Rozwiązania i wskazówki do wybranych przykładów z Z01

$\lim\limits_{n\to\infty}\frac{1}{n^p}=\begin{cases}0 & p>0\\ 1 & p=0\\ \infty & p<0\end{cases}$	$\lim\limits_{n\to\infty} q^n=\begin{cases}\infty & q>1\\ 1 & q=1\\ 0 & \lvert q\rvert<1\end{cases}$
$a>0,\ \lim\limits_{n\to\infty}\sqrt[n]{a}=1$	$\lim\limits_{n\to\infty}\sqrt[n]{n}=1$
$\lim\limits_{n\to\infty}\left(1+\frac{A}{a_n}\right)^{a_n}=e^A$	

$\frac{0}{0},\frac{\infty}{\infty},\infty-\infty,0\cdot\infty,\ \ 1^{\infty},0^0,\infty^0$ nieoznaczone

$\frac{1}{\pm\infty}=0,\ \ \frac{1}{0^+}=\infty,\ \ \frac{1}{0^-}=-\infty,\ (0^+)^{\infty}=0,\ \ \frac{\infty}{0^+}=\infty,\ \frac{0}{\infty}=0,\ \infty^{-\infty}=0$

0. Dla podanych ciągów napisać wzory określające wskazane wyrazy

$$a)\ a_2=\frac{2}{2\cdot 2+3}=\frac{2}{7};\ \ a_{2n+1}=\frac{2n+1}{2(2n+1)+3}=\frac{2n+1}{4n+2+3}=\frac{2n+1}{4n+5}$$

$$b)\ a_2=\frac{\sqrt{2^2+2}+1}{2\cdot 2}=\frac{\sqrt{6}+1}{2\cdot 2};\ \ a_{2n}=\frac{\sqrt{(2n)^2+2n}+1}{2(2n)}=\frac{\sqrt{4n^2+2n}+1}{4n}$$

$$c)\ a_3=(3+10)!=13!\ \ a_{m+3}=(m+3+10)!=(m+13)!$$

1. Oblicz granice

$$b)\ \lim_{n\to\infty}\frac{(\sqrt{n^2+n}+n)}{(\sqrt{n+1}+1)}\,|'n'\,do\ najwyższej\ potęgi\ to\ n^{\frac{1}{2}}\,|=\lim_{n\to\infty}\frac{\sqrt{n}\ \ (\sqrt{n+1}+\sqrt{n})}{\sqrt{n}\ \ \left(\sqrt{1+\frac{1}{n}}+\frac{1}{\sqrt{n}}\right)}=\infty$$

$$c)\ \lim_{n\to\infty}\arcsin\left(\frac{n-1}{n+1}\right)=\arcsin 1=\frac{\pi}{2}\quad \text{ponieważ}\quad \lim_{n\to\infty}\frac{n-1}{1+n}=1$$

$$d)\ \lim_{n\to\infty}\frac{\arctan n!}{n+1}\left[\frac{\operatorname{arctg}\infty}{\infty}=\frac{\frac{\pi}{2}}{\infty}\right]=0$$

$$e)\ \lim_{k\to\infty}\left(\sqrt{k+1}-\sqrt{k-1}\right)=\lim_{k\to\infty}\frac{\left(\sqrt{k+1}-\sqrt{k-1}\right)\left(\sqrt{k+1}+\sqrt{k-1}\right)}{\left(\sqrt{k+1}+\sqrt{k-1}\right)}=\lim_{k\to\infty}\frac{k+1-(k-1)}{\left(\sqrt{k+1}+\sqrt{k-1}\right)}=$$

$$=\lim_{k\to\infty}\frac{k+1-k+1}{\left(\sqrt{k+1}+\sqrt{k-1}\right)}=\lim_{k\to\infty}\frac{2}{\left(\sqrt{k+1}+\sqrt{k-1}\right)}=0,$$ mianownik zbiega do nieskończoności.

$$f)\ \lim_{n\to\infty}\frac{3n}{\sqrt{n^2-n}-\sqrt{n^2+n}}=\lim_{n\to\infty}\frac{3n\left(\sqrt{n^2-n}+\sqrt{n^2+n}\right)}{\left(\sqrt{n^2-n}-\sqrt{n^2+n}\right)\left(\sqrt{n^2-n}+\sqrt{n^2+n}\right)}=$$

$$\lim_{n\to\infty}\frac{3n\left(\sqrt{n^2-n}+\sqrt{n^2+n}\right)}{n^2-n-n^2-n}=\lim_{n\to\infty}\frac{3n\left(\sqrt{n^2-n}+\sqrt{n^2+n}\right)}{-2n}=-\infty$$

$$i)\ \lim_{k\to\infty}7^k-6^k-5^k=\lim_{k\to\infty}7^k\left(1-\frac{6^k}{7^k}-\frac{5^k}{7^k}\right)=\lim_{k\to\infty}7^k\left(1-\left(\frac{6}{7}\right)^k-\left(\frac{5}{7}\right)^k\right)=\infty$$

$$j)\ \lim_{n\to\infty} \ln(2n^4-2n+1)-\ln(n^5-2n+1) = \lim_{n\to\infty} \ln\frac{2n^4-2n+1}{n^5-2n+1}$$
$$= \ln\left(\lim_{n\to\infty}\frac{2n^4-2n+1}{n^5-2n+1}\right)\ [\ln 0] = -\infty$$

2. Znajdź granice następujących ciągów $\lim_{n\to\infty} a_n$

$$a)\ \lim_{n\to\infty}\frac{3\cdot 2^{3n-2}-8}{8^{n+1}+16} = \lim_{n\to\infty}\frac{3\cdot 8^n\cdot\frac{1}{4}-8}{8^n\cdot 8+16} = \lim_{n\to\infty}\frac{8^n\left(3\cdot\frac{1}{4}-8\frac{1}{8^n}\right)}{8^n\left(8+16\frac{1}{8^n}\right)} = \frac{3}{32}$$

$$c)\ \lim_{n\to\infty}\frac{(n^4+4)n!+(n-1)!}{n\cdot(n+1)!} = \lim_{n\to\infty}\frac{n!\left[(n^4+4)+\frac{1}{n}\right]}{n!\cdot n\cdot(n+1)} = \infty$$

3. Korzystając z twierdzenia o trzech ciągach oblicz granice $\lim_{n\to\infty} a_n$:

Tw. ***o trzech ciągach***
Niech dla $n > n_0$ będzie: $a_n \le b_n \le c_n$ i $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = q$
To: $\lim_{n\to\infty} b_n = q$

$a)\ \lim_{n\to\infty}\frac{\sqrt[n]{4^n+5^n}}{\sqrt[n]{6^n+7^n}}$ licznik i mianownik oddzielnie

1. $\sqrt[n]{5^n} \le \sqrt[n]{4^n+5^n} \le \sqrt[n]{5^n+5^n} = \sqrt[n]{2\cdot 5^n} \to 5$

2. $\sqrt[n]{7^n} \le \sqrt[n]{6^n+7^n} \le \sqrt[n]{7^n+7^n} = \sqrt[n]{2\cdot 7^n} \to 7$

$$\lim_{n\to\infty}\frac{\sqrt[n]{4^n+5^n}}{\sqrt[n]{6^n+7^n}} = \frac{5}{7}$$

$b)\ \lim_{n\to\infty}\sqrt[n]{5^{n+2}+2^{3n-1}+3^{n+2}} = \sqrt[n]{5^2 5^n+2^{-1}8^n+3^2 3^n}$:

prawa nierówność:

$$\sqrt[n]{5^2 5^n+2^{-1}8^n+3^2 3^n} \le \sqrt[n]{5^2 8^n+2^{-1}8^n+3^2 8^n} = \sqrt[n]{8^n(5^2+1+3^2)} = 8\sqrt[n]{35} \xrightarrow[n\to\infty]{} 8$$

lewa nierówność:

$$8 \xleftarrow[n\to\infty]{} 8\sqrt[n]{2^{-1}} = \sqrt[n]{2^{-1}8^n} \le \sqrt[n]{5^2 5^n+2^{-1}8^n+3^2 3^n}$$

$c)\ \lim_{k\to\infty}\sqrt[k]{k^2+k}$: $\quad 1 \xleftarrow[k\to\infty]{} \sqrt[k]{k^2} \le \sqrt[k]{k^2+k} \le \sqrt[k]{k^2+k^2} = \sqrt[k]{2}\sqrt[k]{k} \xrightarrow[k\to\infty]{} 1$

$d)\ \lim_{n\to\infty}\sqrt[n]{4^n+5\cdot 3^{2n}+2^{n+1}} = \lim_{n\to\infty}\sqrt[n]{4^n+5\cdot 9^n+2\cdot 2^n}$

$$9 = \sqrt[n]{9^n} \le \sqrt[n]{4^n+5\cdot 9^n+2\cdot 2^n} \le \sqrt[n]{9^n+5\cdot 9^n+2\cdot 9^n} = \sqrt[n]{8\cdot 9^n} \to 9$$

$e)\ \lim_{k\to\infty}\frac{k\sin(k!)}{k^2+1}$: $\quad 0 \xleftarrow[k\to\infty]{} \frac{k\cdot(-1)}{k^2+1} \le \frac{k\sin(k!)}{k^2+1} \le \frac{k\cdot 1}{k^2+1} \xrightarrow[k\to\infty]{} 0$

$f)\ \lim\limits_{n\to\infty} \dfrac{4^n + (-1)^n\, 2^n}{3\cdot 4^n + 1}$ $\qquad \dfrac{1}{3} \xleftarrow[n\to\infty]{} \dfrac{4^n - 2^n}{3\cdot 4^n + 1} \le \dfrac{4^n + (-1)^n\, 2^n}{3\cdot 4^n + 1} \le \dfrac{4^n + 2^n}{3\cdot 4^n + 1} \xrightarrow[n\to\infty]{} \dfrac{1}{3}$

$g)\ \lim\limits_{n\to\infty} \dfrac{2n^2 + \sin(n!)}{3n^2 - 4\cos(n!)}$ $\qquad \dfrac{2}{3} \xleftarrow[n\to\infty]{} \dfrac{2n^2 - 1}{3n^2 + 4} \le \dfrac{2n^2 + \sin(n!)}{3n^2 - 4\cos(n!)} \le \dfrac{2n^2 + 1}{3n^2 - 4} \xrightarrow[n\to\infty]{} \dfrac{2}{3}$

$h)\ \lim\limits_{n\to\infty} \dfrac{5^n + (-2)^n}{3^n + 4};$ $\qquad \infty \xleftarrow[n\to\infty]{} \dfrac{5^n - 2^n}{3^n + 4} \le \dfrac{5^n + (-2)^n}{3^n + 4} \le \dfrac{5^n + 2^n}{3^n + 4} \xrightarrow[n\to\infty]{} \infty$

$i)\ \lim\limits_{n\to\infty} \dfrac{n\sin(n^2) - n^3}{n + 1};$ $\qquad -\infty \xleftarrow[n\to\infty]{} \dfrac{-n - n^3}{n + 1} \le \dfrac{n\sin(n^2) - n^3}{n + 1} \le \dfrac{n - n^3}{n + 1} \xrightarrow[n\to\infty]{} -\infty$

$j)\ \lim\limits_{n\to\infty} (n^2 + 2n\cos n);$ $\qquad \infty \xleftarrow[n\to\infty]{} n^2 - 2n \le n^2\cos n + 2n \le n^2 + 2n \xrightarrow[n\to\infty]{} \infty$

$k^*)\ \lim\limits_{n\to\infty} \sqrt[n+1]{3^n + 4^{n+1}}\ ;$

$$4 = \sqrt[n+1]{4^{n+1}} \le \sqrt[n+1]{\frac{1}{3}\cdot 3^{n+1} + 4^{n+1}} \le \sqrt[n+1]{4^{n+1} + 4^{n+1}} \xrightarrow[n\to\infty]{} 4$$

$l^*)\ \lim\limits_{n\to\infty} \sqrt[n+1]{2n + 3};$

$$1 \xleftarrow[n\to\infty]{} \sqrt[n+1]{n + 1} \le \sqrt[n+1]{2n + 3} = \sqrt[n+1]{2n + 2 + 1} = \sqrt[n+1]{2(n + 1) + 1} \le \sqrt[n+1]{3(n + 1)} \xrightarrow[n\to\infty]{} 1$$

4. Oblicz granice :

$a)\ metoda\ I$: $\lim\limits_{n\to\infty} \left(\dfrac{2n - 5}{2n + 7}\right)^{5n} = \lim\limits_{n\to\infty} \left[\dfrac{2n\left(1 - \frac{5}{2n}\right)}{2n\left(1 + \frac{7}{2n}\right)}\right]^{5n} = \lim\limits_{n\to\infty} \left[\dfrac{\left(1 - \frac{5}{2n}\right)}{\left(1 + \frac{7}{2n}\right)}\right]^{5n} =$

$$= \frac{\lim\limits_{n\to\infty} \left(1 - \frac{5}{2n}\right)^{5n}}{\lim\limits_{n\to\infty} \left(1 + \frac{7}{2n}\right)^{5n}} = \frac{e^{\frac{-25}{2}}}{e^{\frac{35}{2}}} = e^{\frac{-25}{2} - \frac{35}{2}} = e^{-30}.$$

$b)\ metoda\ II$: $\lim\limits_{n\to\infty} \left(\dfrac{3n + 2}{3n - 3}\right)^{3n+1} = \lim\limits_{n\to\infty} \left(\dfrac{3n - 3 + 5}{3n - 3}\right)^{3n+1} = \lim\limits_{n\to\infty} \left(1 + \dfrac{5}{3n - 3}\right)^{3n+1} =$

$$= \lim_{n\to\infty} \left(1 + \frac{5}{3n - 3}\right)^{3n-3\cdot\frac{3n+1}{3n-3}} = \lim_{n\to\infty} \left(\left(1 + \frac{5}{3n - 3}\right)^{3n-3}\right)^{\frac{3n+1}{3n-3}} = (e^5)^1 = e^5.$$

ponieważ : $\lim\limits_{n\to\infty} \dfrac{3n + 1}{3n - 3} = 1$

$c)\ \lim_{n\to\infty}\left(\frac{3n^2+2}{3n^2-3}\right)^{3n^2+2n-3} = \lim_{n\to\infty}\left(\frac{3n^2-3+5}{3n^2-3}\right)^{3n^2+2n-3} = \lim_{n\to\infty}\left(1+\frac{5}{3n^2-3}\right)^{3n^2+2n-3} =$

$\lim_{n\to\infty}\left[\left(1+\frac{5}{3n^2-3}\right)^{3n^2-3}\right]^{\frac{3n^2+2n-3}{3n^2-3}} = e^5 \qquad \text{bo} \quad \lim_{n\to\infty}\frac{3n^2+2n-3}{3n^2-3} = 1$

$e)^{*}\ \lim_{n\to\infty} n(\ln(n+1)-\ln n) = \lim_{n\to\infty} n\ln\frac{n+1}{n} = \lim_{n\to\infty} n\ln(1+\frac{1}{n}) = \lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n = \ln e = 1$

$f)^{*}\lim_{n\to\infty}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^{3n} = \lim_{n\to\infty}\left[\left(1+\frac{n^2+n}{n^3}\right)^{\frac{n^3}{n^2+n}}\right]^{3n\cdot\frac{n^2+n}{n^3}} = e^3$

$g)^{*}\lim_{n\to\infty}\left(0{,}99+\frac{1}{n}\right)^n = 0$ ponieważ: $\lim_{n\to\infty}\left(0{,}99+\frac{1}{n}\right) = 0{,}99$, przy $q<1$.