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0.1 Topological preliminaries

0.1.1 Definition of topology

1.2 Definition

(*a*) A collection τ of subsets of a set X is said to be a *topology in* X if τ has the following three properties:

(i) $\varnothing \in \tau$ and $X \in \tau$.
(ii) If $V_i \in \tau$ for $i = 1, \ldots, n$, then $V_1 \cap V_2 \cap \cdots \cap V_n \in \tau$.
(iii) If $\{V_\alpha\}$ is an arbitrary collection of members of τ (finite, countable, or uncountable), then $\bigcup_\alpha V_\alpha \in \tau$.

(*b*) If τ is a topology in X, then X is called a *topological space*, and the members of τ are called the *open sets* in X.

(*c*) If X and Y are topological spaces and if f is a mapping of X into Y, then f is said to be *continuous* provided that $f^{-1}(V)$ is an open set in X for every open set V in Y.

0.1.2 General definitions of closed, closure, compact, neighborhood, Hausdorff space, locally compact, separated and connected sets

2.3 Definitions Let X be a topological space, as defined in Sec. 1.2.

(*a*) A set $E \subset X$ is *closed* if its complement E^c is open. (Hence $\varnothing$ and X are closed, finite unions of closed sets are closed, and arbitrary intersections of closed sets are closed.)

(*b*) The *closure* $\bar{E}$ of a set $E \subset X$ is the smallest closed set in X which contains E. (The following argument proves the existence of $\bar{E}$: The collection Ω of all closed subsets of X which contain E is not empty, since $X \in \Omega$; let $\bar{E}$ be the intersection of all members of Ω.)

(*c*) A set $K \subset X$ is *compact* if every open cover of K contains a finite subcover. More explicitly, the requirement is that if $\{V_\alpha\}$ is a collection of open sets whose union contains K, then the union of some finite subcollection of $\{V_\alpha\}$ also contains K.
In particular, if X is itself compact, then X is called a *compact space*.

(*d*) A *neighborhood* of a point $p \in X$ is any open subset of X which contains p. (The use of this term is not quite standardized; some use

"neighborhood of p" for any set which contains an open set containing p.)

(*e*) X is a *Hausdorff space* if the following is true: If $p \in X$, $q \in X$, and $p \neq q$, then p has a neighborhood U and q has a neighborhood V such that $U \cap V = \varnothing$.

(*f*) X is *locally compact* if every point of X has a neighborhood whose closure is compact.

Open sets remains to be open under arbitrary unions, while closed sets remains to be closed under arbitrary intersections. Imagine the special case of intervals to remember it.

The definition of compact is not technique to prove a set compact but a property.

2.45 Definition Two subsets A and B of a metric space X are said to be *separated* if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty, i.e., if no point of A lies in the closure of B and no point of B lies in the closure of A.

A set $E \subset X$ is said to be *connected* if E is *not* a union of two nonempty separated sets.

2.46 Remark Separated sets are of course disjoint, but disjoint sets need not be separated. For example, the interval $[0, 1]$ and the segment $(1, 2)$ are *not* separated, since 1 is a limit point of $(1, 2)$. However, the segments $(0, 1)$ and $(1, 2)$ *are* separated.

Another definition of closure:

Definition *For a subset E of a topological space X, a point $x \in X$ is called a* **point of closure** *of E provided every neighborhood of x contains a point in E. The collection of points of closure of E is called the* **closure** *of E and denoted by $\overline{E}$.*

0.1.3 Metric definition of neighborhood, limit point, closed, interior point, open, complement, perfect, bounded, dense

2.18 Definition Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X.

(*a*) *A neighborhood* of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$, for some $r > 0$. The number r is called the *radius* of $N_r(p)$.
(*b*) A point p is a *limit point* of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$.
(*c*) If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E.
(*d*) E is *closed* if every limit point of E is a point of E.
(*e*) A point p is an *interior* point of E if there is a neighborhood N of p such that $N \subset E$.
(*f*) E is *open* if every point of E is an interior point of E.
(*g*) The *complement* of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.
(*h*) E is *perfect* if E is closed and if every point of E is a limit point of E.
(*i*) E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.
(*j*) E is *dense in* X if every point of X is a limit point of E, or a point of E (or both).

2.26 Definition If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X, then the *closure* of E is the set $\bar{E} = E \cup E'$.

定义 1 (relatively open). *Suppose $E \subset Y \subset X$, where X is a metric space. Motivated by the idea that Y can also be a metric space, we say E is **open relative** to Y if to each $p \in E$ there is associated an $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$.*

A set may be open relative to Y without being an open subset of X, e.g. $(a, b) \subset \mathbb{R} \subset \mathbb{R}^2$.

定理 1. *Suppose $Y \subset X$. A subset E of Y is open relative to Y iff $E = Y \cap G$ for some open subset G of X.*

theorem 1 can be another definition of **relatively open**.

0.1.4 Some properties in metric spaces

- Every neighborhood is an open set.
- If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.
- A finite point set has no limit points.
- Closed subsets of compact sets are compact.
- Perfect set in $\mathbb{R}^k$ is uncountable.
- If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.
- If $\{K_n\}$ is a decreasing sequence of nonempty compact sets, then $\bigcap_{n=1}^{\infty} K_n$ is not empty.

0.1.5 Examples

2.21 Examples Let us consider the following subsets of R^2:

(*a*) The set of all complex z such that $|z| < 1$.
(*b*) The set of all complex z such that $|z| \leq 1$.
(*c*) A nonempty finite set.
(*d*) The set of all integers.
(*e*) The set consisting of the numbers $1/n$ ($n = 1, 2, 3, \ldots$). Let us note that this set E has a limit point (namely, $z = 0$) but that no point of E is a limit point of E; we wish to stress the difference between having a limit point and containing one.
(*f*) The set of all complex numbers (that is, R^2).
(*g*) The segment (a, b).

Let us note that (*d*), (*e*), (*g*) can be regarded also as subsets of R^1.
Some properties of these sets are tabulated below:

	Closed	*Open*	*Perfect*	*Bounded*
(*a*)	No	Yes	No	Yes
(*b*)	Yes	No	Yes	Yes
(*c*)	Yes	No	No	Yes
(*d*)	Yes	No	No	No
(*e*)	No	No	No	Yes
(*f*)	Yes	Yes	Yes	No
(*g*)	No		No	Yes

In (*g*), we left the second entry blank. The reason is that the segment (a, b) is not open if we regard it as a subset of R^2, but it is an open subset of R^1.

0.1.6 Heine-Borel theorem

定理 2 (Heine-Borel theorem)**.** *The compact subsets of a enclidean space* $\mathbb{R}^n$ *are precisely those that are closed and bounded.*

Moreover, the theorem is true for any locally compact Hausdorff space. Note that metric spaces are locally compact Hausdorff space.

定理 3 (Closed subsets of compact sets are compact.)**.** *Suppoes K is compact and F is closed, in a topological space X. If $F \subset K$ then F is compact.*

证明. If $\{V_\alpha\}$ is an open cover of F and $W = F^c$ then $W \cup \bigcup_\alpha V_\alpha$ covers X; hence there is a finite collection $\{V_{\alpha_i}\}$ such that

$$K \subset W \cup V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$$

Then $F \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. □

推论 1. *If $A \subset B$ and if B has compact closure, so does A.*

定理 4. *X hausdorff, $K \subset X$, K compact, and $p \in K^c$. Then there are open sets U and W such that $p \in U$, $K \subset W$, and $U \cap W = \varnothing$.*

证明. If $q \in K$, the Hausdorff separation axiom implies the existence of disjoint open sets U_q and V_q such that $p \in U_q$ and $q \in V_q$. Since K is compact, there are points $q_1, \ldots, q_n \in K$ such that

$$K \subset V_{q_1} \cup \cdots \cup V_{q_n}$$

Our requirements are then satisfied by the sets

$$U = U_{q_1} \cap \cdots \cap U_{q_n} \qquad W = V_{q_1} \cup \cdots \cup V_{q_n}$$

□

推论 2. *Compact subsets of Hausdorff spaces are closed. (the inverse is not true.)*

推论 3. *If F closed, K compact in a Hausdorff space, then $F \cap K$ is compact.*

2.6 Theorem *If $\{K_\alpha\}$ is a collection of compact subsets of a Hausdorff space and if $\bigcap_\alpha K_\alpha = \varnothing$, then some finite subcollection of $\{K_\alpha\}$ also has empty intersection.*

Use the definition of compact set.

2.7 Theorem *Suppose U is open in a locally compact Hausdorff space X, $K \subset U$, and K is compact. Then there is an open set V with compact closure such that*

$$K \subset V \subset \bar{V} \subset U.$$

Use the definition of compact set.

0.2 General properties

See Royden Chapter 11.

0.2.1 Bases and subbases

Definition *For a topological space $(X, \mathcal{T})$ and a point x in X, a collection of neighborhoods of x, $\mathcal{B}_x$, is called a* **base for the topology at** *x provided for any neighborhood $\mathcal{U}$ of x, there is a set B in the collection $\mathcal{B}_x$ for which $B \subseteq \mathcal{U}$. A collection of open sets $\mathcal{B}$ is called a* **base for the topology** *$\mathcal{T}$ provided it contains a base for the topology at each point.*

Proposition 2 *For a nonempty set X, let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a base for a topology for X if and only if*

(i) $\mathcal{B}$ covers X, that is, $X = \bigcup_{B \in \mathcal{B}} B$.

(ii) if B_1 and B_2 are in $\mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a set B in $\mathcal{B}$ for which $x \in B \subseteq B_1 \cap B_2$.

The unique topology that has $\mathcal{B}$ as its base consists of $\emptyset$ and unions of subcollections of $\mathcal{B}$.

Definition *For a topological space $(X, \mathcal{T})$, a subcollection $\mathcal{S}$ of $\mathcal{T}$ that covers X is called a* **subbase** *for the topology $\mathcal{T}$ provided intersections of finite subcollections of $\mathcal{S}$ are a base for $\mathcal{T}$.*

0.2.2 Separation properties

- **The Tychonoff Separation Property**:
  - For each two points u and v in X, there is a neighborhood of u that does not contain v and a neighborhood of v that does not contain u.
- **The Hausdorff Separation Property**:
  - Each two points in X can be separated by disjoint neighborhoods.
- **The Regular Separation Property**:
  - The Tychonoff separation property holds and, moreover, each closed set and point not in the set can be separated by disjoint neighborhoods.

- **The Normal Separation Property**:
  - The Tychonoff separation property holds and, moreover, each two disjoint closed sets can be separated by disjoint neighborhoods.

We have defined what we mean by a neighborhood of a point in a topological space. For a subset K of a topological space X, by a **neighborhood of K** we mean an open set that contains K. We say that two disjoint subsets A and B of X can be **separated by disjoint neighborhoods** provided there are neighborhoods of A and B, respectively, that are disjoint. For a topological space X, we consider the following four separation properties:

The Tychonoff Separation Property For each two points u and v in X, there is a neighborhood of u that does not contain v and a neighborhood of v that does not contain u.

The Hausdorff Separation Property Each two points in X can be separated by disjoint neighborhoods.

The Regular Separation Property The Tychonoff separation property holds and, moreover, each closed set and point not in the set can be separated by disjoint neighborhoods.

The Normal Separation Property The Tychonoff separation property holds and, moreover, each two disjoint closed sets can be separated by disjoint neighborhoods.

We naturally call a topological space Tychonoff, Hausdorff, regular, or normal, provided it satisfies the respective separation property.

命题 1. *A topological space X is a Tychonoff space iff every set consisting of a single point is closed.*

命题 2. *Every metric space is normal.*

命题 3. *Let X be Tychoneff. Then X is normal iff whenever $\mathcal{U}$ is a neighborhood of a closed subset F of X, there is another neighborhood of F whose closure is contained in $\mathcal{U}$, that is, there is an open set $\mathcal{O}$ for which*

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}$$

0.2.3 Countablity and Separability

Definition *A topological space X is said to be* **first countable** *provided there is a countable base at each point. The space X is said to be* **second countable** *provided there is a countable base for the topology.*

It is clear that a second countable space is first countable.

Example Every metric space X is first countable since for $x \in X$, the countable collection of open balls $\{B(x, 1/n)\}_{n=1}^{\infty}$ is a base at x for the topology induced by the metric.

In a topological space that is not first countable, it is possible for a point to be a point of closure of a set and yet no sequence in the set converges to the point.

Definition *A subset E of topological space X is said to be* **dense** *in X provided every open set in X contains a point of E. We call X* **separable** *provided it has a countable dense subset.*

The Urysohn Metrization Theorem *Let X be a second countable topological space. Then X is metrizable if and only if it is normal.*

0.2.4 Strong, weak topology, induced topology, homeomorphism

Definition *Given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ for a set X, if $\mathcal{T}_2 \subseteq \mathcal{T}_1$, we say that $\mathcal{T}_2$ is* **weaker** *than $\mathcal{T}_1$ and that $\mathcal{T}_1$ is* **stronger** *than $\mathcal{T}_2$.*

Definition *Let X be a nonempty set and consider a collection of mappings $\mathcal{F} = \{f_\alpha : X \to X_\alpha\}_{\alpha \in \Lambda}$, where each X_α is a topological space. The weakest topology for X that contains the collection of sets*

$$\mathcal{F} = \left\{ f_\alpha^{-1}(\mathcal{O}_\alpha) \;\middle|\; f_\alpha \in \mathcal{F},\ \mathcal{O}_\alpha \text{ open in } X_\alpha \right\}$$

is called the weak topology for X induced by $\mathcal{F}$.

Definition *A continuous mapping from a topological space X to a topological space Y is said to be a* **homeomorphism** *provided it is one-to-one, maps X onto Y, and has a continuous inverse f^{-1} from Y to X.*

0.2.5 Compact and Sequentially Compact topological spaces

Definition *A topological space X is said to be* **compact** *provided every open cover of X has a finite subcover. A subset K of X is called compact provided K, considered as a topological space with the subspace topology inherited from X, is compact.*

In view of the definition of the subspace topology, a subset K of X is compact provided every covering of K by a collection of open subsets of X has a finite subcover.

定义 2 (finite intersection property). *A collection of sets is said to be* ***finite intersection property*** *provided every finite subcollection has nonempty intersection.*

Proposition 14 *A topological space X is compact if and only if every collection of closed subsets of X that possesses the finite intersection property has nonempty intersection.*

Proposition 15 *A closed subset K of a compact topological space X is compact.*

Definition *A topological space X is said to be* **sequentially compact** *provided each sequence in X has a subsequence that converges to a point of X.*

Proposition 17 *Let X be a second countable topological space. Then X is compact if and only if it is sequentially compact.*

Theorem 18 *A compact Hausdorff space is normal.*

Proposition 19 *A continuous one-to-one mapping f of a compact space X onto a Hausdorff space Y is a homeomorphism.*

Additionally, homeomorphism requires continuous inverse, so hypothesis upon space is neccessary.

Proposition 20 *The continuous image of a compact topological space is compact.*

Regard a compact set as a compact topological space...

Corollary 21 *A continuous real-valued function on a compact topological space takes a maximum and minimum functional value.*

定义 3 (countably compact). *A topological space is said to be* ***countably compact*** *provided every countable open cover has a finite subcover.*

0.2.6 Separate, Connected, Intermediate value property

定义 4 (separate). *Two nonempty open subsets of a topological spaces X are said to* ***separate*** *X if they are disjoint and their union is X.*

定义 5 (connected). *A topological space which cannot be separated by such a pair is said to be* ***connected****.*

Connectness is preserved under continuous mapping.

Proposition 22 *Let f be a continuous mapping of a connected space X to a topological space Y. Then its image $f(Y)$ is connected.*

For a set C of real number, the following are equivalent:

- C is an interval.
- C is convex.

- C is connected.

Definition *A topological space X is said to have the intermediate value property provided the image of any continuous real-valued function on X is an interval.*

Proposition 23 *A topological space has the intermediate value property if and only if it is connected.*

0.3 Three Fundamental Theorems