Advanced Topics in OR

Lecture Notes 4

Stochastic Order Relations

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Applications of Variability Orderings

Definition: X: nonnegative random variable

X is *new better than used in expectation* (NBUE) if

$$E\left[X-a \mid X>a\right] \le E\left[X\right] \quad \text{for all } a \ge 0$$

X is *new worse than used in expectation* (NWUE) if

$$E\left[X-a \mid X>a\right] \ge E\left[X\right] \quad \text{for all } a \ge 0$$

X is NBUE → the expected additional life of any used item is less (greater) than or equal to the expected life of a new item.

Proposition 9.6.1: If F is an NBUE distribution having mean μ, then $F \le_V \exp(\mu)$

Reversed in NWUE

Applications of Variability Orderings

Proof: If X has distribution F, then

$$E\left[X-a \middle| X>a\right]=\int_0^\infty P\left\{X-a>x \middle| X>a\right\}dx$$

$$=\int_0^\infty \frac{\bar{F}(a+x)}{\bar{F}(a)}dx$$

$$=\int_a^\infty \frac{\bar{F}(y)}{\bar{F}(a)}dy$$

Hence for F NBUE with mean μ, we have

$$\int_a^\infty \frac{\bar{F}(y)}{\bar{F}(a)}dy \le \mu \quad \Rightarrow \quad \frac{\bar{F}(a)}{\int_a^\infty \bar{F}(y)\,dy} \ge \frac{1}{\mu}$$

Applications of Variability Orderings

$$\Longrightarrow \quad \int_0^c \left(\frac{\bar{F}(a)}{\int_a^\infty \bar{F}(y)\,dy} \right) da \ge \frac{c}{\mu}$$

Let $\quad x = \int_a^\infty \bar{F}(y)\,dy \qquad dx = -\bar{F}(a)$

$$\Longrightarrow \quad -\int_\mu^{x(c)} \frac{dx}{x} \ge \frac{c}{\mu} \qquad x(c) = \int_c^\infty \bar{F}(y)\,dy$$

$$\Longrightarrow \quad -\log\left(\int_c^\infty \frac{\bar{F}(y)\,dy}{\mu} \right) \ge \frac{c}{\mu}$$

or $\quad \int_c^\infty \bar{F}(y)\,dy \le \mu e^{-c/\mu}$

Applications of Variability Orderings

Comparison of G/G/1 queues

X_n, $n \geq 1$: interarrival times, iid

S_n, $n \geq 1$: service times, iid

D_n, $n \geq 1$: delay in the queue

$$\Rightarrow \quad D_1 = 0$$

$$D_{n+1} = \max\left\{0, D_n + S_n - X_{n+1}\right\}$$

Theorem 9.6.2: Two G/G/1 systems, i = 1, 2. If

(i) $E\left[X_n^{(1)}\right] = E\left[X_n^{(2)}\right]$ (ii) $X_n^{(1)} \geq_V X_n^{(2)}$ $S_n^{(1)} \geq_V S_n^{(2)}$

then $D_n^{(1)} \geq_V D_n^{(2)}$ for all n

Applications of Variability Orderings

Comparison of G/G/1 queues

Proof: The proof is by induction. It is obvious for $n = 1$.

Assume it for n

$$\Longrightarrow \quad D_n^{(1)} \geq_V D_n^{(2)} \qquad S_n^{(1)} \geq_V S_n^{(2)} \qquad -X_n^{(1)} \geq_V -X_n^{(2)}$$

By Proposition 9.5.4,

$$D_n^{(1)} + S_n^{(1)} - X_{n+1}^{(1)} \geq_V D_n^{(2)} + S_n^{(2)} - X_{n+1}^{(2)}$$

Since $h(x) = \max(0, x)$ is an increasing convex function, it follows that

$$D_{n+1}^{(1)} \geq_V D_{n+1}^{(2)}$$

Applications of Variability Orderings

Comparison of G/G/1 queues

Corollary 9.6.3: For a G/G/1 queue with $E[S] < E[X]$.

(i) If the interarrival distribution is NBUE with $1/\lambda$, then

$$W_Q \le \frac{\lambda E\left[S^2\right]}{2\left(1-\lambda E[S]\right)}$$

(ii) If the service distribution G is NBUE with $1/\mu$, then

$$W_Q \le \mu\beta\left(1-\beta\right) \quad \text{where} \quad \beta = \int_0^\infty e^{-\mu t(1-\beta)}\, dG(t)$$

Proof: An NBUE distribution is less variable than an exponential distribution with the same mean. (i) we can compare with M/G/1, and (ii) with G/M/1.

Applications of Variability Orderings

A renewal process application

Lemma 9.6.5: F_i, $i \geq 1$, be NBUE, each having mean μ. G $\sim$ exponential with mean μ. Then for each k,

$$\sum_{i=k}^{\infty}\left(F_1 * \cdots * F_i\right)(t) \leq \sum_{i=k}^{\infty} G_{(i)}(t)$$

Proof: The proof is by induction on k.

$k = 1$, let X_1, X_2, … $\sim$ F_i. Let

$$N^*(t) = \max\left\{n, \sum_{1}^{n} X_i \leq t\right\}$$

Applications of Variability Orderings

A renewal process application

$$\Longrightarrow \qquad E\left[\sum_{i=1}^{N^*(t)+1} X_i\right] = E[X]E\left[N^*(t)+1\right]$$

We also have

$$\sum_{i=1}^{N^*(t)+1} X_i = t + \text{time from } t \text{ until } N^* \text{ increases}$$

E[time from t until N^* increases] is equal to the expected excess life of one of the X_i.

By NBUE, $\Longrightarrow$ $E\left[\sum_{i=1}^{N^*(t)+1} X_i\right] \le t + \mu$

Applications of Variability Orderings

A renewal process application

$$\Rightarrow \quad E\left[N^*(t)\right] \le \frac{t}{\mu} = \sum_{i=1}^{\infty} G_{(i)}(t)$$

However,

$$E\left[N^*(t)\right] = \sum_{i=1}^{\infty} P\left\{N^*(t) \ge i\right\}$$

$$= \sum_{i=1}^{\infty} P\left\{X_1 + \cdots + X_i \le t\right\} \quad = \sum_{i=1}^{\infty} \left(F_1 * \cdots * F_i\right)(t)$$

The result is established when $k = 1$.

Assume the result for k.

$$\sum_{i=k+1}^{\infty} \left(F_1 * \cdots * F_i\right)(t) = \sum_{i=k+1}^{\infty} \int_0^t \left(F_1 * \cdots * F_{i-1}\right)(t-x)\, dF_i(x)$$

Applications of Variability Orderings

A renewal process application

$$= \int_0^t \sum_{j=k}^{\infty} \left(F_1 * \cdots * F_j\right)(t-x)\, dF_{j+1}(x) \le \int_0^t \sum_{j=k}^{\infty} G_{(j)}(t-x)\, dF_{j+1}(x)$$

$$= \sum_{j=k}^{\infty} \left(G_{(j)} * F_{j+1}\right)(t) \quad = \left(\left(\sum_{j=k}^{\infty} G_{(j-1)} * F_{j+1}\right) * G\right)(t)$$

$$\le \left(\left(\sum_{j=k}^{\infty} G_{(j)}\right) * G\right)(t) \quad = \sum_{j=k}^{\infty} G_{(j+1)}(t) \quad = \sum_{j=k+1}^{\infty} G_{(j)}(t)$$

The result is established.

Applications of Variability Orderings

A renewal process application

$\{N_F(t), t \geq 0\}$: renewal process $\sim F$.

Theorem 9.6.4: F is NBUE with mean μ, then $N_F(t) \leq_V N(t)$, where $N(t)$ is a Poisson process with rate $1/\mu$.

Proof: We must show that for all $k \geq 1$

$$\sum_{i=k}^{\infty} P\{N_F(t) \geq i\} \leq \sum_{i=k}^{\infty} P\{N(t) \geq i\}$$

Then Lemma 9.6.5 indicates the result.

Remark: Whether we can directly prove $\sum_{i=k}^{\infty} F_{(i)}(t) \leq \sum_{i=k}^{\infty} G_{(i)}(t)$

Applications of Variability Orderings

A renewal process application

Remark: The result holds for $k = 1$.

For $k + 1$, we can reach

$$\sum_{i=k+1}^{\infty} F_{(i)} \le \left(\sum_{j=k}^{\infty} G_{(j-1)} * F_{j+1} \right) * G$$

Without Lemma 9.6.5, we may not have

$$\sum_{j=k}^{\infty} G_{(j-1)} * F_{j+1} \le \sum_{j=k}^{\infty} G_{(j)}$$

A Branching Process Application

Lemma 9.6.7:

$X_1, X_2, \ldots$: sequence of nonnegative i.i.d.

$Y_1, Y_2, \ldots$: sequence of nonnegative i.i.d.

M and N: integer-valued nonnegative random variables

$$X_i \geq_V Y_i \quad , \quad N \geq_V M \qquad \Rightarrow \qquad \sum_{i=1}^{N} X_i \geq_V \sum_{i=1}^{M} Y_i$$

Proof: We first show that
$$\sum_{i=1}^{N} X_i \geq_V \sum_{i=1}^{M} X_i$$

Let h be an increasing convex function. We need to show

$$E\left[h\left(\sum_{i=1}^{N} X_i\right)\right] \geq E\left[h\left(\sum_{i=1}^{M} X_i\right)\right]$$

A Branching Process Application

Then, we need to show that the following function is increasing and convex in n

$$g(n) = E\left[h\left(X_1 + \cdots + X_n\right)\right]$$

Because h is increasing and each X_i is nonnegative, g is clearly increasing.

For the convexity, we need to show $g(n + 1) - g(n)$ is increasing in n.

Let $S_n = \sum_1^n X_i$

Note that $g(n+1) - g(n) = E\left[h\left(S_n + X_{n+1}\right) - h\left(S_n\right)\right]$

Now $E\left[h\left(S_n + X_{n+1}\right) - h\left(S_n\right) \middle| S_n = t\right] = E\left[h\left(t + X_{n+1}\right) - h(t)\right] = f(t)$

A Branching Process Application

Since h is convex, it follows that $f(t)$ is increasing in t.

Since S_n increasing in n, we see that $E[f(S_n)]$ increases in n

But $E\left[f\left(S_n\right)\right] = g\left(n+1\right) - g\left(n\right)$

We have proven that $\sum_{1}^{N} X_i \geq_V \sum_{1}^{M} X_i$

We need to show $\sum_{1}^{M} X_i \geq_V \sum_{1}^{M} Y_i$

Or equivalently to show $E\left[h\left(\sum_{1}^{M} X_i\right)\right] \geq E\left[h\left(\sum_{1}^{M} Y_i\right)\right]$

A Branching Process Application

But $$E\left[h\left(\sum_{1}^{M} X_i\right)\middle| M=m\right] = E\left[h\left(\sum_{1}^{m} X_i\right)\right]$$ by independent

$$\geq E\left[h\left(\sum_{1}^{m} Y_i\right)\right]$$ since $$\sum_{1}^{m} X_i \geq_V \sum_{1}^{m} Y_i$$

$$= E\left[h\left(\sum_{1}^{M} Y_i\right)\middle| M=m\right]$$

The result follows by taking expectations of both sides of the above

A Branching Process Application

Two branching processes

F_1: the number of offspring per individual in the process 1

F_2: the number of offspring per individual in the process 2

Suppose that $F_1 \geq_V F_2$

$Z_n^{(i)}$: the size of the *n*th generation of the *i*th process

Theorem 9.6.6: If $Z_0^{(i)} = 1$, $i = 1, 2$, then $Z_n^{(1)} \geq_V Z_n^{(2)}$ for all n.

Proof: The proof is by induction on n. It is true for $n = 0$. Suppose it for n. Now

$$Z_{n+1}^{(1)} = \sum_{j=1}^{Z_n^{(1)}} X_j \qquad Z_{n+1}^{(2)} = \sum_{j=1}^{Z_n^{(2)}} Y_j$$

A Branching Process Application

X_j: the number of offspring of the jth person of the nth generation, $\sim F_1$

Y_j: $\sim F_2$

Since $X_j \geq_V Y_j$ (by the hypothesis)

$Z_n^{(1)} \geq_V Z_n^{(2)}$ (by the induction hypothesis)

the result follows from Lemma 9.6.7

Corollary 9.6.8: Let μ_1 and μ_2 denote the mean F_1 and F_2. If $Z_0^{(i)} = 1$, $\mu_1 = \mu_2 = \mu$, and $F_1 \geq_V F_2$, then

$$P\left\{Z_n^{(1)} = 0\right\} \geq P\left\{Z_n^{(2)} = 0\right\} \quad \text{for all } n$$

A Branching Process Application

Proof: From Theorem 9.6.6, we have $Z_n^{(1)} \geq_V Z_n^{(2)}$

From Proposition 9.5.1, $\sum_{i=2}^{\infty} P\left\{Z_n^{(1)} \geq i\right\} \geq \sum_{i=2}^{\infty} P\left\{Z_n^{(2)} \geq i\right\}$

Since $E\left[Z_n^{(1)}\right] = \sum_{i=1}^{\infty} P\left\{Z_n^{(1)} \geq i\right\} = \mu^n$

we have $\mu^n - P\left\{Z_n^{(1)} \geq 1\right\} \geq \mu^n - P\left\{Z_n^{(2)} \geq 1\right\}$

or $P\left\{Z_n^{(2)} \geq 1\right\} \geq P\left\{Z_n^{(1)} \geq 1\right\}$

which proves the result.

Associated Random Variables

The set of random variables $X_1, \ldots, X_n$ is said to be associated if for all increasing functions f and g

$$E\left[f(\boldsymbol{X})g(\boldsymbol{X})\right] \geq E\left[f(\boldsymbol{X})\right]E\left[g(\boldsymbol{X})\right]$$

Proposition 7.2.1: If f and g are both increasing functions,

$$\Longrightarrow \quad E\left[f(X)g(X)\right] \geq E\left[f(X)\right]E\left[g(X)\right]$$

Proposition 9.7.1: Independent random variables are associated.

Proof: Suppose that $X_1, \ldots, X_n$ are independent. The proof is by induction.

From Proposition 7.2.1, the result follows when $n = 1$.

Associated Random Variables

Assume the result for $n-1$. We have

$$
\begin{aligned}
E\left[f(\boldsymbol{X})g(\boldsymbol{X})\middle|X_n=x\right] \\
&= E\left[f(X_1,\cdots,X_{n-1},x)g(X_1,\cdots,X_{n-1},x)\middle|X_n=x\right] \\
&= E\left[f(X_1,\cdots,X_{n-1},x)g(X_1,\cdots,X_{n-1},x)\right] \\
&\geq E\left[f(X_1,\cdots,X_{n-1},x)\right]E\left[g(X_1,\cdots,X_{n-1},x)\right] \\
&= E\left[f(\boldsymbol{X})\middle|X_n=x\right]E\left[g(\boldsymbol{X})\middle|X_n=x\right]
\end{aligned}
$$

Hence

$$E\left[f(\boldsymbol{X})g(\boldsymbol{X})\middle|X_n\right]\geq E\left[f(\boldsymbol{X})\middle|X_n\right]E\left[g(\boldsymbol{X})\middle|X_n\right]$$

Associated Random Variables

$$\Longrightarrow \quad E\left[f(\boldsymbol{X})g(\boldsymbol{X})\right] \geq E\left[E\left[f(\boldsymbol{X})|X_n\right]E\left[g(\boldsymbol{X})|X_n\right]\right]$$

Since $E\left[f(\boldsymbol{X})|X_n\right]$ and $E\left[g(\boldsymbol{X})|X_n\right]$ are both increasing functions of X_n, Proposition 7.2.1 yields the result.

Increasing functions of associated random variables are also associated

$\Longrightarrow$ Increasing functions of independent random variables are also associated

Example 9.7(A): A system composed of n components

$$X_i = \begin{cases} 1 & \text{working} \\ 0 & \text{failed} \end{cases}$$

Associated Random Variables

$$P\{X_i = 1\} = p_i$$

Subsets $C_1, \ldots, C_r$ $\qquad |C_1| + \cdots + |C_r| = n$

The system works if and only if at least one of the components in each of these subsets is working.

Let $\quad S = \begin{cases} 1 & \text{the system works} \\ 0 & \text{otherwise} \end{cases}$

Then $\quad S = \prod_{i=1}^{r} Y_i \quad$ where $\quad Y_i = \max_{j \in C_i} X_j$

As $Y_1, \ldots, Y_r$ are all increasing functions of the independent random variables $X_1, \ldots, X_n$, they are associated.

Associated Random Variables

$$\Rightarrow \quad P\{S=1\} = E[S] = E\left[\prod_{i=1}^{r} Y_i\right] \geq E[Y_1]E\left[\prod_{i=2}^{r} Y_i\right] \geq \cdots \geq \prod_{i=1}^{r} E[Y_i]$$

Since Y_i is equal to 1 if at least one of the components in C_i works, we have

$$P\{\text{system works}\} \geq \prod_{i=1}^{r}\left\{1 - \prod_{j \in C_i}\left(1 - p_j\right)\right\}$$

The easiest way of showing that random variables are associated is by representing each of them as an increasing function of a specified set of independent random variables

Associated Random Variables

Definition: The stochastic process $\{X(t), t \geq 0\}$ is said to be associated if for all n and $t_1, \ldots, t_n$ and the random variables $X(t_1), \ldots, X(t_n)$ are associated.

Example 9.7(B): Any stochastic process $\{X(t), t \geq 0\}$ having independent increments and $X(0) = 0$, such as a Poisson process or a Brownian motion process, is associated.

Let $t_1 < t_2 < \ldots < t_n$. Then as $X(t_1), \ldots, X(t_n)$ are all increasing functions of the independent random variables $X(t_i) - X(t_{i-1})$, $i = 1, \ldots, n$ (where $t_0 = 0$), it follows that they are associated.

Associated Random Variables

Definition: Markov process $\{X_n, n \geq 0\}$ is said to be stochastically monotone if

$$P\left\{X_n \leq a \middle| X_{n-1} = x\right\}$$

is a decreasing function of x for all n and a.

Proposition 9.7.2: A stochastically monotone Markov process is associated.

Proof: Let $F_{n,x}$ denote the conditional distribution function of X_n given that $X_{n-1} = x$.

$U_1, \ldots, U_n$: independent uniform (0, 1) random variables

X_0: specified by the process

For $i = 1, \ldots, n$, $\quad X_i = F_{i,X_{i-1}}^{-1}(U_i) = \inf\left\{x : U_i \leq F_{i,X_{i-1}}(x)\right\}$

Associated Random Variables

$F_{i,X_{i-1}}(x)$ is decreasing in X_{i-1}

$\Longrightarrow$ X_i is increasing in X_{i-1}

$F_{i,X_{i-1}}(x)$ is increasing in x

$\Longrightarrow$ X_i is increasing in U_i

Hence, $X_1 = F_{1,X_0}^{-1}(U_1)$ is an increasing function of X_0 and U_1

$X_2 = F_{2,X_1}^{-1}(U_2)$ is an increasing function of X_1 and U_2

thus is an increasing function of X_0, U_1 and U_2

As each of the X_i, $i = 1, \ldots, n$ is an increasing function of the independent random variables X_0, $U_1, \ldots, U_n$, it follows that $X_1, \ldots, X_n$ are associated.

Associated Random Variables

Example 9.7(C): Bayesian theory

Λ: a random variable

$X_1, \ldots, X_n$: independent, $\sim F_\lambda$

If $F_\lambda(x)$ is decreasing in λ for all x, then $X_1, \ldots, X_n$ are associated

$\Downarrow$

$U_1, \ldots, U_n$: independent uniform (0, 1) random variables

Define $X_1, \ldots, X_n$ recursively by $X_i = F_\Lambda^{-1}(U_i)$

As the X_i are increasing functions of the independent random variables $\Lambda, U_1, \ldots, U_n$, they are associated.