpha+x-1} e^{-2y}}{x!}, \quad y > 0, x = 0, 1, 2, \dots$$

Conditional distributions:

- $Y | X = x \sim \Gamma(\alpha + x, 1/2)$
- $X | Y = y \sim \text{Poisson}(y)$

Gibbs algorithm:

- (1) Generate $Y_i | X_{i-1} \sim \Gamma(\alpha + X_{i-1}, 1/2)$
- (2) Generate $X_i | Y_i \sim \text{Poisson}(Y_i)$

10.29.4 11.4 Modern Bayesian Methods

Hierarchical Bayes

Définition 37: Hierarchical Bayes Model

$$X | \theta \sim f(x | \theta)$$

$$\Theta | \gamma \sim h(\theta | \gamma)$$

$$\Gamma \sim \psi(\gamma)$$

where γ is a **hyperparameter**.

Advantages:

- More flexible prior specification
- Partial pooling of information
- Automatic sensitivity analysis

Posterior:

$$k(\theta | \mathbf{x}) = \frac{\int f(\mathbf{x} | \theta)h(\theta | \gamma)\psi(\gamma)d\gamma}{\int \int f(\mathbf{x} | \theta)h(\theta | \gamma)\psi(\gamma)d\gamma d\theta}$$

MCMC Implementation: Use Gibbs sampler with conditionals:

- $\Theta_i | \mathbf{x}, \gamma_{i-1} \sim g(\theta | \mathbf{x}, \gamma_{i-1})$
- $\Gamma_i | \mathbf{x}, \theta_i \sim g(\gamma | \mathbf{x}, \theta_i)$

11.4.1 Empirical Bayes

Définition 38: Empirical Bayes

$$X | \theta \sim f(x | \theta)$$

$$\Theta | \gamma \sim h(\theta | \gamma)$$

Estimate $\hat{\gamma}$ from marginal likelihood:

$$m(\mathbf{x} | \gamma) = \int f(\mathbf{x} | \theta)h(\theta | \gamma)d\theta$$

Use posterior $k(\theta | \mathbf{x}, \hat{\gamma})$ for inference.

Procedure:

- (1) Maximize $m(\mathbf{x} | \gamma)$ to get $\hat{\gamma}$
- (2) Substitute $\hat{\gamma}$ into posterior
- (3) Make inference using $k(\theta | \mathbf{x}, \hat{\gamma})$

Example: Poisson-Gamma

$$X_i | \lambda \sim \text{iid Poisson}(\lambda)$$

$$\Lambda | b \sim \Gamma(1, b)$$

MLE: $\hat{b} = \bar{x}$

Empirical Bayes estimator: $\hat{\lambda} = \bar{x}$ (equals MLE!)

10.29.5 Problem-Solving Strategies

(1) Choosing Priors

- Conjugate priors: For analytical convenience
- Informative priors: When strong prior knowledge exists
- Noninformative priors: For "objective" analysis
- Sensitivity analysis: Check robustness to prior choice

(2) Computational Methods

- Analytical: When conjugate priors available
- Simple Monte Carlo: When easy to sample from posterior
- Gibbs sampler: For complex multiparameter problems
- Other MCMC: When Gibbs sampler difficult

(3) Model Selection and Validation

- Bayes factors: For model comparison
- Posterior predictive checks: For model validation
- Cross-validation: For predictive performance
- Sensitivity analysis: For robustness

(4) Practical Considerations

- Computational cost: Balance accuracy vs. efficiency
- Convergence diagnosis: Essential for MCMC
- Multiple chains: Check convergence
- Effective sample size: Ensure adequate precision

10.29.6 Key Distributions and Results

Conjugate Families

- Beta-Binomial: $\text{Beta}(\alpha, \beta) + \text{Binomial} \rightarrow \text{Beta}(\alpha + y, \beta + n - y)$
- Gamma-Poisson: $\Gamma(\alpha, \beta) + \text{Poisson} \rightarrow \Gamma(\alpha + \sum x_i, \beta / (n\beta + 1))$
- Normal-Normal: Both likelihood and prior normal

Improper Priors

- Uniform: $h(\theta) \propto 1$
- Jeffreys': $h(\theta) \propto \sqrt{I(\theta)}$
- Scale-invariant: $h(\sigma) \propto 1/\sigma$

Asymptotic Properties

- Bernstein-von Mises: Posterior \rightarrow Normal around MLE
- Consistency: Posterior concentrates on true value
- Efficiency: Posterior variance \rightarrow Cramér-Rao bound

10.29.7 Applications and Extensions

Modern Applications

- (1) Machine learning: Bayesian neural networks, Gaussian processes
- (2) Biostatistics: Clinical trials, personalized medicine
- (3) Economics: Bayesian econometrics, decision theory
- (4) Engineering: Reliability analysis, quality control

Advanced Topics

- Variational Bayes: Approximate inference
- Reversible jump MCMC: Variable dimension problems
- Particle filters: Dynamic models
- Approximate Bayesian computation (ABC): Intractable likelihoods

This chapter provides the foundation for Bayesian statistical analysis, from basic conjugate models to modern computational methods, enabling sophisticated probabilistic modeling and inference.

10.30 Convergence of Random Variables

See All of Statistic Chapter 5.

10.30.1 Converges in probability, distribution, quadratic mean

5.2 Types of Convergence

The two main types of convergence are defined as follows.

5.1 Definition. Let X_1, X_2, \dots be a sequence of random variables and let X be another random variable. Let F_n denote the CDF of X_n and let F denote the CDF of X .

1. X_n converges to X in probability, written $X_n \xrightarrow{P} X$, if, for every $\epsilon > 0$,

$$\mathbb{P}(|X_n - X| > \epsilon) \rightarrow 0 \quad (5.1)$$

as $n \rightarrow \infty$.

2. X_n converges to X in distribution, written $X_n \rightsquigarrow X$, if

$$\lim_{n \rightarrow \infty} F_n(t) = F(t) \quad (5.2)$$

at all t for which F is continuous.

When the limiting random variable is a point mass, we change the notation slightly. If $\mathbb{P}(X = c) = 1$ and $X_n \xrightarrow{P} X$ then we write $X_n \xrightarrow{P} c$. Similarly, if $X_n \rightsquigarrow X$ we write $X_n \rightsquigarrow c$.

There is another type of convergence which we introduce mainly because it is useful for proving convergence in probability.

5.2 Definition. X_n converges to X in quadratic mean (also called convergence in L_2), written $X_n \xrightarrow{\text{qm}} X$, if

$$\mathbb{E}(X_n - X)^2 \rightarrow 0 \quad (5.3)$$

as $n \rightarrow \infty$.

Again, if X is a point mass at c we write $X_n \xrightarrow{\text{qm}} c$ instead of $X_n \xrightarrow{\text{qm}} X$.

Their relationships

5.4 Theorem. *The following relationships hold:*

- (a) $X_n \xrightarrow{\text{qm}} X$ implies that $X_n \xrightarrow{P} X$.
- (b) $X_n \xrightarrow{P} X$ implies that $X_n \rightsquigarrow X$.
- (c) If $X_n \rightsquigarrow X$ and if $\mathbb{P}(X = c) = 1$ for some real number c , then $X_n \xrightarrow{P} X$.

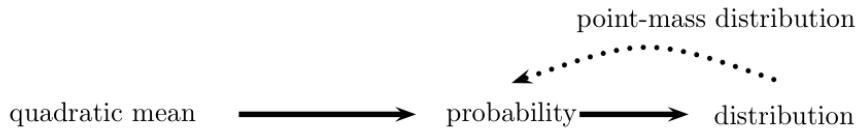


FIGURE 5.2. Relationship between types of convergence.

10.30.2 Some convergence properties that preserved under transformations

5.5 Theorem. *Let X_n, X, Y_n, Y be random variables. Let g be a continuous function.*

- (a) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, then $X_n + Y_n \xrightarrow{P} X + Y$.
- (b) If $X_n \xrightarrow{\text{qm}} X$ and $Y_n \xrightarrow{\text{qm}} Y$, then $X_n + Y_n \xrightarrow{\text{qm}} X + Y$.
- (c) If $X_n \rightsquigarrow X$ and $Y_n \rightsquigarrow c$, then $X_n + Y_n \rightsquigarrow X + c$.
- (d) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, then $X_n Y_n \xrightarrow{P} XY$.
- (e) If $X_n \rightsquigarrow X$ and $Y_n \rightsquigarrow c$, then $X_n Y_n \rightsquigarrow cX$.
- (f) If $X_n \xrightarrow{P} X$, then $g(X_n) \xrightarrow{P} g(X)$.
- (g) If $X_n \rightsquigarrow X$, then $g(X_n) \rightsquigarrow g(X)$.

Parts (c) and (e) are known as **Slutzky's theorem**. It is worth noting that $X_n \rightsquigarrow X$ and $Y_n \rightsquigarrow Y$ does not in general imply that $X_n + Y_n \rightsquigarrow X + Y$.

10.30.3 Almost Sure and L^1 Convergence

We say that X_n converges almost surely to X , written $X_n \xrightarrow{as} X$ if

$$\mathbb{P}(\{s : X_n(s) \rightarrow X(s)\}) = 1$$

We say that X_n converges in L^1 to X , written $X_n \xrightarrow{L^1} X$ if

$$\mathbb{E}|X_n - X| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

5.17 Theorem. Let X_n and X be random variables. Then:

- (a) $X_n \xrightarrow{\text{as}} X$ implies that $X_n \xrightarrow{\text{P}} X$.
- (b) $X_n \xrightarrow{\text{qm}} X$ implies that $X_n \xrightarrow{L_1} X$.
- (c) $X_n \xrightarrow{L_1} X$ implies that $X_n \xrightarrow{\text{P}} X$.

The weak law of large numbers says that \bar{X}_n converges to $\mathbb{E}(X_1)$ in probability. The strong law asserts that this is also true almost surely.

5.18 Theorem (The Strong Law of Large Numbers). Let X_1, X_2, \dots be IID. If $\mu = \mathbb{E}|X_1| < \infty$ then $\bar{X}_n \xrightarrow{\text{as}} \mu$.

A sequence X_n is **asymptotically uniformly integrable** if

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}(|X_n| I(|X_n| > M)) = 0.$$

5.19 Theorem. If $X_n \xrightarrow{\text{P}} b$ and X_n is asymptotically uniformly integrable, then $\mathbb{E}(X_n) \rightarrow b$.

10.31 常见分布

• 二项分布 (Binomial Distribution)

- 记作: $X \sim B(n, p)$

- PMF:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \dots, n$$

- MGF:

$$M_X(t) = (1 - p + pe^t)^n$$

- 特征函数:

$$\varphi_X(t) = (1 - p + pe^{it})^n$$

• 负二项分布 (Negative Binomial Distribution)

- 记作: $X \sim NB(r, p)$

- PMF:

$$P(X = k) = \binom{k+r-1}{k} p^r (1-p)^k, \quad k = 0, 1, 2, \dots$$

- MGF:

$$M_X(t) = \left(\frac{p}{1 - (1-p)e^t} \right)^r, \quad t < -\ln(1-p)$$

- 特征函数:

$$\varphi_X(t) = \left(\frac{p}{1 - (1-p)e^{it}} \right)^r$$

- 几何分布 (Geometric Distribution)

- 记作: $X \sim Geo(p)$

- PMF:

$$P(X = k) = p(1-p)^{k-1}, \quad k = 1, 2, \dots$$

- MGF:

$$M_X(t) = \frac{pe^t}{1 - (1-p)e^t}, \quad t < -\ln(1-p)$$

- 特征函数:

$$\varphi_X(t) = \frac{pe^{it}}{1 - (1-p)e^{it}}$$

- 多项分布 (Multinomial Distribution)

- 记作: $\mathbf{X} \sim Mult(n, \mathbf{p})$, 其中 $\mathbf{p} = (p_1, p_2, \dots, p_k)$

- PMF:

$$P(\mathbf{X} = \mathbf{x}) = \frac{n!}{x_1!x_2!\cdots x_k!} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

, 其中 $\sum_{i=1}^k x_i = n$

- MGF:

$$M_{\mathbf{X}}(\mathbf{t}) = (p_1 e^{t_1} + p_2 e^{t_2} + \cdots + p_k e^{t_k})^n$$

- 特征函数:

$$\varphi_{\mathbf{X}}(\mathbf{t}) = (p_1 e^{it_1} + p_2 e^{it_2} + \cdots + p_k e^{it_k})^n$$

- 超几何分布 (Hypergeometric Distribution)

- 记作: $X \sim HGeom(N, K, n)$

- PMF:

$$P(X = k) = \frac{\binom{K}{k} \binom{N-K}{n-k}}{\binom{N}{n}}, \quad \max(0, n+K-N) \leq k \leq \min(n, K)$$

- MGF: 无简单表达式

- 特征函数: 无简单表达式

- 泊松分布 (Poisson Distribution)

- 记作: $X \sim Pois(\lambda)$

- PMF:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \dots$$

- MGF:

$$M_X(t) = e^{\lambda(e^t - 1)}$$

- 特征函数:

$$\varphi_X(t) = e^{\lambda(e^{it} - 1)}$$

• 伽马分布 (Gamma Distribution)

- 记作: $X \sim Gamma(\alpha, \beta)$

- PDF:

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0$$

- CDF: $F(x) = \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)}$, $x > 0$, 其中 $\gamma(\alpha, y)$ 是不完全伽马函数

- MGF:

$$M_X(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha}, \quad t < \beta$$

- 特征函数:

$$\varphi_X(t) = \left(1 - \frac{it}{\beta}\right)^{-\alpha}$$

• 卡方分布 (Chi-Square Distribution)

- 记作: $X \sim \chi^2(k)$, 是自由度为 k 的卡方分布

- PDF:

$$f(x) = \frac{1}{2^{k/2} \Gamma(k/2)} x^{k/2-1} e^{-x/2}, \quad x > 0$$

- CDF: $F(x) = \frac{\gamma(k/2, x/2)}{\Gamma(k/2)}$, $x > 0$

- MGF:

$$M_X(t) = (1 - 2t)^{-k/2}, \quad t < 1/2$$

- 特征函数:

$$\varphi_X(t) = (1 - 2it)^{-k/2}$$

• 贝塔分布 (Beta Distribution)

- 记作: $X \sim Beta(\alpha, \beta)$

- PDF:

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}, \quad 0 < x < 1$$

- CDF: $F(x) = \frac{B_x(\alpha, \beta)}{B(\alpha, \beta)}$, $0 \leq x \leq 1$, 其中 $B_x(\alpha, \beta)$ 是不完全贝塔函数

- MGF: 无简单表达式
- 特征函数: 无简单表达式

• 正态分布 (Normal Distribution)

- 记作: $X \sim N(\mu, \sigma^2)$

- PDF:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

- CDF: $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$

- MGF:

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

- 特征函数:

$$\varphi_X(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$$

• t 分布 (Student's t-Distribution)

- 记作: $X \sim t(n)$, 其中 n 是自由度

- PDF:

$$f(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}, \quad x \in \mathbb{R}$$

- CDF: 无简单表达式

- MGF: 不存在

- 特征函数: 无简单表达式

• F 分布 (F-Distribution)

- 记作: $X \sim F(d_1, d_2)$, 其中 d_1, d_2 是自由度

- PDF:

$$g_1(w) = \frac{\Gamma[(r_1 + r_2)/2] (r_1/r_2)^{r_1/2}}{\Gamma(r_1/2) \Gamma(r_2/2)} \frac{w^{r_1/2-1}}{(1 + r_1 w/r_2)^{(r_1+r_2)/2}} \quad 0 < w < \infty$$

- CDF: 无简单表达式

- MGF: 不存在

- 特征函数: 无简单表达式

This is the Chapter 6 of All of Statistics.

10.32 Models, Statistical Inference and Learning

10.32.1 Motivation

A typical statistical inference question: Given a sample $X_1, \dots, X_n \sim F$ how do we infer F ?

10.32.2 What is statistical model

A **statistical model** \mathfrak{F} is a set of distributions (or densities or regression functions). A **parametric model** is a set \mathfrak{F} that can be parameterized by a finite number of parameters. For example, if we assume that the data come from a Normal distribution, then the model is

$$\mathfrak{F} = \left\{ f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(x - \mu)^2\right\}, \quad \mu \in \mathbb{R}, \sigma > 0 \right\}. \quad (6.1)$$

This is a two-parameter model. We have written the density as $f(x; \mu, \sigma)$ to show that x is a value of the random variable whereas μ and σ are parameters.

In general, a parametric model takes the form

$$\mathfrak{F} = \{f(x; \theta) : \theta \in \Theta\}$$

where θ is an unknown parameter (or vector) taking values in the **parameter space** Θ .

Nonparametric model

A nonparametric model is a set \mathfrak{F} that cannot be parameterized by a finite number of parameters. For example, $\mathfrak{F}_{\text{ALL}} = \{\text{all CDF's}\}$ is nonparametric.

10.32.3 Examples of parameter spaces

6.1 Example (One-dimensional Parametric Estimation). Let X_1, \dots, X_n be independent Bernoulli(p) observations. The problem is to estimate the parameter p . ■

6.2 Example (Two-dimensional Parametric Estimation). Suppose that $X_1, \dots, X_n \sim F$ and we assume that the PDF $f \in \mathfrak{F}$ where \mathfrak{F} is given in (6.1). In this case there are two parameters, μ and σ . The goal is to estimate the parameters from the data. If we are only interested in estimating μ , then μ is the parameter of interest and σ is a nuisance parameter. ■

6.3 Example (Nonparametric estimation of the CDF). Let X_1, \dots, X_n be independent observations from a CDF F . The problem is to estimate F assuming only that $F \in \mathfrak{F}_{\text{ALL}} = \{\text{all CDF's}\}$. ■

6.4 Example (Nonparametric density estimation). Let X_1, \dots, X_n be independent observations from a CDF F and let $f = F'$ be the PDF. Suppose we want to estimate the PDF f . It is not possible to estimate f assuming only that $F \in \mathfrak{F}_{\text{ALL}}$. We need to assume some smoothness on f . For example, we might assume that $f \in \mathfrak{F} = \mathfrak{F}_{\text{DENS}} \cap \mathfrak{F}_{\text{SOB}}$ where $\mathfrak{F}_{\text{DENS}}$ is the set of all probability density functions and

$$\mathfrak{F}_{\text{SOB}} = \left\{ f : \int (f''(x))^2 dx < \infty \right\}.$$

The class $\mathfrak{F}_{\text{SOB}}$ is called a **Sobolev space**; it is the set of functions that are not “too wiggly.” ■

Remarque 50. We first discuss nonparametric inference. Two dominant approaches to statistical inference are **frequentist inference** and **Bayesian inference**.

10.32.4 Notations

If $\mathfrak{F} = \{f(x; \theta) : \theta \in \Theta\}$ is a parametric model, we write

$$\mathbb{P}_\theta(X \in A) = \int_A f(x; \theta) dx \quad \text{and} \quad \mathbb{E}_\theta(r(X)) = \int r(x) f(x; \theta) dx$$

The subscript θ indicates that the probability or expectation is w.r.t. $f(x; \theta)$; it does not mean we are averaging over θ . Similarly, write \mathbb{V}_θ for the variance.

10.33 Fundamental Concepts in Inference

Many inferential problems can be identified as one of three types:

- Estimation
- Confidence sets
- Hypothesis testing

10.33.1 Point Estimation

Point estimation refers to provide a single "best guess" of some quantity of interest, such as cdf, pdf, regression function....

Denote a point estimate of θ by $\hat{\theta}$ or $\hat{\theta}_n$.

Remarque 51. Note that θ is fixed, unknown quantity, $\hat{\theta}$ depends on the data so is a r.v.

$$\text{bias}(\hat{\theta}_n) = \mathbb{E}_{\theta}(\hat{\theta}_n) - \theta$$

Say $\hat{\theta}_n$ unbiased if $\mathbb{E}(\hat{\theta}_n) = \theta$.

Note. Unbiasedness is considered less important than before.

Say $\hat{\theta}_n$ consistent if $\hat{\theta}_n \xrightarrow{P} \theta$.

The distribution of $\hat{\theta}_n$ is called the **sampling distribution**. The standard deviation of $\hat{\theta}_n$ is called the **standard error**, denoted by se:

$$\text{se} = \text{se}(\hat{\theta}_n) = \sqrt{\mathbb{V}(\hat{\theta}_n)}$$

Often, se depends on the unknown F . The estimated se is denoted by $\hat{\text{se}}$.

6.8 Example. Let $X_1, \dots, X_n \sim \text{Bernoulli}(p)$ and let $\hat{p}_n = n^{-1} \sum_i X_i$. Then $\mathbb{E}(\hat{p}_n) = n^{-1} \sum_i \mathbb{E}(X_i) = p$ so \hat{p}_n is unbiased. The standard error is $\text{se} = \sqrt{\mathbb{V}(\hat{p}_n)} = \sqrt{p(1-p)/n}$. The estimated standard error is $\hat{\text{se}} = \sqrt{\hat{p}(1-\hat{p})/n}$.

■

The **mean squared error** (MSE) is defined by

$$\text{MSE} := \mathbb{E}_{\theta}(\hat{\theta}_n - \theta)^2$$

We have

$$\text{MSE} = \text{bias}^2(\hat{\theta}_n) + \mathbb{V}_{\theta}(\hat{\theta}_n).$$

6.10 Theorem. If $\text{bias} \rightarrow 0$ and $\text{se} \rightarrow 0$ as $n \rightarrow \infty$ then $\hat{\theta}_n$ is consistent, that is, $\hat{\theta}_n \xrightarrow{P} \theta$.

PROOF. If $\text{bias} \rightarrow 0$ and $\text{se} \rightarrow 0$ then, by Theorem 6.9, $\text{MSE} \rightarrow 0$. It follows that $\hat{\theta}_n \xrightarrow{\text{qm}} \theta$. (Recall Definition 5.2.) The result follows from part (b) of Theorem 5.4. ■

6.12 Definition. An estimator is **asymptotically Normal** if

$$\frac{\hat{\theta}_n - \theta}{\text{se}} \rightsquigarrow N(0, 1). \quad (6.8)$$

Which means the estimator is not always asymptotically Normal. But point estimators often have this property.

10.33.2 Confidence Sets

A $1 - \alpha$ **confidence interval** for a parameter θ is an interval $C_n = (a, b)$ where $a = a(X_1, \dots, X_n)$ and $b = b(X_1, \dots, X_n)$ are functions of the data such that

$$\mathbb{P}_{\theta}(\theta \in C_n) \geq 1 - \alpha, \quad \text{for all } \theta \in \Theta. \quad (6.9)$$

In words, (a, b) traps θ with probability $1 - \alpha$. We call $1 - \alpha$ the **coverage** of the confidence interval.

Note. Note that C_n is random and θ is fixed.

Warning! There is much confusion about how to interpret a confidence interval. A confidence interval is not a probability statement about θ since θ is a fixed quantity, not a random variable. Some texts interpret confidence intervals as follows: if I repeat the experiment over and over, the interval will contain the parameter 95 percent of the time. This is correct but useless since we rarely repeat the same experiment over and over. A better interpretation is this:

On day 1, you collect data and construct a 95 percent confidence interval for a parameter θ_1 . On day 2, you collect new data and construct a 95 percent confidence interval for an unrelated parameter θ_2 . On day 3, you collect new data and construct a 95 percent confidence interval for an unrelated parameter θ_3 . You continue this way constructing confidence intervals for a sequence of unrelated parameters $\theta_1, \theta_2, \dots$. Then 95 percent of your intervals will trap the true parameter value. There is no need to introduce the idea of repeating the same experiment over and over.

Remarque 52.

Example

6.15 Example. In the coin flipping setting, let $C_n = (\hat{p}_n - \epsilon_n, \hat{p}_n + \epsilon_n)$ where $\epsilon_n^2 = \log(2/\alpha)/(2n)$. From Hoeffding's inequality (4.4) it follows that

$$\mathbb{P}(p \in C_n) \geq 1 - \alpha$$

for every p . Hence, C_n is a $1 - \alpha$ confidence interval. ■

10.33.3 Construct Normal-based Confidence Interval

Point estimators often have a limite Normal distribution, meaning that $\hat{\theta}_n \sim N(\theta, \hat{s}^2)$. In this case we can sonstrcut (approximate) confidence intervals as follows.

6.16 Theorem (Normal-based Confidence Interval). Suppose that $\hat{\theta}_n \approx N(\theta, \hat{se}^2)$.

Let Φ be the CDF of a standard Normal and let $z_{\alpha/2} = \Phi^{-1}(1 - (\alpha/2))$, that is, $\mathbb{P}(Z > z_{\alpha/2}) = \alpha/2$ and $\mathbb{P}(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$ where $Z \sim N(0, 1)$.

Let

$$C_n = (\hat{\theta}_n - z_{\alpha/2} \hat{se}, \hat{\theta}_n + z_{\alpha/2} \hat{se}). \quad (6.10)$$

Then

$$\mathbb{P}_{\theta}(\theta \in C_n) \rightarrow 1 - \alpha. \quad (6.11)$$

Examples

6.17 Example. Let $X_1, \dots, X_n \sim \text{Bernoulli}(p)$ and let $\hat{p}_n = n^{-1} \sum_{i=1}^n X_i$. Then $\mathbb{V}(\hat{p}_n) = n^{-2} \sum_{i=1}^n \mathbb{V}(X_i) = n^{-2} \sum_{i=1}^n p(1-p) = n^{-2} np(1-p) = p(1-p)/n$. Hence, $se = \sqrt{p(1-p)/n}$ and $\hat{se} = \sqrt{\hat{p}_n(1-\hat{p}_n)/n}$. By the Central Limit Theorem, $\hat{p}_n \approx N(p, \hat{se}^2)$. Therefore, an approximate $1 - \alpha$ confidence interval is

$$\hat{p}_n \pm z_{\alpha/2} \hat{se} = \hat{p}_n \pm z_{\alpha/2} \sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}.$$

Compare this with the confidence interval in example 6.15. The Normal-based interval is shorter but it only has approximately (large sample) correct coverage. ■

Example 4.2.2 (Large Sample Confidence Interval for the Mean μ). Suppose X_1, X_2, \dots, X_n is a random sample on a random variable X with mean μ and variance σ^2 , but, unlike the last example, the distribution of X is not normal. However, from the above discussion we know that the distribution of Z_n , (4.2.4), is approximately $N(0, 1)$. Hence

$$1 - \alpha \approx P_{\mu} \left(-z_{\alpha/2} < \frac{\bar{X} - \mu}{S/\sqrt{n}} < z_{\alpha/2} \right).$$

Using the same algebraic derivation as in the last example, we obtain

$$1 - \alpha \approx P_{\mu} \left(\bar{X} - z_{\alpha/2} \frac{S}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2} \frac{S}{\sqrt{n}} \right). \quad (4.2.5)$$

Again, letting \bar{x} and s denote the realized values of the statistics \bar{X} and S , respectively, after the sample is drawn, an approximate $(1 - \alpha)100\%$ confidence interval for μ is given by

$$(\bar{x} - z_{\alpha/2} s / \sqrt{n}, \bar{x} + z_{\alpha/2} s / \sqrt{n}). \quad (4.2.6)$$

This is called a **large sample** confidence interval for μ . ■

10.33.4 Hypothesis Testing

In **hypothesis testing**, we start with some default theory — called a **null hypothesis** — and we ask if the data provide sufficient evidence to reject the theory. If not we retain the null hypothesis.²

²The term “retaining the null hypothesis” is due to Chris Genovese. Other terminology is “accepting the null” or “failing to reject the null.”

6.18 Example (Testing if a Coin is Fair). Let

$$X_1, \dots, X_n \sim \text{Bernoulli}(p)$$

be n independent coin flips. Suppose we want to test if the coin is fair. Let H_0 denote the hypothesis that the coin is fair and let H_1 denote the hypothesis that the coin is not fair. H_0 is called the **null hypothesis** and H_1 is called the **alternative hypothesis**. We can write the hypotheses as

$$H_0 : p = 1/2 \quad \text{versus} \quad H_1 : p \neq 1/2.$$

It seems reasonable to reject H_0 if $T = |\hat{p}_n - (1/2)|$ is large. When we discuss hypothesis testing in detail, we will be more precise about how large T should be to reject H_0 . ■

10.34 Parametric Inference

See All of Statistic Chap 9

We now turn our attention to parametric models, that is, models of the form

$$\mathfrak{F} = \left\{ f(x; \theta) : \theta \in \Theta \right\} \quad (9.1)$$

where the $\Theta \subset \mathbb{R}^k$ is the parameter space and $\theta = (\theta_1, \dots, \theta_k)$ is the parameter. The problem of inference then reduces to the problem of estimating the parameter θ .

There are two methods for estimating θ , the method of moments and the method of maximum likelihood.

10.34.1 Parameter of Interest

When estimating a parameter θ , we may only be interested in a function $T(\theta)$, called the **parameter of interest**. Other components of θ are **nuisance**

parameters. For instance, if $X \sim N(\mu, \sigma^2)$ and we want to estimate μ , then $\mu = T(\theta)$ is the parameter of interest, with σ being the nuisance parameter. As an example, if $X_1, \dots, X_n \sim \text{Normal}(\mu, \sigma^2)$ and we want to know the fraction τ of the population with test scores larger than 1, then $\tau = T(\mu, \sigma) = 1 - \Phi((1 - \mu)/\sigma)$.

10.34.2 The Method of Moment

Suppose that the parameter $\theta = (\theta_1, \dots, \theta_k)$ has k components. For $1 \leq j \leq k$, define the j^{th} moment

$$\alpha_j \equiv \alpha_j(\theta) = \mathbb{E}_\theta(X^j) = \int x^j dF_\theta(x)$$

and the j^{th} sample moment

$$\hat{\alpha}_j = \frac{1}{n} \sum_{i=1}^n X_i^j.$$

Définition 39: method of moments estimator

The **method of moments estimator** $\hat{\theta}_n$ is defined to be the value of θ such that

$$\begin{aligned} \alpha_1(\hat{\theta}_n) &= \hat{\alpha}_1 \\ \alpha_2(\hat{\theta}_n) &= \hat{\alpha}_2 \\ &\vdots \\ \alpha_k(\hat{\theta}_n) &= \hat{\alpha}_k. \end{aligned} \tag{10.5}$$

Example 1

Let $X_1, \dots, X_n \sim b(1, p)$. Then $\alpha_1(p) = \alpha_1 = \mathbb{E}_p(X) = p$, and $\hat{\alpha}_1 = n^{-1} \sum_{i=1}^n X_i$. By eq. (10.5),

$$\hat{p}_n = \alpha_1(\hat{p}_n) = \hat{\alpha}_1 = n^{-1} \sum_{i=1}^n X_i$$

Example 2

Let $X_1, \dots, X_n \sim N(\mu, \sigma^2)$. Then $\alpha_1 = \mathbb{E}_\theta(X_1) = \mu, \alpha_2 = \mathbb{E}_\theta(X_1^2) = \sigma^2 + \mu^2$. We need to solve the equations

$$\begin{aligned}\hat{\mu} &= n^{-1} \sum_{i=1}^n X_i \\ \hat{\sigma}^2 + \hat{\mu}^2 &= n^{-1} \sum_{i=1}^n X_i^2\end{aligned}$$

The solution is

$$\hat{\mu} = \bar{X}_n \quad \hat{\sigma}^2 = n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$$

Théorème 55: Theorem 9.6

Let $\hat{\theta}_n$ denote the method of moments estimator. Under appropriate conditions on the model, the following statements hold:

- (1) The estimate $\hat{\theta}_n$ exists with probability tending to 1.
- (2) The estimate is consistent: $\hat{\theta}_n \xrightarrow{P} \theta$.
- (3) The estimate is asymptotically Normal:

$$\sqrt{n} (\hat{\theta}_n - \theta) \rightsquigarrow N(0, \Sigma)$$

where

$$\Sigma = g \mathbb{E}_\theta (YY^T) g^T$$

$$Y = (X, X^2, \dots, X^k)^T, g = (g_1, \dots, g_k) \text{ and } g_j = \partial \alpha_j^{-1}(\theta) / \partial \theta$$

10.34.3 Maximum Likelihood Method

Let X_1, \dots, X_n i.i.d. with pdf $f(x; \theta)$.

Définition 40: Likelihood function

The **likelihood function** is defined by

$$\mathcal{L}_n(\theta) = \prod_{i=1}^n f(X_i; \theta)$$

The **log-likelihood function** is defined by $\ell_n(\theta) = \log \mathcal{L}_n(\theta)$.

The likelihood function is just the **joint density of the data** (x_1, \dots, x_n) , except that we treat it is a function of the parameter θ . Thus, $\mathcal{L}_n : \Theta \rightarrow [0, \infty)$.

Définition 41: maximum likelihood estimator

The **maximum likelihood estimator MLE**, denoted by $\hat{\theta}_n$, is the value of θ that maximizes $\mathcal{L}_n(\theta)$.

Note. Often, it is easier to work with the log-likelihood.

Remarque 53. If we multiply $\mathcal{L}_n(\theta)$ by any positive constant c (not depending on θ) then this will not change the mle. Hence, we shall often drop constants in the likelihood function.

Example 1

Let $X_1, \dots, X_n \sim N(\mu, \sigma^2)$. The parameter is $\theta = (\mu, \sigma)$, and the likelihood function (ignoring some constants) is:

$$\begin{aligned}\mathcal{L}_n(\mu, \sigma) &= \prod_i \frac{1}{\sigma} \exp \left\{ -\frac{1}{2\sigma^2} (X_i - \mu)^2 \right\} \\ &= \sigma^{-n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_i (X_i - \mu)^2 \right\} \\ &= \sigma^{-n} \exp \left\{ -\frac{nS^2}{2\sigma^2} \right\} \exp \left\{ -\frac{n(\bar{X} - \mu)^2}{2\sigma^2} \right\}\end{aligned}$$

where $\bar{X} = n^{-1} \sum_i X_i$ is the sample mean and $S^2 = n^{-1} \sum_i (X_i - \bar{X})^2$. The last equality above follows from the fact that $\sum_i (X_i - \mu)^2 = nS^2 + n(\bar{X} - \mu)^2$, which can be verified by writing $\sum_i (X_i - \mu)^2 = \sum_i (X_i - \bar{X} + \bar{X} - \mu)^2$ and then expanding the square.

The log-likelihood is

$$\ell(\mu, \sigma) = -n \log \sigma - \frac{nS^2}{2\sigma^2} - \frac{n(\bar{X} - \mu)^2}{2\sigma^2}$$

Solving the equations

$$\frac{\partial \ell(\mu, \sigma)}{\partial \mu} = 0 \quad \text{and} \quad \frac{\partial \ell(\mu, \sigma)}{\partial \sigma} = 0$$

we conclude that $\hat{\mu} = \bar{X}$ and $\hat{\sigma} = S$. It can be verified that these are indeed global maxima of the likelihood.

Example 2: A Hard Example

Here is an example that many people find confusing. Let $X_1, \dots, X_n \sim \text{Unif}(0, \theta)$. Recall that

$$f(x; \theta) = \begin{cases} 1/\theta & 0 \leq x \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

Consider a fixed value of θ . Suppose $\theta < X_i$ for some i . Then, $f(X_i; \theta) = 0$ and hence $\mathcal{L}_n(\theta) = \prod_i f(X_i; \theta) = 0$. It follows that $\mathcal{L}_n(\theta) = 0$ if any $X_i > \theta$. Therefore, $\mathcal{L}_n(\theta) = 0$ if $\theta < X_{(n)}$ where $X_{(n)} = \max\{X_1, \dots, X_n\}$.

Now consider any $\theta \geq X_{(n)}$. For every X_i we then have that $f(X_i; \theta) = 1/\theta$ so that $\mathcal{L}_n(\theta) = \prod_i f(X_i; \theta) = \theta^{-n}$. In conclusion,

$$\mathcal{L}_n(\theta) = \begin{cases} \left(\frac{1}{\theta}\right)^n & \theta \geq X_{(n)} \\ 0 & \theta < X_{(n)} \end{cases}$$

Now $\mathcal{L}_n(\theta)$ is strictly decreasing over the interval $[X_{(n)}, \infty)$. Hence, $\hat{\theta}_n = X_{(n)}$.

10.34.4 Properties of Maximum Likelihood Estimators

The MLE $\hat{\theta}_n$ possesses many properties that make it an appealing choice choice of estimator

- (1) The MLE is **consistent**: $\hat{\theta}_n \xrightarrow{P} \theta_*$ where θ_* denotes the true value of the parameter θ ;
- (2) The MLE is **equivariant**: if $\hat{\theta}_n$ is the MLE of θ then $g(\hat{\theta}_n)$ is the MLE of $g(\theta)$;⁴
- (3) The MLE is **asymptotically Normal**: $(\hat{\theta} - \theta_*) / \hat{s}_e \rightsquigarrow N(0, 1)$; also, the estimated standard error s_e can often be computed analytically;
- (4) The MLE is **asymptotically optimal** or **efficient**: roughly, this means that among all well-behaved estimators, the MLE has the smallest variance, at least for large samples;
- (5) The MLE is approximately the Bayes estimator. (This point will be explained later.)

⁴when g admits an inverse.

In sufficiently complicated problems, the MLE will no longer be a good estimator.

We focus on the simpler situations where the MLE works well, assuming that the model satisfies certain regularity conditions, which are essentially smoothness conditions on $f(x; \theta)$.

10.34.5 Consistency of Maximum Likelihood Estimators

Consistency means that the mle converges in probability to the true value.

Définition 42: Kullback-Leibler distance

If f and g are pdf's, define the **Kullback-Leibler distance**^a between f and g to be

$$D(f, g) = \int f(x) \log \left(\frac{f(x)}{g(x)} \right) dx$$

It can be shown that $D(f, g) \geq 0$ and $D(f, f) = 0$. For any $\theta, \psi \in \Theta$ write $D(\theta, \psi)$ to mean $D(f(x; \theta), f(x; \psi))$.

^aThis is not a distance in the formal sense because $D(f, g)$ is not symmetric.

We will say that the model \mathfrak{F} is **identifiable** if $\theta \neq \psi$ implies that $D(\theta, \psi) > 0$. This means that different values of the parameter correspond to different distributions.

Théorème 56

Let θ_* denote the true value of θ . Define

$$M_n(\theta) = \frac{1}{n} \sum_i \log \frac{f(X_i; \theta)}{f(X_i; \theta_*)}$$

and $M(\theta) = -D(\theta_*, \theta)$. Suppose that

$$\sup_{\theta \in \Theta} |M_n(\theta) - M(\theta)| \xrightarrow{\text{P}} 0$$

and that, for every $\epsilon > 0$,

$$\sup_{\theta: |\theta - \theta_*| \geq \epsilon} M(\theta) < M(\theta_*) .$$

Let $\hat{\theta}_n$ denote the MLE. Then $\hat{\theta}_n \xrightarrow{\text{P}} \theta_*$.

10.34.6 Asymptotic Normality

10.35 The standard machine: approximate Borel function by simple functions

If we want to verify a property that holds for general Borel-measurable function f , we can follow four steps as below.

- Verify the property when f is indicator function.
- Verify the property when f is nonnegative simple function.
- Verify the property when f is Borel-measurable function
- Verify the property when f is General Borel-measurable function

An example is as follow.

10.35.1 Example

Théorème 57

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let g be a Borel-measurable function on \mathbb{R} . Then

$$\mathbb{E}|g(X)| = \int_{\mathbb{R}} |g(x)| d\mu_X(x)$$

and if this quantity is finite, then

$$\mathbb{E}g(X) = \int_{\mathbb{R}} g(x) d\mu_X(x)$$

证明. Step 1. Indicator functions. (Omitted)

Step 2. Nonnegative simple functions. (Trivial because of the linearity)

Step 3. Nonnegative Borel-measurable functions. Let $g(x)$ be an arbitrary nonnegative Borel-measurable function defined on \mathbb{R} . For each positive integer n , define the sets

$$B_{k,n} = \left\{ x; \frac{k}{2^n} \leq g(x) < \frac{k+1}{2^n} \right\}, \quad k = 0, 1, 2, \dots, 4^n - 1$$

Remarque 54. 这样的定义是为了保证在 $n \rightarrow \infty$ 时，每个划分越来越细，同时 $\bigcup_{k=1}^{4^n-1} B_{k,n} \rightarrow [0, +\infty)$ 。而且对于任意 n , $\{B_{k,n+1}\}$ 是 $\{B_{k,n}\}$ 的加细 ($B_{k,n}$ 的划分点都包含在 $B_{k,n+1}$ 的划分点集中)

For each fixed n , the sets $B_{0,n}, B_{1,n}, \dots, B_{4^n-1,n}$ correspond to the partition

$$0 < \frac{1}{2^n} < \frac{2}{2^n} < \cdots < \frac{4^n}{2^n} = 2^n.$$

At the next stage $n+1$, the partition points include all those at stage n and new partition points at the midpoints between the old ones. Because of this fact, the simple functions

$$g_n(x) = \sum_{k=0}^{4^n-1} \frac{k}{2^n} \mathbb{I}_{B_{k,n}}(x)$$

satisfy $0 \leq g_1 \leq g_2 \leq \cdots \leq g$. Furthermore, these functions become more and more accurate approximations of g as n becomes larger; indeed $\lim_{n \rightarrow \infty} g_n(x) = g(x)$ for every $x \in \mathbb{R}$. From Step 2, we know that

$$\mathbb{E}g_n(X) = \int_{\mathbb{R}} g_n(x) d\mu_X(x)$$

for every n . Letting $n \rightarrow \infty$ and using the Monotone Convergence Theorem, on both sides of the equation, we obtain

$$\mathbb{E}g(X) = \lim_{n \rightarrow \infty} \mathbb{E}g_n(X) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} g_n(x) d\mu_X(x) = \int_{\mathbb{R}} g(x) d\mu_X(x)$$

This proves when g is a nonnegative Borel-measurable function.

Step 4. General Borel-measurable function. (consider $g = g^+ - g^-$ where g^+ and g^- are both nonnegative Borel-measurable functions)

□

10.36 引言：我们为什么要学习方差分析？

同学们好！在之前的学习中，我们已经掌握了如何对一个或两个总体的均值进行假设检验（例如，使用 t 检验）。但是，在现实世界中，我们经常需要比较的不是两个，而是三个、四个甚至更多的组。比如：

- **农业：**比较四种不同肥料对小麦产量的影响。
- **医学：**评估三种不同药物对降低血压的效果。
- **制造业：**检验五条不同生产线的次品率是否存在显著差异。

面对这些“多组比较”的问题，如果我们两两使用 t 检验，不仅计算繁琐，更重要的是会大大增加犯“第一类错误”的概率。想象一下，如果有 5 个组，你需要做 $C_5^2 = 10$ 次 t 检验，每次检验的显著性水平为 $\alpha = 0.05$ ，那么至少犯一次错误的概率将远大于 0.05。

为了解决这个问题，统计学大师费希尔（R.A. Fisher）开发了一种强大的工具——方差分析（Analysis of Variance, ANOVA）。它的核心思想非常巧妙：通过分析数据的总变异来源，来推断各组的均值是否存在显著差异。

本章，我们将系统地学习方差分析与回归分析，而今天的重点是单因子方差分析。

10.37 §8.1 单因子方差分析 (One-Way ANOVA)

10.37.1 8.1.1 核心问题与关键示例

定义：什么是“单因子”？

在方差分析中，我们把研究中影响我们观察结果的“原因”或“条件”称为因子（Factor）。而因子的不同状态或类别，称为水平（Level）。

- **单因子方差分析：**指的是我们只关心一个因素对结果的影响。

关键示例：三种鸡饲料的增重效果比较

为了让大家更好地理解，我们贯穿始终地使用教材中的这个经典例子（例 8.1.1）。

- **研究目的：**比较三种不同饲料对雏鸡增肥效果的影响。
- **因子 (Factor)：**饲料类型（记为因子 A ）。这是我们研究的唯一影响因素。
- **水平 (Levels)：**因子 A 有 3 个水平，分别是：
 - A_1 : 鱼粉为主的饲料
 - A_2 : 槐米粉为主的饲料
 - A_3 : 莜蓿粉为主的饲料
- **试验数据：**选择了 24 只雏鸡，随机均分为三组，每组 8 只，分别喂养三种饲料。60 天后，记录它们的质量（单位：g）。如下表所示：

我们的核心问题是：这三种饲料的增重效果真的有区别吗？还是说，观察到的数据差异仅仅是由随机波动引起的？这就是方差分析要回答的问题。

表 10.1: 鸡饲料试验数据

饲料水平		鸡的质量 (g)							
A_1	1073	1009	1060	1001	1002	1012	1009	1028	
A_2	1107	1092	990	1109	1090	1074	1122	1001	
A_3	1093	1029	1080	1021	1022	1032	1029	1048	

10.37.2 8.1.2 单因子方差分析的统计模型

为了用数学语言来描述这个问题，我们需要建立一个统计模型。这个模型是进行一切分析的基础。

模型的三大基本假定

在进行方差分析前，我们必须做出以下三个重要的假定。这些假定是模型成立的前提，在实际应用中需要进行检验。

- (1) 正态性 (Normality): 每一个水平下的观测数据都来自一个正态分布总体。即，对于第 i 个水平， $Y_{ij} \sim N(\mu_i, \sigma^2)$ 。
- (2) 方差齐性 (Homoscedasticity): 所有总体的方差都是相同的，即 $\sigma_1^2 = \sigma_2^2 = \dots = \sigma_r^2 = \sigma^2$ 。
- (3) 独立性 (Independence): 所有观测值都是相互独立的。这通常通过实验的随机化设计来保证。

模型的建立：效应模型

我们将第 i 个水平下的第 j 次观测值记为 y_{ij} 。这个观测值可以被分解为几个部分。

首先，一个直观的模型是：

$$y_{ij} = \mu_i + \varepsilon_{ij}$$

其中， μ_i 是第 i 个水平的真实平均效应（比如，长期使用 A_1 饲料的鸡的平均体重），而 ε_{ij} 是随机误差，它代表了除了因子水平之外，其他所有偶然因素

(如个体差异、测量误差等) 带来的影响。根据我们的假定, ε_{ij} 相互独立, 且都服从 $N(0, \sigma^2)$ 。

为了更好地分析“差异”, 我们引入总均值和水平效应的概念, 将模型进一步深化。

- **总均值 μ :** 所有水平真实均值的平均值, $\mu = \frac{1}{r} \sum_{i=1}^r \mu_i$ 。它代表了本次试验所有条件下的一个“一般水平”。
- **水平效应 a_i :** 第 i 个水平的特殊效应, 定义为该水平的均值与总均值的离差, 即 $a_i = \mu_i - \mu$ 。它反映了第 i 个处理相比于“一般水平”的优劣程度。

一个重要的性质是: 所有水平效应的总和为零, $\sum_{i=1}^r a_i = 0$ 。

由此, 我们可以得到单因子方差分析的效应模型 (Effects Model):

$$y_{ij} = \mu + a_i + \varepsilon_{ij}$$

这个模型非常优雅地告诉我们: 任何一次观测值 = 整体平均水平 + 该处理的特殊效应 + 随机误差。

假设检验的数学表达

有了模型, 我们就可以把最初模糊的问题“三种饲料效果是否相同”, 转化为一个精确的数学假设。

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_r$$

$$H_1 : \mu_1, \mu_2, \dots, \mu_r \text{ 不全相等}$$

利用效应模型, 这个假设可以等价地写成:

$$H_0 : a_1 = a_2 = \cdots = a_r = 0$$

$$H_1 : a_1, a_2, \dots, a_r \text{ 不全为 } 0$$

原假设 H_0 成立, 意味着所有水平的特殊效应都是 0, 因子 A 的不同水平之间没有显著差异。反之, H_1 成立则意味着至少有一个水平的效应与其他水平不同。

10.37.3 8.1.3 思想核心: 平方和分解 (Sum of Squares Decomposition)

方差分析的精髓在于, 它将数据总的变异“分解”为来自不同源头的部分。

三种变异的来源

我们先定义一些基本符号，假设有 r 个水平，每个水平有 m 次观测。

- $\bar{y}_{i\cdot}$: 第 i 组的样本均值, $\bar{y}_{i\cdot} = \frac{1}{m} \sum_{j=1}^m y_{ij}$
- $\bar{y}_{..}$: 所有数据的总样本均值, $\bar{y}_{..} = \frac{1}{rm} \sum_{i=1}^r \sum_{j=1}^m y_{ij}$

总的变异，可以用每个数据点到总均值的离差平方和来度量，称为总平方和 (Total Sum of Squares, SST)。

$$\text{SST} = \sum_{i=1}^r \sum_{j=1}^m (y_{ij} - \bar{y}_{..})^2$$

SST 反映了全部数据的离散程度。

现在，我们来看这个总变异是由什么构成的。通过一个简单的代数恒等式：

$$(y_{ij} - \bar{y}_{..}) = (y_{ij} - \bar{y}_{i\cdot}) + (\bar{y}_{i\cdot} - \bar{y}_{..})$$

将这个式子两边平方再求和，经过推导可以得到一个非常关键的定理。

定理：平方和分解定理

$$\sum_{i=1}^r \sum_{j=1}^m (y_{ij} - \bar{y}_{..})^2 = \sum_{i=1}^r \sum_{j=1}^m (y_{ij} - \bar{y}_{i\cdot})^2 + m \sum_{i=1}^r (\bar{y}_{i\cdot} - \bar{y}_{..})^2$$

这个公式可以简洁地写成：

$$\text{SST} = \text{SSE} + \text{SSA}$$

我们来解读一下这个核心公式：

- **SST (Total Sum of Squares)**: 总离差平方和。
- **SSE (Error Sum of Squares)**: 误差平方和，也叫组内平方和。

$$\text{SSE} = \sum_{i=1}^r \sum_{j=1}^m (y_{ij} - \bar{y}_{i\cdot})^2$$

它度量的是每个组内部的数据波动。因为组内所有成员接受的处理是相同的，所以这种波动只能归因于随机误差。

- **SSA (Factor A Sum of Squares)**: 因子 A 平方和，也叫组间平方和。

$$\text{SSA} = m \sum_{i=1}^r (\bar{y}_{i\cdot} - \bar{y}_{..})^2$$

它度量的是各组的均值与总均值之间的差异。这种差异可能来自两方面：一是因子 A 的水平效应（如果 H_0 为假），二是随机误差。

知识脉络解读：这个分解是方差分析的灵魂！它把一个复杂的问题（总变异 SST）拆解成了两个有明确来源的部分：

- (1) **组内变异 SSE**: 纯粹由随机性导致。
- (2) **组间变异 SSA**: 可能由“处理效应 + 随机性”导致。

接下来，我们的逻辑就是：如果组间变异“显著地”大于组内变异，我们就有理由相信，这种超出部分是由处理效应引起的，从而拒绝原假设 H_0 。

10.37.4 方差分析表与 F 检验

如何衡量“显著地大”？我们需要构建一个检验统计量。

均方 (Mean Square)

为了消除样本量对平方和大小的影响，我们用平方和除以其对应的自由度 (degrees of freedom, df)，得到均方 (Mean Square, MS)。

- 因子 A 的均方 (MSA): $MSA = \frac{SSA}{df_A} = \frac{SSA}{r-1}$
- 误差均方 (MSE): $MSE = \frac{SSE}{df_E} = \frac{SSE}{r(m-1)}$

MSA 可以看作是组间变异的平均度量，MSE 可以看作是组内变异的平均度量，并且可以证明，MSE 是总体方差 σ^2 的无偏估计，即 $E(MSE) = \sigma^2$ 。

F-统计量的构建

我们构造如下的 F 统计量：

$$F = \frac{MSA}{MSE}$$

这个统计量的直观意义是：

$$F = \frac{\text{(可能由处理效应 + 随机误差引起的) 组间变异}}{\text{(仅由随机误差引起的) 组内变异}}$$

- 如果原假设 H_0 为真（即 $a_i = 0$ ），那么 MSA 和 MSE 都只反映了随机误差，它们的期望是相等的，所以 F 的值应该在 1 附近。
- 如果原假设 H_0 为假（即 a_i 不全为 0），那么 MSA 的期望会大于 MSE 的期望（因为它包含了处理效应），所以 F 的值会倾向于变得比较大。

可以证明，在 H_0 成立的条件下，该统计量服从 F 分布：

$$F = \frac{MSA}{MSE} \sim F(r-1, r(m-1))$$

其中 $df_1 = r - 1$ 是第一自由度（分子的自由度）， $df_2 = r(m - 1)$ 是第二自由度（分母的自由度）。

方差分析表 (ANOVA Table)

我们通常把所有的计算结果整理在一张标准化的表格里，这就是方差分析表。

表 10.2: 单因子方差分析表

变异来源	平方和 (SS)	自由度 (df)	均方 (MS)	F 统计量
组间 (因子 A)	SSA	$r - 1$	MSA	$F = \frac{MSA}{MSE}$
组内 (误差)	SSE	$r(m - 1)$	MSE	
总计	SST	$rm - 1$		

决策规则：给定显著性水平 α (如 0.05)，我们去查 F 分布表，得到一个临界值 $F_\alpha(r - 1, r(m - 1))$ 。

- 如果计算得到的 $F > F_\alpha(r - 1, r(m - 1))$ ，则拒绝 H_0 ，认为各水平的效应有显著差异。
- 如果 $F \leq F_\alpha(r - 1, r(m - 1))$ ，则不拒绝 H_0 ，尚无足够证据认为各水平效应有差异。

在软件输出中，我们通常直接看 P-value。如果 P-value $< \alpha$ ，则拒绝 H_0 。

10.38 计算实践：鸡饲料案例全流程分析

理论是指导，实践是目的。现在我们来对鸡饲料数据进行一次完整的方差分析。

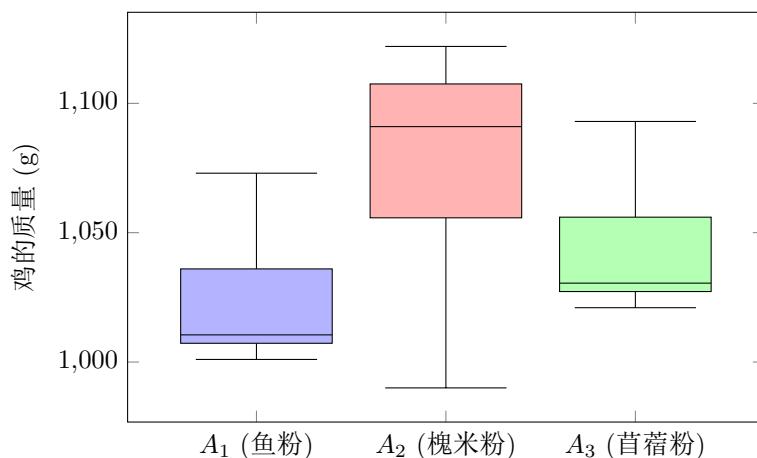
10.38.1 数据回顾与预处理

首先，让我们回顾一下我们的数据：

表 10.3: 鸡饲料试验数据回顾

饲料水平	鸡的质量 (g)							
A_1 (鱼粉)	1073	1009	1060	1001	1002	1012	1009	1028
A_2 (槐米粉)	1107	1092	990	1109	1090	1074	1122	1001
A_3 (苜蓿粉)	1093	1029	1080	1021	1022	1032	1029	1048

三种饲料条件下鸡质量的箱线图



从箱线图中，我们可以直观地看到三种饲料组的分布情况。 A_2 组（槐米粉）的均值似乎高于其他两组，但我们需要通过方差分析来确定这种差异是否具有统计显著性。

10.38.2 第一步：计算基本统计量

首先，我们需要计算各组的总和、均值，以及所有数据的总和、总均值。

- 因子 A 的水平数 $r = 3$ 。
- 每个水平的重复数 $m = 8$ 。
- 总观测数 $n = rm = 24$ 。

数据汇总表：

表 10.4: 鸡饲料试验的基本统计量

饲料水平	样本量	总和	均值	样本方差
A_1 (鱼粉)	8	8194	1024.25	902.5
A_2 (槐米粉)	8	8585	1073.125	1793.8
A_3 (苜蓿粉)	8	8354	1038.00	942.9
总计	24	25083	1045.125	-

各组总和 $T_{i\cdot} = \sum_{j=1}^8 y_{ij}$ 与均值 $\bar{y}_{i\cdot} = T_{i\cdot}/8$:

$$T_{1\cdot} = 1073 + 1009 + 1060 + 1001 + 1002 + 1012 + 1009 + 1028 = 8194$$

$$\bar{y}_{1\cdot} = 8194/8 = 1024.25$$

$$T_{2\cdot} = 1107 + 1092 + 990 + 1109 + 1090 + 1074 + 1122 + 1001 = 8585$$

$$\bar{y}_{2\cdot} = 8585/8 = 1073.125$$

$$T_{3\cdot} = 1093 + 1029 + 1080 + 1021 + 1022 + 1032 + 1029 + 1048 = 8354$$

$$\bar{y}_{3\cdot} = 8354/8 = 1038$$

总和 $T = \sum T_{i\cdot}$ 与总均值 $\bar{y}_{..} = T/n$:

$$T = 8194 + 8585 + 8354 = 25083$$

$$\bar{y}_{..} = 25083/24 \approx 1045.125$$

10.38.3 第二步：计算平方和 (SS)

这是计算的核心部分。为了简化手算，通常会使用所谓的“修正项”法，这里我们直接使用定义式来突出概念。

原始数据及均值再次提示：

- 总均值 $\bar{y}_{..} = 1045.125$
- 组 1 均值 $\bar{y}_{1\cdot} = 1024.25$
- 组 2 均值 $\bar{y}_{2\cdot} = 1073.125$
- 组 3 均值 $\bar{y}_{3\cdot} = 1038$

- 总平方和 (SST):

$$\begin{aligned} SST &= \sum_{i=1}^3 \sum_{j=1}^8 (y_{ij} - \bar{y}_{..})^2 \\ &= (1073 - 1045.125)^2 + (1009 - 1045.125)^2 + \dots + (1048 - 1045.125)^2 \\ &= 28638.96 \end{aligned}$$

- 组间平方和 (SSA):

$$\begin{aligned} SSA &= m \sum_{i=1}^3 (\bar{y}_{i\cdot} - \bar{y}_{..})^2 \\ &= 8 \times [(1024.25 - 1045.125)^2 + (1073.125 - 1045.125)^2 + (1038 - 1045.125)^2] \\ &= 8 \times [(-20.875)^2 + (28.0)^2 + (-7.125)^2] \\ &= 8 \times [435.76 + 784 + 50.76] = 8 \times 1270.52 = 10164.16 \end{aligned}$$

- 组内平方和 (SSE): 我们可以利用 $SST = SSA + SSE$ 来计算。

$$\begin{aligned} SSE &= SST - SSA \\ &= 28638.96 - 10164.16 = 18474.8 \end{aligned}$$

10.38.4 第三步：构建方差分析表并进行 F 检验

现在，我们将所有计算结果填入方差分析表。

已计算的关键值汇总：

- 总平方和: $SST = 28638.96$
- 组间平方和: $SSA = 10164.16$
- 组内平方和: $SSE = 18474.8$

- 自由度:

- 组间自由度 $df_A = r - 1 = 3 - 1 = 2$
- 组内自由度 $df_E = r(m - 1) = 3(8 - 1) = 21$
- 总自由度 $df_T = n - 1 = 24 - 1 = 23$ (检查: $2 + 21 = 23$)

- 均方 (MS):

- $MSA = SSA / df_A = 10164.16 / 2 = 5082.08$
- $MSE = SSE / df_E = 18474.8 / 21 = 879.75$

- F 统计量:

$$\circ F = MSA/MSE = 5082.08/879.75 \approx 5.78$$

完整的方差分析表：

表 10.5: 鸡饲料案例的方差分析表

变异来源	平方和 (SS)	自由度 (df)	均方 (MS)	F 统计量
饲料 (组间)	10164.16	2	5082.08	5.78
误差 (组内)	18474.80	21	879.75	
总计	28638.96	23		

10.38.5 第四步：统计决策与结论

- 确定显著性水平: 我们取 $\alpha = 0.05$ 。
- 查找临界值: 我们需要查找 F 分布表中对应于 $F_{0.05}(2, 21)$ 的临界值。查表可得, $F_{0.05}(2, 21) = 3.47$ 。
- 比较与决策: 我们计算得到的 F 值为 5.78。因为 $5.78 > 3.47$, 所以我们的检验统计量落在了拒绝域中。因此, 我们拒绝原假设 H_0 。

结论: 在 $\alpha = 0.05$ 的显著性水平下, 我们有充分的统计证据表明, 三种饲料配方对雏鸡的增重效果存在显著差异, 效果不全相同。

10.38.6 分析后的思考

方差分析告诉我们“有差异”, 但它没有告诉我们“谁和谁有差异”。我们只知道这三种饲料不全相同, 但具体是 A_2 显著优于 A_1 和 A_3 吗? 还是三者两两之间都有显著差异? 要回答这些更具体的问题, 就需要进行多重比较 (Multiple Comparisons) 或事后检验 (Post-Hoc Tests), 比如 Tukey's HSD 检验或者 Bonferroni 校正等。这通常是方差分析之后的下一步探索。

10.39 §8.2 多重比较 (Multiple Comparisons)

10.39.1 引言: F 检验之后, 我们该做什么?

同学们, 在上一节的鸡饲料案例中, F 检验告诉我们一个重要的结论: 三种饲料的增重效果不全相同。这是一个“总体性”的结论。但这并没有回答我们更关心的问题:

- 究竟是哪几种饲料之间存在差异?
- 是 A_2 比 A_1 好? 还是 A_2 比 A_3 好? 或者两者都成立?
- A_1 和 A_3 之间有差异吗?

为了回答这些问题, 我们需要在 F 检验显著后, 进行所谓的多重比较或事后检验 (Post-hoc test)。它的目的是在控制总体错误率的前提下, 对所有处理均值进行两两比较。

一个常见的误区是: 为什么不直接对每一对都做 t 检验呢? 我们在引言中提过, 这样做会急剧增加犯第一类错误的概率。多重比较方法就是为了解决这个问题而设计的。

10.39.2 单对均值差的置信区间

在深入多重比较方法前, 我们首先要掌握如何对指定的一对均值差 (如 $\mu_i - \mu_j$) 进行区间估计。这是 t 检验思想在方差分析框架下的延伸。

Théorème 58: 均值差的 t 分布

在方差分析的基本假定下, 对于任意两组 i 和 j , 有

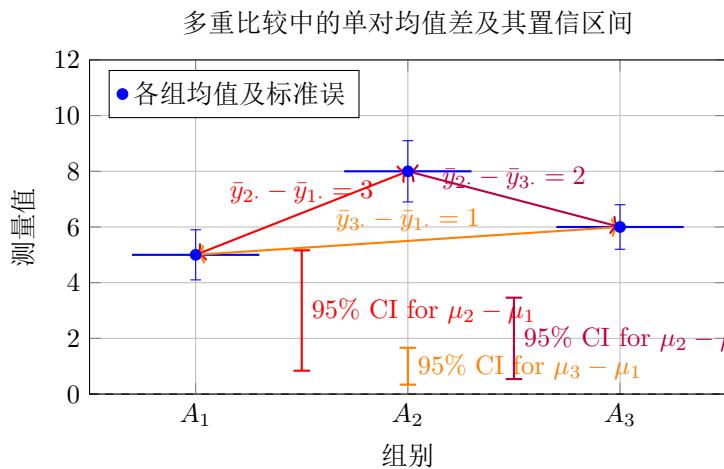
$$\frac{(\bar{y}_{i\cdot} - \bar{y}_{j\cdot}) - (\mu_i - \mu_j)}{\sqrt{(\frac{1}{m_i} + \frac{1}{m_j})MSE}} \sim t(f_e)$$

其中 f_e 是误差自由度, m_i, m_j 是两组的样本量, MSE 是均方误差。

根据此定理, 我们可以推导出 $\mu_i - \mu_j$ 的 $1 - \alpha$ 置信区间:

$$\left[(\bar{y}_{i\cdot} - \bar{y}_{j\cdot}) \pm t_{\alpha/2}(f_e) \sqrt{MSE(\frac{1}{m_i} + \frac{1}{m_j})} \right] \quad (10.6)$$

注意: 这个公式与我们之前学的双样本 t 检验非常相似, 但关键区别在于, 这里的方差估计 MSE 使用了所有组的信息, 因此更稳健。



图解说明：

- 图中展示了三个组 (A_1, A_2, A_3) 的样本均值及其标准误（误差条）。
- 三条彩色箭头表示三对可能的均值差： $\bar{y}_{2\cdot} - \bar{y}_{1\cdot}$, $\bar{y}_{3\cdot} - \bar{y}_{1\cdot}$, $\bar{y}_{2\cdot} - \bar{y}_{3\cdot}$ 。
- 图下方的三条线段表示对应的 95% 置信区间。注意到：
 - $\mu_2 - \mu_1$ 的置信区间不包含 0，说明这两组的差异在统计上是显著的。
 - $\mu_3 - \mu_1$ 的置信区间包含 0，说明这两组的差异在统计上不显著。
 - $\mu_2 - \mu_3$ 的置信区间不包含 0，说明这两组的差异在统计上是显著的。

然而，如果我们对所有可能的组合都使用这个公式，就会遇到第一类错误率累积的问题。这正是多重比较方法要解决的核心。

10.39.3 Tukey's HSD (Honestly Significant Difference) 方法

Tukey's HSD 法，简称 T 法，是当各组样本容量相等时最常用的一种多重比较方法。

核心思想与统计量

T 法引入了一个新的统计量，称为学生化极差统计量 (Studentized Range Statistic)，记为 q 。它的核心思想是，我们不再孤立地看某两组均值的差，而是看所有组均值中的最大值与最小值之差。我们计算一个“诚实显著性差异” (HSD) 的临界值，任何两组均值差的绝对值如果超过这个临界值，就被认为是显著的。

计算步骤 (以样本量相等为例)

(1) 确定参数:

- 显著性水平 α (例如 0.05)
- 处理组数量 r
- 误差自由度 $df_E = r(m - 1)$

(2) 查找临界值: 查阅学生化极差分布表 (q-table), 找到临界值 $q_\alpha(r, df_E)$ 。

(3) 计算 HSD 临界值:

$$HSD = q_\alpha(r, df_E) \sqrt{\frac{MSE}{m}} \quad (10.7)$$

其中 MSE 是我们从 ANOVA 表中得到的均方误差, m 是每组的样本量。

(4) 进行比较: 比较任意两组的均值差 $|\bar{y}_i - \bar{y}_j|$ 与 HSD 的大小。

- 如果 $|\bar{y}_i - \bar{y}_j| \geq HSD$, 则我们宣布 μ_i 和 μ_j 之间存在显著差异。
- 如果 $|\bar{y}_i - \bar{y}_j| < HSD$, 则我们认为两组均值之间没有显著差异。

应用: 鸡饲料案例的多重比较

让我们回到鸡饲料的例子:

- $\alpha = 0.05, r = 3, m = 8, df_E = 21, MSE = 879.75$ 。
- 查 q 表得 $q_{0.05}(3, 21) \approx 3.58$ 。
- 计算 HSD:

$$HSD = 3.58 \times \sqrt{\frac{879.75}{8}} \approx 3.58 \times 10.486 = 37.54$$

• 比较均值差:

- $|\bar{y}_1 - \bar{y}_2| = |1024.25 - 1073.125| = 48.875 > 37.54$ (显著)
- $|\bar{y}_1 - \bar{y}_3| = |1024.25 - 1038| = 13.75 < 37.54$ (不显著)
- $|\bar{y}_2 - \bar{y}_3| = |1073.125 - 1038| = 35.125 < 37.54$ (不显著)

多重比较结论: 通过 Tukey's HSD 检验, 我们可以更精确地指出: 饲料 A_2 (槐米粉) 的增重效果显著优于饲料 A_1 (鱼粉)。而 A_1 与 A_3 (苜蓿粉), 以及 A_2 与 A_3 之间的效果差异在统计上并不显著。这为我们选择饲料提供了更具体的指导。

10.39.4 Scheffé 法 (S 法)

当各组样本容量不相等时, Tukey's HSD 不再适用。此时, 我们可以使用更为通用的 Scheffé 法。S 法是所有多重比较方法中最保守的, 即最不容易发现显著差异, 但它的优点是可以用于检验样本均值的任意线性组合, 功能更强大。

理论基础

Scheffé 法基于 F 分布, 其理论基础是: 对于任意的线性组合 $L = \sum_{i=1}^r c_i \mu_i$ (其中 $\sum c_i = 0$), 有:

$$P\left(\frac{(L - \hat{L})^2}{\text{MSE} \cdot \sum_{i=1}^r \frac{c_i^2}{m_i}} < (r-1)F_\alpha(r-1, f_e), \forall L\right) = 1 - \alpha \quad (10.8)$$

其中 $\hat{L} = \sum_{i=1}^r c_i \bar{y}_{i\cdot}$ 。

对于均值差 $\mu_i - \mu_j$, 我们可以将其表示为线性组合, 其中 $c_i = 1, c_j = -1$, 其他 $c_k = 0$ 。

检验方法与计算步骤

对于任意两组 i 和 j 的均值差, 我们计算一个临界值 S_{ij} :

$$S_{ij} = \sqrt{(r-1)F_\alpha(r-1, f_e)} \cdot \sqrt{\text{MSE}\left(\frac{1}{m_i} + \frac{1}{m_j}\right)} \quad (10.9)$$

其中, $F_\alpha(r-1, f_e)$ 是 F 分布在上 α 分位点的值。

决策规则:

- 如果 $|\bar{y}_{i\cdot} - \bar{y}_{j\cdot}| \geq S_{ij}$, 则认为 μ_i 和 μ_j 之间存在显著差异。
- 否则, 认为两者无显著差异。

同样地, Scheffé 法也可以用来构造置信区间。对于任意两组 i 和 j , $\mu_i - \mu_j$ 的 $(1 - \alpha)$ 置信区间为:

$$(\bar{y}_{i\cdot} - \bar{y}_{j\cdot}) \pm S_{ij} \quad (10.10)$$

示例计算: 样本量不等的情况

假设有三个组, 样本量分别为 $m_1 = 5, m_2 = 8, m_3 = 6$, 组均值分别为 $\bar{y}_1 = 10.2, \bar{y}_2 = 12.5, \bar{y}_3 = 9.8$, $\text{MSE} = 4.5$, 误差自由度 $f_e = 16$ 。

使用 Scheffé 法进行多重比较 ($\alpha = 0.05$):

(1) 查找 F 分布临界值: $F_{0.05}(2, 16) = 3.63$

(2) 计算三对比较的 Scheffé 临界值:

$$\begin{aligned} S_{12} &= \sqrt{(3-1) \times 3.63} \times \sqrt{4.5 \times \left(\frac{1}{5} + \frac{1}{8}\right)} \\ &= \sqrt{7.26} \times \sqrt{4.5 \times 0.325} \\ &= 2.695 \times 1.208 = 3.256 \end{aligned}$$

$$\begin{aligned} S_{13} &= \sqrt{(3-1) \times 3.63} \times \sqrt{4.5 \times \left(\frac{1}{5} + \frac{1}{6}\right)} \\ &= 2.695 \times \sqrt{4.5 \times 0.367} \\ &= 2.695 \times 1.284 = 3.460 \end{aligned}$$

$$\begin{aligned} S_{23} &= \sqrt{(3-1) \times 3.63} \times \sqrt{4.5 \times \left(\frac{1}{8} + \frac{1}{6}\right)} \\ &= 2.695 \times \sqrt{4.5 \times 0.292} \\ &= 2.695 \times 1.146 = 3.088 \end{aligned}$$

(3) 计算各对均值差的绝对值并与相应的 S_{ij} 比较:

$$|\bar{y}_{1\cdot} - \bar{y}_{2\cdot}| = |10.2 - 12.5| = 2.3 < S_{12} = 3.256 \quad (\text{不显著})$$

$$|\bar{y}_{1\cdot} - \bar{y}_{3\cdot}| = |10.2 - 9.8| = 0.4 < S_{13} = 3.460 \quad (\text{不显著})$$

$$|\bar{y}_{2\cdot} - \bar{y}_{3\cdot}| = |12.5 - 9.8| = 2.7 < S_{23} = 3.088 \quad (\text{不显著})$$

多重比较结论: 根据 Scheffé 法, 在 $\alpha = 0.05$ 的显著性水平下, 三组均值之间的差异均不显著。注意到, 如果使用单对 t 检验, 可能会得出不同结论, 因为 Scheffé 法考虑了多重比较问题, 更为保守。

与 Tukey 法的比较

当样本容量相等时, Tukey 法和 Scheffé 法都可以使用。一般来说, Tukey 法更有效 (检验功效更高), 而 Scheffé 法则更为保守。这是因为:

- Tukey 法: 专门针对均值间的两两比较设计, 临界值基于学生化极差分布。
- Scheffé 法: 能够处理任意线性组合的比较, 不仅限于两两比较, 因此必须“更为”保守”来控制总体错误率。

判断选择：

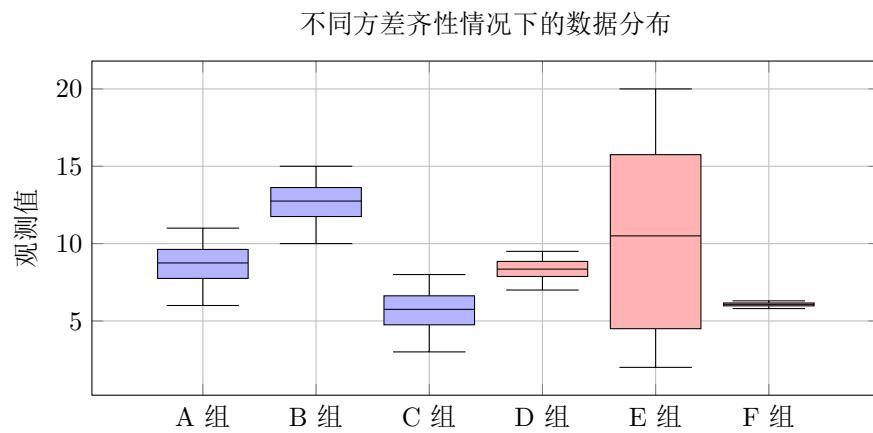
- 如果只关心均值的两两比较且样本量相等，选择 Tukey 法。
- 如果样本量不等或需要检验复杂的线性组合（如“第一组均值是否等于第二组和第三组均值的平均”），选择 Scheffé 法。

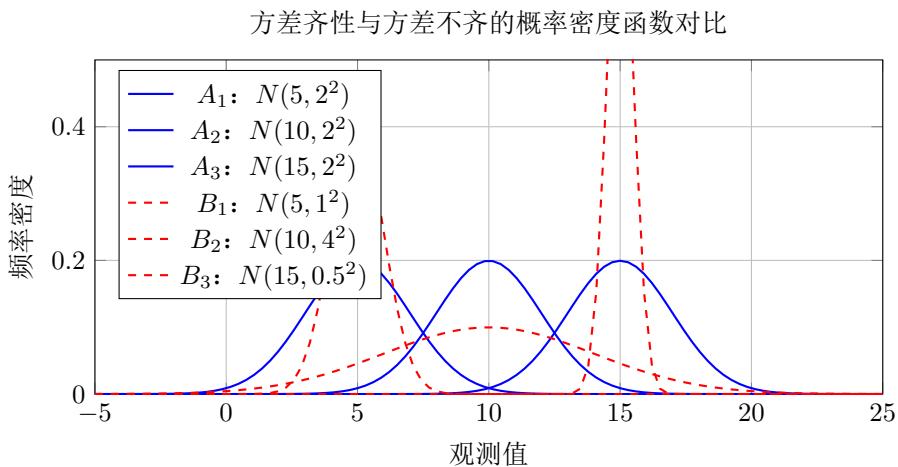
10.40 §8.3 方差齐性检验 (Test for Homogeneity of Variances)

10.40.1 回顾：为什么要做这个检验？

我们再三强调，方差分析和 t 检验都有一个重要的前提假定：方差齐性，即所有比较的组都来自方差相等的总体。

理论研究表明，F 检验对于偏离正态性的情况具有一定的“稳健性”（Robustness），即影响不大。但是，F 检验对于方差不齐的情况则比较敏感。如果各组方差相差很大，那么我们算出的 F 值和 P 值的可靠性就会大大降低，可能导致我们做出错误的结论。因此，在进行方差分析之前，进行一次方差齐性检验，是严谨的数据分析流程中必不可少的一步。





图解说明：

- 上图中，A、B、C三组具有相似的方差（箱线图的高度相近），符合方差齐性假设；而D、E、F三组的方差相差很大，违反了方差齐性假设。
- 下图展示了方差齐性和方差不齐两种情况下的概率密度函数。蓝色实线表示三个均值不同但方差相同的正态分布，符合方差齐性假设；红色虚线表示三个均值不同且方差也不同的正态分布，违反了方差齐性假设。
- 当方差不齐时，我们无法确定组间差异是由均值不同还是由方差不同引起的，这会导致F检验的结果不可靠。

10.40.2 Bartlett 检验

Bartlett 检验是一种常用的方差齐性检验方法，它适用于各组样本量相等或不等的场合，但要求每个样本量最好不低于 5。

检验统计量与步骤

- 原假设与备择假设：

$$H_0: \sigma_1^2 = \sigma_2^2 = \dots = \sigma_r^2$$

$$H_1: \text{至少有两个 } \sigma_i^2 \text{ 不相等}$$

检验的核心是构造 Bartlett 统计量，其计算步骤如下：

- (1) 对于第 i 组 ($i = 1, \dots, r$)，计算其样本量 m_i 和样本方差 s_i^2 。令 $f_i = m_i - 1$ 为其自由度。

(2) 计算合并方差 (Pooled Variance) s_p^2 :

$$s_p^2 = \frac{\sum_{i=1}^r (m_i - 1)s_i^2}{\sum_{i=1}^r (m_i - 1)} = \frac{\sum_{i=1}^r f_i s_i^2}{\sum f_i} \quad (10.11)$$

(3) 计算 Bartlett 统计量 M^2 (在某些文献中也记为 B 或 χ^2):

$$M^2 = (\sum f_i) \ln(s_p^2) - \sum_{i=1}^r f_i \ln(s_i^2) \quad (10.12)$$

(4) 计算修正系数 C:

$$C = 1 + \frac{1}{3(r-1)} \left(\sum_{i=1}^r \frac{1}{f_i} - \frac{1}{\sum f_i} \right) \quad (10.13)$$

(5) 最终的检验统计量为 $B = M^2/C$ 。在原假设 H_0 成立时, 该统计量近似服从自由度为 $r-1$ 的卡方分布, 即 $B \sim \chi^2(r-1)$ 。

修正的 Bartlett 检验

当样本容量较小时, 上述 Bartlett 检验的近似效果可能不够理想。此时可以采用修正的 Bartlett 检验, 该检验将统计量转化为 F 统计量:

Théorème 59: 修正的 Bartlett 检验

对于样本容量较小的情况, 我们可以构造以下统计量:

$$B' = \frac{f_2 \cdot B \cdot C}{r-1} \cdot \frac{1}{A - B \cdot C} \quad (10.14)$$

其中:

$$f_2 = \frac{r+1}{(C-1)^2} \quad (10.15)$$

$$A = \frac{f_2}{r-1-C+\frac{2}{f_2}} \quad (10.16)$$

在原假设 H_0 成立的条件下, B' 近似服从自由度为 $(r-1, f_2)$ 的 F 分布, 即 $B' \sim F(r-1, f_2)$ 。

决策规则:

- 若 $B' \geq F_{1-\alpha}(r-1, f_2)$, 则拒绝 H_0 , 认为各总体方差不全相等;
- 若 $B' < F_{1-\alpha}(r-1, f_2)$, 则接受 H_0 , 认为各总体方差相等。

计算实例: 以下是绿茶叶酸含量数据的方差齐性检验。

表 10.6: 四种产地绿茶叶酸含量数据

样本编号	产地 1	产地 2	产地 3	产地 4
1	5.23	5.42	5.87	5.31
2	5.65	5.68	5.54	5.85
3	5.08	5.35	5.32	5.45
4	5.45	5.51	5.78	6.12
5	5.27	5.29	5.92	5.95
均值 \bar{y}_i	5.34	5.45	5.69	5.74
方差 s_i^2	0.231	0.153	0.277	0.346

我们收集了四种不同产地的绿茶叶样本，每种产地取 5 个样本，测量其中的茶多酚含量（单位：

已知：

- 四组数据的样本方差分别为： $s_1^2 = 0.231$, $s_2^2 = 0.153$, $s_3^2 = 0.277$, $s_4^2 = 0.346$
- 每组样本量均为 $m_i = 5$, 因此自由度 $f_i = 4$, 总自由度 $\sum f_i = 16$

首先，计算合并方差 s_p^2 :

$$\begin{aligned}
 s_p^2 &= \frac{\sum_{i=1}^4 f_i s_i^2}{\sum f_i} \\
 &= \frac{4 \times 0.231 + 4 \times 0.153 + 4 \times 0.277 + 4 \times 0.346}{16} \\
 &= \frac{0.924 + 0.612 + 1.108 + 1.384}{16} \\
 &= \frac{4.028}{16} \\
 &= 0.252
 \end{aligned}$$

接下来，计算 Bartlett 统计量 M^2 :

$$\begin{aligned}
 M^2 &= (\sum f_i) \ln(s_p^2) - \sum_{i=1}^4 f_i \ln(s_i^2) \\
 &= 16 \times \ln(0.252) - [4 \times \ln(0.231) + 4 \times \ln(0.153) + 4 \times \ln(0.277) + 4 \times \ln(0.346)] \\
 &= 16 \times (-1.378) - [4 \times (-1.466) + 4 \times (-1.877) + 4 \times (-1.283) + 4 \times (-1.061)] \\
 &= -22.048 - [-5.864 - 7.508 - 5.132 - 4.244] \\
 &= -22.048 - [-22.748] \\
 &= -22.048 + 22.748 \\
 &= 0.700
 \end{aligned}$$

计算修正系数 C :

$$\begin{aligned}
 C &= 1 + \frac{1}{3(4-1)} \left(\sum_{i=1}^4 \frac{1}{4} - \frac{1}{16} \right) \\
 &= 1 + \frac{1}{9} \left(1 - \frac{1}{16} \right) \\
 &= 1 + \frac{1}{9} \cdot \frac{15}{16} \\
 &= 1 + \frac{15}{144} \\
 &= 1.104
 \end{aligned}$$

计算修正后的 Bartlett 统计量:

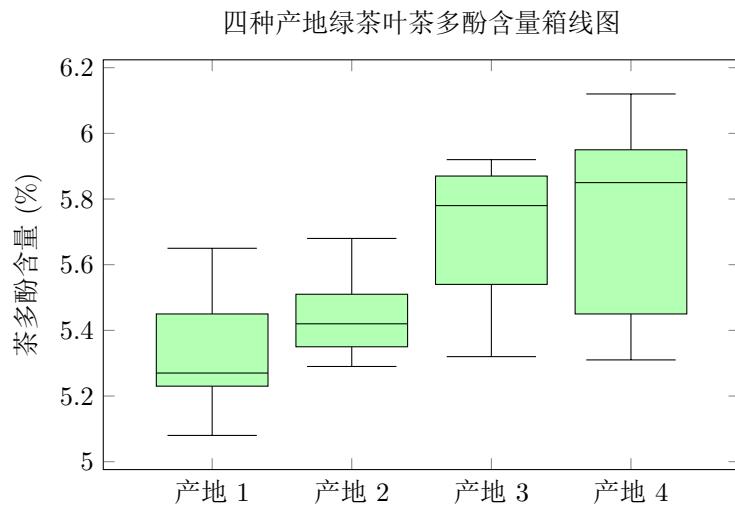
$$B = \frac{M^2}{C} = \frac{0.700}{1.104} = 0.634$$

对于样本容量较小的情况，计算修正的 F 统计量:

$$\begin{aligned}
 f_2 &= \frac{4+1}{(1.104-1)^2} = \frac{5}{0.104^2} = 462.5 \\
 A &= \frac{462.5}{4-1-1.104+\frac{2}{462.5}} = \frac{462.5}{1.9} = 243.4 \\
 B' &= \frac{462.5 \times 0.634 \times 1.104}{3(243.4 - 0.634 \times 1.104)} = \frac{323.4}{728.0} = 0.444
 \end{aligned}$$

查表得 $F_{0.95}(3, 462.5) \approx F_{0.95}(3, \infty) = 2.61$ 。由于 $B' = 0.444 < 2.61$ ，故接受原假设 H_0 ，认为四个水平下的方差间无显著差异，满足方差齐性假设。

进一步绘制箱线图可以直观地观察四组数据的方差相似性:



从图中可以看出，四组数据的箱体高度（表示四分位距 IQR）相对接近，支持了方差齐性的结论。虽然产地 4 的数据分散程度略大，但整体上四组数据的方差差异不足以拒绝方差齐性假设。

10.40.3 Hartley 检验（最大方差比检验）

当各组样本容量相等时，可以使用 Hartley 提出的最大方差比检验（F-max test）。这是一种简便易行的方法。

检验统计量

设有 r 个总体，每个总体有 m 个观测值（样本容量相等）， s_i^2 是第 i 个样本的方差。Hartley 统计量定义为：

$$H = \frac{\max_{1 \leq i \leq r} s_i^2}{\min_{1 \leq i \leq r} s_i^2} \quad (10.17)$$

即最大样本方差与最小样本方差之比。

决策规则：

- 若 $H > H_\alpha(r, m - 1)$ ，则拒绝 H_0 ，认为各总体方差不全相等；
- 若 $H \leq H_\alpha(r, m - 1)$ ，则接受 H_0 ，认为各总体方差相等。

其中 $H_\alpha(r, m - 1)$ 是显著性水平为 α 时 Hartley 分布的临界值。

计算实例：考虑鸡饲料试验中的三个组，我们可以计算其样本方差：

$$s_1^2 = \frac{1}{7} \sum_{j=1}^8 (y_{1j} - \bar{y}_{1\cdot})^2 = 902.5$$

$$s_2^2 = \frac{1}{7} \sum_{j=1}^8 (y_{2j} - \bar{y}_{2\cdot})^2 = 1793.8$$

$$s_3^2 = \frac{1}{7} \sum_{j=1}^8 (y_{3j} - \bar{y}_{3\cdot})^2 = 942.9$$

Hartley 统计量为：

$$H = \frac{\max(s_1^2, s_2^2, s_3^2)}{\min(s_1^2, s_2^2, s_3^2)} = \frac{1793.8}{902.5} = 1.99$$

查 Hartley 分布表得 $H_{0.05}(3, 7) = 8.44$ 。由于 $H = 1.99 < 8.44$ ，故接受原假设 H_0 ，认为三个饲料水平下的方差无显著差异。

10.40.4 方差不齐时的处理方法

当方差齐性检验的结果为“方差不齐”时，我们有几种处理选择：

- (1) **数据变换：**通过适当的变换使得变换后的数据满足方差齐性假定。常用的变换包括：
 - 对数变换: $Y' = \log(Y)$ 或 $Y' = \ln(Y)$
 - 平方根变换: $Y' = \sqrt{Y}$
 - 倒数变换: $Y' = 1/Y$
 - Box-Cox 变换: $Y' = \frac{Y^\lambda - 1}{\lambda}$ (当 $\lambda \neq 0$) 或 $Y' = \ln(Y)$ (当 $\lambda = 0$)
- (2) **使用稳健的方差分析方法：**例如 Welch's ANOVA，它是标准 ANOVA 的一种修正，不要求方差齐性。
- (3) **使用非参数方法：**如 Kruskal-Wallis 检验，它不需要数据满足正态性和方差齐性的假定。
- (4) **独立进行双样本比较：**但要注意多重比较问题，需要进行显著性水平的调整。

知识脉络解读：8.2 节和 8.3 节分别是方差分析的“后处理”和“前处理”。8.3 节的方差齐性检验是前提诊断，确保我们的分析方法是可靠的；8.2 节的多重比较是结果深化，让我们的结论更具体、更有指导意义。

10.41 §8.4 一元线性回归分析 (Simple Linear Regression)

10.41.1 引言：从“差异”到“关系”

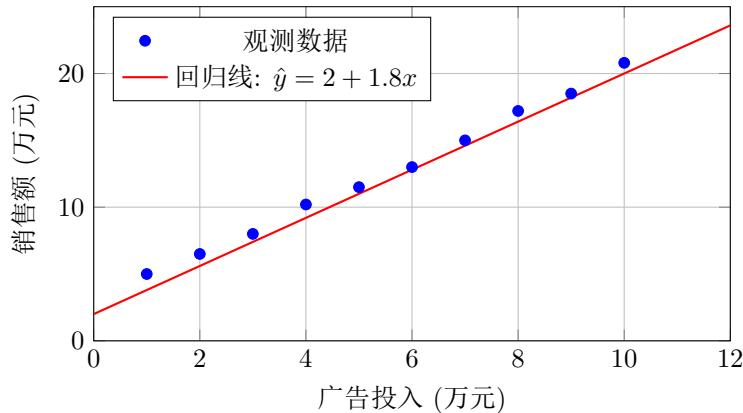
同学们，方差分析帮助我们判断不同类别（因子水平）之间是否存在均值上的显著差异。现在，我们要进入一个新的领域：回归分析。

回归分析探讨的是变量之间的关系。我们不再是比較离散的组别，而是研究一个或多个自变量（Independent Variable, x ）的变化如何影响另一个因变量（Dependent Variable, y ）的变化。

- **核心问题：**我们能否用一个数学模型来描述并量化 x 和 y 之间的依赖关系？
- **最终目的：**利用这个模型，通过已知的 x 值来预测未知的 y 值。

例如：广告投入 (x) 与销售额 (y) 的关系；房价 (y) 与房屋面积 (x) 的关系。本节我们从最简单，也是最基础的一元线性回归开始。

广告投入与销售额关系的散点图



10.41.2 一元线性回归模型

模型设定

我们假设变量 y 与 x 之间的关系可以用一条直线来近似描述。当然，现实中的数据点不会完美地落在一条直线上，总会有随机的波动。因此，我们建立如下模型：

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

其中：

- y_i 是第 i 个观测的因变量值。
- x_i 是第 i 个观测的自变量值。
- β_0 是截距 (Intercept)，表示当 $x = 0$ 时， y 的期望值。
- β_1 是斜率 (Slope)，表示 x 每增加一个单位， y 的期望值平均增加的数量。这是我们最关心的参数，它代表了 x 对 y 的影响强度和方向。
- ε_i 是随机误差项，代表了除 x 之外所有能影响 y 的未被观测的因素。

模型基本假定 (与方差分析类似)：

- (1) 误差项的期望为 0, $E(\varepsilon_i) = 0$ 。
- (2) 误差项的方差为常数 (方差齐性), $\text{Var}(\varepsilon_i) = \sigma^2$ 。
- (3) 不同的误差项之间相互独立, $\text{Cov}(\varepsilon_i, \varepsilon_j) = 0$ for $i \neq j$ 。
- (4) (用于区间估计和假设检验) 误差项服从正态分布, $\varepsilon_i \sim N(0, \sigma^2)$ 。

参数估计：最小二乘法 (Least Squares Method)

我们如何根据样本数据 $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ 来估计出真实的 β_0 和 β_1 呢？最小二乘法的思想是：找到一条直线 $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$ ，使得所有观测点到这条直线的垂直距离的平方和最小。这个平方和称为残差平方和 (Sum of Squared Residuals, SSE)。

$$Q(\beta_0, \beta_1) = \sum_{i=1}^n (y_i - \hat{y}_i)^2 = \sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i))^2$$

通过对 Q 分别求关于 β_0 和 β_1 的偏导数并令其为 0，可以解得 β_0 和 β_1 的估计值 $\hat{\beta}_0$ 和 $\hat{\beta}_1$ ：

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{L_{xy}}{L_{xx}} \\ \hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x}\end{aligned}$$

其中 L_{xx} 和 L_{xy} 是常用的简记符号。得到的回归方程为 $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$ 。

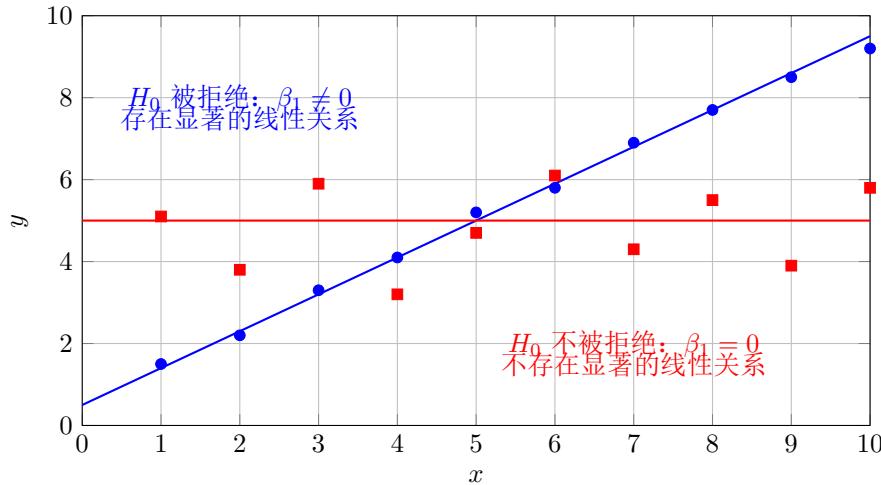
10.41.3 回归方程的显著性检验

我们找到了最佳的直线，但这条直线真的有用吗？ x 和 y 之间真的存在线性关系吗？或者观察到的关系仅仅是抽样造成的偶然？这就是回归的显著性检验要回答的问题。

核心是检验斜率 β_1 是否显著不为 0。

$$H_0 : \beta_1 = 0 \quad \text{vs} \quad H_1 : \beta_1 \neq 0$$

回归方程显著性检验的两种可能情况



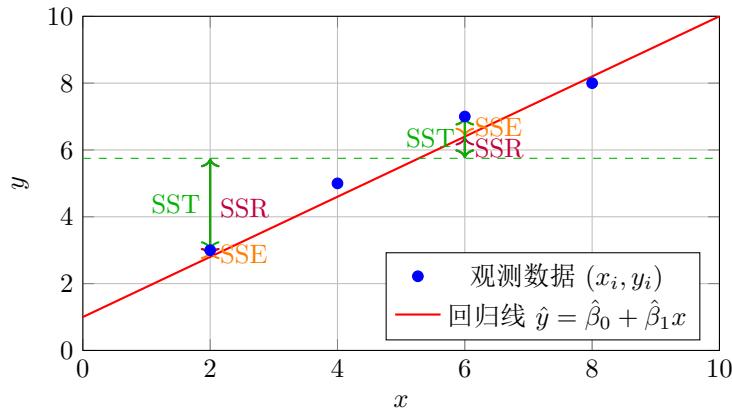
检验方法一：F 检验（回归的方差分析）

与 ANOVA 类似，我们也可以对回归进行平方和分解。

$$\text{SST} = \text{SSR} + \text{SSE}$$

- **SST (总平方和):** $\sum(y_i - \bar{y})^2$ 。度量了 y 的总变异。
- **SSR (回归平方和):** $\sum(\hat{y}_i - \bar{y})^2$ 。度量了能被回归模型解释的 y 的变异。
- **SSE (误差平方和/残差平方和):** $\sum(y_i - \hat{y}_i)^2$ 。度量了模型不能解释的 y 的变异。

回归中的平方和分解



图中关系说明：

- **SST**: 从观测点 (x_i, y_i) 到总均值 \bar{y} 的距离, 表示总变异。
- **SSE**: 从观测点 (x_i, y_i) 到拟合值 \hat{y}_i 的距离, 表示未被解释的变异。
- **SSR**: 从拟合值 \hat{y}_i 到总均值 \bar{y} 的距离, 表示被回归解释的变异。
- **关键关系**: $SST = SSE + SSR$, 这种分解是回归分析和方差分析共有的核心思想。

我们可以构建一个类似于 ANOVA 的方差分析表:

表 10.7: 一元线性回归的方差分析表

变异来源	平方和 (SS)	自由度 (df)	均方 (MS)	F 统计量
回归 (Regression)	SSR	1	$MSR = SSR/1$	$F = \frac{MSR}{MSE}$
误差 (Error)	SSE	$n - 2$	$MSE = SSE/(n-2)$	
总计 (Total)	SST	$n - 1$		

在 $H_0 : \beta_1 = 0$ 成立的条件下, $F = \frac{MSR}{MSE} \sim F(1, n - 2)$ 。如果 F 值足够大 (大于临界值, 或 P-value 足够小), 我们就拒绝 H_0 , 认为回归关系显著。

检验方法二: t 检验

对于一元线性回归, 检验 $H_0 : \beta_1 = 0$ 也可以用 t 检验。检验统计量为:

$$t = \frac{\hat{\beta}_1 - 0}{SE(\hat{\beta}_1)} = \frac{\hat{\beta}_1}{\hat{\sigma}/\sqrt{L_{xx}}}$$

其中 $SE(\hat{\beta}_1)$ 是 $\hat{\beta}_1$ 的标准误, $\hat{\sigma} = \sqrt{MSE}$ 。在 H_0 成立时, $t \sim t(n - 2)$ 。

知识脉络解读: 在一元线性回归中, F 检验和 t 检验是等价的! 可以证明 $F = t^2$ 。所以用哪种方法结论都一样。但在多元回归中 (多个 x), F 检验用于检验整个模型的显著性, 而 t 检验用于检验单个回归系数的显著性, 两者的作用就分开了。

10.41.4 估计与预测

当回归方程被检验为显著后, 它就成了一个有用的工具。我们可以用它来做两件重要的事情:

- **估计 (Estimation):** 估计在某个给定的 x_0 值下, 所有 y 的平均值 $E(y_0) = \beta_0 + \beta_1 x_0$ 。
- **预测 (Prediction):** 预测在某个给定的 x_0 值下, 一个单独的 y 的观测值 y_0 。

均值的置信区间

对于 $E(y_0)$, 它的点估计是 $\hat{y}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0$ 。其 $1 - \alpha$ 的置信区间为:

$$\hat{y}_0 \pm t_{\alpha/2, n-2} \cdot \hat{\sigma} \sqrt{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{L_{xx}}}$$

个体的预测区间

对于单个值 y_0 , 它的点预测也是 \hat{y}_0 。但由于单个值比均值有更大的不确定性 (需要考虑个体的随机误差 ε_0), 所以它的预测区间会更宽。其 $1 - \alpha$ 的预测区间为:

$$\hat{y}_0 \pm t_{\alpha/2, n-2} \cdot \hat{\sigma} \sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{L_{xx}}}$$

** 注意关键差异 **: 预测区间的根号下多了一个“1”, 这代表了个体本身的不确定性。

重要观察: 无论是置信区间还是预测区间, 其宽度都依赖于 $(x_0 - \bar{x})^2$ 。这意味着, 当 x_0 离样本均值 \bar{x} 越远, 我们的估计和预测就越不准, 区间就越宽。这提醒我们, 用回归模型进行“外推”(即预测远离我们数据范围的点) 是需要非常谨慎的。

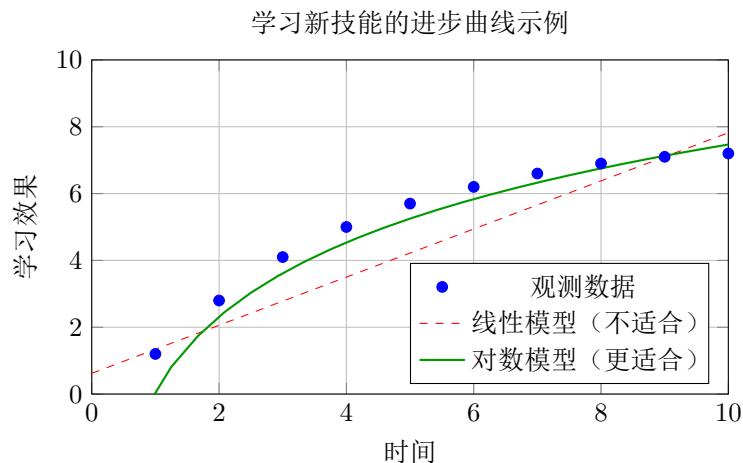
10.42 §8.5 一元非线性回归分析

10.42.1 当关系不是一条直线时

到目前为止, 我们讨论的回归都是线性的。但在现实世界中, 很多变量之间的关系并非简单的直线。例如:

- 药效随时间的变化 (先增强后减弱)
- 学习新技能的进步速度 (先快后慢, 趋于平缓)
- 物质的衰变过程 (指数下降)

当散点图呈现出明显的曲线趋势时, 强行使用线性模型会产生巨大的误差, 得到无用的结论。此时, 我们就需要进入非线性回归的领域。



10.4.2.2 非线性回归的数学模型

非线性回归模型的一般形式为：

$$y_i = f(x_i; \beta_1, \beta_2, \dots, \beta_p) + \varepsilon_i, \quad i = 1, 2, \dots, n \quad (10.18)$$

其中：

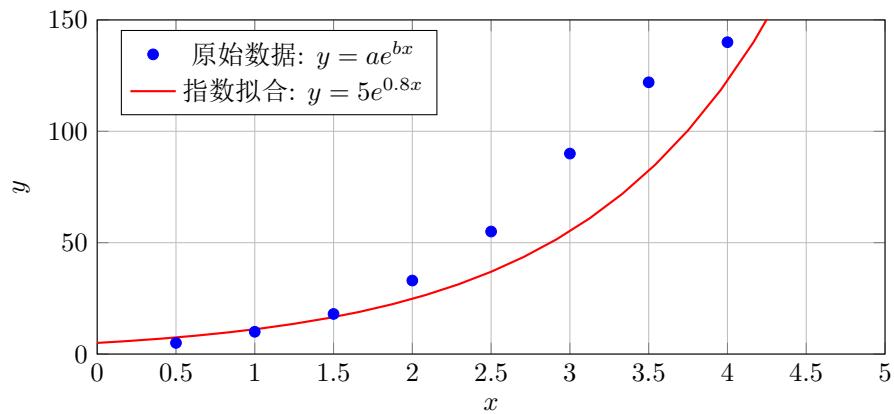
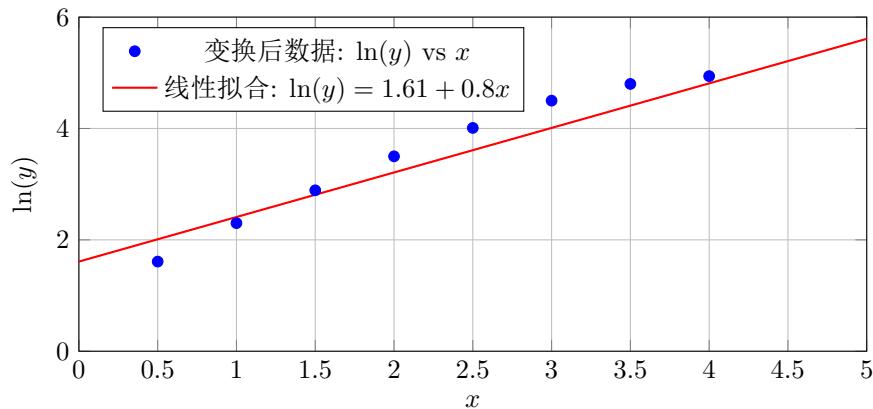
- $f(x_i; \beta_1, \beta_2, \dots, \beta_p)$ 是一个非线性函数
- $\beta_1, \beta_2, \dots, \beta_p$ 是待估计的参数
- ε_i 是随机误差项，假定 $\varepsilon_i \sim N(0, \sigma^2)$

直接估计非线性模型的参数通常比较复杂，需要采用迭代方法求解。然而，许多常见的非线性模型可以通过变量变换转化为线性模型，这是一种被称为“线性化”的强大技术。

10.4.2.3 核心策略：“掰弯”化”直”

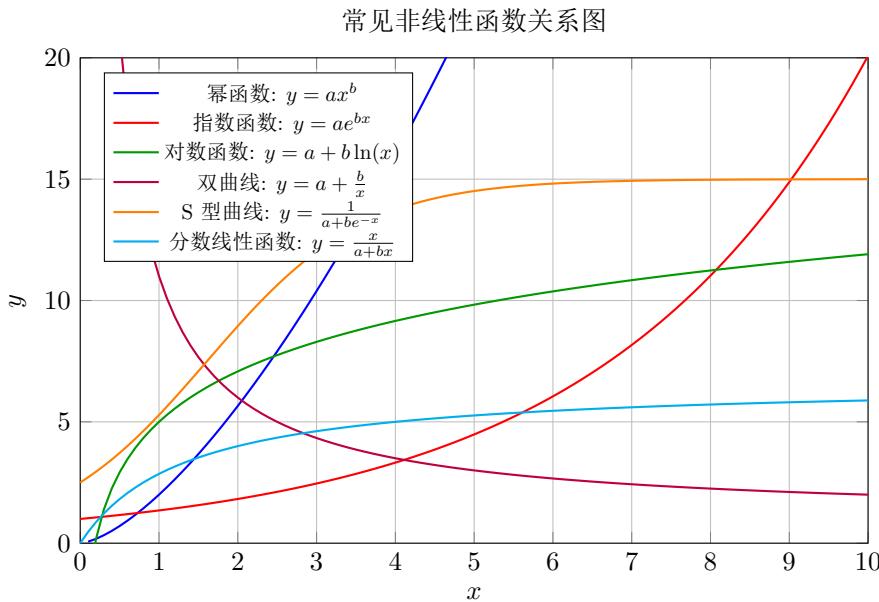
处理非线性回归的一个非常强大且常用的策略是：通过变量变换，将非线性的关系转化为我们已经熟悉的线性关系，然后使用标准的线性回归方法进行分析。这个过程就像是把弯曲的尺子掰直了再用。

非线性关系（指数函数）的线性化变换示例

变换后的线性关系: $\ln(y) = \ln(a) + bx$ 

常见可线性化的曲线模型与变换公式

以下是一些常见的、可以通过变换化为线性模型的曲线类型及其详细变换方法:



不同非线性函数的特点：

- **幂函数 ($y = ax^b$):** 当 $b > 1$ 时曲线上凸，当 $0 < b < 1$ 时曲线下凸。适用于描述物理量的标度关系。
- **指数函数 ($y = ae^{bx}$):** 增长率与函数值成正比，适合描述具有“复利效应”的现象，如人口增长、细胞繁殖。
- **对数函数 ($y = a + b \ln(x)$):** 随 x 增大， y 增长速率减慢，适合描述边际递减现象。
- **双曲线 ($y = a + \frac{b}{x}$):** 随 x 增大迅速接近水平渐近线，适合描述衰减过程。
- **S型曲线 ($y = \frac{1}{a+be^{-x}}$):** 有上下限的增长过程，如技术扩散、学习曲线。
- **分数线性函数 ($y = \frac{x}{a+bx}$):** 适合描述反应速率、吸附过程等。

这些非线性关系通过适当的变量变换，都可以转化为线性关系进行分析。例如，对指数函数取对数后，可得到线性关系： $\ln(y) = \ln(a) + bx$ 。

10.43 本章总结与知识网络

同学们，第八章我们学习了两个强大的统计工具：方差分析和回归分析。它们看似不同，但内在联系紧密。

- **核心思想的共性：**两者都运用了变异分解的思想。通过将总变异 (SST) 分解为模型可以解释的部分 (SSA 或 SSR) 和模型不能解释的残差部分

表 10.8: 常见的可线性化非线性模型及其变换方法

函数名称	原始模型	变量变换	线性化后模型	参数关系
双曲线	$y = a + \frac{b}{x}$	$u = \frac{1}{x}, v = y$	$v = a + bu$	直接对应
幂函数	$y = ax^b$	$u = \ln(x), v = \ln(y)$	$v = \ln(a) + bu$	$\alpha = \ln(a), \beta = b$
指数函数	$y = ae^{bx}$	$u = x, v = \ln(y)$	$v = \ln(a) + bu$	$\alpha = \ln(a), \beta = b$
对数函数	$y = a + b \ln(x)$	$u = \ln(x), v = y$	$v = a + bu$	直接对应
S型曲线	$y = \frac{1}{a+be^{-x}}$	$u = e^{-x}, v = \frac{1}{y}$	$v = a + bu$	直接对应
修正指数曲线	$y = a + be^{cx}$	$y_{i+1} - y_i = k(y_i - y_{i-1})$	特殊处理	$k = e^c$
分数线性函数	$y = \frac{x}{a+bx}$	$u = x, v = \frac{x}{y}$	$v = a + bu$	直接对应

(SSE)，来构建检验统计量 (F 统计量)。

- 研究问题的侧重：

- 方差分析：处理的自变量是定性/分类型的（如不同的饲料、不同的产地）。它关心的是“不同组之间，均值有无差异？”
- 回归分析：处理的自变量是定量/连续型的（如温度、面积、投入资本）。它关心的是“因变量如何随着自变量的变化而变化？它们的关系是怎样的？”

实际上，方差分析可以被看作是一种特殊的回归分析，其中的分类型自变量可以用“虚拟变量”来表示。掌握了这两个工具，你们就具备了分析实验数据和探寻变量关系的核心能力。

10.43.1 关键公式汇总

单因子方差分析

$$\begin{aligned} \text{SST} &= \sum_{i=1}^r \sum_{j=1}^m (y_{ij} - \bar{y}_{..})^2 && \text{总平方和} \\ \text{SSA} &= m \sum_{i=1}^r (\bar{y}_{i\cdot} - \bar{y}_{..})^2 && \text{组间平方和} \\ \text{SSE} &= \sum_{i=1}^r \sum_{j=1}^m (y_{ij} - \bar{y}_{i\cdot})^2 && \text{组内平方和} \\ F &= \frac{\text{MSA}}{\text{MSE}} = \frac{\text{SSA}/(r-1)}{\text{SSE}/[r(m-1)]} && \text{F 统计量} \end{aligned}$$

多重比较

$$\begin{aligned} \text{HSD} &= q_\alpha(r, df_E) \cdot \sqrt{\frac{\text{MSE}}{m}} && \text{Tukey's HSD 临界值} \\ S_{ij} &= \sqrt{(r-1)F_\alpha(r-1, f_e)} \cdot \sqrt{\text{MSE}\left(\frac{1}{m_i} + \frac{1}{m_j}\right)} && \text{Scheffé 临界值} \end{aligned}$$

方差齐性检验

$$\begin{aligned} H &= \frac{\max_{1 \leq i \leq r} s_i^2}{\min_{1 \leq i \leq r} s_i^2} && \text{Hartley 统计量} \\ B &= \frac{M^2}{C} = \frac{(\sum f_i) \ln(s_p^2) - \sum_{i=1}^r f_i \ln(s_i^2)}{C} && \text{Bartlett 统计量} \end{aligned}$$

线性回归

$$\begin{aligned} \hat{\beta}_1 &= \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{L_{xy}}{L_{xx}} && \text{斜率估计} \\ \hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} && \text{截距估计} \\ R^2 &= 1 - \frac{\sum (y_i - \hat{y}_i)^2}{\sum (y_i - \bar{y})^2} = \frac{\text{SSR}}{\text{SST}} && \text{决定系数} \end{aligned}$$

希望这份讲义能帮助大家更好地理解和掌握本章内容。请务必多加练习，将理论与实际问题联系起来！

10.44 双因子方差分析 (Two-Way ANOVA)

10.44.1 引言

在实际研究中，我们常常需要同时考察两个因素对结果的影响。例如，研究不同肥料（因子 A）和不同灌溉方式（因子 B）对作物产量的影响。双因子方差分析（Two-Way ANOVA）正是用来解决这类问题的统计方法。

10.44.2 双因子方差分析模型

表 10.9: 双因子方差分析中的符号说明

符号	含义
y_{ijk}	第 i 个 A 水平、第 j 个 B 水平下的第 k 次观测值
$\bar{y}_{ij\cdot}$	第 i 个 A 水平、第 j 个 B 水平下的平均值 ($\frac{1}{n} \sum_{k=1}^n y_{ijk}$)
$\bar{y}_{i\cdot\cdot}$	第 i 个 A 水平下的总平均值 ($\frac{1}{bn} \sum_{j=1}^b \sum_{k=1}^n y_{ijk}$)
$\bar{y}_{\cdot j\cdot}$	第 j 个 B 水平下的总平均值 ($\frac{1}{an} \sum_{i=1}^a \sum_{k=1}^n y_{ijk}$)
\bar{y}_{\dots}	总体平均值 ($\frac{1}{abn} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n y_{ijk}$)

假设有因子 A (a 个水平) 和因子 B (b 个水平)，每个处理组合有 n 次重复观测。我们用 y_{ijk} 表示第 i 个 A 水平、第 j 个 B 水平下的第 k 次观测值。双因子方差分析的数学模型为：

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$$

其中：

- μ : 总体均值，可以通过所有观测值的平均 \bar{y}_{\dots} 估计
- α_i : 因子 A 的第 i 个水平的主效应， $\sum_{i=1}^a \alpha_i = 0$ ，可以通过 $\hat{\alpha}_i = \bar{y}_{i\cdot\cdot} - \bar{y}_{\dots}$ 估计
- β_j : 因子 B 的第 j 个水平的主效应， $\sum_{j=1}^b \beta_j = 0$ ，可以通过 $\hat{\beta}_j = \bar{y}_{\cdot j\cdot} - \bar{y}_{\dots}$ 估计
- $(\alpha\beta)_{ij}$: A 和 B 的交互效应， $\sum_{i=1}^a (\alpha\beta)_{ij} = \sum_{j=1}^b (\alpha\beta)_{ij} = 0$ ，可以通过 $\widehat{(\alpha\beta)}_{ij} = \bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\dots}$ 估计

表 10.10: 双因子方差分析数据结构示例 ($a = 3, b = 2, n = 2$)

因子 A	因子 B		行均值
	B_1	B_2	
A_1	y_{111}, y_{112}	y_{121}, y_{122}	$\bar{y}_{1..}$
	$\bar{y}_{11..}$	$\bar{y}_{12..}$	
A_2	y_{211}, y_{212}	y_{221}, y_{222}	$\bar{y}_{2..}$
	$\bar{y}_{21..}$	$\bar{y}_{22..}$	
A_3	y_{311}, y_{312}	y_{321}, y_{322}	$\bar{y}_{3..}$
	$\bar{y}_{31..}$	$\bar{y}_{32..}$	
列均值	$\bar{y}_{.1..}$	$\bar{y}_{.2..}$	$\bar{y}_{...}$

- ε_{ijk} : 随机误差, 假定 $\varepsilon_{ijk} \sim N(0, \sigma^2)$ 且相互独立, 可以通过 $\hat{\varepsilon}_{ijk} = y_{ijk} - \bar{y}_{ij..}$ 估计

10.44.3 平方和分解

双因子方差分析中, 总平方和 SST 可以分解为:

$$SST = SSA + SSB + SSAB + SSE$$

其中:

- SSA: 因子 A 的平方和, 反映 A 的主效应
- SSB: 因子 B 的平方和, 反映 B 的主效应
- SSAB: 交互作用的平方和, 反映 A 和 B 的交互效应
- SSE: 误差平方和, 反映随机误差

计算公式如下：

$$\begin{aligned} \text{SSA} &= bn \sum_{i=1}^a (\bar{y}_{i..} - \bar{y}...)^2 \\ \text{SSB} &= an \sum_{j=1}^b (\bar{y}_{.j.} - \bar{y}...)^2 \\ \text{SSAB} &= n \sum_{i=1}^a \sum_{j=1}^b (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}...)^2 \\ \text{SSE} &= \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (y_{ijk} - \bar{y}_{ij.})^2 \end{aligned}$$

各项自由度为：

$$df_A = a - 1$$

$$df_B = b - 1$$

$$df_{AB} = (a - 1)(b - 1)$$

$$df_E = ab(n - 1)$$

$$df_T = abn - 1$$

10.44.4 检验的构建

在双因子方差分析中，我们关心三个基本假设：

- (1) H_0^A : 因子 A 没有主效应，即 $\alpha_1 = \alpha_2 = \dots = \alpha_a = 0$
- (2) H_0^B : 因子 B 没有主效应，即 $\beta_1 = \beta_2 = \dots = \beta_b = 0$
- (3) H_0^{AB} : 因子 A 和 B 之间没有交互效应，即 $(\alpha\beta)_{ij} = 0$ 对所有 i, j

分别计算：

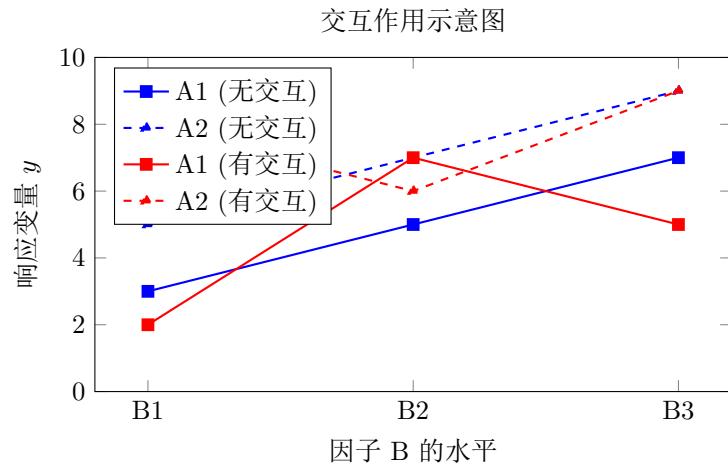
$$\begin{aligned} F_A &= \frac{\text{MSA}}{\text{MSE}} = \frac{\text{SSA}/df_A}{\text{SSE}/df_E} \sim F(df_A, df_E) \\ F_B &= \frac{\text{MSB}}{\text{MSE}} = \frac{\text{SSB}/df_B}{\text{SSE}/df_E} \sim F(df_B, df_E) \\ F_{AB} &= \frac{\text{MSAB}}{\text{MSE}} = \frac{\text{SSAB}/df_{AB}}{\text{SSE}/df_E} \sim F(df_{AB}, df_E) \end{aligned}$$

10.44.5 交互作用的理解

交互作用是双因子方差分析中最重要的概念之一。当交互作用显著时，一个因子的效应取决于另一个因子处于什么水平。例如，某种肥料的效果可能在湿润条件下很好，但在干旱条件下效果不佳。

表 10.11: 双因子方差分析表

变异来源	平方和 (SS)	自由度 (df)	均方 (MS)	F 值	P 值
因子 A	SSA	$a - 1$	MSA	F_A	P_A
因子 B	SSB	$b - 1$	MSB	F_B	P_B
交互作用 $A \times B$	SSAB	$(a - 1)(b - 1)$	MSAB	F_{AB}	P_{AB}
误差	SSE	$ab(n - 1)$	MSE		
总计	SST	$abn - 1$			



图中，蓝线表示无交互作用的情况：两条线平行，表示因子 A 的两个水平下，响应变量 y 随着因子 B 水平变化的趋势相同。红线表示有交互作用的情况：两条线不平行甚至交叉，表示因子 A 的不同水平下，响应变量 y 随因子 B 变化的趋势不同。

10.44.6 案例分析：农作物产量研究

考虑一个研究不同肥料类型（因子 A, 3 个水平）和灌溉频率（因子 B, 2 个水平）对小麦产量的影响的实验。每个处理组合重复 4 次，得到如下数据（单位：kg/plot）：

经过方差分析计算，我们得到：

表 10.12: 小麦产量数据 (kg/plot)

重复	肥料 A1		肥料 A2		肥料 A3	
	灌溉 B1	灌溉 B2	灌溉 B1	灌溉 B2	灌溉 B1	灌溉 B2
1	32.4	40.3	37.2	45.3	29.8	42.6
2	30.1	38.9	35.5	44.1	28.7	41.8
3	33.6	41.2	38.1	46.5	30.2	43.5
4	31.5	39.5	36.8	44.9	29.5	42.3
平均值	31.9	40.0	36.9	45.2	29.5	42.6

表 10.13: 小麦产量的双因子方差分析表

变异来源	SS	df	MS	F	P
肥料 (A)	148.63	2	74.31	52.7	< 0.001
灌溉 (B)	648.45	1	648.45	460.3	< 0.001
交互作用 (A×B)	19.22	2	9.61	6.8	0.006
误差	25.34	18	1.41		
总计	841.64	23			

结论:

- 肥料类型对产量有显著影响 ($F = 52.7, p < 0.001$)
- 灌溉频率对产量有显著影响 ($F = 460.3, p < 0.001$)
- 肥料类型和灌溉频率之间存在显著的交互作用 ($F = 6.8, p = 0.006$)

交互作用的存在表明: 不同肥料的效果取决于灌溉条件。从数据可以看出, 肥料 A2 在两种灌溉条件下都产量最高, 但在高频灌溉 (B2) 条件下, 三种肥料的差异变得更加明显。

10.44.7 无重复情况

当每个处理组合只有一次观测（无重复）时，我们无法估计交互效应和误差项。此时，通常假设交互效应不存在，并用交互效应的平方和代替误差平方和。这种情况下的方差分析称为无重复双因子方差分析。

10.44.8 结语

双因子方差分析是单因子方差分析的自然扩展，它允许我们同时考察两个因素及其交互作用对响应变量的影响。这种分析方法在农业、医学、心理学、工业等领域有广泛应用。

在实际研究中，我们可能需要考虑三个或更多因素的影响，这就引出了多因子方差分析（Multi-Way ANOVA）。随着因素数量的增加，交互效应的解释和计算会变得更加复杂，通常需要借助统计软件来完成。

- ¹ Section 8.3: 4, 9, 15, 21 (ed7:4, 7, 12, 18)
- ² Section 9.2: 1, 5, 8 (ed7: 2, 5, 7)
- ³ Section 9.5: 7, 10 (ed7: 6, 9)

Exercice 2

8.3.4. Let X_1, \dots, X_n and Y_1, \dots, Y_m follow the location model

$$\begin{aligned} X_i &= \theta_1 + Z_i, \quad i = 1, \dots, n \\ Y_i &= \theta_2 + Z_{n+i}, \quad i = 1, \dots, m, \end{aligned}$$

where Z_1, \dots, Z_{n+m} are iid random variables with common pdf $f(z)$. Assume that $E(Z_i) = 0$ and $\text{Var}(Z_i) = \theta_3 < \infty$.

- (a) Show that $E(X_i) = \theta_1$, $E(Y_i) = \theta_2$, and $\text{Var}(X_i) = \text{Var}(Y_i) = \theta_3$.
- (b) Consider the hypotheses of Example 8.3.1, i.e.,

$$H_0 : \theta_1 = \theta_2 \text{ versus } H_1 : \theta_1 \neq \theta_2.$$

Show that under H_0 , the test statistic T given in expression (8.3.4) has a limiting $N(0, 1)$ distribution.

- (c) Using part (b), determine the corresponding large sample test (decision rule) of H_0 versus H_1 . (This shows that the test in Example 8.3.1 is asymptotically correct.)

(a) 显然

(b)

$$T = \sqrt{\frac{nm}{n+m}}(\bar{X} - \bar{Y}) \left\{ (n+m-2)^{-1} \left[\sum_1^n (X_i - \bar{X})^2 + \sum_1^m (Y_i - \bar{Y})^2 \right] \right\}^{-1/2}$$

设 $\bar{Z} = \frac{1}{n} \sum_{i=1}^n Z_i$, $\bar{W} = \frac{1}{m} \sum_{i=1}^m Z_{n+i}$, 于是

$$T = \sqrt{\frac{nm}{n+m}}(\bar{Z} - \bar{W}) \left\{ (n+m-2)^{-1} \left[\sum_{i=1}^n (Z_i - \bar{Z})^2 + \sum_{i=1}^m (Z_{n+i} - \bar{W})^2 \right] \right\}^{-1/2}$$

$\text{Var}(\bar{Z}) = \frac{1}{n}\theta_3$, $\text{Var}(\bar{W}) = \frac{1}{m}\theta_3$. Then $\text{Var}(\bar{Z} - \bar{W}) = (\frac{1}{n} + \frac{1}{m})\theta_3$. By CLT,

$$\frac{\bar{Z} - \bar{W}}{\sqrt{(\frac{1}{n} + \frac{1}{m})\theta_3}} \xrightarrow{\mathcal{D}} N(0, 1)$$

Thus $\sqrt{\frac{nm}{n+m}}(\bar{Z} - \bar{W}) \xrightarrow{\mathcal{D}} \sqrt{\theta_3}N(0, 1)$. By the Law of Large Numbers, as $n \rightarrow \infty$, $S_Z^2 = \frac{1}{n-1} \sum_{i=1}^n (Z_i - \bar{Z})^2 \xrightarrow{\mathcal{P}} \text{Var}(Z_i) = \theta_3$. And $S_W^2 = \frac{1}{m-1} \sum_{i=1}^m (Z_{n+i} - \bar{W})^2 \xrightarrow{\mathcal{P}} \text{Var}(Z_{n+i}) = \theta_3$. Thus $S_p^2 = \frac{(n-1)S_Z^2 + (m-1)S_W^2}{n+m-2} \xrightarrow{\mathcal{P}} \theta_3$.

By Slutsky's theorem, $\sqrt{\frac{nm}{n+m}} \frac{\bar{Z} - \bar{W}}{\sqrt{\theta_3}} \cdot \frac{\sqrt{\theta_3}}{\sqrt{S_p^2}} \xrightarrow{\mathcal{D}} N(0, 1)$.

(c) Since, under H_0 , the test statistic T asymptotically follows a standard normal distribution $N(0, 1)$, we can use the quantiles of the standard normal distribution to construct a large sample test.

We reject H_0 at level α if $|T| > z_{\alpha/2}$.

Exercice 3

8.3.9. Let X_1, X_2, \dots, X_n be iid $N(\theta_1, \theta_2)$. Show that the likelihood ratio principle for testing $H_0 : \theta_2 = \theta'_2$ specified, and θ_1 unspecified, against $H_1 : \theta_2 \neq \theta'_2$, θ_1 unspecified, leads to a test that rejects when $\sum_1^n (x_i - \bar{x})^2 \leq c_1$ or $\sum_1^n (x_i - \bar{x})^2 \geq c_2$, where $c_1 < c_2$ are selected appropriately.

$$L(\theta_1, \theta_2) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\theta_2}} \exp\left(-\frac{(x_i - \theta_1)^2}{2\theta_2}\right) = (2\pi\theta_2)^{-n/2} \exp\left(-\frac{1}{2\theta_2} \sum_{i=1}^n (x_i - \theta_1)^2\right)$$

The likelihood ratio test statistic Λ is defined as:

$$\Lambda = \frac{\sup_{\boldsymbol{\theta} \in \Theta_0} L(\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta} \in \Theta} L(\boldsymbol{\theta})}$$

Let

$$\frac{\partial \ln L}{\partial \theta_1} = -\frac{1}{2\theta_2} \sum_{i=1}^n 2(x_i - \theta_1)(-1) = \frac{1}{\theta_2} \sum_{i=1}^n (x_i - \theta_1) = 0$$

$$\frac{\partial \ln L}{\partial \theta_2} = -\frac{n}{2\theta_2} + \frac{1}{2\theta_2^2} S^2 = 0$$

Then $\hat{\theta}_1 = \bar{x}$, $\hat{\theta}_2 = \frac{1}{n} S^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$.

$$L(\hat{\theta}_1, \hat{\theta}_2) = (2\pi)^{-n/2} \left(\frac{S^2}{n}\right)^{-n/2} \exp\left(-\frac{nS^2}{2S^2}\right) = (2\pi)^{-n/2} \left(\frac{S^2}{n}\right)^{-n/2} \exp\left(-\frac{n}{2}\right)$$

Under H_0 , the likelihood function becomes:

$$L(\theta_1, \theta'_2) = (2\pi\theta'_2)^{-n/2} \exp\left(-\frac{1}{2\theta'_2} \sum_{i=1}^n (x_i - \theta_1)^2\right)$$

We need to maximize this with respect to θ_1 . As before, the MLE for θ_1 is $\hat{\theta}_{1,0} = \bar{x}$. So, the maximized likelihood under H_0 is:

$$L(\hat{\theta}_{1,0}, \theta'_2) = L(\bar{x}, \theta'_2) = (2\pi\theta'_2)^{-n/2} \exp\left(-\frac{1}{2\theta'_2} \sum_{i=1}^n (x_i - \bar{x})^2\right)$$

$$L(\hat{\theta}_{1,0}, \theta'_2) = (2\pi\theta'_2)^{-n/2} \exp\left(-\frac{S^2}{2\theta'_2}\right)$$

Therefore the likelihood ratio statistic Λ is

$$\Lambda = \left(\frac{S^2}{n\theta'_2}\right)^{n/2} \exp\left(\frac{n}{2} - \frac{S^2}{2\theta'_2}\right)$$

which is a increasing function of S^2 . Then $\Lambda \geq c$ iff $S^2 \geq c'$ for some c' . By Student's theorem, we have $\frac{1}{\theta'_2} S^2 \sim \chi^2(n-1)$. Then the rejection region is

$$\frac{1}{\theta'_2} S^2 \leq \chi^2_{\alpha/2, n-1} \text{ or } \frac{1}{\theta'_2} S^2 \geq \chi^2_{1-\alpha/2, n-1}$$

i.e.

$$S^2 \leq \theta'_2 \chi^2_{\alpha/2, n-1} \text{ or } S^2 \geq \theta'_2 \chi^2_{1-\alpha/2, n-1}$$

Exercice 4

8.3.15. Let X_1, X_2, \dots, X_n and Y_1, Y_2, \dots, Y_n be independent random samples from two normal distributions $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, respectively, where σ^2 is the common but unknown variance.

- (a) Find the likelihood ratio Λ for testing $H_0 : \mu_1 = \mu_2 = 0$ against all alternatives.
- (b) Rewrite Λ so that it is a function of a statistic Z which has a well-known distribution.
- (c) Give the distribution of Z under both null and alternative hypotheses.

(a). $L(\mu, \mu_0, \sigma) = (2\pi)^{-\frac{n}{2}} \sigma^{-n} \exp \left\{ -\frac{1}{2\sigma^2} \left[\sum_{i=1}^n (x_i - \mu)^2 + \sum_{i=1}^n (y_i - \mu_0)^2 \right] \right\}$

$\hat{\mu} = \sup_{\theta \in \Theta_0} L(\theta) / \sup_{\theta \in \Theta} L(\theta)$

For $\theta \in \Theta_0$, $L(\theta) = (2\pi)^{-\frac{n}{2}} \sigma^{-n} \exp \left\{ -\frac{1}{2\sigma^2} \left[\sum_{i=1}^n (x_i^2 + \frac{n}{n-1} y_i^2) \right] \right\}$

$L(\theta) = -\frac{1}{2\sigma^2} \left[\sum_{i=1}^n (x_i^2 + y_i^2) \right] - 2n \log \sigma + \text{Const.}$

$\frac{\partial L}{\partial \sigma} = -\frac{1}{\sigma^3} \left[\sum_{i=1}^n (x_i^2 + y_i^2) \right] - \frac{2n}{\sigma} = 0$

$\Rightarrow \hat{\sigma}_0^2 = \frac{1}{2n} \sum_{i=1}^n (x_i^2 + y_i^2)$

For $\theta \in \Theta$, $L(\theta) = -\frac{1}{2\sigma^2} \left[\sum_{i=1}^n (x_i^2 - \mu_0^2) + (y_i^2 - \mu_0^2) \right] - 2n \log \sigma + \text{Const}$

$\frac{\partial L(\theta)}{\partial \mu_0} = -\frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu_0) = 0 \Rightarrow \hat{\mu}_0 = \bar{x}$

$\frac{\partial L(\theta)}{\partial \mu_0} = 0 \Rightarrow \hat{\mu}_0 = \bar{y}$

$\frac{\partial L(\theta)}{\partial \sigma} = 0 \Rightarrow \hat{\sigma}^2 = \frac{1}{2n} \sum_{i=1}^n (x_i - \hat{\mu}_0)^2 + (y_i - \hat{\mu}_0)^2$

$= \frac{1}{2n} \sum_{i=1}^n (x_i - \bar{x})^2 + (y_i - \bar{y})^2$

$\Rightarrow L(0, 0, \hat{\sigma}_0^2) = \frac{\left(\frac{1}{2n} \sum_{i=1}^n (x_i^2 + y_i^2) \right)^{n/2}}{\left(\frac{1}{2n} \sum_{i=1}^n (x_i - \bar{x})^2 + (y_i - \bar{y})^2 \right)^{n/2}} \exp \left\{ -\frac{1}{2\hat{\sigma}_0^2} \right\}$

$L(\hat{\mu}_0, \hat{\mu}_0, \hat{\sigma}^2) = \left(\frac{\sum_{i=1}^n x_i^2 + \sum_{i=1}^n y_i^2}{\sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{i=1}^n (y_i - \bar{y})^2} \right)^{-n}$

(b). Denote $S_x^2 := \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$
 $S_y^2 := \frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})^2$
 $\text{Then } \sum_{i=1}^n x_i^2 = \sum_{i=1}^n (x_i - \bar{x})^2 + n\bar{x}^2 = (n-1)S_x^2 + n\bar{x}^2$
 $\sum_{i=1}^n y_i^2 = (n-1)S_y^2 + n\bar{y}^2$
 $\Rightarrow \Lambda = \left(1 + \frac{n(\bar{x}^2 + \bar{y}^2)}{(n-1)(S_x^2 + S_y^2)} \right)^{-n} \text{ where } \bar{x} \sim N(\mu_1, \frac{\sigma^2}{n})$
 $\bar{y} \sim N(\mu_2, \frac{\sigma^2}{n})$
Under H_0 , $\bar{x} \sim N(0, \frac{\sigma^2}{n})$, $\bar{y} \sim N(0, \frac{\sigma^2}{n})$
 $\text{Then } \Lambda = \left(1 + \frac{[\bar{x}/(\sigma/\sqrt{n})]^2 + [\bar{y}/(\sigma/\sqrt{n})]^2}{(n-1)S_x^2/\sigma^2 + (n-1)S_y^2/\sigma^2} \right)^{-n}$
where $\bar{x}/(\sigma/\sqrt{n}) \sim N(0, 1)$, $\bar{y}/(\sigma/\sqrt{n}) \sim N(0, 1)$
By Student's t-thm, $(n-1)S_x^2/\sigma^2 \sim \chi^2(n-1)$, $(n-1)S_y^2/\sigma^2 \sim \chi^2(n-1)$
 $S_x^2, \bar{x}, S_y^2, \bar{y}$ are independent
 $\Rightarrow Z := \frac{\{\bar{x}/(\sigma/\sqrt{n})^2 + \bar{y}/(\sigma/\sqrt{n})^2\}^{1/2}}{[(n-1)S_x^2/\sigma^2 + (n-1)S_y^2/\sigma^2]^{1/2}}$
The numerator $\sim \chi^2(2)$ while
where $[\bar{x}/(\sigma/\sqrt{n})]^2 + [\bar{y}/(\sigma/\sqrt{n})]^2 \sim \chi^2(2)$
 $(n-1)S_x^2/\sigma^2 + (n-1)S_y^2/\sigma^2 \sim \chi^2(2n-2)$
Under H_0 , $Z \sim F(2, 2n-2)$
Under H_1 , this is a hard problem.
Under H_1 , $Z \sim F(2, 2n-2, \lambda)$, where $\lambda := \frac{n(\mu_1^2 + \mu_2^2)}{\sigma^2}$.
Claim that, $Z \sim F(2, 2n-2, \lambda)$.

Under H_1 ,

$$Z = \frac{\{[\bar{X}/(\sigma/\sqrt{n})]^2 + [\bar{Y}/(\sigma/\sqrt{n})]^2\}/2}{[(n-1)S_x^2/\sigma^2 + (n-1)S_y^2/\sigma^2]/(2n-2)}$$

where $\bar{X}/(\sigma/\sqrt{n}) \sim N(\mu_1/(\sigma/\sqrt{n}), 1)$ and $\bar{Y}/(\sigma/\sqrt{n}) \sim N(\mu_2/(\sigma/\sqrt{n}), 1)$. By the definition of noncentral chi-square distribution,

$$[\bar{X}/(\sigma/\sqrt{n})]^2 + [\bar{Y}/(\sigma/\sqrt{n})]^2 \sim \chi^2(2, \lambda)$$

where $\lambda = \frac{\mu_1^2 + \mu_2^2}{(\sigma/\sqrt{n})^2} = \frac{n(\mu_1^2 + \mu_2^2)}{\sigma^2}$. Thus $Z \sim F(2, 2n-2, \lambda)$.

Exercice 5

8.3.21. Suppose X_1, \dots, X_n is a random sample on X which has a $N(\mu, \sigma_0^2)$ distribution, where σ_0^2 is known. Consider the two-sided hypotheses

$$H_0 : \mu = 0 \text{ versus } H_1 : \mu \neq 0.$$

Show that the test based on the critical region $C = \{|\bar{X}| > \sqrt{\sigma_0^2/n} z_{\alpha/2}\}$ is an unbiased level α test.

Under H_0 , $\mathbb{P}_{H_0}(\mathbf{X} \in C) = \mathbb{P}_{H_0}\left(\left|\frac{\bar{X}}{\sqrt{\sigma_0^2/n}}\right| > z_{\alpha/2}\right) = \alpha.$

$$\bar{X} = \mu + \sqrt{\sigma_0^2/n} \cdot Z \sim N\left(\mu, \frac{\sigma_0^2}{n}\right)$$

where $Z \sim N(0, 1)$. Then

$$\begin{aligned} C &= \{|\bar{X}| > \sqrt{\sigma_0^2/n}z_{\alpha/2}\} \\ &= \{|\mu + \sqrt{\sigma_0^2/n}Z| > \sqrt{\sigma_0^2/n}z_{\alpha/2}\} \\ &\stackrel{c:=\mu/\sqrt{\sigma_0^2/n}}{=} \{Z > z_{\alpha/2} - c\} \cup \{Z < -z_{\alpha/2} - c\} \end{aligned}$$

For any $\mu \neq 0$, we have $c \neq 0$. Then

$$\begin{aligned} \mathbb{P}_\mu(\mathbf{X} \in C) &= \mathbb{P}(Z > z_{\alpha/2} - c) + \mathbb{P}(Z < -z_{\alpha/2} - c) \\ &= \mathbb{P}(Z > z_{\alpha/2}) + \mathbb{P}(z_{\alpha/2} - c < Z \leq z_{\alpha/2}) + \mathbb{P}(Z < -z_{\alpha/2}) - \mathbb{P}(-z_{\alpha/2} - c \leq Z < -z_{\alpha/2}) \\ &= \alpha + \mathbb{P}(z_{\alpha/2} - c < Z \leq z_{\alpha/2}) - \mathbb{P}(-z_{\alpha/2} - c \leq Z < -z_{\alpha/2}) \\ &= \alpha + \mathbb{P}(z_{\alpha/2} - c < Z \leq z_{\alpha/2}) - \mathbb{P}(z_{\alpha/2} < Z \leq z_{\alpha/2} + c) \\ &\geq \alpha \end{aligned}$$

Thus the test based on the critical region C is an unbiased level α test.

Exercice 6

9.2.1. Consider the T -statistic that was derived through a likelihood ratio for testing the equality of the means of two normal distributions having common variance in Example 8.3.1. Show that T^2 is exactly the F -statistic of expression (9.2.11).

$$\begin{aligned} T &= \sqrt{\frac{nm}{n+m}}(\bar{X} - \bar{Y}) \left\{ (n+m-2)^{-1} \left[\sum_1^n (X_i - \bar{X})^2 + \sum_1^m (Y_i - \bar{Y})^2 \right] \right\}^{-1/2} \\ T^2 &= \frac{nm}{n+m}(\bar{X} - \bar{Y})^2 \frac{(n+m-2)}{\sum_1^n (X_i - \bar{X})^2 + \sum_1^m (Y_i - \bar{Y})^2} \end{aligned}$$

In (9.2.11),

$$F = \frac{Q_4/(b-1)}{Q_3/(n-b)}$$

where

$$Q_3 = \sum_{j=1}^b \sum_{i=1}^{n_j} (x_{ij} - \bar{x}_{\cdot j})^2$$

$$Q_4 = \sum_{j=1}^b n_j (\bar{x}_{\cdot j} - \bar{x}_{\cdot \cdot})^2$$

Let $j = 2$, and $x_{i1} = X_i$ for $i = 1, \dots, n$, $x_{i2} = Y_i$ for $i = 1, \dots, m$. Then

$$\bar{x}_{\cdot 1} = \bar{X}, \quad \bar{x}_{\cdot 2} = \bar{Y}$$

$$\bar{x}_{..} = \frac{n\bar{x}_{\cdot 1} + m\bar{x}_{\cdot 2}}{n+m} = \frac{n\bar{X} + m\bar{Y}}{n+m}$$

Then

$$Q_4 = n \left(\bar{X} - \frac{n\bar{X} + m\bar{Y}}{n+m} \right)^2 + m \left(\bar{Y} - \frac{n\bar{X} + m\bar{Y}}{n+m} \right)^2 = \frac{mn}{m+n} (\bar{X} - \bar{Y})^2$$

$$Q_3 = \sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^m (Y_i - \bar{Y})^2$$

$$b-1 = 1, \quad n+m-b = m+n-2$$

Thus

$$T^2 = F$$

for $j = 2$, and $x_{i1} = X_i$ for $i = 1, \dots, n$, $x_{i2} = Y_i$ for $i = 1, \dots, m$.

Exercice 7

9.2.5. Using the notation of this section, assume that the means satisfy the condition that $\mu = \mu_1 + (b-1)d = \mu_2 - d = \mu_3 - d = \dots = \mu_b - d$. That is, the last $b-1$ means are equal but differ from the first mean μ_1 , provided that $d \neq 0$. Let independent random samples of size a be taken from the b normal distributions with common unknown variance σ^2 .

- (a) Show that the maximum likelihood estimators of μ and d are $\hat{\mu} = \bar{X}_{..}$ and

$$\hat{d} = \frac{\sum_{j=2}^b \bar{X}_{\cdot j} / (b-1) - \bar{X}_{\cdot 1}}{b}.$$

- (b) Using Exercise 9.2.4, find Q_6 and $Q_7 = cd^2$ so that, when $d = 0$, Q_7/σ^2 is $\chi^2(1)$ and

$$\sum_{i=1}^a \sum_{j=1}^b (X_{ij} - \bar{X}_{..})^2 = Q_3 + Q_6 + Q_7.$$

- (c) Argue that the three terms in the right-hand member of part (b), once divided by σ^2 , are independent random variables with chi-square distributions, provided that $d = 0$.

- (d) The ratio $Q_7/(Q_3 + Q_6)$ times what constant has an F -distribution, provided that $d = 0$? Note that this F is really the square of the two-sample T used to test the equality of the mean of the first distribution and the common mean of the other distributions, in which the last $b-1$ samples are combined into one.

(a)

$$(a) \quad \bar{x}_{ij} \sim N(\mu_j, \sigma^2), \quad i=1, \dots, a, \quad j=1, \dots, b$$

$$\begin{aligned} L(\mu, d) &= \prod_{j=1}^b \prod_{i=1}^a (2\pi)^{-\frac{1}{2}} \sigma^{-1} \exp \left\{ -\frac{1}{2\sigma^2} (x_{ij} - \mu_j)^2 \right\} \\ &= \prod_{j=1}^b (2\pi)^{-\frac{ab}{2}} \sigma^{-ab} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^a (x_{ij} - \mu_j)^2 \right\} \\ &= (2\pi)^{-ab/2} \sigma^{-ab} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{j=1}^b \sum_{i=1}^a (x_{ij} - \mu_j)^2 \right\} \\ &\stackrel{\text{且 } \mu_1 = \mu - (b-1)d, \mu_2 = \dots = \mu_b = \mu + d}{=} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{j=1}^b \sum_{i=1}^a (x_{ij} - \mu)^2 \right\} \end{aligned}$$

$$l(\mu, d) = -\frac{1}{2\sigma^2} \left[\sum_{j=1}^b \sum_{i=1}^a (x_{ij} - \mu - d)^2 + \sum_{i=1}^a (x_{ii} - \mu + (b-1)d)^2 \right]$$

$$\frac{\partial l}{\partial \mu} = \frac{1}{2\sigma^2} \left[\sum_{j=1}^b \sum_{i=1}^a 2(x_{ij} - \mu - d) + \sum_{i=1}^a 2(x_{ii} - \mu + (b-1)d) \right] = 0$$

$$\frac{\partial l}{\partial d} = \frac{1}{2\sigma^2} \left[\sum_{j=1}^b \sum_{i=1}^a 2(x_{ij} - \mu - d) + \cancel{2(b-1) \sum_{i=1}^a (x_{ii} - \mu + (b-1)d)} \right] = 0$$

$$\Rightarrow \sum_{j=1}^b \sum_{i=1}^a x_{ij} - [a(b-1) + a]\mu - [a(b-1) + a(b-1)]d = 0$$

$$-2b \sum_{i=1}^a x_{ii} + \sum_{j=1}^b \sum_{i=1}^a x_{ij} - [a(b-1) + a(b-1)]\mu - [a(b-1) + a(b-1)]d = 0$$

$$\Rightarrow \hat{\mu} = \frac{1}{ab} \sum_{j=1}^b \sum_{i=1}^a x_{ij} = \bar{x}_{..}$$

$$\hat{d} = \frac{1}{ab(b-1)} \left[\sum_{j=1}^b \sum_{i=1}^a x_{ij} - 2(b-1) \sum_{i=1}^a x_{ii} \right] = \frac{\sum_{j=2}^b \bar{x}_{ij}/(b-1) - \bar{x}_{..}}{b}$$

(b)(c)

(b) $Q_3 = \sum_{j=1}^b \sum_{i=1}^a (x_{ij} - \bar{x}_{..})^2$

$$\begin{aligned} \sum_{i=1}^a \sum_{j=1}^b (x_{ij} - \bar{x}_{..})^2 &= \sum_{i=1}^a \sum_{j=1}^b (\bar{x}_{ij} - \bar{x}_{..} + \bar{x}_{..} - \bar{x}_{ij})^2 \\ &= \sum_{i=1}^a \sum_{j=1}^b (\bar{x}_{ij} - \bar{x}_{..})^2 + 2 \underbrace{\sum_{i=1}^a \sum_{j=1}^b (\bar{x}_{ij} - \bar{x}_{..})(\bar{x}_{..} - \bar{x}_{ij})}_{=0} + \sum_{i=1}^a \sum_{j=1}^b (\bar{x}_{..} - \bar{x}_{ij})^2 \\ &= Q_7 + a \sum_{j=1}^b (\bar{x}_{..} - \bar{x}_{ij})^2 \end{aligned}$$

$$Q_7 = c d^2 = c \cdot \frac{1}{b} \left(\sum_{j=1}^b \bar{x}_{..} - \bar{x}_{ij} \right)^2$$

when $d=0$, $x_{ij} \sim N(\mu, \sigma^2) \Rightarrow \bar{x}_{ij} \sim N(\mu, \frac{\sigma^2}{a}) \Rightarrow \sum_{j=1}^b \bar{x}_{ij} \sim N(a\mu, \frac{b\sigma^2}{a})$

$$\Rightarrow \frac{1}{b} \sum_{j=1}^b \bar{x}_{..} \sim N(\mu, \frac{1}{ab} \sigma^2) \Rightarrow \frac{1}{b} \sum_{j=1}^b \bar{x}_{..} - \bar{x}_{ij} \sim N(0, \frac{b}{ab} \sigma^2)$$

$$\Rightarrow \left(\frac{1}{b} \sum_{j=1}^b \bar{x}_{..} - \bar{x}_{ij} \right)^2 / \left(\frac{b}{ab} \sigma^2 \right) \sim \chi^2(1)$$

$$\Rightarrow \frac{c}{\sigma^2} = \frac{a(b-1)}{b^2} \Rightarrow c = \frac{ab(b-1)}{\sigma^2}$$

$$\Rightarrow Q_7 = \frac{a(b-1)}{b} \left(\sum_{j=1}^b \bar{x}_{..} - \bar{x}_{ij} \right)^2$$

$$Q_6 = a \sum_{j=1}^b (\bar{x}_{..} - \bar{x}_{ij})^2 - Q_7$$

$$\sum_{j=1}^b (\bar{x}_{..} - \bar{x}_{ij})^2 \stackrel{(a=9, b=4)}{=} \sum_{j=2}^b (\bar{x}_{..} - \bar{x}_{..})^2 + \frac{b-1}{b} (\bar{x}_{..} - \bar{x}_{..})^2 \quad \text{其中 } \bar{x}_{..} = \frac{1}{b} \sum_{j=1}^b \bar{x}_{..}$$

$$\Rightarrow Q_6 = a \sum_{j=1}^b (\bar{x}_{..} - \bar{x}_{ij})^2 - Q_7$$

$$= a \sum_{j=1}^b \left(\bar{x}_{..} - \frac{1}{b-1} \sum_{j=2}^b \bar{x}_{..} \right)^2 + \underbrace{\frac{a(b-1)}{b} (\bar{x}_{..} - \frac{1}{b-1} \sum_{j=2}^b \bar{x}_{..})^2}_{= Q_1} - Q_7$$

$$= a \sum_{j=1}^b \left(\bar{x}_{..} - \frac{1}{b-1} \sum_{j=2}^b \bar{x}_{..} \right)^2$$

(c). when $d=0$, $\frac{Q_7}{\sigma^2} \sim \chi^2(1)$. Consider $\frac{Q_3}{\sigma^2}$, $x_{ij} \sim N(\mu, \sigma^2) \Rightarrow \bar{x}_{ij} \sim \bar{x}_{..}$

$$\Rightarrow \bar{x}_{ij} \sim N(\mu, \frac{\sigma^2}{a})$$

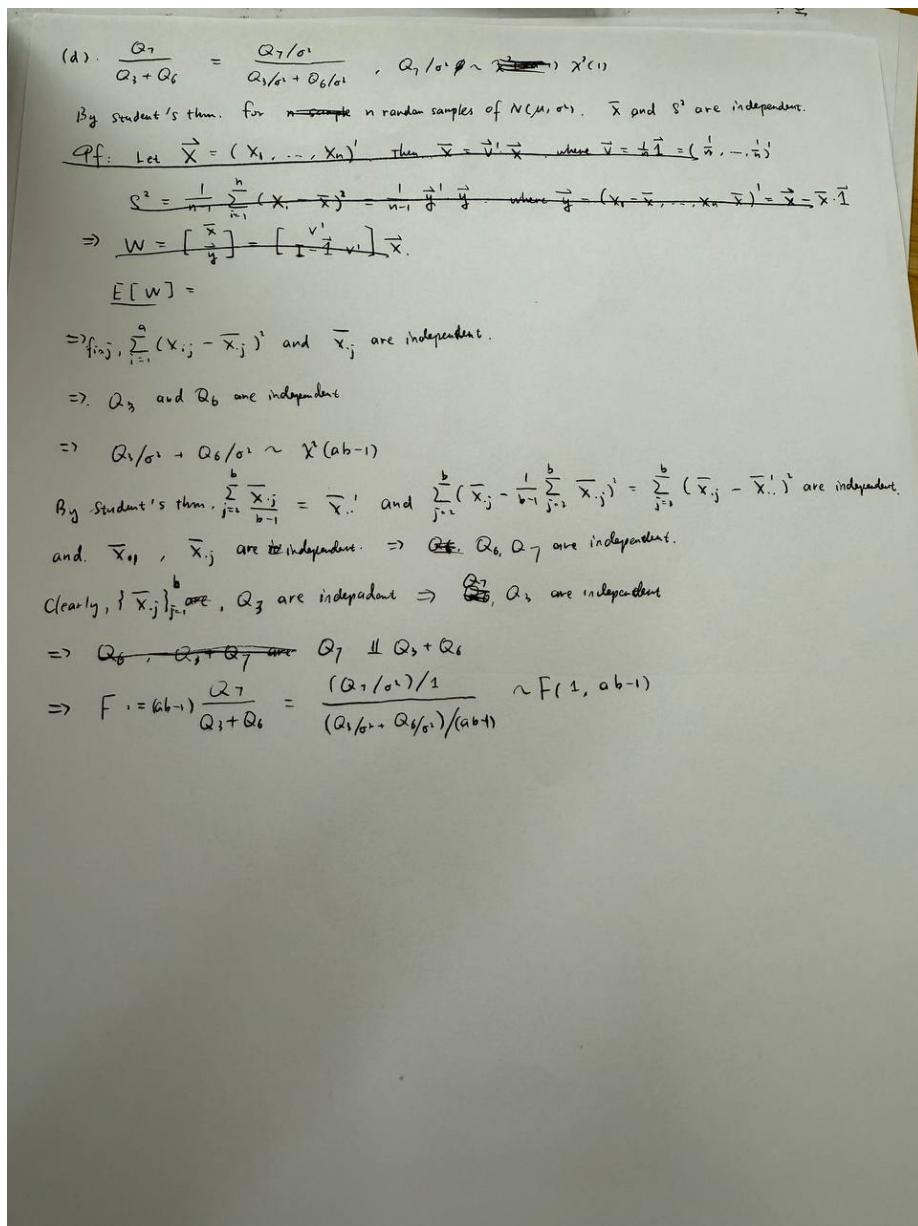
By Student's t-thm, $\sum_{i=1}^a (x_{ij} - \bar{x}_{ij})^2 / \sigma^2 \sim \chi^2(a-1)$

$$\Rightarrow Q_3 / \sigma^2 = \sum_{j=1}^b \sum_{i=1}^a (x_{ij} - \bar{x}_{ij})^2 / \sigma^2 \sim \chi^2((a-1)b)$$

Consider $\frac{Q_6}{\sigma^2}$, by student's t-thm, $\bar{x}_{..} \sim \frac{1}{a} \sum_{i=1}^a x_{ij} \sim N(\mu, \frac{\sigma^2}{a})$,

$$Q_6 / \sigma^2 = \sum_{j=1}^b (\bar{x}_{..} - \bar{x}_{..})^2 / (b/a) \sim \chi^2(b-1)$$

(d)



Exercice 8

9.2.8. The driver of a diesel-powered automobile decided to test the quality of three types of diesel fuel sold in the area based on mpg. Test the null hypothesis that the three means are equal using the following data. Make the usual assumptions and take $\alpha = 0.05$.

Brand A:	38.7	39.2	40.1	38.9
Brand B:	41.9	42.3	41.3	
Brand C:	40.8	41.2	39.5	38.9

To test the null hypothesis that the three means are equal, we can use a one-way Analysis of Variance (ANOVA).

Given data: Brand A: 38.7, 39.2, 40.1, 38.9

Brand B: 41.9, 42.3, 41.3

Brand C: 40.8, 41.2, 39.5, 38.9, 40.3

(1) State the null and alternative hypotheses.

$H_0 : \mu_A = \mu_B = \mu_C$ (The means of mpg for the three brands of diesel fuel are equal)

$H_1 :$ At least one mean is different

(2) Calculate the sample means for each brand.

$$\hat{x}_A = (38.7 + 39.2 + 40.1 + 38.9)/4 = 156.9/4 = 39.225$$

$$\hat{x}_B = (41.9 + 42.3 + 41.3)/3 = 125.5/3 = 41.833$$

$$\hat{x}_C = (40.8 + 41.2 + 39.5 + 38.9 + 40.3)/5 = 200.7/5 = 40.14$$

(3) Calculate the overall mean (grand mean). Total number of observations

$$N = 4 + 3 + 5 = 12 \quad \text{Sum of all observations} = 38.7 + 39.2 + 40.1 + 38.9 + 41.9 + 42.3 + 41.3 + 40.8 + 41.2 + 39.5 + 38.9 + 40.3 = 483.1$$

$$\bar{x} = 483.1/12 = 40.258$$

(4) Calculate the Sum of Squares Between (SSB) treatments. $n_A = 4$, $n_B = 3$, $n_C = 5$

$$SSB = n_A(\hat{x}_A - \bar{x})^2 + n_B(\hat{x}_B - \bar{x})^2 + n_C(\hat{x}_C - \bar{x})^2$$

$$SSB = 4(39.225 - 40.258)^2 + 3(41.833 - 40.258)^2 + 5(40.14 - 40.258)^2$$

$$SSB = 4(-1.033)^2 + 3(1.575)^2 + 5(-0.118)^2$$

$$SSB = 4(1.067089) + 3(2.480625) + 5(0.013924)$$

$$SSB = 4.268356 + 7.441875 + 0.06962 = 11.779851$$

- (5) Calculate the Sum of Squares Within (SSW) treatments (Error Sum of Squares). For Brand A: $(38.7 - 39.225)^2 + (39.2 - 39.225)^2 + (40.1 - 39.225)^2 + (38.9 - 39.225)^2$

$$= (-0.525)^2 + (-0.025)^2 + (0.875)^2 + (-0.325)^2$$

$$= 0.275625 + 0.000625 + 0.765625 + 0.105625 = 1.1475$$

$$\text{For Brand B: } (41.9 - 41.833)^2 + (42.3 - 41.833)^2 + (41.3 - 41.833)^2$$

$$= (0.067)^2 + (0.467)^2 + (-0.533)^2$$

$$= 0.004489 + 0.218089 + 0.284089 = 0.506667$$

$$\text{For Brand C: } (40.8 - 40.14)^2 + (41.2 - 40.14)^2 + (39.5 - 40.14)^2 + (38.9 - 40.14)^2 + (40.3 - 40.14)^2$$

$$= (0.66)^2 + (1.06)^2 + (-0.64)^2 + (-1.24)^2 + (0.16)^2$$

$$= 0.4356 + 1.1236 + 0.4096 + 1.5376 + 0.0256 = 3.532$$

$$SSW = 1.1475 + 0.506667 + 3.532 = 5.186167$$

- (6) Calculate the Degrees of Freedom. Degrees of freedom between groups (df_B) = $k - 1 = 3 - 1 = 2$ (where k is the number of groups) Degrees of freedom within groups (df_W) = $N - k = 12 - 3 = 9$ Total degrees of freedom (df_T) = $N - 1 = 12 - 1 = 11$

- (7) Calculate the Mean Squares. Mean Square Between (MSB)

$$= SSB/df_B = 11.779851/2 = 5.8899255$$

Mean Square Within (MSW)

$$= SSW/df_W = 5.186167/9 = 0.57624077$$

- (8) Calculate the F-statistic.

$$F = MSB/MSW = 5.8899255/0.57624077 = 10.2212$$

- (9) Determine the critical F-value. Using $\alpha = 0.05$, $df_1 = 2$, and $df_2 = 9$. From the F-distribution table, the critical F-value is approximately $F_{0.05, 2, 9} = 4.26$.
- (10) Make a decision. Since the calculated F-statistic (10.2212) is greater than the critical F-value (4.26), we reject the null hypothesis.

Conclusion: At a 0.05 level of significance, there is sufficient evidence to conclude that at least one of the mean mpg for the three types of diesel fuel is different.

Exercice 9

9.5.7. With the background of the two-way classification with $c > 1$ observations per cell, determine the distribution of the mles of α_i, β_j , and γ_{ij} .

Let's revisit the log-likelihood function:

$$\ln L = -\frac{abc}{2} \ln(2\pi) - \frac{abc}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c (X_{ijk} - (\mu + \alpha_i + \beta_j + \gamma_{ij}))^2$$

To find the MLEs, we need to maximize this function with respect to $\mu, \alpha_i, \beta_j, \gamma_{ij}$, and σ^2 . This involves taking partial derivatives with respect to each parameter and setting them to zero. We'll also impose the standard identifiability constraints to ensure a unique solution:

- (1) $\sum_{i=1}^a \alpha_i = 0$
- (2) $\sum_{j=1}^b \beta_j = 0$
- (3) $\sum_{i=1}^a \gamma_{ij} = 0$ for all j
- (4) $\sum_{j=1}^b \gamma_{ij} = 0$ for all i

Let $P_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij}$. The terms involving the parameters only appear in the sum of squares part. Minimizing the sum of squares is equivalent to maximizing the log-likelihood for these parameters.

$$S = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c (X_{ijk} - P_{ijk})^2$$

1. Derivative with respect to μ :

$$\frac{\partial S}{\partial \mu} = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c 2(X_{ijk} - (\mu + \alpha_i + \beta_j + \gamma_{ij}))(-1) = 0$$

$$\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c X_{ijk} = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c (\mu + \alpha_i + \beta_j + \gamma_{ij})$$

$$abc\bar{X}_{...} = abc\mu + bc \sum_i \alpha_i + ac \sum_j \beta_j + c \sum_i \sum_j \gamma_{ij}$$

Applying the constraints ($\sum_i \alpha_i = 0$, $\sum_j \beta_j = 0$, and $\sum_i \sum_j \gamma_{ij} = \sum_j (\sum_i \gamma_{ij}) = \sum_j 0 = 0$):

$$abc\bar{X}_{...} = abc\hat{\mu} + 0 + 0 + 0$$

$$\hat{\mu} = \bar{X}_{...}$$

2. Derivative with respect to α_i (for a specific i_0):

$$\frac{\partial S}{\partial \alpha_{i_0}} = \sum_{j=1}^b \sum_{k=1}^c 2(X_{i_0 j k} - (\mu + \alpha_{i_0} + \beta_j + \gamma_{i_0 j}))(-1) = 0$$

$$\sum_{j=1}^b \sum_{k=1}^c X_{i_0 j k} = \sum_{j=1}^b \sum_{k=1}^c (\mu + \alpha_{i_0} + \beta_j + \gamma_{i_0 j})$$

$$bc\bar{X}_{i_0 ..} = bc\hat{\mu} + bc\hat{\alpha}_{i_0} + c \sum_j \hat{\beta}_j + c \sum_j \hat{\gamma}_{i_0 j}$$

Applying the constraints ($\sum_j \hat{\beta}_j = 0$ and $\sum_j \hat{\gamma}_{i_0 j} = 0$ for a given i_0):

$$bc\bar{X}_{i_0 ..} = bc\hat{\mu} + bc\hat{\alpha}_{i_0} + 0 + 0$$

Substituting $\hat{\mu} = \bar{X}_{...}$:

$$bc\bar{X}_{i_0 ..} = bc\bar{X}_{...} + bc\hat{\alpha}_{i_0}$$

$$\hat{\alpha}_{i_0} = \bar{X}_{i_0 ..} - \bar{X}_{...}$$

This applies for any i_0 , so $\hat{\alpha}_i = \bar{X}_{i ..} - \bar{X}_{...}$.

3. Derivative with respect to β_j (for a specific j_0):

By symmetry with α_i :

$$\frac{\partial S}{\partial \beta_{j_0}} = \sum_{i=1}^a \sum_{k=1}^c 2(X_{i j_0 k} - (\mu + \alpha_i + \beta_{j_0} + \gamma_{i j_0}))(-1) = 0$$

$$\sum_{i=1}^a \sum_{k=1}^c X_{i j_0 k} = \sum_{i=1}^a \sum_{k=1}^c (\mu + \alpha_i + \beta_{j_0} + \gamma_{i j_0})$$

$$ac\bar{X}_{.j_0 ..} = ac\hat{\mu} + c \sum_i \hat{\alpha}_i + ac\hat{\beta}_{j_0} + c \sum_i \hat{\gamma}_{i j_0}$$

Applying the constraints ($\sum_i \hat{\alpha}_i = 0$ and $\sum_i \hat{\gamma}_{i j_0} = 0$ for a given j_0):

$$ac\bar{X}_{.j_0 ..} = ac\hat{\mu} + 0 + ac\hat{\beta}_{j_0} + 0$$

Substituting $\hat{\mu} = \bar{X}_{...}$:

$$ac\bar{X}_{.j_0..} = ac\bar{X}_{...} + ac\hat{\beta}_{j_0}$$

$$\hat{\beta}_{j_0} = \bar{X}_{.j_0..} - \bar{X}_{...}$$

This applies for any j_0 , so $\hat{\beta}_j = \bar{X}_{.j..} - \bar{X}_{...}$.

4. Derivative with respect to γ_{ij} (for a specific i_0, j_0):

$$\begin{aligned} \frac{\partial S}{\partial \gamma_{i_0 j_0}} &= \sum_{k=1}^c 2(X_{i_0 j_0 k} - (\mu + \alpha_{i_0} + \beta_{j_0} + \gamma_{i_0 j_0}))(-1) = 0 \\ \sum_{k=1}^c X_{i_0 j_0 k} &= \sum_{k=1}^c (\mu + \alpha_{i_0} + \beta_{j_0} + \gamma_{i_0 j_0}) \\ c\bar{X}_{i_0 j_0..} &= c\hat{\mu} + c\hat{\alpha}_{i_0} + c\hat{\beta}_{j_0} + c\hat{\gamma}_{i_0 j_0} \end{aligned}$$

Substituting the MLEs for $\hat{\mu}, \hat{\alpha}_{i_0}, \hat{\beta}_{j_0}$:

$$c\bar{X}_{i_0 j_0..} = c(\bar{X}_{...}) + c(\bar{X}_{i_0..} - \bar{X}_{...}) + c(\bar{X}_{.j_0..} - \bar{X}_{...}) + c\hat{\gamma}_{i_0 j_0}$$

Divide by c :

$$\begin{aligned} \bar{X}_{i_0 j_0..} &= \bar{X}_{...} + \bar{X}_{i_0..} - \bar{X}_{...} + \bar{X}_{.j_0..} - \bar{X}_{...} + \hat{\gamma}_{i_0 j_0} \\ \bar{X}_{i_0 j_0..} &= \bar{X}_{i_0..} + \bar{X}_{.j_0..} - \bar{X}_{...} + \hat{\gamma}_{i_0 j_0} \\ \hat{\gamma}_{i_0 j_0} &= \bar{X}_{i_0 j_0..} - \bar{X}_{i_0..} - \bar{X}_{.j_0..} + \bar{X}_{...} \end{aligned}$$

This applies for any i_0, j_0 , so $\hat{\gamma}_{ij} = \bar{X}_{ij..} - \bar{X}_{i..} - \bar{X}_{.j..} + \bar{X}_{...}$.

5. Derivative with respect to σ^2 :

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{abc}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c (X_{ijk} - (\mu + \alpha_i + \beta_j + \gamma_{ij}))^2 = 0$$

Setting to zero and replacing parameters with their MLEs (since the value of σ^2 that maximizes the likelihood also maximizes it for the other parameters fixed at their MLEs):

$$\hat{\sigma}^2 = \frac{1}{abc} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c (X_{ijk} - (\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\gamma}_{ij}))^2$$

Since $\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\gamma}_{ij} = \bar{X}_{ij..}$, we get:

$$\hat{\sigma}^2 = \frac{1}{abc} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c (X_{ijk} - \bar{X}_{ij..})^2$$

Distribution of the MLEs

Since the X_{ijk} are assumed to be independent and identically distributed $N(\mu + \alpha_i + \beta_j + \gamma_{ij}, \sigma^2)$, and all the MLEs $\hat{\mu}, \hat{\alpha}_i, \hat{\beta}_j, \hat{\gamma}_{ij}$ are linear combinations of these normal random variables, they will also be normally distributed.

Let $E[\hat{\theta}]$ and $Var(\hat{\theta})$ denote the expected value and variance of an estimator $\hat{\theta}$.

1. Distribution of $\hat{\mu}$:

$$\begin{aligned}\hat{\mu} &= \frac{1}{abc} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c X_{ijk} \\ E[\hat{\mu}] &= \frac{1}{abc} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c E[X_{ijk}] = \frac{1}{abc} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c (\mu + \alpha_i + \beta_j + \gamma_{ij})\end{aligned}$$

Using the constraints: $\sum \alpha_i = 0, \sum \beta_j = 0, \sum_i \gamma_{ij} = 0, \sum_j \gamma_{ij} = 0$.

$$E[\hat{\mu}] = \frac{1}{abc} (abc\mu + bc \sum_i \alpha_i + ac \sum_j \beta_j + c \sum_i \sum_j \gamma_{ij}) = \frac{1}{abc} (abc\mu + 0 + 0 + 0) = \mu$$

$$Var(\hat{\mu}) = Var\left(\frac{1}{abc} \sum X_{ijk}\right) = \frac{1}{(abc)^2} \sum Var(X_{ijk}) = \frac{abc\sigma^2}{(abc)^2} = \frac{\sigma^2}{abc}$$

So, $\hat{\mu} \sim N\left(\mu, \frac{\sigma^2}{abc}\right)$.

2. Distribution of $\hat{\alpha}_i$:

$$\hat{\alpha}_i = \bar{X}_{i..} - \bar{X}_{...}$$

$$\begin{aligned}E[\hat{\alpha}_i] &= E[\bar{X}_{i..}] - E[\bar{X}_{...}] \\ E[\bar{X}_{i..}] &= \frac{1}{bc} \sum_{j=1}^b \sum_{k=1}^c E[X_{ijk}] = \frac{1}{bc} \sum_{j=1}^b \sum_{k=1}^c (\mu + \alpha_i + \beta_j + \gamma_{ij}) \\ &= \frac{1}{bc} (bc\mu + bc\alpha_i + c \sum_j \beta_j + c \sum_j \gamma_{ij})\end{aligned}$$

Using constraints: $\sum_j \beta_j = 0$ and $\sum_j \gamma_{ij} = 0$.

$$E[\bar{X}_{i..}] = \frac{1}{bc} (bc\mu + bc\alpha_i) = \mu + \alpha_i$$

So, $E[\hat{\alpha}_i] = (\mu + \alpha_i) - \mu = \alpha_i$.

For the variance of $\hat{\alpha}_i$, we use $Var(Y_1 - Y_2) = Var(Y_1) + Var(Y_2) - 2Cov(Y_1, Y_2)$.

$$Var(\bar{X}_{i..}) = \frac{\sigma^2}{bc}$$

$$Var(\bar{X}_{...}) = \frac{\sigma^2}{abc}$$

$$Cov(\bar{X}_{i..}, \bar{X}_{...}) = Cov\left(\frac{1}{bc} \sum_{j=1}^b \sum_{k=1}^c X_{ijk}, \frac{1}{abc} \sum_{p=1}^a \sum_{q=1}^b \sum_{r=1}^c X_{pqr}\right)$$

The terms common to both sums are those where $p = i$. So there are bc common terms.

$$Cov(\bar{X}_{i..}, \bar{X}_{...}) = \frac{1}{bc} \frac{1}{abc} \sum_{j=1}^b \sum_{k=1}^c Var(X_{ijk}) = \frac{1}{abc^2} \sum_{j=1}^b \sum_{k=1}^c \sigma^2 = \frac{bc\sigma^2}{abc^2} = \frac{\sigma^2}{abc}$$

$$Var(\hat{\alpha}_i) = \frac{\sigma^2}{bc} + \frac{\sigma^2}{abc} - 2 \frac{\sigma^2}{abc} = \frac{\sigma^2}{bc} - \frac{\sigma^2}{abc} = \sigma^2 \left(\frac{a-1}{abc} \right)$$

So, $\hat{\alpha}_i \sim N(\alpha_i, \sigma^2 \frac{a-1}{abc})$.

3. Distribution of $\hat{\beta}_j$:

By symmetry with $\hat{\alpha}_i$:

$$E[\hat{\beta}_j] = \beta_j$$

$$Var(\hat{\beta}_j) = \sigma^2 \left(\frac{1}{ac} - \frac{1}{abc} \right) = \sigma^2 \left(\frac{b-1}{abc} \right)$$

So, $\hat{\beta}_j \sim N(\beta_j, \sigma^2 \frac{b-1}{abc})$.

4. Distribution of $\hat{\gamma}_{ij}$:

$$\hat{\gamma}_{ij} = \bar{X}_{ij.} - \bar{X}_{i..} - \bar{X}_{.j.} + \bar{X}_{...}$$

$$E[\hat{\gamma}_{ij}] = E[\bar{X}_{ij.}] - E[\bar{X}_{i..}] - E[\bar{X}_{.j.}] + E[\bar{X}_{...}]$$

$$E[\bar{X}_{ij.}] = \mu + \alpha_i + \beta_j + \gamma_{ij}$$

So, $E[\hat{\gamma}_{ij}] = (\mu + \alpha_i + \beta_j + \gamma_{ij}) - (\mu + \alpha_i) - (\mu + \beta_j) + \mu = \gamma_{ij}$.

Calculating the variance of $\hat{\gamma}_{ij}$ is more complex due to the multiple covariance terms. However, since it is a linear combination of normally distributed random variables, it is also normally distributed. Its variance can be shown to be:

$$Var(\hat{\gamma}_{ij}) = \sigma^2 \left(\frac{1}{c} - \frac{1}{bc} - \frac{1}{ac} + \frac{1}{abc} \right) = \sigma^2 \frac{(a-1)(b-1)}{abc}$$

So, $\hat{\gamma}_{ij} \sim N(\gamma_{ij}, \sigma^2 \frac{(a-1)(b-1)}{abc})$.

5. Distribution of $\hat{\sigma}^2$:

$$\hat{\sigma}^2 = \frac{1}{abc} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^c (X_{ijk} - \bar{X}_{ij.})^2 = \frac{SSE}{abc}$$

We know that $\frac{SSE}{\sigma^2}$ follows a chi-squared distribution with degrees of freedom equal to $ab(c - 1)$.

$$\frac{abc\hat{\sigma}^2}{\sigma^2} \sim \chi_{ab(c-1)}^2$$

These derivations show how the MLEs are obtained by solving the system of equations derived from setting the partial derivatives of the log-likelihood function to zero, incorporating the identifiability constraints. The distributions then follow directly from the properties of linear combinations of normal random variables and the properties of the sum of squared errors.

Exercice 10

9.5.10. We wish to compare compressive strengths of concrete corresponding to $a = 3$ different drying methods (treatments). Concrete is mixed in batches that are just large enough to produce three cylinders. Although care is taken to achieve uniformity, we expect some variability among the $b = 5$ batches used to obtain the following compressive strengths. (There is little reason to suspect interaction, and hence only one observation is taken in each cell.) Data are also in the data set `sec95set2.rda`.

Treatment	Batch				
	B_1	B_2	B_3	B_4	B_5
A_1	52	47	44	51	42
A_2	60	55	49	52	43
A_3	56	48	45	44	38

- (a) Use the 5% significance level and test $H_A : \alpha_1 = \alpha_2 = \alpha_3 = 0$ against all alternatives.
- (b) Use the 5% significance level and test $H_B : \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$ against all alternatives.

In this problem, we have a two-way ANOVA without replication, as there is only one observation per cell ($c = 1$).

The model for this scenario, given that there's little reason to suspect interaction, is typically:

$$X_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}$$

where X_{ij} is the observation for the i -th treatment and j -th batch, μ is the overall mean, α_i is the effect of the i -th treatment, β_j is the effect of the j -th batch, and ϵ_{ij} are i.i.d. $N(0, \sigma^2)$ random errors.

We have $a = 3$ treatments (drying methods) and $b = 5$ batches.

The data are:

First, let's calculate the necessary sums and means.

Treatment	B_1	B_2	B_3	B_4	B_5
A_1	52	47	44	51	42
A_2	60	55	49	52	43
A_3	56	48	45	44	38

Row Means (Treatment Means):

$$\bar{X}_{1\cdot} = (52 + 47 + 44 + 51 + 42)/5 = 236/5 = 47.2$$

$$\bar{X}_{2\cdot} = (60 + 55 + 49 + 52 + 43)/5 = 259/5 = 51.8$$

$$\bar{X}_{3\cdot} = (56 + 48 + 45 + 44 + 38)/5 = 231/5 = 46.2$$

Column Means (Batch Means):

$$\bar{X}_{\cdot 1} = (52 + 60 + 56)/3 = 168/3 = 56$$

$$\bar{X}_{\cdot 2} = (47 + 55 + 48)/3 = 150/3 = 50$$

$$\bar{X}_{\cdot 3} = (44 + 49 + 45)/3 = 138/3 = 46$$

$$\bar{X}_{\cdot 4} = (51 + 52 + 44)/3 = 147/3 = 49$$

$$\bar{X}_{\cdot 5} = (42 + 43 + 38)/3 = 123/3 = 41$$

Grand Mean: Total sum of all observations = $236 + 259 + 231 = 726$

Number of observations $N = a \times b = 3 \times 5 = 15$

$$\bar{X}_{..} = 726/15 = 48.4$$

Now, we calculate the sums of squares:

Total Sum of Squares (SST_{Total}):

$$SST_{Total} = \sum_{i=1}^a \sum_{j=1}^b (X_{ij} - \bar{X}_{..})^2$$

$$SST_{Total} = (52 - 48.4)^2 + (47 - 48.4)^2 + \dots + (38 - 48.4)^2$$

$$SST_{Total} = (3.6)^2 + (-1.4)^2 + (-4.4)^2 + (2.6)^2 + (-6.4)^2 + (11.6)^2 + (6.6)^2 + (0.6)^2 + (3.6)^2 + (-5.4)^2 + (7.6)^2 + (-0.4)^2$$

$$SST_{Total} = 12.96 + 1.96 + 19.36 + 6.76 + 40.96 + 134.56 + 43.56 + 0.36 + 12.96 + 29.16 + 57.76 + 0.16 + 11.56 + 19.36 + 10.24$$

$$SST_{Total} = 500.6$$

Sum of Squares for Treatments (SSTreatments or SSA):

$$SSA = b \sum_{i=1}^a (\bar{X}_{i\cdot} - \bar{X}_{..})^2$$

$$SSA = 5 \times [(47.2 - 48.4)^2 + (51.8 - 48.4)^2 + (46.2 - 48.4)^2]$$

$$SSA = 5 \times [(-1.2)^2 + (3.4)^2 + (-2.2)^2]$$

$$SSA = 5 \times [1.44 + 11.56 + 4.84]$$

$$SSA = 5 \times 17.84 = 89.2$$

Sum of Squares for Batches (SSBatches or SSB):

$$SSB = a \sum_{j=1}^b (\bar{X}_{.j} - \bar{X}_{..})^2$$

$$SSB = 3 \times [(56 - 48.4)^2 + (50 - 48.4)^2 + (46 - 48.4)^2 + (49 - 48.4)^2 + (41 - 48.4)^2]$$

$$SSB = 3 \times [57.76 + 2.56 + 5.76 + 0.36 + 54.76]$$

$$SSB = 3 \times 121.2 = 363.6$$

Sum of Squares for Error (SSE): For a two-way ANOVA without replication,

$$SSE = SST_{Total} - SSA - SSB$$

$$SSE = 500.6 - 89.2 - 363.6 = 47.8$$

Degrees of Freedom:

$$df_{Total} = N - 1 = 15 - 1 = 14$$

$$df_{Treatments} = a - 1 = 3 - 1 = 2$$

$$df_{Batches} = b - 1 = 5 - 1 = 4$$

$$df_{Error} = (a - 1)(b - 1) = (3 - 1)(5 - 1) = 2 \times 4 = 8$$

Check: $df_{Treatments} + df_{Batches} + df_{Error} = 2 + 4 + 8 = 14 = df_{Total}$.

Mean Squares:

$$MS_{Treatments} = SSA / df_{Treatments} = 89.2 / 2 = 44.6$$

$$MS_{Batches} = SSB / df_{Batches} = 363.6 / 4 = 90.9$$

$$MS_{Error} = SSE / df_{Error} = 47.8 / 8 = 5.975$$

F-statistics:

- (1) Test $H_A : \alpha_1 = \alpha_2 = \alpha_3 = 0$ (No treatment effect)

$$F_A = MS_{Treatments}/MS_{Error}$$

$$F_A = 44.6/5.975 = 7.464$$

Critical F-value for F_A :

$$F_{0.05, df_{Treatments}, df_{Error}} = F_{0.05, 2, 8}$$

Using an F-table,

$$F_{0.05, 2, 8} \approx 4.46$$

Decision: Since

$$F_A = 7.464 > 4.46$$

we reject the null hypothesis H_A . Conclusion for (a): At the 5% significance level, there is sufficient evidence to conclude that there is a significant difference in the compressive strengths of concrete corresponding to the three different drying methods.

- (2) Test $H_B : \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$ (No batch effect)

$$F_B = MS_{Batches}/MS_{Error}$$

$$F_B = 90.9/5.975 = 15.213$$

Critical F-value for F_B :

$$F_{0.05, df_{Batches}, df_{Error}} = F_{0.05, 4, 8}$$

Using an F-table,

$$F_{0.05, 4, 8} \approx 3.84$$

Decision: Since

$$F_B = 15.213 > 3.84$$

we reject the null hypothesis H_B . Conclusion for (b): At the 5% significance level, there is sufficient evidence to conclude that there is a significant difference in the compressive strengths of concrete corresponding to the five different batches.

ANOVA Table Summary:

Source of Variation	DF	SS	MS	F	P-value (approx)
Treatments (A)	2	89.2	44.6	7.464	< 0.05
Batches (B)	4	363.6	90.9	15.213	< 0.05
Error	8	47.8	5.975		
Total	14	500.6			

第十一章 高等概率论

11.1 高等概率论笔记

见 [prob_2024.pdf](#).

参考书与练习见 [prob_work.pdf](#).

11.2 Markov Chain

见：随机过程基础 Basic Stochastic Processes (Zdzisław Brzeźniak, Tomasz Zastawniak) (Z-Library)

Exercise 5.8 (random walk)

Suppose that $S = \mathbb{Z}$. Let η_n , $n \geq 1$ be a sequence of independent identically distributed random variables with $P(\eta_1 = 1) = p$ and $P(\eta_1 = -1) = q = 1 - p$. Define $\xi_n = \sum_{i=1}^n \eta_i$ for $n \geq 1$ and $\xi_0 = 0$. Show that ξ_n is a Markov chain with transition probabilities

$$p(j|i) = \begin{cases} p, & \text{if } j = i + 1, \\ q, & \text{if } j = i - 1, \\ 0, & \text{otherwise.} \end{cases}$$

ξ_n , $n \geq 0$, is called a *random walk* starting at 0. Replacing $\xi_0 = 0$ with $\xi_0 = i$, we get a random walk starting at i .

Hint $\xi_{n+1} = \xi_n + \eta_{n+1}$. Are ξ_n and η_{n+1} independent?

To show that ξ_n is a Markov chain, we need to demonstrate that for any $n \geq 0$ and any possible values i_0, i_1, \dots, i_n, j , the conditional probability of $\xi_{n+1} = j$ given $\xi_0 = i_0, \xi_1 = i_1, \dots, \xi_n = i_n$ depends only on i_n . That is, we need to show:

$$P(\xi_{n+1} = j | \xi_n = i_n, \dots, \xi_0 = i_0) = P(\xi_{n+1} = j | \xi_n = i_n)$$

We are given $\xi_{n+1} = \xi_n + \eta_{n+1}$. So, $P(\xi_{n+1} = j | \xi_n = i_n, \dots, \xi_0 = i_0) = P(\xi_n + \eta_{n+1} = j | \xi_n = i_n, \dots, \xi_0 = i_0)$. Substitute $\xi_n = i_n$ into the condition:

$$P(i_n + \eta_{n+1} = j | \xi_n = i_n, \dots, \xi_0 = i_0) = P(\eta_{n+1} = j - i_n | \xi_n = i_n, \dots, \xi_0 = i_0)$$

Since η_{n+1} is independent of $\xi_n, \xi_{n-1}, \dots, \xi_0$ (because ξ_k for $k \leq n$ are sums of η_i for $i \leq n$, and η_{n+1} is independent of η_1, \dots, η_n), we can write:

$$P(\eta_{n+1} = j - i_n | \xi_n = i_n, \dots, \xi_0 = i_0) = P(\eta_{n+1} = j - i_n)$$

Now, let's consider the right side of the Markov property: $P(\xi_{n+1} = j | \xi_n = i_n)$.

Using $\xi_{n+1} = \xi_n + \eta_{n+1}$:

$$P(\xi_{n+1} = j | \xi_n = i_n) = P(\xi_n + \eta_{n+1} = j | \xi_n = i_n) = P(i_n + \eta_{n+1} = j | \xi_n = i_n)$$

Since η_{n+1} is independent of ξ_n :

$$P(i_n + \eta_{n+1} = j | \xi_n = i_n) = P(\eta_{n+1} = j - i_n)$$

Therefore, we have shown that $P(\xi_{n+1} = j | \xi_n = i_n, \dots, \xi_0 = i_0) = P(\xi_{n+1} = j | \xi_n = i_n)$, which means ξ_n is a Markov chain.

Now, let's determine the transition probabilities $p(j | i) = P(\xi_{n+1} = j | \xi_n = i)$. As derived above, $p(j | i) = P(\eta_{n+1} = j - i)$.

Exercise 5.9

For the random walk ξ_n defined in Exercise 5.8 prove that

$$P(\xi_n = j | \xi_0 = i) = \binom{n}{\frac{n+j-i}{2}} p^{\frac{n+j-i}{2}} q^{\frac{n-i}{2}} \quad (5.19)$$

if $n + j - i$ is an even non-negative integer, and $P(\xi_n = j | \xi_0 = i) = 0$ otherwise.

Hint Use induction. Note that $\binom{n}{\frac{n+j-i}{2}} p^{\frac{n+j-i}{2}}$ equals 0 if $|j - i| \geq n + 1$.

证明. Use Chapman-Kolomogorov equation:

$$\begin{aligned} p_{n+1}(j | i) &= \sum_{s \in S} p_n(j | s) p_1(s | i) \\ &= p_n(j | s) \cdot p + p_n(j | i-1) \cdot q \end{aligned}$$

Induction yields the conclusion. \square

Here are two important identities.

Exercise 5.17

For $|x| < 1$ and $j, i \in S$ define

$$P_{ji}(x) = \sum_{n=0}^{\infty} p_n(j|i)x^n, \quad (5.38)$$

$$F_{ji}(x) = \sum_{n=1}^{\infty} f_n(j|i)x^n, \quad (5.39)$$

where $f_n(j|i) = P(\xi_n = j, \xi_k \neq j, k = 1, \dots, n-1 | \xi_0 = i)$. Show that the power series in (5.38)–(5.39) are absolutely convergent for $|x| < 1$ and that

$$P_{ji}(x) = F_{ji}(x)P_{jj}(x), \text{ if } j \neq i, \quad (5.40)$$

$$P_{ii}(x) = 1 + F_{ii}(x)P_{ii}(x). \quad (5.41)$$

Hint Note that $|p_n(j|i)| \leq 1$, so the radius of convergence of the power series (5.38) is ≥ 1 .

Exercise 5.19

Show that a state j is recurrent if and only if $\sum_n p_n(j|j) = \infty$. Deduce that the state j is transient if and only if

$$\sum_n p_n(j|j) < \infty. \quad (5.42)$$

Show that if j is transient, then for each $i \in S$

$$\sum_n p_n(j|i) < \infty. \quad (5.43)$$

Hint If j is recurrent, then $F_{jj}(x) \rightarrow \sum_n f_n(j|j) = 1$ as $x \nearrow 1$. Use (5.41) in conjunction with Abel's Lemma.

(a) $P_{jj}(x) \rightarrow \sum_n p_n(j|j)$. And by (5.41), $P_{jj}(x) = \frac{1}{1 - F_{jj}(x)}$. Then j is recurrent iff $F_{jj}(x) \rightarrow 1$ iff $P_{jj}(x) \rightarrow \infty$ iff $\sum_n p_n(j|j) = \infty$.

(b) is similar.

(c) if j is transient, by (5.40),

$$\begin{aligned} \underbrace{P_{ji}(x)}_{\rightarrow \sum_n p_n(j|i)} &= \underbrace{F_{ji}(x)}_{\rightarrow \sum_n f_n(j|i) \leq 1} \underbrace{P_{jj}(x)}_{\rightarrow \sum_n p_n(j|j)} \\ &\rightarrow \sum_n p_n(j|i) \end{aligned}$$

Exercise 5.22

Consider a symmetric random walk on \mathbb{Z} . Show that 0 is a null-recurrent state. Can you deduce whether other states are positive-recurrent or null-recurrent?

Hint State 0 is null-recurrent if and only if $\sum_n n f_n(0|0) = \infty$. As in Exercise 5.18, $\sum_n n f_n(0|0) = \lim_{x \nearrow 1} F'_{00}(x)$, where F_{00} is defined by (5.39).

$$p_n(0 | 0) = \begin{cases} \binom{2k}{k} 2^{-2k} & n = 2k \\ 0 & n = 2k - 1 \end{cases}$$

Then

$$P_{00}(x) = \sum_{k=1}^{\infty} \binom{2k}{k} \left(\frac{x}{2}\right)^{2k} = (1-x^2)^{-1/2}$$

Thus

$$\begin{aligned} F_{00}(x) &= 1 - \frac{1}{P_{00}(x)} = 1 - (1-x^2)^{1/2} \\ F'_{00}(x) &= \frac{x}{\sqrt{1-x^2}} \xrightarrow{x \rightarrow 1} \infty \end{aligned}$$

Definition 5.8

Suppose that $\xi_n, n \in \mathbb{N}$, is a Markov chain on a state space S . Let $i \in S$. We say that i is a *periodic state* if and only if the greatest common divisor (gcd) of all $n \in \mathbb{N}^*$, where $\mathbb{N}^* = \{1, 2, 3, \dots\}$, such that $p_n(i|i) > 0$ is ≥ 2 . Otherwise, the state i is called *aperiodic*. In both cases, the gcd is denoted by $d(i)$ and is called the *period* of the state i . Thus, i is periodic if and only if $d(i) \geq 2$. A state i which is positive recurrent and aperiodic is called *ergodic*.

11.3 Poisson 分布

Let η_1, η_2, \dots be a sequence of independent random variables, each having the same exponential distribution of rate λ . For example, the times between the emissions of radioactive particles (or between calls made at a telephone exchange) have this property. We put

$$\xi_n = \eta_1 + \dots + \eta_n,$$

To study this conjecture recall the construction of a Poisson process $N(t)$ based on a sequence of independent random variables η_1, η_2, \dots , all having the same exponential distribution. We shall try to represent $N(t+s) - N(t)$ in a similar way.

Let us put

$$\eta_1^t := \xi_{N(t)+1} - t, \quad \eta_n^t := \eta_{N(t)+n}, \quad n = 2, 3, \dots,$$

see Figure 6.3. These are the times between the jumps of $N(t+s) - N(t)$. Then we define

$$\begin{aligned}\xi_n^t &= \eta_1^t + \dots + \eta_n^t, \\ N^t(s) &= \max\{n : \xi_n^t \leq s\}.\end{aligned}$$

Definition 6.5

We say that $N(t)$, where $t \geq 0$, is a *Poisson process* if

$$N(t) = \max\{n : t \geq \xi_n\}.$$

Theorem 6.1

For any fixed $t \geq 0$

$$N^t(s) = N(t+s) - N(t), \quad s \geq 0$$

is a Poisson process independent of $N(t)$ with the same probability law as $N(s)$.

Theorem 6.2

For any $0 \leq t_1 \leq t_2 \leq \dots \leq t_n$ the increments

$$N(t_1), N(t_2) - N(t_1), N(t_3) - N(t_2), \dots, N(t_n) - N(t_{n-1})$$

are independent and have the same probability distribution as

$$N(t_1), N(t_2 - t_1), N(t_3 - t_2), \dots, N(t_n - t_{n-1}).$$

Definition 6.7

We say that a stochastic process $\xi(t)$, where $t \in T$, has *independent increments* if

$$\xi(t_1) - \xi(t_0), \dots, \xi(t_n) - \xi(t_{n-1})$$

are independent for any $t_0 < t_1 < \dots < t_n$ such that $t_0, t_1, \dots, t_n \in T$.

Definition 6.8

A stochastic process $\xi(t)$, where $t \in T$, is said to have *stationary increments* if for any $s, t \in T$ the probability distribution of $\xi(t+h) - \xi(s+h)$ is the same for each h such that $s+h, t+h \in T$.

11.4 Brown Motion (Wiener process)

the trajectories of real smoke particles. Meanwhile, the density $f_{W(t)}$ of the random variable $W(t)$ is very smooth, given by the exponential function

$$f_{W(t)}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}},$$

which is a solution of the *diffusion equation*

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}$$

Definition 6.9

The *Wiener process* (or *Brownian motion*) is a stochastic process $W(t)$ with values in \mathbb{R} defined for $t \in [0, \infty)$ such that

- 1) $W(0) = 0$ a.s.;
- 2) the sample paths $t \mapsto W(t)$ are a.s. continuous;
- 3) for any finite sequence of times $0 < t_1 < \dots < t_n$ and Borel sets $A_1, \dots, A_n \subset \mathbb{R}$

$$\begin{aligned} & P\{W(t_1) \in A_1, \dots, W(t_n) \in A_n\} \\ &= \int_{A_1} \cdots \int_{A_n} p(t_1, 0, x_1) p(t_2 - t_1, x_1, x_2) \cdots \\ & \quad \cdots p(t_n - t_{n-1}, x_{n-1}, x_n) dx_1 \cdots dx_n, \end{aligned}$$

where

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} \tag{6.5}$$

defined for any $x, y \in \mathbb{R}$ and $t > 0$ is called the *transition density*.

Exercise 6.19

Show that

$$f_{W(t)}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

is the probability density of $W(t)$ and find the expectation and variance of $W(t)$.

Hint The density of $W(t)$ can be obtained from condition 3) of Definition 6.9 written for a single time t and a single Borel set. You will need the formula

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

to compute the integrals in the expressions for the expectation and variance.

$$\mathbb{P}(W(t) \leq x) = \int_{-\infty}^x p(t, 0, x_1) dx_1 = \int_{-\infty}^x \frac{1}{\sqrt{2\pi t}} \underbrace{e^{-(0-x_1)^2/(2t)}}_{=e^{-x_1^2/(2t)}} dx_1$$

Remark 6.1

The results of Exercise 6.19 mean that $W(t)$ has the normal distribution with mean 0 and variance t .

Exercise 6.20

Show that

$$E(W(s)W(t)) = \min \{s, t\}.$$

Hint The joint density of $W(s)$ and $W(t)$ will be needed. It can be found from condition 3) of Definition 6.9 written for two times s and t and two Borel sets.

WLOG, assume that $s > t$, then

$$\mathbb{P}(W(s) \leq x, W(t) \leq y) = \int_{-\infty}^x \int_{-\infty}^y \frac{1}{\sqrt{2\pi t}} e^{-x_1^2/(2t)} \cdot \frac{1}{\sqrt{2\pi(s-t)}} e^{-(x_2-x_1)^2/(2(s-t))} dx_2 dx_1$$

Then

$$f_{W(s), W(t)}(x, y) = \frac{1}{2\pi\sqrt{t(s-t)}} e^{-x^2/(2t)} \cdot e^{-(x-y)^2/(2(s-t))}$$

$$\begin{aligned}
\mathbb{E}(W(s)W(t)) &= \int_{\mathbb{R}^2} xy \cdot f_{W(s), W(t)}(x, y) dx dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} xy \cdot \frac{1}{2\pi\sqrt{t(s-t)}} e^{-x^2/(2t)} e^{-(x-y)^2/(2(s-t))} dx dy \\
&= \int_{\mathbb{R}} \frac{1}{2\pi\sqrt{t(s-t)}} xe^{-x^2/(2t)} \underbrace{\int_{\mathbb{R}} e^{-(x-y)^2/(2(s-t))} dy}_{=\sqrt{2\pi(s-t)} \cdot x} dx \\
&= \frac{1}{\sqrt{2\pi t}} \underbrace{\int_{\mathbb{R}} x^2 e^{-x^2/(2t)} dx}_{=\sqrt{2\pi t^{3/2}}} \\
&= t
\end{aligned}$$

Exercise 6.21**Show that**

$$E(|W(t) - W(s)|^2) = |t - s|.$$

$$\begin{aligned}
\mathbb{E}(|W(t) - W(s)|^2) &= \mathbb{E}(W(t)^2) + \mathbb{E}(W(s)^2) - 2\mathbb{E}(W(t)W(s)) \\
&= \int_{\mathbb{R}} x^2 \cdot \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)} dx + \int_{\mathbb{R}} y^2 \cdot \frac{1}{\sqrt{2\pi s}} e^{-y^2/(2s)} dy - 2\min\{t, s\} \\
&= t + s - 2\min\{t, s\} \\
&= |t - s|
\end{aligned}$$

Exercise 6.22Compute the characteristic function $E(\exp(i\lambda W(t)))$ for any $\lambda \in \mathbb{R}$.**Hint** Use the density of $W(t)$ found in Exercise 6.19.

$$\begin{aligned}
\mathbb{E}(\exp\{i\lambda W(t)\}) &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t) + i\lambda x} dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{2t}(x^2 - 2it\lambda x + (it\lambda)^2) - \frac{t\lambda^2}{2}} dx \\
&= e^{-t\lambda^2/2}
\end{aligned}$$

Exercise 6.23

Find $E(W(t)^4)$.

Hint This can be done, for example, by expressing the expectation in terms of the density of $W(t)$ and computing the resulting integral, or by computing the fourth derivative of the characteristic function of $W(t)$ at 0. The second method is more efficient.

$$\begin{aligned}\mathbb{E}(W(t)^4) &= [\partial_\lambda^4 \mathbb{E}(\exp\{i\lambda W(t)\})]_{\lambda=0} \\ &= [\partial_\lambda^4 (e^{-t\lambda^2/2})]_{\lambda=0} \\ &= 3t^2\end{aligned}$$

Definition 6.10

We call $W(t) = (W^1(t), \dots, W^n(t))$ an *n-dimensional Wiener process* if $W^1(t), \dots, W^n(t)$ are independent \mathbb{R} -valued Wiener processes.

Exercise 6.24

For a two-dimensional Wiener process $W(t) = (W^1(t), W^2(t))$ find the probability that $|W(t)| < R$, where $R > 0$ and $|x|$ is the Euclidean norm of $x = (x^1, x^2)$ in \mathbb{R}^2 , i.e. $|x|^2 = (x^1)^2 + (x^2)^2$.

Hint Express the probability in terms of the joint density of $W^1(t)$ and $W^2(t)$. Independence means that the joint density of $W^1(t)$ and $W^2(t)$ is the product of their respective densities, which are known from Exercise 6.19. It is convenient to use polar coordinates to compute the resulting integral over a disc.

$$\begin{aligned}\mathbb{P}(|W(t)| < R) &= \int_{x^2+y^2 < R} \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)} \cdot \frac{1}{\sqrt{2\pi t}} e^{-y^2/(2t)} dx dy \\ &= \frac{1}{2\pi t} \int_0^{2\pi} d\theta \int_0^R e^{-r^2/(2t)} dr \\ &= 1 - e^{-R^2/(2t)}\end{aligned}$$

Proposition 6.2

For any $0 \leq s < t$ the increment $W(t) - W(s)$ has the normal distribution with mean 0 and variance $t - s$.

Corollary 6.1

Proposition 6.2 implies that $W(t)$ has stationary increments.

Proposition 6.3

For any $0 = t_0 \leq t_1 \leq \dots \leq t_n$ the increments

$$W(t_1) - W(t_0), \dots, W(t_n) - W(t_{n-1})$$

are independent.

证明. It suffices to show that $W(s) - W(t)$ and $W(u) - W(r)$ are uncorrelated,
i.e. $\mathbb{E}[(W(s) - W(t))(W(u) - W(r))] = 0$. \square

Corollary 6.2

For any $0 \leq s < t$ the increment $W(t) - W(s)$ is independent of the σ -field

$$\mathcal{F}_s = \sigma \{W(r) : 0 \leq r \leq s\}.$$

Exercise 6.25

Show that $W(t)$ is a martingale with respect to the filtration \mathcal{F}_t .

Hint Take advantage of the fact that $W(t) - W(s)$ is independent of \mathcal{F}_s if $s < t$.

Solution 6.25

For any $0 \leq s < t$

$$\begin{aligned} E(W(t)|\mathcal{F}_s) &= E(W(t) - W(s)|\mathcal{F}_s) + E(W(s)|\mathcal{F}_s) \\ &= E(W(t) - W(s)) + W(s) \\ &= W(s), \end{aligned}$$

since $W(t) - W(s)$ is independent of \mathcal{F}_s by Corollary 6.2, $W(s)$ is \mathcal{F}_s -measurable
and $E(W(t)) = E(W(s)) = 0$.

Exercise 6.26

Show that $|W(t)|^2 - t$ is a martingale with respect to the filtration \mathcal{F}_t .

Hint Once again, use the fact that $W(t) - W(s)$ is independent of \mathcal{F}_s if $s < t$.

For $0 \leq s < t$,

$$\begin{aligned}\mathbb{E}[|W(t)|^2 - t | \mathcal{F}_s] &= \mathbb{E}(|W(t) - W(s)|^2 + 2W(s)(W(t) - W(s)) + |W(s)|^2 - t | \mathcal{F}_s) \\ &= \mathbb{E}(|W(t) - W(s)|^2) + 2W(s) \underbrace{\mathbb{E}[W(s) - W(t)]}_{=0} + |W(s)|^2 - t \\ &= |t - s| + |W(s)|^2 - t \\ &= |W(s)|^2 - s\end{aligned}$$

Theorem 6.3

A stochastic process $W(t), t \geq 0$, is a Wiener process if and only if the following conditions hold:

- 1) $W(0) = 0$ a.s.;
- 2) the sample paths $t \mapsto W(t)$ are continuous a.s.;
- 3) $W(t)$ has stationary independent increments;
- 4) the increment $W(t) - W(s)$ has the normal distribution with mean 0 and variance $t - s$ for any $0 \leq s < t$.

Exercise 6.27

Show that for any $T > 0$

$$V(t) = W(t + T) - W(T)$$

is a Wiener process if $W(t)$ is.

Hint Are the increments of $V(t)$ independent? What is their distribution? Does $V(t)$ have continuous paths? Is it true that $V(0) = 0$?

- (i) $V(0) = W(T + 0) - W(T) = 0$.
- (ii) $t \mapsto V(t) = W(t + T) - W(t)$ is continuous since $t \mapsto t + T \mapsto W(t + T)$ is continuous.
- (iii) for $t_1 - s_1 = t_2 - s_2$, where $t_1 > s_1, t_2 > s_2$. The increment $V(t_1) - V(s_1) = W(t_1 + T) - W(s_1 + T)$ have the same pdf as $W(t_2 + T) - W(s_2 + T) = V(t_2) - V(s_2)$. Also the increments of $V(t)$ are independent as $W(t_1 + T) - W(s_1 + T)$ and $W(t_2 + T) - W(s_2 + T)$ are independent.
- (iv) the increment $V(t) - V(s) = W(t + T) - W(s + T)$ has the same pdf as $W(t) - W(s)$, which $\sim N(0, t - s)$.

Theorem 6.4 (Lévy's martingale characterization)

Let $W(t), t \geq 0$, be a stochastic process and let $\mathcal{F}_t = \sigma(W_s, s \leq t)$ be the filtration generated by it. Then $W(t)$ is a Wiener process if and only if the following conditions hold:

- 1) $W(0) = 0$ a.s.;
- 2) the sample paths $t \mapsto W(t)$ are continuous a.s.;
- 3) $W(t)$ is a martingale with respect to the filtration \mathcal{F}_t ;
- 4) $|W(t)|^2 - t$ is a martingale with respect to \mathcal{F}_t .

Exercise 6.28

Let $c > 0$. Show that $V(t) = \frac{1}{c}W(c^2t)$ is a Wiener process if $W(t)$ is.

Hint Is $V(t)$ a martingale? With respect to which filtration? Is $|V(t)|^2 - t$ a martingale? Are the paths of $V(t)$ continuous? Is it true that $V(0) = 0$?

Solution 6.28

It is clear that $V(0) = \frac{1}{c}W(0) = 0$ a.s. and the paths $t \mapsto V(t) = \frac{1}{c}W(c^2t)$ are a.s. continuous. We shall verify that $V(t)$ and $|V(t)|^2 - t$ are martingales with respect to the filtration

$$\begin{aligned}\mathcal{G}_t &= \sigma\{V(s) : 0 \leq s \leq t\} \\ &= \sigma\{W(c^2s) : 0 \leq s \leq t\} \\ &= \sigma\{W(s) : 0 \leq s \leq c^2t\} \\ &= \mathcal{F}_{c^2t}.\end{aligned}$$

Indeed, if $s < t$, then $c^2s < c^2t$, so

$$\begin{aligned}E(V(t)|\mathcal{G}_s) &= E\left(\frac{1}{c}W(c^2t)|\mathcal{F}_{c^2s}\right) \\ &= \frac{1}{c}E(W(c^2t)|\mathcal{F}_{c^2s}) \\ &= \frac{1}{c}W(c^2s) = V(s)\end{aligned}$$

and

$$\begin{aligned}E(|V(t)|^2 - t|\mathcal{G}_s) &= E\left(\frac{1}{c^2}|W(c^2t)|^2 - t|\mathcal{F}_{c^2s}\right) \\ &= \frac{1}{c^2}E(|W(c^2t)|^2 - c^2t|\mathcal{F}_{c^2s}) \\ &= \frac{1}{c^2}(|W(c^2s)|^2 - c^2s) \\ &= |V(s)|^2 - s,\end{aligned}$$

since $W(t)$ and $|W(t)|^2 - t$ are martingales with respect to the filtration \mathcal{F}_t . It follows by Levy's martingale characterization that $V(t)$ is a Wiener process.

6.3.3 Sample Paths

Let

$$0 = t_0^n < t_1^n < \dots < t_n^n = T,$$

where

$$t_i^n = \frac{iT}{n},$$

be a partition of the interval $[0, T]$ into n equal parts. We denote by

$$\Delta_i^n W = W(t_{i+1}^n) - W(t_i^n)$$

the corresponding increments of the Wiener process $W(t)$.

Exercise 6.29

Show that

$$\lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} (\Delta_i^n W)^2 = T \quad \text{in } L^2.$$

Hint You need to show that

$$\lim_{n \rightarrow \infty} E \left(\left[\sum_{i=0}^{n-1} (\Delta_i^n W)^2 - T \right]^2 \right) = 0.$$

Use the independence of increments to simplify the expectation. What are the expectations of $\Delta_i^n W$, $(\Delta_i^n W)^2$ and $(\Delta_i^n W)^4$?

The next theorem on the variation of the paths of $W(t)$ is a consequence of the result in Exercise 6.29. First, let us recall that the variation of a function is defined as follows.

$$\begin{aligned}\mathbb{E}[\Delta_i^n W] &= \mathbb{E}[W(t_{i+1}^n)] - \mathbb{E}[W(t_i^n)] = 0 - 0 = 0 \\ \mathbb{E}[(\Delta_i^n W)^2] &= t_{i+1}^n - t_i^n = \frac{T}{n} \\ \mathbb{E}[(\Delta_i^n W)^4] &= 3(t_{i+1}^n - t_i^n)^2 = \frac{3T^2}{n^2}\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E} \left(\left[\sum_{i=0}^{n-1} (\Delta_i^n W)^2 - T \right]^2 \right) &= \mathbb{E} \left[\sum_{i=0}^{n-1} (\Delta_i^n W)^4 + 2 \sum_{i \neq j} (\Delta_i^n W)^2 (\Delta_j^n W)^2 \right] - T^2 \\ &= \sum \mathbb{E}(\Delta_i^n W)^4 + 2 \sum \mathbb{E}[(\Delta_i^n W)^2] \mathbb{E}[(\Delta_j^n W)^2] - T^2 \\ &= \frac{3n}{n^2} T^2 + 2 \cdot \frac{n(n-1)}{2} \frac{T}{n} \frac{T}{n} - T^2 \\ &= \frac{2T^2}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty\end{aligned}$$

Definition 6.11

The *variation* of a function $f : [0, T] \rightarrow \mathbb{R}$ is defined to be

$$\limsup_{\Delta t \rightarrow 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|,$$

where $t = (t_0, t_1, \dots, t_n)$ is a partition of $[0, T]$, i.e. $0 = t_0 < t_1 < \dots < t_n = T$, and where

$$\Delta t = \max_{i=0, \dots, n-1} |t_{i+1} - t_i|.$$

Theorem 6.5

The variation of the paths of $W(t)$ is infinite a.s.

Proof

Consider the sequence of partitions $t^n = (t_0^n, t_1^n, \dots, t_n^n)$ of $[0, T]$ into n equal parts. Then

$$\sum_{i=0}^{n-1} |\Delta_i^n W|^2 \leq \left(\max_{i=0, \dots, n-1} |\Delta_i^n W| \right) \sum_{i=0}^{n-1} |\Delta_i^n W|.$$

Since the paths of $W(t)$ are a.s. continuous on $[0, T]$,

$$\lim_{n \rightarrow \infty} \left(\max_{i=0, \dots, n-1} |\Delta_i^n W| \right) = 0 \quad \text{a.s.}$$

By Exercise 6.29 there is a subsequence $t^{n_k} = (t_0^{n_k}, t_1^{n_k}, \dots, t_{n_k}^{n_k})$ of partitions such that

$$\lim_{k \rightarrow \infty} \sum_{i=0}^{n_k-1} |\Delta_i^{n_k} W|^2 = T \quad \text{a.s.}$$

This is because every sequence of random variables convergent in L^2 has a subsequence convergent a.s. It follows that

$$\lim_{k \rightarrow \infty} \sum_{i=0}^{n_k-1} |\Delta_i^{n_k} W| = \infty \quad \text{a.s.},$$

while

$$\lim_{k \rightarrow \infty} \Delta t^{n_k} = \lim_{k \rightarrow \infty} \frac{T}{n_k} = 0,$$

which proves the theorem. \square

Theorem 6.5 has important consequences for the theory of stochastic integrals presented in the next chapter. This is because an integral of the form

$$\int_0^T f(t) dW(t)$$

cannot be defined pathwise (that is, separately for each $\omega \in \Omega$) as the Riemann-Stieltjes integral if the paths have infinite variation. It turns out that an intrinsically stochastic approach will be needed to tackle such integrals, see Chapter 7.

Exercise 6.30

Show that $W(t)$ is a.s. non-differentiable at $t = 0$.

Hint By Exercise 6.28 $V_c(t) = \frac{1}{c}W(c^2t)$ is a Wiener process for any $c > 0$. Deduce that the probability

$$P\left\{\frac{|W(t)|}{t} > cM \text{ for some } t \in [0, \frac{1}{c^2}] \right\}$$

is the same for each $c > 0$. What is the probability that the limit of $\frac{|W(t)|}{t}$ exists as $t \searrow 0$, then?

Solution 6.30

We claim that, with probability 1, for any positive integer n there is a $t \in [0, \frac{1}{n^4}]$ such that $\frac{|W(t)|}{t} > n$. This condition implies that $W(t)$ is not differentiable at $t = 0$.

Let us put

$$A_n = \left\{ \frac{|W(t)|}{t} > n \text{ for some } t \in [0, \frac{1}{n^4}] \right\}$$

By Exercise 6.28

$$V_n(t) = \frac{1}{n^2}W(n^4t)$$

is a Brownian motion for any n . Therefore

$$\begin{aligned} P(A_n) &\geq P\left\{\frac{|W(1/n^4)|}{1/n^4} > n\right\} \\ &= P\left\{\frac{|V(1/n^4)|}{1/n^4} > n\right\} \\ &= P\left\{|V(1)| > \frac{1}{n}\right\} \rightarrow 1 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Since A_1, A_2, \dots is a contracting sequence of events,

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} P(A_n) = 1,$$

which proves the claim.

Exercise 6.31

Show that for any $t \geq 0$ the Wiener process $W(t)$ is a.s. non-differentiable at t .

Hint $V_t(s) = W(s+t) - W(t)$ is a Wiener process for any $t \geq 0$.

$W(t)$ is a.s. non-differentiable at $t = 0$. For $t > 0$, let $V_t(s) = W(s+t) - W(t)$, which is a Wiener process, then $V_t(s)$ is a.s. non-differentiable at $s = 0$. Thus $W(s+t)$ is a.s. non-differentiable at $s = 0$, i.e. $W(t)$ is a.s. non-differentiable at t .

Theorem 6.6

With probability 1 the Wiener process $W(t)$ is non-differentiable at any $t \geq 0$.

Doob's Maximal L^2 Inequality for Brownian Motion

Theorem 6.7 (Doob's maximal L^2 inequality)

For any $t > 0$

$$E \left(\max_{s \leq t} |W(s)|^2 \right) \leq 4E|W(t)|^2. \quad (6.6)$$

Proof

For $t > 0$ and $n \in \mathbb{N}$ we define

$$M_k^n = \left| W\left(\frac{kt}{2^n}\right) \right|, \quad 0 \leq k \leq 2^n. \quad (6.7)$$

Then, by Jensen's inequality, $M_k^n, k = 0, \dots, 2^n$, is a non-negative square integrable submartingale with respect to the filtration $\mathcal{F}_k^n = \mathcal{F}_{\frac{k}{2^n}}$, so by Theorem 4.1

$$E \left(\max_{k \leq 2^n} |M_k^n|^2 \right) \leq 4E|M_{2^n}^n|^2 = 4E|W(t)|^2.$$

Since $W(t)$ has a.s. continuous paths,

$$\lim_{n \rightarrow \infty} \max_{k \leq 2^n} |M_k^n|^2 = \max_{s \leq t} |W(s)|^2 \quad \text{a.s.}$$

Moreover, since $M_k^n = M_{2k}^{n+1}$, the sequence $\sup_{k \leq 2^n} |M_k^n|^2, n \in \mathbb{N}$, is increasing. Hence by the Lebesgue monotone convergence theorem $\max_{s \leq t} |W(s)|^2$ is an integrable function and

$$E \left(\max_{s \leq t} |W(s)|^2 \right) = \lim_{n \rightarrow \infty} E \left(\max_{k \leq 2^n} |M_k^n|^2 \right) \leq 4E|W(t)|^2,$$

completing the proof. \square

Exercise 6.35

Show that $e^{W(t)}e^{-\frac{t}{2}}$ is a martingale. (It is called the *exponential martingale*.)

Hint What is the expectation of $e^{W(t)-W(s)}$ for $s < t$? By independence it is equal to the conditional expectation of $e^{W(t)-W(s)}$ given \mathcal{F}_s . This will give the martingale condition.

Exercise 6.36

Compute $E(W(s)|W(t))$ for $0 \leq s < t$.

Hint You want to find a Borel function F such that $E(W(s)|W(t)) = F(W(t))$, i.e.

$$\int_{\{W(t) \in A\}} W(s) dP = \int_{\{W(t) \in A\}} F(W(t)) dP.$$

Either side of this equality can be transformed using the joint density of $W(t)$ and $W(s)$.

第十二章 数学分析进阶

第十三章 微分几何

13.1 基础知识

2 基础知识

定理 2.1: 点到平面距离公式

点 $P(x_0, y_0, z_0)$ 到直线 $Ax + By + Cz + D = 0$ 的距离为 $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

定理 2.2: 混合积

$(a \times b) \cdot c$ 称为向量 a, b, c 的混合积, 记为 (a, b, c) , 若记它们的坐标为 $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), c = (c_1, c_2, c_3)$, 则混合积 $(a, b, c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, 其绝对值表示这三个向量 a, b, c 张成的平行六面体的体积.

定理 2.3: 异面直线距离公式

设 l_1, l_2 是两条异面直线, M_1, M_2 分别在直线 l_1, l_2 上, 则两直线距离 $d = \frac{|(\overrightarrow{M_1 M_2}, v_1, v_2)|}{|v_1 \times v_2|}$, 其中 v_1, v_2 分别为 l_1, l_2 的方向向量.

注释 2.1: 异面直线的公垂线存在唯一, 至于其距离也可以采用如下方法计算: 在两条直线上各取一点, 计算这两点的距离, 是一个二元二次型, 求出最小值和取等条件, 便能得到距离和公垂线方程.

定理 2.4: 向量外积

设 $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$ 为两个向量, 则它们的外积为 $a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$, 表示这两个向量张成的平行四边形的面积.

注释 2.2: 这个性质, 可以用来计算点到直线的距离.

13.1.1 判断曲面类型

见《解析几何》·吕林根·第四版

二次曲面的一般方程为

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0$$

适当选取坐标系，二次曲面的方程总可以化为下列五个方程之一：

- (1) $a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{44} = 0, a_{11}a_{22}a_{33} \neq 0.$
- (2) $a_{11}x^2 + a_{22}y^2 + 2a_{34}z = 0, a_{11}a_{22}a_{34} \neq 0.$
- (3) $a_{11}x^2 + a_{22}y^2 + a_{44} = 0, a_{11}a_{22} \neq 0.$
- (4) $a_{11}x^2 + 2a_{24}y = 0, a_{11}a_{24} \neq 0.$
- (5) $a_{11}x^2 + a_{44} = 0, a_{11} \neq 0.$

通过适当选取坐标系，二次曲面的方程总可以化成以下十七种形式之一：

常见的二次曲面

椭球面	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
单叶双曲面	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
双叶双曲面	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$
二次锥面	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$
椭圆抛物面	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$
双曲抛物面 ¹	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$
椭圆柱面	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
双曲柱面	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
抛物柱面	$x^2 = 2py$

¹也叫马鞍面

其余的不常见的二次曲面

虚椭球面

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$$

点（虚母线二次锥面）

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$$

虚椭圆柱面

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$$

交于一条实直线的一对共轭虚平面

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$$

一对相交平面

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

一对平行平面

$$x^2 = a^2$$

一对平行的共轭虚平面

$$x^2 = -a^2$$

一对重合平面

$$x^2 = 0$$

13.1.2 直母线

例题 3.2 设 S 是马鞍面, 证明: (1) 同族中的任意两条直母线异面; (2) 异族中的任意两条直母线相交; (3) 同族中的全体直母线平行于同一平面.

证明. 设马鞍面的方程为 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z, a > 0, b > 0$. 它的两族直母线为

$$(I) \begin{cases} x = au + av, \\ y = bu - bv, \\ z = 2uv, \end{cases} \text{ 及 } (II) \begin{cases} x = av + au, \\ y = -bv + bu, \\ z = 2vu, \end{cases}$$

(1) 在 (I) 族中任取两条直母线 l_1, l_2 , 对应的参数为 $u_1, u_2, u_1 \neq u_2, l_1, l_2$ 分别经过点 $M_1(au_1, bu_1, 0), M_2(au_2, bu_2, 0)$ 方向向量是 $v_1 = (a, -b, 2u_1), v_2 = (a, -b, 2u_2)$, 显然两方向不共线, 计算混合积

$$(\overrightarrow{M_1 M_2}, v_1, v_2) = \begin{vmatrix} a(u_2 - u_1) & b(u_2 - u_1) & 0 \\ a & -b & 2u_1 \\ a & -b & 2u_2 \end{vmatrix} = -4ab(u_2 - u_1)^2 \neq 0.$$

所以 (I) 中任意两条直母线 l_1, l_2 异面. 同理可得 (II) 中任意两条直母线也异面.

(2) 在两族直母线中分别任取一条, 记为 l_1, l_2 , 对应的参数为 u_1, v_2, l_1, l_2 分别经过点 $M_1(au_1, bu_1, 0), M_2(av_2, -bv_2, 0)$, 方向向量是 $v_1 = (a, -b, 2u_1), v_2 = (a, b, 2v_2)$. 显然两方向不共线, 即它们不可能平行.

计算混合积

$$(\overrightarrow{M_1 M_2}, v_1, v_2) = \begin{vmatrix} a(v_2 - u_1) & -b(v_2 + u_1) & 0 \\ a & -b & 2u_1 \\ a & b & 2v_2 \end{vmatrix} = 0.$$

所以异族中的任意两条直母线 l_1, l_2 相交.

(3) 由于 (I) 中任意直母线的方向向量为 $v_1 = (a, -b, 2u)$, 它平行于平面 $bx + ay = 0$, 所以 (I) 中所有直母线平行于平面 $bx + ay = 0$.

由于 (II) 中任意直母线的方向向量为 $v_2 = (a, b, 2v)$, 它平行于平面 $bx - ay = 0$, 所以 (II) 中所有直母线平行于平面 $bx - ay = 0$. \square

注释 3.2: 直母线的方程可以采用因式分解的方法来求, 写成 $\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 2z$, 让左边的一项为右边某一项的 k 倍即可, 写出两个方程, 就是需要的直线.

进一步, 它存两族直母线:

$$\begin{cases} \frac{x}{a} + \frac{y}{b} + 2\lambda = 0, \\ \lambda\left(\frac{x}{a} - \frac{y}{b}\right) + z = 0, \end{cases} \quad \begin{cases} \lambda\left(\frac{x}{a} + \frac{y}{b}\right) + z = 0, \\ x - \frac{y}{b} + 2\lambda = 0, \end{cases}$$

其中, λ 是任意实数.

13.1.3 与二次曲面交线为圆的平面

例题 3.4 求与椭圆抛物面 $10x^2 + 2y^2 = z$ 的交线为圆的平面.

证明. 因为椭圆抛物面 $10x^2 + 2y^2 = z$ 开口朝 z 轴方向, 交线为圆, 所以平面的法向量不会平行于 xOy 坐标面, 可设所求平面为 $\Pi: Ax + By + z + D = 0$.

由于空间的圆一定是某球面与平面的交线, 所以该圆可设为球面 $x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d = 0$ 与平面 Π 的交线. 交线向 xOy 坐标面的投影柱面是相同的, 而它们的方程分别为

$$\begin{aligned} 10x^2 + 2y^2 + Ax + By + D &= 0, \\ (1 + A^2)x^2 + (1 + B^2)y^2 + 2ABxy \\ + 2(AD + a - cA)x + 2(BD + b - cB)y + D^2 - 2cD + d &= 0, \end{aligned}$$

比较它们的系数得到 $AB = 0, \frac{1 + A^2}{10} = \frac{1 + B^2}{2}$, 于是 $B = 0, A = \pm 2$. 平面方程: $\pm 2x + z = k$.

该平面要与椭圆抛物面相交, 将平面方程代入椭圆抛物面方程中得 $10x^2 + 2y^2 \pm 2x = k$, 该方程有解, 经配方得到 k 满足: $k > -\frac{1}{10}$. \square

注释 3.4: 这是计算平面与二次曲面交线为圆的标准流程, 圆就是平面和球的交线, 引入一个球消元对比系数即可, 实际上利用平移不改变形状这一性质, 我们可以直接设 $z = Ax + By$ 代入, 然后对比系数时候把 A, B 解出来就可以了, 这样子最快最省事, 但是要绝对严谨和保证不扣分, 还是应该算完整.

13.2 曲面第一基本形式

参考微分几何例题详解和习题汇编 · 陈维恒.

正则参数曲面² S 是满足 $\mathbf{r}_u \times \mathbf{r}_v \neq 0$ 的连续可微映射

$$\mathbf{r}: D \subset E^2 \rightarrow E^3 \quad (u, v) \mapsto (x(u, v), y(u, v), z(u, v))$$

S 在每一点处有确定的标架 $\{\mathbf{r}(u, v); \mathbf{r}_u(u, v), \mathbf{r}_v(u, v), \mathbf{n}(u, v)\}$, 其中

$$\mathbf{n}(u, v) = \frac{\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)}{|\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)|}$$

称一个变换 $(\tilde{u}, \tilde{v}) \rightarrow (u, v)$ 为正则参数变换, 若

$$\frac{\partial(u, v)}{\partial(\tilde{u}, \tilde{v})} \neq 0$$

而且有足够的可微性.

$$I = d\mathbf{r}(u, v) \cdot d\mathbf{r}(u, v) = E(du)^2 + 2Fdudv + G(dv)^2$$

其中 $E = \mathbf{r}_u \cdot \mathbf{r}_u, G = \mathbf{r}_v \cdot \mathbf{r}_v, F = \mathbf{r}_u \cdot \mathbf{r}_v$.

13.2.1 直纹面

直纹面可以表示为

$$\mathbf{r} = \mathbf{a}(u) + v\mathbf{l}(u)$$

其中 $\mathbf{a}(u)$ 是直纹面的准现, $\mathbf{l}(u)$ 是直母线的方向向量.

² 正则曲面是二维流形的例子, 曲面的每一个正则参数表示给出了一个局部坐标系

可展曲面

可展曲面是一种特殊的直纹面. 它的切平面沿每一条直母线是不变的. 曲面 S 是可展曲面的充要条件是

$$(\mathbf{a}'(u), \mathbf{l}(u), \mathbf{l}'(u)) = 0$$

13.2.2 例题

参数化曲面

直纹面的参数方程

Exercice 1

例题 3.2 写出单叶双曲面 $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ 作为直纹面的参数方程.

直接强行因式分解即可.

验证曲面的正则性

例题 3.3 在球面 $\Sigma : x^2 + y^2 + z^2 = 1$ 上, 命 $N = (0, 0, 1)$, $S = (0, 0, -1)$. 对于赤道平面上的任意一点 $p = (u, v, 0)$, 可以作唯一的一条直线经过 N, p 两点, 它与球面有唯一的交点, 记为 p' .

(1) 证明: 点 p' 的坐标是

$$x = \frac{2u}{u^2 + v^2 + 1}, \quad y = \frac{2v}{u^2 + v^2 + 1}, \quad z = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1},$$

并且它给出了球面上去掉北极 N 的剩余部分的正则参数表示;

验证曲面的可定向性

- (2) 求球面上去掉南极 S 的剩余部分的类似的正则参数表示;
- (3) 求上面两种正则参数表示在公共部分的参数变换;
- (4) 证明球面是可定向曲面.

判断正则参数曲面是球面的一部分

Exercice 2

例题 3.5 证明：一个正则参数曲面是球面的一部分的充分必要条件是，它所有的法线都经过一个固定点。

必要性显然，下面考虑充分性：假定曲面 S 的所有法线都经过一个固定点 \mathbf{c}_0 ，即存在函数 $\lambda(u, v)$ 使得

$$\mathbf{r}(u, v) + \lambda(u, v)\mathbf{n}(u, v) = \mathbf{c}_0 \quad (13.1)$$

下面证明 $\lambda(u, v)$ 是常值函数。将 eq. (13.1) 分别对 u, v 求导数得到

$$\mathbf{r}_u + \lambda\mathbf{n}_u + \lambda_u\mathbf{n} = 0 \quad \mathbf{r}_v + \lambda\mathbf{n}_v + \lambda_v\mathbf{n} = 0 \quad (13.2)$$

因为 $\mathbf{r}_u, \mathbf{r}_v$ 是切向量，又因为单位向量函数 \mathbf{n} 的偏导数必定与其它自身正交，因此用 \mathbf{r} 与 eq. (13.2) 做内积得到 $\lambda_u = \lambda_v = 0$ 。这说明 $\lambda(u, v) = \lambda_0$ ，于是

$$\mathbf{r}(u, v) - \mathbf{c}_0 = \lambda_0\mathbf{n}(u, v)$$

因此 $|\mathbf{r}(u, v) - \mathbf{c}_0|^2 = \lambda_0^2$ 。

旋转面的充要条件

Exercice 3

例题 3.6 证明：旋转面的法线必定与旋转轴平行或相交；反过来，如果一个正则参数曲面的所有法线都与一条固定的直线相交，则它必定是旋转面。

必要性显然。因为旋转面的参数方程为

$$\mathbf{r} = \mathbf{r}(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$$

其中 $f(u) > 0, f'^2(u) + g'(u) > 0$ 。

充分性的证明需要发挥一点几何想象力。不妨假定曲面 S 的所有法线都经过 z -轴，用一个通过 z -轴，并且与 Oxz 平面的夹角为 v 的平面截曲面 S ，其截线的参数方程可以假设为 $(u \cos v, u \sin v, g(u, v))$ ，这就是说该截线上的点到 z -轴的距离是 u ，到 Oxy 平面的距离是 $g(u, v)$ ，这里 u 是该截线上的参数， v 是任意的固定值。现在要证明：函数 $g(u, v)$ 与 v 无关，因而该曲面 S 就是一个旋转面。³

³很明显，当 v 变化时，上面的截线就扫出曲面 S 。

已知参数方程求第一基本形式

Exercice 4

例题 3.10 设球面的参数方程是 (参看例题 3. 3 (1))

$$\mathbf{r} = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right),$$

求它的第一基本形式.

直接计算 $E = \mathbf{r}_u \cdot \mathbf{r}_u, F = \mathbf{r}_u \cdot \mathbf{r}_v, G = \mathbf{r}_v \cdot \mathbf{r}_v$.

在球面上求与经线成固定角的轨线方程

例题 3.12 在球面上求与经线相交成固定角的轨线的方程.

解 由于现在要求的是与经线相交成固定角的轨线, 因此将球面看作旋转面写出它的参数方程比较方便. 设球面的参数方程是

$$\mathbf{r}(u, v) = (a \cos u \cos v, a \cos u \sin v, a \sin u),$$

对它求偏导数得到

$$\begin{aligned}\mathbf{r}_u &= (-a \sin u \cos v, -a \sin u \sin v, a \cos u), \\ \mathbf{r}_v &= (-a \cos u \sin v, a \cos u \cos v, 0),\end{aligned}$$

因此

$$E = \mathbf{r}_u \cdot \mathbf{r}_u = a^2, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v = 0, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v = a^2 \cos^2 u.$$

根据球面的参数方程, 其经线是 u -曲线, 它的切方向是 $(\delta u, \delta v) = (1, 0)$.

假定球面上的曲线 C 的切方向是 (du, dv) , 它与经线的夹角余弦是

$$\begin{aligned}\cos \theta &= \frac{Edu + Fdv}{\sqrt{E} \sqrt{E(du)^2 + 2Fdu dv + G(dv)^2}} \\ &= \frac{du}{\sqrt{(du)^2 + \cos^2 u(dv)^2}},\end{aligned}$$

所以曲线 C 所满足的微分方程是

$$c^2(du)^2 = (du)^2 + \cos^2 u(dv)^2, \quad \text{常数 } c^2 \geq 1,$$

解方程得到

$$\begin{aligned}v &= \pm \sqrt{c^2 - 1} \int \frac{du}{\cos u} = \pm \sqrt{c^2 - 1} \ln \left(\frac{1 + \sin u}{\cos u} \right) + c_1 \\ &= \pm \sqrt{c^2 - 1} \ln \left(\tan \left(\frac{u}{2} + \frac{\pi}{4} \right) \right) + c_1.\end{aligned}$$

正交化参数曲面网

例题 3.13 改写曲面

$$\mathbf{r} = (u \cos v, u \sin v, u + v)$$

的参数方程, 使得它的参数曲线网是正交曲线网.

先写成第一基本形式, 然后再配方换元.

求单参数平面族的包络

Exercice 5

求单参数平面族的包络:

$$x \cos \alpha + y \sin \alpha - z \sin \alpha = 1$$

命

$$F(x, y, z, \alpha) = x \cos \alpha + y \sin \alpha - z \sin \alpha - 1$$

则

$$F_\alpha(x, y, z, \alpha) = -x \sin \alpha + y \cos \alpha - z \cos \alpha$$

将方程组 $F = 0, F_\alpha = 0$ 中的参数 α 消去得到

$$x^2 + (y - z)^2 = 1$$

这是一张柱面，属于可展曲面的一种。写成参数方程的形式是

$$\mathbf{r} = (\cos u, \sin u + v, v) = (\cos u, \sin u, 0) + v(0, 1, 1)$$

13.3 曲面第二基本形式

曲面第一基本形式描写曲面上与度量有关的性质，曲面第二基本形式描写曲面的形状。

设曲面 S 的参数方程是 $\mathbf{r} = \mathbf{r}(u, v)$ ，其单位法向量为 $\mathbf{n}(u, v)$ ，则曲面的第二基本形式为

$$\mathrm{II} = L(du)^2 + 2M dudv + N(dv)^2$$

其中

$$L = \mathbf{r}_{uu} \cdot \mathbf{n} = -\mathbf{r}_u \cdot \mathbf{n}_u, \quad N = \mathbf{r}_{vv} \cdot \mathbf{n} = -\mathbf{r}_v \cdot \mathbf{n}_v$$

$$M = \mathbf{r}_{uv} \cdot \mathbf{n} = -\mathbf{r}_u \cdot \mathbf{n}_v = -\mathbf{r}_v \cdot \mathbf{n}_u$$

在计算系数 L, M, N 时，必须用单位法向量 $\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$ 。初学者容易犯用 $\mathbf{r}_u \times \mathbf{r}_v$ 代替 \mathbf{n} 的错误。

法曲率：

$$\kappa_n = \frac{L(\mathrm{d}u)^2 + 2M \mathrm{d}u \mathrm{d}v + N(\mathrm{d}v)^2}{E(\mathrm{d}u)^2 + 2F \mathrm{d}u \mathrm{d}v + G(\mathrm{d}v)^2} = \frac{\mathrm{II}}{\mathrm{I}}$$

正则参数曲面在任意一个固定点，其法曲率必定在两个彼此正交的切方向上分别取最大值和最小值。曲面在一个固定点处沿各个切方向的法曲率的最大值和最小值称为曲面在该点的主曲率，记为 κ_1, κ_2 ，达到这最大值和最小值的切方向称为曲面在该点的主方向。这个事实可以通过直接计算来证实（参看例题 4.5）。若曲面在 p 点的两个彼此正交的主方向单位向量是 e_1, e_2 ，对应的主曲率是 κ_1, κ_2 ，则曲面在点 p 沿着与主方向 e_1 的夹角为 θ 的切方向的法曲率是

$$\kappa_n(\theta) = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta,$$

这就是著名的 Euler 公式。

Gauss 曲率

13.3.1 求解主曲率和主方向

7. 求主曲率和主方向的具体做法如下：设有实数 λ 和非零切向量 $d\mathbf{r}$ 使得

$$W(d\mathbf{r}) = -d\mathbf{n} = \lambda d\mathbf{r},$$

即

$$-(\mathbf{n}_u du + \mathbf{n}_v dv) = \lambda(\mathbf{r}_u du + \mathbf{r}_v dv),$$

用 \mathbf{r}_u 和 \mathbf{r}_v 分别去点乘上面的式子，则得

$$(\lambda E - L)du + (\lambda F - M)dv = 0, \quad (\lambda F - M)du + (\lambda G - N)dv = 0.$$

因为 (du, dv) 是上面的线性方程组的非零解，故特征值（也就是主曲率） κ_1, κ_2 满足二次方程（特征方程）

$$\begin{vmatrix} \lambda E - L & \lambda F - M \\ \lambda F - M & \lambda G - N \end{vmatrix} = 0, \quad \text{即 } \lambda^2 - 2\lambda H + K = 0,$$

其中

$$H = \frac{EN - 2FM + GL}{2(EG - F^2)}, \quad K = \frac{LN - M^2}{EG - F^2}.$$

根据二次方程的根与系数的关系，得知 $2H = \kappa_1 + \kappa_2$, $K = \kappa_1 \kappa_2$, 所以称 H 为平均曲率，称 K 为 Gauss 曲率。

将 κ_1 和 κ_2 逐次代替前面的线性方程组中的 λ , 解出的 (du, dv) 就是相应的主方向。

13.3.2 直接求主方向

8. 直接求主方向的方法：将前面的线性方程组改写为

$$\lambda(Edu + Fdv) - (Ldu + Mdv) = 0, \quad \lambda(Fdu + Gdv) - (Mdu + Ndv) = 0,$$

由于 $(\lambda, -1) \neq 0$ 满足上面的线性方程，所以必须有

$$\begin{vmatrix} Edu + Fdv & Ldu + Mdv \\ Fdu + Gdv & Mdu + Ndv \end{vmatrix} = 0,$$

展开后重新整理得到便于记忆的形式：

$$\begin{vmatrix} (dv)^2 & -dudv & (du)^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0.$$

这个二次方程的解 (du, dv) 就是主方向。

13.3.3 渐进曲线

法曲率为零的切方向称为渐近方向。曲面只在双曲点和抛物点有渐近方向。曲面上其切方向处处是曲面的渐近方向的曲线称为曲面上的渐近曲线。渐近曲线的微分方程是

$$L(du)^2 + 2M du dv + N(dv)^2 = 0$$

13.3.4 椭圆点、抛物点、双曲点

12. 曲面上的点按照 Gauss 曲率的符号不同分为椭圆点、抛物点和双曲点三类。曲面在这些点附近的形状如同近似曲面。具体情形如下列表格所示：

点型、Dupin 标形和近似曲面

点型	Gauss 曲率	Dupin 标形	近似曲面	渐近方向
椭圆点	$K > 0$	椭圆	椭圆抛物面	无
双曲点	$K < 0$	两对共轭 双曲线	双曲抛物面	两个
抛物点 (非平点)	$K = 0$	一对平行直线	抛物柱面	一个
抛物点 (平点)	$K = 0$	无	无	任意的 切方向

13.3.5 例题

计算法曲率, 主曲率, Gauss 曲率, 平均曲率, 主方向

例题 4.15 求双曲抛物面 $\mathbf{r} = (a(u+v), b(u-v), 2uv)$ 的 Gauss 曲率 K , 平均曲率 H , 主曲率 κ_1, κ_2 和它们所对应的主方向.

解 对双曲抛物面的参数方程求偏导数得到

$$\begin{aligned}\mathbf{r}_u &= (a, b, 2v), \quad \mathbf{r}_v = (a, -b, 2u), \\ \mathbf{r}_u \times \mathbf{r}_v &= (2b(u+v), -2a(u-v), -2ab), \\ \mathbf{n} &= \frac{1}{A} (b(u+v), -a(u-v), -ab),\end{aligned}$$

其中 $A = \sqrt{b^2(u+v)^2 + a^2(u-v)^2 + a^2b^2} = \frac{1}{2}\sqrt{EG - F^2}$,

$$\mathbf{r}_{uu} = (0, 0, 0), \quad \mathbf{r}_{uv} = (0, 0, 2), \quad \mathbf{r}_{vv} = (0, 0, 0).$$

直接计算得到

$$\begin{aligned} E &= a^2 + b^2 + 4v^2, \quad F = a^2 - b^2 + 4uv, \quad G = a^2 + b^2 + 4u^2, \\ L &= 0, \quad M = \frac{-2ab}{A}, \quad N = 0. \end{aligned}$$

因此,

$$\begin{aligned} H &= \frac{EN - 2FM + GL}{2(EG - F^2)} = -\frac{FM}{EG - F^2} = \frac{ab(a^2 - b^2 + 4uv)}{2A^3}, \\ K &= \frac{LN - M^2}{EG - F^2} = -\frac{M^2}{EG - F^2} = -\frac{a^2b^2}{A^4}. \end{aligned}$$

解方程

$$\begin{aligned} \lambda^2 - 2H\lambda + K &= \lambda^2 + \frac{2FM}{EG - F^2}\lambda - \frac{M^2}{EG - F^2} = 0, \\ \left(\lambda + \frac{FM}{EG - F^2}\right)^2 &= \frac{M^2}{EG - F^2} + \frac{F^2M^2}{(EG - F^2)^2} = \frac{M^2EG}{(EG - F^2)^2}, \end{aligned}$$

得到

$$\begin{aligned} \kappa_1 &= -\frac{M(F + \sqrt{EG})}{EG - F^2} = \frac{M}{F - \sqrt{EG}} \\ &= \frac{ab}{2A^3} \left(a^2 - b^2 + 4uv + \sqrt{(a^2 + b^2 + 4u^2)(a^2 + b^2 + 4v^2)} \right), \\ \kappa_2 &= -\frac{M(F - \sqrt{EG})}{EG - F^2} = \frac{M}{F + \sqrt{EG}} \\ &= \frac{ab}{2A^3} \left(a^2 - b^2 + 4uv - \sqrt{(a^2 + b^2 + 4u^2)(a^2 + b^2 + 4v^2)} \right). \end{aligned}$$

将 $\lambda = \kappa_1$ 和 $\lambda = \kappa_2$ 分别代入

$$\frac{du}{dv} = -\frac{\lambda F - M}{\lambda E - L} = -\frac{F}{E} + \frac{1}{\lambda} \cdot \frac{M}{E},$$

便得到对应的主方向. 直接计算得到, 对应于 κ_1 的主方向是

$$\frac{du}{dv} = -\frac{F}{E} + \frac{F - \sqrt{EG}}{M} \cdot \frac{M}{E} = -\frac{\sqrt{G}}{\sqrt{E}} = -\sqrt{\frac{a^2 + b^2 + 4u^2}{a^2 + b^2 + 4v^2}},$$

对应于 κ_2 的主方向是

$$\frac{du}{dv} = -\frac{F}{E} + \frac{F + \sqrt{EG}}{M} \cdot \frac{M}{E} = \frac{\sqrt{G}}{\sqrt{E}} = \sqrt{\frac{a^2 + b^2 + 4u^2}{a^2 + b^2 + 4v^2}}.$$

注记 尽管计算比较繁琐，但是本题属于常规的计算，没有特别的困难。需要指出的是，在求第二基本形式的系数时， n 是 $r_u \times r_v$ 的单位化。在做题时这个地方容易出错，需要特别小心。另外，在计算时充分利用 $L = N = 0$ 的特殊性，采用符号计算，直到化简以后才用相应的表达式代入，这就排除了繁杂的计算过程。这是值得注意的技巧。 $L = N \equiv 0$ 正好是在曲面上取渐近曲线网为参数曲线网的充分必要条件，本题的一般公式适用于这种情形。

求脐点

脐点（Umbilic Point）是微分几何中的一个重要概念，主要用于研究曲面的二阶几何性质（如曲率）。它是指曲面的主曲率在该点处相等的点。

例题 4.18 求曲面 $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$ 的脐点，其中 $\alpha > \beta > \gamma > 0$.

13.4 曲面论基本定理

为了说明 I, II 构成曲面的完全的不变量系统，需要曲面论的基本公式，也就是曲面的自然标架场的求导公式。它们在曲面的理论中扮演基本的角色，相当于曲线论中的 Frenet 公式。

Einstein 和式约定：

3. 曲面论的基本公式和基本方程涉及二元函数的两次以上偏导数, 不可避免的要用求和表达式. 关于曲面的各种量的原有记号 (Gauss 记号) 显得不适用了, 必须引进带指标的记号, 并且采用 Einstein 和式约定使繁复的表达式得以简化.

原有记号和带指标记号的对应如表所示:

Gauss 记号	u	v	\mathbf{r}_u	\mathbf{r}_v	E	F	G	L	M	N
张量记号	u^1	u^2	\mathbf{r}_1	\mathbf{r}_2	g_{11}	g_{12}	g_{22}	b_{11}	b_{12}	b_{22}

规定希腊字母 $\alpha, \beta, \gamma, \dots$ 作为指标的取值范围为 $\{1, 2\}$. 规定拉丁字母 $i, j, k, l, a, b, c, \dots$ 作为指标的取值范围为 $\{1, 2, 3\}$.

这样, $\mathbf{r}(u, v)$ 记成 $\mathbf{r}(u^1, u^2)$, 并且

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial \mathbf{r}}{\partial u^1} = \mathbf{r}_1, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial \mathbf{r}}{\partial u^2} = \mathbf{r}_2.$$

相应地,

$$\begin{aligned} d\mathbf{r} &= \sum_{\alpha=1}^2 \mathbf{r}_\alpha du^\alpha = \mathbf{r}_\alpha du^\alpha, \\ I &= |d\mathbf{r}|^2 = \sum_{\alpha, \beta=1}^2 (\mathbf{r}_\alpha \cdot \mathbf{r}_\beta) du^\alpha du^\beta = \sum_{\alpha, \beta=1}^2 g_{\alpha\beta} du^\alpha du^\beta = g_{\alpha\beta} du^\alpha du^\beta, \end{aligned}$$

其中 $\mathbf{r}_\alpha \cdot \mathbf{r}_\beta = g_{\alpha\beta}$. 第二基本形式是

$$II = \sum_{\alpha, \beta=1}^2 (\mathbf{r}_{\alpha\beta} \cdot \mathbf{n}) du^\alpha du^\beta = \sum_{\alpha, \beta=1}^2 b_{\alpha\beta} du^\alpha du^\beta = b_{\alpha\beta} du^\alpha du^\beta,$$

其中 $\mathbf{r}_{\alpha\beta} \cdot \mathbf{n} = b_{\alpha\beta}$.

13.4.1 Gauss 绝妙定理

Théorème 1: Gauss 绝妙定理

曲面的 Gauss 曲率是曲面在保长变换下的不变量.

事实上, 由 Gauss 方程可知

$$K = \frac{b_{11}b_{22} - (b_{12})^2}{g_{11}g_{22} - (g_{12})^2} = \frac{R_{1212}}{g_{11}g_{22} - (g_{12})^2}$$

因此曲面的 Gauss 曲率是由它的第一基本形式完全确定的. 在 $g_{12} = F = 0$ 的情形, Gauss 曲率用曲面的第一基本形式的表达式是

$$K = -\frac{1}{\sqrt{EG}} \left(\left(\frac{(\sqrt{E})_v}{\sqrt{G}} \right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}} \right)_u \right)$$

在曲面的等温参数系下, $E = G = \lambda^2, F = 0$, 则 Gauss 曲率的表达式是

$$K = -\frac{1}{\lambda^2} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \log \lambda$$

Remarque 55. Gauss 绝妙定理是微分几何学发展过程中的里程碑, 开创了内蕴几何学的新时代, 进而引发了 Riemann 几何学.

13.5 极小曲面

参见彭家贵《微分几何教材》

Définition 1: 极小曲面

\mathbb{E}^3 中平均曲率处处恒为零的曲面称为极小曲面.

13.5.1 判定极小曲面的准则

第一基本形式系数

$$E = x_u \cdot x_u = \cosh^2 u, \quad F = x_u \cdot x_v = 0$$

$$G = x_v \cdot x_v = \cosh^2 u$$

$$L = x_{uu} \cdot n = -1, \quad M = x_{uv} \cdot n = 0, \quad N = x_{vv} \cdot n = 1$$

中的曲率函数

$$H = \frac{1}{2} \cdot \frac{LG - 2MF + NE}{EG - F^2} = 0$$

即曲面为极小曲面.

13.6 测地曲率和测地线

通常我们把欧式平面看作是二维平直空间 (Gauss 曲率为零), 而把给定第一基本形式的抽象曲面称为二维弯曲空间. 本章的目标就是研究二维弯曲空间中的几何学.

13.6.1 Motivation

对于正则参数曲面 S 满足方程 $\mathbf{r} = \mathbf{r}(u^1, u^2)$. C 是曲面 S 上的一条曲线, 它的方程是 $u^\alpha = u^\alpha(s), \alpha = 1, 2$, 其中 s 是曲线 C 的弧长参数. 那么 C 的参数方程为

$$\mathbf{r} = \mathbf{r}(s) = \mathbf{r}(u^1(s), u^2(s))$$

我们的目的是沿着曲线 C 建立一个新的正交标架场 $\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, 兼顾 C 和 S , 定义如下

$$\mathbf{e}_1 = \boldsymbol{\alpha}(s) \quad \mathbf{e}_3 = \mathbf{n}(s) \quad \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{n}(s) \times \boldsymbol{\alpha}(s)$$

直观上, \mathbf{e}_2 是将曲线 C 的切向量 $\mathbf{e}_1 = \boldsymbol{\alpha}$ 绕着 S 的单位法向量 \mathbf{n} 正向旋转 90° 得到的. 改标架场沿着曲线 C 的运动公式为

$$\begin{cases} \frac{d\mathbf{r}(s)}{ds} = & \mathbf{e}_1, \\ \frac{d\mathbf{e}_1}{ds} = & \kappa_g \mathbf{e}_2 + \kappa_n \mathbf{e}_3, \\ \frac{d\mathbf{e}_2}{ds} = -\kappa_g \mathbf{e}_1 & + \tau_g \mathbf{e}_3, \\ \frac{d\mathbf{e}_3}{ds} = -\kappa_n \mathbf{e}_1 & - \tau_g \mathbf{e}_2, \end{cases}$$

其中 $\kappa_n \mathbf{e}_3$ 是曲线 C 的曲率向量在曲面 S 的法向量上的正交投影, 故 κ_n 恰好是曲面 S 上的曲线 C 的法曲率; $\kappa_g \mathbf{e}_2$ 是曲线 C 的曲率向量在曲面 S 的切平面上的正交投影。这里的 κ_g 的计算公式是

$$\begin{aligned} \kappa_g &= \frac{d^2\mathbf{r}(s)}{ds^2} \cdot \mathbf{e}_2 = \mathbf{r}''(s) \cdot (\mathbf{n}(s) \times \mathbf{r}'(s)) \\ &= (\mathbf{n}(s), \mathbf{r}'(s), \mathbf{r}''(s)), \end{aligned}$$

把最后的式子展开得到

$$\kappa_g = \sqrt{g_{11}g_{22} - (g_{12})^2} \left| \begin{array}{cc} \frac{du^1}{ds} & \frac{d^2u^1}{ds^2} + \Gamma_{\alpha\beta}^1 \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \\ \frac{du^2}{ds} & \frac{d^2u^2}{ds^2} + \Gamma_{\alpha\beta}^2 \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \end{array} \right|.$$

曲线 C 作为曲面 S 内的曲线的测地曲率 κ_g 和它作为空间曲线的曲率 κ 的关系式是

$$\kappa_g = \kappa \cos \tilde{\varphi}, \quad \kappa^2 = \kappa_g^2 + \kappa_n^2$$

这里 $\tilde{\varphi}$ 是曲线 C 的次法向量和曲面 S 的单位法向量之间的夹角.

沿曲线 C 的上述正交标架场的运动公式中的 τ_g 不是属于曲面的内蕴几何学的量, 它的计算公式是

$$\tau_g = \frac{1}{\sqrt{g_{11}g_{22} - (g_{12})^2}} \begin{vmatrix} \left(\frac{du^2}{ds}\right)^2 & -\frac{du^1}{ds} \frac{du^2}{ds} & \left(\frac{du^1}{ds}\right)^2 \\ g_{11} & g_{12} & g_{22} \\ b_{11} & b_{12} & b_{22} \end{vmatrix}$$

称为曲线的测地挠率. 实际上, 测地挠率 τ_g 和法曲率 κ_n 的性质相同, 都是曲面 S 在任意一点的切方向的函数, 与曲线 C 本身弯曲性无关.

13.6.2 测地线

Définition 2: 测地线

在曲面 S 上测地曲率恒等于零的曲线称为曲面 S 上的测地线. 曲面 S 上的测地线是属于曲面 S 的内蕴几何学的概念.

因为平面曲线的测地曲率就是它的相对曲率, 因此平面上的测地线就是该平面上的直线. 由此可见, 曲面上的测地线的概念是平面上的直线概念的推广.

运动学的观点

曲面 S 上的测地线 C 作为 S 的外围空间 E^3 中的曲线的特征是: 或者曲线 C 本身是直线, 或者它的主法向量处处是曲面 S 的法向量. 从运动学观点来看, 测地线 C 的特征是: 如果在曲面 S 上运动的质点 p 只受到将它约束在曲面 S 上的力的作用 (即作用力的方向垂直于曲面 S), 则点 p 的轨迹 C 是曲面 S 上的测地线.

最短弧长

曲面 S 上的测地线 C 所满足的内在特征是: 对于它在曲面 S 内的任意一个有固定端点的变分 $C_t (-\varepsilon < t < \varepsilon)$ 而言, $C (= C_0)$ 的弧长是变分曲线 C_t 的弧长的临界值.

13.6.3 曲面分类

13. Gauss 曲率为常数 K 的曲面的第一基本形式可以表示成

$$\text{当 } K > 0 \text{ 时, } I = (du)^2 + \cos^2(\sqrt{K}u)(dv)^2,$$

$$\text{当 } K = 0 \text{ 时, } I = (du)^2 + (dv)^2,$$

$$\text{当 } K < 0 \text{ 时, } I = (du)^2 + \cosh^2(\sqrt{-K}u)(dv)^2.$$

因此, 有相同常数 Gauss 曲率的曲面彼此都可以建立保长对应.

13.6.4 Gauss-Bonnet 公式 & 定理

16. Gauss-Bonnet 公式: 假定曲线 C 是有向曲面 S 上的一条由 n 段光滑曲线组成的分段光滑简单闭曲线, 它所包围的区域 D 是曲面 S 的一个单连通区域, 则

$$\oint_C \kappa_g ds + \iint_D K d\sigma = 2\pi - \sum_{i=1}^n \alpha_i,$$

其中 κ_g 是曲线 C 的测地曲率, K 是曲面 S 的 Gauss 曲率, α_i 表示曲线 C 在角点 $s = s_i$ 的外角.

17. Gauss-Bonnet 定理: 假定 S 是有界的无边有向闭曲面, 则

$$\iint_D K d\sigma = 2\pi\chi(S) = 4\pi(1 - g),$$

其中 $\chi(S)$ 是曲面 S 的 Euler 示性数, g 是曲面 S 的亏格.

18. 利用 Gauss-Bonnet 公式可以证明: 当单位向量 X 围绕落在曲面 S 上的分段光滑简单闭曲线 C 平行移动一周后再回到出发点时与初始单位向量 X 会有一个夹角, 如果曲线 C 所围成的区域 D 是单连通的, 则所转过的角度恰好是曲面 S 的 Gauss 曲率 K 在曲线 C 所围成的单连通区域 D 上的积分. 这个事实说明弯曲空间和平直空间的差异.

13.7 空间曲线

13.7.1 正则曲线, 弧长参数, 单位切向量, 曲率, 主法向量, 次法向量, 挠率, Frenet 公式

对于 \mathbb{R}^3 中的正则曲线 ($\mathbf{r}'(t) \neq 0$) $C : \mathbf{r} = \mathbf{r}(t), a \leq t \leq b$, 引进参数 s 使得

$$s = s(t) = \int_a^t |\mathbf{r}'(t)| dt$$

s 被称为曲线 C 的 (弧长参数). 一个显然的事实是, 正则曲线 $C : \mathbf{r} = \mathbf{r}(t)$ 的参数 t 为正则参数的特征是 $|\mathbf{r}'(t)| \equiv 1$. 一般的计算中, t 未必是弧长参数.

关于空间曲线的理论, 最重要的是沿曲线 $\mathbf{r} = \mathbf{r}(t)$ 定义的 Frenet 标架 $\{\mathbf{r}; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ 和 Frenet 公式.

假定空间曲线 C 的参数方程为 $\mathbf{r} = \mathbf{r}(s)$, 其中 s 是弧长参数, 那么它的单位切向量:

$$\boldsymbol{\alpha}(s) = \mathbf{r}'(s)$$

曲率:

$$\kappa(s) = |\boldsymbol{\alpha}'(s)| = |\mathbf{r}''(s)|$$

曲率非零时, 主法向量 (这样的定义是自然的):

$$\boldsymbol{\beta}(s) = \frac{\boldsymbol{\alpha}'(s)}{|\boldsymbol{\alpha}'(s)|} = \frac{\mathbf{r}''(s)}{|\mathbf{r}''(s)|}$$

次法向量:

$$\boldsymbol{\gamma}(s) = \boldsymbol{\alpha}(s) \times \boldsymbol{\beta}(s)$$

挠率⁴

$$\tau(s) = -\boldsymbol{\gamma}'(s) \cdot \boldsymbol{\beta}(s)$$

曲线的 Frenet 公式是

$$\begin{cases} \mathbf{r}'(s) &= \boldsymbol{\alpha}(s), \\ \boldsymbol{\alpha}'(s) &= \kappa(s)\boldsymbol{\beta}(s), \\ \boldsymbol{\beta}'(s) &= -\kappa(s)\boldsymbol{\alpha}(s) + \tau(s)\boldsymbol{\gamma}(s), \\ \boldsymbol{\gamma}'(s) &= -\tau(s)\boldsymbol{\beta}(s), \end{cases}$$

上式都是关于弧长参数 $s = \int_a^t |\mathbf{r}'(t)| dt$ 的导数.

⁴挠率一开始的定义就是使得 $\boldsymbol{\gamma}'(s) = -\tau(s)\boldsymbol{\beta}(s)$ 的 s 函数, 下面这个只是计算式.

Frenet 公式推导

首先根据 $\alpha(s)$ 定义知道 $\mathbf{r}'(s) = \alpha(s)$. 其次根据 $\kappa(s), \beta(s)$ 的定义知道

$$\kappa(s)\beta(s) = \mathbf{r}''(s) = \alpha'(s)$$

根据 $\tau(s)$ 最原本的定义可知

$$\gamma'(s) = -\tau(s)\beta(s)$$

接下来待定系数求解 $\beta'(s) = a\alpha(s) + b\beta(s) + c\gamma(s)$, 分别与 $\alpha(s), \beta(s), \gamma(s)$ 作内积得到

$$a = \beta'(s) \cdot \alpha(s) \stackrel{\alpha(s) \perp \beta(s)}{=} -\beta(s) \cdot \alpha'(s) = -\kappa(s)$$

$$b = \beta'(s) \cdot \beta(s) = 0$$

$$c = \beta'(s) \cdot \gamma(s) \stackrel{\beta(s) \perp \gamma(s)}{=} -\beta(s) \cdot \underbrace{\gamma'(s)}_{=-\tau(s)\beta(s)} = \tau(s)$$

从而

$$\beta'(s) = -\kappa(s)\alpha(s) + \tau(s)\gamma(s)$$

我们完成了 Frenet 公式的推导证明!

13.7.2 转化为关于 t 的方程

在曲线的一般参数下, 设曲线的参数方程是 $\mathbf{r} = \mathbf{r}(t)$, 则它的单位切向量是

$$\alpha(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

假定曲线的弧长参数是 s , 则 $s'(t) = |\mathbf{r}'(t)|$, 所以

$$\mathbf{r}'(t) = \frac{d\mathbf{r}(t)}{ds} \frac{ds}{dt} = \alpha(t)s'(t)$$

因此

$$\begin{aligned} \mathbf{r}''(t) &= \frac{d\alpha(t)}{dt} s'(t) + \alpha(t)s''(t) \\ &= \frac{d\alpha(t)}{ds} (s'(t))^2 + \alpha(t)s''(t) \\ &= \kappa(t)\beta(t) (s'(t))^2 + \alpha(t)s''(t), \end{aligned}$$

故

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \kappa(t) (s'(t))^3 \alpha(t) \times \beta(t) = \kappa(t) (s'(t))^3 \gamma(t)$$

由此得到曲线的曲率是

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{(s'(t))^3} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

次法向量是

$$\gamma(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}$$

这样，曲线的主法向量是

$$\beta(t) = \gamma(t) \times \alpha(t)$$

再利用 Frenet 公式得到

$$\begin{aligned}\tau(t) &= -\left(\gamma'(t) \frac{dt}{ds}\right) \cdot \beta(t) = -\frac{1}{|\mathbf{r}'(t)|} \gamma'(t) \cdot \beta(t) \\ &= \frac{(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t))}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}\end{aligned}$$

此时，单位切向量，主法向量和次法向量的导数是

$$\begin{aligned}\alpha'(t) &= \frac{d\alpha}{ds} \cdot s'(t) = \kappa(t)s'(t)\beta(t), \\ \beta'(t) &= \frac{d\beta}{ds} \cdot s'(t) = -\kappa(t)s'(t)\alpha(t) + \tau(t)s'(t)\gamma(t), \\ \gamma'(t) &= \frac{d\gamma}{ds} \cdot s'(t) = -\tau(t)s'(t)\beta(t).\end{aligned}$$

Théorème 2: 空间曲线基本定理

给定两个连续可微函数 $\kappa(s), \tau(s)$ 其中 $\kappa(s) > 0$ ，则在三维欧式空间中存在一条空间曲线，以 s 为弧长参数，以 $\kappa(s)$ 为曲率，以 $\tau(s)$ 为挠率，并且这样的曲线的形状是完全确定的。

Remarque 56. 在给定 $\kappa(s), \tau(s)$ 的情况下，Frenet 公式构成了向量 $\mathbf{r}, \alpha, \beta, \gamma$ 的微分方程组，可以求解。

13.7.3 切触阶

两条相交曲线在交点附近的接近程度是用所谓的切触阶来刻画的。

5. 两条相交的曲线在交点附近的接近程度是用所谓的切触阶来刻画的. 设曲线 C_1 和 C_2 相交于点 p_0 , 在 C_1 和 C_2 上各取一点 p_1 和 p_2 , 使得曲线 C_1 在点 p_0 和 p_1 之间的弧长是 Δs , C_2 在点 p_0 和 p_2 之间的弧长也是 Δs , 若有正整数 n 使得

$$\lim_{\Delta s \rightarrow 0} \frac{|p_1 p_2|}{(\Delta s)^n} = 0, \quad \lim_{\Delta s \rightarrow 0} \frac{|p_1 p_2|}{(\Delta s)^{n+1}} \neq 0,$$

则称曲线 C_1 和 C_2 在交点 p_0 处有 n 阶切触.

容易证明: 设曲线 $r_1(s)$ 和 $r_2(s)$ 都以 s 为它们的弧长参数, 且 $r_1(s_0) = r_2(s_0)$, 则它们在 $s = s_0$ 处有 n 阶切触的充分必要条件是

$$r_1^{(i)}(s_0) = r_2^{(i)}(s_0), \quad \forall 1 \leq i \leq n; \quad r_1^{(n+1)}(s_0) \neq r_2^{(n+1)}(s_0).$$

若一条曲线 C 和一个曲面 Σ 相交, 同样能够用切触阶来刻画曲线和曲面的接近程度. 设交点是 p_0 . 在曲线 C 上取一点 p_1 , 把曲线 C 上从点 p_0 到点 p_1 的弧长记为 Δs , 把点 p_1 到曲面 Σ 的距离最近的点记为 p_2 , 若有正整数 n 使得

$$\lim_{\Delta s \rightarrow 0} \frac{|p_1 p_2|}{(\Delta s)^n} = 0, \quad \lim_{\Delta s \rightarrow 0} \frac{|p_1 p_2|}{(\Delta s)^{n+1}} \neq 0,$$

则称曲线 C 和曲面 Σ 在交点 p_0 处有 n 阶切触.

13.7.4 平面曲线

平面曲线可以看作空间曲线的特例, 即 $\tau(s) \equiv 0$ 的空间曲线. 空间曲线求曲率 $\kappa(s) = |\mathbf{r}''(s)|$ 的公式照用. 特别的是, 平面本身有定向, 将其单位切向量正向 (逆时针) 旋转 90° 便得到法向量 (唯一确定). 于是

$$\alpha(s) = (x'(s), y'(s))$$

正向旋转 90° 得到⁵

$$\beta(s) = (-y'(s), x'(s))$$

相对曲率

$$\kappa_r(s) = \alpha'(s) \cdot \beta(s) = x'(s)y''(s) - y'(s)x''(s)$$

相对曲率 κ_r 与曲率 κ 的关系是 $\kappa_r = \pm \kappa$. 正号表示曲线朝着 $\beta(s)$ 的方向弯曲, 负号表示曲线的主法向量为 $-\beta(s)$.

⁵不一定是主法向量

13.7.5 例题

已知参数方程求曲线方程

已知一般方程求参数方程

已知曲线方程求曲线曲率、挠率、Frenet 标架

已知参数方程，直接套公式爆算：

例题 2.4 求圆螺旋线 $\mathbf{r} = (a \cos t, a \sin t, bt)$ 的曲率、挠率和它的 Frenet 标架，其中 a, b 是常数，且 $a > 0$.

已知一般方程：

例题 2.5 求曲线 C

$$\begin{cases} x^2 + y^2 + z^2 = 1, \\ x^2 + y^2 = x \end{cases}$$

在 $(0, 0, 1)$ 处的曲率 κ , 挠率 τ 和 Frenet 标架.

直接通过对方程求导，求解出 $\mathbf{r}'(0), \mathbf{r}''(0), \mathbf{r}'''(0)$.

解 这是例题 2.2 考虑过的曲线. 解此题的方法有两种. 一种方法是把该曲线在点 $(0, 0, 1)$ 的邻域内的部分用参数方程表示出来，然后按照例题 2.4 的办法进行计算. 但是，有时候用参数方程表示两个曲面的交线比较复杂，涉及解函数方程. 因此，我们在此介绍第二种方法.

求解新曲线的曲率、挠率、Frenet 标架

例题 2.6 已知 $C : \mathbf{r} = \mathbf{r}(s)$ 是一条正则参数曲线， s 是它的弧长参数，其曲率 $\kappa(s) > 0$ 和挠率 $\tau(s) > 0$, $\{\mathbf{r}(s); \alpha(s), \beta(s), \gamma(s)\}$ 是沿曲线 C 的 Frenet 标架场. 作一条新的曲线 \tilde{C} :

$$\tilde{\mathbf{r}}(s) = \int_{s_0}^s \beta(s) ds.$$

求曲线 \tilde{C} 的曲率 $\tilde{\kappa}$ 、挠率 $\tilde{\tau}$ 和 Frenet 标架场 $\{\tilde{\mathbf{r}}(s); \tilde{\alpha}(s), \tilde{\beta}(s), \tilde{\gamma}(s)\}$.

例题 2.7 在上面例题关于曲线 C 的假设下, 求它的切线的球面标线 \tilde{C} :

$$\tilde{r}(s) = \alpha(s)$$

的曲率 $\tilde{\kappa}$ 、挠率 $\tilde{\tau}$ 和 Frenet 标架场.

特定曲线满足曲率挠率关系式

例题 2.9 设曲线 $r = r(s)$ 的曲率 $\kappa(s)$ 和挠率 $\tau(s)$ 都不为零, s 是弧长参数. 如果该曲线落在一个球面上, 则它的曲率和挠率必满足关系式

$$\left(\frac{1}{\kappa(s)}\right)^2 + \left(\frac{1}{\tau(s)} \frac{d}{ds} \left(\frac{1}{\kappa(s)}\right)\right)^2 = \text{常数}.$$

例题 2.10 假定 $r = r(s)$ 是以 s 为弧长参数的正则参数曲线, 它的挠率不为零, 曲率不是常数, 并且下面的关系式成立:

$$\left(\frac{1}{\kappa(s)}\right)^2 + \left(\frac{1}{\tau(s)} \frac{d}{ds} \left(\frac{1}{\kappa(s)}\right)\right)^2 = R_0^2 = \text{常数},$$

证明该曲线落在一个球面上.

已知曲率挠率, 求解曲线参数方程

例题 2.11 求曲率和挠率分别是常数 $\kappa_0 > 0, \tau_0$ 的曲线的参数方程.

根据曲线论基本定理具有这样常数曲率 $\kappa_0 > 0$ 和挠率 τ_0 的曲线必定是圆螺旋线

$$r = (a \cos t, a \sin t, bt) \quad a > 0, b \text{ 是常数}$$

Remarque 57. 本题也可以根据 Frenet 标架直接求解微分方程组.

求密切圆

例题 2.12 假定 $C : \mathbf{r} = \mathbf{r}(s)$ 是以 s 为弧长参数的正则参数曲线, 它的曲率处处不为零. 求与它在 $s = s_0$ 处有最高切触阶的圆周.

解 此题的关键在于如何表示三维欧氏空间中的圆周. 假定圆心的位置向量是 \mathbf{c} , 半径是 R , 要写出它的参数方程需要在它所在的平面上取两个彼此正交的单位向量, 设为 \mathbf{a}, \mathbf{b} , 此时该圆周的参数方程是

$$\tilde{\mathbf{r}}(s) = \mathbf{c} + R \left(\cos \frac{s}{R} \mathbf{a} + \sin \frac{s}{R} \mathbf{b} \right),$$

显然 s 是该圆周的弧长参数.

为简单起见, 设 $s_0 = 0$, 并且记曲线 C 在该点的 Frenet 标架是 $\{\mathbf{r}_0; \alpha_0, \beta_0, \gamma_0\}$, 曲率是 κ_0 , 挠率是 τ_0 . 对于与已知曲线在 $s = 0$ 处相交的圆周, 设圆心的位置向量是 \mathbf{c} , 取 $\mathbf{a} = (\mathbf{r}_0 - \mathbf{c})/R$, 其中 $R = |\mathbf{r}_0 - \mathbf{c}|$, \mathbf{b} 是与 \mathbf{a} 正交的单位向量. 因此该圆周 \tilde{C} 的参数方程是

$$\tilde{\mathbf{r}}(s) = \mathbf{c} + R \left(\cos \frac{s}{R} \mathbf{a} + \sin \frac{s}{R} \mathbf{b} \right),$$

其中 s 是该圆周的弧长参数. 两条相交的曲线在交点处有 n 阶切触的条件是它们的参数方程在该点有相同的直到 n 阶的关于弧长参数的各阶导数. 已知

$$\mathbf{r}'(0) = \alpha_0, \quad \mathbf{r}''(0) = \kappa(0)\beta_0,$$

以下记 $\kappa_0 = \kappa(0)$. 然而关于圆周 \tilde{C} 有

$$\begin{aligned} \tilde{\mathbf{r}}(0) &= \mathbf{r}_0, & \tilde{\mathbf{r}}'(0) &= \mathbf{b}, \\ \tilde{\mathbf{r}}''(0) &= -\frac{1}{R}\mathbf{a} = \frac{\mathbf{c} - \mathbf{r}_0}{|\mathbf{c} - \mathbf{r}_0|^2}, \end{aligned}$$

于是由条件

$$\tilde{\mathbf{r}}'(0) = \mathbf{r}'(0), \quad \tilde{\mathbf{r}}''(0) = \mathbf{r}''(0)$$

得到

$$\mathbf{b} = \alpha_0, \quad R = \frac{1}{\kappa_0}, \quad \mathbf{c} = \mathbf{r}_0 + \frac{1}{\kappa_0}\beta_0.$$

所以它是落在曲线 C 在点 $s = 0$ 的密切平面上、半径为 $1/\kappa_0$ 、与曲线 C 在点 $s = 0$ 处相切的圆周, 它与曲线 C 在 $s = 0$ 处至少有二阶切触, 称为原曲线在该点的曲率圆, 或密切圆.

求密切球

例题 2.13 设 $C: \mathbf{r} = \mathbf{r}(s)$ 是曲率和挠率都不为零的正则参数曲线, s 是弧长参数, 求与曲线 C 在点 $s = s_0$ 处有最高阶切触的球面.

求渐伸线和渐缩线参数方程

例题 2.15 如果在曲线 C_1 和 C_2 之间存在一个对应, 使得曲线 C_1 在任意一点的切线恰好是曲线 C_2 在对应点的法线, 则称曲线 C_2 是 C_1 的渐伸线, 同时称曲线 C_1 是 C_2 的渐缩线. 设正则参数曲线 C 的参数方程是 $\mathbf{r}(s)$, s 是弧长参数, 求曲线 C 的渐伸线的参数方程.

解 设

$$\mathbf{r}_1(s) = \mathbf{r}(s) + \lambda(s)\boldsymbol{\alpha}(s)$$

是曲线 C 的渐伸线, 因此曲线 C 的切向量 $\boldsymbol{\alpha}(s)$ 应该是曲线 $\mathbf{r}_1(s)$ 的法向量. 对上式求导得到

$$\mathbf{r}'_1(s) = (1 + \lambda'(s))\boldsymbol{\alpha}(s) + \lambda(s)\kappa(s)\boldsymbol{\beta}(s),$$

将上式两边与 $\boldsymbol{\alpha}(s)$ 作点乘得到

$$1 + \lambda'(s) = \mathbf{r}'_1(s) \cdot \boldsymbol{\alpha}(s) = 0,$$

因此

$$\lambda(s) = c - s,$$

故所求的曲线 C 的渐伸线是

$$\mathbf{r}_1(s) = \mathbf{r}(s) + (c - s)\boldsymbol{\alpha}(s),$$

其中 c 是任意常数.

如果曲线 C 的参数是一般参数 t , 则曲线 C 的渐伸线是

$$\mathbf{r}_1(t) = \mathbf{r}(t) + (c - s(t))\boldsymbol{\alpha}(t).$$

曲线的渐伸线可以看作该曲线的切线族的正交轨线，而渐伸线的表达式可以解释为：将一条软线沿曲线放置，把一端固定，另一端慢慢离开原曲线，并且把软线抻直，使软线抻直的部分始终保持为原曲线的切线，则这另一端描出的曲线就是原曲线的渐伸线。

例题 2.16 设正则参数曲线 C 的参数方程是 $\mathbf{r}(s)$, s 是弧长参数，则 C 的渐缩线的参数方程是

$$\mathbf{r}_1(s) = \mathbf{r}(s) + \frac{1}{\kappa(s)} \boldsymbol{\beta}(s) - \frac{1}{\kappa(s)} \left(\tan \int \tau(s) ds \right) \boldsymbol{\gamma}(s).$$

注记 如果 C 是平面曲线，则 C 的渐缩线是

$$\mathbf{r}_1(s) = \mathbf{r}(s) + \frac{1}{\kappa(s)} \boldsymbol{\beta}(s).$$

所以 C 的渐缩线是 C 的曲率中心的轨迹。此时，曲率 κ 可以换成相对曲率 κ_r ，而主法向量 $\boldsymbol{\beta}$ 应该相应地换成单位切向量 $\boldsymbol{\alpha}$ 绕正向旋转 90° 得到的法向量。

第十四章 计算数学

14.1 常微分方程数值解

见 Fiddie 讲义 11,12,13

14.1.1 单步法

把高阶微分方程改写为一阶微分方程组

我们可以把高阶微分方程写成一阶微分方程组的形式。例如，考虑问题

$$\frac{d^3y}{dt^3} = g\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right),$$

初值条件是

$$y(0) = c_1, \quad y'(0) = c_2, \quad y''(0) = c_3.$$

我们定义 $\mathbf{w} = (w_1, w_2, w_3)$ 如下： $w_1 = y$, $w_2 = y'$, $w_3 = y''$, 那么这个问题可以改写成

$$\mathbf{w}' = \begin{pmatrix} w_2 \\ w_3 \\ g(t, w_1, w_2, w_3) \end{pmatrix},$$

其中初值 $\mathbf{w}(0) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$.

由于常微分方程组问题以及高阶微分方程的问题可以由一维一阶常微分方程情形推广，我们重点研究一维情形。

基于Taylor展开的方法

单步法就是形如下式的方法:

$$y_{n+1} = y_n + h\Psi(t_n, y_n, h),$$

其中 $y_n \approx y(t_n)$, 而 $h = t_{n+1} - t_n$ 是步长. 由于 y_{n+1} 只由前一时刻的信息决定, 所以我们把它叫做**单步法(one-step method)**.

若真解充分光滑, 我们有Taylor展开

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2}y''(x) + \cdots + \frac{h^p}{p!}y^{(p)}(x) + \frac{h^{p+1}}{(p+1)!}y^{(p+1)}(\xi).$$

忽略余项, 我们就能得到**基于Taylor展开的迭代公式**

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{3!}y'''_n + \cdots + \frac{h^p}{p!}y_n^{(p)}.$$

基于Taylor展开的方法

下面要计算 y 的各阶导数. 由于 $y' = f(t, y)$, 利用复合函数求导公式, 可得

$$\begin{aligned} y'' &= f_t + f_y y' = f_t + f_y f, \\ y''' &= (f_{tt} + f_{ty} y') + f_y(f_t + f_y y') + (f_{ty} + f_{yy} y')f \\ &= (f_{tt} + 2f_{ty} f + f^2 f_{yy}) + (f_t + f_y f)f_y. \end{aligned}$$

类似我们可以去计算更高阶的导数.

对于自治系统, f 与 t 无关, 那么上述迭代公式会极大地简化.

当 $n = 1$ 时, 这个迭代格式就是

$$y_{n+1} = y_n + hf(t_n, y_n),$$

这就是**显式Euler方法**.

自治系统就是

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$$

其中 $\mathbf{f}(t, \mathbf{y}) = \mathbf{f}(\mathbf{y})$ 与 t 无关.

二阶Runge-Kutta法: 基于数值积分

假如对 $y' = f(t, y)$ 关于 $[t_n, t_{n+1}]$ 积分, 有

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t)) dt.$$

这里涉及到一个积分, 我们考虑用数值积分处理. 采用**中点公式**处理右端积分, 我们可以得到如下迭代公式:

$$y_{n+1} - y_n = hf\left(t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right).$$

但是, 用什么值去逼近 $y\left(t_n + \frac{h}{2}\right)$ 呢? 我们可以考虑用步长为 $\frac{h}{2}$ 的显式Euler方法计算它, 于是就有

$$\begin{aligned} k_1 &= hf(t_n, y_n), \\ k_2 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{1}{2}k_1\right), \\ y_{n+1} &= y_n + k_2. \end{aligned}$$

这就是一种特殊的二阶Runge-Kutta方法, 叫做**变形的Euler方法或中点方法**.

二阶Runge-Kutta法: 基于Taylor展开

对于一般的二阶Runge-Kutta法, 我们可以如下推导. 从Taylor展开入手

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \frac{h^3}{6}y'''(t) + \dots, \quad (1)$$

记 $f = f(t, y)$, $y = y(t)$. 前面已经推导了

$$\begin{aligned} y'' &= f_t + f_y f, \\ y''' &= (f_{tt} + 2f_{ty}f + f^2f_{yy}) + (f_t + f_y f)f_y. \end{aligned}$$

现在, 把它代入(1)式可得

$$\begin{aligned} y(t+h) &= y + hf + \frac{h^2}{2}(f_t + ff_y) + O(h^3) \\ &= y + \frac{1}{2}hf + \frac{1}{2}h[f + hf_t + hff_y] + O(h^3). \end{aligned}$$

注意到, 根据Taylor展开,

$$f(t+h, y+hf) = f + hf_t + hff_y + O(h^2),$$

所以把它代入中括号里面的项可得

$$y(t+h) = y + \frac{1}{2}hf + \frac{1}{2}hf(t+h, y+hf) + O(h^3).$$

二阶Runge-Kutta法: 基于Taylor展开

$$y(t+h) = y + \frac{1}{2}hf + \frac{1}{2}hf(t+h, y+hf) + O(h^3).$$

现在, 忽略余项, 我们就得到了如下的迭代公式:

$$y_{n+1} = y_n + \frac{1}{2}(K_1 + K_2),$$

其中

$$K_1 = hf(t_n, y_n),$$

$$K_2 = hf(t_n + h, y_n + F_1).$$

这就是**改进的Euler方法**.

注: 不同的书对这些格式的命名并不一致, 这里以林成森《数值计算方法》为准.

14.1.2 单步法的相容性、收敛性、稳定性

我们用数值方法求解常微分方程的时候, 需要假设问题是“适定的”, 即真解存在唯一. 而我们有如下的Picard-Lindelöf定理:

Theorem (Picard-Lindelöf定理)

设 $G \subset \mathbb{R}^{n+1}$ 是一个区域, $f : G \rightarrow \mathbb{R}^n$ 是连续函数, 满足 *Lipschitz* 条件

$$\|f(t, u) - f(t, v)\| \leq L\|u - v\|, \forall (t, u), (t, v) \in G.$$

其中 $L > 0$ 是 *Lipschitz* 常数. 则对任意初值 $(t_0, u_0) \in G$, 存在区间 $[t_0 - a, t_0 + a]$ ($a > 0$), 使得初值问题

$$u' = f(t, u), \quad u(t_0) = u_0$$

存在唯一解, 此时称这个初值问题是**适定的**.

假如初值问题

$$\begin{cases} y' = f(t, y), & a \leq t \leq b, \\ y(a) = \eta \end{cases} \quad (*)$$

的显式单步法为

$$\begin{aligned} y_{n+1} &= y_n + h\Phi(t_n, y_n, h), \quad n = 0, 1, \dots, N-1, \\ y_0 &= \eta, \end{aligned}$$

其中 $h = \frac{b-a}{N}$, $t_n = a + nh$. 我们希望

$$\lim_{h \rightarrow 0} \left[\frac{y(t+h) - y(t)}{h} - \Phi(t, y(t), h) \right] = 0,$$

这样, 当 Φ 关于 h 连续的时候, 有

$$y'(t) = \Phi(t, y(t), 0) = f(t, y).$$

这就是单步法(*)的**相容条件**. 相容的方法至少是一阶方法.

收敛性

如果 $f(t, y)$ 在区域 $R = \{(t, y) | a \leq t \leq b\}$ 连续, 关于 y 是 Lipschitz 的, 并且固定 $t \in [a, b]$, 记 $t = nh$, 有

$$\lim_{h \rightarrow 0} y_n = y(t),$$

那么称单步法(*)收敛.

Theorem

若 $\Phi(t, y, h)$ 关于 $a \leq t \leq b, 0 < h \leq h_0, y \in \mathbb{R}$ 满足 Lipschitz 条件, 则单步法(*)收敛 \Leftrightarrow 相容.

稳定性

Definition (稳定性)

若存在 $h_0 > 0$ 与 $C > 0$ 使得对任意的初始值 y_0, \tilde{y}_0 , 用单步法

$$\begin{aligned} y_{n+1} &= y_n + h\Phi(t_n, y_n, h), \quad n = 0, 1, \dots, N-1, \\ y_0 &= \eta, \end{aligned}$$

得到的对应解是 y_n, \tilde{y}_n , 对所有的 $0 < h \leq h_0$, 恒有

$$|y_n - \tilde{y}_n| \leq C|y_0 - \tilde{y}_0|, \quad nh \leq b - a,$$

则称单步法是稳定的.

Theorem

若 $\Phi(t, y, h)$ 对于 $a \leq t \leq b, 0 < h \leq h_0$ 以及一切实数 y , 关于 y 满足 Lipschitz 条件, 则上述单步法是稳定的.

绝对稳定

Definition (绝对稳定)

给定微分方程和步长 h , 若由单步法计算 y_n 时有大小为 δ 的误差, 而引起其后值 $y_m (m > n)$ 的变化小于 δ (即 $\tilde{y}_n = y_n + \delta$ 可以推出 $|\tilde{y}_m - y_m| < |\delta|$), 则称单步法是绝对稳定的.

我们一般只考虑模型问题

$$y' = \mu y$$

的稳定性, 这门课程限定 $\mu \in \mathbb{R}$ 的情形. 若对所有 $\mu h \in (\alpha, \beta)$, 单步法都绝对稳定, 则称 (α, β) 为绝对稳定区间.

书上推导过的一些方法的绝对稳定区间:

方法	数值格式	绝对稳定区间
Euler方法	$y_{n+1} = y_n + hf(t_n, y_n)$	$(-2, 0)$
梯形方法	$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$	$(-\infty, 0)$
RK4方法	详见第11讲	$(-2.78, 0)$

例题

稳定性

Example (林成森书, 习题10.19)

证明初值问题 相容

$$y' = \frac{1}{1+y^2}, \quad 0 \leq t \leq 1$$

的Euler方法收敛且稳定.

$$\begin{aligned} f(y) &= \frac{1}{1+y^2} & y_{n+1} &= y_n + h f(y_n) & \bar{f}(t, h, y) &= f(y) \\ f'(y) &= -\frac{2y}{(1+y^2)^2} & |f'(y)| &\leq \frac{1+y^2}{(1+y^2)^2} = \frac{1}{1+y^2} \leq 1 \\ \{y_k\}_{k=0}^n \text{ 由 } y_0 \text{ 生成} & & \{\tilde{y}_k\}_{k=0}^n \text{ 由 } \tilde{y}_0 \text{ 生成} & & nh &= 1 \\ |y_{k+1} - \tilde{y}_{k+1}| &= |y_k + h f(y_k) - (\tilde{y}_k + h f(\tilde{y}_k))| \\ &\leq |y_k - \tilde{y}_k| + h |f(y_k) - f(\tilde{y}_k)| \\ &\leq |y_k - \tilde{y}_k| + h |y_k - \tilde{y}_k| = (1+h) |y_k - \tilde{y}_k| \\ \Rightarrow |y_n - \tilde{y}_n| &\leq (1+h)^n |y_0 - \tilde{y}_0| \leq (e^h)^n |y_0 - \tilde{y}_0| = e^n |y_0 - \tilde{y}_0| \\ \Rightarrow \text{Euler 方法收敛.} & \end{aligned}$$

Example (林成森书, 习题10.21)

求中点方法

$$y_{n+1} = y_n + h f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n)\right)$$

的绝对稳定区间.

证明: 考虑模型问题 $y' = \mu y$ $f(y) = \mu y$ 则原方法为

$$\begin{aligned} y_{n+1} &= y_n + \mu h \left(y_n + \frac{h}{2} \mu y_n \right) \\ &= \left(1 + \mu h + \frac{\mu^2 h^2}{2} \right) y_n \end{aligned}$$

要使得中点方法绝对稳定 只需

$$\begin{aligned} \left| 1 + \mu h + \frac{\mu^2 h^2}{2} \right| &< 1 \\ \Leftrightarrow -2 &< \mu h + \frac{\mu^2 h^2}{2} < 0 \Leftrightarrow \mu h \in (-2, 0) \end{aligned}$$

14.1.3 多步法

考虑以下形式的数值方法

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_j y_{n+j} + h \sum_{j=0}^k \beta_j f(t_{n+j}, y_{n+j}), \quad n = 0, 1, \dots$$

其中 a_j 和 b_j 是实常数, h 是步长。这个公式被称为线性多步法。当 $b_k = 0$ 时, 该方法是显式的; 否则, 它是隐式的。

改写为

$$\sum_{j=0}^k a_j y_{n+j} = h \sum_{j=0}^k b_j f(t_{n+j}, y_{n+j}), \quad n = 0, 1, \dots \quad (14.1)$$

其中 $a_k \neq 0$.

初值问题

$$y' = f(t, y), a \leq t \leq b,$$

$$y(a) = \eta$$

的线性 k 步法的一般公式是

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j f(t_{n+j}, y_{n+j}), n = 0, 1, \dots, N - k.$$

其中 $h = \frac{b-a}{N}$, 且 α_j, β_j 是常数.

用它来算 $\{y_n\}$ 需要 k 个初始值 y_0, y_1, \dots, y_{k-1} , 初值问题只能给出 y_0 , 而 y_1, \dots, y_{k-1} 需要用单步法来算出来.

若 $\beta_k = 0$, 则 y_{n+k} 可直接计算, 此时这个格式是显式的, 否则是隐式的.

设 $y(t)$ 是初值问题的解, 对

$$y'(t) = f(t, y(t))$$

在区间 $[t_n, t_{n+1}]$ 两端积分可得

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t)) dt.$$

右边的积分可用数值积分逼近, 得到的迭代公式叫做 **Adams 格式**.

比如, 如果用 $k+1$ 个点 $\{(t_i, f_i)\}_{i=n-k}^n$ 的 Newton 外插公式来逼近 $f(t, y(t))$, 得到

$$p_k(t) = \sum_{m=0}^k (-1)^m \binom{-s}{m} \nabla^m f_n,$$

其中 $f_m \triangleq f(t_m, y_m)$ 为 $f(t_m, y(t_m))$ 的近似值. 这个公式叫做 **Adams-Basford 公式**.

Example

用数值积分的方法构造初值问题 $y' = f(t, y)$, $y(0) = y_0$ 的二步二阶 Adams 显式格式.

解: 对 $y'(t) = f(t, y(t))$ 在区间 $[t_n, t_{n+1}]$ 两端积分可得

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t)) dt.$$

记 $f_n = f(t_n, y_n)$. 用 2 个点 (t_n, f_n) 与 (t_{n-1}, f_{n-1}) 对 $f(t, y(t))$ 作 Newton 插值多项式, 可得

$$f(t, y(t)) = f_n + \frac{f_n - f_{n-1}}{t_n - t_{n-1}}(t - t_n) + O(h^2),$$

所以

$$\begin{aligned} \int_{t_n}^{t_{n+1}} f(t, y(t)) dt &= f_n \cdot (t_{n+1} - t_n) + \frac{1}{2}(t_{n+1} - t_n)^2 \cdot \frac{f_n - f_{n-1}}{t_n - t_{n-1}} + O(h^3) \\ &= h f_n + \frac{1}{2}h(f_n - f_{n-1}) = h \left(\frac{3}{2}f_n - f_{n-1} \right) + O(h^3). \end{aligned}$$

故二步二阶 Adams 显式格式是(“二阶”可用局部截断误差的定义来说明)

$$y_{n+1} = y_n + h \left(\frac{3}{2}f_n - f_{n-1} \right).$$

Example (林成森书, 习题 10.30)

用待定系数法导出 Milne 方法校正公式 $y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n+1} + 4f_n + f_{n-1})$.

解: 对 $y'(t) = f(t, y(t))$ 在区间 $[t_{n-1}, t_{n+1}]$ 两端积分可得

$$y(t_{n+1}) - y(t_{n-1}) = \int_{t_{n-1}}^{t_{n+1}} f(t, y(t)) dt.$$

我们假设 f_n 为 $f(t_n, y(t_n))$, 以及

$$\int_{t_{n-1}}^{t_{n+1}} f(t, y(t)) dt \approx h(Af_{n+1} + Bf_n + Cf_{n-1}).$$

对数值积分公式求系数可以用待定系数法, 我们令 $f(t, y(t)) = 1, t - t_n, (t - t_n)^2$ 使得上面的约等号变成等号, 可得

$$\begin{cases} 2h = h(A + B + C), \\ 0 = h(Ah - Ch), \\ \frac{2}{3}h^3 = h(Ah^2 + Ch^2). \end{cases} \Rightarrow \begin{cases} A = \frac{1}{3}, \\ B = \frac{4}{3}, \\ C = \frac{1}{3}. \end{cases}$$

对于线性 k 步法 eq. (14.1), 考虑

$$\rho(\lambda) = a_k \lambda^k + a_{k-1} \lambda^{k-1} + \cdots + a_1 \lambda + a_0,$$

$$\sigma(\lambda) = b_k \lambda^k + b_{k-1} \lambda^{k-1} + \cdots + b_1 \lambda + b_0,$$

Théorème 1: 相容条件

线性 k 步法相容的充要条件是

$$\rho(1) = 0, \rho'(1) = \sigma(1).$$

Definition (稳定性)

假设 $f(t, y)$ 在一个区域 R 中连续, 且关于 y 满足 Lipschitz 条件, 若存在 $C > 0$ 与 h_0 使得当 $0 < h \leq h_0$ 时, (*) 的任何两个解 y_n 与 \tilde{y}_n 满足

$$\max_{nh \leq b-a} |y_n - \tilde{y}_n| \leq C \max_{0 \leq j \leq k-1} |y_j - \tilde{y}_j|,$$

则称多步法(*)**稳定**.

Theorem (根条件)

线性 k 步法(*)稳定的充要条件是 $\rho(\lambda)$ 满足 **特征根条件**: 即 $\rho(\lambda)$ 的所有根都在单位圆中, 且在单位圆周上的根只能是单根.

注: 这个条件又称为**弱根条件**. 满足弱根条件的稳定性是**弱稳定性**.

注: 还有**强根条件**: 特征多项式 $\rho(\lambda)$ 的全部根除了 $\lambda = 1$ 以外都落在单位圆内. 满足强根条件的稳定性叫**强稳定性**.

Example

Adams 显式公式

$$y_{n+1} = y_n + h \sum_{j=0}^k \beta_{kj} f_{n-j}$$

的特征多项式是 $\rho(\lambda) = \lambda^{k+1} - \lambda^k$, 只有一个根 1, 其余根都是 0, 则这个格式是强稳定的.

Example

Milne 方法校正公式

$$y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n+1} + 4f_n + f_{n-1})$$

的特征多项式是 $\rho(\lambda) = \lambda^2 - 1$, 它的特征根为 ± 1 , 所以是弱稳定但不是强稳定的.

多步法的收敛性

Definition (收敛性)

设 $f(t, y)$ 在 $R = \{(t, y) : a \leq t \leq b, y \in \mathbb{R}\}$ 连续, 关于 y 满足Lipschitz条件, 若对任意 $t \in [a, b]$, 当 $h \rightarrow 0$ 而 $a + nh = t_n = t$ 固定时, (*)的解 y_n 收敛于原问题的真解 $y(t)$, 则称 k 步法(*)收敛.

Theorem (收敛 \Rightarrow 相容)

若(*)收敛, 则(*)相容.

Theorem (收敛 \Rightarrow 稳定)

若(*)收敛, 则(*)稳定.

Theorem (相容+稳定 \Rightarrow 收敛)

若 k 步法(*)相容且稳定, 则(*)收敛.

多步法的绝对稳定区间

讨论典型微分方程

$$y' = \mu y$$

那么线性 k 步法为

$$\sum_{j=0}^k a_j y_{n+j} = \mu h \sum_{j=0}^k b_j y_{n+j}. \quad (**)$$

它是齐次常系数线性差分方程, 特征方程是

$$\rho(\lambda) - \mu h \sigma(\lambda) = 0.$$

把差分方程的精确解记为 y_n .

Definition

对给定的 μ, h , 若特征方程

$$\rho(\lambda) - \mu h \sigma(\lambda) = 0$$

的所有根 λ_r 的模都小于1, 则称 k 步法(**)关于 μh 绝对稳定. 若对所有 $\mu h \in (\alpha, \beta)$, (**)都绝对稳定, 称 (α, β) 为绝对稳定区间.

例题

Exemple 45. 判断求解常微分方程初值问题 $y' = f(t, y), y(t_0) = y_0$ 的线性多步法

$$y_{n+1} - y_{n-1} = \frac{h}{3}(3f_{n+1} - f_n + 4f_{n-1})$$

是否收敛，并说明理由。

首先判断零稳定性。该方法的特征多项式为 $\rho(z) = z^2 - 1$, 其根为 $z_1 = 1, z_2 = -1$, 均在单位圆上, 且绝对值为 1 的根是单根, 故该方法是零稳定的。

接下来判断相容性。 $\rho(1) = 0, \sigma(1) = \frac{1}{3}(3 - 1 + 4) = 2$, 故 $\frac{\sigma(1)}{\rho'(1)} = \frac{2}{2} = 1$. 所以该方法收敛。

14.2 数值积分误差估计

好的, 这是一份关于数值积分主要公式的备考资料, 包括公式、误差分析、代数精度以及简要的推导思路。

数值积分备考资料

14.2.1 引言

数值积分 (Numerical Quadrature) 用于逼近定积分 $\int_a^b f(x) dx$ 的值, 特别是当被积函数 $f(x)$ 的原函数很难或无法用初等函数表达时, 或者当 $f(x)$ 是以离散数据点形式给出时。

14.2.2 一、牛顿-柯特斯公式 (Newton-Cotes Formulas)

牛顿-柯特斯公式的基本思想是用一个次数为 n 的插值多项式 $P_n(x)$ 来近似 $f(x)$, 然后计算插值多项式的积分。

$$\int_a^b f(x) dx \approx \int_a^b P_n(x) dx$$

节点 x_i 在积分区间 $[a, b]$ 内等距分布。

1. 梯形公式 (Trapezoidal Rule)

使用 1 次插值多项式 (连接 $(a, f(a))$ 和 $(b, f(b))$ 的直线)。

基本梯形公式 (2 个节点: $x_0 = a, x_1 = b$)

- 估计值:

$$T(f) = \frac{b-a}{2} [f(a) + f(b)]$$

- 误差分析 (截断误差):

$$E_T(f) = \int_a^b f(x) dx - T(f) = -\frac{(b-a)^3}{12} f''(\xi), \quad \xi \in (a, b)$$

- 代数精度: 1 (对于次数 ≤ 1 的多项式, 结果精确)

- 简要推导思路:

(1) 用线性插值多项式 $P_1(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$ 近似 $f(x)$ 。

(2) 计算 $\int_a^b P_1(x) dx$ 得到 $T(f)$ 。

(3) 误差项可以通过泰勒展开或插值多项式的误差公式推导。对于 $f(x) - P_1(x) = \frac{f''(\eta_x)}{2!}(x-a)(x-b)$, 积分后得到。

复合梯形公式 ($n+1$ 个节点, n 个子区间) 将 $[a, b]$ 分成 n 个等长的子区间 $[x_i, x_{i+1}]$, 每个子区间长度 $h = \frac{b-a}{n}$, $x_i = a + ih$.

- 估计值:

$$T_n(f) = h \left[\frac{1}{2} f(x_0) + \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2} f(x_n) \right]$$

- 误差分析:

$$E_{T_n}(f) = -\frac{(b-a)h^2}{12} f''(\xi) = -\frac{(b-a)^3}{12n^2} f''(\xi), \quad \xi \in (a, b)$$

- 简要推导思路: 将各子区间上的基本梯形公式误差累加起来。 $\sum_{i=0}^{n-1} -\frac{h^3}{12} f''(\xi_i) = -\frac{h^3}{12} \sum f''(\xi_i)$ 。根据中值定理, 存在 $\xi \in (a, b)$ 使得 $nf''(\bar{\xi}) = \sum f''(\xi_i)$, 则误差为 $-\frac{h^3}{12} nf''(\bar{\xi}) = -\frac{(b-a)h^2}{12} f''(\bar{\xi})$ 。

2. 辛普森公式 (Simpson's Rule)

使用 2 次插值多项式。通常考虑区间 $[a, b]$ 和中点 $m = (a+b)/2$ 。

基本辛普森公式 (3 个节点: $x_0 = a, x_1 = (a+b)/2, x_2 = b$) 令 $h = (b-a)/2$ (即每个小子区间的长度)。

- 估计值:

$$S(f) = \frac{h}{3} [f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)] = \frac{b-a}{6} [f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)]$$

- 误差分析:

$$E_S(f) = \int_a^b f(x) dx - S(f) = -\frac{h^5}{90} f^{(4)}(\xi) = -\frac{(b-a)^5}{2880} f^{(4)}(\xi), \quad \xi \in (a, b)$$

- 代数精度: 3 (对于次数 ≤ 3 的多项式, 结果精确)。比预期的 2 次要高, 这是因为选取节点的方式对称, 使得三次项的误差积分为 0。

- 简要推导思路:

- (1) 用过点 $(a, f(a)), (\frac{a+b}{2}, f(\frac{a+b}{2})), (b, f(b))$ 的二次插值多项式 $P_2(x)$ 近似 $f(x)$ 。
- (2) 计算 $\int_a^b P_2(x) dx$ 得到 $S(f)$ 。
- (3) 误差项可以通过更高阶的泰勒展开 (Peano 核方法) 或插值多项式误差 $\int_a^b \frac{f^{(3)}(\eta_x)}{3!} (x-a)(x-\frac{a+b}{2})(x-b) dx$ 推导, 但更标准的方法是使用 Hermite 插值或特定技巧显示其为 $f^{(4)}$ 。一个常用方法是先对 $f(x)$ 在 $(a+b)/2$ 处泰勒展开到 4 阶。

复合辛普森公式 (n 个子区间, n 必须为偶数, n+1 个节点) 将 $[a, b]$ 分成 n 个等长的子区间 (n 为偶数), 每个子区间长度 $h = \frac{b-a}{n}$, $x_i = a + ih$ 。相当于应用 $n/2$ 次基本辛普森公式。

- 估计值:

$$S_n(f) = \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

- 误差分析:

$$E_{S_n}(f) = -\frac{(b-a)h^4}{180} f^{(4)}(\xi) = -\frac{(b-a)^5}{180n^4} f^{(4)}(\xi), \quad \xi \in (a, b)$$

- 简要推导思路:

累加 $n/2$ 个基本辛普森公式的误差。每个基本辛普森公式作用的区间长度是 $2h$ 。所以单个误差是 $-\frac{(2h)^5}{2880} f^{(4)}(\xi_i) = -\frac{h^5}{90} f^{(4)}(\xi_i)$ 。总共 $n/(2h) \times (LengthOfOneSimpsonApplication) = (b-a)/(2h)$ 个这样的应用。

总误差是

$$\sum_{k=1}^{n/2} -\frac{(2h)^5}{2880} f^{(4)}(\xi_k) = \frac{n}{2} \left(-\frac{32h^5}{2880} f^{(4)}(\bar{\xi}) \right) = \frac{n}{2} \left(-\frac{h^5}{90} f^{(4)}(\bar{\xi}) \right) = -\frac{nh^5}{180} f^{(4)}(\bar{\xi}) = -\frac{(b-a)h^4}{180} f^{(4)}(\bar{\xi})$$

14.2.3 二、高斯求积公式 (Gaussian Quadrature)

高斯求积公式的目标是选择节点 x_i 和权重 w_i 使得求积公式对于尽可能高次的多项式是精确的。对于 n 个节点，高斯求积公式可以达到 $2n - 1$ 的代数精度。通用形式：

$$\int_a^b \omega(x) f(x) dx \approx \sum_{i=1}^n w_i f(x_i)$$

其中 $\omega(x)$ 是权函数。节点 x_i 是与权函数 $\omega(x)$ 在区间 $[a, b]$ 上相关的 n 次正交多项式的根。

- 高斯-勒让德求积 (Gauss-Legendre Quadrature) 这是最常见的高斯求积，其中权函数 $\omega(x) = 1$ ，积分区间通常标准化为 $[-1, 1]$ 。

$$\int_{-1}^1 f(x) dx \approx \sum_{i=1}^n w_i f(x_i)$$

- 节点 x_i : 是 n 次勒让德多项式 $P_n(x)$ 的根。
- 权重 w_i : $w_i = \frac{2}{(1-x_i^2)[P'_n(x_i)]^2}$ 。
- 估计值 (例子):

$$\begin{aligned} * n = 1: & x_1 = 0, w_1 = 2. \int_{-1}^1 f(x) dx \approx 2f(0) \text{ (中点公式)} \\ * n = 2: & x_{1,2} = \pm \frac{1}{\sqrt{3}}, w_{1,2} = 1. \int_{-1}^1 f(x) dx \approx f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}) \\ * n = 3: & x_1 = 0, x_{2,3} = \pm \sqrt{\frac{3}{5}}. w_1 = \frac{8}{9}, w_{2,3} = \frac{5}{9}. \int_{-1}^1 f(x) dx \approx \end{aligned}$$

$$\frac{8}{9}f(0) + \frac{5}{9}f(-\sqrt{\frac{3}{5}}) + \frac{5}{9}f(\sqrt{\frac{3}{5}})$$

- 误差分析:

$$E_G(f) = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi) = \frac{f^{(2n)}(\xi)}{(2n)!} \int_{-1}^1 \prod_{i=1}^n (x-x_i)^2 dx, \quad \xi \in (-1, 1)$$

(对于一般的区间 $[a, b]$ ，需要做线性变换 $x = \frac{b-a}{2}t + \frac{a+b}{2}$ ，然后误差项会包含 $(\frac{b-a}{2})^{2n+1}$ 因子) $E_G(f) = \frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3(\text{const})} f^{(2n)}(\xi)$

- 代数精度: $2n - 1$

- 简要推导思路:

(1) 目标: 选择 n 个节点 x_i 和 n 个权重 w_i (共 $2n$ 个自由参数)，使得公式对次数尽可能高的多项式精确。

(2) 令 $f(x)$ 是一个次数 $\leq 2n - 1$ 的多项式。作带余除法 $f(x) = q(x)\phi_n(x) + r(x)$ ，其中 $\phi_n(x)$ 是 n 次正交多项式 (其根为 x_i)， $\deg(q), \deg(r) \leq n - 1$ 。

(3) $\int_a^b \omega(x) f(x) dx = \int_a^b \omega(x) q(x)\phi_n(x) dx + \int_a^b \omega(x) r(x) dx = \int_a^b \omega(x) r(x) dx$
(由于 $\phi_n(x)$ 与所有次数 $< n$ 的多项式正交)。

(4) $\sum w_i f(x_i) = \sum w_i (q(x_i)\phi_n(x_i) + r(x_i)) = \sum w_i r(x_i)$ (因为 x_i 是 $\phi_n(x)$ 的根)。

(5) 问题转化为选择 w_i 使得 $\int_a^b \omega(x)r(x)dx = \sum w_i r(x_i)$ 对所有次数 $\leq n-1$ 的 $r(x)$ 精确。这可以通过构造一个 $n-1$ 次的插值多项式 $L(x)$ 使得 $L(x_i) = r(x_i)$, 然后令 $w_i = \int_a^b \omega(x)l_i(x)dx$ (其中 $l_i(x)$ 是拉格朗日基函数)。

14.2.4 三、切比雪夫求积公式 (Chebyshev Quadrature)

高斯-切比雪夫求积 (Gauss-Chebyshev Quadrature)

这是高斯求积的一种特殊情况, 权函数 $\omega(x) = \frac{1}{\sqrt{1-x^2}}$, 积分区间 $[-1, 1]$ 。

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx \approx \sum_{i=1}^n w_i f(x_i)$$

- 节点 x_i : 是 n 次第一类切比雪夫多项式 $T_n(x)$ 的根: $x_i = \cos\left(\frac{(2i-1)\pi}{2n}\right)$, for $i = 1, \dots, n$.

- 权重 w_i : 非常特殊, 所有权重都相等 $w_i = \frac{\pi}{n}$.

- 估计值:

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx \approx \frac{\pi}{n} \sum_{i=1}^n f\left(\cos\left(\frac{(2i-1)\pi}{2n}\right)\right)$$

- 误差分析:

$$E_{GC}(f) = \frac{2\pi}{2^{2n}(2n)!} f^{(2n)}(\xi), \quad \xi \in (-1, 1)$$

- 代数精度: $2n - 1$

- 简要推导思路: 同高斯求积一般思路, 使用切比雪夫多项式作为正交多项式系。

下面给出误差的推导

设 $f(x)$ 在 $[-1, 1]$ 上充分光滑, 则可以将其展开为 Chebyshev 多项式级数:

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$$

其中 $T_k(x)$ 是第 k 阶 Chebyshev 多项式, a_k 是对应的系数。

将上述展开式代入积分:

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \int_{-1}^1 \frac{\sum_{k=0}^{\infty} a_k T_k(x)}{\sqrt{1-x^2}} dx = \sum_{k=0}^{\infty} a_k \int_{-1}^1 \frac{T_k(x)}{\sqrt{1-x^2}} dx$$

利用 Chebyshev 多项式的正交性：

$$\int_{-1}^1 \frac{T_k(x)T_j(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0, & k \neq j \\ \pi, & k = j = 0 \\ \frac{\pi}{2}, & k = j \neq 0 \end{cases}$$

可以得到

$$\int_{-1}^1 \frac{T_k(x)}{\sqrt{1-x^2}} dx = \begin{cases} \pi, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

因此，

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = a_0 \pi$$

另一方面，考虑数值积分公式：

$$\frac{\pi}{n} \sum_{i=1}^n f \left(\cos \left(\frac{(2i-1)\pi}{2n} \right) \right) = \frac{\pi}{n} \sum_{i=1}^n \sum_{k=0}^{\infty} a_k T_k \left(\cos \left(\frac{(2i-1)\pi}{2n} \right) \right)$$

利用 Chebyshev 多项式的离散正交性：

$$\sum_{i=1}^n T_k \left(\cos \left(\frac{(2i-1)\pi}{2n} \right) \right) T_j \left(\cos \left(\frac{(2i-1)\pi}{2n} \right) \right) = \begin{cases} 0, & k \neq j, k+j < 2n \\ \frac{n}{2}, & k = j \neq 0, k+j < 2n \\ n, & k = j = 0 \end{cases}$$

可以得到

$$\frac{\pi}{n} \sum_{i=1}^n T_k \left(\cos \left(\frac{(2i-1)\pi}{2n} \right) \right) = \begin{cases} \pi, & k = 0 \\ 0, & 0 < k < 2n \end{cases}$$

因此，

$$\frac{\pi}{n} \sum_{i=1}^n f \left(\cos \left(\frac{(2i-1)\pi}{2n} \right) \right) = \pi \sum_{k=0}^{\infty} a_k \frac{1}{n} \sum_{i=1}^n T_k \left(\cos \left(\frac{(2i-1)\pi}{2n} \right) \right) = \pi a_0 + \pi \sum_{k=2n, 4n, \dots} a_k$$

所以误差为

$$E_{GC}(f) = \int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx - \frac{\pi}{n} \sum_{i=1}^n f \left(\cos \left(\frac{(2i-1)\pi}{2n} \right) \right) = -\pi \sum_{k=2n, 4n, \dots} a_k$$

当 f 充分光滑时，有

$$|a_k| \leq \frac{M}{k!}$$

其中 M 为常数。因此，误差可以表示为

$$E_{GC}(f) = \frac{2\pi}{2^{2n}(2n)!} f^{(2n)}(\xi), \quad \xi \in (-1, 1)$$

其中 $\xi \in (-1, 1)$.

14.2.5 四、其他常见公式 (提及)

- 中点公式 (Midpoint Rule): $n = 1$ 的开型牛顿-柯特斯公式, 或 $n = 1$ 的高斯-勒让德公式(经过调整)。 $\int_a^b f(x)dx \approx (b-a)f\left(\frac{a+b}{2}\right)$. 误差: $\frac{(b-a)^3}{24}f''(\xi)$ 。代数精度: 1.
- 龙贝格积分 (Romberg Integration): 对复合梯形公式使用理查森外推法, 逐步提高精度。
- 自适应求积 (Adaptive Quadrature): 根据被积函数在不同子区间的行为自动调整步长或子区间划分, 以在满足给定精度要求的情况下最小化计算量。

14.2.6 五、代数精度总结

公式名称	基本型节点数	代数精度	复合型误差阶 (h)
梯形公式	2	1	$O(h^2)$
辛普森公式	3	3	$O(h^4)$
高斯-勒让德 (n 点)	n	$2n - 1$	(通常不直接用复合)
高斯-切比雪夫 (n 点)	n	$2n - 1$	(通常不直接用复合)
(狭义) 切比雪夫 (n 点)	n	n	(若存在)
中点公式	1	1	$O(h^2)$ (复合型)

14.2.7 误差分析推导关键点

- 牛顿-柯特斯:

- 基于插值多项式的误差公式: $f(x) - P_n(x) = \frac{f^{(n+1)}(\eta_x)}{(n+1)!} \prod_{i=0}^n (x - x_i)$ 。
- 积分 $\int_a^b (f(x) - P_n(x))dx$ 。如果 $\int_a^b \prod(x - x_i)dx \neq 0$, 误差主项包含 $f^{(n+1)}$ 。
- 对于对称节点 (如辛普森, 中点), 有时积分 $\int_a^b \prod(x - x_i)dx = 0$, 使得误差阶比预期高一阶 (即 $f^{(n+2)}$)。这需要使用更高阶的埃尔米特 (Hermite) 插值或 Peano 核方法。
- Peano 核方法: 误差 $E(f) = \int_a^b K(t)f^{(k+1)}(t)dt$ 对于所有次数 $\leq k$

的多项式 $P_k(x)$, $E(P_k) = 0$ 。 $K(t) = \frac{1}{k!} E_x[(x-t)_+^k]$ 。

- 高斯求积:

- 利用正交多项式的性质。关键在于 n 次正交多项式 $\phi_n(x)$ 与所有次数小于 n 的多项式关于权函数 $\omega(x)$ 正交。
- 误差基于 $2n-1$ 次 Hermite 插值多项式 $H_{2n-1}(x)$, 它在节点 x_i 处与 $f(x)$ 和 $f'(x)$ 都一致 (但高斯公式本身不使用 f' 的值)。实际推导更直接: 构造一个在节点 x_i 处取值为 $f(x_i)$ 的 $n-1$ 次多项式 $L_{n-1}(x)$ 。那么 $f(x) - L_{n-1}(x)$ 在 x_i 处为 0。误差 $E(f) = \int_a^b \omega(x)(f(x) - L_{n-1}(x))dx$ 。 $f(x) - L_{n-1}(x) = \frac{f^{(n)}(\eta_x)}{n!} \prod(x-x_i)$ 。但高斯公式误差是关于 $f^{(2n)}$ 的, 这是因为节点 x_i 的特殊选择 (正交多项式的根)。
- 一个更标准的误差推导是考虑一个在 x_i 处与 f 值相同的 $2n-1$ 次 Hermite 插值多项式 $H(x)$, 它在 x_i 处满足 $H(x_i) = f(x_i)$ 和 $H'(x_i) = f'(x_i)$ (尽管公式不直接用 f')。积分 $H(x)$ 得到高斯公式权重。误差 $\int \omega(x)(f(x) - H(x))dx = \int \omega(x) \frac{f^{(2n)}(\eta)}{(2n)!} \prod(x-x_i)^2 dx$ 。

这份资料应该能帮助你复习数值积分的主要内容。祝你考试顺利!

14.3 正交多项式

- Legendre 多项式: $P_n(x)$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]$$

- Laguerre 多项式: $L_n(x)$

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^n)$$

- Hermite 多项式: $H_n(x)$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$$

- Chebyshev 多项式: $T_n(x)$

$$T_n(x) = \cos(n \arccos(x))$$

我们关心的是 Legendre 多项式和 Chebyshev 多项式。

14.3.1 Legendre 多项式

Legendre 多项式是以 $[-1, 1]$ 为定义域, 以 1 为权函数的正交多项式。满足递推关系

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad P_0(x) = 1, P_1(x) = x$$

14.3.2 Chebyshev 多项式

Chebyshev 多项式是以 $[-1, 1]$ 为定义域, 以 $1/\sqrt{1-x^2}$ 为权函数的正交多项式。满足递推关系

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad T_0(x) = 1, T_1(x) = x$$