Advanced Topics in OR

Lecture Notes 2

Stochastic Order Relations

Zhao Xiaobo
Department of IE
Tsinghua University
Beijing 100084, China
Tel. 010-62784898
Email. xbzhao@tsinghua.edu.cn

Coupling

$\{X(t),\, t \geq 0\}$: a birth and death process

Proposition 9.2.3: $\{X(t),\, t \geq 0\}$ stochastically increasing in $X(0)$.

⇨ $E[f(X(t_1), \ldots, X(t_n)) \mid X(t_1) = i]$: is increasing in i for all $t_1, \ldots, t_n$ and increasing functions f.

Proof: $\{X_1(t),\, t \geq 0\}$: birth and death process, $X_1(0) = i + 1$

$\{X_2(t),\, t \geq 0\}$: birth and death process, $X_2(0) = i$

Either $X_1(t)$ is always larger than $X_2(t)$ or else they are equal at some time.

Let T be the first time they become equal

$$T = \begin{cases} \infty & \text{if } X_1(t) > X_2(t) \text{ for all } t \\ \text{1st } t & X_1(t) = X_2(t) \text{ otherwise} \end{cases}$$

Coupling

If $T < \infty$, their continuation after T has the same probability structure. We define a third process

$$X_3(t) = \begin{cases} X_1(t) & \text{if } t < T \\ X_2(t) & \text{if } t \geq T \end{cases}$$

$\{X_3(t)\}$: a birth and death process, with $X_3(0) = i + 1$

Since

$$X_1(t) > X_2(t) \qquad \text{for} \quad t < T$$

$$\Longrightarrow \qquad X_3(t) \geq X_2(t) \qquad \text{for all} \quad t$$

Coupling

Proposition 9.2.4: $P\{X(t) \geq j \mid X(0) = 0\}$ increases in t for all j.

Proof: For $s < t$

$$P\left\{X(t) \geq j \middle| X(0) = 0\right\}$$

$$= \sum_i P\left\{X(t) \geq j \middle| X(0) = 0, X(t-s) = i\right\} P\left\{X(t-s) = i \middle| X(0) = 0\right\}$$

$$= \sum_i P\left\{X(t) \geq j \middle| X(t-s) = i\right\} P_{0i}(t-s)$$

$$= \sum_i P\left\{X(s) \geq j \middle| X(0) = i\right\} P_{0i}(t-s)$$

$$\geq \sum_i P\left\{X(s) \geq j \middle| X(0) = 0\right\} P_{0i}(t-s)$$

$$= P\left\{X(s) \geq j \middle| X(0) = 0\right\} \sum_i P_{0i}(t-s) \quad = P\left\{X(s) \geq j \middle| X(0) = 0\right\}$$

Coupling

Remark: A nice qualitative property about the transition probabilities of a birth and death process

It is often quite difficult to determine explicitly the values $P_{0j}(t)$ for fixed t, it is a simple matter to obtain the limiting probabilities P_j.

From Proposition 9.2.4, we have

$$P\{X(t) \geq j \mid X(0)=0\} \leq \lim_{t\to\infty} P\{X(t) \geq j \mid X(0)=0\} = \sum_{i=j}^{\infty} P_i$$

$\Longrightarrow$ implies that $X(t)$ is stochastically smaller than the random variable having the limiting distribution. Supply a bound on the distribution of $X(t)$.

Coupling

P_{ij}^n transition probability of finite-state irreducible MC

Result: For a finite-state ergodic MC, there must exist N, $\varepsilon > 0$, such that

$$P_{ij}^N > \varepsilon \qquad \text{for all } i, j$$

Consider two independent MCs

$\{X_n, n \geq 0\}$ with $P\{X_0 = i\} = 1$

$\{X'_n, n \geq 0\}$ with $P\{X'_0 = j\} = \pi_j$, stationary probabilities

$$\pi_j = \sum_{i=1}^{M} \pi_i P_{ij} \qquad \sum_{i=1}^{M} \pi_j = 1$$

Coupling

T: the first time both processes are in the same state

$$T = \min\left\{n : X_n = X_n^{'}\right\}$$

$$T > mN \Rightarrow X_N \neq X_N^{'}, X_{2N} \neq X_{2N}^{'}, \cdots, X_{mN} \neq X_{mN}^{'}$$

$$\Longrightarrow P\{T > mN\} \leq P(A_1)P\left(A_2 \middle| A_1\right)\cdots P\left(A_m \middle| A_1, \cdots, A_{m-1}\right)$$

where A_i is the event that $X_{iN} \neq X_{iN}^{'}$

The probability that the two chains will both be in state j a time N in the future is at least ε^2.

Thus, they will be in the same state is at least $M\varepsilon^2$.

Coupling

$$\Longrightarrow \quad P\left(A_k \middle| A_1, \cdots, A_{k-1}\right) \le 1 - M\varepsilon^2$$

$$\Longrightarrow \quad P\{T > mN\} \le \left(1 - M\varepsilon^2\right)^m = (1-\alpha)^m$$

Define a third MC by

$$\bar{X}_n = \begin{cases} X_n' & \text{if } n \le T \\ X_n & \text{if } n \ge T \end{cases}$$

$\{\bar{X}_n, n \ge 0\}$ with transition probabilities P_{ij}

$$P\{\bar{X}_n = j\} =$$

$$P\{\bar{X}_n = j | T \le n\} P\{T \le n\} + P\{\bar{X}_n = j | T > n\} P\{T > n\}$$

Coupling

$$= P\left\{\bar{X}_n = j \middle| T \le n\right\} P\{T \le n\} + P\left\{\bar{X}_n = j, T > n\right\}$$

Similarly

$$P_{ij}^n = P\left\{X_n = j\right\} = P\left\{X_n = j \middle| T \le n\right\} P\{T \le n\} + P\left\{X_n = j, T > n\right\}$$

Hence

$$P_{ij}^n - P\left\{\bar{X}_n = j\right\} = P\left\{X_n = j, T > n\right\} - P\left\{\bar{X}_n = j, T > n\right\}$$

implying that $\left|P_{ij}^n - P\left\{\bar{X}_n = j\right\}\right| \le P\{T > n\} \le (1-\alpha)^{n/N-1}$

Since $P\left\{\bar{X}_n = j\right\} = \pi_j$

$\Longrightarrow$ $\left|P_{ij}^n - \pi_j\right| \le \dfrac{\beta^n}{1-\alpha}$ where $\beta = (1-\alpha)^{1/N}$

Coupling

X has a larger hazard rate function than *Y*, if

$$\lambda_X(t) \geq \lambda_Y(t) \qquad \text{for all } t \geq 0$$

Since

$$P\{X > t+s \mid X > t\} = \exp\left\{-\int_t^{t+s} \lambda(x)\,dy\right\}$$

$$\Longrightarrow \quad P\{X > t+s \mid X > t\} \leq P\{Y > t+s \mid Y > t\}$$

or

$$X_t \leq_{st} Y_t \qquad \text{for all } t \geq 0$$

where X_t and Y_t are the remaining lives of a *t*-unit-old item having the same distributions as *X* and *Y*

Coupling

Useful in comparing counting processes

A delayed renewal process

G: first renewal, $\sim \lambda_G(t)$

F: other interarrivals, $\sim \lambda_F(t)$

Let

$$\max\left\{\max_{0\le s\le t}\lambda_F(s), \max_{0\le s\le t}\lambda_G(s)\right\}\le \mu(t)$$

The delayed renewal process can be generated by a random sampling from a nonhomogeneous Poisson process with $\mu(t)$

$S_1, S_2, \cdots$ Times at which events occur, $\{N(t), t \ge 0\}$, $\sim \mu(t)$

Coupling

Define a counting process. Let

$$I_i = \begin{cases} 1 & \text{if an event occurs at time } S_i \\ 0 & \text{otherwise} \end{cases}$$

Given $S_1, S_2, \cdots$ take $P\{I_1 = 1\} = \dfrac{\lambda_G(S_1)}{\mu(S_1)}$

and for $i > 1$,

$$P\{I_i = 1 | I_1, \cdots, I_{i-1}\} = \begin{cases} \dfrac{\lambda_G(S_i)}{\mu(S_i)} & \text{if } I_1 = \cdots = I_{i-1} = 0 \\ \dfrac{\lambda_F(S_i - S_j)}{\mu(S_i)} & \text{if } j = \max\{k : k < i, I_k = 1\} \end{cases}$$

Coupling

$A(t)$: the age of the counting process at time t

$A(S_1) = S_1$

$A(S_2) = S_2$ if $I_1 = 0$; $\quad A(S_2) = S_2 - S_1$ if $I_1 = 1$

Then

$$P\{I_i = 1 | I_1, \cdots, I_{i-1}\} = \begin{cases} \dfrac{\lambda_G(S_i)}{\mu(S_i)} & \text{if } A(S_i) = S_i \\[2ex] \dfrac{\lambda_F(S_i - S_j)}{\mu(S_i)} & \text{if } A(S_i) < S_i \end{cases}$$

We claim that the counting process defined by the I_i, $i \geq 1$, constitutes the desired delayed renewal process.

See the analysis below.

Coupling

Given the past, the probability intensity

$$P\left\{\text{event in } (t, t+h) \middle| \text{history up to } t\right\}$$

$$= P\left\{\text{an event in } (t, t+h)\text{, and is counted} \middle| \text{history up to } t\right\}$$

$$= \left(\mu(t)h + o(h)\right) P\left\{\text{it is counted} \middle| \text{history up to } t\right\}$$

$$= \begin{cases} \left[\mu(t)h + o(h)\right]\dfrac{\lambda_G(t)}{\mu(t)} = \lambda_G(t)h + o(h) & \text{if } A(t) = t \\ \left[\mu(t)h + o(h)\right]\dfrac{\lambda_F(A(t))}{\mu(t)} = \lambda_F(A(t))h + o(h) & \text{if } A(t) < t \end{cases}$$

The probability of an event at any time t depends only on the age at that time and is equal to $\lambda_G(t)$ if the age is t and to $\lambda_F(A(t))$ otherwise.

Coupling

Theorem (Blackwell's Theorem)

Let $\{N^*(t), t \geq 0\}$ denote a renewal process $\sim$ F. Then

$$m(t+a) - m(t) \to \frac{a}{\mu} \quad as \quad t \to \infty$$

where $m(t) = E[N^*(t)]$

Proof: Assumption: $\lambda_F(t)$ is bounded away from 0 and ∞.

There is $0 < \lambda_1 < \lambda_2 < \infty$ such that

$$\lambda_1 < \lambda_F(t) < \lambda_2 \quad \text{for all } t$$

The same assumption for $\lambda_G(t)$.

Coupling

$S_1, S_2, \cdots$ A Poisson process with λ_2

$I_1^*, I_2^*, \cdots$ Generated according to G and $\mu(t) = \lambda_2$

$\Longrightarrow \{N_0(t), t \geq 0\}$ A delayed renewal process

$I_1, I_2, \cdots$ Generated according to $G = F$ and $\mu(t) = \lambda_2$

$\Longrightarrow \{N(t), t \geq 0\}$ A renewal process

Let

$$N = \min\{i : I_i = I_i^* = 1\}$$

It follows that

$$P\{I_i = I_i^* = 1 \mid I_1, \cdots, I_{i-1}, I_1^*, \cdots, I_{i-1}^*\} \geq \left(\frac{\lambda_1}{\lambda_2}\right)^2$$

Coupling

$$\Longrightarrow \quad P\{N<\infty\}=1$$

Define a third sequence by

$$\bar{I}_i = \begin{cases} I_i^* & \text{for } i \le N \\ I_i & \text{for } i \ge N \end{cases}$$

Letting $N(t, t+a) = N(t+a) - N(t)$, and similarly for $\bar{N}$

$$\Longrightarrow \quad E\left[\bar{N}(t,t+a)\right]=$$

$$E\left[\bar{N}(t,t+a)\middle|S_N \le t\right]P\{S_N \le t\}+E\left[\bar{N}(t,t+a)\middle|S_N > t\right]P\{S_N > t\}$$

$$=E\left[N(t,t+a)\middle|S_N \le t\right]P\{S_N \le t\}+E\left[\bar{N}(t,t+a)\middle|S_N > t\right]P\{S_N > t\}$$

$$=E\left[N(t,t+a)\right]+\left(E\left[\bar{N}(t,t+a)\middle|S_N > t\right]-E\left[N(t,t+a)\middle|S_N > t\right]\right)P\{S_N > t\}$$

Coupling

We take $G = F_e$ $\quad F_e(t) = \int_0^t \bar{F}(y)\,dy \Big/ \mu$

(By Theorem 3.5.2) $\quad E\left[\bar{N}(t, t+a)\right] = \dfrac{a}{\mu}$

Also, it easily follows that

$$E\left[N(t, t+a) \middle| S_N > t\right] \le \lambda_2 a$$

$$E\left[\bar{N}(t, t+a) \middle| S_N > t\right] \le \lambda_2 a$$

$$P\left\{S_N > t\right\} \to 0 \quad \text{as } t \to \infty$$

$\Downarrow$

$$E\left[\bar{N}(t, t+a)\right] - E\left[N(t, t+a)\right] \to 0 \quad \text{as } t \to \infty$$

$\Rightarrow$ $\quad E\left[N(t, t+a)\right] \to \dfrac{a}{\mu}$

Coupling

Monontonicity results

$\{N(t), t \geq 0\}$: counting process

Lemma 9.3.1: $\{N(t), t \geq 0\} \sim F$ decreasing failure rate.

$\{N_y(t), t \geq 0\}$ a delayed renewal process with initial H_y and F.

$$\left(N(T_1), \cdots, N(T_n)\right) \geq_{st} \left(N_y(T_1), \cdots, N_y(T_n)\right)$$

Proof: $N^* = \{N^*(t), t \leq y\}$: the first y time units of a renewal process, $\sim F$.

N_y the continuation of N^* from time y onward.

$A^*(y)$: age at time y of N^*

A Poisson process with rate $\mu = \lambda(0)$, and S_1, S_2, … times events occur.

Coupling

Use the Poisson process to generate a counting process N

$$P\left\{I_i = 1 \middle| I_1, \cdots, I_{i-1}\right\} = \frac{\lambda\left(A(S_i)\right)}{\mu}$$

The generated counting process is a renewal process $\sim F$

Define another counting process with $\bar{I}_i$ and $\bar{A}(t)$

$\bar{A}(t)$: the time at t since the last event, or $t + A^*(y)$ if no events

If $I_i = 0$, then $\bar{I}_i = 0$

If $I_i = 1$, then $\bar{I}_i = \begin{cases} 1 & \text{with probability } \lambda\left(\bar{A}(S_i)\right) / \lambda\left(A(S_i)\right) \\ 0 & \text{otherwise} \end{cases}$

Note that $\lambda\left(\bar{A}(S_i)\right) \le \lambda\left(A(S_i)\right)$

Coupling

Hence

$$P\left\{\bar{I}_i = 1 \middle| \bar{I}_1, \cdots, \bar{I}_{i-1}, I_1, \cdots, I_{i-1}\right\} = P\left\{I_i = 1 \middle| I_1, \cdots, I_{i-1}\right\} P\left\{\bar{I}_i = 1 \middle| I_i = 1\right\}$$

$$= \frac{\lambda\left(\bar{A}\left(S_i\right)\right)}{\mu}$$

⇒ The generated counting process $\bar{I}_i$ is a delayed renewal process with initial H_y, because $\bar{A}(0) = A^*(y)$

Since events of N_y can only occur at time points where events of N, it follows that

$$N(T) \geq N_y(T) \qquad \text{for all sets } T$$

Coupling

Proposition 9.3.2: Monotonicity Properties of DFR renewal process

$\{N(t),\, t \geq 0\}$: renewal process with DFR

$A(t)$: age at time t

$Y(t)$: excess at time t

$\Rightarrow$ Both $A(t)$ and $Y(t)$ increase stochastically in t

Proof: We need to show

$$P\{A(t+y) > a\} \geq P\{A(t) > a\}$$

$A(t)$: age at time t of the renewal process N

$A(t+y)$: age at time t of the renewal process N_y

Coupling

Letting $T = [t - a, t]$, from Lemma 9.3.1,

$$P\{N(T) \geq 1\} \geq P\{N_y(T) \geq 1\}$$

$$\Longrightarrow \quad P\{A(t) \leq a\} \geq P\{A(t+y) \leq a\}$$

The proof for the excess is similar except letting $T = [t, t + a]$

Bounds on renewal function of DFR

Corollary 9.3.3: F is a DFR with the first two moments

$$\mu_1 = \int x dF(x) \qquad \mu_2 = \int x^2 dF(x)$$

$$\Longrightarrow \quad \frac{t}{\mu_1} \leq m(t) \leq \frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1 \qquad \bar{F}(t) \geq \exp\left(-\frac{t}{\mu_1} - \frac{\mu_2}{2\mu_1^2} + 1\right)$$

Coupling

Proof: Let X_1, X_2, … be the interarrival times

$$\sum_{i=1}^{N(t)+1} X_i = t + Y(t)$$

where $Y(t)$ is the excess at t. By Wald's equation

$$\mu_1\left(m(t)+1\right) = t + E\left[Y(t)\right]$$

From Proposition 9.3.2, $E[Y(t)]$ is increasing in t.

On the other hand $E\left[Y(0)\right] = \mu_1$ $\quad \lim_{t\to\infty} E\left[Y(t)\right] = \frac{\mu_2}{2\mu_1}$

$$\Longrightarrow \quad t + \mu_1 \le \mu_1\left(m(t)+1\right) \le t + \frac{\mu_2}{2\mu_1}$$

$$\Longrightarrow \quad \frac{t}{\mu_1} \le m(t) \le \frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1$$

Coupling

From renewal theory $$m(t) = \sum_{n=1}^{\infty} F_n(t)$$

$$m'(t) = \sum_{n=1}^{\infty} F_n'(t)\,dt = \sum_{n=1}^{\infty} P\{n\text{th renewal occurs in } (t, t+dt\} + o(dt)$$

$$= P\{\text{a renewal occurs in } (t, t+dt\} + o(dt)$$

Since $\lambda(t)$ is decreasing in t $\Longrightarrow$ $m'(t) = E\left[\lambda\left(A(t)\right)\right] \geq \lambda(t)$

$$m(t) \geq \int_0^t \lambda(s)\,ds$$

Since $$\bar{F}(t) = \exp\left(-\int_0^t \lambda(s)\,ds\right)$$

$$\Longrightarrow \quad \bar{F}(t) \geq e^{-m(t)}$$