Advanced Topics in OR

Lecture Notes 1

Stochastic Order Relations

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Stochastically Larger

Random variable X is stochastically larger than random variable Y, written $X \geq_{st} Y$, if

$$P\{X > a\} \geq P\{Y > a\} \qquad \text{for all } a$$

$X \sim$ distribution F
$Y \sim$ distribution G $\Rightarrow$ $\bar{F}(a) \geq \bar{G}(a) \quad \text{for all } a$

Lemma 9.1.1. If $X \geq_{st} Y$, then $\mathrm{E}[X] \geq \mathrm{E}[Y]$

Proof: If X and Y are nonnegative, then

$$E[X] = \int_0^\infty P\{X > a\}\, da \geq \int_0^\infty P\{Y > a\}\, da = E[Y]$$

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For a general random variable Z

$$Z = Z^{+} - Z^{-}$$

It holds that

$$X \geq_{st} Y \quad \Rightarrow \quad X^{+} \geq_{st} Y^{+} \quad \text{and} \quad X^{-} \leq_{st} Y^{-}$$

$$\Rightarrow \quad E[X] = E\left[X^{+}\right] - E\left[X^{-}\right] \geq E\left[Y^{+}\right] - E\left[Y^{-}\right] = E[Y]$$

Proposition 9.1.2. $X \geq_{st} Y$, iff E[$f(X)$] $\geq$ E[$f(Y)$] for all increasing functions f.

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Proof: Fisrt show that $X \geq_{st} Y \rightarrow f(X) \geq_{st} f(Y)$ for all increasing functions f.

Letting $f^{-1}(a) = \inf\{x : f(x) \geq a\}$

$$P\{f(X) > a\} = P\left\{X > f^{-1}(a)\right\} \geq P\left\{Y > f^{-1}(a)\right\} = P\{f(Y) > a\}$$

Suppose that $E[f(X)] \geq E[f(Y)]$ for all increasing functions f. For any a, let f_a denote the increasing function

$$f_a(x) = \begin{cases} 1 & \text{if } x > a \\ 0 & \text{if } x \leq a \end{cases} \quad \Longrightarrow \quad E\left[f_a(X)\right] = P\{X > a\}$$

$$E\left[f_a(Y)\right] = P\{Y > a\}$$

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Example 9.1(A): ***Increasing and Decreasing Failure Rate.***

X: a nonnegative random variable, distribution F, density f.
Failure (or hazard) rate function of X

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)}$$

X is an increasing failure rate (IFR) random variable if

$$\lambda(t) \uparrow t$$

X is an decreasing failure rate (DFR) random variable if

$$\lambda(t) \downarrow t$$

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Example 9.1(A): ***Increasing and Decreasing Failure Rate.***

Think of X as the life of some item

$\lambda(t)dt$: Probability that a t-unit-old item fails in the interval $(t, t + dt)$

X is IFR (DFR): the old item is the more (less) likely it is to fail in a small time dt.

Suppose the item has survived to time t, and let X_t denote its additional life from t onward.

$$\bar{F}_t(a) = P\{X_t > a\} = P\{X - t > a \mid X > t\} = \frac{\bar{F}(t+a)}{\bar{F}(t)}$$

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Proposition 9.1.3:

X is IFR $\longleftrightarrow$ X_t is stochastically decreasing in t

X is DFR $\longleftrightarrow$ X_t is stochastically increasing in t

Proof: The hazard rate function of X_t

$$F_t(a) = 1 - \bar{F}_t(a) = 1 - \frac{1 - F(t+a)}{1 - F(t)} = \frac{F(t+a) - F(t)}{\bar{F}(t)}$$

$$\Downarrow$$

$$f_t(a) = \frac{f(t+a)}{\bar{F}(t)}$$

$$\Longrightarrow \quad \lambda_t(a) = \frac{f_t(a)}{\bar{F}_t(a)} = \frac{f(t+a)}{\bar{F}(t)} \frac{\bar{F}(t)}{\bar{F}(t+a)} = \lambda(t+a)$$

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Proof: We have

$$\bar{F}_t(s) = \exp\left\{-\int_0^s \lambda_t(a)\,da\right\} = \exp\left\{-\int_t^{t+s} \lambda(y)\,dy\right\}$$

If $\lambda(y)$ is increasing (decreasing), then

$\bar{F}_t(s)$ is decreasing (increasing)

If $\bar{F}_t(s)$ is decreasing (increasing), then

$\lambda(y)$ is increasing (decreasing)

Stochastically Larger

Proposition 9.1.5:

if F_α is a DFR distribution for all $0 < \alpha < \infty$ and G a distribution function on $(0, \infty)$, then F is DFR, where

$$F(t) = \int_0^\infty F_\alpha(t)\, dG(\alpha)$$

Proof:

$$\lambda_F(t) = \frac{\frac{d}{dt}F(t)}{\bar{F}(t)} = \frac{\int_0^\infty f_\alpha(t)\, dG(\alpha)}{\bar{F}(t)}$$

$$\frac{d}{dt}\lambda_F(t) = \frac{\bar{F}(t)\int f_\alpha^{'}(t)\, dG(\alpha) + \left(\int f_\alpha(t)\, dG(\alpha)\right)^2}{\bar{F}^2(t)}$$

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Proof:

Since $\bar{F}(t) = \int \bar{F}_\alpha(t)\,dG(\alpha)$

We need to show

$$\left(\int f_\alpha(t)\,dG(\alpha)\right)^2 \le \left(\int -f_\alpha^{'}(t)\,dG(\alpha)\right)\left(\int \bar{F}_\alpha(t)\,dG(\alpha)\right)$$

Lemma 9.1.4: The Cauchy-Schwarz Inequality

For any distribution G and functions $h(t)$, $k(t)$, $t \ge 0$,

$$\left(\int h(t)k(t)\,dG(t)\right)^2 \le \left(\int h^2(t)\,dG(t)\right)\left(\int k^2(t)\,dG(t)\right)$$

Stochastically Larger

Proof: Letting $h(\alpha)=\left(\bar{F}_\alpha(t)\right)^{1/2}$ $k(\alpha)=\left(-f_\alpha^{'}(t)\right)^{1/2}$

Applying the Cauchy-Schwarz Inequality

$$\left(\int\left(-\bar{F}_\alpha(t)\,f_\alpha^{'}(t)\right)^{1/2}dG(\alpha)\right)^2 \le \int \bar{F}_\alpha(t)\,dG(\alpha)\int -f_\alpha^{'}(t)\,dG(\alpha)$$

It suffices to show

$$\left(\int f_\alpha(t)\,dG(\alpha)\right)^2 \le \left(\int\left(-\bar{F}_\alpha(t)\,f_\alpha^{'}(t)\right)^{1/2}dG(\alpha)\right)^2$$

F_α is DFR

$$\Rightarrow \quad 0 \ge \frac{d}{dt}\frac{f_\alpha(t)}{\bar{F}_\alpha(t)} = \frac{\bar{F}_\alpha(t)\,f_\alpha^{'}(t)+f_\alpha^2(t)}{\bar{F}_\alpha^2(t)}$$

$$\Rightarrow \quad -\bar{F}_\alpha(t)\,f_\alpha^{'}(t) \ge f_\alpha^2(t)$$

Coupling

Lemma 9.2.1: Let F and G be continuous functions. If X has distribution F then the random variable $G^{-1}(F(X))$ has distribution G.

Proof:

$$P\left\{G^{-1}\left(F\left(X\right)\right)\leq a\right\}=P\left\{F\left(X\right)\leq G\left(a\right)\right\}$$

$$=P\left\{X\leq F^{-1}\left(G\left(a\right)\right)\right\}$$

$$=F\left(F^{-1}\left(G\left(a\right)\right)\right)$$

$$=G\left(a\right)$$

Coupling

Proposition 9.2.2: If F and G are distributions such that $\bar{F}(a) \geq \bar{G}(a)$, then there exist random variables X and Y having distributions F and G respectively such that

$$P\{X \geq Y\} = 1$$

Proof: Let $X \sim F$ Define $Y = G^{-1}(F(X)) \sim G$

Because $F \leq G$, it follows that $F^{-1} \geq G^{-1}$

$$\Longrightarrow \quad Y = G^{-1}\left(F(X)\right) \leq F^{-1}\left(F(X)\right) = X$$

Coupling

Example 9.2(A): *Stochastic Ordering of Vectors*

Let $X_1, \ldots, X_n$ be independent and $Y_1, \ldots, Y_n$ be independent. If $X_i \geq_{st} Y_i$, then for any increasing f

$$f(X_1, \cdots, X_n) \geq_{st} f(Y_1, \cdots, Y_n)$$

Proof: Use Proposition 9.2.2 to generate independent $Y_1^*, \cdots, Y_n^*$

$Y_i^* \sim$ the distribution of Y_i and $Y_i^* \leq X_i$

f is increasing $\Longrightarrow$ $f(X_1, \cdots, X_n) \geq f\left(Y_1^*, \cdots, Y_n^*\right)$

$$\Longrightarrow \quad f\left(Y_1^*, \cdots, Y_n^*\right) > a \Rightarrow f(X_1, \cdots, X_n) > a$$

$$\Downarrow$$

$$P\{f(Y_1, \cdots, Y_n) > a\} = P\left\{f\left(Y_1^*, \cdots, Y_n^*\right) > a\right\} \leq P\{f(X_1, \cdots, X_n) > a\}$$

Coupling

Example 9.2(B): *Stochastic Ordering of Poisson Random Variables*

A Poisson variable is stochastically increasing in its mean. Let N denote a Poisson random variable with mean λ.

For any p, $0 < p < 1$, let I_1, I_2, … be independent of each other and of N and such that

$$I_j = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

Then $\sum_{j=1}^{N} I_j$ is Poisson with mean λp

Since $\sum_{j=1}^{N} I_j \leq N$ the result follows.

Coupling

Definition: *Random vectors* $\underline{X} = \left(X_1, \cdots, X_n\right) \qquad \underline{Y} = \left(Y_1, \cdots, Y_n\right)$

$\underline{X} \geq_{st} \underline{Y}$ if, for all increasing functions f,

$$E\left[f\left(\underline{X}\right)\right] \geq E\left[f\left(\underline{Y}\right)\right]$$

Stochastic process $\{X(t), t \geq 0\}$ is greater than stochastic process $\{Y(t), t \geq 0\}$ if

$\left(X\left(t_1\right), \cdots, X\left(t_n\right)\right) \geq_{st} \left(Y\left(t_1\right), \cdots, Y\left(t_n\right)\right)$ for all n, t_1, …, t_n

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If $\underline{X}$ and $\underline{Y}$ are vectors of independent components such that $X_i \geq_{st} Y_i$, then $\underline{X} \geq_{st} \underline{Y}$

Counterexample when independency assumption is dropped?

Coupling

Example 9.2(C): *Comparing Renewal Process*

$N_i = \{N_i(t),\, t \geq 0\}$, two renewal process with F and G

If $\bar{F} \geq \bar{G}$ then $\{N_1(t), t \geq 0\} \leq_{st} \{N_2(t), t \geq 0\}$

Proof: Let $X_1, X_2, \ldots \sim F, \rightarrow N_1^* \sim N_1$

Generate $Y_1, Y_2, \ldots \sim G, \rightarrow N_2^* \sim N_2$

$Y_i \leq X_i \Longrightarrow N_1^*(t) \leq N_2^*(t)$ for all t

Coupling

Example 9.2(D):

Let $X_1, X_2, \ldots$ a sequence $\sim$ Bernoulli random variables.

Let $p_i = P\{X_i = 1\}$. If $p_i \geq p$ for all i, then with probability 1,

$$\liminf_n \sum_{i=1}^{n} \frac{X_i}{n} \geq p$$

Proof: Let X_i, $i \geq 1$, a sequence

Y_i, $i \geq 1$, $\sim$ Bernoulli random variables, $P\{Y_i = 1\} = p$ and $X_i \geq Y_i$ for all i.

U_i, $i \geq 1$, $\sim U(0, 1)$

Coupling

Proof: For $i = 1, \ldots, n$, set

$$X_i = \begin{cases} 1 & \text{if } U_i \le p_i \\ 0 & \text{otherwise} \end{cases} \qquad Y_i = \begin{cases} 1 & \text{if } U_i \le p \\ 0 & \text{otherwise} \end{cases}$$

Since $p \le p_i$, it follows that $Y_i \le X_i$

$$\Rightarrow \quad \liminf_n \sum_{i=1}^{n} \frac{X_i}{n} \ge \liminf_n \sum_{i=1}^{n} \frac{Y_i}{n}$$

From the strong law of large numbers, with probability 1,

$$\liminf_n \sum_{i=1}^{n} \frac{Y_i}{n} = p$$

Coupling

Example 9.2(E): *Bounds on the Coupon Collector's Problem*

m distinct types of coupons

P_j: probability of type j collected

N: number of coupons to collect all types

$i_1, \ldots, i_m$: a permutation of 1, …, m

T_j: additional coupons after having $i_1, \ldots, i_{j-1}$,

0 or $\sim$ geometric

$$N = \sum_{j=1}^{m} T_j \qquad E[N] = \sum_{j=1}^{m} \frac{P\{i_j \text{ is the last of } i_1, \cdots, i_j\}}{P_{i_j}}$$

Coupling

X_j: ~ exponential with rate 1

$\Longrightarrow$ X_j/P_j: ~ exponential with rate P_j

$$P\left\{i_j \text{ is the last of } i_1,\cdots,i_j\right\} = P\left\{X_{i_j}/P_{i_j} = \max\left(X_{i_1}/P_{i_1},\cdots,X_{i_j}/P_{i_j}\right)\right\}$$

Renumber the coupon types so that $P_1 \le \ldots \le P_m$

$\Longrightarrow$

$$P\{j \text{ is the last of } 1,\cdots,j\} = P\left\{X_j/P_j = \max_{1\le i\le j} X_i/P_i\right\}$$

$$\le P\left\{X_j/P_j = \max_{1\le i\le j} X_i/P_j\right\}$$

$$= \frac{1}{j}$$

Coupling

On the other hand

$$P\{\, j \text{ is the last of } m, \cdots, j\} \geq \frac{1}{m-j+1}$$

$$\sum_{j=1}^{m} \frac{1}{m-j+1} \frac{1}{P_j} \leq E[N] \leq \sum_{j=1}^{m} \frac{1}{j} \frac{1}{P_j}$$

Example 9.2(F): *A Bin Packing Problem*

n items with weights U(0, 1), are put into bins with capacity one.

Analyze $E[B]$, the expected number of bins needed.

Coupling

N_i: the number of items in bin i

W_i: initial item in bin i, not fit in bin i – 1

$$N_i \overset{d}{=} \max\left\{ j : W_i + U_1 + \cdots + U_{j-1} \le 1 \right\}$$

A_{i-1}: unused capacity in bin i – 1

$$P\left\{ W_i > x \middle| A_{i-1} \right\} = P\left\{ U > x \middle| U > A_{i-1} \right\}$$

Since $P\left\{ U > x \middle| U > A_{i-1} \right\} > P\left\{ U > x \right\}$

⟹ W_i is stochastically larger than U

⟹ $$N_i \le_{st} \max\left\{ j : U_1 + \cdots + U_j \le 1 \right\}$$

A renewal process with U(0, 1)

Coupling

The number of bins needed

$$B = \min\left\{ m : \sum_{i=1}^{m} N_i \geq n \right\}$$

X_i : renewal process N(1) with U(0, 1)

$\Rightarrow$ where $N = \min\left\{ m : \sum_{i=1}^{m} X_i \geq n \right\}$

By Wald's equation,

$$E\left[\sum_{i=1}^{N} X_i \right] = E[N]E[X_i]$$

we can show (exercise) $E[X_i] = e - 1$

Coupling

Since $\sum_{i=1}^{N} X_i \geq n$

$$\Longrightarrow \qquad E[N] \geq \frac{n}{e-1}$$

By using $B \geq_{st} N$, we have $\quad E[B] \geq \frac{n}{e-1}$

If the weights $\sim$ arbitrary distribution F on [0, 1], then

$$E[B] \geq \frac{n}{m(1)}$$

where $m(1)$ is the expected number of renewals by time 1