Part II: Stochastic Programming (Spring 2021) **Homework: 08**

Lecture 2: Uncertainty and Modeling Issues

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P1.

Let x represent the first-stage production of a given good. Let ξ be the demand for the same good. A typical second stage would consist of selling as much as possible, namely, $\min(\xi, x)$. Obtain a closed form expression for the recourse function $E_\xi[min(\xi, x)]$ in the following cases of ξ:

- (a) Poisson distribution,
- (b) A normal distribution.

Solution:

- (a) Suppose that $\xi \sim P(\lambda)$

$$E_\xi[min(\xi, x)] = \sum_{\xi=0}^{x} \xi e^{-\lambda} \frac{\lambda^\xi}{\xi!} + \sum_{\xi=x+1}^{\infty} x e^{-\lambda} \frac{\lambda^\xi}{\xi!}$$

- (b) Suppose that $\xi \sim N(\mu, \sigma^2)$

$$\begin{aligned} E_\xi[min(\xi, x)] &= \frac{1}{\sqrt{2\pi}\sigma}\left(\int_{-\infty}^{x} \xi e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} d\xi + \int_{x}^{\infty} x e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} d\xi\right) \\ &= \frac{\sigma}{\sqrt{2\pi}}\left(1 - e^{-\frac{(\frac{x-\mu}{\sigma})^2}{2}}\right) \frac{1}{\sqrt{2\pi}\sigma}[\mu\phi(x) + x(1-\phi(x))] \end{aligned}$$

P2.

Consider an airplane with x seats. Assume passengers with reservations show up with probability 0.90, independently of each other.

- (a) Let $x = 40$. If 42 passengers receive a reservation, what is the probability that at least one is denied seat.
- (b) Let $x = 50$. How many reservations can be accepted under the constraint that the probability of seating all passengers who arrive for the flight is greater than 90% ?

Solution:

- (a) Donate Y as the number of customer shows up.

$$P = P\{Y = 41\} + P\{Y = 42\} = 42 * 0.9^{41} \times 0.1 + 0.9^{42}$$

- (b)

$$P\{Y = y\} = C_{50}^{y} p^{y}(1-p)^{50-y}$$
$$E[Y] = \sum_{0}^{50} C_{50}^{y} y p^{y}(1-p)^{50-y}$$

P3.

Show that VaR is not a coherent risk measure.

Solution:

- The coherent risk measure has the following property according to the wikipedia: Sub-additivity.
- Sub-additivity: If Z_1, $Z_2 \in \mathcal{L}$, then $\varrho(Z_1 + Z_2) \leq \varrho(Z_1) + \varrho(Z_2)$, i.e., the risk of two portfolios together cannot get any worse than adding the two risks separately: this is the diversification principle.
- VaR obviously doesn't respect the sub-additivity property.
- According to the definition of the VaR: $VaR_\alpha(\xi)$=$min\{t|P(\xi \leq t) \geq \alpha\}$.
- As a simple example to demonstrate the non-coherence of value-at-risk consider looking at the VaR of a portfolio at 95% confidence over the next year of two default-able zero coupon bonds that mature in 1 years time denominated in our numeraire currency.

  Assume the following:

  - The current yield on the two bonds is 0%
  - The two bonds are from different issuers
  - Each bond has a 4% probability of defaulting over the next year
  - The event of default in either bond is independent of the other
  - Upon default the bonds have a recovery rate of 30%

  Under these conditions the 95% VaR for holding either of the bonds is 0 since the probability of default is less than 5%. However if we held a portfolio that consisted of 50% of each bond by value then the 95% VaR is 35% (= 0.5*0.7 + 0.5*0) since the probability of at least one of the bonds defaulting is 7.84% (= 1 - 0.96*0.96) which exceeds 5%. This violates the sub-additivity property showing that VaR is not a coherent risk measure.

P4.

Prove that CVaR is a coherent risk measure.

Solution:

- Let $F(x)$ be a probability distribution of a random variable X.

$$F(x) = ProbX \leq x$$

 and for some probability $q \in (0,1)$ let's define the q-quantile

$$x_q = inf\{x|F(x) \geq q\}$$

- If $F()$ is continuous we have $F(x_q) = q$, while if $F()$ is discontinuous in x_q and therefore $ProbX = x_q > 0$ we may have $F(x_q) = ProbX \leq xq > q$. Our definition of q-Tail Mean x_q as the "expected value of the distribution in the q-quantile" must take this fact into account.
- Definition: For a random variable X and for a specified level of probability q, let's define the q-Tail Mean:

$$\begin{aligned}\overline{x_q} &\equiv \frac{1}{q}E\{X1_{X\leq x_q}\} + (1 - \frac{F(x_q)}{q})x_q \\ &= \frac{1}{q}E\{X1^q_{X\leq x_q}\}\end{aligned}$$

 where in the last expression we introduced

$$1^q_{X\leq x_q} = 1_{X\leq x_q} + \frac{q - F(x_q)}{ProbX = x_q}1^q_{X=x_q}$$

- The second term in the sum is zero if $ProbX = x_q = 0$. In what follows we will make use of the following properties:

$$\begin{aligned}E\{X1^q_{X\leq x_q}\} = q \\ 0 \leq 1^q_{X\leq x_q} \leq 1\end{aligned}$$

- The only thing to show is in the case $X = x_q$:

$$1^q_{X\leq x_q}|_{X=x_q} = 1 + \frac{q - F(x_q)}{ProbX = x_q} = \frac{q - F(x_q)}{ProbX = x_q^-} \in [0,1]$$

P5.

Prove that CVaR is an upper bound of VaR.

Solution:

$$\begin{aligned}CVaR_\alpha = E(X|X \geq q_\alpha) = \frac{E(X1_{X\geq x_q})}{Prob\{X \geq x_q\}} \\ &= \frac{E(X - q_\alpha + q_\alpha 1_{X\geq q_\alpha})}{Prob\{X \geq x_q\}} \\ &= \frac{E((X - q_\alpha)1_{X\geq x_q})}{ProbX \geq x_q} + q_\alpha \frac{E(1_{X\geq x_q})}{Prob\{X \geq x_q\}} \\ &= \frac{E((X - q_\alpha)^+)}{\alpha} + q_\alpha \\ &\geq q_\alpha \\ &= VaR_\alpha\end{aligned}$$