Part II: Stochastic Programming

Lecture 1: Introduction and Examples

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What is Stochastic Programming (SP)

- Study models and methods for optimal decision making in problems involving uncertain data.
  - ***Stochastic***: opposed to deterministic and means that some data are random
  - ***Programming***: various parts of the problem can be modeled as linear or nonlinear mathematical programs
- Uncertainty is usually characterized by a probability distribution on the parameters.
- Probability distributions governing the data are known or can be estimated.

Course Objectives

- Build intuition on how to model uncertainty into mathematical programs
- Understand
  - which changes uncertainty brings into the decision process
  - what difficulties uncertainty may bring
  - what problems are solvable
- Learn how to solve stochastic programming problems
- Develop intuition for research areas in stochastic programming

Textbook and Grading

- Textbook
  - ***Introduction to Stochastic Programming*** by John R. Birge and François Louveaux, second edition, 2011 (electronic version available on Tsinghua library website)
- Reference
  - ***Lectures on Stochastic Programming: Modeling and Theory*** by Alexander Shapiro, Alexander Shapiro and Andrzej Ruszczynski, second edition, 2014
- Grading
  - Homework: 30%
  - Final exam: 70%

Tentative Course Schedule

Week	Date	Topic	Notes
9	April 20	Introduction to stochastic programming and examples	
10	April 27	Uncertainty and modeling issues	
11	May 4	-	Holiday, no class
12	May 11	Basic properties and theory	
13	May 18	Benders decomposition	
14	May 25	L-shaped method	
15	June 1	Solution methods for stochastic integer programs	
16	June 8	Value of information and the stochastic solution	
17-18	To be determined	-	Final exam

Example 1: The farmer's problem

- The problem: A farmer has to decide in winter how to allocate 500 acres land in the coming year for raising wheat, corn, and sugar beets.
- Parameters
  - *Planting costs*: wheat \$150/acre; corn \$230/acre; beets \$260/acre.
  - *Yields* (mean): wheat 2.5 tons/acre; core 3 tons/acre; beets 20 tons/acre.
  - *Prices*: wheat \$170/ton to sell, \$238/ton to buy; corn \$150/ton to sell, \$210/ton to buy; beets 36\$/ton up to 6000 ton (quota), 10\$/ton if over the quota.
- Livestock requirements: wheat 200 tons, corn 240 tons.

Data

	Wheat	Corn	Sugar Beets
Yield (T/acre)	2.5	3	20
Planting cost ($/acre)	150	230	260
Selling price ($/T)	170	150	36 under 6000 T 10 above 6000 T
Purchase price ($/T)	238	210	–
Minimum requirement (T)	200	240	–
Total available land: 500 acres			

Variables

x_1 = acres of land devoted to wheat,
x_2 = acres of land devoted to corn,
x_3 = acres of land devoted to sugar beets,
w_1 = tons of wheat sold,
y_1 = tons of wheat purchased,
w_2 = tons of corn sold,
y_2 = tons of corn purchased,
w_3 = tons of sugar beets sold at the favorable price,
w_4 = tons of sugar beets sold at the lower price.

Linear Programming Formulation

$$\begin{aligned} \min\ & 150x_1 + 230x_2 + 260x_3 + 238y_1 - 170w_1 \\ & + 210y_2 - 150w_2 - 36w_3 - 10w_4 \\ \text{s. t.}\ & x_1 + x_2 + x_3 \leq 500 \,,\ 2.5\,x_1 + y_1 - w_1 \geq 200 \,, \\ & 3\,x_2 + y_2 - w_2 \geq 240 \,,\ w_3 + w_4 \leq 20x_3, w_3 \leq 6000 \,, \\ & x_1, x_2, x_3,\ y_1, y_2,\ w_1, w_2, w_3, w_4 \geq 0 \,. \end{aligned}$$

binding constraint

Culture	Wheat	Corn	Sugar Beets
Surface (acres)	120	80	300
Yield (T)	300	240	6000
Sales (T)	100	–	6000
Purchase (T)	–	–	–
Overall profit: \$118,600			

Scenario Solutions

Random Factor: Yield variations due to *Weather*: +/- 20% of the mean

Scenario Approach

- A – *Good weather* - Assume +20%

SOLUTION:	WHEAT	CORN	BEETS
ACRES (x_i)=	183	67	250
YIELD =	550	240	6000

PROFIT= $167,667 per season

- B – *Bad weather* - Assume -20%

SOLUTION:	WHEAT	CORN	BEETS
ACRES (x_i)=	100	25	375
YIELD =	200	60	6000

PROFIT= $59,950 per season

- But how would you allocate the land before you know the weather condition?

Stochastic Linear Program (SLP)

- Allocate the land before knowing the future weather
  - Suppose each scenario occurs equally likely (probability = 1/3 each)
  - Consider all scenarios in a single mathematical program
- Formulation: minimizing expected costs.

$$
\begin{aligned}
\min\ & 150x_1 + 230x_2 + 260x_3 \\
& - \tfrac{1}{3}(170w_{11} - 238y_{11} + 150w_{21} - 210y_{21} + 36w_{31} + 10w_{41}) \\
& - \tfrac{1}{3}(170w_{12} - 238y_{12} + 150w_{22} - 210y_{22} + 36w_{32} + 10w_{42}) \\
& - \tfrac{1}{3}(170w_{13} - 238y_{13} + 150w_{23} - 210y_{23} + 36w_{33} + 10w_{43}) \\
\text{s.t. } & x_1 + x_2 + x_3 \le 500\ ,\ 3x_1 + y_{11} - w_{11} \ge 200\ , \\
& 3.6x_2 + y_{21} - w_{21} \ge 240\ ,\ w_{31} + w_{41} \le 24x_3\ ,\ w_{31} \le 6000\ , \\
& 2.5x_1 + y_{12} - w_{12} \ge 200\ ,\ 3x_2 + y_{22} - w_{22} \ge 240\ , \\
& w_{32} + w_{42} \le 20x_3\ ,\ w_{32} \le 6000\ ,\ 2x_1 + y_{13} - w_{13} \ge 200, \\
& 2.4x_2 + y_{23} - w_{23} \ge 240,\ w_{33} + w_{43} \le 16x_3\ , \\
& w_{33} \le 6000,\ x, y, w \ge 0\ .
\end{aligned}
$$

Stochastic Solution

- SOLUTION

		WHEAT	CORN	BEETS
ACRES (x_i)	=	170	80	250
YIELD (Low)	=	340	192	4000
YIELD (Mean)	=	425	240	5000
YIELD (High)	=	510	288	6000

binding constraint

EXPECTED PROFIT = $108,390/season (Recourse Problem (RP))

- Expected Value of Perfect Information (EVPI)
  - Expected profit of perfect information (Wait-and-See): WS = (1/3)(167,667 + 118,600 + 59,950) = $115,406
  - EVPI = WS – RP = $7,016
  - measures the value of knowing the future with certainty

- Value of the Stochastic Solution (VSS)
  - RP minus expected value of using solution with means (result of ignoring uncertainty) (EMS = $107,240)
  - VSS = RP – EMS = $1,150
  - assesses the value of knowing and using distributions on future outcomes

Observations

- In the deterministic (single-scenario) linear program, two constraints are binding
- In the stochastic linear program, two constraints are binding but in different scenarios
- No single-scenario linear program can reproduce the stochastic program solution
- The alternative future scenarios create a non-linear adjustment for risk/uncertainty (even without including risk aversion)

General Formulation of SLP

- Key steps: make decision x now, observe an uncertain outcome $\boldsymbol{\xi}(\omega)$, make a recourse decision $y(\omega)$.

- Formulation (two-stage SLP with recourse):

$$\min\ c^T x + E_{\boldsymbol{\xi}}[Q(x,\boldsymbol{\xi})]$$
$$s.t. Ax = b, x \geq 0$$

where $Q(x,\boldsymbol{\xi}) = \min\{\boldsymbol{q}^T y \mid Wy = \boldsymbol{h} - \boldsymbol{T}x, y \geq 0\}$,
$\mathcal{Q}(x) = E_{\boldsymbol{\xi}}[Q(x,\boldsymbol{\xi})]$ is the recourse function and $\boldsymbol{\xi}$ consists of random components of $\boldsymbol{q}, (W), \boldsymbol{h}, \boldsymbol{T}$.

- In the farm example, the randomness is in $\boldsymbol{T}$ (yield).

Example 2: News Vendor Problem

- Buy x newspapers at cost c, sell at price q
- Return unsold ones at return value $r < c$
- Demand is random ξ with cumulative distribution $F(\xi)$
- Formulate the problem to minimize cost (maximum profit)

$$
\begin{aligned}
&\min cx + \mathcal{Q}(x) \\
&s.t.\ 0 \le x \le u
\end{aligned}
$$

where

$$\mathcal{Q}(x) = E_{\xi}[Q(x, \xi)]$$

and

$$
\begin{aligned}
Q(x, \xi) = \min\ & -qy(\xi) - rw(\xi) \\
s.t.\ & y(\xi) \le \xi, \\
& y(\xi) + w(\xi) \le x, \\
& y(\xi), w(\xi) \ge 0
\end{aligned}
$$

Solution to the News Vendor Problem

- Solution to the second stage:

$$y^*(\xi) = \min(\xi, x)\,,$$
$$w^*(\xi) = \max(x - \xi, 0)\,.$$

Hence

$$\begin{aligned} Q(x,\xi) &= -q\min(\xi,x) - r\max(x-\xi,0) \\ &= -q[\min(\xi - x, 0) + x] - r\max(x-\xi,0) \\ &= -qx + (q-r)\max(x-\xi,0) \end{aligned}$$

and

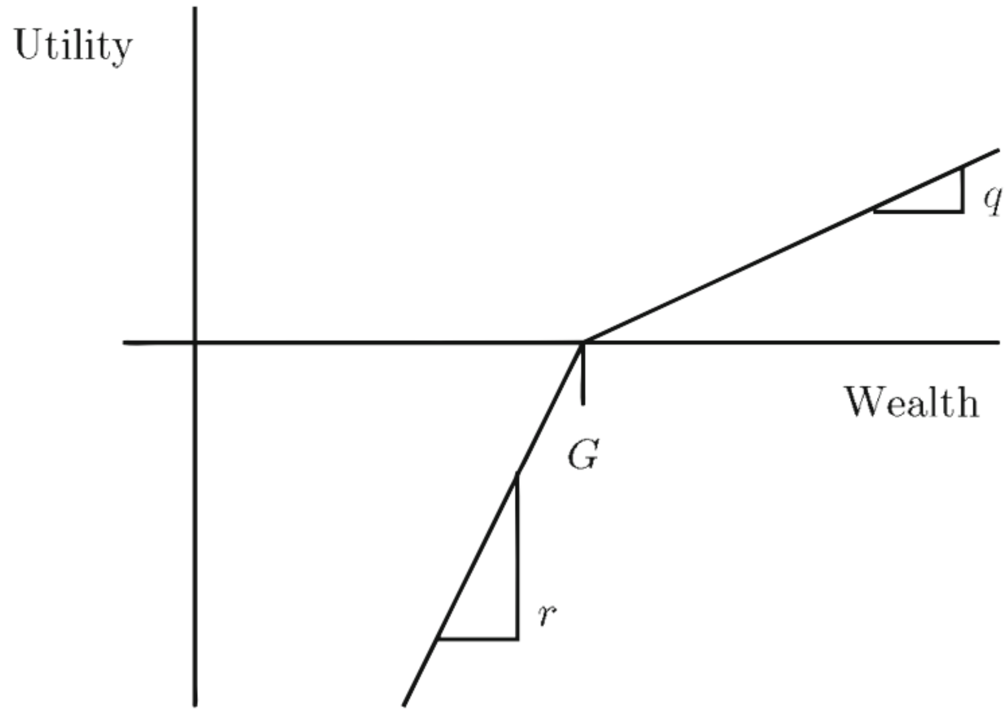
$$\mathcal{Q}(x) = E_{\xi}[Q(x,\xi)] = -qx + (q-r)\int_0^x F(\xi)d\xi$$

- Solution to the first stage: first order necessary condition:

$$c + \mathcal{Q}'(x) = 0$$
$$c - q + (q-r)F(x) = 0 \Leftrightarrow F(x) = \frac{q-c}{q-r}$$

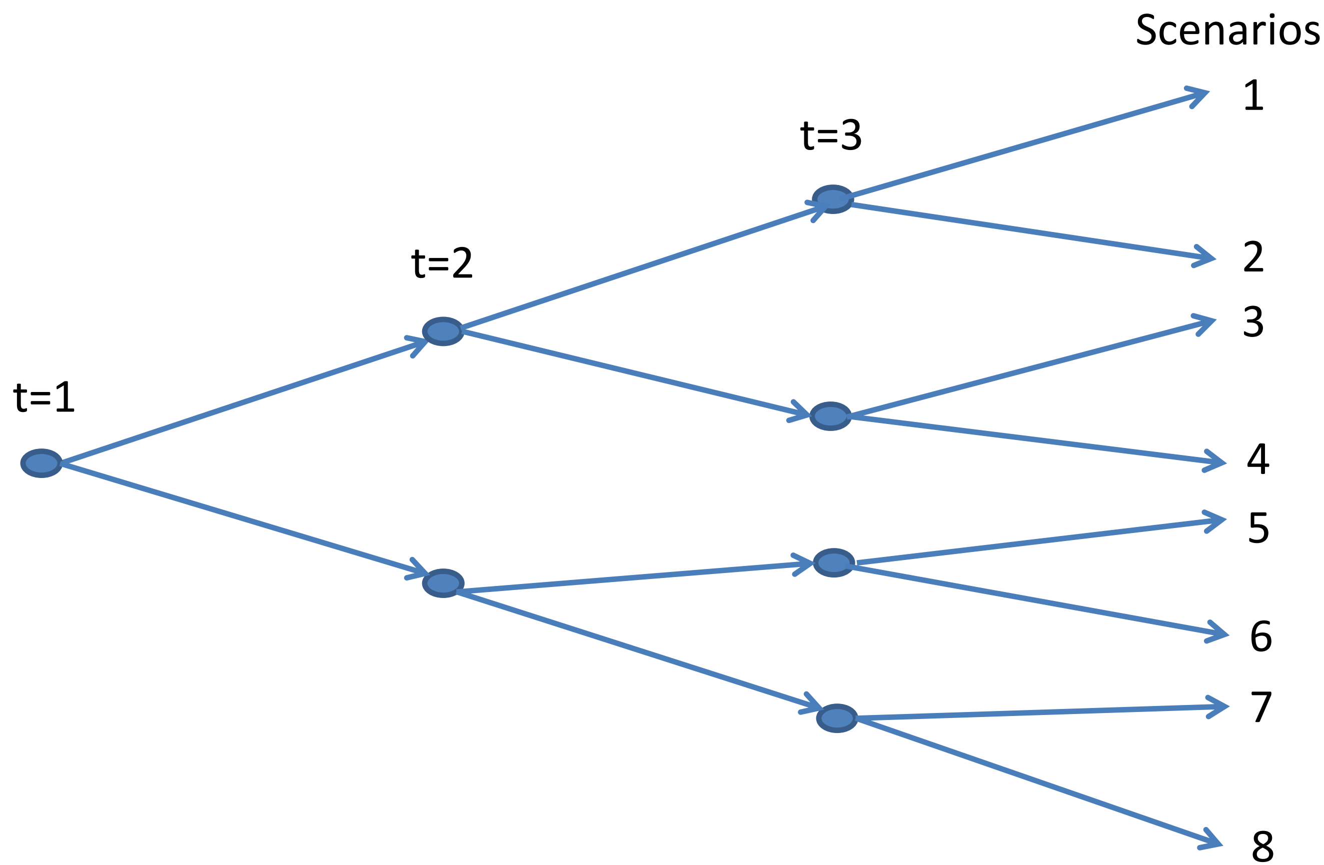
Example 3: Financial Planning

- Objective: to accumulate $\$G$ in H investment periods from now, with initial wealth $\$b$ to invest in I investments.
- Assumption:
  - uncertain return on each investment in each period
  - concave (risk-averse) utility (piecewise linear)



Scenario Approach

- Tree of scenarios with three periods



Notation

- Notation
  - S: scenario set
  - $p(s)$: probability of scenario $s \in S$
  - $\xi(i,t,s)$: return of investment i in period t under scenario s
  - $x(i,t,s)$: investment in i in period t under scenario s
  - s_t: realization of return at time t under scenario s
  - $S^t_{s_{1,}\cdots,s_{t-1}}$: group of scenarios having the same history up to time t
  - $J(s,t)$: equals to $\{s_{1,}\cdots, s_{t-1}\}$ such that s $\in S^t_{s_{1,}\cdots,s_{t-1}}$

Formulation without Transaction Fees

$$\max z = \sum_s p(s)(qy(s) - rw(s))$$

Utility

$$\text{s. t. } \sum_{i=1}^{I} x(i,1,s) = b \ , \ \forall s \in S \ ,$$

Initial investments

$$\sum_{i=1}^{I} \xi(i,t,s)x(i,t-1,s) - \sum_{i=1}^{I} x(i,t,s) = 0 \ , \ \forall s \in S \ ,$$

$$t = 2, \ldots, H \ ,$$

Flow conservation

$$\sum_{i=1}^{I} \xi(i,H,s)x(i,H,s) - y(s) + w(s) = G \ ,$$

Deficit and surplus

$$\left(\sum_{s' \in S^t_{J(s,t)}} p(s')x(i,t,s') \right) - \left(\sum_{s' \in S^t_{J(s,t)}} p(s') \right) x(i,t,s) = 0 \ ,$$

$$\forall 1 \le i \le I \ , \ \forall 1 \le t \le H \ , \ \forall s \in S \ ,$$

$$x(i,t,s) \ge 0 \ , \ y(s) \ge 0 \ , \ w(s) \ge 0 \ ,$$

$$\forall \ 1 \le i \le I \ , \ \forall \ 1 \le t \le H \ , \ \forall \ s \in S \ ,$$

Nonanticipativity constraints: forces all decisions within the same group at time *t* to be the same

Optimal Solution

Period, Scenario	Stock	Bonds
1,1-8	41.5	13.5
2,1-4	65.1	2.17
2,5-8	36.7	22.4
3,1-2	83.8	0.00
3,3-4	0.00	71.4
3,5-6	0.00	71.4
3,7-8	64.0	0.00

Summary

- Three application problems
  - The farmer's problem (two-stage, discrete distribution)
  - News vendor problem (two-stage, continuous distribution)
  - Financial planning (multi-stage, discrete distribution)

- Key concepts
  - Uncertainty
  - Recourse
  - Expected Value of Perfect Information (EVPI)
  - Value of Stochastic Solution (VSS)
  - Nonanticipativity constraints

Homework

- Reading:
  - Sections 1.1-1.3 of textbook
- Exercise:
  - Page 18, Problems 3, 4 and 6
  - ***Due on April 27 (Tue), before 10:00 am***
  - ***Late submission will NOT be accepted!***