实变函数

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学号：23363017

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第 1 次作业 (截止时间: 3月12日23:59)

1. 设 $\{f_k(x)\}$ 是定义在 $E\subset\mathbb{R}^n$ 上的实值函数列. 试将下列的点集 A 用集族 $\mathscr{A}$ 中的点集经过至多三重的可列并或可列交运算后表示出来.

   (1) $A=\left\{x\in E:\sup\limits_k f_k(x)\leqslant a\right\},\quad \mathscr{A}=\left\{\{x\in E:f_k(x)\leqslant t\}:t\in\mathbb{R},k\geqslant 1\right\};$

   (2) $A=\left\{x\in E:\inf\limits_k f_k(x)\geqslant a\right\},\quad \mathscr{A}=\left\{\{x\in E:f_k(x)\geqslant t\}:t\in\mathbb{R},k\geqslant 1\right\};$

   (3) $A=\left\{x\in E:\sup\limits_k f_k(x)> a\right\},\quad \mathscr{A}=\left\{\{x\in E:f_k(x)> t\}:t\in\mathbb{R},k\geqslant 1\right\};$

   (4) $A=\left\{x\in E:\inf\limits_k f_k(x)< a\right\},\quad \mathscr{A}=\left\{\{x\in E:f_k(x)< t\}:t\in\mathbb{R},k\geqslant 1\right\};$

   (5) $A=\left\{x\in E:\sup\limits_k f_k(x)\geqslant a\right\},\quad \mathscr{A}=\left\{\{x\in E:f_k(x)> t\}:t\in\mathbb{R},k\geqslant 1\right\};$

   (6) $A=\left\{x\in E:\sup\limits_k f_k(x)\geqslant a\right\},\quad \mathscr{A}=\left\{\{x\in E:f_k(x)\geqslant t\}:t\in\mathbb{R},k\geqslant 1\right\};$

   (7) $A=\left\{x\in E:\varlimsup\limits_{k\to\infty} f_k(x)\geqslant a\right\},\quad \mathscr{A}=\left\{\{x\in E:f_k(x)> t\}:t\in\mathbb{R},k\geqslant 1\right\};$

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   (11) $A=\left\{x\in E:\sup\limits_k f_k(x)=+\infty\right\},\quad \mathscr{A}=\left\{\{x\in E:f_k(x)\geqslant t\}:t\in\mathbb{R},k\geqslant 1\right\};$

   (12) $A=\left\{x\in E:\lim\limits_{k\to\infty} f_k(x)=+\infty\right\},\quad \mathscr{A}=\left\{\{x\in E:f_k(x)\geqslant t\}:t\in\mathbb{R},k\geqslant 1\right\}.$

2. 教材第一章习题第 6 题. 原题有个小错误, 请自行识别并更正.
3. 教材第一章习题第 9 题. 原题中的圆指的是圆周. 再分别考察 A_n 为开圆盘和闭圆盘的情形下的上下极限.
4. 教材第一章习题第 13 题.
5. 教材第一章习题第 14 题.
6. 教材第一章习题第 15 题. 若正确, 予以证明; 若不正确, 举出反例.
7. 教材第一章习题第 21 题, 并增加

   (4) 对任意 $A\subset B\subset X$, 有 $f(B\setminus A)=f(B)\setminus f(A)$.
8. 教材第一章习题第 26 题. 原题有误, $C\supset A$, $B\supset D$ 应为 $A\supset B$, $C\supset D$.

习题 1. 设 $\{f_k(x)\}$ 是定义在 $E\subset\mathbb{R}^n$ 上的实值函数列. 试将下列的点集 A 用集族 $\mathscr{A}$ 中的点集经过至多三重的可列并或可列交运算后表示出来.

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(3) $A=\{x\in E:\sup_k f_k(x)> a\},\quad \mathscr{A}=\{\{x\in E:f_k(x)> t\}:t\in\mathbb{R},k\geqslant 1\}$

(4) $A=\{x\in E:\inf_k f_k(x)<a\}$, $\mathscr{A}=\{\{x\in E:f_k(x)<t\}:t\in\mathbb{R},k\geqslant 1\}$

(5) $A=\{x\in E:\sup_k f_k(x)\geqslant a\}$, $\mathscr{A}=\{\{x\in E:f_k(x)>t\}:t\in\mathbb{R},k\geqslant 1\}$

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(7) $A=\left\{x\in E:\overline{\lim}_{k\to\infty} f_k(x)\geqslant a\right\}$, $\mathscr{A}=\{\{x\in E:f_k(x)>t\}:t\in\mathbb{R},k\geqslant 1\}$

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解答. *(1)*

$$A=\{x\in E:\sup_k f_k(x)\leq a\}=\bigcap_{k=1}^{\infty}\{x\in E:f_k(x)\leq a\}$$

(2)

$$A=\{x\in E:\inf_k f_k(x)\geq a\}=\bigcup_{k=1}^{\infty}\left\{x\in E:\int_k f_k(x)\geq a\right\}$$

(3) 取*(1)*两边的补集

$$A=\{x\in E:\sup_k f_k(x)>a\}=A_1^c=\left(\bigcap_{k=1}^{\infty}\{x\in E:f_k(x)\leq a\}\right)^c=\bigcup_{k=1}^{\infty}\{x\in E:f_k(x)>a\}$$

(4) 取*(2)*两边的补集

$$A=\{x\in E:\inf_k f_k(x)<a\}=A_2^c=\left(\bigcup_{k=1}^{\infty}\{x\in E:f_k(x)<a\}\right)^c=\bigcup_{k=1}^{\infty}\{x\in E:f_k(x)<a\}$$

(5) 对于任意 $x\in A$，$\sup_k f_k(x)\geq a$ 意味着对于任意 $\epsilon>0$，存在 k 使得 $f_k(x)>a-\epsilon$。用集合语言表示，

$$A=\{x\in E:\sup_k f_k(x)\geq a\}=\bigcap_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\left\{x\in E:f_k(x)>a-\frac{1}{n}\right\}$$

(6) 对于任意 $x\in A$，$\sup_k f_k(x)\geq a$ 意味着对于任意 $\epsilon>0$，存在 k 使得 $f_k(x)\geq a-\epsilon$。用集合语言表示，

$$A=\{x\in E:\sup_k f_k(x)\geq a\}=\bigcap_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\left\{x\in E:f_k(x)\geq a-\frac{1}{n}\right\}$$

(7) 对于任意 $x \in A$,

$$\limsup_{k\to\infty} f_k(x) \ge a \iff \lim_{n\to\infty} \sup_{k\ge n} f_k(x) \ge a \iff \lim_{n\to\infty} g_n(x) \ge a$$

其中 $g_n := \sup_{k\ge n} f_k$ 且

$$\{x \in E : \lim_{n\to\infty} g_n(x) \ge a\} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{x \in E : g_n(x) > a - \frac{1}{m}\right\}$$

$$\left\{x \in E : g_n(x) > a - \frac{1}{m}\right\} = \left\{x \in E : \sup_{k\ge n} f_k(x) > a - \frac{1}{m}\right\} = \bigcup_{k=n}^{\infty} \left\{x \in E : f_k(x) > a - \frac{1}{m}\right\}$$

由于 $\bigcup_{k=n}^{\infty} \left\{x \in E : f_k(x) > a - \frac{1}{m}\right\}$ 包含在 $\bigcup_{k=n-1}^{\infty} \left\{x \in E : f_k(x) > a - \frac{1}{m}\right\}$ 中，我们有

$$\bigcup_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left\{x \in E : f_k(x) > a - \frac{1}{m}\right\} = \bigcup_{n=1}^{\infty} \left\{x \in E : f_k(x) > a - \frac{1}{m}\right\}$$

因此

$$\{x \in E : \limsup_{k\to\infty} f_k(x) \ge a\} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{x \in E : f_k(x) > a - \frac{1}{m}\right\}$$

(8) 类似于(7)

$$\{x \in E : \limsup_{k\to\infty} f_k(x) \ge a\} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{x \in E : f_k(x) \ge a - \frac{1}{m}\right\}$$

(9)

$$\begin{aligned}
\{x \in E : \liminf_{k\to\infty} f_k(x) \ge a\} &= \{x \in E : \lim_{n\to\infty} \inf_{k\ge n} f_k(x) \ge a\} \\
&=: \{x \in E : \lim_{n\to\infty} g_n(x) \ge a\} \\
&= \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{x \in E : \inf_{k\ge n} f_k(x) \ge a - \frac{1}{m}\right\} \\
&= \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left\{x \in E : f_k(x) \ge a - \frac{1}{m}\right\}
\end{aligned}$$

(10) 类似于(9)

$$\{x \in R : \liminf_{k\to\infty} f_k(x) \ge a\} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left\{x \in E : f_k(x) > a - \frac{1}{m}\right\}$$

(11)

$$\{x\in R:\sup_k f_k(x)=+\infty\}=\bigcap_{m=1}^{\infty}\{x\in R:\sup_k f_k(x)\geq m\}$$
$$\overset{(6)}{=}\bigcap_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\left\{x\in E:f(x)\geq m-\frac{1}{n}\right\}$$
$$=\bigcap_{m=1}^{\infty}\bigcup_{k=1}^{\infty}\{x\in E:f(x)\geq m\}$$

(12)

$$\{x\in R:\lim_{k\to\infty}f_k(x)=\infty\}=\bigcap_{k=1}^{\infty}\{x\in E:f_k(x)\geq m\}$$

6\. 若$\lim\limits_{n\to\infty}f_n(x)=f(x)\,(x\in E)$，对于任意实数 c 用简写 $E(f>c)$ 和 $E(f\geqslant c)$ 表示 $\{x\in E$

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$|f>c\}$ 和 $\{x\in E\mid f\geqslant c\}$，并令 $E_{n,k}=E\left(f_n>c-\frac{1}{k}\right)$，试证 $\lim\limits_{n\to\infty}E_{n,k}$ 存在，并且

$$E(f\geqslant c)=\bigcap_{k=1}^{\infty}\lim_{n\to\infty}E_{n,k}.$$

习题 2. *Find an error of the question above.*

Errata: If $\lim_{n\to\infty}f_n(x)=f(x)$ for $x\in E$, $E_{n,k}:=\left\{x\in E:f_n(x)>c-\frac{1}{k}\right\}$, show that for each fixed k, $\lim_{n\to\infty}E_{n,k}$ exists and

$$\{x\in E:f(x)\geq c\}=\bigcap_{k=1}^{\infty}\lim_{n\to\infty}E_{n,k}$$

证明. Fix k then $\lim_{n\to\infty}E_{n,k}$ exists iff $\limsup_{n\to\infty}E_{n,k}$ and $\liminf_{n\to\infty}E_{n,k}$ exists and equals.

$$\limsup_{n\to\infty}E_{n,k}=\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}E_{n,k}=\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\left\{x\in E:f_n(x)>c-\frac{1}{k}\right\}=\left\{x\in E:\limsup_{n\to\infty}f_n(x)>c-\frac{1}{k}\right\}$$

$$\liminf_{n\to\infty}E_{n,k}=\left\{x\in E:\liminf_{n\to\infty}f_n(x)>c-\frac{1}{k}\right\}$$

Since $\lim_{n\to\infty}f_n(x)=f(x)$ exists, $\limsup_{n\to\infty}f_n(x)=\liminf_{n\to\infty}f_n(x)$ thus $\limsup_{n\to\infty}E_{n,k}=\liminf_{n\to\infty}E_{n,k}$. Hence $\lim_{n\to\infty}E_{n,k}$ exists and equals to

$\{x \in E : f(x) > c - \frac{1}{k}\}$. Obviously, $\{x \in E : f(x) \geq c\} = \bigcap_{k=1}^{\infty} \lim_{n\to\infty} E_{n,k}$.

□

习题 3. *9.* 设 A_n 是平面上以 $\left(\frac{(-1)^n}{n}, 0\right)$ 为心半径为 1 的圆，求 $\underline{\lim}_{n\to\infty} A_n$ 和 $\overline{\lim}_{n\to\infty} A_n$.

解答. *(1)*

$$A_n = \left\{(x,y) : \left(x - \frac{(-1)^n}{n}\right)^2 + y^2 = 1\right\}$$

Then

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty}(\varnothing) = \varnothing$$

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k = \{(x,y) \in \mathbb{R}^2 : \exists \text{ infinite many } k_j \text{ such that } (x,y) \in A_{k_j}\}$$

If $(x,y) \in A_m \cap A_n, m \neq n$ *then* $(x,y) \notin A_k$ *for* $k \neq m, k \neq n$. *Therefore* $\limsup_{n\to\infty} A_n = \varnothing$.

(2)

$$A_n = \left\{(x,y) : \left(x - \frac{(-1)^n}{n}\right)^2 + y^2 \leq 1\right\}$$

Then

$$\liminf_{n\to\infty} A_n = \{(x,y) \in \mathbb{R}^2 : \exists K, s.t.\ (x,y) \in A_k \text{ for all } k > K\}$$

$$\limsup_{n\to\infty} A_n = \{(x,y) \in \mathbb{R}^2 : \exists \text{ infinite many } k_j \text{ such that } (x,y) \in A_{k_j}\}$$

Claim that

$$\liminf_{n\to\infty} A_n = \{(x,y) : x^2 + y^2 < 1\}$$

$$\limsup_{n\to\infty} A_n = \{(x,y) : x^2 + y^2 \leq 1\} \setminus \{(0,1),(0,-1)\}$$

(3)

$$A_n = \left\{(x,y) : \left(x - \frac{(-1)^n}{n}\right)^2 + y^2 < 1\right\}$$

Then

$$\liminf_{n\to\infty} A_n = \{(x,y) \in \mathbb{R}^2 : \exists K, s.t.\ x \in A_k \text{ for all } k > K\}$$

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Claim that

$$\liminf_{n\to\infty} A_n = \{(x,y) : x^2+y^2<1\}$$

$$\limsup_{n\to\infty} A_n = \{(x,y) : x^2+y^2\le 1\}\setminus\{(0,1),(0,-1)\}$$

习题 4. *13.* 设 f,g 是定义在集 E 上的实函数，证明：

$$\{x \mid x\in E, f(x)>g(x)\} = \bigcup_{k=1}^{\infty}\left(E\left(f>r_k\right)\cap E\left(g<r_k\right)\right),$$

其中 $\{r_k\}$ 是有理数 $\mathbf{Q}$ 的一个排序.

解答. *Denote that*

$$A=\{x\in E : f(x)>g(x)\},\quad B=\bigcup_{k=1}^{\infty}(\{x\in E : f(x)>r_k\}\cap\{x\in E : g(x)<r_k\})$$

For $x\in A$, we have $f(x)>g(x)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $r_k\in(g(x),f(x))$ for some k, thus $x\in\{x\in E : f(x)>r_k\}\cap\{x\in E : g(x)<r_k\}\subset B$. Therefore $A\subset B$.

For $x\in B$, we have $f(x)>r_k, g(x)<r_k$ for some k then $f(x)<g(x)$, i.e. $x\in\{x\in E : f(x)>g(x)\}$. Therefore $A\supset B$. Hence $A=B$.

习题 5. *14.* 设函数 $\{f_n\}$ 在点集 E 上处处收敛于 f，并且 $|f(x)|<\infty(x\in E)$. 证明：对任意的 $\varepsilon>0$，集列极限 $\lim_{n\to\infty} E\left(|f_n-f|>\varepsilon\right)$ 存在且为空集.

证明. For any $\epsilon>0$, denote that

$$A_n=\{x\in E : |f_n(x)-f(x)|>\epsilon\}$$

Since $f_n\to f$ on E pointwisely and $|f(x)|<\infty, \forall x\in E$,we know that for any fixed $x\in E$, we have $\lim_{n\to\infty} f_n(x)=f(x)$ i.e. there exists $N>0$ such that $|f_n(x)-f(x)|<\epsilon$ for all $n>N$.

$$\liminf_{n\to\infty} A_n = \{x\in E : \exists K, \text{s.t. } x\in A_k \text{ for all } k>K\}=\varnothing$$

$$\limsup_{n\to\infty} A_n = \{x\in E : \exists \text{ infinite many } k_j \text{ such that } x\in A_{k_j}\}=\varnothing$$

Therefore $\lim_{n\to\infty} A_n$ exists and equals to $\varnothing$. □

习题 6. *15.* 设 $A_1,A_2,\cdots,A_k,\cdots$ 是函数 f 定义域的子集，试问等式

$$f\left(\bigcup_{k=1}^{\infty} A_k\right)=\bigcup_{k=1}^{\infty} f\left(A_k\right) \text{ 和} f\left(\bigcap_{k=1}^{\infty} A_k\right)=\bigcap_{k=1}^{\infty} f\left(A_k\right)$$

是否正确?

$$f\left(\bigcup_{k=1}^{\infty} A_k\right) = \bigcup_{k=1}^{\infty} f(A_k) \qquad f\left(\bigcap_{k=1}^{\infty} A_k\right) \subset \bigcap_{k=1}^{\infty} f(A_k)$$

Since $A_k \subset \bigcup_{k=1}^{\infty} A_k$ for any k, $f(A_k) \subset f\left(\bigcup_{k=1}^{\infty} A_k\right)$. Therefore $\bigcup_{k=1}^{\infty} f(A_k) \subset f\left(\bigcup_{k=1}^{\infty} A_k\right)$.

If $x \in f\left(\bigcup_{k=1}^{\infty} A_k\right) = \{f(y) : y \in \bigcup_{k=1}^{\infty} A_k\}$, then there exists $y \in \bigcup_{k=1}^{\infty} A_k$ such that $f(y) = x$. By the definition of $\bigcup_{k=1}^{\infty}$, we have $y \in A_n$ for some $n \in \mathbb{N}$ thus $x \in f(A_n) \subset \bigcup_{k=1}^{\infty} f(A_k)$. Therefore $f\left(\bigcup_{k=1}^{\infty} A_k\right) \subset \bigcup_{k=1}^{\infty} f(A_k)$. Hence $f\left(\bigcup_{k=1}^{\infty} A_k\right) = \bigcup_{k=1}^{\infty} f(A_k)$.

For $x \in f\left(\bigcap_{k=1}^{\infty} A_k\right)$, there exists $y \in \bigcap_{k=1}^{\infty} A_k$ such that $x = f(y)$. For any $n \in \mathbb{N}$, $y \in \bigcap_{k=1}^{\infty} A_k \subset A_n$ then $x \in f(A_n)$. Therefore $x \in \bigcap_{k=1}^{\infty} f(A_k)$, i.e. $f\left(\bigcap_{k=1}^{\infty} f(A_k)\right) \subset \bigcap_{k=1}^{\infty} f(A_k)$.

习题 7. *21.* 设 $f : X \to Y$,则下面的命题互相等价:

(1) f 是 X 到 $f(X)$ 的一一映射;

(2) 对任意的 $A, B \subset X$ 有 $f(A \cap B) = f(A) \cap f(B)$;

(3) 对满足 $A \cap B = \varnothing$ 的 $A, B \subset X$ 有 $f(A) \cap f(B) = \varnothing$.

(4) 对任意 $A \subset B \subset X$ 有 $f(B \setminus A) = f(B) \setminus f(A)$.

(1) means f is one-one. Since $f : X \to f(X)$ is defined to be onto, then (1) means if $x \neq y$ for $x, y \in X$ then $f(x) \neq f(y)$. In other words, if $f(x) = f(y)$ then $x = y$.

证明. **(1) ⇒ (2)** for any $x \in A \cap B$, we have $x \in A$ and $x \in B$ then $f(x) \in f(A)$ and $f(x) \in f(B)$ thus $x \in f(A) \cap f(B)$. Therefore $f(A \cap B) \subset f(A) \cap f(B)$. It suffices to show that $f(A) \cap f(B) \subset f(A \cap B)$. We argue by contradiction. If there is an element $y \in f(A) \cap f(B)$ not containd in $f(A \cap B)$, then there exists $a \in A$ and $b \in B$ such that $f(a) = y$, $f(b) = y$. Since f is one-one, we have $a = b$. Therefore $a = b \in A \cap B$, $y = f(a) \in f(A \cap B)$, which is a contradiction. Hence $f(A) \cap f(B) \subset f(A \cap B) \Rightarrow f(A) \cap f(B) = f(A \cap B)$.

(2) ⇒ (3) Trivial.

(3) ⇒ (4) For fixed A and B, $A \subset B \subset X$, subtract $f(A)$ from $f(B)$. Since $A \cap (B \setminus A) = \varnothing$, then we have $f(A) \cap f(B \setminus A) = \varnothing$. But $f(B) = f(A \cup (B \setminus A))$, so $f(B) = f(A) \sqcup f(B \setminus A)$ i.e. $f(B) \setminus f(A) = f(B \setminus A)$.

(4) $\Rightarrow$ (1) Pick $x_1, x_2 \in X$, $x_1 \neq x_2$, and let $A = \{x_1\}, B = \{x_1, x_2\}$, then $f(B \setminus A) = f(B) \setminus f(A)$ means $f(x_2) = f(\{x_1, x_2\} \setminus \{x_1\}) = f(\{x_1, x_2\}) \setminus f(\{x_1\})$, which implies $f(\{x_2\}) \cap f(\{x_1\}) = \varnothing$ i.e. $f(x_1) \neq f(x_2)$.

□

26. 论断"若 $A \sim C, B \sim D$,并且 $A \supset C, B \supset D$,则 $(A \backslash B) \sim (C \backslash D)$"是否正确？请予证明或举出反例.

习题 8. *Errata:* 原题有误，$C \supset A, B \supset D$ 应改为 $A \supset B, C \supset D$.

解答. *$(A \setminus B) \sim (C \setminus D)$ does not always hold. A counterexample is as follows.*

Let A, B, C, D be groups, B is a subgroup of A and D is a subgroup of C. The relation $\sim$ on groups is defined to be group isomorphism. Then we have $A \cong C$ and $B \cong D$. Since $\setminus$ is set minus, $A \setminus B$ or $C \setminus D$ is not a group! (without the identity) Therefore $(A \setminus B) \not\sim (C \setminus D)$.