Part II: Stochastic Programming (Spring 2021) **Homework: 08**

Applications of Network Models

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P1.

- Euler's Theorem: A connected graph G possesses an Euler tour (Euler path) if and only if G contains exactly zero (exactly two) nodes of odd degree.

Solution:

- If G contains zero node of odd degree, then it can be easily proved by mathematical induction.
- Under the former condition, when it refers to two node of odd degree, then we can add a edge between these two points, and transform G to a graph that consists of even degree. Thus if choosing these two points as the origin node and end node, there must exist a path that each edge in G be traveled once.

P2.

- Notation: Property 1: The optimum traveling salesman tour does not intersect itself.

Solution:

- Suppose that the optimum traveling salesman tour contains two edges intersect, and the nodes of the two edge are (a, b) and (c, d).
  In that case, the optimum tour can be improved by changing the path to (a, d) and (b, c) since the the sum of the diagonal lengths in the quadrilateral is greater than the sum of the opposite sides.

Problem 6.1

- (a) If $l(e, f) = -2$, the shortest path from a to j is $a \to b \to f \to e \to h \to j$, and its value equals to $3+7-2+1+2 = 11$.
- (b) The wrong point of this approach is that this action would make different effects on different number of links of paths. For example, there are two paths A and B, with actual length 5 and 6, and the number of links of them is 2 and 1. Then, we have the processed length of them by this action: A=5+2×2=9 and B=6+2=8. Noted that the action changes the relative length of different paths, hence, the optimal solution would not be true.

- (c) If $l(e, f) = -8$, the shortest path from a to j is $-\infty$, because there exists a negative cycle $f \rightarrow e \rightarrow h \rightarrow f$ with length -1, then we could always pass this cycle to reduce the sum of length of the path. Therefore, the minimum distance between a and j is $-\infty$.
- (d) If there is a path consists of more than $n-1$ links in the graph with n nodes, then we have that there exists one node which appears at least twice. However, the goal is minimizing the length of the path. Then we have that the length of this cycle is negative, otherwise, it would not appear in the minimum path.
- (e) When there are some negative cycles in a graph, we could find at least one negative number in the diagonal elements of the final matrix $d^(n)$.
- (f) Initialization:

$$D^0 = \begin{bmatrix} 0 & \infty & 3 & \infty & \infty \\ 4 & 0 & \infty & 1 & 2 \\ 3 & 8 & 0 & 2 & 6 \\ \infty & 1 & \infty & 0 & 4 \\ \infty & -3 & 6 & 4 & 0 \end{bmatrix}, \quad P^0 = \begin{bmatrix} - & 1 & 1 & 1 & 1 \\ 2 & - & 2 & 2 & 2 \\ 3 & 3 & - & 3 & 3 \\ 4 & 4 & 4 & - & 4 \\ 5 & 5 & 5 & 5 & - \end{bmatrix} \tag{1}$$

Node 1:

$$D^1 = \begin{bmatrix} 0 & \infty & 3 & \infty & \infty \\ 4 & 0 & 7^+ & 1 & 2 \\ 3 & 8 & 0 & 2 & 6 \\ \infty & 1 & \infty & 0 & 4 \\ \infty & -3 & 6 & 4 & 0 \end{bmatrix}, \quad P^1 = \begin{bmatrix} - & 1 & 1 & 1 & 1 \\ 2 & - & 1^+ & 2 & 2 \\ 3 & 3 & - & 3 & 3 \\ 4 & 4 & 4 & - & 4 \\ 5 & 5 & 5 & 5 & - \end{bmatrix} \tag{2}$$

Node 2:

$$D^2 = \begin{bmatrix} 0 & \infty & 3 & \infty & \infty \\ 4 & 0 & 7 & 1 & 2 \\ 3 & 8 & 0 & 2 & 6 \\ 5^+ & 1 & 8^+ & 0 & 3^+ \\ 1^+ & -3 & 4^+ & -2^+ & -1^+ \end{bmatrix}, \quad P^2 = \begin{bmatrix} - & 1 & 1 & 1 & 1 \\ 2 & - & 1 & 2 & 2 \\ 3 & 3 & - & 3 & 3 \\ 2^+ & 4 & 1^+ & - & 2^+ \\ 2^+ & 5 & 1^+ & 2^+ & 2^+ \end{bmatrix} \tag{3}$$

Node 3:

$$D^3 = \begin{bmatrix} 0 & 11^+ & 3 & 5^+ & 9^+ \\ 4 & 0 & 7 & 1 & 2 \\ 3 & 8 & 0 & 2 & 6 \\ 5 & 1 & 8 & 0 & 3 \\ 1 & -3 & 4 & -2 & -1 \end{bmatrix}, \quad P^3 = \begin{bmatrix} - & 3^+ & 1 & 3^+ & 3^+ \\ 2 & - & 1 & 2 & 2 \\ 3 & 3 & - & 3 & 3 \\ 2 & 4 & 1 & - & 2 \\ 2 & 5 & 1 & 2 & 2 \end{bmatrix} \tag{4}$$

Node 4:

$$D^4 = \begin{bmatrix} 0 & 6^+ & 3 & 5 & 8^+ \\ 4 & 0 & 7 & 1 & 2 \\ 3 & 3^+ & 0 & 2 & 5^+ \\ 5 & 1 & 8 & 0 & 3 \\ 1 & -3 & 4 & -2 & -1 \end{bmatrix}, \quad P^4 = \begin{bmatrix} - & 4^+ & 1 & 3 & 2^+ \\ 2 & - & 1 & 2 & 2 \\ 3 & 4^+ & - & 3 & 2^+ \\ 2 & 4 & 1 & - & 2 \\ 2 & 5 & 1 & 2 & 2 \end{bmatrix} \tag{5}$$

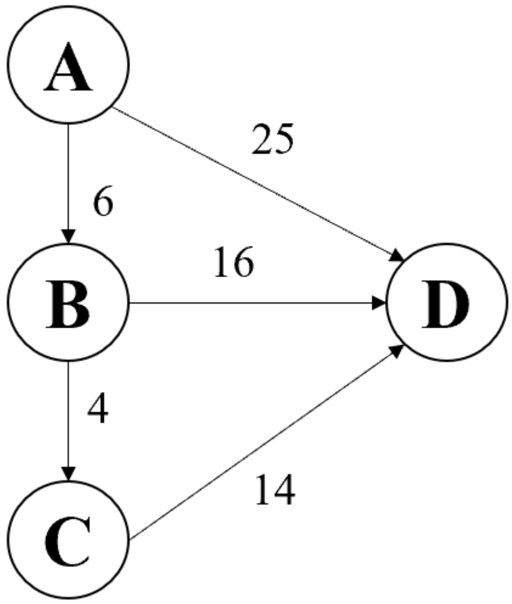
Node 5:

$$D^5 = \begin{bmatrix} 0 & 5^+ & 3 & 5 & 7^+ \\ 3^+ & -1^+ & 6^+ & 0^+ & 1^+ \\ 3 & 2^+ & 0 & 2 & 4^+ \\ 4^+ & 0^+ & 7^+ & 0 & 2^+ \\ 0^+ & -4^+ & 3^+ & -3^+ & -2^+ \end{bmatrix}, \quad P^5 = \begin{bmatrix} - & 5^+ & 1 & 3 & 2^+ \\ 2^+ & 5^+ & 1^+ & 2^+ & 2^+ \\ 3 & 5^+ & - & 3 & 2^+ \\ 2^+ & 5^+ & 1^+ & - & 2^+ \\ 2^+ & 5^+ & 1^+ & 2^+ & 2^+ \end{bmatrix} \tag{6}$$

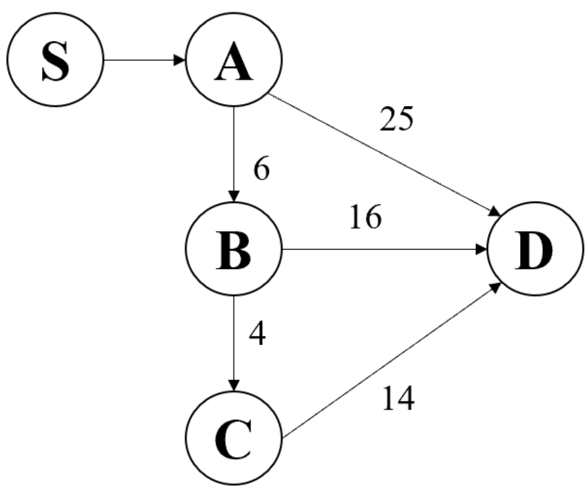
- (g). If there exists some in the graph, we could change the step of making node closed, ensure that nodes could not become closed until all labels are permanent.

Problem 6.3

- (a)



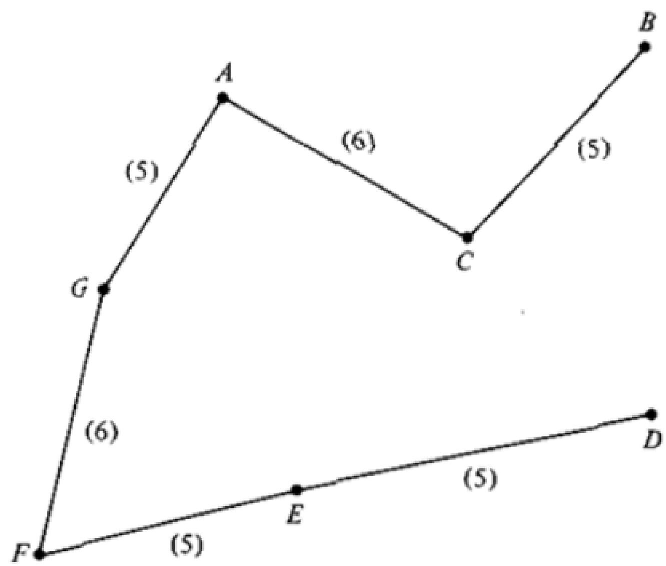
- (b)



- (c) The preferred route in part (a) is choice 2, and the shortest expected travel time is 6+16=22 minutes. The preferred route in part (b) is choice 2, and the shortest expected travel time is 22+5=27 minutes.

Problem 6.4

- Noted that the optimal road network should contain a minimum spanning tree to minimize the quantity Z:



- Then the sum of length of this minimize spanning tree is 32 miles, and the upper limit is 34 miles, which means that the minimize spanning tree meets this constrain. Hence, it is the optimal road network.

Problem 6.6

- (a)

$$\begin{aligned}\sum_{i\in N} P_i &= \sum_{i\in N}\{(indegreeofi) - (outdegreeofi)\}\\ &= \sum_{i\in N}(indegreeofi) - \sum_{i\in N}(outdegreeofi)\\ &= \sum_{i\in N}\sum_{(k,i)\in A} 1 - \sum_{i\in N}\sum_{(i,k)\in A} 1 = 0\end{aligned}$$

- (b) In order to have a directed Euler tour, we must have $P_i^{'} = 0$ for all nodes. Parallel to the i undirected version, we add artificial arcs (i,j) between supply nodes $i \in S$ and demand nodes $j \in D$. Unlike the undirected version, where one additional arc was sufficient to make any odd node even, here it may be necessary to add many arcs to a node whose $|P_i|$ is large. In order to minimize the total length of arcs added, we construct $\sum_{i\in S} P_i$ minimum distance paths between the supply nodes and demand nodes. In order to ensure $P_i^{'} = 0$ for all nodes, we require $\sum_{j\in D} x_{ij} = P_i$, $\forall i \in S$, which implies that

$$P_i^{'} = P_i - \text{ outdegree of new artificial arcs} = P_i - x_{ij} = 0\ .$$
$$P_j^{'} = P_j + \text{ indegree of new artificial arcs} = P_j + x_{ij} = 0\ .$$

- (c)
  - Step 1:
    $S = \{b, d, g\}$ with $P_b = P_d = P_g = 1$, and $D = \{a, e\}$ with $P_a = -2, P_e = -1$. By inspection,

    $$d(b,a) = 5, d(b,e) = 17d(d,a) = 14, d(d,e) = 3d(g,a) = 20, d(g,e) = 9$$

  - Step 2:

    $$\begin{aligned} minimize \quad & z = 5x_{ba} + 17x_{be} + 14x_{da} + 3x_{de} + 20x_{ga} + 9x_{ge} \\ subjectto \quad & x_{ba} + x_{be} = 1 \\ & x_{da} + x_{de} = 1 \\ & x_{ga} + x_{ge} = 1 \\ & x_{ba} + x_{da} + x_{ga} = 2 \\ & x_{be} + x_{de} + x_{ge} = 1 \\ & x_{ij} \in \{0, 1, 2, ...\} \end{aligned}$$

  - Step3:
    $b \rightarrow a \rightarrow c \rightarrow d \rightarrow c \rightarrow f \rightarrow g \rightarrow d \rightarrow e \rightarrow g \rightarrow d \rightarrow e \rightarrow b \rightarrow a \rightarrow d \rightarrow e \rightarrow b \rightarrow a \rightarrow b$ is one possible tour.

- (d) The suggested method forces us to traverse every undirected arc twice (once in each direction), which may not be optimal.