Problem 1: Two-horse race

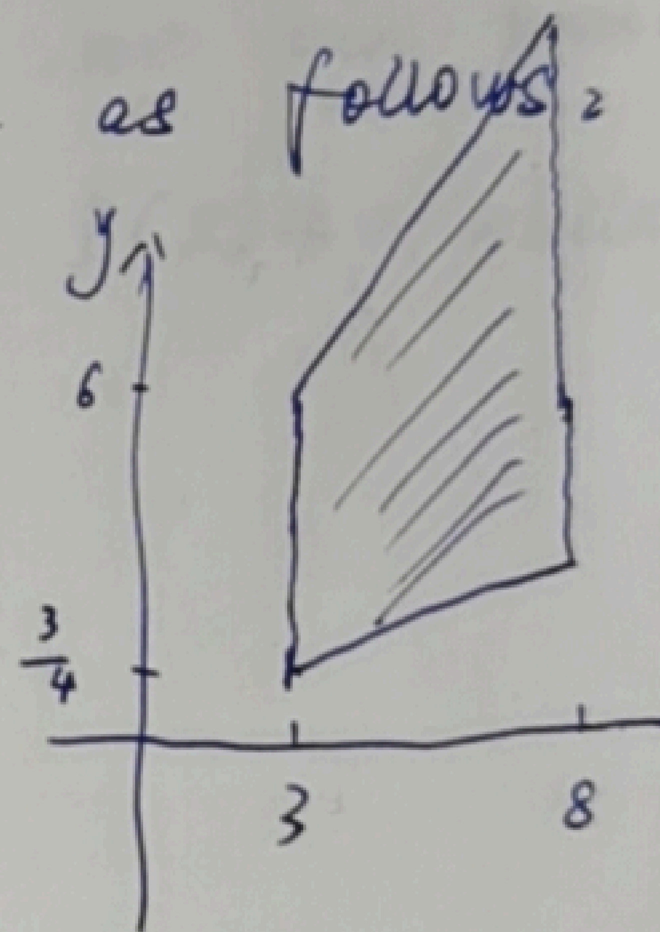
(a) The conditional pdf of U given $V=v$ is:

$$f_{U|V}(u|v) = \frac{f(u,v)}{f_V(v)}$$

The marginal pdf of V given $V=v$ is:

$$f_V(v) = \int_{-\infty}^{+\infty} f(u,v)\,du$$

(b) The density graph is showed as follows:



The unity pdf is:

$$f(x,y) = f_{Y|X}(y|x)\,f_X(x) = \frac{4}{35x}$$

case 1: $y \in [\frac{3}{4}, 2]$

$$f_Y(y) = \int_3^{4y} f_{X,Y}(x,y)\,dx = \frac{4}{35}\ln\frac{4y}{3}$$

case 2: $y \in [2, 6]$

$$f_Y(y) = \int_6^8 f_{X,Y}(x,y)\,dx = \frac{4}{35}\ln\frac{8}{3}$$

case 3: $y \in [6, 16]$

$$f_Y(y) = \int_{\frac{y}{2}}^{8} f_{X,Y}(x,y)\,dx = \frac{4}{35}\ln\frac{16}{y}$$

(c)

$$f_{X|Y}(x|y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}=\begin{cases}\dfrac{\frac{4}{35x}}{\frac{4}{35}\ln\frac{4}{3}y}=\dfrac{1}{x\ln\frac{4}{3}y} & y\in[\frac{3}{4},2]\\ \dfrac{1}{x\cdot\ln\frac{8}{3}} & y\in[2,6]\\ \dfrac{1}{x\cdot\ln\frac{16}{y}} & y\in[6,16]\end{cases}$$

(d) They are not independent since we can promote a counterexemple that when $y=\frac{3}{4}$, $P(S)\neq 0$ while $P(S|y=\frac{3}{4})=0$

(e) $P=\int_3^8\int_{x-1}^{x+1}\frac{4}{35x}\,dx\,dy=\frac{8}{35}\ln\frac{8}{3}$

(f) $P=1-\int_6^8\int_x^{2x}\frac{4}{35x}\,dx\,dy=\frac{27}{35}$

Problem 2. Cell phones.

(a) Donate X_{ij} as the spending time of the j^{th} call in i^{th} day. $(i=1,2,\cdots 30, j=1,2,\cdots N_i)$.

N_i as the number of calls received in i^{th} day.

The the total spending time should be:

$$X^* = \sum_{i=1}^{30}\sum_{j=1}^{N_i} X_{ij} \qquad N_i \sim PP(3) \quad X_{ij} \sim E(\tfrac{1}{5})$$

$$E(X^*) = E\left[\sum_{i=1}^{30}\sum_{j=1}^{N_i} X_{ij}\right] = 30\, E\left[\sum_{j=1}^{N_i} X_{ij}\right] = 30\, E[X_{ij}]\, E[N_i] = 450$$

$$\mathrm{var}(X^*) = \mathrm{var}\left[\sum_{i=1}^{30}\sum_{j=1}^{N_i} X_{ij}\right] = \sum_{i=1}^{30} \mathrm{var}\left[\sum_{j=1}^{N_i} X_{ij}\right]$$

(since independence)

$$= \sum_{i=1}^{30} \left(E(N_i)\,\mathrm{var}X_{ij} + [E(X_{ij})]^2\,\mathrm{var}N_i \right)$$

$$= \sum_{i=1}^{30} (3\times 25 + 5^2\times 3) = 4500$$

(b) Donate N_{ij} as the number of calls accept in i^{th} day of j^{th} month. ($i=1,\sim 30$, $j=1,\sim 12$)

$$\cancel{E(N_{ij})=}\ N^* = \sum_{j=1}^{12}\sum_{i=1}^{30} N_{ij}$$

$$E(N^*) = 360\,E(N_{ij}) = 1080 \qquad Var(N^*) = 360\,var(N_{ij}) = 1080$$

$$P(1190 \le N^* \le 1210) = P\left(\frac{110}{\sqrt{1080}} \le \frac{N^*-1080}{\sqrt{1080}} \le \frac{130}{\sqrt{1080}}\right).$$

$$= \Phi\left(\frac{130}{\sqrt{1080}}\right) - \Phi\left(\frac{110}{\sqrt{1080}}\right)$$

(c) $T = \sum_{i=1}^{4} T_i$, is an Erlang distribution

$$f_{T_4}(t) = \begin{cases} \dfrac{\lambda^4 t^3 e^{-\lambda t}}{3!} & t \ge 0 \\ 0 & \text{otherwise} \end{cases} \qquad \lambda = \frac{1}{5}$$

(d) Donate X as the number of phone calls.

$$P\{X=k\} = \begin{cases} e^{-\frac{1}{5}\times 20} = e^{-4} & \cancel{k=1}\quad k=1 \\ \int_0^{20} e^{-\frac{1}{5}x}\frac{1}{5}e^{-\frac{1}{5}(20-x)} = \frac{4}{5}e^{-4} & k=2 \\ \int_0^{20}\int_0^{20-x} \frac{1}{5}e^{-\frac{1}{5}x}\frac{1}{5}e^{-\frac{1}{5}y}\frac{1}{5}e^{-\frac{1}{5}(20-x-y)}\,dx\,dy = \frac{8}{5}e^{-4} & k=3. \\ & | \\ \frac{4^{n-1}}{5(n-1)}e^{-4} & k=n \end{cases}$$

(e) $X^* = \sum_{i=1}^{30}\sum_{j=1}^{n_i} x_{i,j}$

For plan i, we would pay $= D_i + C_i \cancel{\max}(X^* - N_i)^+$

To minimize the objective function, we can choose the expected/mean value of x^* as the N_i,

f) ~~Plan 2 will be better~~

If the sample is large enough, then the parameter N_i may need some adjustments.

So the adviser's statement is wrong.

Problem 3: Dogs in the Woods.

(a) $p(x) = \frac{1}{15} \quad (30 \le x \le 45)$.

(b) $$P = \frac{C_3^2 C_3^1 \cdot \frac{2}{3} \cdot \frac{1}{3} + C_3^1 C_3^2 \cdot \frac{1}{3} \cdot \frac{2}{3}}{C_6^3} = \frac{1}{5}$$

(c) The joint pdf is $f_{X,Y}(x,y) = \frac{1}{15^2} \quad (x,y) \in [30,45]^2$

Donate $T = X+Y$ as the total amount of calories.

$$P(T \le t) = \begin{cases} \frac{1}{15^2} \cdot \frac{1}{2}(t-60)^2 = \frac{1}{450}(t-60)^2 & 60 \le t < 75 \\ \cancel{\tfrac{1}{2}}\, 1 - \frac{1}{15^2} \cdot \frac{1}{2}(90-t)^2 = 1 - \frac{1}{450}(90-t)^2 & 75 \le t \le 90 \\ 1 & t > 90 \end{cases}$$

Thus, $$p_T(t) = \begin{cases} \frac{1}{225}(t-60) & 60 \le t \le 75 \\ \frac{1}{225}(90-t) & 75 < t \le 90 \end{cases}$$

$$\Rightarrow E(T) = \int_{60}^{90} p_T(t) \cdot t = \frac{1115}{9}$$

(d) $P = \dfrac{C_2^2}{C_4^2} = \dfrac{1}{6}$

(e) $\sigma_0^2 = 2\sigma^2 = 150.$ $\qquad P = \dfrac{C_3^1}{C_6^2} = \dfrac{1}{5}$

$\Rightarrow \sigma^{*2} = \sigma_0^2(1-P) = 120.$

(f) $\sigma_0^2 = var(T+S) = \sigma_T^2 + \sigma_S^2 = \dfrac{75}{2}$

$P = 1 \times \dfrac{1}{3} = \dfrac{1}{3}$ $\qquad \sigma^{*2} = \sigma_0^2(1-P) = 25$