# 1

# Univariate Stochastic Orders

In this chapter we study stochastic orders that compare the "location" or the "magnitude" of random variables. The most important and common orders that are considered in this chapter are the usual stochastic order $\leq_{\rm st}$, the hazard rate order $\leq_{\rm hr}$, and the likelihood ratio order $\leq_{\rm lr}$. Some variations of these orders, and some related orders, are also examined in this chapter.

## 1.A The Usual Stochastic Order

### 1.A.1 Definition and equivalent conditions

Let $X$ and $Y$ be two random variables such that

$$P\{X > x\} \leq P\{Y > x\} \quad \text{for all } x \in (-\infty, \infty). \tag{1.A.1}$$

Then $X$ is said to be *smaller than $Y$ in the usual stochastic order* (denoted by $X \leq_{\rm st} Y$). Roughly speaking, (1.A.1) says that $X$ is less likely than $Y$ to take on large values, where "large" means any value greater than $x$, and that this is the case for all $x$'s. Note that (1.A.1) is the same as

$$P\{X \leq x\} \geq P\{Y \leq x\} \quad \text{for all } x \in (-\infty, \infty). \tag{1.A.2}$$

It is easy to verify (by noting that every closed interval is an infinite intersection of open intervals) that $X \leq_{\rm st} Y$ if, and only if,

$$P\{X \geq x\} \leq P\{Y \geq x\} \quad \text{for all } x \in (-\infty, \infty). \tag{1.A.3}$$

In fact, we can recast (1.A.1) and (1.A.3) in a seemingly more general, but actually an equivalent, way as follows:

$$P\{X \in U\} \leq P\{Y \in U\} \quad \text{for all upper sets } U \subseteq (-\infty, \infty). \tag{1.A.4}$$

(In the univariate case, that is on the real line, a set $U$ is an upper set if, and only if, it is an open or a closed right half line.) In the univariate case the equivalence of (1.A.4) with (1.A.1) and (1.A.3) is trivial, but in Chapter 6 it will be seen that the generalizations of each of these three conditions to the multivariate case yield different definitions of stochastic orders.

Still another way of rewriting (1.A.1) or (1.A.3) is the following:

$$E[I_U(X)] \leq E[I_U(Y)] \quad \text{for all upper sets } U \subseteq (-\infty, \infty), \tag{1.A.5}$$

where $I_U$ denotes the indicator function of $U$. From (1.A.5) it follows that if $X \leq_{\rm st} Y$, then

$$E\Big[\sum_{i=1}^{m} a_i I_{U_i}(X)\Big] - b \leq E\Big[\sum_{i=1}^{m} a_i I_{U_i}(Y)\Big] - b \tag{1.A.6}$$

for all $a_i \geq 0$, $i = 1, 2, \ldots, m$, $b \in (-\infty, \infty)$, and $m \geq 0$. Given an increasing function $\phi$, it is possible, for each $m$, to define a sequence of $U_i$'s, a sequence of $a_i$'s, and a $b$ (all of which may depend on $m$), such that as $m \to \infty$ then (1.A.6) converges to

$$E[\phi(X)] \leq E[\phi(Y)], \tag{1.A.7}$$

provided the expectations exist. It follows that $X \leq_{\rm st} Y$ *if, and only if,* (1.A.7) *holds for all increasing functions* $\phi$ *for which the expectations exist.*

The expressions $\int_x^\infty P\{X > y\}\mathrm{d}y$ and $\int_x^\infty P\{Y > y\}\mathrm{d}y$ are used extensively in Chapters 2, 3, and 4. It is of interest to note that $X \leq_{\rm st} Y$ if, and only if,

$$\int_x^\infty P\{Y > y\}\mathrm{d}u - \int_x^\infty P\{X > y\}\mathrm{d}y \quad \text{is decreasing in } x \in (-\infty, \infty). \tag{1.A.8}$$

If $X$ and $Y$ are discrete random variables taking on values in $\mathbb{N}$, then we have the following. Let $p_i = P\{X = i\}$ and $q_i = P\{Y = i\}$, $i \in \mathbb{N}$. Then $X \leq_{\rm st} Y$ if, and only if,

$$\sum_{j=-\infty}^{i} p_j \geq \sum_{j=-\infty}^{i} q_j, \quad i \in \mathbb{N},$$

or, equivalently, $X \leq_{\rm st} Y$ if, and only if,

$$\sum_{j=i}^{\infty} p_j \leq \sum_{j=i}^{\infty} q_j, \quad i \in \mathbb{N}.$$

### 1.A.2 A characterization by construction on the same probability space

An important characterization of the usual stochastic order is the following theorem (here $=_{\rm st}$ denotes equality in law).

**Theorem 1.A.1.** *Two random variables $X$ and $Y$ satisfy $X \leq_{\text{st}} Y$ if, and only if, there exist two random variables $\hat{X}$ and $\hat{Y}$, defined on the same probability space, such that*

$$\hat{X} =_{\text{st}} X, \tag{1.A.9}$$

$$\hat{Y} =_{\text{st}} Y, \tag{1.A.10}$$

*and*

$$P\{\hat{X} \leq \hat{Y}\} = 1. \tag{1.A.11}$$

*Proof.* Obviously (1.A.9), (1.A.10), and (1.A.11) imply that $X \leq_{\text{st}} Y$. In order to prove the necessity part of Theorem 1.A.1, let $F$ and $G$ be, respectively, the distribution functions of $X$ and $Y$, and let $F^{-1}$ and $G^{-1}$ be the corresponding right continuous inverses (see Note on page 1). Define $\hat{X} = F^{-1}(U)$ and $\hat{Y} = G^{-1}(U)$ where $U$ is a uniform $[0, 1]$ random variable. Then it is easy to see that $\hat{X}$ and $\hat{Y}$ satisfy (1.A.9) and (1.A.10). From (1.A.2) it is seen that (1.A.11) also holds. □

Theorem 1.A.1 is a special case of a more general result that is stated in Section 6.B.2.

From (1.A.2) and Theorem 1.A.1 it follows that the random variables $X$ and $Y$, with the respective distribution functions $F$ and $G$, satisfy $X \leq_{\text{st}} Y$ if, and only if,

$$F^{-1}(u) \leq G^{-1}(u), \quad \text{for all } u \in (0, 1). \tag{1.A.12}$$

Another way of restating Theorem 1.A.1 is the following. We omit the obvious proof of it.

**Theorem 1.A.2.** *Two random variables $X$ and $Y$ satisfy $X \leq_{\text{st}} Y$ if, and only if, there exist a random variable $Z$ and functions $\psi_1$ and $\psi_2$ such that $\psi_1(z) \leq \psi_2(z)$ for all $z$ and $X =_{\text{st}} \psi_1(Z)$ and $Y =_{\text{st}} \psi_2(Z)$.*

In some applications, when the random variables $X$ and $Y$ are such that $X \leq_{\text{st}} Y$, one may wish to construct a $\hat{Y}$ $[\hat{X}]$ on the probability space on which $X$ $[Y]$ is defined, such that $\hat{Y} =_{\text{st}} Y$ and $P\{X \leq \hat{Y}\} = 1$ $[\hat{X} =_{\text{st}} X$ and $P\{\hat{X} \leq Y\} = 1]$. This is always possible. Here we will show how this can be done when the distribution function $F$ $[G]$ of $X$ $[Y]$ is absolutely continuous. When this is the case, $F(X)$ $[G(Y)]$ is uniformly distributed on $[0, 1]$, and therefore $\hat{Y} = G^{-1}(F(X))$ $[\hat{X} = F^{-1}(G(Y))]$ is the desired construction $\hat{Y}$ $[\hat{X}]$.

### 1.A.3 Closure properties

Using (1.A.1) through (1.A.11) it is easy to prove each of the following closure results. The following notation will be used: For any random variable $Z$ and an event $A$, let $[Z|A]$ denote any random variable that has as its distribution the conditional distribution of $Z$ given $A$.

**Theorem 1.A.3.** (a) *If* $X \leq_{\mathrm{st}} Y$ *and* $g$ *is any increasing [decreasing] function, then* $g(X) \leq_{\mathrm{st}} [\geq_{\mathrm{st}}]\ g(Y)$.

(b) *Let* $X_1, X_2, \ldots, X_m$ *be a set of independent random variables and let* $Y_1, Y_2, \ldots, Y_m$ *be another set of independent random variables. If* $X_i \leq_{\mathrm{st}} Y_i$ *for* $i = 1, 2, \ldots, m$*, then, for any increasing function* $\psi : \mathbb{R}^m \to \mathbb{R}$*, one has*

$$\psi(X_1, X_2, \ldots, X_m) \leq_{\mathrm{st}} \psi(Y_1, Y_2, \ldots, Y_m).$$

*In particular,*

$$\sum_{j=1}^{m} X_j \leq_{\mathrm{st}} \sum_{j=1}^{m} Y_j.$$

*That is, the usual stochastic order is closed under convolutions.*

(c) *Let* $\{X_j,\ j = 1, 2, \ldots\}$ *and* $\{Y_j,\ j = 1, 2, \ldots\}$ *be two sequences of random variables such that* $X_j \to_{\mathrm{st}} X$ *and* $Y_j \to_{\mathrm{st}} Y$ *as* $j \to \infty$*, where "*$\to_{\mathrm{st}}$*" denotes convergence in distribution. If* $X_j \leq_{\mathrm{st}} Y_j$*,* $j = 1, 2, \ldots$*, then* $X \leq_{\mathrm{st}} Y$.

(d) *Let* $X$*,* $Y$*, and* $\Theta$ *be random variables such that* $[X \big| \Theta = \theta] \leq_{\mathrm{st}} [Y \big| \Theta = \theta]$ *for all* $\theta$ *in the support of* $\Theta$*. Then* $X \leq_{\mathrm{st}} Y$*. That is, the usual stochastic order is closed under mixtures.*

In the next result and in the sequel we define $\sum_{j=1}^{0} a_j \equiv 0$ for any sequence $\{a_j,\ j = 1, 2, \ldots\}$.

**Theorem 1.A.4.** *Let* $\{X_j,\ j = 1, 2, \ldots\}$ *be a sequence of nonnegative independent random variables, and let* $M$ *be a nonnegative integer-valued random variable which is independent of the* $X_i$*'s. Let* $\{Y_j,\ j = 1, 2, \ldots\}$ *be another sequence of nonnegative independent random variables, and let* $N$ *be a nonnegative integer-valued random variable which is independent of the* $Y_i$*'s. If* $X_i \leq_{\mathrm{st}} Y_i$*,* $i = 1, 2, \ldots$*, and if* $M \leq_{\mathrm{st}} N$*, then*

$$\sum_{j=1}^{M} X_j \leq_{\mathrm{st}} \sum_{j=1}^{N} Y_j.$$

Another related result is given next.

**Theorem 1.A.5.** *Let* $\{X_j,\ j = 1, 2, \ldots\}$ *be a sequence of nonnegative independent and identically distributed random variables, and let* $M$ *be a positive integer-valued random variable which is independent of the* $X_i$*'s. Let* $\{Y_j,\ j = 1, 2, \ldots\}$ *be another sequence of independent and identically distributed random variables, and let* $N$ *be a positive integer-valued random variable which is independent of the* $Y_i$*'s. Suppose that for some positive integer* $K$ *we have that*

$$\sum_{j=1}^{K} X_j \leq_{\mathrm{st}} [\geq_{\mathrm{st}}]\ Y_1$$

*and*

$$M \leq_{\text{st}} [\geq_{\text{st}}] \ KN,$$

*then*

$$\sum_{j=1}^{M} X_j \leq_{\text{st}} [\geq_{\text{st}}] \ \sum_{j=1}^{N} Y_j.$$

*Proof.* The assumptions yield

$$\sum_{i=1}^{M} X_i \leq_{\text{st}} [\geq_{\text{st}}] \ \sum_{i=1}^{KN} X_i = \sum_{i=1}^{N} \ \sum_{j=K(i-1)+1}^{Ki} X_j \leq_{\text{st}} [\geq_{\text{st}}] \ \sum_{i=1}^{N} Y_i. \qquad \square$$

Consider now a family of distribution functions $\{G_\theta, \ \theta \in \mathcal{X}\}$ where $\mathcal{X}$ is a subset of the real line $\mathbb{R}$. Let $X(\theta)$ denote a random variable with distribution function $G_\theta$. For any random variable $\Theta$ with support in $\mathcal{X}$, and with distribution function $F$, let us denote by $X(\Theta)$ a random variable with distribution function $H$ given by

$$H(y) = \int_{\mathcal{X}} G_\theta(y)\mathrm{d}F(\theta), \quad y \in \mathbb{R}.$$

The following result is a generalization of both parts (a) and (c) of Theorem 1.A.3.

**Theorem 1.A.6.** *Consider a family of distribution functions $\{G_\theta, \ \theta \in \mathcal{X}\}$ as above. Let $\Theta_1$ and $\Theta_2$ be two random variables with supports in $\mathcal{X}$ and distribution functions $F_1$ and $F_2$, respectively. Let $Y_1$ and $Y_2$ be two random variables such that $Y_i =_{\text{st}} X(\Theta_i)$, $i = 1, 2$; that is, suppose that the distribution function of $Y_i$ is given by*

$$H_i(y) = \int_{\mathcal{X}} G_\theta(y)\mathrm{d}F_i(\theta), \quad y \in \mathbb{R}, \ i = 1, 2.$$

*If*

$$X(\theta) \leq_{\text{st}} X(\theta') \quad \textit{whenever } \theta \leq \theta', \tag{1.A.13}$$

*and if*

$$\Theta_1 \leq_{\text{st}} \Theta_2, \tag{1.A.14}$$

*then*

$$Y_1 \leq_{\text{st}} Y_2. \tag{1.A.15}$$

*Proof.* Note that, by (1.A.13), $P\{X(\theta) > y\}$ is increasing in $\theta$ for all $y$. Thus

$$\begin{aligned} P\{Y_1 > y\} &= \int_{\mathcal{X}} P\{X(\theta) > y\}\mathrm{d}F_1(\theta) \\ &\leq \int_{\mathcal{X}} P\{X(\theta) > y\}\mathrm{d}F_2(\theta) \\ &= P\{Y_2 > y\}, \quad \text{for all } y, \end{aligned}$$

where the inequality follows from (1.A.14) and (1.A.7). Thus (1.A.15) follows from (1.A.1). $\square$

Note that, using the notation that is introduced below before Theorem 1.A.14, (1.A.13) can be rewritten as $\{X(\theta),\ \theta \in \mathcal{X}\} \in \mathrm{SI}$.

The following example shows an application of Theorem 1.A.6 in the area of Bayesian imperfect repair; a related result is given in Example 1.B.16.

*Example 1.A.7.* Let $\Theta_1$ and $\Theta_2$ be two random variables with supports in $\mathcal{X} = (0, 1]$ and distribution functions $F_1$ and $F_2$, respectively. For some survival function $\overline{K}$, define

$$\overline{G}_\theta = \overline{K}^{1-\theta}, \quad \theta \in (0, 1],$$

and let $X(\theta)$ have the survival function $\overline{K}^{1-\theta}$. Note that (1.A.13) holds because $\overline{K}^{1-\theta}(y) \le \overline{K}^{1-\theta'}(y)$ for all $y$ whenever $0 < \theta \le \theta' \le 1$. Thus, if $\Theta_1 \le_{\mathrm{st}} \Theta_2$ then $Y_i$, with survival function $\overline{H}_i$ defined by

$$\overline{H}_i(y) = \int_0^1 \overline{K}^{1-\theta}(y)\mathrm{d}F_i(\theta), \quad y \in \mathbb{R},\ i = 1, 2,$$

satisfy $Y_1 \le_{\mathrm{st}} Y_2$.

### 1.A.4 Further characterizations and properties

Clearly, if $X \le_{\mathrm{st}} Y$ then $EX \le EY$. However, as the following result shows, if two random variables are ordered in the usual stochastic order and have the same expected values, they must have the same distribution.

**Theorem 1.A.8.** *If $X \le_{\mathrm{st}} Y$ and if $E[h(X)] = E[h(Y)]$ for some strictly increasing function $h$, then $X =_{\mathrm{st}} Y$.*

*Proof.* First we prove the result when $h(x) = x$. Let $\hat{X}$ and $\hat{Y}$ be as in Theorem 1.A.1. If $P\{\hat{X} < \hat{Y}\} > 0$, then $EX = E\hat{X} < E\hat{Y} = EY$, a contradiction to the assumption $EX = EY$. Therefore $X =_{\mathrm{st}} \hat{X} = \hat{Y} =_{\mathrm{st}} Y$. Now let $h$ be some strictly increasing function. Observe that if $X \le_{\mathrm{st}} Y$, then $h(X) \le_{\mathrm{st}} h(Y)$ and therefore from the above result we have that $h(X) =_{\mathrm{st}} h(Y)$. The strict monotonicity of $h$ yields $X =_{\mathrm{st}} Y$. □

Other results that give conditions, involving stochastic orders, which imply stochastic equalities, are given in Theorems 3.A.43, 3.A.60, 4.A.69, 5.A.15, 6.B.19, 6.G.12, 6.G.13, and 7.A.14–7.A.16.

As was mentioned above, if $X \le_{\mathrm{st}} Y$, then $EX \le EY$. It is easy to find counterexamples which show that the converse is false. However, $X \le_{\mathrm{st}} Y$ implies other moment inequalities (for example, $EX^3 \le EY^3$). Thus one may wonder whether $X \le_{\mathrm{st}} Y$ can be characterized by a collection of moment inequalities. Brockett and Kahane [109, Corollary 1] showed that there exist no finite number of moment inequalities which imply $X \le_{\mathrm{st}} Y$. In fact, they showed it for many other stochastic orders that are studied later in this book.

In order to state the next characterization we define the following class of bivariate functions:

$$\mathcal{G}_{\mathrm{st}} = \{\phi : \mathbb{R}^2 \to \mathbb{R} : \phi(x, y) \text{ is increasing in } x \text{ and decreasing in } y\}.$$

**Theorem 1.A.9.** *Let $X$ and $Y$ be independent random variables. Then $X \leq_{\rm st} Y$ if, and only if,*

$$\phi(X,Y) \leq_{\rm st} \phi(Y,X) \quad \text{for all } \phi \in \mathcal{G}_{\rm st}. \tag{1.A.16}$$

*Proof.* Suppose that (1.A.16) holds. The function $\phi$ defined by $\phi(x,y) \equiv x$ belongs to $\mathcal{G}_{\rm st}$. Therefore $X \leq_{\rm st} Y$.

In order to prove the "only if" part, suppose that $X \leq_{\rm st} Y$. Let $\phi \in \mathcal{G}_{\rm st}$ and define $\psi(x,y) = \phi(x,-y)$. Then $\psi$ is increasing on $\mathbb{R}^2$. Since $X$ and $Y$ are independent it follows that $X$ and $-Y$ are independent and also that $-X$ and $Y$ are independent. Since $X \leq_{\rm st} Y$ it follows (for example, from Theorem 1.A.1) that $-Y \leq_{\rm st} -X$. Therefore, by Theorem 1.A.3(b), we have

$$\psi(X,-Y) \leq_{\rm st} \psi(Y,-X),$$

that is,

$$\phi(X,Y) \leq_{\rm st} \phi(Y,X). \qquad \square$$

The next result is a similar characterization. In order to state it we need the following notation: Let $\phi_1$ and $\phi_2$ be two bivariate functions. Denote $\Delta\phi_{21}(x,y) = \phi_2(x,y) - \phi_1(x,y)$. The proof of the following theorem is omitted.

**Theorem 1.A.10.** *Let $X$ and $Y$ be two independent random variables. Then $X \leq_{\rm st} Y$ if, and only if,*

$$E\phi_1(X,Y) \leq E\phi_2(X,Y)$$

*for all $\phi_1$ and $\phi_2$ which satisfy that, for each $y$, $\Delta\phi_{21}(x,y)$ decreases in $x$ on $\{x \leq y\}$; for each $x$, $\Delta\phi_{21}(x,y)$ increases in $y$ on $\{y \geq x\}$; and $\Delta\phi_{21}(x,y) \geq -\Delta\phi_{21}(y,x)$ whenever $x \leq y$.*

Another similar characterization is given in Theorem 4.A.36.

Let $X$ and $Y$ be two random variables with distribution functions $F$ and $G$, respectively. Let $\mathcal{M}(F,G)$ denote the Fréchet class of bivariate distributions with fixed marginals $F$ and $G$. Abusing notation we write $(\hat{X},\hat{Y}) \in \mathcal{M}(F,G)$ to mean that the jointly distributed random variables $\hat{X}$ and $\hat{Y}$ have the marginal distribution functions $F$ and $G$, respectively. The Fortret-Mourier-Wasserstein distance between the finite mean random variables $X$ and $Y$ is defined by

$$d(X,Y) = \inf_{(\hat{X},\hat{Y}) \in \mathcal{M}(F,G)} \{E|\hat{Y} - \hat{X}|\}. \tag{1.A.17}$$

**Theorem 1.A.11.** *Let $X$ and $Y$ be two finite mean random variables such that $EX \leq EY$. Then $X \leq_{\rm st} Y$ if, and only if, $d(X,Y) = EY - EX$.*

*Proof.* Suppose that $d(X,Y) = EY - EX$. The infimum in (1.A.17) is attained for some $(\hat{X},\hat{Y})$, and we have $E|\hat{Y} - \hat{X}| = E(\hat{Y} - \hat{X})$. Therefore $P\{\hat{X} \leq \hat{Y}\} = 1$, and from Theorem 1.A.1 it follows that $X \leq_{\rm st} Y$.

Conversely, suppose that $X \leq_{\rm st} Y$. Let $\hat{X}$ and $\hat{Y}$ be as in Theorem 1.A.1. Then, for any $(X',Y') \in \mathcal{M}(F,G)$ we have that $E|Y' - X'| \geq |EY' - EX'| = E\hat{Y} - E\hat{X}$. Therefore $d(X,Y) = EY - EX$. $\square$

A simple sufficient condition which implies the usual stochastic order is described next. The following notation will be used. Let $a(x)$ be defined on $I$, where $I$ is a subset of the real line. The number of sign changes of $a$ in $I$ is defined by

$$S^-(a) = \sup S^-[a(x_1), a(x_2), \ldots, a(x_m)], \tag{1.A.18}$$

where $S^-(y_1, y_2, \ldots, y_m)$ is the number of sign changes of the indicated sequence, zero terms being discarded, and the supremum in (1.A.18) is extended over all sets $x_1 < x_2 < \cdots < x_m$ such that $x_i \in I$ and $m < \infty$. The proof of the next theorem is simple and therefore it is omitted.

**Theorem 1.A.12.** *Let $X$ and $Y$ be two random variables with (discrete or continuous) density functions $f$ and $g$, respectively. If*

$$S^-(g - f) = 1 \quad \textit{and the sign sequence is } -, +,$$

*then $X \le_{\rm st} Y$.*

Let $X_1$ be a nonnegative random variable with distribution function $F_1$ and survival function $\overline{F}_1 \equiv 1 - F_1$. Define the Laplace transform of $X_1$ by

$$\varphi_{X_1}(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda x} \mathrm{d}F_1(x), \quad \lambda > 0,$$

and denote

$$\overline{a}_\lambda^{X_1}(n) = \frac{(-1)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \left[ \frac{1 - \varphi_{X_1}(\lambda)}{\lambda} \right], \quad n \ge 0,\ \lambda > 0,$$

and

$$\overline{\alpha}_\lambda^{X_1}(n) = \lambda^n \overline{a}_\lambda^{X_1}(n-1), \quad n \ge 1,\ \lambda > 0.$$

Similarly, for a nonnegative random variable $X_2$ with distribution function $F_2$ and survival function $\overline{F}_2 \equiv 1 - F_2$, define $\overline{\alpha}_\lambda^{X_2}(n)$. It can be shown that $\overline{\alpha}_\lambda^{X_1}$ and $\overline{\alpha}_\lambda^{X_2}$ are discrete survival functions (see the proof of the next theorem); denote the corresponding discrete random variables by $N_\lambda(X_1)$ and $N_\lambda(X_2)$. The following result gives a Laplace transform characterization of the order $\le_{\rm st}$.

**Theorem 1.A.13.** *Let $X_1$ and $X_2$ be two nonnegative random variables, and let $N_\lambda(X_1)$ and $N_\lambda(X_2)$ be as described above. Then*

$$X_1 \le_{\rm st} X_2 \Longleftrightarrow N_\lambda(X_1) \le_{\rm st} N_\lambda(X_2) \quad \textit{for all } \lambda > 0.$$

*Proof.* First suppose that $X_1 \le_{\rm st} X_2$. Select a $\lambda > 0$. Let $Z_1, Z_2, \ldots,$ be independent exponential random variables with mean $1/\lambda$. It can be shown that $\overline{\alpha}_\lambda^{X_1}(n) = P\{\sum_{i=1}^n Z_i \le X_1\}$ and that $\overline{\alpha}_\lambda^{X_2}(n) = P\{\sum_{i=1}^n Z_i \le X_2\}$. It thus follows that $N_\lambda(X_1) \le_{\rm st} N_\lambda(X_2)$.

Now suppose that $N_\lambda(X_1) \le_{\rm st} N_\lambda(X_2)$ for all $\lambda > 0$. Select an $x > 0$. Thus $\overline{\alpha}_{n/x}^{X_1}(n) \le \overline{\alpha}_{n/x}^{X_2}(n)$. Letting $n \to \infty$, one obtains $\overline{F}_1(x) \le \overline{F}_2(x)$ for all continuity points $x$ of $F_1$ and $F_2$. Therefore, $X_1 \le_{\rm st} X_2$ by (1.A.1). $\square$

The implication $\Longrightarrow$ in Theorem 1.A.13 can be generalized as follows. A family of random variables $\{Z(\theta),\ \theta \in \Theta\}$ ($\Theta$ is a subset of the real line) is said to be stochastically increasing in the usual stochastic order (denoted by $\{Z(\theta),\ \theta \in \Theta\} \in \mathrm{SI}$) if $Z(\theta) \le_{\mathrm{st}} Z(\theta')$ whenever $\theta \le \theta'$. Recall from Theorem 1.A.3(a) that if $X_1 \le_{\mathrm{st}} X_2$, then $g(X_1) \le_{\mathrm{st}} g(X_2)$ for any increasing function $g$. The following result gives a stochastic generalization of this fact.

**Theorem 1.A.14.** *If $\{Z(\theta),\ \theta \in \Theta\} \in \mathrm{SI}$ and if $X_1 \le_{\mathrm{st}} X_2$, where $X_k$ and $Z(\theta)$ are independent for $k = 1, 2$ and $\theta \in \Theta$, then $Z(X_1) \le_{\mathrm{st}} Z(X_2)$.*

Note that Theorem 1.A.14 is a restatement of Theorem 1.A.6.

Let $X$ be a random variable and denote by $X_{(-\infty,a]}$ the truncation of $X$ at $a$, that is, $X_{(-\infty,a]}$ has as its distribution the conditional distribution of $X$ given that $X \le a$. $X_{(a,\infty)}$ is similarly defined. It is simple to prove the following result. Results that are stronger than this are contained in Theorems 1.B.20, 1.B.55, and 1.C.27.

**Theorem 1.A.15.** *Let $X$ be any random variable. Then $X_{(-\infty,a]}$ and $X_{(a,\infty)}$ are increasing in $a$ in the sense of the usual stochastic order.*

An interesting example in which truncated random variables are compared is the following.

*Example 1.A.16.* Let $X^{(1)}, X^{(2)}, \ldots, X^{(n)}$ be independent and identically distributed random variables. For a fixed $t$, let $X^{(1)}_{(t,\infty)}, X^{(2)}_{(t,\infty)}, \ldots, X^{(n)}_{(t,\infty)}$ be the corresponding truncations, and assume that they are also independent and identically distributed. Then

$$\left(\max\left\{X^{(1)}, X^{(2)}, \ldots, X^{(n)}\right\}\right)_{(t,\infty)} \le_{\mathrm{st}} \max\left\{X^{(1)}_{(t,\infty)}, X^{(2)}_{(t,\infty)}, \ldots, X^{(n)}_{(t,\infty)}\right\},$$

where $\left(\max\left\{X^{(1)}, X^{(2)}, \ldots, X^{(n)}\right\}\right)_{(t,\infty)}$ denotes the corresponding truncation of $\max\left\{X^{(1)}, X^{(2)}, \ldots, X^{(n)}\right\}$. The proof consists of a straightforward verification of (1.A.2) for the compared random variables.

Let $\phi_1$ and $\phi_2$ be two functions that satisfy $\phi_1(x) \le \phi_2(x)$ for all $x \in \mathbb{R}$, and let $X$ be a random variable. Then, clearly, $\phi_1(X) \le \phi_2(X)$ almost surely. From Theorem 1.A.1 we thus obtain the following result.

**Theorem 1.A.17.** *Let $X$ be a random variable and let $\phi_1$ and $\phi_2$ be two functions that satisfy $\phi_1(x) \le \phi_2(x)$ for all $x \in \mathbb{R}$. Then*

$$\phi_1(X) \le_{\mathrm{st}} \phi_2(X).$$

*In particular, if $\phi$ is a function that satisfies $x \le [\ge]\ \phi(x)$ for all $x \in \mathbb{R}$, then $X \le_{\mathrm{st}} [\ge_{\mathrm{st}}]\ \phi(X)$.*

*Remark 1.A.18.* The set of all distribution functions on $\mathbb{R}$ is a lattice with respect to the order $\le_{\mathrm{st}}$. That is, if $X$ and $Y$ are random variables with distributions $F$ and $G$, then there exist random variables $Z$ and $W$ such that $Z \le_{\mathrm{st}} X$, $Z \le_{\mathrm{st}} Y$, $W \ge_{\mathrm{st}} X$, and $W \ge_{\mathrm{st}} Y$. Explicitly, $Z$ has the survival function $\min\{\overline{F}, \overline{G}\}$ and $W$ has the survival function $\max\{\overline{F}, \overline{G}\}$.

The next four theorems give conditions under which the corresponding spacings are ordered according to the usual stochastic order. Let $X_1, X_2, \ldots, X_m$ be any random variables with the corresponding order statistics $X_{(1)} \le X_{(2)} \le \cdots \le X_{(m)}$. Define the corresponding spacings by $U_{(i)} = X_{(i)} - X_{(i-1)}$, $i = 2, 3, \ldots, m$. When the dependence on $m$ is to be emphasized, we will denote the spacings by $U_{(i:m)}$.

**Theorem 1.A.19.** *Let* $X_1, X_2, \ldots, X_m, X_{m+1}$ *be independent and identically distributed* IFR (DFR) *random variables. Then*

$$(m-i+1)U_{(i:m)} \ge_{\rm st} [\le_{\rm st}]\ (m-i)U_{(i+1:m)}, \quad i = 2, 3, \ldots, m-1,$$

*and*

$$(m-i+2)U_{(i:m+1)} \ge_{\rm st} [\le_{\rm st}]\ (m-i+1)U_{(i:m)}, \quad i = 2, 3, \ldots, m.$$

The proof of Theorem 1.A.19 is not given here. A stronger version of the DFR part of Theorem 1.A.19 is given in Theorem 1.B.31. Some of the conclusions of Theorem 1.A.19 can be obtained under different conditions. These are stated in the next two theorems. Again, the proofs are not given. In the next two theorems we take $X_{(0)} \equiv 0$, and thus $U_{(1)} = X_{(1)}$. For the following theorem recall from page 1 the definition of Schur concavity.

**Theorem 1.A.20.** *Let* $X_1, X_2, \ldots, X_m$ *be nonnegative random variables with an absolutely continuous joint distribution function. If the joint density function of* $X_1, X_2, \ldots, X_m$ *is Schur concave* (*Schur convex*)*, then*

$$(m-i+1)U_{(i:m)} \ge_{\rm st} [\le_{\rm st}]\ (m-i)U_{(i+1:m)}, \quad i = 1, 2, \ldots, m-1.$$

**Theorem 1.A.21.** *Let* $X_1, X_2, \ldots, X_m$ *be independent exponential random variables with possibly different parameters. Then*

$$(m-i+1)U_{(i:m)} \le_{\rm st} (m-i)U_{(i+1:m)}, \quad i = 1, 2, \ldots, m-1.$$

**Theorem 1.A.22.** *Let* $X_1, X_2, \ldots, X_m$ *be independent and identically distributed random variables with a finite support, and with an increasing* [*decreasing*] *density function over that support. Then*

$$U_{(i:m)} \ge_{\rm st} [\le_{\rm st}]\ U_{(i+1:m)}, \quad i = 2, 3, \ldots, m-1.$$

The proof of Theorem 1.A.22 uses the likelihood ratio order, and therefore it is deferred to Section 1.C, Remark 1.C.3.

Note that any absolutely continuous DFR random variable has a decreasing density function. Thus we see that the assumption in the DFR part of Theorem 1.A.19 is stronger than the assumption in the decreasing part of Theorem 1.A.22, but the conclusion in the DFR part of Theorem 1.A.19 is stronger than the conclusion in the decreasing part of Theorem 1.A.22. It is of interest to compare Theorems 1.A.19–1.A.22 with Theorems 1.B.31 and 1.C.42.

From Theorem 1.A.1 it is obvious that if $X_{(1)} \le X_{(2)} \le \cdots \le X_{(m)}$ are the order statistics corresponding to the random variables $X_1, X_2, \ldots, X_m$, then $X_{(1)} \le_{\text{st}} X_{(2)} \le_{\text{st}} \cdots \le_{\text{st}} X_{(m)}$. Now let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(m)}$ be the order statistics corresponding to the random variables $X_1, X_2, \ldots, X_m$, and let $Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(m)}$ be the order statistics corresponding to the random variables $Y_1, Y_2, \ldots, Y_m$. As usual, for any distribution function $F$, we let $\overline{F} \equiv 1 - F$ denote the corresponding survival function.

**Theorem 1.A.23.** (a) *Let $X_1, X_2, \ldots, X_m$ be independent random variables with distribution functions $F_1, F_2, \ldots, F_m$, respectively. Let $Y_1, Y_2, \ldots, Y_m$ be independent and identically distributed random variables with a common distribution function $G$. Then $X_{(i)} \le_{\text{st}} Y_{(i)}$ for all $i = 1, 2, \ldots, m$ if, and only if,*

$$\prod_{j=1}^{m} F_j(x) \ge G^m(x) \quad \text{for all } x;$$

*that is, if, and only if, $X_{(m)} \le_{\text{st}} Y_{(m)}$.*

(b) *Let $X_1, X_2, \ldots, X_m$ be independent random variables with survival functions $\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_m$, respectively. Let $Y_1, Y_2, \ldots, Y_m$ be independent and identically distributed random variables with a common survival function $\overline{G}$. Then $X_{(i)} \ge_{\text{st}} Y_{(i)}$ for all $i = 1, 2, \ldots, m$ if, and only if,*

$$\prod_{j=1}^{m} \overline{F}_j(x) \ge \overline{G}^m(x) \quad \text{for all } x;$$

*that is, if, and only if, $X_{(1)} \ge_{\text{st}} Y_{(1)}$.*

The proof of Theorem 1.A.23 is not given here.

More comparisons of order statistics in the usual stochastic order can be found in Theorem 6.B.23 and in Corollary 6.B.24.

The following neat example compares a sum of independent heterogeneous exponential random variables with an Erlang random variable; it is of interest to compare it with Examples 1.B.5 and 1.C.49. We do not give the proof here.

*Example 1.A.24.* Let $X_i$ be an exponential random variable with mean $\lambda_i^{-1} > 0$, $i = 1, 2, \ldots, m$, and assume that the $X_i$'s are independent. Let $Y_i$, $i = 1, 2, \ldots, m$, be independent, identically distributed, exponential random variables with mean $\eta^{-1}$. Then

$$\sum_{i=1}^{m} X_i \ge_{\text{st}} \sum_{i=1}^{m} Y_i \Longleftrightarrow \sqrt[m]{\lambda_1 \lambda_2 \cdots \lambda_m} \le \eta.$$

The next example may be compared with Examples 1.B.6, 1.C.51, and 4.A.45.

*Example 1.A.25.* Let $X_i$ be a binomial random variable with parameters $n_i$ and $p_i$, $i = 1, 2, \ldots, m$, and assume that the $X_i$'s are independent. Let $Y$ be a binomial random variable with parameters $n$ and $p$ where $n = \sum_{i=1}^m n_i$. Then

$$\sum_{i=1}^m X_i \geq_{\rm st} Y \Longleftrightarrow p \leq \sqrt[n]{p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}},$$

and

$$\sum_{i=1}^m X_i \leq_{\rm st} Y \Longleftrightarrow 1-p \leq \sqrt[n]{(1-p_1)^{n_1}(1-p_2)^{n_2}\cdots(1-p_m)^{n_m}}.$$

The following example gives necessary and sufficient conditions for the comparison of normal random variables; it is generalized in Example 6.B.29. See related results in Examples 3.A.51 and 4.A.46.

*Example 1.A.26.* Let $X$ be a normal random variable with mean $\mu_X$ and variance $\sigma_X^2$, and let $Y$ be a normal random variable with mean $\mu_Y$ and variance $\sigma_Y^2$. Then $X \leq_{\rm st} Y$ if, and only if, $\mu_X \leq \mu_Y$ and $\sigma_X^2 = \sigma_Y^2$.

*Example 1.A.27.* Let the random variable $X$ have a unimodal density, symmetric about 0. Then

$$(X+a)^2 \leq_{\rm st} (X+b)^2 \quad \text{whenever } |a| \leq |b|.$$

*Example 1.A.28.* Let $\boldsymbol{X}$ have a multivariate normal density with mean vector $\mathbf{0}$ and variance-covariance matrix $\boldsymbol{\Sigma}_1$. Let $\boldsymbol{Y}$ have a multivariate normal density with mean vector $\mathbf{0}$ and variance-covariance matrix $\boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2$, where $\boldsymbol{\Sigma}_2$ is a nonnegative definite matrix. Then

$$\|\boldsymbol{X}\|^2 \leq_{\rm st} \|\boldsymbol{Y}\|^2,$$

where $\|\cdot\|$ denotes the Euclidean norm.

The next result involves the total time on test (TTT) transform and the observed TTT random variable. Let $F$ be the distribution function of a nonnegative random variable, and suppose, for simplicity, that 0 is the left endpoint of the support of $F$. The TTT transform associated with $F$ is defined by

$$H_F^{-1}(u) = \int_0^{F^{-1}(u)} \overline{F}(x)\mathrm{d}x, \quad 0 \leq u \leq 1, \tag{1.A.19}$$

where $\overline{F} \equiv 1-F$ is the survival function associated with $F$. The inverse, $H_F$, of the TTT transform is a distribution function. If the mean $\mu = \int_0^\infty x\mathrm{d}F(x) = \int_0^\infty \overline{F}(x)\mathrm{d}x$ is finite, then $H_F$ has support in $[0, \mu]$. If $X$ has the distribution function $F$, then let $X_{\rm ttt}$ be any random variable that has the distribution $H_F$. The random variable $X_{\rm ttt}$ is called the *observed total time on test.*

**Theorem 1.A.29.** *Let $X$ and $Y$ be two nonnegative random variables. Then*

$$X \leq_{\rm st} Y \Longrightarrow X_{\rm ttt} \leq_{\rm st} Y_{\rm ttt}.$$

See related results in Theorems 3.B.1, 4.A.44, 4.B.8, 4.B.9, and 4.B.29.

### 1.A.5 Some properties in reliability theory

Recall from page 1 the definitions of the IFR, DFR, NBU, and NWU properties. The next result characterizes random variables that have these properties by means of the usual stochastic order. The statements in the next theorem follow at once from the definitions. Recall from Section 1.A.3 that for any random variable $Z$ and an event $A$ we denote by $[Z|A]$ any random variable that has as its distribution the conditional distribution of $Z$ given $A$.

**Theorem 1.A.30.** (a) *The random variable $X$ is* IFR [DFR] *if, and only if,* $[X - t|X > t] \geq_{\rm st} [\leq_{\rm st}] [X - t'|X > t']$ *whenever $t \leq t'$.*
(b) *The nonnegative random variable $X$ is* NBU [NWU] *if, and only if,* $X \geq_{\rm st} [\leq_{\rm st}] [X - t|X > t]$ *for all $t > 0$.*

Note that if $X$ is the lifetime of a device, then $[X - t|X > t]$ is the residual life of such a device with age $t$. Theorem 1.A.30(a), for example, characterizes IFR and DFR random variables by the monotonicity of their residual lives with respect to the order $\leq_{\rm st}$. Theorem 1.A.30 should be compared to Theorem 1.B.38, where a similar characterization is given. Some multivariate analogs of Theorem 1.A.30(a) are used in Section 6.B.6 to introduce some multivariate IFR notions.

For a nonnegative random variable $X$ with a finite mean, let $A_X$ denote the corresponding asymptotic equilibrium age. That is, if the distribution function of $X$ is $F$, then the distribution function $F_{\rm e}$ of $A_X$ is defined by

$$F_{\rm e}(x) = \frac{1}{EX}\int_0^x \overline{F}(y)\mathrm{d}y, \quad x \geq 0, \tag{1.A.20}$$

where $\overline{F} \equiv 1 - F$ is the corresponding survival function. Recall from page 1 the definitions of the NBUE and the NWUE properties. The following result is immediate.

**Theorem 1.A.31.** *The nonnegative random variable $X$ with finite mean is* NBUE [NWUE] *if, and only if,* $X \geq_{\rm st} [\leq_{\rm st}] A_X$.

Another characterization of NBUE random variables is the following. Recall from Section 1.A.4 the definition of the observed total time on test random variable $X_{\rm ttt}$.

**Theorem 1.A.32.** *Let $X$ be a nonnegative random variable with finite mean $\mu$. Then $X$ is* NBUE *if, and only if,*

$$X_{\rm ttt} \geq_{\rm st} \mathcal{U}(0, \mu),$$

*where $\mathcal{U}(0, \mu)$ denotes a uniform random variable on $(0, \mu)$.*

Let $X$ be a nonnegative random variable with finite mean and distribution function $F$, and let $A_X$ be the corresponding asymptotic equilibrium age having the distribution function $F_{\rm e}$ given in (1.A.20). The requirement

$$X \geq_{\rm st} [A_X - t | A_X > t] \quad \text{for all } t \geq 0, \tag{1.A.21}$$

has been used in the literature as a way to define an aging property of the lifetime $X$. It turns out that this aging property is equivalent to the new better than used in convex ordering (NBUC) notion that is defined in (4.A.31) in Chapter 4.

## 1.B The Hazard Rate Order

### 1.B.1 Definition and equivalent conditions

If $X$ is a random variable with an absolutely continuous distribution function $F$, then the hazard rate of $X$ at $t$ is defined as $r(t) = ({\rm d}/{\rm d}t)(-\log(1-F(t)))$. The hazard rate can alternatively be expressed as

$$r(t) = \lim_{\Delta t \downarrow 0} \frac{P\{t < X \leq t + \Delta t | X > t\}}{\Delta t} = \frac{f(t)}{\overline{F}(t)}, \quad t \in \mathbb{R}, \tag{1.B.1}$$

where $\overline{F} \equiv 1 - F$ is the survival function and $f$ is the corresponding density function. As can be seen from (1.B.1), the hazard rate $r(t)$ can be thought of as the intensity of failure of a device, with a random lifetime $X$, at time $t$. Clearly, the higher the hazard rate is the smaller $X$ should be stochastically. This is the motivation for the order discussed in this section.

Let $X$ and $Y$ be two nonnegative random variables with absolutely continuous distribution functions and with hazard rate functions $r$ and $q$, respectively, such that

$$r(t) \geq q(t), \quad t \in \mathbb{R}. \tag{1.B.2}$$

Then $X$ is said to be *smaller than $Y$ in the hazard rate order* (denoted as $X \leq_{\rm hr} Y$).

Although the hazard rate order is usually applied to random lifetimes (that is, nonnegative random variables), definition (1.B.2) may also be used to compare more general random variables. In fact, even the absolute continuity, which is required in (1.B.2), is not really needed. It is easy to verify that (1.B.2) holds if, and only if,

$$\frac{\overline{G}(t)}{\overline{F}(t)} \quad \text{increases in } t \in (-\infty, \max(u_X, u_Y)) \tag{1.B.3}$$

(here $a/0$ is taken to be equal to $\infty$ whenever $a > 0$). Here $F$ denotes the distribution function of $X$ and $G$ denotes the distribution function of $Y$, and $u_X$ and $u_Y$ denote the corresponding right endpoints of the supports of $X$ and of $Y$. Equivalently, (1.B.3) can be written as

$$\overline{F}(x)\overline{G}(y) \geq \overline{F}(y)\overline{G}(x) \quad \text{for all } x \leq y. \tag{1.B.4}$$

Thus (1.B.3) or (1.B.4) can be used to define the order $X \le_{\mathrm{hr}} Y$ even if $X$ and/or $Y$ do not have absolutely continuous distributions. A useful further condition, which is equivalent to $X \le_{\mathrm{hr}} Y$ when $X$ and $Y$ have absolutely continuous distributions with densities $f$ and $g$, respectively, is the following:

$$\frac{f(x)}{\overline{F}(y)} \ge \frac{g(x)}{\overline{G}(y)} \quad \text{for all } x \le y. \tag{1.B.5}$$

Rewriting (1.B.4) as

$$\frac{\overline{F}(t+s)}{\overline{F}(t)} \le \frac{\overline{G}(t+s)}{\overline{G}(t)} \quad \text{for all } s \ge 0 \text{ and all } t,$$

it is seen that $X \le_{\mathrm{hr}} Y$ if, and only if,

$$P\{X - t > s \big| X > t\} \le P\{Y - t > s \big| Y > t\} \quad \text{for all } s \ge 0 \text{ and all } t; \tag{1.B.6}$$

that is, if, and only if, the residual lives of $X$ and $Y$ at time $t$ are ordered in the sense $\le_{\mathrm{st}}$ for all $t$. Equivalently, (1.B.6) can be written as

$$[X \big| X > t] \le_{\mathrm{st}} [Y \big| Y > t] \quad \text{for all } t. \tag{1.B.7}$$

Substituting $u = \overline{F}^{-1}(t)$ in (1.B.3) shows that $X \le_{\mathrm{hr}} Y$ if, and only if,

$$\frac{\overline{G}\,\overline{F}^{-1}(u)}{u} \ge \frac{\overline{G}\,\overline{F}^{-1}(v)}{v} \quad \text{for all } 0 < u \le v < 1.$$

Simple manipulations show that the latter condition is equivalent to

$$\frac{1 - FG^{-1}(1-u)}{u} \le \frac{1 - FG^{-1}(1-v)}{v} \quad \text{for all } 0 < u \le v < 1. \tag{1.B.8}$$

For discrete random variables that take on values in $\mathbb{N}$ the definition of $\le_{\mathrm{hr}}$ can be written in two different ways. Let $X$ and $Y$ be such random variables. We denote $X \le_{\mathrm{hr}} Y$ if

$$\frac{P\{X = n\}}{P\{X \ge n\}} \ge \frac{P\{Y = n\}}{P\{Y \ge n\}}, \quad n \in \mathbb{N}. \tag{1.B.9}$$

Equivalently, $X \le_{\mathrm{hr}} Y$ if

$$\frac{P\{X = n\}}{P\{X > n\}} \ge \frac{P\{Y = n\}}{P\{Y > n\}}, \quad n \in \mathbb{N}.$$

The discrete analog of (1.B.4) is that (1.B.9) holds if, and only if,

$$P\{X \ge n_1\}P\{Y \ge n_2\} \ge P\{X \ge n_2\}P\{Y \ge n_1\} \quad \text{for all } n_1 \le n_2. \tag{1.B.10}$$

In a similar manner (1.B.3) and (1.B.5) can be modified in the discrete case. Unless stated otherwise, we consider only random variables with absolutely continuous distribution functions in the following sections.

### 1.B.2 The relation between the hazard rate and the usual stochastic orders

By setting $x = -\infty$ in (1.B.4) (or $n_1 = -\infty$ in (1.B.10)), and then using (1.A.1), we obtain the following result.

**Theorem 1.B.1.** *If $X$ and $Y$ are two random variables such that $X \leq_{\rm hr} Y$, then $X \leq_{\rm st} Y$.*

### 1.B.3 Closure properties and some characterizations

Let $\phi$ be a strictly increasing function with inverse $\phi^{-1}$. If $X$ has the survival function $\overline{F}$, then $\phi(X)$ has the survival function $\overline{F}\phi^{-1}$. Similarly, if $Y$ has the survival function $\overline{G}$, then $\phi(Y)$ has the survival function $\overline{G}\phi^{-1}$. If $X \leq_{\rm hr} Y$, then from (1.B.3) it follows that

$$\frac{\overline{G}\phi^{-1}(t)}{\overline{F}\phi^{-1}(t)} \quad \text{increases in } t \text{ over } \{t : \overline{G}\phi^{-1}(t) > 0\}.$$

We have thus shown an important special case of the next theorem. When $\phi$ is *just* increasing (rather than *strictly* increasing) the result is still true, but the above simple argument is no longer sufficient for its proof.

**Theorem 1.B.2.** *If $X \leq_{\rm hr} Y$, and if $\phi$ is any increasing function, then $\phi(X) \leq_{\rm hr} \phi(Y)$.*

In general, if $X_1 \leq_{\rm hr} Y_1$ and $X_2 \leq_{\rm hr} Y_2$, where $X_1$ and $X_2$ are independent random variables and $Y_1$ and $Y_2$ are also independent random variables, then it is not necessarily true that $X_1 + X_2 \leq_{\rm hr} Y_1 + Y_2$. However, if these random variables are IFR, then it is true. This is shown in Theorem 1.B.4, but first we state and prove the following lemma, which is of independent interest.

**Lemma 1.B.3.** *If the random variables $X$ and $Y$ are such that $X \leq_{\rm hr} Y$ and if $Z$ is an* IFR *random variable independent of $X$ and $Y$, then*

$$X + Z \leq_{\rm hr} Y + Z. \tag{1.B.11}$$

*Proof.* Denote by $f_W$ and $\overline{F}_W$ the density function and the survival function of any random variable $W$. Note that, for $x \leq y$,

$$
\begin{aligned}
&\overline{F}_{X+Z}(x)\overline{F}_{Y+Z}(y) - \overline{F}_{X+Z}(y)\overline{F}_{Y+Z}(x) \\
&= \int_v \int_{u \geq v} \big[f_X(u)\overline{F}_Z(x-u)f_Y(v)\overline{F}_Z(y-v) \\
&\qquad\qquad\qquad + f_X(v)\overline{F}_Z(x-v)f_Y(u)\overline{F}_Z(y-u)\big]\mathrm{d}u\mathrm{d}v \\
&- \int_v \int_{u \geq v} \big[f_X(u)\overline{F}_Z(y-u)f_Y(v)\overline{F}_Z(x-v) \\
&\qquad\qquad\qquad + f_X(v)\overline{F}_Z(y-v)f_Y(u)\overline{F}_Z(x-u)\big]\mathrm{d}u\mathrm{d}v \\
&= \int_v \int_{u \geq v} \big[\overline{F}_X(u)f_Y(v) - f_X(v)\overline{F}_Y(u)\big] \\
&\qquad\qquad \times \big[\overline{F}_Z(y-v)f_Z(x-u) - f_Z(y-u)\overline{F}_Z(x-v)\big]\mathrm{d}u\mathrm{d}v,
\end{aligned}
$$

where the second equality is obtained by integration by parts with respect to $u$ and by collection of terms. Since $X \leq_{\rm hr} Y$ it follows from (1.B.5) that the expression within the first set of brackets in the last integral is nonpositive. Since $Z$ is IFR it can be verified that the quantity in the second pair of brackets in the last integral is also nonpositive. Therefore, the integral is nonnegative. This proves (1.B.11). □

The above proof is very similar to the proof that a convolution of two independent IFR random variables is IFR. In fact, this convolution result can be shown to be a consequence of Lemma 1.B.3; see Corollary 1.B.39 in Section 1.B.5.

**Theorem 1.B.4.** *Let $(X_i, Y_i)$, $i = 1, 2, \ldots, m$, be independent pairs of random variables such that $X_i \leq_{\rm hr} Y_i$, $i = 1, 2, \ldots, m$. If $X_i, Y_i$, $i = 1, 2, \ldots, m$, are all* IFR, *then*

$$\sum_{i=1}^{m} X_i \leq_{\rm hr} \sum_{i=1}^{m} Y_i.$$

*Proof.* Repeated application of (1.B.11), using the closure property of IFR under convolution, yields the desired result. □

The following neat example compares a sum of independent heterogeneous exponential random variables with an Erlang random variable; it is of interest to compare it with Examples 1.A.24 and 1.C.49. We do not give the proof here.

*Example 1.B.5.* Let $X_i$ be an exponential random variable with mean $\lambda_i^{-1} > 0$, $i = 1, 2, \ldots, m$, and assume that the $X_i$'s are independent. Let $Y_i$, $i = 1, 2, \ldots, m$, be independent, identically distributed, exponential random variables with mean $\eta^{-1}$. Then

$$\sum_{i=1}^{m} X_i \geq_{\rm hr} \sum_{i=1}^{m} Y_i \Longleftrightarrow \sqrt[m]{\lambda_1 \lambda_2 \cdots \lambda_m} \leq \eta.$$

The next example may be compared with Examples 1.A.25, 1.C.51, and 4.A.45.

*Example 1.B.6.* Let $X_i$ be a binomial random variable with parameters $n_i$ and $p_i$, $i = 1, 2, \ldots, m$, and assume that the $X_i$'s are independent. Let $Y$ be a binomial random variable with parameters $n$ and $p$ where $n = \sum_{i=1}^m n_i$. Then

$$\sum_{i=1}^{m} X_i \ge_{\mathrm{hr}} Y \Longleftrightarrow p \le \frac{n}{\sum_{i=1}^m (n_i/p_i)},$$

and

$$\sum_{i=1}^{m} X_i \le_{\mathrm{hr}} Y \Longleftrightarrow 1 - p \le \frac{n}{\sum_{i=1}^m (n_i/(1-p_i))}.$$

A hazard rate order comparison of random sums is given in the following result.

**Theorem 1.B.7.** *Let $\{X_i,\ i = 1, 2, \ldots\}$ be a sequence of nonnegative* IFR *independent random variables. Let $M$ and $N$ be two discrete positive integer-valued random variables such that $M \le_{\mathrm{hr}} N$ (in the sense of* (1.B.9) *or* (1.B.10))*, and assume that $M$ and $N$ are independent of the $X_i$'s. Then*

$$\sum_{i=1}^{M} X_i \le_{\mathrm{hr}} \sum_{i=1}^{N} X_i.$$

The hazard rate order (unlike the usual stochastic order; see Theorem 1.A.3(d)) does not have the property of being simply closed under mixtures. However, under quite strong conditions the order $\le_{\mathrm{hr}}$ is closed under mixtures. This is shown in the next theorem.

**Theorem 1.B.8.** *Let $X$, $Y$, and $\Theta$ be random variables such that $[X\big|\Theta = \theta] \le_{\mathrm{hr}} [Y\big|\Theta = \theta']$ for all $\theta$ and $\theta'$ in the support of $\Theta$. Then $X \le_{\mathrm{hr}} Y$.*

*Proof.* Select a $\theta$ and a $\theta'$ in the support of $\Theta$. Let $\overline{F}(\cdot\big|\theta)$, $\overline{G}(\cdot\big|\theta)$, $\overline{F}(\cdot\big|\theta')$, and $\overline{G}(\cdot\big|\theta')$ be the survival functions of $[X\big|\Theta = \theta]$, $[Y\big|\Theta = \theta]$, $[X\big|\Theta = \theta']$, and $[Y\big|\Theta = \theta']$, respectively. For simplicity assume that these random variables have densities which we denote by $f(\cdot\big|\theta)$, $g(\cdot\big|\theta)$, $f(\cdot\big|\theta')$, and $g(\cdot\big|\theta')$, respectively. It is sufficient to show that for $\alpha \in (0, 1)$ we have

$$\frac{\alpha f(t\big|\theta) + (1-\alpha) f(t\big|\theta')}{\alpha \overline{F}(t\big|\theta) + (1-\alpha)\overline{F}(t\big|\theta')} \ge \frac{\alpha g(t\big|\theta) + (1-\alpha) g(t\big|\theta')}{\alpha \overline{G}(t\big|\theta) + (1-\alpha)\overline{G}(t\big|\theta')} \quad \text{for all } t \ge 0. \tag{1.B.12}$$

This is an inequality of the form

$$\frac{a+b}{c+d} \ge \frac{w+x}{y+z},$$

where all eight variables are nonnegative and by the assumptions of the theorem they satisfy

$$\frac{a}{c} \geq \frac{w}{y}, \quad \frac{a}{c} \geq \frac{x}{z}, \quad \frac{b}{d} \geq \frac{w}{y}, \quad \text{and} \quad \frac{b}{d} \geq \frac{x}{z}.$$

It is easy to verify that the latter four inequalities imply the former one, completing the proof of the theorem. □

It should be pointed out, however, that mixtures, of distributions which are ordered by the hazard rate order, are ordered by the usual stochastic order. That is, if $X$, $Y$, and $\Theta$ are random variables such that $[X\big|\Theta = \theta] \leq_{\rm hr} [Y\big|\Theta = \theta]$ for all $\theta$ in the support of $\Theta$, then $X \leq_{\rm st} Y$. This follows from a (conditional) application of Theorem 1.B.1, combined with the fact that the usual stochastic order is closed under mixtures (Theorem 1.A.3(d)).

In order to state the next characterization we define the following class of bivariate functions.

$$\mathcal{G}_{\rm hr} = \big\{\phi : \mathbb{R}^2 \to \mathbb{R} : \phi(x,y) \text{ is increasing in } x, \text{ for each } y, \text{ on } \{x \geq y\}, \\ \text{and is decreasing in } y, \text{ for each } x, \text{ on } \{y \geq x\}\big\}.$$

**Theorem 1.B.9.** *Let $X$ and $Y$ be independent random variables. Then $X \leq_{\rm hr} Y$ if, and only if,*

$$\phi(X,Y) \leq_{\rm st} \phi(Y,X) \quad \textit{for all } \phi \in \mathcal{G}_{\rm hr}. \tag{1.B.13}$$

*Proof.* Suppose that (1.B.13) holds. Select an $x$ and a $y$ such that $x \geq y$. Let $\phi(u,v) = I_{\{u \geq x, v \geq y\}}$, where $I_A$ denotes the indicator function of the set $A$. It is easy to see that $\phi(u,v)$ is increasing in $u$. In addition, for a fixed $u$ and $v$ such that $v \geq u$, we have that $\phi(u,v) = 1$ if $u \geq x$ and $\phi(u,v) = 0$ if $u < x$. Therefore, $\phi \in \mathcal{G}_{\rm hr}$. Hence,

$$\overline{F}(y)\overline{G}(x) = E\phi(Y,X) \geq E\phi(X,Y) = \overline{F}(x)\overline{G}(y) \quad \text{whenever } x \geq y,$$

where $\overline{F}$ and $\overline{G}$ are the survival functions of $X$ and $Y$, respectively. Therefore, by (1.B.4), $X \leq_{\rm hr} Y$.

Conversely, assume that $X \leq_{\rm hr} Y$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be an increasing function and let $\phi \in \mathcal{G}_{\rm hr}$. Denote $a(x,y) = \psi(\phi(x,y)) - \psi(\phi(y,x))$. For simplicity assume that $a$ is differentiable and that $X$ and $Y$ have densities that we denote by $f$ and $g$, respectively (otherwise, approximation arguments can be used). Then

$$\begin{aligned} Ea(X,Y) &= \int_{y=-\infty}^{\infty} \int_{x \geq y} a(x,y)[f(x)g(y) - f(y)g(x)]\mathrm{d}x\mathrm{d}y \\ &= \int_{y=-\infty}^{\infty} \int_{x \geq y} \frac{\partial}{\partial x} a(x,y)\Big[\overline{F}(x)g(y) - f(y)\overline{G}(x)\Big]\mathrm{d}x\mathrm{d}y \leq 0, \end{aligned}$$

where the second equality follows from integration by parts, and the inequality follows from $X \le_{\rm hr} Y$, the fact that $a(x,y)$ increases in $x$ for all $x \ge y$, and (1.B.5). □

The next result is a similar characterization. It uses the notation of Theorem 1.A.10, and their comparison is of interest. The proof of the following theorem is omitted.

**Theorem 1.B.10.** *Let $X$ and $Y$ be two independent random variables. Then $X \le_{\rm hr} Y$ if, and only if,*

$$E\phi_1(X,Y) \le E\phi_2(X,Y)$$

*for all $\phi_1$ and $\phi_2$ such that, for each $x$, $\Delta\phi_{21}(x,y)$ increases in $y$ on $\{y \ge x\}$, and such that $\Delta\phi_{21}(x,y) \ge -\Delta\phi_{21}(y,x)$ whenever $x \le y$.*

A further similar characterization is given in Theorem 4.A.36. The next result describes another characterization of the order $\le_{\rm hr}$.

**Theorem 1.B.11.** *Let $X$ and $Y$ be two, absolutely continuous or discrete, independent random variables. Then $X \le_{\rm hr} Y$ if, and only if,*

$$[X \big| \min(X,Y) = z] \le_{\rm hr} [Y \big| \min(X,Y) = z] \quad \textit{for all } z. \tag{1.B.14}$$

*Also, $X \le_{\rm hr} Y$ if, and only if,*

$$[X \big| \min(X,Y) = z] \le_{\rm st} [Y \big| \min(X,Y) = z] \quad \textit{for all } z. \tag{1.B.15}$$

*Proof.* First suppose that $X$ and $Y$ are absolutely continuous. Denote the survival functions of $X$ and $Y$ by $\overline{F}$ and $\overline{G}$, respectively, and denote the corresponding density functions by $f$ and $g$. Then

$$P[X > x \big| \min(X,Y) = z] = \begin{cases} 1, & \text{if } x < z, \\ \frac{\overline{F}(x)g(z)}{f(z)\overline{G}(z)+g(z)\overline{F}(z)}, & \text{if } x \ge z, \end{cases} \tag{1.B.16}$$

and

$$P[Y > x \big| \min(X,Y) = z] = \begin{cases} 1, & \text{if } x < z, \\ \frac{\overline{G}(x)f(z)}{f(z)\overline{G}(z)+g(z)\overline{F}(z)}, & \text{if } x \ge z. \end{cases} \tag{1.B.17}$$

Therefore

$$\frac{P[Y > x | \min(X,Y) = z]}{P[X > x | \min(X,Y) = z]} = \begin{cases} 1, & \text{if } x < z, \\ \frac{\overline{G}(x)}{\overline{F}(x)} \cdot \frac{f(z)}{g(z)}, & \text{if } x \ge z. \end{cases} \tag{1.B.18}$$

If $X \le_{\rm hr} Y$, then $\frac{\overline{G}(z)}{\overline{F}(z)} \cdot \frac{f(z)}{g(z)} \ge 1$, and $\frac{\overline{G}(x)}{\overline{F}(x)}$ is increasing in $x$. Thus (1.B.18) is increasing in $x$, and (1.B.14) follows. Obviously (1.B.15) follows from (1.B.14).

Now suppose that (1.B.15) holds. Then from (1.B.16) and (1.B.17) we get that $\overline{F}(x)g(z) \le \overline{G}(x)f(z)$ for all $x \ge z$. Therefore $X \le_{\rm hr} Y$ by (1.B.5).

The proof when $X$ and $Y$ are discrete is similar. □

Some related characterizations are given in the next result.

**Theorem 1.B.12.** *Let $X$ and $Y$ be two independent random variables. The following conditions are equivalent:*

(a) $X \le_{\rm hr} Y$.
(b) $E[\alpha(X)]E[\beta(Y)] \le E[\alpha(Y)]E[\beta(X)]$ *for all functions $\alpha$ and $\beta$ for which the expectations exist and such that $\beta$ is nonnegative and $\alpha/\beta$ and $\beta$ are increasing.*
(c) *For any two increasing functions $a$ and $b$ such that $b$ is nonnegative, if $E[a(X)b(X)] = 0$, then $E[a(Y)b(Y)] \ge 0$.*

*Proof.* Assume (a). Let $\alpha$ and $\beta$ be as in (b). Define $\phi_1(x,y) = \alpha(x)\beta(y)$ and $\phi_2(x,y) = \alpha(y)\beta(x)$. Then $\Delta\phi_{21}(x,y) = \phi_2(x,y) - \phi_1(x,y) = \beta(x)\beta(y) \cdot [\alpha(y)/\beta(y) - \alpha(x)/\beta(x)]$, which is increasing in $y$. Note that $\Delta\phi_{21}(x,y) + \Delta\phi_{21}(y,x) = 0$. Condition (b) now follows from Theorem 1.B.10.

Assume (b). By taking, for some $u \le v$, $\alpha(x) = I_{(v,\infty)}(x)$ and $\beta(x) = I_{(u,\infty)}(x)$ in (b) one obtains (1.B.4), from which (a) follows.

Assume (b). Let $a$ and $b$ be two increasing functions such that $b$ is nonnegative and such that $E[a(X)b(X)] = 0$. Define $\beta(x) = b(x)$ and $\alpha(x) = a(x)b(x)$. Substitution in (b) yields $E[a(Y)b(Y)] \ge 0$; that is, (c) holds.

Assume (c). Let $\alpha$ and $\beta$ be as in (b). Denote $c = E[\alpha(X)]/E[\beta(X)]$. Define $a(x) = \alpha(x)/\beta(x) - c$ and $b(x) = \beta(x)$. Then $E[a(X)b(X)] = 0$. So, by (c), $E[a(Y)b(Y)] \ge 0$. But the latter reduces to $E[\alpha(X)]E[\beta(Y)] \le E[\alpha(Y)]E[\beta(X)]$, and this establishes (b). □

*Example 1.B.13.* Let $\{N(t),\ t \ge 0\}$ be a nonhomogeneous Poisson process with mean function $\Lambda$ (that is, $\Lambda(t) \equiv E[N(t)]$, $t \ge 0$). Let $T_1, T_2, \ldots$ be the successive epoch times, and let $X_n \equiv T_n - T_{n-1}$, $n = 1, 2 \ldots$ (where $T_0 \equiv 0$), be the corresponding inter-epoch times. The survival function of $T_n$ is given by $P\{T_n > t\} = \sum_{i=0}^{n-1} \frac{(\Lambda(t))^i}{i!} \cdot \mathrm{e}^{-\Lambda(t)}$, $t \ge 0$, $n = 1, 2, \ldots$. It is easy to verify that $\frac{P\{T_{n+1}>t\}}{P\{T_n>t\}}$ is increasing in $t \ge 0$, $n = 1, 2, \ldots$, and thus, by (1.B.3),

$$T_n \le_{\rm hr} T_{n+1}, \quad n = 1, 2, \ldots. \tag{1.B.19}$$

A result that is stronger than (1.B.19) is given in Example 1.C.47.

If we denote by $F_n$ the distribution function of $T_n$, then

$$\begin{aligned} P\{X_{n+1} > t\} &= \int_0^\infty P\{T_{n+1} - T_n > t \big| T_n = u\} \mathrm{d}F_n(u) \\ &= \int_0^\infty \exp\{-[\Lambda(t+u) - \Lambda(u)]\} \mathrm{d}F_n(u) \\ &= E[\exp\{-[\Lambda(t+T_n) - \Lambda(T_n)]\}, \quad n = 0, 1, \ldots. \end{aligned}$$

Fix $t_1 \le t_2$ and let $\alpha(x) \equiv \exp\{-[\Lambda(t_2+x) - \Lambda(x)]\}$ and $\beta(x) \equiv \exp\{-[\Lambda(t_1+x) - \Lambda(x)]\}$. Note that if $\Lambda$ is concave, then $\alpha(x)/\beta(x)$ is increasing. Thus, by Theorem 1.B.12(b), if $\Lambda$ is concave, then

$$\frac{P\{X_{n+1} > t_2\}}{P\{X_n > t_2\}} = \frac{E[\alpha(T_n)]}{E[\alpha(T_{n-1})]} \geq \frac{E[\beta(T_n)]}{E[\beta(T_{n-1})]} = \frac{P\{X_{n+1} > t_1\}}{P\{X_n > t_1\}},$$
$$n = 1, 2, \ldots.$$

It follows, by (1.B.3), that

$$X_n \leq_{\text{hr}} X_{n+1}, \quad n = 1, 2, \ldots.$$

It can be shown in a similar manner that if $\Lambda$ is convex, then $X_n \geq_{\text{hr}} X_{n+1}$, $n = 1, 2, \ldots$.

As another example of the use of Theorem 1.B.12 consider an increasing convex function $H$ such that $H(0) = 0$. Let $X$ and $Y$ be nonnegative random variables such that $X \leq_{\text{hr}} Y$. Then

$$\frac{E[H(X)]}{E[X]} \leq \frac{E[H(Y)]}{E[Y]}.$$

Rather than using Theorem 1.B.12, one can also obtain the above inequality from (2.B.5) in Chapter 2, and from the fact that the hazard rate order implies the hmrl order (which is discussed there).

Other characterizations of the order $\leq_{\text{hr}}$ can be found in Theorems 2.A.6 and 5.A.22.

Consider now a family of distribution functions $\{G_\theta,\ \theta \in \mathcal{X}\}$ where $\mathcal{X}$ is a subset of the real line. As in Section 1.A.3 let $X(\theta)$ denote a random variable with distribution function $G_\theta$. For any random variable $\Theta$ with support in $\mathcal{X}$, and with distribution function $F$, let us denote by $X(\Theta)$ a random variable with distribution function $H$ given by

$$H(y) = \int_{\mathcal{X}} G_\theta(y) \mathrm{d}F(\theta), \quad y \in \mathbb{R}.$$

The following result generalizes both Theorems 1.B.2 and 1.B.8, just as Theorem 1.A.6 generalized both parts (a) and (c) of Theorem 1.A.3.

**Theorem 1.B.14.** *Consider a family of distribution functions $\{G_\theta,\ \theta \in \mathcal{X}\}$ as above. Let $\Theta_1$ and $\Theta_2$ be two random variables with supports in $\mathcal{X}$ and distribution functions $F_1$ and $F_2$, respectively. Let $Y_1$ and $Y_2$ be two random variables such that $Y_i =_{\text{st}} X(\Theta_i)$, $i = 1, 2$, that is, suppose that the survival function of $Y_i$ is given by*

$$\overline{H}_i(y) = \int_{\mathcal{X}} \overline{G}_\theta(y) \mathrm{d}F_i(\theta), \quad y \in \mathbb{R},\ i = 1, 2.$$

*If*

$$X(\theta) \leq_{\text{hr}} X(\theta') \quad \textit{whenever } \theta \leq \theta', \tag{1.B.20}$$

*and if*

$$\Theta_1 \leq_{\text{hr}} \Theta_2, \tag{1.B.21}$$

*then*

$$Y_1 \leq_{\text{hr}} Y_2. \tag{1.B.22}$$

*Proof.* Assumption (1.B.20) means that $\overline{G}_\theta(y)$ is TP$_2$ (totally positive of order 2) as a function of $\theta \in \mathcal{X}$ and of $y \in \mathbb{R}$ (that is, $\overline{G}_\theta(y)\overline{G}_{\theta'}(y') \geq \overline{G}_\theta(y')\overline{G}_{\theta'}(y)$ whenever $y \leq y'$ and $\theta \leq \theta'$). Assumption (1.B.21) means that $\overline{F}_i(\theta)$, as a function of $i \in \{1,2\}$ and of $\theta \in \mathcal{X}$, is TP$_2$. Also, from Theorem 1.B.1 it follows that $\overline{G}_\theta(y)$ is increasing in $\theta$. Therefore, by Theorem 2.1 of Joag-Dev, Kochar, and Proschan [259], $\overline{H}_i(y)$ is TP$_2$ in $i \in \{1,2\}$ and $y \in \mathbb{R}$. That gives (1.B.22). $\square$

The following example shows an interesting and useful application of Theorem 1.B.14

*Example 1.B.15.* Let $\{X_n^i,\ n \geq 0\}$ be a Markov chain with state space $\{1, 2, \ldots, M\}$ ($M$ can be infinity) and transition matrix $\boldsymbol{P}$, which starts from state $i$; that is, $X_0^i = i$. If $X_1^i \leq_{\text{hr}} X_1^{i'}$ for all $i \leq i'$, then

(a) $I_1 \leq_{\text{hr}} I_2$ implies that $X_n^{I_1} \leq_{\text{hr}} X_n^{I_2}$ for all $n \geq 0$, and
(b) $X_n^1 \leq_{\text{hr}} X_{n'}^1$ whenever $n \leq n'$.

In order to prove (a), first note that the result is trivial for $n = 0$. Suppose that the result is true for $n = k$. Define $Y(i) = X_1^i$. By the Markov property, we have $X_{k+1}^i =_{\text{st}} Y(X_k^i)$ for all $i$. By induction, $X_k^{I_1} \leq_{\text{hr}} X_k^{I_2}$. In particular, $Y(X_k^i) \leq_{\text{hr}} Y(X_k^{i'})$ for all $i \leq i'$. Therefore, from Theorem 1.B.14 we get $X_{k+1}^{I_1} = Y(X_k^{I_1}) \leq_{\text{hr}} Y(X_k^{I_2}) = X_{k+1}^{I_2}$.

In order to prove (b), note that $X_0^1 = 1 \leq_{\text{hr}} X_1^1$. So, by (a) and the Markov property we have $X_n^1 \leq_{\text{hr}} X_n^{X_1^1} =_{\text{st}} X_{n+1}^1$.

The following example shows an application of Theorem 1.B.14 in the area of Bayesian imperfect repair.

*Example 1.B.16.* Let $\Theta_1$ and $\Theta_2$ be two random variables as in Example 1.A.7. Let $\overline{G}_\theta$, $X(\theta)$, $Y_1$, and $Y_2$ also be as in Example 1.A.7. Note that (1.B.20) holds because $\overline{K}^{1-\theta'}(y)/\overline{K}^{1-\theta}(y)$ is increasing in $y$ whenever $0 < \theta \leq \theta' \leq 1$. Thus, if $\Theta_1 \leq_{\text{hr}} \Theta_2$, then $Y_1 \leq_{\text{hr}} Y_2$.

It is of interest to compare Example 1.B.16 to Example 5.B.13 which deals with random minima and maxima.

The next example deals with the same proportional hazard model as in Example 1.B.16; however, for convenience we change the notation.

*Example 1.B.17.* Let $\Theta$ and $X$ be two nonnegative random variables with distribution function $F$ and $G$, respectively. Let $Y$ have the survival function $\overline{H}$ defined as

$$\overline{H}(y) = \int_0^\infty \overline{G}^\theta(y)\mathrm{d}F(\theta), \quad y \geq 0.$$

Suppose that $G$ is absolutely continuous with hazard rate function $r$. Then $H$ is also absolutely continuous, and we denote its hazard rate function by $q$. We will now show that if $E\Theta \leq 1$, then $X \leq_{\text{hr}} Y$. In order to see it, write $\overline{H}(y) = M(\log \overline{G}(y))$, where $M$ is the moment generating function of $\Theta$. Differentiating $-\log \overline{H}(y)$ we obtain

$$\begin{aligned} q(y) = -\frac{\mathrm{d}}{\mathrm{d}y} \log \overline{H}(y) &= r(y) \frac{M'(\log \overline{G}(y))}{M(\log \overline{G}(y))} \\ &= r(y) \frac{E\Theta \mathrm{e}^{\Theta \log \overline{G}(y)}}{E\mathrm{e}^{\Theta \log \overline{G}(y)}} \le r(y) \frac{E\Theta E\mathrm{e}^{\Theta \log \overline{G}(y)}}{E\mathrm{e}^{\Theta \log \overline{G}(y)}} = r(y) E\Theta \le r(y), \end{aligned}$$

where the first inequality follows from Chebyshev's Inequality (that is, $\mathrm{Cov}(\Theta, \mathrm{e}^{\Theta \log \overline{G}(y)}) \le 0$), and the second inequality follows from $E\Theta \le 1$. The stated result now follows from (1.B.2).

The following result gives a Laplace transform characterization of the order $\le_{\mathrm{hr}}$. It should be compared with Theorem 1.A.13.

**Theorem 1.B.18.** *Let $X_1$ and $X_2$ be two nonnegative random variables, and let $N_\lambda(X_1)$ and $N_\lambda(X_2)$ be as described in Theorem* 1.A.13*. Then*

$$X_1 \le_{\mathrm{hr}} X_2 \Longleftrightarrow N_\lambda(X_1) \le_{\mathrm{hr}} N_\lambda(X_2) \quad \textit{for all } \lambda > 0,$$

*where the notation $N_\lambda(X_1) \le_{\mathrm{hr}} N_\lambda(X_2)$ is in the sense of* (1.B.9).

*Proof.* First suppose that $X_1 \le_{\mathrm{hr}} X_2$. Denote

$$\Gamma_\lambda(n, x) = \lambda \mathrm{e}^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}, \quad n \ge 1, \ x \ge 0.$$

Let $\overline{\alpha}_\lambda^{X_1}(n) = P\{N_\lambda(X_1) \ge n\}$ and $\overline{\alpha}_\lambda^{X_2}(n) = P\{N_\lambda(X_2) \ge n\}$ be as in the proof of Theorem 1.A.13. Then it can be verified that

$$\overline{\alpha}_\lambda^{X_1}(n) = \int_0^\infty \Gamma_\lambda(n, x) \overline{F}_1(x) \mathrm{d}x \quad \text{and} \quad \overline{\alpha}_\lambda^{X_2}(n) = \int_0^\infty \Gamma_\lambda(n, x) \overline{F}_2(x) \mathrm{d}x,$$

where $\overline{F}_1$ and $\overline{F}_2$ are the survival functions corresponding to $X_1$ and $X_2$. For $n_1 \le n_2$, some computation yields

$$\begin{aligned} &\overline{\alpha}_\lambda^{X_1}(n_1)\overline{\alpha}_\lambda^{X_2}(n_2) - \overline{\alpha}_\lambda^{X_1}(n_2)\overline{\alpha}_\lambda^{X_2}(n_1) \\ &\quad = \int_{y=0}^\infty \int_{x=0}^y [\Gamma_\lambda(n_1, x)\Gamma_\lambda(n_2, y) - \Gamma_\lambda(n_1, y)\Gamma_\lambda(n_2, x)] \\ &\qquad\qquad \times \left[\overline{F}_1(x)\overline{F}_2(y) - \overline{F}_1(y)\overline{F}_2(x)\right] \mathrm{d}x \mathrm{d}y. \end{aligned}$$

It is not hard to verify that if $x \le y$ and $n_1 \le n_2$, then $[\Gamma_\lambda(n_1, x)\Gamma_\lambda(n_2, y) - \Gamma_\lambda(n_1, y)\Gamma_\lambda(n_2, x)] \ge 0$. Also, using (1.B.4) it is seen that $X_1 \le_{\mathrm{hr}} X_2$ implies $\left[\overline{F}_1(x)\overline{F}_2(y) - \overline{F}_1(y)\overline{F}_2(x)\right] \ge 0$ for $x \le y$. Thus, from (1.B.10) it is seen that $N_\lambda(X_1) \le_{\mathrm{hr}} N_\lambda(X_2)$.

Now suppose that $N_\lambda(X_1) \le_{\rm hr} N_\lambda(X_2)$ for every $\lambda > 0$. Define $c(n,\lambda) = \overline{\alpha}_\lambda^{X_1}(n)/\overline{\alpha}_\lambda^{X_2}(n)$. It can be shown that $c(n,\lambda)$ increases in $\lambda$ and decreases in $n$. Thus, $c(n,n/x) \ge c(n,n/y)$ whenever $x \le y$. Letting $n \to \infty$ shows that $\overline{F}_1(x)/\overline{F}_2(x) \ge \overline{F}_1(y)/\overline{F}_2(y)$ for all continuity points $x$ and $y$ of $F_1$ and $F_2$ such that $x \le y$. Thus, from (1.B.3) it is seen that $X_1 \le_{\rm hr} X_2$. $\square$

The implication $\Longrightarrow$ in Theorem 1.B.18 can be generalized in the same manner that Theorem 1.A.14 generalizes the implication $\Longrightarrow$ in Theorem 1.A.13. We will not state the result here since it is equivalent to Theorem 1.B.14.

A related result is the following.

**Theorem 1.B.19.** *Let $X_1, X_2, \ldots, X_m$, $\Theta_1$, and $\Theta_2$ be independent nonnegative random variables. Define*

$$N_j(t) = \sum_{i=1}^{n} I_{[\Theta_j X_i]}(t), \quad t \ge 0,\ j = 1,2,$$

*where*

$$I_{[\Theta_j X_i]}(t) = \begin{cases} 1 & \text{if} \quad \Theta_j X_i > t, \\ 0 & \text{if} \quad \Theta_j X_i \le t. \end{cases}$$

*If $\Theta_1 \le_{\rm hr} \Theta_2$ then $N_1(t) \le_{\rm hr} N_2(t)$ in the sense of* (1.B.9) *for all $t \ge 0$.*

The following easy-to-prove result strengthens Theorem 1.A.15. An even stronger result appears in Theorem 1.C.27.

**Theorem 1.B.20.** *Let $X$ be any random variable. Then $X_{(-\infty,a]}$ and $X_{(a,\infty)}$ are increasing in $a$ in the sense of the hazard rate order.*

In Theorem 1.A.17 it was seen that if $\phi$ is a function which satisfies that $\phi(x) \le x$ for all $x \in \mathbb{R}$, then $\phi(X) \le_{\rm st} X$. The order $\le_{\rm hr}$ does not satisfy such a general property. However, we have the following easy-to-prove result.

**Theorem 1.B.21.** *Let $X$ be a nonnegative* IFR *random variable, and let $a \le 1$ be a positive constant. Then $aX \le_{\rm hr} X$.*

In fact, a necessary and sufficient condition for a nonnegative random variable $X$, with survival function $\overline{F}$, to satisfy $aX \le_{\rm hr} X$ for all $0 < a < 1$, is that $\log \overline{F}(\mathrm{e}^x)$ is concave in $x \ge 0$.

In the next result it is shown that a random variable, whose distribution is the mixture of two distributions of hazard rate ordered random variables, is bounded from below and from above, in the hazard rate order sense, by these two random variables.

**Theorem 1.B.22.** *Let $X$ and $Y$ be two random variables with distribution functions $F$ and $G$, respectively. Let $W$ be a random variable with the distribution function $pF + (1-p)G$ for some $p \in (0,1)$. If $X \le_{\rm hr} Y$, then $X \le_{\rm hr} W \le_{\rm hr} Y$.*

*Proof.* Let $u_X$, $u_Y$, and $u_W$ denote the right endpoints of the supports of the corresponding random variables, and note that $\max(u_X, u_W) = \max(u_X, u_Y)$. Now, if $X \leq_{\text{hr}} Y$, then

$$\frac{p\overline{F}(t) + (1-p)\overline{G}(t)}{\overline{F}(t)} = p + (1-p)\frac{\overline{G}(t)}{\overline{F}(t)}$$

is increasing in $t \in (-\infty, \max(u_X, u_W))$. Therefore, by (1.B.3), $X \leq_{\text{hr}} W$. The proof that $W \leq_{\text{hr}} Y$ is similar. $\square$

*Example 1.B.23.* For a nonnegative random variable $X$ with density function $f$, and for a nonnegative function $w$ such that $E[w(X)]$ exists, define $X^w$ as the random variable with the so-called *weighted* density function $f_w$ given by

$$f_w(x) = \frac{w(x)f(x)}{E[w(X)]}, \quad x \geq 0.$$

Similarly, for another nonnegative random variable $Y$ with density function $g$, such that $E[w(Y)]$ exists, define $Y^w$ as the random variable with the density function $g_w$ given by

$$g_w(x) = \frac{w(x)g(x)}{E[w(Y)]}, \quad x \geq 0.$$

We will show that if $w$ is increasing, then

$$X \leq_{\text{hr}} Y \Longrightarrow X^w \leq_{\text{hr}} Y^w. \tag{1.B.23}$$

In order to do this, first note that the hazard rate function $r_{X^w}$ of $X^w$ is given by

$$r_{X^w}(x) = \frac{w(x)r_X(x)}{E[w(X)\big|X > x]}, \quad x \geq 0,$$

where $r_X$ is the hazard rate function of $X$. Similarly, the hazard rate function $r_{Y^w}$ of $Y^w$ is given by

$$r_{Y^w}(x) = \frac{w(x)r_Y(x)}{E[w(Y)\big|Y > x]}, \quad x \geq 0,$$

where $r_Y$ is the hazard rate function of $Y$. Now, from $X \leq_{\text{hr}} Y$ it follows that $[X\big|X > x] \leq_{\text{hr}} [Y\big|Y > x]$ for all $x \geq 0$. Next, using Theorem 1.B.2 and the monotonicity of $w$, we get that $[w(X)\big|X > x] \leq_{\text{hr}} [w(Y)\big|Y > x]$, and therefore, by Theorem 1.B.1, $E[w(X)\big|X > x] \leq E[w(Y)\big|Y > x]$. Combining this inequality with $r_X \geq r_Y$, it is seen that $r_{X^w} \geq r_{Y^w}$.

The above random variables are also studied in Example 1.C.59.

In particular, taking $w$ to be the identity function $w(x) = x$, we see from (1.B.23) that the hazard rate ordering of $X$ and $Y$ implies the hazard rate ordering of the corresponding spread (or length-biased) random variables. See Example 8.B.12 for another result involving spreads.

Analogous to the result in Remark 1.A.18, it can be shown that the set of all distribution functions on $\mathbb{R} \cup \{\infty\}$ is a lattice with respect to the order $\leq_{\rm hr}$.

The following example may be compared to Examples 1.C.48, 2.A.22, 3.B.38, 4.B.14, 6.B.41, 6.D.8, 6.E.13, and 7.B.13.

*Example 1.B.24.* Let $X$ and $Y$ be two absolutely continuous nonnegative random variables with survival functions $\overline{F}$ and $\overline{G}$, respectively. Denote $\Lambda_1 = -\log \overline{F}$, $\Lambda_2 = -\log \overline{G}$, and $\lambda_i = \Lambda_i'$, $i = 1, 2$. Consider two nonhomogeneous Poisson processes $N_1 = \{N_1(t),\ t \geq 0\}$ and $N_2 = \{N_2(t),\ t \geq 0\}$ with mean functions $\Lambda_1$ and $\Lambda_2$ (see Example 1.B.13), respectively. Let $T_{i,1}, T_{i,2}, \ldots$ be the successive epoch times of process $N_i$, $i = 1, 2$. Note that $X =_{\rm st} T_{1,1}$ and $Y =_{\rm st} T_{2,1}$.

It turns out that the hazard rate ordering of the first two epoch times implies the hazard rate ordering of all the corresponding later epoch times; that is, it will be shown below that if $X \leq_{\rm hr} Y$, then $T_{1,n} \leq_{\rm hr} T_{2,n}$, $n \geq 1$.

The survival function $\overline{F}_{1,n}$ of $T_{1,n}$ is given by

$$\overline{F}_{1,n}(t) = P(T_{1,n} > t) = \sum_{j=0}^{n-1} \frac{(\Lambda_1(t))^j}{j!} \mathrm{e}^{-\Lambda_1(t)} = \overline{\Gamma}_n(\Lambda_1(t)), \quad t \geq 0, \quad (1.B.24)$$

where $\overline{\Gamma}_n$ is the survival function of the gamma distribution with scale parameter 1 and shape parameter $n$. The corresponding density function $f_{1,n}$ is given by

$$f_{1,n}(t) = \gamma_n(\Lambda_1(t))\lambda_1(t), \quad t \geq 0,$$

where $\gamma_n$ is the density function associated with $\overline{\Gamma}_n$. The corresponding hazard rate function $r_{F_{1,n}}$ is given by

$$r_{F_{1,n}}(t) = r_{\Gamma_n}(\Lambda_1(t))\lambda_1(t), \quad t \geq 0,$$

where $r_{\Gamma_n}$ is the hazard rate function associated with $\overline{\Gamma}_n$. Similarly,

$$r_{F_{2,n}}(t) = r_{\Gamma_n}(\Lambda_2(t))\lambda_2(t), \quad t \geq 0.$$

If $X \leq_{\rm hr} Y$, then

$$r_{F_{1,n}}(t) = r_{\Gamma_n}(\Lambda_1(t))\lambda_1(t) \geq r_{\Gamma_n}(\Lambda_2(t))\lambda_2(t) = r_{F_{2,n}}(t), \quad t \geq 0,$$

where the inequality follows from $\lambda_1(t) \geq \lambda_2(t)$, $\Lambda_1(t) \geq \Lambda_2(t)$, and the fact that the hazard rate function of the gamma distribution described above is increasing.

Now let $X_{i,n} \equiv T_{i,n} - T_{i,n-1}$, $n \geq 1$ (where $T_{i,0} \equiv 0$), be the inter-epoch times of the process $N_i$, $i = 1, 2$. Again, note that $X =_{\rm st} X_{1,1}$ and $Y =_{\rm st} X_{2,1}$. It turns out that, under some conditions, the hazard rate ordering of the first two inter-epoch times implies the hazard rate ordering of all the corresponding later inter-epoch times. Explicitly, it will be shown below that if $X \le_{\rm hr} Y$, and if $\overline{F}$ and $\overline{G}$ are logconvex (that is, $X$ and $Y$ are DFR), and if

$$\frac{\lambda_2(t)}{\lambda_1(t)} \quad \text{is increasing in } t \ge 0, \tag{1.B.25}$$

then $X_{1,n} \le_{\rm hr} X_{2,n}$ for each $n \ge 1$.

For the purpose of this proof let us denote $F$ by $F_1$, and $G$ by $F_2$. Let $\overline{G}_{i,n}$ denote the survival function of $X_{i,n}$, $i = 1, 2$. The stated result is obvious for $n = 1$, so let us fix an $n \ge 2$. Then, from (7) in Baxter [62] we obtain

$$\overline{G}_{i,n}(t) = \int_0^\infty \lambda_i(s) \frac{\Lambda_i^{n-2}(s)}{(n-2)!} \overline{F}_i(s+t)\mathrm{d}s, \quad t \ge 0,\ i \in \{1, 2\}. \tag{1.B.26}$$

Condition (1.B.25) means that

$$\lambda_i(t) \text{ is TP}_2 \text{ (totally positive of order 2) in } (i, t).$$

Condition (1.B.25) also implies that $\Lambda_2(t)/\Lambda_1(t)$ is increasing in $t \ge 0$, that is, $\Lambda_i(t)$ is $\mathrm{TP}_2$ in $(i, t)$. Since a product of $\mathrm{TP}_2$ kernels is $\mathrm{TP}_2$ we get that

$$\lambda_i(t) \frac{\Lambda_i^{n-2}(t)}{(n-2)!} \quad \text{is TP}_2 \text{ in } (i, t).$$

The assumption $F_1 \le_{\rm hr} F_2$ implies that

$$\overline{F}_i(s+t) \text{ is TP}_2 \text{ in } (i, s) \text{ and in } (i, t).$$

Finally, the logconvexity of $\overline{F}_1$ and of $\overline{F}_2$ means that

$$\overline{F}_i(s+t) \text{ is TP}_2 \text{ in } (s, t).$$

Thus, by Theorem 5.1 on page 123 of Karlin [275], we get that $\overline{G}_{i,n}(t)$ is $\mathrm{TP}_2$ in $(i, t)$; that is, $X_{1,n} \le_{\rm hr} X_{2,n}$.

The inequality $X_{1,n} \le_{\rm hr} X_{2,n}$, $n \ge 1$, can also be obtained under slightly weaker assumptions, namely, that $X \le_{\rm hr} Y$, that (1.B.25) holds, and that either $X$ or $Y$ is DFR; see Hu and Zhuang [245].

*Example 1.B.25.* Let $X_1$, $X_2$, $Y_1$, and $Y_2$ be independent, nonnegative random variables such that $X_1 =_{\rm st} X_2$ and $Y_1 =_{\rm st} Y_2$. Denote by $\lambda_X$ and $\lambda_Y$ the hazard rate functions of $X_1$ and $Y_1$, respectively. If $X_1 \le_{\rm hr} Y_1$, and if $\lambda_Y/\lambda_X$ is decreasing on $[0, 1)$, then

$$\min\{\max(X_1, X_2), \max(Y_1, Y_2)\} \le_{\rm hr} \min\{\max(X_1, Y_1), \max(X_2, Y_2)\}.$$

### 1.B.4 Comparison of order statistics

Let $X_1, X_2, \dots, X_m$ be random variables. As usual denote the corresponding order statistics by $X_{(1)} \le X_{(2)} \le \cdots \le X_{(m)}$. When we want to emphasize the dependence on $m$, we denote the order statistics by $X_{(1:m)} \le X_{(2:m)} \le \cdots \le X_{(m:m)}$. The following three theorems compare the order statistics in the hazard rate order.

**Theorem 1.B.26.** *If $X_1, X_2, \dots, X_m$ are independent random variables, then $X_{(k)} \le_{\mathrm{hr}} X_{(k+1)}$ for $k = 1, 2, \dots, m-1$.*

A relatively simple proof of Theorem 1.B.26 can be obtained using the likelihood ratio order which is discussed in the next section. Therefore the proof of this theorem will be given there in Remark 1.C.40.

Theorem 12.5 in Cramer and Kamps [136] extends Theorem 1.B.26 to the so called sequential order statistics.

Further comparisons of order statistics are given in the next two theorems.

**Theorem 1.B.27.** *Let $X_1, X_2, \dots, X_m$ be independent random variables. If $X_j \le_{\mathrm{hr}} X_m$ for all $j = 1, 2, \dots, m-1$, then $X_{(k-1:m-1)} \le_{\mathrm{hr}} X_{(k:m)}$ for $k = 2, 3, \dots, m$.*

**Theorem 1.B.28.** *If $X_1, X_2, \dots, X_m$ are independent random variables, then $X_{(k:m-1)} \ge_{\mathrm{hr}} X_{(k:m)}$ for $k = 1, 2, \dots, m-1$.*

From Theorem 1.B.27 it follows that if $X_1, X_2, \dots, X_m$ are independent random variables, then

$$X_{(1:1)} \ge_{\mathrm{hr}} X_{(1:2)} \ge_{\mathrm{hr}} \cdots \ge_{\mathrm{hr}} X_{(1:m)}. \tag{1.B.27}$$

One may wonder what kind of results of this type hold without the independence assumption. Since $X_{(1:1)} \ge X_{(1:2)} \ge \cdots \ge X_{(1:m)}$ a.s., it follows from Theorem 1.A.1 that $X_{(1:1)} \ge_{\mathrm{st}} X_{(1:2)} \ge_{\mathrm{st}} \cdots \ge_{\mathrm{st}} X_{(1:m)}$ hold without any (independence) assumption. However, a counterexample in the literature shows that (1.B.27) does not always hold. We now describe some conditions under which (1.B.27) holds.

Let $\boldsymbol{X} = (X_1, X_2, \dots, X_m)$ be a random vector with a partially differentiable survival function $\overline{F}$. The function $R = -\log \overline{F}$ is called the hazard function of $\boldsymbol{X}$, and the vector $\boldsymbol{r}_{\boldsymbol{X}}$ of partial derivatives, defined by

$$\begin{aligned} \boldsymbol{r}_{\boldsymbol{X}}(\boldsymbol{x}) &= \big(r_{\boldsymbol{X}}^{(1)}(\boldsymbol{x}), r_{\boldsymbol{X}}^{(2)}(\boldsymbol{x}), \dots, r_{\boldsymbol{X}}^{(m)}(\boldsymbol{x})\big) \\ &= \Big(\frac{\partial}{\partial x_1} R(\boldsymbol{x}), \frac{\partial}{\partial x_2} R(\boldsymbol{x}), \dots, \frac{\partial}{\partial x_m} R(\boldsymbol{x})\Big), \end{aligned} \tag{1.B.28}$$

for all $\boldsymbol{x} \in \{\boldsymbol{x} : \overline{F}(\boldsymbol{x}) > 0\}$, is called the hazard gradient of $\boldsymbol{X}$; see Johnson and Kotz [264] and Marshall [381]. Note that $r_{\boldsymbol{X}}^{(i)}(\boldsymbol{x})$ can be interpreted as the conditional hazard rate of $X_i$ evaluated at $x_i$, given that $X_j > x_j$ for all $j \ne i$. That is,

$$r_{\boldsymbol{X}}^{(i)}(\boldsymbol{x}) = \frac{f_i(x_i|X_j > x_j, j \neq i)}{\overline{F}_i(x_i|X_j > x_j, j \neq i)},$$

where $f_i(\cdot|X_j > x_j, j \neq i)$ and $\overline{F}_i(\cdot|X_j > x_j, j \neq i)$ are the conditional density and survival functions of $X_i$, given that $X_j > x_j$ for all $j \neq i$. For convenience, here and below we set $r_{\boldsymbol{X}}^{(i)}(\boldsymbol{x}) = \infty$ for all $\boldsymbol{x} \in \{\boldsymbol{x} : \overline{F}(\boldsymbol{x}) = 0\}$.

For any subset $P \subseteq \{1, 2, \ldots, m\}$ define

$$Y_P = \min_{i \in P} X_i.$$

Denote

$$1_P(i) = \begin{cases} 0 & \text{if } i \notin P, \\ 1 & \text{if } i \in P, \end{cases}$$

$$\mathbf{1}_P = (1_P(1), 1_P(2), \ldots, 1_P(m)), \quad \text{and} \quad \mathbf{1}_{P^c} = \mathbf{1} - \mathbf{1}_P,$$

where $\mathbf{1} = (1, 1, \ldots, 1)$, and $P^c$ denotes the complement of $P$ in $\{1, 2, \ldots, m\}$. Also denote

$$\infty \cdot 1_{P^c}(i) = \begin{cases} 0 & \text{if } i \notin P^c, \\ \infty & \text{if } i \in P^c, \end{cases}$$

and $\infty \cdot \mathbf{1}_{P^c} = (\infty \cdot 1_{P^c}(1), \infty \cdot 1_{P^c}(2), \ldots, \infty \cdot 1_{P^c}(m))$. Then the survival function $\overline{G}_P$ of $Y_P$ can be expressed as

$$\overline{G}_P(t) = \overline{F}(t \cdot \mathbf{1}_P - \infty \cdot \mathbf{1}_{P^c}), \quad t \in \mathbb{R}.$$

**Theorem 1.B.29.** *Let $(X_1, X_2, \ldots, X_m)$ be a random vector with an absolutely continuous distribution function. Let $P$ and $Q$ be two subsets of $\{1, 2, \ldots, m\}$ such that $P \subset Q$. If*

$$r^{(i)}(t \cdot \mathbf{1}_P - \infty \cdot \mathbf{1}_{P^c}) \leq r^{(i)}(t \cdot \mathbf{1}_Q - \infty \cdot \mathbf{1}_{Q^c}), \quad t \in \mathbb{R},\ i \in P, \tag{1.B.29}$$

*then*

$$Y_P \geq_{\text{hr}} Y_Q.$$

A sufficient condition for (1.B.29) is that

$$r^{(i)}(x_1, x_2, \ldots, x_m) \text{ is increasing in } x_j, \quad j \neq i,\ i = 1, 2, \ldots, m.$$

This is easily seen to be equivalent to the requirement that

$$\frac{\overline{F}(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_m)}{\overline{F}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m)} \text{ is decreasing in } x_j,\ j \neq i, \text{ whenever } x_i \leq x_i',\ i = 1, 2, \ldots, m. \tag{1.B.30}$$

Condition (1.B.30) means that $\overline{F}$ is $\text{RR}_2$ (reverse regular of order 2) in pairs; see Karlin [275]. In particular, it holds when $X_1, X_2, \ldots, X_m$ are independent. Karlin and Rinott [279] showed that some multivariate normal distributions, as well as the Dirichlet distribution, are $\mathrm{RR}_2$ in pairs. So Theorem 1.B.29 applies to these distributions.

When $(X_1, X_2, \ldots, X_m)$ has an exchangeable distribution function, then the corresponding multivariate hazard function $R$ is permutation symmetric. Therefore each $r^{(i)}$ can be expressed by means of $r^{(1)}$ as follows

$$\begin{aligned} r^{(i)}&(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m) \\ &= r^{(1)}(x_i, x_2, \ldots, x_{i-1}, x_1, x_{i+1}, \ldots, x_m), \quad i = 2, 3, \ldots, m. \end{aligned}$$

**Corollary 1.B.30.** *Let $(X_1, X_2, \ldots, X_m)$ be a random vector with an absolutely continuous exchangeable distribution function. If*

$$r^{(1)}(\underbrace{t, t, \ldots, t}_{i \text{ times}}, \underbrace{-\infty, -\infty, \ldots, -\infty}_{m-i \text{ times}}) \le r^{(1)}(\underbrace{t, t, \ldots, t}_{i+1 \text{ times}}, \underbrace{-\infty, -\infty, \ldots, -\infty}_{m-i-1 \text{ times}}),$$
$$t \in \mathbb{R}, \; i = 1, 2, \ldots, m-1, \tag{1.B.31}$$

*then*

$$X_{(1:1)} \ge_{\mathrm{hr}} X_{(1:2)} \ge_{\mathrm{hr}} \cdots \ge_{\mathrm{hr}} X_{(1:m)}. \tag{1.B.32}$$

If (1.B.31) is not imposed, then (1.B.32) need not be true; this follows from a counterexample in the literature.

The following result strengthens the DFR part of Theorem 1.A.19. Recall that the spacings that correspond to the random variables $X_1, X_2, \ldots, X_m$ are denoted by $U_{(i)} = X_{(i)} - X_{(i-1)}$, $i = 2, 3, \ldots, m$, where the $X_{(i)}$'s are the corresponding order statistics. When the dependence on $m$ is to be emphasized, we will denote the spacings by $U_{(i:m)}$.

**Theorem 1.B.31.** *Let $X_1, X_2, \ldots, X_m, X_{m+1}$ be independent and identically distributed, absolutely continuous,* DFR *random variables. Then*

$$(m-i+1)U_{(i:m)} \le_{\mathrm{hr}} (m-i)U_{(i+1:m)}, \quad i = 2, 3, \ldots, m-1, \tag{1.B.33}$$
$$(m-i+2)U_{(i:m+1)} \le_{\mathrm{hr}} (m-i+1)U_{(i:m)}, \quad i = 2, 3, \ldots, m, \tag{1.B.34}$$

*and*

$$U_{(i:m)} \le_{\mathrm{hr}} U_{(i+1:m+1)}, \quad i = 2, 3, \ldots, m. \tag{1.B.35}$$

Note that (1.B.33)–(1.B.35) can be summarized as

$$(m-j+1)U_{(j:m)} \le_{\mathrm{hr}} (n-i+1)U_{(i:n)} \quad \text{whenever } i-j \ge \max\{0, n-m\}.$$

Theorem 1.B.31 is a simple consequence of Theorem 1.C.45 below. It is of interest to compare Theorem 1.B.31 to Theorems 1.A.19 and 1.A.22.

A comparison of such normalized spacings from two different samples is described next. Here $U_{(i:m)}$ denotes, as before, the $i$th spacing that corresponds to the sample $X_1, X_2, \ldots, X_m$, and $V_{(j:n)}$ denotes the $j$th spacing that corresponds to the sample $Y_1, Y_2, \ldots, Y_n$. It is of interest to compare the next result with Theorem 1.C.45.

**Theorem 1.B.32.** *For positive integers $m$ and $n$, let $X_1, X_2, \ldots, X_m$ be independent identically distributed random variables with an absolutely continuous common distribution function, and let $Y_1, Y_2, \ldots, Y_n$ be independent identically distributed random variables with a possibly different absolutely continuous common distribution function. If $X_1 \le_{\rm hr} Y_1$, and if either $X_1$ or $Y_1$ is* DFR, *then*

$$(m-j+1)U_{(j:m)} \le_{\rm st} (n-i+1)V_{(i:n)} \quad \textit{whenever } i-j \ge \max\{0, n-m\}.$$

The hazard rate order is closed under the operation of taking minima, as the next result shows.

**Theorem 1.B.33.** *Let $(X_i, Y_i)$, $i = 1, 2, \ldots, m$, be independent pairs of random variables such that $X_i \le_{\rm hr} Y_i$, $i = 1, 2, \ldots, m$. Then*

$$\min\{X_1, X_2, \ldots, X_m\} \le_{\rm hr} \min\{Y_1, Y_2, \ldots, Y_m\}.$$

*Proof.* Clearly, it is enough to show the result when $m = 2$. For simplicity assume that $X_1$, $X_2$, $Y_1$, and $Y_2$ have hazard rate functions $r_1$, $r_2$, $q_1$, and $q_2$, respectively. Then it is very easy to see that the hazard rate function of $\min\{X_1, X_2\}$ is $r_1 + r_2$ and the hazard rate function of $\min\{Y_1, Y_2\}$ is $q_1 + q_2$. By the assumptions of the theorem (see (1.B.2)) $r_1(t) \ge q_1(t)$ and $r_2(t) \ge q_2(t)$ for all $t \ge 0$. Therefore $r_1(t) + r_2(t) \ge q_1(t) + q_2(t)$ for all $t \ge 0$, that is, $\min\{X_1, X_2\} \le_{\rm hr} \min\{Y_1, Y_2\}$. □

If the $X_i$'s in Theorem 1.B.33 are identically distributed and if the $Y_i$'s in Theorem 1.B.33 are also identically distributed, then all order statistics (and not just the minima) corresponding to the $X_i$'s and the $Y_i$'s can be compared in the hazard rate order. This is shown in the following result.

**Theorem 1.B.34.** *Let $(X_i, Y_i)$, $i = 1, 2, \ldots, m$, be independent pairs of absolutely continuous random variables such that $X_i \le_{\rm hr} Y_i$, $i = 1, 2, \ldots, m$. Suppose that the $X_i$'s are identically distributed and that the $Y_i$'s are identically distributed. Then*

$$X_{(k:m)} \le_{\rm hr} Y_{(k:m)}, \quad k = 1, 2, \ldots, m. \tag{1.B.36}$$

If the $X_i$'s or the $Y_i$'s in Theorem 1.B.34 are not identically distributed, then the conclusion (1.B.36) need not hold. However, the following result, from Chapter 16 by Boland and Proschan in [515], gives conditions under which (1.B.36) holds.

**Proposition 1.B.35.** *Let $X_1, X_2, \ldots, X_m$ [respectively, $Y_1, Y_2, \ldots, Y_m$] be $m$ independent (not necessarily identically distributed) absolutely continuous random variables, all with support $(a, b)$ for some $a < b$. If $X_i \le_{\rm hr} Y_j$ for all $i$ and $j$, then $X_{(k:m)} \le_{\rm hr} Y_{(k:m)}$, $k = 1, 2, \ldots, m$.*

A result which is stronger than Proposition 1.B.35, but which uses Proposition 1.B.35 in its proof, is the following.

**Theorem 1.B.36.** *Let $X_1, X_2, \dots, X_m$ be $m$ independent (not necessarily identically distributed) random variables, and let $Y_1, Y_2, \dots, Y_n$ be other $n$ independent (not necessarily identically distributed) random variables, all having absolutely continuous distributions with support $(a, b)$ for some $a < b$. If $X_i \le_{\rm hr} Y_j$ for all $i$ and $j$, then*

$$X_{(j:m)} \le_{\rm hr} Y_{(i:n)} \quad \textit{whenever } i - j \ge \max\{0, n - m\}.$$

The proof of Theorem 1.B.36 uses the likelihood ratio order which is discussed in the next section. Therefore the proof will be given in Remark 1.C.41.

The following example describes an interesting instance in which the two maxima are ordered in the hazard rate order. It may be compared with Example 3.B.32.

*Example 1.B.37.* Let $Y_1, Y_2, \dots, Y_m$ be independent exponential random variables with hazard rates $\lambda_1, \lambda_2, \dots, \lambda_m$, respectively. Let $X_1, X_2, \dots, X_m$ be independent and identically distributed exponential random variables with hazard rate $\overline{\lambda} = \sum_{i=1}^m \lambda_i / m$. Then

$$X_{(m:m)} \le_{\rm hr} Y_{(m:m)}. \tag{1.B.37}$$

Let $Z_1, Z_2, \dots, Z_m$ be independent and identically distributed exponential random variables with hazard rate $\tilde{\lambda} = \left(\prod_{i=1}^m \lambda_i\right)^{1/m}$. Then

$$Z_{(m:m)} \le_{\rm hr} Y_{(m:m)}. \tag{1.B.38}$$

In fact, from the arithmetic-geometric mean inequality ($\overline{\lambda} \ge \tilde{\lambda}$) and Proposition 1.B.35, it follows that (1.B.38) implies (1.B.37).

### 1.B.5 Some properties in reliability theory

The order $\le_{\rm hr}$ can be trivially (but beneficially) used to characterize IFR random variables. The next result lists several such characterizations. Recall from Section 1.A.3 that for any random variable $Z$ and an event $A$ we denote by $[Z \big| A]$ any random variable that has as its distribution the conditional distribution of $Z$ given $A$.

**Theorem 1.B.38.** *The random variable $X$ is* IFR [DFR] *if, and only if, one of the following equivalent conditions holds (when the support of the distribution function of $X$ is bounded, condition* (iii) *does not have a simple* DFR *analog):*

(i) $[X - t \big| X > t] \ge_{\rm hr} [\le_{\rm hr}]\ [X - t' \big| X > t']$ *whenever* $t \le t'$.
(ii) $X \ge_{\rm hr} [\le_{\rm hr}]\ [X - t \big| X > t]$ *for all* $t \ge 0$ *(when $X$ is a nonnegative random variable).*
(iii) $X + t \le_{\rm hr} X + t'$ *whenever* $t \le t'$.

Note that if $X$ is the lifetime of a device, then $[X - t | X > t]$ is the residual life of such a device with age $t$. Theorem 1.B.38(i), for example, characterizes IFR random variables by the monotonicity of their residual lives with respect to the order $\le_{\rm hr}$. Some multivariate analogs of conditions (i) and (ii) of Theorem 1.B.38 are used in Section 6.D.3 to introduce a multivariate IFR notion.

Part (iii) of Theorem 1.B.38 can be used to prove the closure under convolution property of IFR random variables:

**Corollary 1.B.39.** *Let $X$ and $Y$ be two independent* IFR *random variables. Then $X + Y$ has an* IFR *distribution.*

*Proof.* From Theorem 1.B.38(iii) it follows that $X + t \le_{\rm hr} X + t'$ whenever $t \le t'$. Also, $Y$ is independent of $X+t$ and of $X+t'$ for all $t$ and $t'$, respectively. From Lemma 1.B.3 it now follows that $X + Y + t \le_{\rm hr} X + Y + t'$ whenever $t \le t'$. Thus, again from Theorem 1.B.38(iii), it follows that $X + Y$ is IFR. □

Recall from (1.A.20) that for a nonnegative random variable $X$ with a finite mean we denote by $A_X$ the corresponding asymptotic equilibrium age. Recall from page 1 the definitions of the DMRL and the IMRL properties. The following result is immediate.

**Theorem 1.B.40.** *The nonnegative random variable $X$ with finite mean is* DMRL [IMRL] *if, and only if, $X \ge_{\rm hr} [\le_{\rm hr}] A_X$.*

### 1.B.6 The reversed hazard order

If $X$ is a random variable with an absolutely continuous distribution function $F$, then the reversed hazard rate of $X$ at the point $t$ is defined as $\tilde{r}(t) = ({\rm d}/{\rm d}t)(\log F(t))$. One interpretation of the reversed hazard rate at time $t$ is the following. Suppose that $X$ is nonnegative with distribution function $F$. Then $X$ can be thought of as the lifetime of some device. Given that the device has already failed by time $t$, then the probability that it survived up to time $t - \varepsilon$ (for a small $\varepsilon > 0$) is approximately $\varepsilon \cdot \tilde{r}(t)$. Some of the results regarding the hazard rate order have analogs when the hazard rate is replaced by the reversed hazard rate.

Let $X$ and $Y$ be two random variables with absolutely continuous distribution functions and with reversed hazard rate functions $\tilde{r}$ and $\tilde{q}$, respectively, such that

$$\tilde{r}(t) \le \tilde{q}(t), \quad t \in \mathbb{R}. \tag{1.B.39}$$

Then $X$ is said to be *smaller than $Y$ in the reversed hazard rate order* (denoted as $X \le_{\rm rh} Y$).

In fact, the absolute continuity, which is required in (1.B.39), is not really needed. It easy to verify that (1.B.39) holds if, and only if,

$$\frac{G(t)}{F(t)} \quad \text{increases in } t \in (\min(l_X, l_Y), \infty) \tag{1.B.40}$$

(here $a/0$ is taken to be equal to $\infty$ whenever $a > 0$). Here $F$ denotes the distribution function of $X$ and $G$ denotes the distribution function of $Y$, and $l_X$ and $l_Y$ denote the corresponding left endpoints of the supports of $X$ and of $Y$. Equivalently, (1.B.40) can be written as

$$F(x)G(y) \geq F(y)G(x) \quad \text{for all } x \leq y. \tag{1.B.41}$$

Thus (1.B.40) or (1.B.41) can be used to define the order $X \leq_{\mathrm{rh}} Y$ even if $X$ and/or $Y$ do not have absolutely continuous distributions. The analog of (1.B.5) for the reversed hazard order when $X$ and $Y$ have densities $f$ and $g$, respectively, is that $X \leq_{\mathrm{rh}} Y$ if, and only if,

$$\frac{f(y)}{F(x)} \leq \frac{g(y)}{G(x)} \quad \text{for all } x \leq y. \tag{1.B.42}$$

Another condition that is equivalent to $X \leq_{\mathrm{rh}} Y$ is

$$\frac{GF^{-1}(u)}{u} \leq \frac{GF^{-1}(v)}{v} \quad \text{for all } 0 < u \leq v < 1.$$

Finally, another condition that is equivalent to $X \leq_{\mathrm{rh}} Y$ is

$$P\{X - t \leq -s \big| X \leq t\} \geq P\{Y - t \leq -s \big| Y \leq t\} \quad \text{for all } s \geq 0 \text{ and all } t,$$

or, equivalently,

$$[X \big| X \leq t] \leq_{\mathrm{st}} [Y \big| Y \leq t] \quad \text{for all } t. \tag{1.B.43}$$

For discrete random variables $X$ and $Y$ that take on values in $\mathbb{N}$, we denote $X \leq_{\mathrm{rh}} Y$ if

$$\frac{P\{X = n\}}{P\{X \leq n\}} \leq \frac{P\{Y = n\}}{P\{Y \leq n\}}, \quad n \in \mathbb{N}. \tag{1.B.44}$$

A useful relationship between the hazard rate and the reversed hazard rate orders is described in the following theorem.

**Theorem 1.B.41.** *Let $X$ and $Y$ be two continuous random variables with supports $(l_X, u_X)$ and $(l_Y, u_Y)$, respectively. Then*

$$X \leq_{\mathrm{hr}} Y \Longrightarrow \phi(X) \geq_{\mathrm{rh}} \phi(Y)$$

*for any continuous function $\phi$ which is strictly decreasing on $(l_X, u_Y)$. Also,*

$$X \leq_{\mathrm{rh}} Y \Longrightarrow \phi(X) \geq_{\mathrm{hr}} \phi(Y)$$

*for any such function $\phi$.*

Using Theorem 1.B.41 it is easy to obtain the following analogs of results regarding the order $\leq_{\mathrm{hr}}$.

**Theorem 1.B.42.** *If $X$ and $Y$ are two random variables such that $X \le_{\rm rh} Y$, then $X \le_{\rm st} Y$.*

**Theorem 1.B.43.** *If $X \le_{\rm rh} Y$, and if $\phi$ is any increasing function, then $\phi(X) \le_{\rm rh} \phi(Y)$.*

**Lemma 1.B.44.** *If the random variables $X$ and $Y$ are such that $X \le_{\rm rh} Y$, and if $Z$ is a random variable independent of $X$ and $Y$ and has decreasing reversed hazard rate, then*

$$X + Z \le_{\rm rh} Y + Z.$$

**Theorem 1.B.45.** *Let $(X_i, Y_i)$, $i = 1, 2, \ldots, m$, be independent pairs of random variables such that $X_i \le_{\rm rh} Y_i$, $i = 1, 2, \ldots, m$. If $X_i, Y_i$, $i = 1, 2, \ldots, m$, all have decreasing reversed hazard rates, then*

$$\sum_{i=1}^{m} X_i \le_{\rm rh} \sum_{i=1}^{m} Y_i.$$

**Theorem 1.B.46.** *Let $X$, $Y$, and $\Theta$ be random variables such that $[X \big| \Theta = \theta] \le_{\rm rh} [Y \big| \Theta = \theta']$ for all $\theta$ and $\theta'$ in the support of $\Theta$. Then $X \le_{\rm rh} Y$.*

In order to state a bivariate characterization result for the order $\le_{\rm rh}$ we define the following class of bivariate functions:

$$\begin{aligned} \mathcal{G}_{\rm rh} = \{\phi : \mathbb{R}^2 \to \mathbb{R} : \phi(x, y) &\text{ is increasing in } x, \text{ for each } y, \text{ on } \{x \le y\}, \\ &\text{and is decreasing in } y, \text{ for each } x, \text{ on } \{y \le x\}\}. \end{aligned}$$

The proof of the next result (Theorem 1.B.47) is similar to the proof of Theorem 1.B.9.

**Theorem 1.B.47.** *Let $X$ and $Y$ be independent random variables. Then $X \le_{\rm rh} Y$ if, and only if,*

$$\phi(X, Y) \le_{\rm st} \phi(Y, X) \quad \text{for all } \phi \in \mathcal{G}_{\rm rh}.$$

The next result uses the notation of Theorem 1.A.10.

**Theorem 1.B.48.** *Let $X$ and $Y$ be two independent random variables. Then $X \le_{\rm rh} Y$ if, and only if,*

$$E\phi_1(X, Y) \le E\phi_2(X, Y)$$

*for all $\phi_1$ and $\phi_2$ such that, for each $y$, $\Delta\phi_{21}(x, y)$ decreases in $x$ on $\{x \le y\}$, and such that $\Delta\phi_{21}(x, y) \ge -\Delta\phi_{21}(y, x)$ whenever $x \le y$.*

A further similar characterization is given in Theorem 4.A.36.

The following result is an analog of Theorem 1.B.11.

**Theorem 1.B.49.** *Let $X$ and $Y$ be two independent random variables. Then $X \le_{\rm rh} Y$ if, and only if,*

$$[X \big| \max(X,Y)=z] \le_{\rm rh} [Y \big| \max(X,Y)=z] \quad \text{for all } z. \tag{1.B.45}$$

*Also, $X \le_{\rm rh} Y$ if, and only if,*

$$[X \big| \max(X,Y)=z] \le_{\rm st} [Y \big| \max(X,Y)=z] \quad \text{for all } z. \tag{1.B.46}$$

*Proof.* First suppose that $X$ and $Y$ are absolutely continuous. Denote the distribution functions of $X$ and $Y$ by $F$ and $G$, respectively, and denote the corresponding density functions by $f$ and $g$. Then

$$P[X \le x \big| \max(X,Y)=z] = \begin{cases} \frac{F(x)g(z)}{f(z)G(z)+g(z)F(z)}, & \text{if } x \le z, \\ 1, & \text{if } x > z, \end{cases} \tag{1.B.47}$$

and

$$P[Y \le x \big| \max(X,Y)=z] = \begin{cases} \frac{G(x)f(z)}{f(z)G(z)+g(z)F(z)}, & \text{if } x \le z, \\ 1, & \text{if } x > z. \end{cases} \tag{1.B.48}$$

Therefore

$$\frac{P[Y \le x | \max(X,Y)=z]}{P[X \le x | \max(X,Y)=z]} = \begin{cases} \frac{G(x)}{F(x)} \cdot \frac{f(z)}{g(z)}, & \text{if } x \le z, \\ 1, & \text{if } x > z. \end{cases} \tag{1.B.49}$$

If $X \le_{\rm rh} Y$, then $\frac{G(x)}{F(x)}$ is increasing in $x$, and $\frac{G(z)}{F(z)} \cdot \frac{f(z)}{g(z)} \le 1$. Thus (1.B.49) is increasing in $x$, and (1.B.45) follows. Obviously (1.B.46) follows from (1.B.45).

Now suppose that (1.B.46) holds. Then from (1.B.47) and (1.B.48) we get that $F(x)g(z) \ge G(x)f(z)$ for all $x \le z$. Therefore $X \le_{\rm rh} Y$ by (1.B.42).

The proof when $X$ and $Y$ are discrete is similar. □

The following result is an analog of Theorem 1.B.12.

**Theorem 1.B.50.** *Let $X$ and $Y$ be two independent random variables. The following conditions are equivalent:*

(a) $X \le_{\rm rh} Y$.
(b) $E[\alpha(X)]E[\beta(Y)] \ge E[\alpha(Y)]E[\beta(X)]$ *for all functions $\alpha$ and $\beta$ for which the expectations exist and such that $\beta$ is nonnegative and $\alpha/\beta$ and $\beta$ are decreasing.*
(c) *For any increasing function $a$ and a nonnegative decreasing function $b$, if $E[a(Y)b(Y)] = 0$, then $E[a(X)b(X)] \le 0$.*

*Example 1.B.51.* Let $X$ and $Y$ be two random variables with support $[c,d]$, where $c < 0 < d$, and suppose that $E[Y] > 0$. Let $u$ be an increasing differentiable concave function, corresponding to the utility function of a risk-averse individual. Let $k_X$ be a value which maximizes $g_X(k) \equiv E[u(kX)]$, and similarly let $k_Y$ be a value which maximizes $g_Y(k) \equiv E[u(kY)]$. Theorem 1.B.50(c) can be used to prove that if $X \le_{\rm rh} Y$, then $k_X \le k_Y$. In order to see it, first note that the result is trivial if $k_X = -\infty$ or if $k_Y = \infty$. Thus, let us assume that $k_X$ and $k_Y$ are finite. Note that then $k_X$ and $k_Y$ satisfy $E[Xu'(k_X X)] = 0$ and $E[Yu'(k_Y Y)] = 0$, where $u'$ denotes the derivative of $u$. Also note that from the assumption $E[Y] > 0$ it follows that $k_Y > 0$. Without loss of generality let $k_Y = 1$. Thus $E[Yu'(Y)] = 0$, and using the concavity of $u$ the assertion would follow if we show that $E[Xu'(X)] \le 0$. But this follows from Theorem 1.B.50(c).

Consider now a family of distribution functions $\{G_\theta,\ \theta \in \mathcal{X}\}$ where $\mathcal{X}$ is a subset of the real line. As in Section 1.A.3 let $X(\theta)$ denote a random variable with distribution function $G_\theta$. For any random variable $\Theta$ with support in $\mathcal{X}$, and with distribution function $F$, let us denote by $X(\Theta)$ a random variable with distribution function $H$ given by

$$H(y) = \int_{\mathcal{X}} G_\theta(y) \mathrm{d}F(\theta), \quad y \in \mathbb{R}.$$

The following result generalizes Theorem 1.B.43, just as Theorem 1.A.6 generalized Theorem 1.A.3(a). The proof of the next theorem is similar to the proof of Theorem 1.B.14 and is therefore omitted.

**Theorem 1.B.52.** *Consider a family of distribution functions* $\{G_\theta,\ \theta \in \mathcal{X}\}$ *as above. Let* $\Theta_1$ *and* $\Theta_2$ *be two random variables with supports in* $\mathcal{X}$ *and distribution functions* $F_1$ *and* $F_2$*, respectively. Let* $Y_1$ *and* $Y_2$ *be two random variables such that* $Y_i =_{\rm st} X(\Theta_i)$, $i = 1, 2$*; that is, suppose that the distribution function of* $Y_i$ *is given by*

$$H_i(y) = \int_{\mathcal{X}} G_\theta(y) \mathrm{d}F_i(\theta), \quad y \in \mathbb{R},\ i = 1, 2.$$

*If*

$$X(\theta) \le_{\rm rh} X(\theta') \quad \textit{whenever } \theta \le \theta',$$

*and if*

$$\Theta_1 \le_{\rm rh} \Theta_2,$$

*then*

$$Y_1 \le_{\rm rh} Y_2.$$

The following result, which is the "reversed hazard analog" of Theorem 1.B.18, gives a Laplace transform characterization of the order $\le_{\rm rh}$.

**Theorem 1.B.53.** *Let* $X_1$ *and* $X_2$ *be two nonnegative random variables, and let* $N_\lambda(X_1)$ *and* $N_\lambda(X_2)$ *be as described in Theorem* 1.A.13*. Then*

$$X_1 \le_{\rm rh} X_2 \Longleftrightarrow N_\lambda(X_1) \le_{\rm rh} N_\lambda(X_2) \quad \textit{for all } \lambda > 0,$$

*where the notation* $N_\lambda(X_1) \le_{\rm rh} N_\lambda(X_2)$ *is in the sense of* (1.B.44).

The implication $\Longrightarrow$ in Theorem 1.B.53 can be generalized in the same manner that Theorem 1.A.14 generalizes the implication $\Longrightarrow$ in Theorem 1.A.13. We will not state the result here since it is equivalent to Theorem 1.B.52.

The reversed hazard analog of Theorem 1.B.19 is the following.

**Theorem 1.B.54.** *Let $X_1, X_2, \ldots, X_m$, $\Theta_1$, and $\Theta_2$ be independent nonnegative random variables. Define $N_j(t)$ for $t \geq 0$ and $j = 1, 2$ as in Theorem* 1.B.19. *If $\Theta_1 \leq_{\rm rh} \Theta_2$, then $N_1(t) \leq_{\rm rh} N_2(t)$ in the sense of* (1.B.44) *for all $t \geq 0$.*

The reversed hazard analog of Theorem 1.B.20 is the following.

**Theorem 1.B.55.** *Let $X$ be any random variable. Then $X_{(-\infty,a]}$ and $X_{(a,\infty)}$ are increasing in $a$ in the sense of the reversed hazard order.*

Analogous to the result in Remark 1.A.18, it can be shown that the set of all distribution functions on $\mathbb{R} \cup \{-\infty\}$ is a lattice with respect to the order $\leq_{\rm rh}$.

The reversed hazard analog of Theorem 1.B.26 is the following.

**Theorem 1.B.56.** *If $X_1, X_2, \ldots, X_m$ are independent random variables, then $X_{(k)} \leq_{\rm rh} X_{(k+1)}$ for $k = 1, 2, \ldots, m-1$.*

The reversed hazard analog of Theorem 1.B.27 is the following.

**Theorem 1.B.57.** *Let $X_1, X_2, \ldots, X_m$ be independent random variables. If $X_m \leq_{\rm rh} X_j$ for all $j = 1, 2, \ldots, m-1$, then $X_{(k-1:m-1)} \leq_{\rm rh} X_{(k:m)}$ for $k = 2, 3, \ldots, m$.*

The reversed hazard analog of Theorem 1.B.28 is the following.

**Theorem 1.B.58.** *If $X_1, X_2, \ldots, X_m$ are independent random variables, then $X_{(k:m-1)} \geq_{\rm rh} X_{(k:m)}$ for $k = 1, 2, \ldots, m-1$.*

The reversed hazard analogs of Theorems 1.B.33, 1.B.34, and 1.B.36 are the following results.

**Theorem 1.B.59.** *Let $(X_i, Y_i)$, $i = 1, 2, \ldots, m$, be independent pairs of random variables such that $X_i \leq_{\rm rh} Y_i$, $i = 1, 2, \ldots, m$. Then*

$$\max\{X_1, X_2, \ldots, X_m\} \leq_{\rm rh} \max\{Y_1, Y_2, \ldots, Y_m\}.$$

**Theorem 1.B.60.** *Let $(X_i, Y_i)$, $i = 1, 2, \ldots, m$, be independent pairs of absolutely continuous random variables such that $X_i \leq_{\rm rh} Y_i$, $i = 1, 2, \ldots, m$. Suppose that the $X_i$'s are identically distributed and that the $Y_i$'s are identically distributed. Then*

$$X_{(k:m)} \leq_{\rm rh} Y_{(k:m)}, \quad k = 1, 2, \ldots, m.$$

**Theorem 1.B.61.** *Let $X_1, X_2, \ldots, X_m$ be $m$ independent (not necessarily identically distributed) random variables, and let $Y_1, Y_2, \ldots, Y_n$ be other $n$ independent (not necessarily identically distributed) random variables, all having absolutely continuous distributions with support $(a, b)$ for some $a < b$. If $X_i \le_{\mathrm{rh}} Y_j$ for all $i$ and $j$, then*

$$X_{(j:m)} \le_{\mathrm{rh}} Y_{(i:n)} \quad \text{whenever } i - j \ge \max\{0, n-m\}.$$

Finally, the reversed hazard analog of Theorem 1.B.38 is the following.

**Theorem 1.B.62.** *The random variable $X$ with support $(a, b)$, for some $-\infty \le a < b \le \infty$, has decreasing [increasing] reversed hazard rate if, and only if, one of the following equivalent conditions holds:*

(i) $[X - t \big| X < t] \ge_{\mathrm{rh}} [\le_{\mathrm{rh}}] \; [X - t' \big| X < t']$ *whenever* $a < t \le t' < b$.
(ii) $X \le_{\mathrm{rh}} [\ge_{\mathrm{rh}}] \; [X - t \big| X < t]$ *for all* $t \in (a, b)$ (*when $X$ is a nonpositive random variable*).
(iii) $X + t \le_{\mathrm{rh}} [\ge_{\mathrm{rh}}] \; X + t'$ *whenever* $a < t \le t' < b$.

**Corollary 1.B.63.** *Let $X$ and $Y$ be two independent random variables with decreasing reversed hazard rates. Then $X+Y$ has a decreasing reversed hazard rate.*

## 1.C The Likelihood Ratio Order

### 1.C.1 Definition

Let $X$ and $Y$ be continuous [discrete] random variables with densities [discrete densities] $f$ and $g$, respectively, such that

$$\frac{g(t)}{f(t)} \quad \text{increases in } t \text{ over the union of the supports of } X \text{ and } Y \tag{1.C.1}$$

(here $a/0$ is taken to be equal to $\infty$ whenever $a > 0$), or, equivalently,

$$f(x)g(y) \ge f(y)g(x) \quad \text{for all } x \le y. \tag{1.C.2}$$

Then $X$ is said to be *smaller than $Y$ in the likelihood ratio order* (denoted by $X \le_{\mathrm{lr}} Y$). By integrating (1.C.2) over $x \in A$ and $y \in B$, where $A$ and $B$ are measurable sets in $\mathbb{R}$, it is seen that (1.C.2) is equivalent to

$$P\{X \in A\}P\{Y \in B\} \ge P\{X \in B\}P\{Y \in A\}$$
$$\text{for all measurable sets } A \text{ and } B \text{ such that } A \le B, \tag{1.C.3}$$

where $A \le B$ means that $x \in A$ and $y \in B$ imply that $x \le y$. Note that condition (1.C.3) does not directly involve the underlying densities, and thus it applies uniformly to continuous distributions, or to discrete distributions, or even to mixed distributions.

At a first glance (1.C.1) or (1.C.2) or (1.C.3) seem to be unintuitive technical conditions. However, it turns out that in many situations they are very easy to verify, and this is one of the major reasons for the usefulness and importance of the order $\le_{\text{lr}}$. It is also easy to verify by a simple differentiation (at least when $X$ and $Y$ have the same support) that

$$X \le_{\text{lr}} Y \Longleftrightarrow GF^{-1} \text{ is convex.} \tag{1.C.4}$$

Here $F$ and $G$ are the distribution functions of $X$ and $Y$, respectively.

### 1.C.2 The relation between the likelihood ratio and the hazard and reversed hazard orders

Note that from (1.C.1) it follows (in the continuous case) that

$$\int_{t=x}^{y}\int_{t'=y}^{\infty} f(t)g(t')\mathrm{d}t'\mathrm{d}t \ge \int_{t=y}^{\infty}\int_{t'=x}^{y} f(t')g(t)\mathrm{d}t'\mathrm{d}t \quad \text{for all } x \le y,$$

which, in turn, implies that

$$\int_{x}^{\infty} f(t)\mathrm{d}t \int_{y}^{\infty} g(t')\mathrm{d}t' \ge \int_{x}^{\infty} g(t)\mathrm{d}t \int_{y}^{\infty} f(t')\mathrm{d}t' \quad \text{for all } x \le y,$$

that is, (1.B.4). We thus have shown a part of the following result. The other parts of the next theorem are proven similarly (recall that the discrete versions of the orders $\le_{\text{hr}}$ and $\le_{\text{rh}}$ are defined in (1.B.9) and (1.B.44), respectively).

**Theorem 1.C.1.** *If $X$ and $Y$ are two continuous or discrete random variables such that $X \le_{\text{lr}} Y$, then $X \le_{\text{hr}} Y$ and $X \le_{\text{rh}} Y$ (and therefore $X \le_{\text{st}} Y$).*

*Remark 1.C.2.* Neither of the orders $\le_{\text{hr}}$ and $\le_{\text{rh}}$ (even if both hold simultaneously) implies the order $\le_{\text{lr}}$. In order to see it let $X$ be a uniform random variable over the set $\{1, 2, 3, 4\}$ and let $Y$ have the probabilities $P\{Y = 1\} = .1$, $P\{Y = 2\} = .3$, $P\{Y = 3\} = .2$, and $P\{Y = 4\} = .4$. Then it is not true that $X \le_{\text{lr}} Y$, however, in this case we have that $X \le_{\text{hr}} Y$ and also that $X \le_{\text{rh}} Y$.

*Remark 1.C.3.* Using Theorem 1.C.1 we can now give a proof of Theorem 1.A.22. Let $F$ and $f$ denote, respectively, the distribution function and the density function of $X_1$. Given $X_{(i-1:m)} = u$ and $X_{(i+1:m)} = v$, the conditional density of $U_{(i:m)}$ at the point $w$ is $\frac{f(u+w)}{F(v)-F(u)}$, $0 \le w \le v-u$, and the conditional density of $U_{(i+1:m)}$ at the point $w$ is $\frac{f(v-w)}{F(v)-F(u)}$, $0 \le w \le v-u$. Since $f$ is increasing [decreasing] it is seen that, conditionally, $U_{(i:m)} \ge_{\text{lr}} [\le_{\text{lr}}]\ U_{(i+1:m)}$, and therefore, by Theorem 1.C.1, $U_{(i:m)} \ge_{\text{st}} [\le_{\text{st}}]\ U_{(i+1:m)}$. Theorem 1.A.22 now follows from Theorem 1.A.3(d).

Although neither of the orders $\le_{\rm hr}$ and $\le_{\rm rh}$ implies the order $\le_{\rm lr}$ (see Remark 1.C.2), the following result gives a simple condition under which this is actually the case. The proof is immediate and is therefore omitted.

**Theorem 1.C.4.** *Let $X$ and $Y$ be two random variables with distribution functions $F$ and $G$, (discrete or continuous) hazard rate functions $r$ and $q$, and (discrete or continuous) reversed hazard rate functions $\tilde{r}$ and $\tilde{q}$, respectively.*

(a) *If $X \le_{\rm hr} Y$ and if $\frac{q(t)}{r(t)}$ increases in $t$, then $X \le_{\rm lr} Y$.*
(b) *If $X \le_{\rm rh} Y$ and if $\frac{\tilde{q}(t)}{\tilde{r}(t)}$ increases in $t$, then $X \le_{\rm lr} Y$.*

### 1.C.3 Some properties and characterizations

The usual stochastic order has the useful and important constructive property described in Theorem 1.A.1. There is no analogous property associated with the likelihood ratio order. Therefore it is of importance to understand better the relationship between the orders $\le_{\rm st}$ and $\le_{\rm lr}$. We already know from Theorems 1.C.1 and 1.B.1 that the likelihood ratio order implies the usual stochastic order. The following result characterizes the likelihood ratio order by means of the order $\le_{\rm st}$. It says that $X \le_{\rm lr} Y$ if, and only if, for any interval $I$, the conditional distribution of $X$, given that $X \in I$, is stochastically smaller than the conditional distribution of $Y$, given that $Y \in I$.

As in Section 1.A.3, $[Z \big| A]$ denotes any random variable that has as its distribution the conditional distribution of $Z$ given $A$. It is of interest to contrast the next result with (1.B.7) and (1.B.43).

**Theorem 1.C.5.** *The two random variables $X$ and $Y$ satisfy $X \le_{\rm lr} Y$ if, and only if,*

$$[X \big| a \le X \le b] \le_{\rm st} [Y \big| a \le Y \le b] \quad \textit{whenever } a \le b. \tag{1.C.5}$$

*Proof.* Suppose that (1.C.5) holds. Select an $a$ and a $b$ such that $a < b$. Then

$$\frac{P\{u \le X \le b\}}{P\{a \le X \le b\}} \le \frac{P\{u \le Y \le b\}}{P\{a \le Y \le b\}} \quad \text{whenever } u \in [a, b].$$

It follows then that

$$\frac{P\{a \le X < u\}}{P\{u \le X \le b\}} \ge \frac{P\{a \le Y < u\}}{P\{u \le Y \le b\}} \quad \text{whenever } u \in [a, b].$$

That is,

$$\frac{P\{a \le X < u\}}{P\{a \le Y < u\}} \ge \frac{P\{u \le X \le b\}}{P\{u \le Y \le b\}} \quad \text{whenever } u \in [a, b].$$

In particular, for $u < b \le v$,

$$\frac{P\{u \le X < b\}}{P\{u \le Y < b\}} \ge \frac{P\{b \le X \le v\}}{P\{b \le Y \le v\}}.$$

Therefore, when $X$ and $Y$ are continuous random variables,

$$\frac{P\{a \le X < u\}}{P\{a \le Y < u\}} \ge \frac{P\{b \le X \le v\}}{P\{b \le Y \le v\}} \quad \text{whenever } a < u \le b \le v.$$

Now let $a \to u$ and $b \to v$ to obtain (1.C.2). The proof for discrete random variables is similar.

Conversely, suppose that $X \le_{\rm lr} Y$, then clearly, $[X\big|a \le X \le b] \le_{\rm lr} [Y\big|a \le Y \le b]$ whenever $a < b$ (see also Theorem 1.C.6). From Theorems 1.C.1 and 1.B.1 we obtain (1.C.5). $\square$

The likelihood ratio order is preserved under general truncations of the involved random variables. This is stated in the next theorem, the proof of which follows directly from (1.C.2).

**Theorem 1.C.6.** *If $X$ and $Y$ are two random variables such that $X \le_{\rm lr} Y$, then for any measurable set $A \subseteq \mathbb{R}$ we have $[X\big|X \in A] \le_{\rm lr} [Y\big|Y \in A]$.*

By combining Theorems 1.C.5 and 1.C.6 it is seen that $X \le_{\rm lr} Y$ if, and only if,

$$[X\big|X \in A] \le_{\rm st} [Y\big|Y \in A] \quad \text{for all measurable sets } A \subseteq \mathbb{R}. \tag{1.C.6}$$

In fact, one can take (1.C.6) as the definition of the likelihood ratio order. The advantage of this approach is that it does not directly involve the underlying densities, and thus, similarly to condition (1.C.3), it applies uniformly to continuous distributions, or to discrete distributions, or even to mixed distributions.

Using the characterization (1.C.3), it is not hard to obtain the following result.

**Theorem 1.C.7.** *Let $\{X_j,\ j = 1, 2, \dots\}$ and $\{Y_j,\ j = 1, 2, \dots\}$ be two sequences of random variables such that $X_j \to_{\rm st} X$ and $Y_j \to_{\rm st} Y$ as $j \to \infty$. If $X_j \le_{\rm lr} Y_j$, $j = 1, 2, \dots$, then $X \le_{\rm lr} Y$.*

Let $\psi$ be a strictly monotone increasing [decreasing] differentiable function with inverse $\psi^{-1}$. If $X$ has the density function $f$, then $\psi(X)$ has the density function $(f\psi^{-1})/(\psi'(\psi^{-1}))$. Similarly, if $Y$ has the density function $g$, then $\psi(Y)$ has the density function $(g\psi^{-1})/(\psi'(\psi^{-1}))$. If $X \le_{\rm lr} Y$, then from (1.C.1) it follows that $\frac{(f\psi^{-1})(u)/(\psi'(\psi^{-1}(u)))}{(g\psi^{-1})(u)/(\psi'(\psi^{-1}(u)))}$ decreases [increases] over the unions of the supports of $\psi(X)$ and $\psi(Y)$. We have thus proved an important special case of Theorem 1.C.8 below. For discrete random variables the result is proven in a similar manner. When $\psi$ is *just* monotone (rather than *strictly* monotone) the result is still true, but the preceding simple argument is no longer sufficient for its proof.

**Theorem 1.C.8.** *If $X \le_{\rm lr} Y$ and $\psi$ is any increasing [decreasing] function, then $\psi(X) \le_{\rm lr} [\ge_{\rm lr}]\ \psi(Y)$.*

If $X_1 \le_{\rm lr} Y_1$ and $X_2 \le_{\rm lr} Y_2$, where $X_1$ and $X_2$ are independent random variables, and $Y_1$ and $Y_2$ are also independent random variables, then it is not necessarily true that $X_1 + X_2 \le_{\rm lr} Y_1 + Y_2$. However, if these random variables have logconcave densities, then it is true. In fact, a slightly stronger result is true:

**Theorem 1.C.9.** *Let $(X_i, Y_i)$, $i = 1, 2, \ldots, m$, be independent pairs of random variables such that $X_i \le_{\rm lr} Y_i$, $i = 1, 2, \ldots, m$. If $X_i, Y_i$, $i = 1, 2, \ldots, m$, all have (continuous or discrete) logconcave densities, except possibly one $X_l$ and one $Y_k$ $(l \ne k)$, then*

$$\sum_{i=1}^{m} X_i \le_{\rm lr} \sum_{i=1}^{m} Y_i.$$

*Proof.* Since a convolution of random variables with logconcave densities has a logconcave density, it is enough to show that if $W_1$, $W_2$, and $Z$ are independent random variables such that $W_1 \le_{\rm lr} W_2$, and $Z$ has a logconcave density function, then $W_1 + Z \le_{\rm lr} W_2 + Z$. We will give the proof for the continuous case; the proof for the discrete case is similar. Let $f_{W_i}$, $f_{W_i+Z}$, $i = 1, 2$, and $f_Z$ denote the density functions of the indicated random variables. Then

$$f_{W_i+Z}(t) = \int_{-\infty}^{\infty} f_Z(t-w) f_{W_i}(w)\, dw, \quad i = 1, 2,\ t \in \mathbb{R}.$$

The assumption $W_1 \le_{\rm lr} W_2$ means that $f_{W_i}(w)$, as a function of $w$ and of $i \in \{1, 2\}$, is $\mathrm{TP}_2$. The logconcavity of $f_Z$ means that $f_Z(t-w)$, as a function of $t$ and of $w$, is $\mathrm{TP}_2$. Therefore, by the basic composition formula (Karlin [275]) we see that $f_{W_i+Z}(t)$ is $\mathrm{TP}_2$ in $i \in \{1, 2\}$ and $t$; that is, $W_1 + Z \le_{\rm lr} W_2 + Z$. □

*Example 1.C.10.* Consider $m$ independent Bernoulli trials with probability $p_i$ of success in the $i$th trial. Let $q(k, \boldsymbol{p})$ denote the probability of $k$ successes, $k = 1, 2, \ldots, m$, where $\boldsymbol{p} = (p_1, p_2, \ldots, p_m)$. Then $q(k+1, \boldsymbol{p})/q(k, \boldsymbol{p})$ is increasing in each $p_i$ for $k = 0, 1, \ldots, m-1$. In order to see it, let $X_i$ be a Bernoulli random variable with probability $p_i$ of success, $i = 1, 2, \ldots, m$, and assume that the $X_i$'s are independent. Similarly, let $Y_i$ be a Bernoulli random variable with probability $p_i'$ of success, $i = 1, 2, \ldots, m$, and assume that the $Y_i$'s are independent. Obviously, the discrete density functions of the $X_i$'s and of the $Y_i$'s are logconcave, and if $\boldsymbol{p} \le \boldsymbol{p}'$, then $X_i \le_{\rm lr} Y_i$, $i = 1, 2, \ldots, m$. The stated result thus follows from Theorem 1.C.9.

For nonnegative random variables, Theorem 1.C.9 can be generalized further by having more $Y_i$'s summed than $X_i$'s. Under the assumptions of Theorem 1.C.9, one then obtains, for $m \le n$, that

$$\sum_{i=1}^{m} X_i \le_{\rm lr} \sum_{i=1}^{n} Y_i.$$

Of course, in this case, for $m+1 \le i \le n$, the $Y_i$'s only need to have logconcave densities—they do not have to have corresponding $X_i$'s to which they need to be comparable in the order $\le_{\rm lr}$. One may expect that the latter inequality can be extended to the following one:

$$\sum_{i=1}^{M} X_i \le_{\rm lr} \sum_{i=1}^{N} Y_i,$$

where $M$ and $N$ are two discrete positive integer-valued random variables, independent of the $X_i$'s and of the $Y_i$'s, respectively, such that $M \le_{\rm lr} N$. Indeed this inequality is true under some additional assumptions on the distributions of the $X_i$'s and the $Y_i$'s that will not be stated here. An important special case is the following theorem.

**Theorem 1.C.11.** *Let $\{X_i,\ i = 1, 2, \dots\}$ be a sequence of nonnegative independent random variables with logconcave densities. Let $M$ and $N$ be two discrete positive integer-valued random variables such that $M \le_{\rm lr} N$, and assume that $M$ and $N$ are independent of the $X_i$'s. Then*

$$\sum_{i=1}^{M} X_i \le_{\rm lr} \sum_{i=1}^{N} X_i.$$

In Pellerey [445] it is claimed that the conclusion of Theorem 1.C.11 holds even under the weaker assumption that $M \le_{\rm hr} N$ (in the sense of (1.B.9) or (1.B.10)). However, there is a mistake in [445] (see Pellerey [446]).

It is of interest to compare Theorem 1.C.11 to the following result, which combines uses of the likelihood ratio and the hazard [reversed hazard] rate orders.

**Theorem 1.C.12.** *Let $\{X_i,\ i = 1, 2, \dots\}$ be a sequence of nonnegative independent random variables that are* IFR *[have decreasing reversed hazard rates]. Let $M$ and $N$ be two discrete positive integer-valued random variables such that $M \le_{\rm lr} N$, and assume that $M$ and $N$ are independent of the $X_i$'s. Then*

$$\sum_{i=1}^{M} X_i \le_{\rm hr} [\le_{\rm rh}] \sum_{i=1}^{N} X_i.$$

Note that the hazard rate part of Theorem 1.C.12 is weaker than Theorem 1.B.7 because of Theorem 1.C.1.

The hazard rate order can be characterized by means of the likelihood ratio order and the appropriate equilibrium age variables. Recall from (1.A.20) that for nonnegative random variables $X$ and $Y$ with finite means we denote by $A_X$ and $A_Y$ the corresponding asymptotic equilibrium ages. The following result is immediate from (1.B.3) and (1.C.1).

**Theorem 1.C.13.** *Let $X$ and $Y$ be two nonnegative random variables with finite positive means. Then $X \le_{\rm hr} Y$ if, and only if, $A_X \le_{\rm lr} A_Y$.*

In light of Theorem 1.C.13 it is of interest to note that the order $\le_{\rm lr}$ can also be used to characterize the hazard rate order as is described in the next theorem. Let $X$ and $Y$ be two nonnegative random variables with finite means and suppose that $X \le_{\rm st} Y$ and that $EX < EY$. Let $F$ and $G$ be the distribution functions of $X$ and of $Y$, respectively. Define the random variable $Z_{X,Y}$ as the random variable that has the density function $h$ given by

$$h(z) = \frac{\overline{G}(z) - \overline{F}(z)}{EY - EX}, \quad z \ge 0. \tag{1.C.7}$$

**Theorem 1.C.14.** *Let $X$ and $Y$ be two nonnegative random variables with finite means such that $X \le_{\rm st} Y$ and such that $EY > EX > 0$. Then*

$$A_X \le_{\rm lr} Z_{X,Y} \Longleftrightarrow A_Y \le_{\rm lr} Z_{X,Y} \Longleftrightarrow X \le_{\rm hr} Y,$$

*where $Z_{X,Y}$ has the density function given in* (1.C.7).

*Proof.* Denote by $f_{\rm e}$ the density function of $A_Y$. Then, using (1.A.20), we obtain

$$\frac{h(x)}{f_{\rm e}(x)} = \frac{EY}{EY - EX}\left(1 - \frac{\overline{F}(x)}{\overline{G}(x)}\right), \quad x \ge 0,$$

and the second stated equivalence follows from (1.C.1) and (1.B.3). The proof of the first equivalence is similar. □

It is of interest to contrast Theorem 1.C.14 with Theorems 2.A.5 and 2.B.3.

The likelihood ratio order enjoys a closure under mixture property which is similar to the closure under mixture property of the hazard rate order stated in Theorem 1.B.8. This is stated next; the proof is similar to the proof of Theorem 1.B.8; we omit the details.

**Theorem 1.C.15.** *Let $X$, $Y$, and $\Theta$ be random variables such that $[X \big| \Theta = \theta] \le_{\rm lr} [Y \big| \Theta = \theta']$ for all $\theta$ and $\theta'$ in the support of $\Theta$. Then $X \le_{\rm lr} Y$.*

As a corollary of Theorem 1.C.15 we obtain the following result.

**Corollary 1.C.16.** *Let $N$ be a positive integer-valued random variable, and let $X_i$, $i = 1, 2, \ldots$, be random variables which are independent of $N$. Let $Y$ be a random variable such that $X_i \le_{\rm lr} Y$, $i = 1, 2, \ldots$. Then $X_N \le_{\rm lr} Y$.*

Consider now a family of (continuous or discrete) density functions $\{g_\theta, \theta \in \mathcal{X}\}$ where $\mathcal{X}$ is a subset of the real line. As in Section 1.A.3 let $X(\theta)$ denote a random variable with density function $g_\theta$. For any random variable $\Theta$ with support in $\mathcal{X}$, and with distribution function $F$, let us denote by $X(\Theta)$ a random variable with density function $h$ given by

$$h(y) = \int_{\mathcal{X}} g_\theta(y) \mathrm{d}F(\theta), \quad y \in \mathbb{R}.$$

The following result generalizes both Theorems 1.C.8 and 1.C.15, just as Theorem 1.A.6 generalized parts (a) and (c) of Theorem 1.A.3.

**Theorem 1.C.17.** *Consider a family of density functions $\{g_\theta,\ \theta \in \mathcal{X}\}$ as above. Let $\Theta_1$ and $\Theta_2$ be two random variables with supports in $\mathcal{X}$ and distribution functions $F_1$ and $F_2$, respectively. Let $Y_1$ and $Y_2$ be two random variables such that $Y_i =_{\mathrm{st}} X(\Theta_i)$, $i = 1, 2$, that is, suppose that the density function of $Y_i$ is given by*

$$h_i(y) = \int_{\mathcal{X}} g_\theta(y) \mathrm{d}F_i(\theta), \quad y \in \mathbb{R},\ i = 1, 2.$$

*If*

$$X(\theta) \leq_{\mathrm{lr}} X(\theta') \quad \textit{whenever } \theta \leq \theta', \tag{1.C.8}$$

*and if*

$$\Theta_1 \leq_{\mathrm{lr}} \Theta_2, \tag{1.C.9}$$

*then*

$$Y_1 \leq_{\mathrm{lr}} Y_2. \tag{1.C.10}$$

*Proof.* We give the proof under the assumption that $\Theta_1$ and $\Theta_2$ are absolutely continuous with density functions $f_1$ and $f_2$, respectively. The proof for the discrete case is similar. Assumption (1.C.8) means that $g_\theta(y)$, as a function of $\theta$ and of $y$, is $\mathrm{TP}_2$. Assumption (1.C.9) means that $f_i(\theta)$, as a function of $i \in \{1, 2\}$ and of $\theta$, is $\mathrm{TP}_2$. Therefore, by the basic composition formula (Karlin [275]) we see that $h_i(y)$ is $\mathrm{TP}_2$ in $i \in \{1, 2\}$ and $y$. That gives (1.C.10). □

A related result is the following; see also Theorems 1.B.19 and 1.B.54.

**Theorem 1.C.18.** *Let $X_1, X_2, \ldots, X_m$, $\Theta_1$, and $\Theta_2$ be independent nonnegative random variables. Define $N_j(t)$ for $t \geq 0$ and $j = 1, 2$ as in Theorem* 1.B.19*. If $\Theta_1 \leq_{\mathrm{lr}} \Theta_2$, then $N_1(t) \leq_{\mathrm{lr}} N_2(t)$ for all $t \geq 0$.*

The following example is an application of Theorem 1.C.17; it may be compared to Examples 1.A.7 and 1.B.16.

*Example 1.C.19.* Let $\Theta_1$ and $\Theta_2$ be two nonnegative random variables with distribution functions $F_1$ and $F_2$, respectively. Let $G$ be some absolutely continuous distribution function, and let $g$ be the corresponding density function. Denote by $X(\theta)$ a random variable with the distribution function $G^\theta$. Define $Y_i = X(\Theta_i)$; that is, the distribution function $H_i$ of $Y_i$ is given by

$$H_i(y) = \int_0^\infty G^\theta(y) \mathrm{d}F_i(\theta), \quad y \in \mathbb{R},\ i = 1, 2.$$

Note that the density function $k_\theta$ of $X(\theta)$ is given by

$$k_\theta(y) = \theta g(y) G^{\theta-1}(y), \quad y \in \mathbb{R}.$$

It is easy to verify that (1.C.8) holds. Thus, by Theorem 1.C.17, if $\Theta_1 \le_{\rm lr} \Theta_2$, then $Y_1 \le_{\rm lr} Y_2$.

Now, denote by $\widetilde{X}(\theta)$ a random variable with the survival function $\overline{G}^\theta$, where $\overline{G} \equiv 1 - G$. Define $\widetilde{Y}_i = \widetilde{X}(\Theta_i)$; that is, the survival function $\overline{\widetilde{H}}_i$ of $\widetilde{Y}_i$ is given by

$$\overline{\widetilde{H}}_i(y) = \int_0^\infty \overline{G}^\theta(y) \mathrm{d}F_i(\theta), \quad y \in \mathbb{R},\ i = 1, 2.$$

Note that the density function $\widetilde{k}_\theta$ of $\widetilde{X}(\theta)$ is given by

$$\widetilde{k}_\theta(y) = \theta g(y) \overline{G}^{\theta-1}(y), \quad y \in \mathbb{R}.$$

It is easy to verify now that $\widetilde{X}(\theta) \ge_{\rm lr} \widetilde{X}(\theta')$ whenever $\theta \le \theta'$. Thus, by an obvious modification of Theorem 1.C.17, if $\Theta_1 \le_{\rm lr} \Theta_2$, then $Y_1 \ge_{\rm lr} Y_2$.

In order to state a bivariate characterization result for the order $\le_{\rm lr}$ we define the following class of bivariate functions:

$$\mathcal{G}_{\rm lr} = \{\phi : \mathbb{R}^2 \to \mathbb{R} : \phi(x, y) \le \phi(y, x) \text{ whenever } x \le y\}.$$

**Theorem 1.C.20.** *Let $X$ and $Y$ be independent random variables. Then $X \le_{\rm lr} Y$ if, and only if,*

$$\phi(X, Y) \le_{\rm st} \phi(Y, X) \quad \textit{for all } \phi \in \mathcal{G}_{\rm lr}. \tag{1.C.11}$$

*Proof.* We give the proof for the absolutely continuous case only; the proof for the discrete case is similar. Suppose that (1.C.11) holds. Select $u$, $v$, $\Delta u > 0$, and $\Delta v > 0$ such that $u \le v$. As before, let $I_A$ denote the indicator function of the set $A$, and define $\phi(x, y) = I_{\{u - \Delta u \le y \le u, v \le x \le v + \Delta v\}}$. Clearly, $\phi \in \mathcal{G}_{\rm lr}$. Hence

$$\begin{aligned} P\{v \le X \le v + \Delta v, u - \Delta u \le Y \le u\} &= E\phi(X, Y) \\ &\le E\phi(Y, X) = P\{v \le Y \le v + \Delta v, u - \Delta u \le X \le u\}. \end{aligned}$$

Dividing both sides by $\Delta u \Delta v$ and letting $\Delta u \to 0$ and $\Delta v \to 0$, we obtain (1.C.2), that is, $X \le_{\rm lr} Y$.

Conversely, suppose that $X \le_{\rm lr} Y$. Let $\phi \in \mathcal{G}_{\rm lr}$ and let $\psi$ be an increasing function. Then

$$\begin{aligned} &E[\psi(\phi(Y, X)) - \psi(\phi(X, Y))] \\ &= \int_y \int_x [\psi(\phi(y, x)) - \psi(\phi(x, y))] f(x) g(y) \mathrm{d}x \mathrm{d}y \\ &= \int_y \int_{y \ge x} [\psi(\phi(y, x)) - \psi(\phi(x, y))][f(x)g(y) - f(y)g(x)] \mathrm{d}y \mathrm{d}x \ge 0. \square \end{aligned}$$

A typical application of Theorem 1.C.20 is shown in the proof of Theorem 6.B.15 in Chapter 6. Another typical application is the following result.

**Theorem 1.C.21.** *Let $X_1, X_2, \ldots, X_m$ be independent random variables such that $X_1 \le_{\rm lr} X_2 \le_{\rm lr} \cdots \le_{\rm lr} X_m$. Let $a_1, a_2, \ldots, a_m$ be constants such that $a_1 \le a_2 \le \cdots \le a_m$. Then*

$$\sum_{i=1}^{m} a_{m-i+1} X_i \le_{\rm st} \sum_{i=1}^{m} a_{\pi_i} X_i \le_{\rm st} \sum_{i=1}^{m} a_i X_i,$$

*where $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)$ denotes any permutation of $(1, 2, \ldots, m)$.*

*Proof.* We only give the proof when $m = 2$; the general case then can be obtained by pairwise interchanges. So, suppose that $X_1 \le_{\rm lr} X_2$ and that $a_1 \le a_2$. Define $\phi$ by $\phi(x, y) = a_1 y + a_2 x$. Then it is easy to verify that $\phi \in \mathcal{G}_{\rm lr}$. Thus, by Theorem 1.C.20, $a_1 X_2 + a_2 X_1 \le_{\rm st} a_1 X_1 + a_2 X_2$. □

The next two results are characterizations similar to the one in Theorem 1.C.20. They use the notation of Theorem 1.A.10, and their comparison is of interest. The proofs of the following two theorems are omitted.

**Theorem 1.C.22.** *Let $X$ and $Y$ be two independent random variables. Then $X \le_{\rm lr} Y$ if, and only if,*

$$E\phi_1(X, Y) \le E\phi_2(X, Y)$$

*for all functions $\phi_1$ and $\phi_2$ that satisfy $\Delta\phi_{21}(x, y) \ge 0$ whenever $x \le y$, and $\Delta\phi_{21}(x, y) \ge -\Delta\phi_{21}(y, x)$ whenever $x \le y$.*

**Theorem 1.C.23.** *Let $X$ and $Y$ be two independent random variables. Then $X \le_{\rm lr} Y$ if, and only if,*

$$\phi_1(X, Y) \le_{\rm st} \phi_2(X, Y)$$

*for all $\phi_1$ and $\phi_2$ that satisfy $\Delta\phi_{21}(x, y) \ge 0$ whenever $x \le y$, and $\phi_1(x, y) \le \phi_2(y, x)$ for all $x$ and $y$ (then, in particular, $\Delta\phi_{21}(x, y) \ge -\Delta\phi_{21}(y, x)$ whenever $x \le y$).*

The next theorem gives a characterization of the likelihood ratio order in the spirit of Theorems 1.B.11 and 1.B.49.

**Theorem 1.C.24.** *Let $X$ and $Y$ be two independent random variables. Then $X \le_{\rm lr} Y$ if, and only if,*

$$\begin{aligned}&[X \,|\, \min(X, Y) = z_1, \max(X, Y) = z_2] \\ &\qquad \le_{\rm lr} [Y \,|\, \min(X, Y) = z_1, \max(X, Y) = z_2] \quad \text{for all } z_1 \le z_2.\end{aligned}$$

*Proof.* First suppose that $X$ and $Y$ are absolutely continuous with density functions $f$ and $g$, respectively. Then

$$\begin{aligned} &P[X = z_1 \big| \min(X,Y) = z_1, \max(X,Y) = z_2] \\ &\quad = 1 - P[X = z_2 \big| \min(X,Y) = z_1, \max(X,Y) = z_2] \\ &\quad = P[Y = z_2 \big| \min(X,Y) = z_1, \max(X,Y) = z_2] \\ &\quad = 1 - P[Y = z_1 \big| \min(X,Y) = z_1, \max(X,Y) = z_2] \\ &\quad = \frac{f(z_1)g(z_2)}{f(z_1)g(z_2) + f(z_2)g(z_1)}, \end{aligned}$$

and the stated result follows.

The proof when $X$ and $Y$ are discrete is similar. □

Another similar characterization is given in Theorem 4.A.36.

The following result gives a Laplace transform characterization of the order $\le_{\rm lr}$. It should be compared with Theorems 1.A.13, 1.B.18, and 1.B.53. The proof is omitted.

**Theorem 1.C.25.** *Let $X_1$ and $X_2$ be two nonnegative random variables, and let $N_\lambda(X_1)$ and $N_\lambda(X_2)$ be as described in Theorem* 1.A.13. *Then*

$$X_1 \le_{\rm lr} X_2 \Longleftrightarrow N_\lambda(X_1) \le_{\rm lr} N_\lambda(X_2) \quad \textit{for all } \lambda > 0.$$

The implication $\Longrightarrow$ in Theorem 1.C.25 can be generalized in the same manner that Theorem 1.A.14 generalizes the implication $\Longrightarrow$ in Theorem 1.A.13. We will not state the result here since it is equivalent to Theorem 1.C.17.

Some interesting simple implications of the likelihood ratio order are described in the following theorem.

**Theorem 1.C.26.** *Let $X$, $Y$, and $Z$ be independent random variables. If $X \le_{\rm lr} Y$, then*

$$\begin{aligned} &[X \big| X + Y = v] \le_{\rm lr} [Y \big| X + Y = v] \quad \textit{for all } v, \\ &[X \big| X + Z = v] \le_{\rm lr} [Y \big| Y + Z = v] \quad \textit{for all } v, \quad \textit{and} \\ &[Z \big| X + Z = v] \ge_{\rm lr} [Z \big| Y + Z = v] \quad \textit{for all } v. \end{aligned}$$

*Proof.* We give only the proof of the first inequality; the proofs of the other two are similar. First suppose that $X$ and $Y$ are absolutely continuous with density functions $f$ and $g$, respectively. Denote the density function of $X+Y$ by $h$. Then the density function of $[Y \big| X + Y = v]$ is given by $\frac{f(v-\cdot)g(\cdot)}{h(v)}$, and the density function of $[X \big| X+Y = v]$ is given by $\frac{f(\cdot)g(v-\cdot)}{h(v)}$. It is now seen that the monotonicity of $g/f$ implies the monotonicity of the ratio of the above two density functions.

The proof when $X$ and $Y$ are discrete is similar. □

The next, easily proven, result is stronger than Theorems 1.A.15, 1.B.20, and 1.B.55.

**Theorem 1.C.27.** *Let $X$ be any random variable. Then $X_{(-\infty,a]}$ and $X_{(a,\infty)}$ are increasing in $a$ in the sense of the likelihood ratio order.*

A similar setting in which the order $\le_{\mathrm{hr}}$ gives rise to the order $\le_{\mathrm{lr}}$ is described in the following result.

**Theorem 1.C.28.** *Let $X$, $Y$, and $T$ be random variables such that $T$ is independent of $(X, Y)$. If $X \le_{\mathrm{hr}} Y$, then*

$$[T \big| T < X] \le_{\mathrm{lr}} [T \big| T < Y].$$

*Proof.* For simplicity assume that $T$ is absolutely continuous with density function $f_T$. Let $\overline{F}_X$ and $\overline{F}_Y$ be the survival functions of $X$ and $Y$. The density function of $[T \big| T < X]$ is proportional to $f_T \overline{F}_X$ and the density function of $[T \big| T < Y]$ is proportional to $f_T \overline{F}_Y$. The stated result now follows from (1.B.3). □

An analog of the remark after Theorem 1.B.21 is the following result; its proof is straightforward.

**Theorem 1.C.29.** *Let $X$ be a nonnegative, absolutely continuous, random variable with the density function $f$. Then $aX \le_{\mathrm{lr}} X$ for all $0 < a < 1$ if, and only if, $\log f(\mathrm{e}^x)$ is concave in $x \ge 0$.*

In the next result it is shown that a random variable, whose distribution is the mixture of two distributions of likelihood ratio ordered random variables, is bounded from below and from above, in the likelihood ratio order sense, by these two random variables.

**Theorem 1.C.30.** *Let $X$ and $Y$ be two random variables with distribution functions $F$ and $G$, respectively. Let $W$ be a random variable with the distribution function $pF + (1-p)G$ for some $p \in (0,1)$. If $X \le_{\mathrm{lr}} Y$, then $X \le_{\mathrm{lr}} W \le_{\mathrm{lr}} Y$.*

*Proof.* Let $A$ and $B$ be two measurable sets such that $A \le B$; see (1.C.3). If $X \le_{\mathrm{lr}} Y$, then

$$\begin{aligned} P\{X \in A\}P\{W \in B\} &= P\{X \in A\}(pP\{X \in B\} + (1-p)P\{Y \in B\}) \\ &\ge P\{X \in B\}(pP\{X \in A\} + (1-p)P\{Y \in A\}) \\ &= P\{X \in B\}P\{W \in A\}, \end{aligned}$$

where the inequality follows from (1.C.3). Thus, by (1.C.3), $X \le_{\mathrm{lr}} W$. The proof that $W \le_{\mathrm{lr}} Y$ is similar. □

Analogous to the result in Remark 1.A.18, it can be shown that some general sets of distribution functions on $\mathbb{R}$ are lattices with respect to the order $\leq_{\mathrm{lr}}$.

Let $X_1, X_2, \ldots, X_m$ be random variables, and let $X_{(k:m)}$ denote the corresponding $k$th order statistic, $k = 1, 2, \ldots, m$.

**Theorem 1.C.31.** *Let $X_1, X_2, \ldots, X_m$ be $m$ independent random variables, all with absolutely continuous distribution functions, all having the same support which is an interval of the real line, and all having differentiable densities.*
(a) *If*
$$X_1 \leq_{\mathrm{lr}} X_2 \leq_{\mathrm{lr}} \cdots \leq_{\mathrm{lr}} X_m,$$
*then*
$$\begin{aligned} X_{(k-1:m)} &\leq_{\mathrm{lr}} X_{(k:m)}, \quad 2 \leq k \leq m, \quad \text{and} \\ X_{(k-1:m-1)} &\leq_{\mathrm{lr}} X_{(k:m)}, \quad 2 \leq k \leq m. \end{aligned}$$
(b) *If*
$$X_1 \geq_{\mathrm{lr}} X_2 \geq_{\mathrm{lr}} \cdots \geq_{\mathrm{lr}} X_m,$$
*then*
$$X_{(k:m)} \leq_{\mathrm{lr}} X_{(k:m-1)}, \quad 1 \leq k \leq m-1.$$

A similar result for a finite population is the following. Consider a finite population of size $N$ which is linearly ordered, and suppose, without loss of generality, that it can be represented as $\{1, 2, \ldots, N\}$. Here let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(m)}$ denote now the order statistics corresponding to a simple random sample of size $m$ from this population.

**Theorem 1.C.32.** *Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(m)}$ be defined as in the preceding paragraph. Then*
$$X_{(1)} \leq_{\mathrm{lr}} X_{(2)} \leq_{\mathrm{lr}} \cdots \leq_{\mathrm{lr}} X_{(m)}.$$

*Proof.* For each $k \in \{1, 2, \ldots, m\}$, let $f_k$ denote the discrete density of $X_{(k)}$. Then
$$f_k(j) = \begin{cases} \frac{\binom{j-1}{k-1}\binom{N-j}{m-k}}{\binom{N}{m}}, & j = k, k+1, \ldots, k+N-m; \\ 0, & \text{otherwise}. \end{cases}$$
Therefore, for $k \in \{1, 2, \ldots, m-1\}$, we have
$$\frac{f_{k+1}(j)}{f_k(j)} = \begin{cases} 0, & j = k; \\ \frac{(m-k)(j-k)}{k(N-j-m+k+1)}, & j = k+1, k+2, \ldots, k+N-m; \\ \infty, & j = k+N-m+1. \end{cases}$$
This is increasing in $j$, and therefore $X_{(k)} \leq_{\mathrm{lr}} X_{(k+1)}$. $\square$

Under some conditions the likelihood ratio order is closed under the formation of order statistics. As above, let $X_{(j:m)}$ denote the $j$th order statistic associated with the random variables $X_1, X_2, \dots, X_m$, and let $Y_{(i:n)}$ denote similarly the $i$th order statistic associated with the random variables $Y_1, Y_2, \dots, Y_n$.

**Theorem 1.C.33.** *Let $X_1, X_2, \dots, X_m$ be $m$ independent random variables, and let $Y_1, Y_2, \dots, Y_n$ be other $n$ independent random variables. If*

$$X_j \le_{\rm lr} Y_i \quad \textit{for all } 1 \le j \le m \textit{ and } 1 \le i \le n,$$

*then*

$$X_{(j:m)} \le_{\rm lr} Y_{(i:n)} \quad \textit{whenever } j \le i \textit{ and } m-j \ge n-i.$$

*Proof.* First we give the proof when $X_1, X_2, \dots, X_m$ and $Y_1, Y_2, \dots, Y_n$ all have absolutely continuous distribution functions. In this proof we use an idea of Chan, Proschan, and Sethuraman [123].

Let $f_j$, $F_j$, and $\overline{F}_j \equiv 1 - F_j$ denote the density, distribution, and survival functions of $X_j$. Similarly, let $g_i$, $G_i$, and $\overline{G}_i$ denote the density, distribution, and survival functions of $Y_i$. The density functions of $X_{(j:m)}$ and $Y_{(i:n)}$ are given by

$$f_{X_{(j:m)}}(t) = \sum_{\boldsymbol{\pi}} f_{\pi_1}(t) F_{\pi_2}(t) \cdots F_{\pi_j}(t) \overline{F}_{\pi_{j+1}}(t) \cdots \overline{F}_{\pi_m}(t),$$

and

$$g_{Y_{(i:n)}}(t) = \sum_{\boldsymbol{\sigma}} g_{\sigma_1}(t) G_{\sigma_2}(t) \cdots G_{\sigma_i}(t) \overline{G}_{\sigma_{i+1}}(t) \cdots \overline{G}_{\sigma_n}(t),$$

where $\sum_{\boldsymbol{\pi}}$ signifies the sum over all permutations $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_m)$ of $(1, 2, \dots, m)$, and $\sum_{\boldsymbol{\sigma}}$ similarly denotes the sum over all permutations $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_n)$ of $(1, 2, \dots, n)$. Write

$$\frac{g_{Y_{(i:n)}}(t)}{f_{X_{(j:m)}}(t)} = \frac{\sum_{\boldsymbol{\sigma}} g_{\sigma_1}(t) G_{\sigma_2}(t) \cdots G_{\sigma_i}(t) \overline{G}_{\sigma_{i+1}}(t) \cdots \overline{G}_{\sigma_n}(t)}{\sum_{\boldsymbol{\pi}} f_{\pi_1}(t) F_{\pi_2}(t) \cdots F_{\pi_j}(t) \overline{F}_{\pi_{j+1}}(t) \cdots \overline{F}_{\pi_m}(t)}. \tag{1.C.12}$$

Now, for any choice of a permutation $\boldsymbol{\pi}$ of $(1, 2, \dots, m)$ and a permutation $\boldsymbol{\sigma}$ of $(1, 2, \dots, n)$ we have

$$\begin{aligned} &\frac{g_{\sigma_1}(t) G_{\sigma_2}(t) \cdots G_{\sigma_i}(t) \overline{G}_{\sigma_{i+1}}(t) \cdots \overline{G}_{\sigma_n}(t)}{f_{\pi_1}(t) F_{\pi_2}(t) \cdots F_{\pi_j}(t) \overline{F}_{\pi_{j+1}}(t) \cdots \overline{F}_{\pi_m}(t)} \\ &\qquad = \frac{g_{\sigma_1}(t)}{f_{\pi_1}(t)} \times \frac{G_{\sigma_2}(t) \cdots G_{\sigma_j}(t)}{F_{\pi_2}(t) \cdots F_{\pi_j}(t)} \times \frac{\overline{G}_{\sigma_{i+1}}(t) \cdots \overline{G}_{\sigma_n}(t)}{\overline{F}_{\pi_{m-n+i+1}}(t) \cdots \overline{F}_{\pi_m}(t)} \\ &\qquad\qquad \times \frac{G_{\sigma_{j+1}}(t) \cdots G_{\sigma_i}(t)}{\overline{F}_{\pi_{j+1}}(t) \cdots \overline{F}_{\pi_{m-n+i}}(t)}. \end{aligned}$$

Since $X_{\pi_1} \le_{\rm lr} Y_{\sigma_1}$ we see from (1.C.1) that the first fraction above is increasing in $t$. From $X_{\pi_k} \le_{\rm lr} Y_{\sigma_k}$ and Theorem 1.C.1 it follows that $X_{\pi_k} \le_{\rm rh} Y_{\sigma_k}$; but that means that $G_{\sigma_k}(t)/F_{\pi_k}(t)$ is increasing in $t$, $k = 2, \ldots, j$, and therefore the second fraction above is increasing in $t$. Similarly, from $X_{\pi_{k+m-n}} \le_{\rm lr} Y_{\sigma_k}$ and Theorem 1.C.1 it also follows that $X_{\pi_{k+m-n}} \le_{\rm hr} Y_{\sigma_k}$; but that means that $\overline{G}_{\sigma_k}(t)/\overline{F}_{\pi_{k+m-n}}(t)$ is increasing in $t$, $k = i+1, \ldots, n$, and therefore the third fraction above is increasing in $t$. The fourth fraction above obviously increases in $t$ too, and thus the whole product increases in $t$.

Note that if $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ are all nonnegative univariate functions, such that $a_j(t)/b_i(t)$ is increasing in $t$ for all $1 \le j \le m$ and $1 \le i \le n$, then $\sum_{j=1}^m a_j(t)/\sum_{i=1}^n b_i(t)$ is also increasing in $t$. It follows from this fact, and from (1.C.12), that $g_{Y_{(i:n)}}(t)/f_{X_{(j:m)}}(t)$ is increasing in $t$, and from (1.C.1) we obtain the stated result.

The result for the case when the random variables do not necessarily have absolutely continuous distribution functions follows from the above proof and the closure of the likelihood ratio order under weak convergence (Theorem 1.C.7). $\square$

Some of the results that are described in the following pages are stated in the literature (see Section 1.E) only for random variables with absolutely continuous distribution functions. However, by the closure of the likelihood ratio order under weak convergence (Theorem 1.C.7) these results are true also for random variables that do not necessarily have absolutely continuous distribution functions.

As a corollary of Theorem 1.C.33 we obtain the following result.

**Corollary 1.C.34.** *Let $X_1, X_2, \ldots, X_m$ be $m$ independent random variables and let $Y_1, Y_2, \ldots, Y_m$ be other $m$ independent random variables. If $X_j \le_{\rm lr} Y_i$, for all choices of $i$ and $j$, then $X_{(k)} \le_{\rm lr} Y_{(k)}$, $k = 1, 2, \ldots, m$.*

*Example 1.C.35.* Let $X$ and $Y$ be two independent random variables. If $X \le_{\rm lr} Y$, then $\min\{X, Y\} \le_{\rm lr} Y$ and $X \le_{\rm lr} \max\{X, Y\}$.

*Example 1.C.36.* Let $X$, $Y$, and $Z$ be three independent random variables. If $X \le_{\rm lr} Y \le_{\rm lr} Z$, then $\min\{X, Y\} \le_{\rm lr} \min\{Y, Z\}$ and $\max\{X, Y\} \le_{\rm lr} \max\{Y, Z\}$.

By letting all the $X_j$'s and $Y_i$'s in Theorem 1.C.33 be identically distributed we obtain the following result.

**Theorem 1.C.37.** *For positive integers $m$ and $n$, let $X_1, X_2, \ldots, X_{\max\{m,n\}}$ be independent identically distributed random variables. Then*

$$X_{(j:m)} \le_{\rm lr} X_{(i:n)} \quad \textit{whenever } j \le i \textit{ and } m - j \ge n - i.$$

In particular, it follows from Theorem 1.C.37 that

$$X_1 \le_{\rm lr} X_{(m:m)}, \quad m = 2, 3, \ldots \tag{1.C.13}$$

and

$$X_1 \geq_{\rm lr} X_{(1:m)}, \quad m = 2, 3, \ldots. \tag{1.C.14}$$

Note that (1.C.13) and (1.C.14) can also be obtained by induction from Example 1.C.35.

The following two corollaries of Theorem 1.C.37 can be compared to Theorems 1.B.27 and 1.B.28.

**Corollary 1.C.38.** *Let $X_1, X_2, \ldots, X_m$ be independent identically distributed random variables. Then $X_{(k-1:m-1)} \leq_{\rm lr} X_{(k:m)}$ for $k = 2, 3, \ldots, m$.*

**Corollary 1.C.39.** *Let $X_1, X_2, \ldots, X_m$ be independent identically distributed random variables. Then $X_{(k:m-1)} \geq_{\rm lr} X_{(k:m)}$ for $k = 1, 2, \ldots, m-1$.*

*Remark 1.C.40.* The likelihood ratio order can be used to provide a proof of Theorem 1.B.26. Let $X_1, X_2, \ldots, X_m$ be independent nonnegative random variables, and let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(m)}$ denote the corresponding order statistics. Fix $s$ and $t$ such that $0 \leq s \leq t$. For $j = 1, 2, \ldots, m$, define $M_j = 1$ if $X_j \leq s$, and $M_j = 0$ if $X_j > s$, and also define $N_j = 1$ if $X_j \leq t$, and $N_j = 0$ if $X_j > t$. Denote $M = \sum_{j=1}^m M_j$ and $N = \sum_{j=1}^m N_j$. Note that, for $j = 1, 2, \ldots, m$, we have

$$\begin{aligned} P\{M < j\} &= P\{X_{(j)} > s\}, \quad \text{and} \\ P\{N < j\} &= P\{X_{(j)} > t\}. \end{aligned}$$

Since $P\{M_j = 1\} = P\{X_j \leq s\} \leq P\{X_j \leq t\} = P\{N_j = 1\}$ it is easily seen that $M_j \leq_{\rm lr} N_j$, $j = 1, 2, \ldots, m$. Also, obviously, $M_j$ and $N_j$ have logconcave discrete density functions. Thus, from Theorem 1.C.9 it is seen that $M \leq_{\rm lr} N$. Therefore, by Theorem 1.C.1, $M \leq_{\rm rh} N$. Thus, from (1.B.44), we get that

$$\frac{P\{N < j\}}{P\{M < j\}} \quad \text{is increasing in } j \geq 1.$$

Therefore, for $k$ such that $1 \leq k \leq m-1$ we have

$$\frac{P\{X_{(k)} > t\}}{P\{X_{(k)} > s\}} = \frac{P\{N < k\}}{P\{M < k\}} \leq \frac{P\{N < k+1\}}{P\{M < k+1\}} = \frac{P\{X_{(k+1)} > t\}}{P\{X_{(k+1)} > s\}}.$$

From (1.B.3) it thus follows that $X_{(k)} \leq_{\rm hr} X_{(k+1)}$.

*Remark 1.C.41.* The likelihood ratio order can be used to provide a proof of Theorem 1.B.36. Let the $X_i$'s and the $Y_j$'s be as in that theorem. Assume that $X_i \leq_{\rm hr} Y_j$ for all $i, j$. We first show that there exists a random variable $Z$ with support $(a, b)$ such that $X_i \leq_{\rm hr} Z \leq_{\rm hr} Y_j$ for all $i, j$. Let $r_{X_i}$ and $r_{Y_j}$ denote the hazard rate functions of the indicated random variables. From the assumption that $X_i \leq_{\rm hr} Y_j$ for all $i, j$ it follows by (1.B.2) that

$$\min\{r_{X_1}(t), r_{X_2}(t), \ldots, r_{X_m}(t)\} \geq \max\{r_{Y_1}(t), r_{Y_2}(t), \ldots, r_{Y_n}(t)\}, \quad t \in (a, b).$$

Let $q$ be a function which satisfies

$$\min\{r_{X_1}(t), r_{X_2}(t), \dots, r_{X_m}(t)\} \geq q(t) \geq \max\{r_{Y_1}(t), r_{Y_2}(t), \dots, r_{Y_n}(t)\}, \quad t \in (a,b);$$

for example, let $q(t) = \min\{r_{X_1}(t), r_{X_2}(t), \dots, r_{X_m}(t)\}$. It can be shown that $q$ is indeed a hazard rate function. Let $Z$ be a random variable with the hazard rate function $q$. Then indeed $X_i \leq_{\rm hr} Z \leq_{\rm hr} Y_j$ for all $i, j$.

Now, let $Z_1, Z_2, \dots, Z_{\max\{m,n\}}$ be independent random variables which are distributed as $Z$. Then, for $j \leq i$ and $m - j \geq n - i$ we have

$$\begin{aligned} X_{(i:m)} &\leq_{\rm hr} Z_{(i:m)} && \text{(by Proposition 1.B.35)} \\ &\leq_{\rm lr} Z_{(j:n)} && \text{(by Theorem 1.C.37)} \\ &\leq_{\rm hr} Y_{(j:n)} && \text{(by Proposition 1.B.35),} \end{aligned}$$

and Theorem 1.B.36 follows from the fact that the likelihood ratio order implies the hazard rate order.

Recall that for a collection $X_1, X_2, \dots, X_m$ of nonnegative random variables, the spacings are defined by $U_{(i)} \equiv X_{(i)} - X_{(i-1)}$, $i = 1, 2, \dots, m$, where $X_{(0)} \equiv 0$. The following result may be compared with Theorems 1.A.19, 1.A.21, and 1.B.31.

**Theorem 1.C.42.** *Let $X_1, X_2, \dots, X_m$ be independent exponential random variables with possibly different parameters. Then*

$$U_{(1)} \leq_{\rm lr} \frac{m-i+1}{m} \cdot U_{(i)}, \quad i = 1, 2, \dots, m.$$

It is worth mentioning that Kochar and Kirmani [313] claimed that if $X_1, X_2, \dots, X_m$ are independent and identically distributed random variables with a common logconvex density, then $U_{(i)} \leq_{\rm lr} ((m-i)/(m-i+1))U_{(i+1)}$ for $i = 1, 2, \dots, m-1$. However, Misra and van der Meulen [396] showed via a counterexample that this is not correct.

For spacings that are not "normalized" we have the following results. We denote by $U_{(i:m)} = X_{(i:m)} - X_{(i-1:m)}$ the $i$th spacing that corresponds to a sample $X_1, X_2, \dots, X_m$ of size $m$.

**Theorem 1.C.43.** *Let $X_1, X_2, \dots, X_m, X_{m+1}$ be independent, identically distributed, nonnegative random variables with a common logconvex density. Then*

$$\begin{aligned} U_{(i:m)} &\leq_{\rm lr} U_{(i+1:m)}, \quad 1 \leq i \leq m-1, \\ U_{(i:m+1)} &\leq_{\rm lr} U_{(i:m)}, \quad 1 \leq i \leq m, \end{aligned}$$

*and*

$$U_{(i:m)} \leq_{\rm lr} U_{(i+1:m+1)}, \quad 1 \leq i \leq m.$$

Note that the three statements of the above theorem can be summarized as

$$U_{(j:m)} \le_{\text{lr}} U_{(i:n)} \quad \text{whenever } i - j \ge \max\{0, n-m\}.$$

We also have the following result.

**Theorem 1.C.44.** *Let $X_1, X_2, \ldots, X_m, X_{m+1}$ be independent, identically distributed, nonnegative random variables with a common logconcave density. Then*

$$U_{(i:m)} \ge_{\text{lr}} U_{(i+1:m+1)}, \quad 1 \le i \le m.$$

A comparison of spacings from two different samples, that is similar to Theorem 1.B.32, is described next. In fact, it will be argued after the next theorem that the next result strengthens Theorem 1.B.31. Here $U_{(i:m)} = X_{(i:m)} - X_{(i-1:m)}$ denotes, as before, the $i$th spacing that corresponds to the sample $X_1, X_2, \ldots, X_m$, and $V_{(j:n)}$ denotes, similarly, the $j$th spacing that corresponds to the sample $Y_1, Y_2, \ldots, Y_n$. Other results which give related comparisons can be found in Theorem 4.B.17 and in Examples 6.B.25 and 6.E.15.

**Theorem 1.C.45.** *For positive integers $m$ and $n$, let $X_1, X_2, \ldots, X_m$ be independent identically distributed random variables with an absolutely continuous common distribution function, and let $Y_1, Y_2, \ldots, Y_n$ be independent identically distributed random variables with a possibly different absolutely continuous common distribution function. If $X_1 \le_{\text{lr}} Y_1$, and if either $X_1$ or $Y_1$ is* DFR*, then*

$$(m-j+1)U_{(j:m)} \le_{\text{hr}} (n-i+1)V_{(i:n)} \quad \textit{whenever } i - j \ge \max\{0, n-m\}.$$

Taking $X_1 =_{\text{st}} Y_1$ in Theorem 1.C.45 it is seen that Theorem 1.B.31 is a consequence of Theorem 1.C.45.

In the following example it is shown that, under the proper conditions, random minima and maxima are ordered in the likelihood ratio order sense; see related results in Examples 3.B.39, 4.B.16, 5.A.24 and 5.B.13.

*Example 1.C.46.* Let $X_1, X_2, \ldots$ be a sequence of absolutely continuous nonnegative independent and identically distributed random variables with a common distribution function $F_{X_1}$ and a common density function $f_{X_1}$. Let $N_1$ and $N_2$ be two positive integer-valued random variables which are independent of the $X_i$'s. Denote $X_{(1:N_j)} \equiv \min\{X_1, X_2, \ldots, X_{N_j}\}$ and $X_{(N_j:N_j)} \equiv \max\{X_1, X_2, \ldots, X_{N_j}\}$, $j = 1, 2$. Then the density function of $X_{(N_j:N_j)}$ is given by

$$f_{X_{(N_j:N_j)}}(x) = \sum_{n=1}^{\infty} nF_{X_1}^{n-1}(x) f_{X_1}(x) P\{N_j = n\}, \quad x \ge 0,\ j = 1, 2.$$

If $N_1 \le_{\text{lr}} N_2$, then $P\{N_j = n\}$ is $\text{TP}_2$ in $n \ge 1$ and $j \in \{1, 2\}$. Also, $nF_{X_1}^{n-1}(x) f_{X_1}(x)$ is $\text{TP}_2$ in $n \ge 1$ and $x \ge 0$. Therefore, by the Basic Composition Formula (Karlin [275]) it follows that $f_{X_{(N_j:N_j)}}(x)$ is $\text{TP}_2$ in $x \ge 0$

and $j \in \{1,2\}$. That is, $X_{(N_1:N_1)} \le_{\rm lr} X_{(N_2:N_2)}$. In a similar fashion it can be shown also that $X_{(1:N_1)} \ge_{\rm lr} X_{(1:N_2)}$.

*Example 1.C.47.* Let $\{N(t),\ t \ge 0\}$ be a nonhomogeneous Poisson process with mean function $\Lambda$ (that is, $\Lambda(t) \equiv E[N(t)]$, $t \ge 0$), and let $T_1, T_2, \ldots$ be the successive epoch times. The survival function of $T_n$ is given by $P\{T_n > t\} = \sum_{i=0}^{n-1} \frac{(\Lambda(t))^i}{i!} \mathrm{e}^{-\Lambda(t)}$, $t \ge 0$, and the density function of $T_n$ is given by $f_n(t) = \lambda(t)\frac{(\Lambda(t))^{(n-1)}}{(n-1)!} \mathrm{e}^{-\Lambda(t)}$, $t \ge 0$, where $\lambda(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t)$, $n = 1, 2, \ldots$. It is easy to verify that $\frac{f_{n+1}(t)}{f_n(t)}$ is increasing in $t \ge 0$, $n = 1, 2, \ldots$, and therefore

$$T_n \le_{\rm lr} T_{n+1}, \quad n = 1, 2, \ldots .$$

Theorem 2.6 on page 182 of Kamps [273] extends Example 1.C.47 (as it extends Theorem 1.C.45) to the so called generalized order statistics. A further extension is described in Franco, Ruiz, and Ruiz [205].

The following example may be compared to Examples 1.B.24, 2.A.22, 3.B.38, 4.B.14, 6.B.41, 6.D.8, 6.E.13, and 7.B.13.

*Example 1.C.48.* Let $X$ and $Y$ be two absolutely continuous nonnegative random variables with survival functions $\overline{F}$ and $\overline{G}$ and density functions $f$ and $g$, respectively. Denote $\Lambda_1 = -\log \overline{F}$, $\Lambda_2 = -\log \overline{G}$, and $\lambda_i = \Lambda_i'$, $i = 1, 2$. Consider two nonhomogeneous Poisson processes $N_1 = \{N_1(t),\ t \ge 0\}$ and $N_2 = \{N_2(t),\ t \ge 0\}$ with mean functions $\Lambda_1$ and $\Lambda_2$ (see Example 1.B.13), respectively. Let $T_{i,1}, T_{i,2}, \ldots$ be the successive epoch times of process $N_i$, $i = 1, 2$. Note that $X =_{\rm st} T_{1,1}$ and $Y =_{\rm st} T_{2,1}$.

It turns out that, under some conditions, the likelihood ratio ordering of the first two epoch times implies the likelihood ratio ordering of all the corresponding later epoch times. Explicitly, it will be shown below that if $X \le_{\rm lr} Y$, and if

$$\frac{\Lambda_2(t)}{\Lambda_1(t)} \quad \text{is increasing in } t \ge 0, \tag{1.C.15}$$

then $T_{1,n} \le_{\rm lr} T_{2,n}$, $n \ge 1$.

From (1.B.24) it is easy to see that the density function $f_{1,n}$ of $T_{1,n}$ is given by

$$f_{1,n}(t) = f(t)\frac{(\Lambda_1(t))^{n-1}}{(n-1)!}, \quad t \ge 0,\ n \ge 1,$$

and that the density function $f_{2,n}$ of $T_{2,n}$ is given by

$$f_{2,n}(t) = g(t)\frac{(\Lambda_2(t))^{n-1}}{(n-1)!}, \quad t \ge 0,\ n \ge 1.$$

Thus,

$$\frac{f_{2,n}(t)}{f_{1,n}(t)} = \frac{g(t)}{f(t)}\left(\frac{\Lambda_2(t)}{\Lambda_1(t)}\right)^{n-1}.$$

Now, if $X \le_{\rm lr} Y$ and (1.C.15) holds, then $f_{2,n}/f_{1,n}$ is increasing and we obtain $T_{1,n} \le_{\rm lr} T_{2,n}$.

Now let $X_{i,n} \equiv T_{i,n} - T_{i,n-1}$, $n \ge 1$ (where $T_{i,0} \equiv 0$), be the inter-epoch times of the process $N_i$, $i = 1, 2$. Again, note that $X =_{\rm st} X_{1,1}$ and $Y =_{\rm st} X_{2,1}$. It turns out that, under some conditions, the likelihood ratio ordering of the first two inter-epoch times implies the likelihood ratio ordering of all the corresponding later inter-epoch times. Explicitly, it will be shown below that if $X \le_{\rm hr} Y$, if $f$ and $g$ are logconvex, and if (1.B.25) holds, then $X_{1,n} \le_{\rm lr} X_{2,n}$ for each $n \ge 1$.

First note that by Theorem 1.C.4 we have $X \le_{\rm lr} Y$. For the purpose of the following proof we denote $f$ by $f_1$ and $g$ by $f_2$. Let $g_{i,n}$ denote the density function of $X_{i,n}$, $i = 1, 2$. The stated result is obvious for $n = 1$, so let us fix an $n \ge 2$. From (1.B.26) we obtain

$$g_{i,n}(t) = \int_0^\infty \lambda_i(s) \frac{\Lambda_i^{n-2}(s)}{(n-2)!} f_i(s+t)\, ds, \quad t \ge 0,\ i = 1, 2.$$

As in Example 1.B.24, we have that

$$\lambda_i(t) \frac{\Lambda_i^{n-2}(t)}{(n-2)!} \quad \text{is TP}_2 \text{ in } (i, t).$$

The assumption $F_1 \le_{\rm lr} F_2$ implies that

$$f_i(s+t) \text{ is TP}_2 \text{ in } (i, s) \text{ and in } (i, t).$$

Finally, the logconvexity of $f_1$ and of $f_2$ means that

$$f_i(s+t) \text{ is TP}_2 \text{ in } (s, t).$$

Thus, by Theorem 5.1 on page 123 of Karlin [275], we get that $g_{i,n}(t)$ is $\text{TP}_2$ in $(i, t)$; that is, $X_{1,n} \le_{\rm lr} X_{2,n}$.

The following neat example compares a sum of independent heterogeneous exponential random variables with an Erlang random variable; it is of interest to compare it with Examples 1.A.24 and 1.B.5. We do not give the proof here.

*Example 1.C.49.* Let $X_i$ be an exponential random variable with mean $\lambda_i^{-1} > 0$, $i = 1, 2, \ldots, m$, and assume that the $X_i$'s are independent. Let $Y_i$, $i = 1, 2, \ldots, m$, be independent, identically distributed, exponential random variables with mean $\eta^{-1}$. Then

$$\sum_{i=1}^m X_i \ge_{\rm lr} \sum_{i=1}^m Y_i \Longleftrightarrow \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_m}{m} \le \eta.$$

A related example is the following. Recall from page 2 the definition of the majorization order $\prec$ among $n$-dimensional vectors. It is of interest to compare the example below with Example 3.B.34.

*Example 1.C.50.* Let $X_i$ be an exponential random variable with mean $\lambda_i^{-1} > 0$, $i = 1, 2, \ldots, m$, and let $Y_i$ be an exponential random variable with mean $\eta_i^{-1} > 0$, $i = 1, 2, \ldots, m$. If $(\lambda_1, \lambda_2, \ldots, \lambda_m) \succ (\eta_1, \eta_2, \ldots, \eta_m)$, then

$$\sum_{i=1}^{m} X_i \geq_{\mathrm{lr}} \sum_{i=1}^{m} Y_i.$$

The next example may be compared with Examples 1.A.25, 1.B.6, and 4.A.45.

*Example 1.C.51.* Let $X_i$ be a binomial random variable with parameters $n_i$ and $p_i$, $i = 1, 2, \ldots, m$, and assume that the $X_i$'s are independent. Let $Y$ be a binomial random variable with parameters $n$ and $p$ where $n = \sum_{i=1}^{m} n_i$. Then

$$\sum_{i=1}^{m} X_i \geq_{\mathrm{lr}} Y \Longleftrightarrow p \leq \frac{n}{\sum_{i=1}^{m}(n_i/p_i)},$$

and

$$\sum_{i=1}^{m} X_i \leq_{\mathrm{lr}} Y \Longleftrightarrow 1 - p \leq \frac{n}{\sum_{i=1}^{m}(n_i/(1-p_i))}.$$

The order $\leq_{\mathrm{lr}}$ can be used to characterize random variables with logconcave densities. The next result lists several such characterizations. It shows that logconcavity can be interpreted as an aging notion in reliability theory by a correct use of the likelihood ratio ordering. This theorem may be compared to Theorem 1.B.38.

**Theorem 1.C.52.** *The random variable $X$ has a logconcave density* (*that is, a Polya frequency of order* 2 (PF$_2$)) *if, and only if, one of the following equivalent conditions holds:*

(i) $[X - t \mid X > t] \geq_{\mathrm{lr}} [X - t' \mid X > t']$ *whenever* $t \leq t'$.
(ii) $X \geq_{\mathrm{lr}} [X - t \mid X > t]$ *for all* $t \geq 0$ (*when $X$ is a nonnegative random variable*).
(iii) $X + t \leq_{\mathrm{lr}} X + t'$ *whenever* $t \leq t'$.

Random variables that satisfy (i) in Theorem 1.C.52 (and hence any of the conditions of that theorem) are said to have the ILR (increasing likelihood ratio) property; see Section 13.D.2 by Righter in [515].

A multivariate extension of parts (i) and (ii) of Theorem 1.C.52 is given in Section 6.E.3.

Another connection between logconcavity and the likelihood ratio order is illustrated in the next result. It is worthwhile to compare the following result with Theorem 6.B.9 in Section 6.B.3.

**Theorem 1.C.53.** *Let $X_1, X_2, \ldots, X_m$ be independent random variables having logconcave density functions. Then*

$$\left[X_i \,\middle|\, \sum_{j=1}^{m} X_j = s\right] \le_{\mathrm{lr}} \left[X_i \,\middle|\, \sum_{j=1}^{m} X_j = s'\right] \quad \textit{whenever } s \le s',\ i = 1, 2, \ldots, m.$$

*Proof.* Since the convolution of logconcave density functions is logconcave, it is sufficient to prove the result for $m = 2$ and $i = 1$. Let $f_1$ and $f_2$ denote the density functions of $X_1$ and $X_2$, respectively. The conditional density of $X_1$, given $X_1 + X_2 = s$, is

$$f_{X_1|X_1+X_2=s}(x_1) = \frac{f_1(x_1)f_2(s-x_1)}{\int f_1(u)f_2(s-u)\mathrm{d}u}.$$

Thus,

$$\frac{f_{X_1|X_1+X_2=s'}(x_1)}{f_{X_1|X_1+X_2=s}(x_1)} = \frac{f_2(s'-x_1)\int f_1(u)f_2(s-u)\mathrm{d}u}{f_2(s-x_1)\int f_1(u)f_2(s'-u)\mathrm{d}u}. \tag{1.C.16}$$

The logconcavity of $f_2$ implies that the expression in (1.C.16)) increases in $x_1$, whenever $s \le s'$. By (1.C.1) the proof is complete. □

Theorems 1.C.52 and 1.C.53 have straightforward discrete analogs, which we do not state here. A few other properties of the order $\le_{\mathrm{lr}}$ can be found in Lemma 13.D.1 in Chapter 13 by Righter, and in (14.B.7) in Chapter 14 by Shanthikumar and Yao, in [515].

An interesting closure property of logconcave density functions is described in the following result.

**Theorem 1.C.54.** *Let $X_1, X_2, \ldots, X_m$ be independent, identically distributed random variables with a common logconcave density function. Then the ith order statistic $X_{(i:m)}$ also has a logconcave density function, $1 \le i \le m$.*

*Proof.* Let $f$, $F$, and $\overline{F}$ denote, respectively, the density, distribution, and survival function of $X_1$. Then the density function of $X_{(i:m)}$ is given by

$$f_{(i:m)}(x) = m\binom{m-1}{i-1}F^{i-1}(x)f(x)\overline{F}^{m-i}(x).$$

Since the logconcavity of $f$ implies the logconcavity of $F$ and of $\overline{F}$, it follows that $f_{(i:m)}$ is logconcave. □

Misra and van der Meulen [396] showed the preservation of logconcavity and logconvexity from the parent density to the density of the corresponding spacings.

The likelihood ratio order can be used to characterize some aging notions in reliability theory. Recall from (1.A.20) that for a nonnegative random variable $X$ with a finite mean we denote by $A_X$ the corresponding asymptotic equilibrium age. Recall from page 1 the definitions of the IFR and the DFR properties. The following result is immediate. It is of interest to contrast it with Theorems 1.A.31 and 1.B.40

**Theorem 1.C.55.** *The nonnegative random variable $X$ with finite mean is* IFR [DFR] *if, and only if,* $X \geq_{\text{lr}} [\leq_{\text{lr}}] A_X$.

An interesting comparison of asymptotic equilibrium ages is described in the next example. Recall from page 1 the definitions of the DMRL property.

*Example 1.C.56.* Let $X$ and $Y$ be two independent nonnegative DMRL random variables with survival functions $\overline{F}$ and $\overline{G}$, density functions $f$ and $g$, and asymptotic equilibrium ages $A_X$ and $A_Y$, respectively. Let $A_{\min\{X,Y\}}$ denote the asymptotic equilibrium age of $\min\{X,Y\}$. Then

$$\min\{A_X, A_Y\} \leq_{\text{lr}} A_{\min\{X,Y\}}.$$

In order to see this, assume, for simplicity, that the supports of $X$ and of $Y$ are $(0,\infty)$. Note that the density function of $\min\{A_X, A_Y\}$ is given by

$$f_{\min\{A_X,A_Y\}}(t) = (EXEY)^{-1}\Big(\overline{F}(t)\int_t^\infty \overline{G}(x)\,dx + \overline{G}(t)\int_t^\infty \overline{F}(x)\mathrm{d}x\Big), \quad t \geq 0,$$

and the density function of $A_{\min\{X,Y\}}$ is given by

$$f_{A_{\min\{X,Y\}}}(t) = \big(E[\min\{X,Y\}]\big)^{-1}\overline{F}(t)\overline{G}(t), \quad t \geq 0.$$

Therefore

$$\frac{f_{A_{\min\{X,Y\}}}(t)}{f_{\min\{A_X,A_Y\}}(t)} = \frac{EXEY}{E[\min\{X,Y\}]}\big(m(t)+l(t)\big)^{-1}, \quad t \geq 0,$$

where $m$ and $l$ are the mean residual life functions of $X$ and of $Y$, given by $m(t) = E[X-t\big|X>t]$ and $l(t) = E[Y-t\big|Y>t]$, $t \geq 0$. The functions $m$ and $l$ are decreasing by the DMRL assumptions, and therefore $\min\{A_X, A_Y\} \leq_{\text{lr}} A_{\min\{X,Y\}}$ by (1.C.1).

In the following example it is shown that if $X$ is increasing in $\Theta$ in the likelihood ratio sense, then the posterior distribution of $\Theta$ is increasing in $X$ in the same sense.

*Example 1.C.57.* Let $X$ be a random variable whose distribution function depends on the real parameter $\Theta$. Denote the prior density function of $\Theta$ by $\pi$, and denote the posterior density function of $\Theta$, given $X = x$, by $\pi^*(\cdot\big|x)$. Also, denote the conditional density of $X$, given $\Theta = \theta$ by $f(\cdot\big|\theta)$, and denote the marginal density of $X$ by $g$. If $X$ is increasing in $\Theta$ in the likelihood ratio sense (that is, if $[X\big|\Theta=\theta] \leq_{\text{lr}} [X\big|\Theta=\theta']$ whenever $\theta \leq \theta'$), then $\Theta$ is increasing in $X$ in the likelihood ratio sense (that is, $[\Theta\big|X=x] \leq_{\text{lr}} [\Theta\big|X=x']$ whenever $x \leq x'$). The proof of this statement is easy by noting that

$$\pi^*(\theta\big|x) = \frac{f(x\big|\theta)\pi(\theta)}{g(x)}.$$

An extension of Example 1.C.57 to the multivariate likelihood ratio order is given in Example 6.E.16.

*Example 1.C.58.* Let $X$ be a random variable whose distribution function depends on the random parameter $\Theta_1$ or, in other circumstances, on the random parameter $\Theta_2$. Denote the prior density functions, of $\Theta_1$ and $\Theta_2$, by $\pi_1$ and $\pi_2$, respectively, and denote the posterior density functions of $\Theta_1$ and $\Theta_2$, given $X = x$, by $\pi_1^*(\cdot\big|x)$ and $\pi_2^*(\cdot\big|x)$, respectively. Also, denote the conditional density of $X$, given $\Theta_1 = \theta$ or $\Theta_2 = \theta$, by $f(\cdot\big|\theta)$, and denote the marginal density of $X$ by $g_1$ or by $g_2$, according to whether $X$ depends on $\Theta_1$ or on $\Theta_2$. Then, for any $x$, we have that

$$\Theta_1 \le_{\mathrm{lr}} \Theta_2 \Longrightarrow [\Theta_1\big|X = x] \le_{\mathrm{lr}} [\Theta_2\big|X = x].$$

The proof of this statement is easy by noting that

$$\pi_i^*(\theta\big|x) = \frac{f(x\big|\theta)\pi_i(\theta)}{g_i(x)}, \quad i = 1, 2.$$

*Example 1.C.59.* Recall from Example 1.B.23 that for a nonnegative random variable $X$ with density function $f$, and for a nonnegative function $w$ such that $E[w(X)]$ exists, we denote by $X^w$ the random variable with the weighted density function $f_w$ given by

$$f_w(x) = \frac{w(x)f(x)}{E[w(X)]}, \quad x \ge 0. \tag{1.C.17}$$

Similarly, for another nonnegative random variable $Y$ with density function $g$, such that $E[w(Y)]$ exists, we denote by $Y^w$ the random variable with the density function $g_w$ given by

$$g_w(x) = \frac{w(x)g(x)}{E[w(Y)]}, \quad x \ge 0. \tag{1.C.18}$$

It is then obvious that $X \le_{\mathrm{lr}} Y \Longrightarrow X^w \le_{\mathrm{lr}} Y^w$.

*Example 1.C.60.* Let $X$ be a nonnegative random variable with density function $f$, and for a nonnegative function $w$ such that $E[w(X)]$ exists, let $X^w$ be the random variable with the weighted density function $f_w$ given in (1.C.17). It is then obvious that if $w$ is increasing [decreasing], then $X \le_{\mathrm{lr}} [\ge_{\mathrm{lr}}]\ X^w$. In particular, the inequality $X \le_{\mathrm{lr}} X^w$ holds when $X^w$ is the length-biased version of $X$; that is, when $w(x) = x$, $x \ge 0$.

*Example 1.C.61.* Let the random variable $X$ have a generalized skew normal distribution with parameters $n$ and $\lambda$, that is, suppose that its density function is given by

$$f(x; n, \lambda) = \frac{\Phi^n(\lambda x)\phi(x)}{C(n, \lambda)}, \quad x \in \mathbb{R},$$

where $\phi$ and $\Phi$ are, respectively, the density and distribution functions of a standard normal random variable, and $C(n, \lambda)$ is given by

$$C(n, \lambda) = \int_{-\infty}^{\infty} \Phi^n(\lambda x)\phi(x)\mathrm{d}x.$$

Let $Y$ have a generalized skew normal distribution with parameters $n_1$ and $\lambda$. It is easy to see that if $\lambda > [<]\ 0$ and $n \le n_1$, then $X \le_{\rm lr} [\ge_{\rm lr}]\ Y$.

### 1.C.4 Shifted likelihood ratio orders

In this subsection we consider only random variables with absolutely continuous distribution functions and interval supports (although it is possible to state and prove analogs of many of the results here also for discrete random variables). So let $X$ and $Y$ be such random variables. Let $l_X$ and $u_X$ be the left and the right endpoints of the support of $X$. Similarly define $l_Y$ and $u_Y$. The values $l_X$, $u_X$, $l_Y$, and $u_Y$ may be infinite. Let $f$ and $g$ denote the density functions of $X$ and $Y$, respectively. Suppose that

$$X - x \le_{\rm lr} Y \quad \text{for each } x \ge 0. \tag{1.C.19}$$

Then $X$ is said to be *smaller than $Y$ in the up shifted likelihood ratio order* (denoted as $X \le_{\rm lr\uparrow} Y$). Rewriting (1.C.19) using (1.C.1) it is seen that $X \le_{\rm lr\uparrow} Y$ if, and only if, for each $x \ge 0$ we have

$$\frac{g(t)}{f(t+x)} \quad \text{is increasing in } t \in (l_X - x, u_X - x) \cup (l_Y, u_Y). \tag{1.C.20}$$

It is readily apparent that

$$X \le_{\rm lr\uparrow} Y \Longrightarrow X \le_{\rm lr} Y.$$

The up shifted likelihood ratio order satisfies some closure properties given in the next theorem.

**Theorem 1.C.62.** (a) *Let $X_1, X_2, \ldots, X_m$ be a set of independent random variables and let $Y_1, Y_2, \ldots, Y_m$ be another set of independent random variables. If $X_i \le_{\rm lr\uparrow} Y_i$ for $i = 1, 2, \ldots, m$, then*

$$\sum_{j=1}^{m} X_j \le_{\rm lr\uparrow} \sum_{j=1}^{m} Y_j.$$

*That is, the up likelihood ratio order is closed under convolutions.*

(b) *Let $\{X_j,\ j = 1, 2, \ldots\}$ and $\{Y_j,\ j = 1, 2, \ldots\}$ be two sequences of random variables such that $X_j \to_{\rm st} X$ and $Y_j \to_{\rm st} Y$ as $j \to \infty$. If $X_j \le_{\rm lr\uparrow} Y_j$, $j = 1, 2, \ldots$, then $X \le_{\rm lr\uparrow} Y$.*

Shanthikumar and Yao [530] proved Theorem 1.C.62(a) by establishing a stochastic monotonicity property of birth and death processes. Hu and Zhu [242] provided a straightforward analytic proof of this result. This result is generalized in Hu, Nanda, Xie, and Zhu [237].

From Theorem 1.C.15 we obtain the following result.

**Theorem 1.C.63.** *Let $X$, $Y$, and $\Theta$ be random variables such that $[X\big|\Theta = \theta] \le_{\mathrm{lr}\uparrow} [Y\big|\Theta = \theta']$ for all $\theta$ and $\theta'$ in the support of $\Theta$. Then $X \le_{\mathrm{lr}\uparrow} Y$.*

Some further properties of the up shifted likelihood ratio order are listed in the following theorems.

**Theorem 1.C.64.** *Let $X$ and $Y$ be two absolutely continuous random variables with interval supports. If $X$ or $Y$ or both have logconcave densities, and if $X \le_{\mathrm{lr}} Y$, then $X \le_{\mathrm{lr}\uparrow} Y$.*

**Theorem 1.C.65.** *Let $X$ and $Y$ be two absolutely continuous random variables with differentiable densities on the respective interval supports. Then $X \le_{\mathrm{lr}\uparrow} Y$ if, and only if, there exists a random variable $Z$ with a logconcave density such that $X \le_{\mathrm{lr}} Z \le_{\mathrm{lr}} Y$.*

**Theorem 1.C.66.** *Let $X$ be an absolutely continuous random variable with an interval support. Then $X \le_{\mathrm{lr}\uparrow} X$ if, and only if, $f$ is logconcave on $(-\infty, \infty)$.*

*Example 1.C.67.* Let $X$ be a random variable with a density function $h$. For each $\theta \in (-\infty, \infty)$, let $X_\theta$ be a random variable with density function $f_\theta$ defined by

$$f_\theta(x) = h(x - \theta), \quad x \in (-\infty, \infty).$$

Then it is easy to see that $X_{\theta_1} \le_{\mathrm{lr}\uparrow} X_{\theta_2}$ whenever $\theta_1 \le \theta_2$ if, and only if, $X \le_{\mathrm{lr}\uparrow} X$; that is, by Theorem 1.C.66, if, and only if, $h$ is logconcave on $(-\infty, \infty)$.

A preservation result of the order $\le_{\mathrm{lr}\uparrow}$ is described next.

**Theorem 1.C.68.** *Let $X$ and $Y$ be two absolutely continuous random variables with interval supports. If $X \le_{\mathrm{lr}\uparrow} Y$ and if the density function of $X$ is increasing [respectively, decreasing] on $(l_Y, u_X)$, then $\phi(X) \le_{\mathrm{lr}\uparrow} \phi(Y)$ for any strictly increasing twice differentiable convex [respectively, concave] function $\phi$ (with first and second derivatives $\phi'$ and $\phi''$) such that $\phi''(x)/(\phi'(x))^2$ is increasing.*

A characterization of the relation $X \le_{\mathrm{lr}\uparrow} Y$ for nonnegative random variables is given next.

**Theorem 1.C.69.** *Let $X$ and $Y$ be two nonnegative absolutely continuous random variables with interval supports; that is, assume that $l_X \ge 0$ and $l_Y \ge 0$. Then $X \le_{\mathrm{lr}\uparrow} Y$ if, and only if,*

$$\big[X - x\big|X > x\big] \le_{\mathrm{lr}} Y \quad \textit{for all } x \in (l_X, u_X).$$

Another shifted likelihood ratio stochastic order is defined next. Let $X$ and $Y$ be two absolutely continuous random variables with support $[0,\infty)$. Suppose that

$$X \le_{\mathrm{lr}} [Y - x \big| Y > x] \quad \text{for all } x \ge 0.$$

Then $X$ is said to be *smaller than $Y$ in the down shifted likelihood ratio order* (denoted as $X \le_{\mathrm{lr}\downarrow} Y$).

Note that in the above definition only nonnegative random variables are compared. This is because for the down shifted likelihood ratio order it is not possible to take an analog of (1.C.19), such as $X \le_{\mathrm{lr}} Y - x$, as a definition. The reason is that here, by taking $x$ very large, it is seen that practically no random variables would satisfy such an order relation. Note that in the definition above, the right-hand side $[Y - x \big| Y > x]$ can take on (as $x$ varies) any value in the right neighborhood of 0. Therefore the support of the compared random variables is restricted here to be $[0,\infty)$.

Let $f$ and $g$ denote the density functions of $X$ and $Y$, respectively. An analog of (1.C.20) is the following:

$$X \le_{\mathrm{lr}\downarrow} Y \Longleftrightarrow \frac{g(t+x)}{f(t)} \text{ is increasing in } t \ge 0 \text{ for all } x \ge 0. \tag{1.C.21}$$

(A discrete version of the down shifted likelihood ratio order is defined and used in Section 6.B.3.)

It is readily apparent that for nonnegative random variables with support $[0,\infty)$ we have

$$X \le_{\mathrm{lr}\downarrow} Y \Longrightarrow X \le_{\mathrm{lr}} Y.$$

We describe now some further properties of the down shifted likelihood ratio order.

**Theorem 1.C.70.** *Let $\{X_j,\ j = 1,2,\ldots\}$ and $\{Y_j,\ j = 1,2,\ldots\}$ be two sequences of random variables, with support $[0,\infty)$, such that $X_j \to_{\mathrm{st}} X$ and $Y_j \to_{\mathrm{st}} Y$ as $j \to \infty$. If $X_j \le_{\mathrm{lr}\downarrow} Y_j$, $j = 1,2,\ldots$, then $X \le_{\mathrm{lr}\downarrow} Y$.*

The following result is an analog of Theorem 1.C.63, however, it does not follow at once from Theorem 1.C.15. Its proof can be found in Lillo, Nanda, and Shaked [361].

**Theorem 1.C.71.** *Let $X$, $Y$, and $\Theta$ be random variables such that $[X \big| \Theta = \theta]$ and $[Y \big| \Theta = \theta]$ are absolutely continuous and have the support $[0,\infty)$ for all $\theta$ in the support of $\Theta$. If $[X \big| \Theta = \theta] \le_{\mathrm{lr}\downarrow} [Y \big| \Theta = \theta']$ for all $\theta$ and $\theta'$ in the support of $\Theta$, then $X \le_{\mathrm{lr}\downarrow} Y$.*

More properties are listed next.

**Theorem 1.C.72.** *Let $X$ and $Y$ be two absolutely continuous random variables with support $[0,\infty)$. If $X$ or $Y$ or both have logconvex densities on $[0,\infty)$, and if $X \le_{\mathrm{lr}} Y$, then $X \le_{\mathrm{lr}\downarrow} Y$.*

**Theorem 1.C.73.** *Let $X$ and $Y$ be two absolutely continuous random variables with differentiable densities on their support $[0, \infty)$. Then $X \le_{\mathrm{lr}\downarrow} Y$ if, and only if, there exists a random variable $Z$ with a logconvex density on $[0, \infty)$ such that $X \le_{\mathrm{lr}} Z \le_{\mathrm{lr}} Y$.*

**Theorem 1.C.74.** *Let $X$ be an absolutely continuous random variable with support $[0, \infty)$. Then $X \le_{\mathrm{lr}\downarrow} X$ if, and only if, $f$ is logconvex on $[0, \infty)$.*

**Theorem 1.C.75.** *Let $X$ and $Y$ be two absolutely continuous random variables with support $[0, \infty)$. If $X \le_{\mathrm{lr}\downarrow} Y$ and if $Y$ has a decreasing density function on $[0, \infty)$, then $\phi(X) \le_{\mathrm{lr}\downarrow} \phi(Y)$ for any strictly increasing twice differentiable convex function $\phi : [0, \infty) \to [0, \infty)$ (with first and second derivatives $\phi'$ and $\phi''$) such that $\phi''(x)/(\phi'(x))^2$ is decreasing.*

*Example 1.C.76.* An interesting family of distribution functions, with associated random variables that are ordered in the down shifted likelihood ratio order, is the Pareto family. Explicitly, for $\theta \in (0, \infty)$, let $X_\theta$ be a random variable with density function $f_\theta$ defined by

$$f_\theta(x) = \theta/(1 + x)^{\theta+1}, \quad x \ge 0.$$

Then, by verifying (1.C.21), it is easy to see that $X_{\theta_1} \le_{\mathrm{lr}\downarrow} X_{\theta_2}$ whenever $\theta_1 \ge \theta_2 > 0$.

Some results that compare order statistics in the shifted likelihood ratio orders are described next. Again, $X_{(j:m)}$ denotes the $j$th order statistic associated with the random variables $X_1, X_2, \ldots, X_m$, and $Y_{(i:n)}$ denotes the $i$th order statistic associated with the random variables $Y_1, Y_2, \ldots, Y_n$. An analog of Theorem 1.C.33 for the order $\le_{\mathrm{lr}\uparrow}$ is the following result. Note that in the following theorem the assumption is stronger than the assumption in Theorem 1.C.33, but so is the conclusion.

**Theorem 1.C.77.** *Let $X_1, X_2, \ldots, X_m$ be $m$ independent random variables, and let $Y_1, Y_2, \ldots, Y_n$ be other $n$ independent random variables, all having absolutely continuous distributions. If $X_j \le_{\mathrm{lr}\uparrow} Y_i$ for all $1 \le j \le m$ and $1 \le i \le n$, then*

$$X_{(j:m)} \le_{\mathrm{lr}\uparrow} Y_{(i:n)} \quad \textit{whenever } j \le i \textit{ and } m - j \ge n - i.$$

*Proof.* Fix an $x \ge 0$ and denote by $(X - x)_{(j:m)}$ the $j$th order statistic among the random variables $X_1 - x, X_2 - x, \ldots, X_m - x$. By assumption we have $X_j - x \le_{\mathrm{lr}\uparrow} Y_i$ for all $1 \le j \le m$ and $1 \le i \le n$. Therefore from Theorem 1.C.33 we get $(X - x)_{(j:m)} \le_{\mathrm{lr}} Y_{(i:n)}$ whenever $j \le i$ and $m - j \ge n - i$. The stated result follows from the fact that $(X - x)_{(j:m)} = X_{(j:m)} - x$. $\square$

For the down shifted likelihood ratio order, the method of proof used in the proof of Theorem 1.C.33 only yields comparisons of minima as described in the following result.

**Theorem 1.C.78.** *Let $X_1, X_2, \ldots, X_m$ be $m$ independent random variables, and let $Y_1, Y_2, \ldots, Y_n$ be other $n$ independent random variables, all having absolutely continuous distributions with support $[0,\infty)$. If $X_j \le_{\mathrm{lr}\downarrow} Y_i$ for all $1 \le j \le m$ and $1 \le i \le n$, then*

$$X_{(1:m)} \le_{\mathrm{lr}\downarrow} Y_{(1:n)} \quad \textit{whenever } m \ge n.$$

Now let $X_1, X_2, \ldots$ be independent and identically distributed random variables. Taking $Y_i =_{\mathrm{st}} X_j$ for all $i$ and $j$ in Theorems 1.C.77 and 1.C.78, and using Theorems 1.C.66 and 1.C.74, we obtain the following analogs of Theorem 1.C.37. Note that in the next theorem (unlike in Theorem 1.C.37) we assume logconcavity or logconvexity of the underlying density function, but the conclusion in part (a) of the next theorem is stronger than the conclusion in Theorem 1.C.37.

**Theorem 1.C.79.** (a) *Let $X_1, X_2, \ldots$ be independent and identically distributed absolutely continuous random variables with an interval support. If the common density function is logconcave, then*

$$X_{(j:m)} \le_{\mathrm{lr}\uparrow} X_{(i:n)} \quad \textit{whenever } j \le i \textit{ and } m-j \ge n-i.$$

(b) *Let $X_1, X_2, \ldots$ be independent and identically distributed absolutely continuous random variables with support $[0,\infty)$. If the common density function is logconvex on $[0,\infty)$, then*

$$X_{(1:m)} \le_{\mathrm{lr}\downarrow} X_{(1:n)} \quad \textit{whenever } m \ge n.$$

## 1.D The Convolution Order

Let $X$ and $Y$ be two random variables such that

$$Y =_{\mathrm{st}} X + U, \tag{1.D.1}$$

where $U$ is a nonnegative random variable, independent of $X$. Then $X$ is said to be *smaller than $Y$ in the convolution order* (denoted as $X \le_{\mathrm{conv}} Y$). Obviously, the convolution order is a partial order. It is equivalent to the information order which is defined for statistical experiments when the underlying parameter is a location parameter.

The convolution order is obviously closed under increasing linear transformations. That is, for any $a \in \mathbb{R}$ and $b \ge 0$ we have

$$X \le_{\mathrm{conv}} Y \Longrightarrow a + bX \le_{\mathrm{conv}} a + bY.$$

The convolution order is obviously also closed under convolutions. That is, let $X_1, X_2, \ldots, X_n$ be a set of independent random variables, and let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables. Then

$$\left(X_j \le_{\rm conv} Y_j,\ j=1,2,\ldots,n\right) \Longrightarrow \sum_{i=1}^n X_i \le_{\rm conv} \sum_{i=1}^n Y_i.$$

It is obvious from Theorem 1.A.2 and (1.D.1) that

$$X \le_{\rm conv} Y \Longrightarrow X \le_{\rm st} Y.$$

For any nonnegative random variable $X$ we denote by $L_X$ its classical Laplace transform, that is,

$$L_X(s) = E[\mathrm{e}^{-sX}], \quad s \ge 0.$$

Recall that a nonnegative function $\phi$ is a Laplace transform of a nonnegative measure on $(0,\infty)$ if, and only if, $\phi$ is completely monotone, that is, all the derivatives $\phi^{(n)}$ of $\phi$ exist, and they satisfy $(-1)^n\phi^{(n)}(x) \ge 0$ for all $x \ge 0$ and $n = 1,2,\ldots$. It follows that for nonnegative random variables we have

$$X \le_{\rm conv} Y \Longleftrightarrow \frac{L_Y(s)}{L_X(s)} \text{ is a completely monotone function in } s \ge 0. \quad (1.D.2)$$

*Example 1.D.1.* Let $X_i$ be an exponential random variable with mean $1/\lambda_i$, $i = 1,2$. If $\lambda_1 > \lambda_2$, then $X_1 \le_{\rm conv} X_2$. To see this, note that the ratio of the Laplace transforms of $X_2$ and $X_1$ at $s$ is equal to $(\lambda_2/\lambda_1)((s+\lambda_1)/(s+\lambda_2))$, and it is easy to verify that this ratio is completely monotone. The result thus follows from (1.D.2).

*Example 1.D.2.* Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed exponential random variables with mean $1/\lambda$ for some $\lambda > 0$. Denote the corresponding order statistics by $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$. Then

$$X_{(i)} \le_{\rm conv} X_{(j)} \quad \text{whenever } 1 \le i < j \le n.$$

To see this, note that

$$X_{(k+1)} =_{\rm st} X_{(k)} + Z_k,$$

where $Z_k$ is an exponential random variable with mean $((n-k)\lambda)^{-1}$, $k = 1,2,\ldots,n-1$, and use the transitivity property of the order $\le_{\rm conv}$.

## 1.E Complements

**Section 1.A:** The usual stochastic order is being used in many areas of applications, but there is no single source where many of the basic results can all be found. Some standard references are the books of Lehmann [342], Marshall and Olkin [383], Ross [475], and Müller and Stoyan [419], where most of the results described in Section 1.A can be found. For example, Theorem 1.A.2 can be found in Marshall and Olkin [383]. The characterization of the usual stochastic order by the monotonicity described in

(1.A.8) is taken from Müller [407], whereas the characterization given in (1.A.12) can be found in Fellman [193]. The comparison of the random sums in Theorem 1.A.5 is motivated by ideas in Pellerey and Shaked [455]; it was communicated to us by Pellerey [444]. The application of the order $\leq_{\rm st}$ in Bayesian imperfect repair (Example 1.A.7) is taken from Lim, Lu, and Park [364]. The result which gives conditions for stochastic equality (Theorem 1.A.8) can be found in Baccelli and Makowski [27] and in Scarsini and Shaked [494]. Lemma 2.1 of Costantini and Pasqualucci [135] with $n = 1$ is an interesting variation of Theorem 1.A.8. The bivariate characterizations in Theorems 1.A.9 and 1.A.10 are taken from Shanthikumar and Yao [532] and from Righter and Shanthikumar [466], respectively. The characterization of the order $\leq_{\rm st}$ by means of the Fortret-Mourier-Wasserstein distance (Theorem 1.A.11) is taken from Adell and de la Cal [3]. The Laplace transform characterization of the order $\leq_{\rm st}$ (Theorem 1.A.13) can be found in Kebir [281] and in Kan and Yi [274]. An extension of Theorem 1.A.13 to more general orders can be found in Nanda [422]. The closure of the order $\leq_{\rm st}$ under a stochastically increasing family of random variables (Theorem 1.A.14) is taken from Shaked and Wong [524]. The condition for the usual stochastic order, given in Theorem 1.A.17, has been communicated to us by Gerchak and He [210]. The comparison of truncated maximum with truncations maximum (Example 1.A.16) can be found in Pellerey and Petakos [453]. The lattice property of the order $\leq_{\rm st}$ (Remark 1.A.18) is given in Müller and Scarsini [418]. The four results that give the stochastic orderings of the spacings, Theorems 1.A.19–1.A.22, can be found in Barlow and Proschan [35], Ebrahimi and Spizzichino [178], Pledger and Proschan [458], and Joag-Dev [258], respectively. The stochastic comparison of order statistics of independent random variables with the order statistics of independent and identically distributed random variables (Theorem 1.A.23) is taken from Ma [371]; it generalizes some previous results in the literature. The stochastic comparison of a sum of independent heterogeneous exponential random variables with a proper Erlang random variable (Example 1.A.24) is taken from Bon and Păltănea [105], where more refined comparisons can also be found. The stochastic comparison of a sum of independent heterogeneous binomial random variables with a proper binomial random variable (Example 1.A.25) is taken from Boland, Singh, and Cukic [102]. The necessary and sufficient conditions for the comparison of normal random variables (Example 1.A.26) are taken from Müller [413]; an extension of this result to Kotz-type distributions is given in Ding and Zhang [168]. The stochastic comparisons of norms, in Examples 1.A.27 and 1.A.28, are taken from Lapidoth and Moser [333]. The TTT transform (1.A.19) is introduced in Barlow, Bartholomew, Bremner, and Brunk [32], and is further studied in Barlow and Doksum [34] and in Barlow and Campo [33]. The observed total time on test random variable $X_{\rm ttt}$ is defined and studied in Li and Shaked [356], where the implication in Theorem 1.A.29 can be found. The characterizations of the NBUE and the NWUE aging notions by means of the usual stochastic order (Theorem 1.A.31) can be found in Whitt [565] and in Fagiuoli and Pellerey [187]. The other characterization, by means of the random variable $X_{\rm ttt}$ (Theorem 1.A.32), is taken from Li and Shaked [356]. The aging notion that is described in (1.A.21) is studied in Mugdadi and Ahmad [402].

Boland, Singh, and Cukic [103] studied an order, called the *stochastic precedence order*, according to which the random variable $X$ is smaller than the random variable $Y$ if $P\{X < Y\} \geq P\{Y < X\}$. If $X$ and $Y$ are independent, then $X \leq_{\rm st} Y$ implies that $X$ is smaller than $Y$ in the stochastic precedence order.

**Section 1.B:** Many of the basic results regarding the hazard rate order can be found in Ross [475] and in Müller and Stoyan [419]. The characterization (1.B.8) can be found in Lehmann and Rojo [345]. The results regarding the preservation of the orders $\leq_{\rm hr}$ and $\leq_{\rm rh}$ under monotone increasing transformations (Theorems 1.B.2 and 1.B.43) can be found in Keilson and Sumita [283]. The closure under convolutions result (Theorem 1.B.4) and the bivariate characterization result (Theorem 1.B.9) are taken from Kijima [291] and Shanthikumar and Yao [532]. A special case of Lemma 1.B.3 can be found in Mukherjee and Chatterjee [403]. The hazard rate order comparison of a sum of independent heterogeneous exponential random variables with a proper Erlang random variable (Example 1.B.5) is taken from Bon and Păltănea [105], where more refined comparisons can also be found. The hazard rate order comparison of a sum of independent heterogeneous binomial random variables with a proper binomial random variable (Example 1.B.6) is taken from Boland, Singh, and Cukic [102]. The hazard rate order comparison of random sums (Theorem 1.B.7) can be found in Pellerey [445]; some related results are Theorem 7.2 of Kijima [291] and Proposition 2.2 of Kebir [282]. The closure under mixtures result (Theorem 1.B.8) can be found in Boland, El-Neweihi, and Proschan [97]; a generalization of it is contained in Nanda, Jain, and Singh [424]. The bivariate characterizations in Theorems 1.B.10 and 1.B.11 are taken from Righter and Shanthikumar [466] and from Cheng and Righter [128], respectively. The characterizations given in Theorem 1.B.12 can be found in Capéraà [118] and in Joag-Dev, Kochar, and Proschan [259]. The hazard rate ordering result regarding the inter-epoch times of a nonhomogeneous Poisson process (Example 1.B.13) is taken from Kochar [309] where other applications of Theorem 1.B.12 can also be found. The hazard rate ordering of the epoch times of a nonhomogeneous Poisson process (1.B.19) can be found in Baxter [62]. The closure property of the order $\leq_{\rm hr}$ under hazard rate ordered mixtures (Theorem 1.B.14) is taken from Shaked and Wong [524]; a related result is Proposition 4.1 of Kebir [282]. The preservation of the order $\leq_{\rm hr}$ under the formation of a proper Markov chain (Example 1.B.15) can essentially be found in Ross, Shanthikumar, and Zhu [478]; they gave a version of this preservation result for the order $\leq_{\rm rh}$. The application of the order $\leq_{\rm hr}$ in Bayesian imperfect repair (Example 1.B.16) is inspired by Lim, Lu, and Park [364], but the result given here is stronger than their Theorem 4.1(iii). The hazard rate order comparison of a proportional hazard mixture with its parent distribution (Example 1.B.17) is taken from Gupta and Gupta [214]. The Laplace transform characterization of the order $\leq_{\rm hr}$ (Theorem 1.B.18) can be found in Kebir [281] and in Kan and Yi [274]. An extension of Theorem 1.B.18 to more general orders can be found in Nanda [422]. The result about the inheritance of the order $\leq_{\rm hr}$, from the mixing scales to the underlying counting processes (Theorem 1.B.19), is essentially taken from Ma [374]. The closure property which is given in Theorem 1.B.21 can be found in Kochar [305]; the necessary and sufficient condition, given after Theorem 1.B.21, is taken from Ma [374]. The result involving the hazard rate comparison of weighted random variables (Example 1.B.23) is taken from Nanda and Jain [423]; see also Bartoszewicz and Skolimowska [51]. The hazard rate comparison of epoch times of nonhomogeneous Poisson processes in Example 1.B.24 can be found in Ahmadi and Arghami [6] and in Belzunce, Lillo, Ruiz, and Shaked [69]; in the latter paper the result is extended to nonhomogeneous pure birth processes. The hazard rate order comparison of inter-epoch times of nonhomogeneous Poisson processes in Example 1.B.24 is taken from Belzunce, Lillo, Ruiz, and Shaked [69], who also obtained a similar result for the more general nonhomogeneous pure birth processes. The hazard rate order comparison of series systems of parallel systems (Example 1.B.25) can be found in Valdés and Zequeira [553]. The proof of Theorem 1.B.26 (given in Remark 1.C.40) is taken from Boland, Shaked, and Shanthikumar [101]. The hazard rate order comparisons of order statistics described in Theorems 1.B.27 and 1.B.28 can be found in Korwar [321]. The conditions that lead to the hazard rate ordering of minima (Theorem 1.B.29 and Corollary 1.B.30) are taken from Navarro and Shaked [430]. The two results that give the hazard rate orderings of the spacings (Theorem 1.B.31) can be found in Kochar and Kirmani [313] and in Khaledi and Kochar [285], whereas the comparison of spacings from two different samples (Theorem 1.B.32) is taken from Khaledi and Kochar [285]; further results can be found in Hu and Wei [240] and in Misra and van der Meulen [396]. The closure property under formations of order statistics (Theorem 1.B.34) is taken from Singh and Vijayasree [537]; see also Lynch, Mimmack, and Proschan [369]. Boland, El-Neweihi, and Proschan [97] show, by a counterexample, that the conclusion of Theorem 1.B.34 need not hold when the $X_i$'s or the $Y_i$'s are not identically distributed. Extensions of Theorem 1.B.34 can be found in Shaked and Shanthikumar [516], in Belzunce, Mercader, and Ruiz [70], and in Hu and Zhuang [247]. The general comparison result, given in Theorem 1.B.36, is taken from Boland, Hu, Shaked, and Shanthikumar [99]; see related results in Franco, Ruiz, and Ruiz [205] and in Hu and Zhuang [247]. The hazard rate order comparisons of maxima of heterogeneous exponential random variables (Example 1.B.37) are taken from Dykstra, Kochar, and Rojo [174] and from Khaledi and Kochar [287]. The closure under convolution property of IFR random variables (Corollary 1.B.39) can be found, for example, in Barlow and Proschan [36, page 100]). The characterizations of the DMRL and the IMRL aging notions by means of the hazard rate order (Theorem 1.B.40) can be found in Brown [111, page 229], in Whitt [565], and in Fagiuoli and Pellerey [187]. The observation that essentially reduces the study of the reversed hazard rate order into the study of the hazard rate order (Theorem 1.B.41) is taken from Nanda and Shaked [428]. The bivariate characterization results for the reversed hazard order (Theorems 1.B.47 and 1.B.49) can be found in Shanthikumar, Yamazaki, and Sakasegawa [529] and in Cheng and Righter [128]. The application of the reversed hazard order in economics, described in Example 1.B.51, is taken from Eeckhoudt and Gollier [180]; further results in this vein can be found in Kijima and Ohnishi [293]. The closure property of the order $\leq_{\mathrm{rh}}$ under reversed hazard rate ordered mixtures (Theorem 1.B.52) is taken from Shaked and Wong [524]; a related result is Proposition 4.1 of Kebir [282]. The Laplace transform characterization of the order $\leq_{\mathrm{rh}}$ (Theorem 1.B.53) is taken from Kebir [281]. The result about the inheritance of the order $\leq_{\mathrm{rh}}$, from the mixing scales to the underlying counting processes (Theorem 1.B.54), is essentially taken from Ma [374]. The results about the reversed hazard rate ordering of order statistics (Theorems 1.B.56 and 1.B.57), and the characterizations of the reversed hazard rate order given in Theorem 1.B.62, can be found in Block, Savits, and Singh [96], whereas the result described in Theorem 1.B.58 is taken from Hu and He [232]. The preservation of the order statistics in the sense of the order $\leq_{\mathrm{rh}}$ (Theorem 1.B.60) can be found in Nanda, Jain, and Singh [426].

An order among nonnegative random variables, which is defined by stipulating the monotonicity of the ratio of the hazard rate functions (when they exist), is studied in Kalashnikov and Rachev [271], Sengupta and Deshpande [500], and Rowell and Siegrist [479]. Equivalently, if $\overline{F}$ and $\overline{G}$ are survival functions, and we denote $R_F = -\log \overline{F}$ and $R_G = -\log \overline{G}$, then the order mentioned above can be defined by requiring that the composition $R_F \circ R_G^{-1}$ be convex on $[0, \infty)$. The notion of the monotonicity of the ratio of hazard rate functions is used in Examples 1.B.24 (see (1.B.25)) and 1.B.25, as well as in Theorem 1.C.4. Sengupta and Deshpande [500] and Rowell and Siegrist [479] also studied the orders defined by stipulating that $R_F \circ R_G^{-1}$ be starshaped or superadditive.

Brown and Shanthikumar [112], Lillo, Nanda, and Shaked [361], Hu and Zhu [242], Di Crescenzo and Longobardi [165], and Belzunce, Ruiz, and Ruiz [74] have introduced and studied various shifted hazard and reversed hazard rate orders. Similar orders which extend the likelihood ratio order are studied in Section 1.C.4.

**Section 1.C:** Again, many of the basic results regarding the likelihood ratio order can be found in Ross [475] and in Müller and Stoyan [419]. Condition (1.C.3) is implicit in Block, Savits, and Shaked [95], and it is explicit in Müller [408]. The relation (1.C.4) is mentioned in Chan, Proschan, and Sethuraman [123]. The sufficient conditions for $X \leq_{\rm lr} Y$, given in Theorem 1.C.4, have been noted in Belzunce, Lillo, Ruiz, and Shaked [69]. The closure property of the likelihood ratio order under conditioning (Theorem 1.C.5) is observed in Whitt [561]. Many variations of Theorem 1.C.5 with respect to general sample spaces can be found in Whitt [561] and in Rüschendorf [485]. The closure under limits property of the order $\leq_{\rm lr}$ (Theorem 1.C.7) is taken from Müller [408]. The result regarding the preservation of the order $\leq_{\rm lr}$ under monotone increasing transformations (Theorem 1.C.8) can be found in Keilson and Sumita [283]. The several closure under convolution results (Theorems 1.C.9, 1.C.11, and 1.C.12) as well as the bivariate characterization result (Theorem 1.C.20) are taken from Shanthikumar and Yao [532]; a related result is Proposition 2.4 of Kebir [282]. A special case of Theorem 1.C.9 can be found in Mukherjee and Chatterjee [403]. The result about the number of successes in independent trials (Example 1.C.10) is statement (7) in Samuels [488], who attributed it to Ghurye and Wallace. The characterization of the order $\leq_{\rm hr}$ by means of the order $\leq_{\rm lr}$, given in Theorem 1.C.14, is taken from Di Crescenzo [164]; a density of the form (1.C.7) can be found in Adell and Lekuona [4, page 773]. The likelihood ratio order comparison of a random random variable with a fixed random variable (Corollary 1.C.16) is a slight generalization of Problem B in Szekli [544, page 22]. The closure property of the order $\leq_{\rm lr}$ under likelihood ratio ordered mixtures (Theorem 1.C.17) is an extension of a result in Kebir [282]. The result about the inheritance of the order $\leq_{\rm lr}$, from the mixing scales to the underlying counting processes (Theorem 1.C.18), is taken from Ma [374]. Example 1.C.19 is inspired by Theorem 4.12 of Asadi and Shanbhag [23], but Example 1.C.19 has weaker assumptions ($\Theta_1$ and $\Theta_2$ need not be degenerate) and stronger conclusions ($Y_1$ and $Y_2$ are ordered in the likelihood ratio order, rather than in the hazard rate order) than the result of Asadi and Shanbhag [23]. The result in Theorem 1.C.21 is a special case of a result in Ross [475]. The bivariate characterizations in Theorems 1.C.22, 1.C.23, and 1.C.24 are taken from Righter and Shanthikumar [466] and from Chapter 13 by Righter in [515]. The Laplace transform characterization of the order $\leq_{\rm lr}$ (Theorem 1.C.25) can be found in Kebir [281]. An extension of Theorem 1.C.25 to more general orders can be found in Nanda [422]. The conditional likelihood ratio orderings, described in Theorem 1.C.26, can be found in Ku and Niu [324] and in Chapter 14 by Shanthikumar and Yao in [515]. The setting in which the order $\leq_{\rm hr}$ gives rise to the order $\leq_{\rm lr}$, as described in Theorem 1.C.28, is essentially taken from Ross, Shanthikumar, and Zhu [478]; they gave a version of this result for the order $\leq_{\rm rh}$. The necessary and sufficient condition for $aX \leq_{\rm lr} X$ (Theorem 1.C.29) can be found in

Hu, Nanda, Xie, and Zhu [237]. The likelihood ratio order comparisons of the order statistics given in Theorem 1.C.31 are taken from Bapat and Kochar [31] and from Hu, Zhu, and Wei [243]; an extension of the first part of Theorem 1.C.31(a) can be found in Ma [373]. The result about the likelihood ratio order comparison of order statistics of a simple random sample from a finite population (Theorem 1.C.32) can be found in Kochar and Korwar [315]. The general result which compares order statistics from two samples of different size (Theorem 1.C.33) is taken from Lillo, Nanda, and Shaked [362]; see related results in Franco, Ruiz, and Ruiz [205] and in Hu and Zhuang [247]. Belzunce and Shaked [78] extended Theorem 1.C.33 to comparison of lifetimes of coherent systems in reliability theory; see also Belzunce, Franco, Ruiz, and Ruiz [66]. The closure property under formation of order statistics (Corollary 1.C.34) can be found in Chan, Proschan, and Sethuraman [123]; a special case of this result can be found in Singh and Vijayasree [537]. The likelihood ratio order comparison of the order statistics given in Theorem 1.C.37 is taken from Raqab and Amin [465]. Theorem 2.6 in Kamps [273, page 182] extends Theorem 1.C.37 to the so called generalized order statistics; see also Korwar [322] and Hu and Zhuang [247]. The special case of Theorem 1.C.37 when $j = i$, is extended in Nanda, Misra, Paul, and Singh [427] to the case when the sample sizes $m$ and $n$ are random. Nanda, Misra, Paul, and Singh [427] also extend the special case of Theorem 1.C.37 when $m = n$, to the case when the common sample size is random. The likelihood ratio order comparison of normalized spacings (Theorem 1.C.42) can be found in Kochar and Korwar [314], whereas the comparisons for nonnormalized spacings (Theorem 1.C.43) are special cases of results in Misra and van der Meulen [396] and in Hu and Zhuang [246, 248]. The comparison of spacings that correspond to random variables with logconcave density (Theorem 1.C.44) is a special case of a result of Hu and Zhuang [246, 248]. The comparison of spacings from two different samples (Theorem 1.C.45) is taken from Khaledi and Kochar [285]; an extension of this result can be found in Franco, Ruiz, and Ruiz [205], and a related result can be found in Belzunce, Mercader, and Ruiz [70]. The results about the likelihood ratio order comparisons of random minima and maxima (Example 1.C.46) are taken from Shaked and Wong [526]; see a related result in Bartoszewicz [49]. The result about the likelihood ratio comparison of the successive epochs of a nonhomogeneous Poisson process (Example 1.C.47) is given in Kochar [307, 309], where it is also shown that it implies the likelihood order comparison of successive record values of a sequence of independent and identically distributed random variables. The likelihood ratio comparisons of epoch and inter-epoch times of nonhomogeneous Poisson processes (Example 1.C.48) are taken from Belzunce, Lillo, Ruiz, and Shaked [69], who also extended them to comparisons of epoch and inter-epoch times of nonhomogeneous pure birth processes. The likelihood ratio order comparison of a sum of independent heterogeneous exponential random variables with a proper Erlang random variable (Example 1.C.49) is a combination of results from Boland, El-Neweihi, and Proschan [98] and from Bon and Păltănea [105], where more refined comparisons can also be found. For instance, the comparison in Example 1.C.50 is given in Boland, El-Neweihi, and Proschan [98]. The likelihood ratio order comparison of a sum of independent heterogeneous binomial random variables with a proper binomial random variable (Example 1.C.51) is taken from Boland, Singh, and Cukic [102]. An interpretation of logconcavity and logconvexity as aging notions can be found in Shaked and Shanthikumar [506], where the proof of parts (i) and (ii) of Theorem 1.C.52 can be found. A proof of (1.C.13) can also be found there. The likelihood ratio ordering of random variables conditioned on their sum (Theorem 1.C.53) is essentially Example 12 of Lehmann [343]. The closure property of logconcave densities under order statistics (Theorem 1.C.54) is a generalization of an observation in Li and Lu [355]. The characterizations of the IFR and the DFR aging notions by means of the likelihood ratio order (Theorem 1.C.55) can be found in Whitt [565]. The likelihood ratio order comparison of the asymptotic equilibrium ages, given in Example 1.C.56, is a special case of a result of Bon and Illayk [104]. The likelihood ratio monotonicity of the parameter in the observation, given the likelihood ratio monotonicity of the observation in the parameter (Example 1.C.57), can be found in Whitt [560], whereas the preservation of the likelihood ratio order of the priors by the posteriors (Example 1.C.58) is given as Remark 3.14 in Spizzichino [539]. The comparison of the weighted random variables (Example 1.C.59) can be found in Bartoszewicz and Skolimowska [51]. An extension of the implication in Example 1.C.59, when $X^w$ and $Y^w$ are the length-biased versions of $X$ and of $Y$, respectively, is given in Hu and Zhuang [244]. An extension of the implication in Example 1.C.59 to multivariate weighted distributions can be found in Jain and Nanda [253]. The result in Example 1.C.60 is taken from Bartoszewicz and Skolimowska [51]; extensions of the inequality $X \le_{\rm lr} X^w$, when $X^w$ is the length-biased version of $X$, are given in Ross [476]. The ordering of generalized skew normal random variables (Example 1.C.61) is taken from Gupta and Gupta [215]. The up shifted likelihood ratio order is introduced in Shanthikumar and Yao [530]. The results described in Section 1.C.4 can mostly be found in Lillo, Nanda, and Shaked [361, 362]. An extension of Theorem 1.C.77 is given in Belzunce, Ruiz, and Ruiz [74]; see also Belzunce and Shaked [78]. Ramos Romero and Sordo Díaz [464] defined an order that is reminiscent of the order $\le_{\rm lr\uparrow}$ as defined in (1.C.19). According to their definition, the nonnegative random variable $X$ is said to be smaller than the nonnegative random variable $Y$ if $aX \le_{\rm lr} Y$ for every $0 < a < 1$.

Lehmann and Rojo [345] used the characterization (1.C.4) in order to define stochastic orders that are stronger than $\le_{\rm lr}$. For example, let $X$ and $Y$ be two random variables with distribution functions $F$ and $G$, respectively, and consider the stipulation that, for a fixed $k$,

$$\frac{\mathrm{d}^n}{\mathrm{d}u^n} GF^{-1}(u) \geq 0 \quad \text{for all } 0 < u < 1 \text{ and all } n = 1, 2, \ldots, k.$$

If $k \geq 3$, then $X$ is stochastically smaller than $Y$ in a sense that is stronger than $\leq_{\mathrm{lr}}$. The order $\leq_{\mathrm{lr}}$ is obtained when $k = 2$. Lehmann and Rojo [345] showed, for example, that if $X_1, X_2, \ldots, X_m$ are independent, identically distributed, then $X_1$ is smaller than $\max\{X_1, X_2, \ldots, X_m\}$, in the above sense, with $k = m$.

Chang [126] considered four exponential random variables $X_1$, $X_2$, $Y_1$, and $Y_2$, with the corresponding rates $\lambda_1$, $\lambda_2$, $\mu_1$, and $\mu_2$, where $X_1$ and $X_2$ are independent, and $Y_1$ and $Y_2$ are independent. He obtained the necessary and sufficient conditions on $\lambda_1$, $\lambda_2$, $\mu_1$, and $\mu_2$, for each of the following results: (i) $X_1 + X_2 \leq_{\mathrm{lr}} Y_1 + Y_2$, (ii) $X_1 + X_2 \geq_{\mathrm{lr}} Y_1 + Y_2$, and (iii) $X_1 + X_2$ and $Y_1 + Y_2$ are not comparable in the likelihood ratio order.

**Section 1.D:** The discussion in this section follows Shaked and Suarez-Llorens [520].

Fagiuoli and Pellerey [185] have introduced an approach that describes a unified point of view regarding some of the orders studied in this chapter and some of the orders studied in Chapters 2, 3, and 4. This approach led Fagiuoli and Pellerey to introduce some families of new orders. Several properties of these orders were studied in Fagiuoli and Pellerey [185], in Nanda, Jain, and Singh [424, 425], and in Hu, Kundu, and Nanda [236]; see also Hesselager [221]. Another general approach that unifies some of the orders studied in this chapter and in Chapter 2 was introduced in Hu, Nanda, Xie, and Zhu [237].

Other orders that are related to the orders $\leq_{\mathrm{st}}$ and $\leq_{\mathrm{lr}}$ have been introduced and studied in Di Crescenzo [163]. Yanagimoto and Sibuya [571], Zijlstra and de Kroon [577], and Shanthikumar and Yao [532], extended the definitions of $X \leq_{\mathrm{st}} Y$, $X \leq_{\mathrm{hr}} Y$, and $X \leq_{\mathrm{lr}} Y$, to jointly distributed random variables $X$ and $Y$; see also Arcones, Kvam, and Samaniego [15]. Ebrahimi and Pellerey [177] have introduced a stochastic order based on a notion of uncertainty and studied its relationship to some of the orders studied in this chapter.