第四节
多元复合函数的求导法则

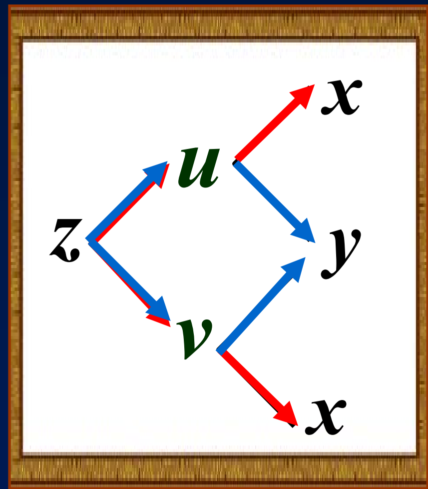
- 一、多元复合函数求导的链式法则
- 二、一阶全微分的形式不变性

一、多元复合函数求导的链式法则

定理8.5 设函数 $u=\varphi(x,y)$和$v=\psi(x,y)$在点(x,y)具有对x及y的偏导数，$z=f(u,v)$在对应点(u,v)处偏导数连续，则复合函 数

$$z=f[\varphi(x,y),\psi(x,y)]$$

在点(x,y)处可导，且



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

若引入记号：

$$f_1' = \frac{\partial f(u,v)}{\partial u}, \quad f_2' = \frac{\partial f(u,v)}{\partial v},$$

$$\varphi_1' = \frac{\partial \varphi(x,y)}{\partial x}, \quad \psi_1' = \frac{\partial \psi(x,y)}{\partial x},$$

$$\varphi_2' = \frac{\partial \varphi(x,y)}{\partial y}, \quad \psi_2' = \frac{\partial \psi(x,y)}{\partial y}$$

则

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = f_1'\varphi_1' + f_2'\psi_1'$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = f_1'\varphi_2' + f_2'\psi_2'$$

证 固定 y, 给 x 以增量 Δx.

相应地, $u=\varphi(x,y)$和 $v=\psi(x,y)$有偏增量:

$$(\Delta u)_x=\varphi(x+\Delta x,y)-\varphi(x,y)$$
$$=\varphi(x+\Delta x,y)-u$$

$$(\Delta v)_x=\psi(x+\Delta x,y)-\psi(x,y)$$
$$=\psi(x+\Delta x,y)-v$$

从而 $z=f(u,v)$获得偏增量:

$$(\Delta z)_x=f[\varphi(x+\Delta x,y),\psi(x+\Delta x,y)]-f[\varphi(x,y),\psi(x,y)]$$

$$=f[u+(\Delta u)_x,v+(\Delta v)_x]-f(u,v)$$

又因 $z = f(u,v)$ 在点 (u,v) 处有连续偏导数，故可 微．

且由定理8.3(可微与偏导数连续的关系)的证明，知

$$\Delta z = \frac{\partial z}{\partial u}\cdot\Delta u + \frac{\partial z}{\partial v}\cdot\Delta v + \varepsilon_1\Delta u + \varepsilon_2\Delta v$$

其中 $\lim\limits_{\substack{\Delta u\to 0\\ \Delta v\to 0}} \varepsilon_i = 0 \quad (i = 1,2)$

特别地，固定 y，即当 $\Delta y = 0,\ \Delta x \neq 0$ 时，有

$$(\Delta z)_x = \frac{\partial z}{\partial u}\cdot(\Delta u)_x + \frac{\partial z}{\partial v}\cdot(\Delta v)_x + \varepsilon_1(\Delta u)_x + \varepsilon_2(\Delta v)_x$$

$\because$ 在 (x,y) 处，$\dfrac{\partial u}{\partial x}$ 和 $\dfrac{\partial v}{\partial x}$ 均存在

$\therefore$ $u = \ (x,y)$ 和 $v = \psi(x,y)$ 在 (x,y) 处均关于 x 连续

$$\therefore \quad \lim_{\Delta x\to 0}(\Delta u)_x = \lim_{\Delta x\to 0}(\Delta v)_x = 0$$

从而 $\lim\limits_{\Delta x\to 0}\varepsilon_1 = \lim\limits_{\Delta x\to 0}\varepsilon_2 = 0$

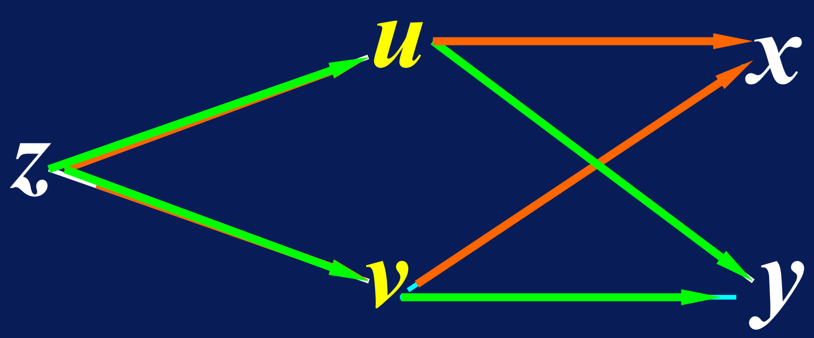
在 $\dfrac{(\Delta z)_x}{\Delta x} = \dfrac{\partial z}{\partial u}\cdot\dfrac{(\Delta u)_x}{\Delta x} + \dfrac{\partial z}{\partial v}\cdot\dfrac{(\Delta v)_x}{\Delta x} + \varepsilon_1\dfrac{(\Delta u)_x}{\Delta x} + \varepsilon_2\dfrac{(\Delta v)_x}{\Delta x}$

中，令 $\Delta x \to 0$, 得

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

类似地，可以证明：$\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y}$

注 1° 复合关系图(结构图)



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

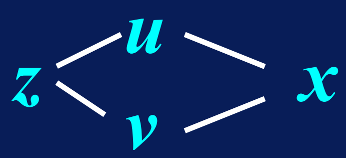
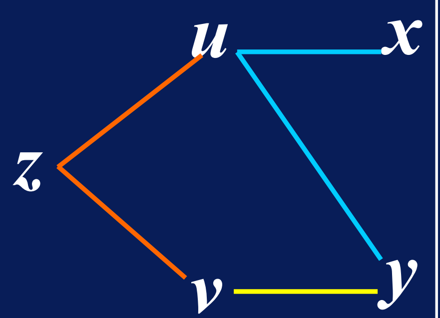
口诀：“项数 = 通向该自变量的路径数”.

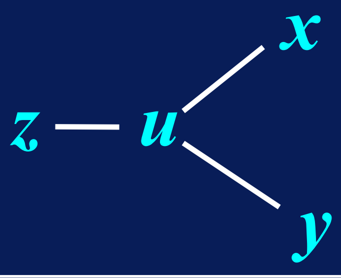
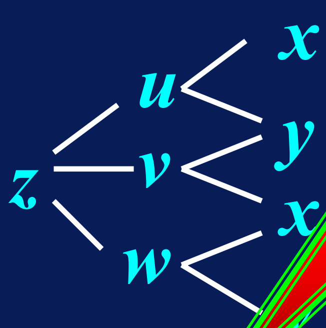
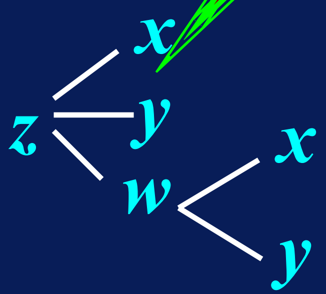
“连线相乘，分线相加”;

“单路全导, 叉路偏导”

2° 其他情形

全导数

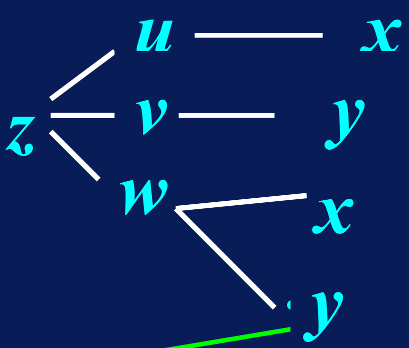
函数关系	结构图	求导公式
$z=f(u,v)$ $u=\varphi(x)$ $v=\psi(x)$		$\dfrac{\mathrm{d}z}{\mathrm{d}x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\mathrm{d}u}{\mathrm{d}x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\mathrm{d}v}{\mathrm{d}x}$
$z=f(u,v)$ $u=\varphi(x,y)$ $v=\psi(y)$		$\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}$ $\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\mathrm{d}v}{\mathrm{d}y}$

函数关系	关系图	求导公式
$z = f(u)$ $u = \varphi(x,y)$		$\dfrac{\partial z}{\partial x} = \dfrac{\mathrm{d}z}{\mathrm{d}u}\dfrac{\partial u}{\partial x}, \quad \dfrac{\partial z}{\partial y} = \dfrac{\mathrm{d}z}{\mathrm{d}u}\dfrac{\partial u}{\partial y}$
$z = f(u,v,w)$ $u = u(x,y)$ $v = v(x,y)$ $w = w(x,y)$		$\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} + \dfrac{\partial z}{\partial w}\dfrac{\partial w}{\partial x}$ $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} + \dfrac{\partial z}{\partial w}\dfrac{\partial w}{\partial y}$
★ $z = f(x,y,w)$ $w = \varphi(x,y)$		$\dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial x}$ $\dfrac{\partial z}{\partial y} = \dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial y}$

变量x一身兼两职

★ $z = f(x, y, w),\quad w = \varphi(x, y)$

即 $\begin{cases} z = f(u, v, w), \\ u = x, \\ v = y, \\ w = \varphi(x, y) \end{cases}$



把复合函数

$z = f[x, y, \varphi(x, y)]$

中的y看作不变，而对x的偏导数

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial x}$$

两者的区别

$$= \frac{\partial z}{\partial u}\cdot 1 + \frac{\partial z}{\partial v}\cdot 0 + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$= \boxed{\frac{\partial f}{\partial x}} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x},$$

把 $z = f(x, y, w)$中的y及w看作不变而对x的偏导数

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3° 若将定理条件: $f(u,v)$ 在点(u,v)偏导数连续减弱为偏导数存在, 则定理结论不一定成立.

如: $z = f(u,v) = \begin{cases} \dfrac{u^2 v}{u^2+v^2}, & u^2+v^2 \neq 0 \\ 0, & u^2+v^2 = 0 \end{cases}$ $\quad u = t, \quad v = t$

可复合为 $z = f(t,t) = \dfrac{t^2 t}{t^2+t^2} = \dfrac{t}{2}$

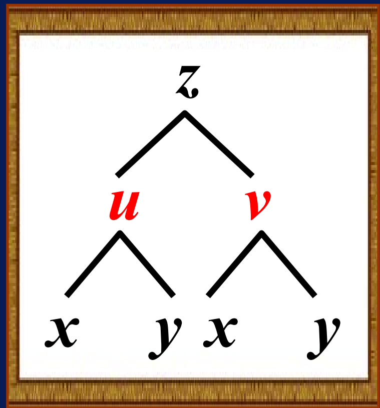
虽然 $\left.\dfrac{\partial z}{\partial u}\right|_{(0,0)} = 0$, $\left.\dfrac{\partial z}{\partial v}\right|_{(0,0)} = 0$ 但 $\dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}$在(0,0)不连续

$$\left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{t=0} = \frac{1}{2} \neq \left(\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}\right)\Big|_{(0,0)} = 0\cdot 1 + 0\cdot 1 = 0$$

1. 中间变量均为多元函数的复合函数

例1 设 $z = e^u \sin v$，$u = xy$，$v = x + y$，求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

解 (方法1) 画出关系



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$ 写出公式

$$= e^u \sin v \cdot y + e^u \cos v \cdot 1$$ 求出各偏导数

$$= e^{xy}[y\sin(x+y) + \cos(x+y)],$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$ 将 x, y 代入

$$= e^u \sin v \cdot x + e^u \cos v \cdot 1$$

$$= e^{xy}[x\sin(x+y) + \cos(x+y)].$$

(方法2) 把 u， v代入，得到复合函数

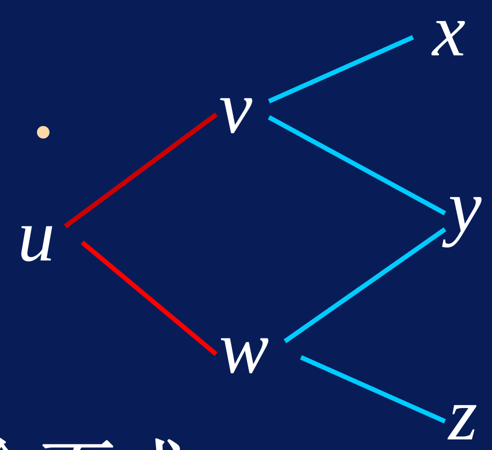
$$z = e^u \sin v \quad = e^{xy} \sin(x+y)$$

$$\frac{\partial z}{\partial x} = e^{xy} y \cdot \sin(x+y) + e^{xy} \cdot \cos(x+y) \cdot 1$$

$$= e^{xy}[y \sin(x+y) + \cos(x+y)]$$

(对x 求偏导数时，暂视 y 为常数)

例2 设 $u = f(\frac{x}{y}, \frac{y}{z})$，其中 f 有一阶连续偏导数，求函数 u 的一阶偏导数．



解 设 $v = \frac{x}{y}, w = \frac{y}{z}$，则函数

由 $u = f(v, w), v = \frac{x}{y}, w = \frac{y}{z}$ 复合而成．

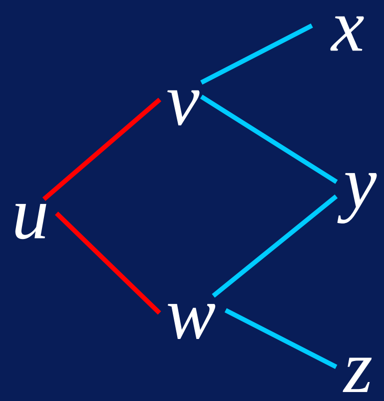
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial f}{\partial v} \cdot \frac{1}{y},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial y} = \frac{\partial f}{\partial v} \cdot \left(-\frac{x}{y^2}\right) + \frac{\partial f}{\partial w} \cdot \frac{1}{z}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial z}=\frac{\partial f}{\partial w}\cdot\left(-\frac{y}{z^2}\right)=-\frac{y}{z^2}\cdot\frac{\partial f}{\partial w}.$$

若使用记号: $\dfrac{\partial f(v,w)}{\partial v}=f_1'$, $\dfrac{\partial f(v,w)}{\partial w}=f_2'$

则上述结果可表示为:



$$\boxed{\begin{aligned}&u=f(v,w),\\ &v=\frac{x}{y},\ w=\frac{y}{z}\end{aligned}}$$

$$\frac{\partial u}{\partial x}=\frac{1}{y}f_1',$$

$$\frac{\partial u}{\partial y}=-\frac{x}{y^2}f_1'+\frac{1}{z}f_2',\quad \frac{\partial u}{\partial z}=-\frac{y}{z^2}f_2'.$$

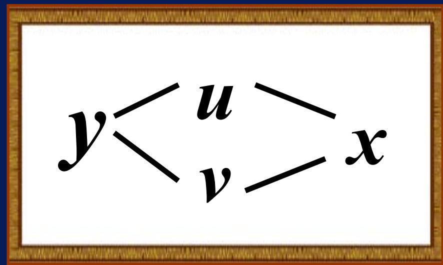
2. 中间变量均为一元函数的复合函数

例3 设 $y = [f(x)]^{\varphi(x)}$，其中 $f(x) > 0$，求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

解 令 $u = f(x), v = \varphi(x)$,

则 $y = [f(x)]^{\varphi(x)}$ 可看作由 $y = u^v$,

$u = f(x), v = \varphi(x)$ 复合而成.



所以 $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\partial y}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial y}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$= vu^{v-1} f'(x) + u^v (\ln u)\varphi'(x)$$

$$= [f(x)]^{\varphi(x)}\left[\frac{\varphi(x)}{f(x)} f'(x) + \varphi'(x)\ln f(x)\right].$$

推广：假设下面所涉及到的函数都可微．

当中间变量多于两个时，**例如:**

$$z = f(u, v, w),$$

$$u = \varphi(t),\ v = \psi(t),\ w = \omega(t)$$

$$\begin{tikzcd} & z \arrow[dl] \arrow[d] \arrow[dr] & \\ u \arrow[d] & v \arrow[d] & w \arrow[d] \\ t & t & t \end{tikzcd}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial w}\cdot\frac{\mathrm{d}w}{\mathrm{d}t}$$

$$= f_1'\varphi' + f_2'\psi' + f_3'\omega'$$

3.中间变量只有一个的复合函数

例4 设 $z = f[xy + \varphi(y)]$，其中f,φ可微，

求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

解(方法1) 令 $u = xy + \varphi(y)$

$$z \,—\, u \begin{array}{l} \nearrow x \\ \searrow y \end{array}$$

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d}\, z}{\mathrm{d}\, u} \cdot \frac{\partial u}{\partial x} = f'(u) \cdot y = y\, f'[xy + \varphi(y)]$$

$$\frac{\partial z}{\partial y} = \frac{\mathrm{d}\, z}{\mathrm{d}\, u} \cdot \frac{\partial u}{\partial y} = f'(u) \cdot [x + \varphi'(y)]$$

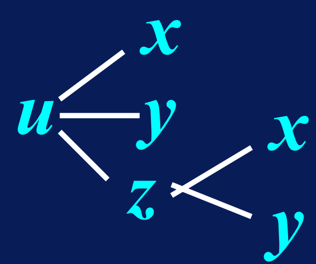
$$= [x + \varphi'(y)]\, f'[xy + \varphi(y)]$$

4.中间变量既有一元函数，又有多元函数的复合函数

例5 设 $u = e^{x^2+y^2+z^2}, z = x^2 \sin y$,求 $\frac{\partial u}{\partial x}$ 及 $\frac{\partial u}{\partial y}$.

解 (方法1) 令 $u = f(x, y, z) = e^{x^2+y^2+z^2}$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}$$



$$= e^{x^2+y^2+z^2} \cdot 2x + e^{x^2+y^2+z^2} \cdot 2z \cdot 2x \sin y$$

$$= 2x(1 + 2x^2 \sin^2 y) e^{x^2+y^2+x^4 \sin^2 y}.$$

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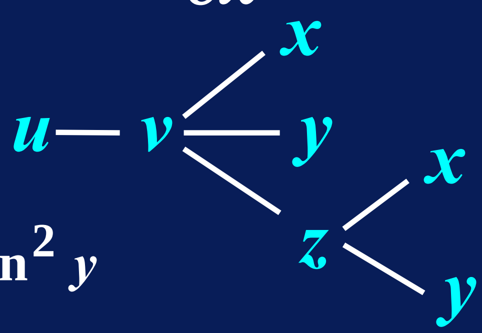
$$\frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}=e^{x^2+y^2+z^2}\cdot 2y+e^{x^2+y^2+z^2}\cdot 2z\cdot x^2\cos y$$

$$=(2y+x^4\sin 2y)e^{x^2+y^2+x^4\sin^2 y}.$$

(方法2)　令 $v=x^2+y^2+z^2$，则 $u=e^v$,, $z=x^2\sin y$.

$$\frac{\partial u}{\partial x}=\frac{\mathrm{d}u}{\mathrm{d}v}\cdot\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=e^v\cdot\left(2x+2z\cdot\frac{\partial z}{\partial x}\right)$$

$$=e^v\cdot(2x+2z\cdot 2x\sin y)$$

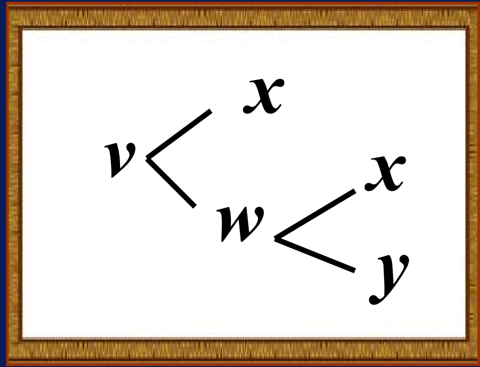


$$=2x(1+2x^2\sin^2 y)e^{x^2+y^2+x^4\sin^2 y}$$

注　对具体函数，用方法2 较简单.

例6 设 $u = xf(x, \frac{y}{x})$, f的二阶偏导数存在．

求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ 及 $\frac{\partial^2 u}{\partial x \partial y}$.



解 令 $w = \frac{y}{x}$, $v = f(x, w)$, $u = xv$.

$$\frac{\partial u}{\partial x} = v + x\frac{\partial v}{\partial x}$$ （乘积求导法则）

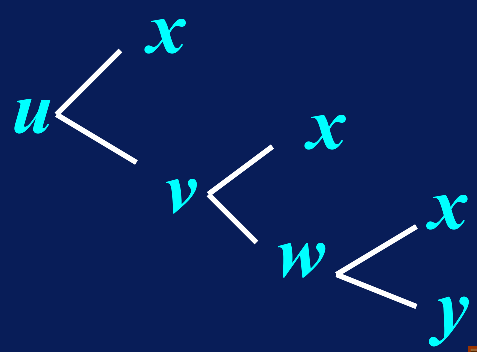
$$= v + x\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}\right)$$

$$= f + x[f_1' + f_2'\cdot(-\frac{y}{x^2})] = f + xf_1' - \frac{y}{x}f_2'.$$

可记 $f = f(x, \frac{y}{x})$, $f_1' = \frac{\partial f}{\partial x}$, $f_2' = \frac{\partial f}{\partial w}$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial w}\cdot\frac{\partial w}{\partial y}$$

$$=x\cdot f_2'\cdot\frac{1}{x}=f_2'$$

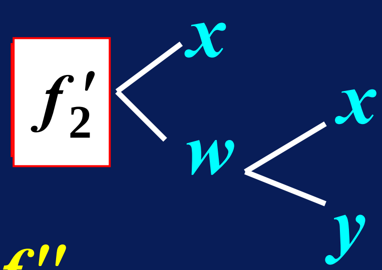


$$u=xv$$
$$v=f(x,w)$$
$$w=\frac{y}{x}$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial y}\left(f+x f_1'-\frac{y}{x}f_2'\right)$$

$$=\cancel{f_2'\cdot\frac{1}{x}}+\ x\cdot f_{12}''\cdot\frac{1}{x}$$

$$-\left(\cancel{\frac{1}{x}f_2'}+\frac{y}{x}f_{22}''\cdot\frac{1}{x}\right)=f_{12}''-\frac{y}{x^2}f_{22}''\,.$$



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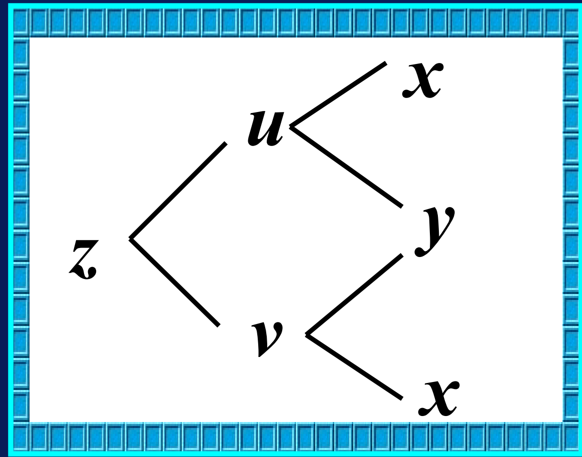
例7 设 $z=f(x,y)$ 有二阶连续偏导数，令

$u=xy,\ v=\dfrac{x}{y},$ 试将方程：



$$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=0$$

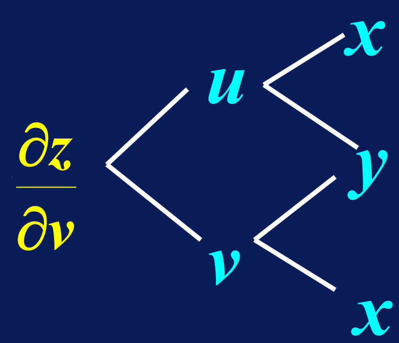
化为关于 u,v 的方程.

解 $z=f(x,y)=f(\sqrt{uv},\sqrt{\dfrac{u}{v}})$

$$=F(u,v)$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}\cdot y+\frac{\partial z}{\partial v}\cdot\frac{1}{y}$$

$$\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}$$



$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right)$$

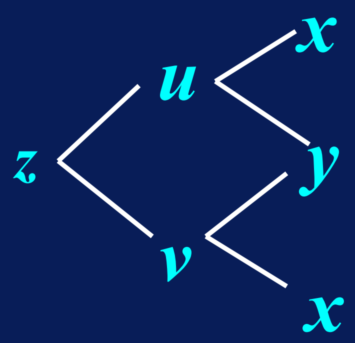
$$= y\cdot\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{y}\cdot\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) = y\cdot\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial v}{\partial x}\right]$$

$$+ \frac{1}{y}\cdot\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}\right]$$

$$\boxed{\frac{\partial u}{\partial x} = y, \quad \frac{\partial v}{\partial x} = \frac{1}{y}}$$

$$= y\cdot\left(\frac{\partial^2 z}{\partial u^2}\cdot y + \frac{\partial^2 z}{\partial u\partial v}\cdot\frac{1}{y}\right) + \frac{1}{y}\cdot\left(\frac{\partial^2 z}{\partial v\partial u}\cdot y + \frac{\partial^2 z}{\partial v^2}\cdot\frac{1}{y}\right)$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} = y^2 \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}$$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

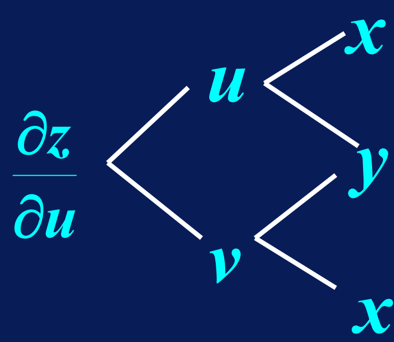
$$= x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}\right)$$

$$= x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) - x\frac{\partial}{\partial y}\left(\frac{1}{y^2}\frac{\partial z}{\partial v}\right)$$

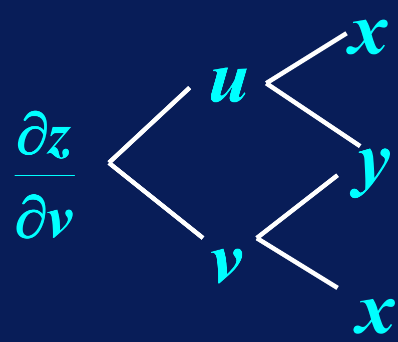
$$\boxed{\begin{gathered} u = xy, \quad v = \frac{x}{y} \\ \frac{\partial u}{\partial y} = x, \quad \frac{\partial v}{\partial y} = -\frac{x}{y^2} \end{gathered}}$$

$$= x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) - x\frac{\partial}{\partial y}\left(\frac{1}{y^2}\frac{\partial z}{\partial v}\right)$$



$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial v}{\partial y}\right]$$

$$- x\left[-\frac{2}{y^3}\frac{\partial z}{\partial v} + \frac{1}{y^2}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)\right]$$



$$= x\left[\frac{\partial^2 z}{\partial u^2}\cdot x + \frac{\partial^2 z}{\partial u\partial v}\cdot\left(-\frac{x}{y^2}\right)\right] + \frac{2x}{y^3}\frac{\partial z}{\partial v}$$

$$-\frac{x}{y^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial y}\right]$$

$$\frac{\partial^2 z}{\partial y^2} = x^2\frac{\partial^2 z}{\partial u^2} - \frac{x^2}{y^2}\frac{\partial^2 z}{\partial u\partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v}$$

$$-\frac{x}{y^2}\left[\frac{\partial^2 z}{\partial v\partial u}\cdot x + \frac{\partial^2 z}{\partial v^2}\cdot\left(-\frac{x}{y^2}\right)\right]$$

$\because$ z有二阶连续偏导数

$\therefore$ $z_{uv} = z_{vu}$

$$= x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2} + \frac{2x}{y^3}\frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2} = y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}$$

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = 4x^2\frac{\partial^2 z}{\partial u\partial v} - \frac{2x}{y}\frac{\partial z}{\partial v}$$

$$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}$$

$$=4x^2\frac{\partial^2 z}{\partial u\partial v}-\frac{2x}{y}\frac{\partial z}{\partial v}$$

$$=4uv\frac{\partial^2 z}{\partial u\partial v}-2v\frac{\partial z}{\partial v}$$

$$u=xy,\quad v=\frac{x}{y}$$

$$x^2=uv,$$

$\therefore$ 当$v\neq 0$时，原方程可化为：

$$2u\frac{\partial^2 z}{\partial u\partial v}-\frac{\partial z}{\partial v}=0.$$

二、一阶全微分形式不变性

设 $z = f(u,v)$ 有连续的偏导数，则

当 u, v 是自变量时，有

$$\mathrm{d}z = \frac{\partial z}{\partial u}\mathrm{d}u + \frac{\partial z}{\partial v}\mathrm{d}v$$

当 u, v 是中间变量时，若 $u = \varphi(x, y), v = \psi(x, y)$

均有连续的偏导数，则

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y$$

$$= (\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x})\mathrm{d}x + (\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y})\mathrm{d}y$$

$$\mathrm{d}z=\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\mathrm{d}x+\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)\mathrm{d}y$$

$$=\frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y\right)+\frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial v}{\partial y}\mathrm{d}y\right)$$

$$=\frac{\partial z}{\partial u}\,\mathrm{d}u+\frac{\partial z}{\partial v}\,\mathrm{d}v$$ —— 一阶 全微分形式不变性

一阶全微分形式不变性的实质:

无论 u , v 是自变量还是中间变量，函数的一阶全微分表达形式都一样，均为

$$\mathrm{d}z=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v.$$

例4 设 $z = f[xy+\varphi(y)]$，其中 f,φ 可微，求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

解(方法2) 由一阶 全微分形式不变性，得

$$\mathrm{d}z = \mathrm{d}\{ f[xy+\varphi(y)] \} \qquad (\text{令 } u = xy+\varphi(y))$$

$$= f'(u)\,\mathrm{d}u = f'[xy+\varphi(y)]\cdot \mathrm{d}[xy+\varphi(y)]$$

$$= f'[xy+\varphi(y)]\cdot[(y\,\mathrm{d}x + x\,\mathrm{d}y) + \varphi'(y)\,\mathrm{d}y]$$

$$= \boxed{y f'[xy+\varphi(y)]}\,\mathrm{d}x + \boxed{[x+\varphi'(y)]\,f'[xy+\varphi(y)]}\,\mathrm{d}y$$

$$\frac{\partial z}{\partial x} = y\,f'[xy+\varphi(y)], \quad \frac{\partial z}{\partial y} = [x+\varphi'(y)]\,f'[xy+\varphi(y)]$$

例5 (方法3)　由一阶全微分的形式不　变性,

$$du = e^{x^2+y^2+z^2} d(x^2+y^2+z^2)$$

$$= e^{x^2+y^2+z^2}(2x\,dx + 2y\,dy + 2z\,dz)$$

$$= e^{x^2+y^2+z^2}(2x\,dx + 2y\,dy + 2z\,\underline{d(x^2\sin y)})$$

$$= e^{x^2+y^2+z^2}[2x\,dx + 2y\,dy + 2z(2x\sin y\,dx + x^2\cos y\,dy)]$$

$$= e^{x^2+y^2+z^2}[2x(1+2z\sin y)dx + (2y+2x^2 z\cos y)dy]$$

$$= e^{x^2+y^2+x^4\sin^2 y}[2x(1+2x^2\sin^2 y)dx$$

$$+ (2y + 2x^4\sin y\cos y)dy]$$

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$$\mathrm{d}u = \underline{\mathrm{e}^{x^2+y^2+x^4\sin^2 y}}[2x(1+2x^2\sin^2 y)\mathrm{d}x$$

$$+\underline{(2y+x^4\sin 2y)\mathrm{d}y]}$$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x(1+2x^2\sin^2 y)\mathrm{e}^{x^2+y^2+x^4\sin^2 y},$$

$$\frac{\partial u}{\partial y} = (2y+x^4\sin 2y)\mathrm{e}^{x^2+y^2+x^4\sin^2 y}.$$

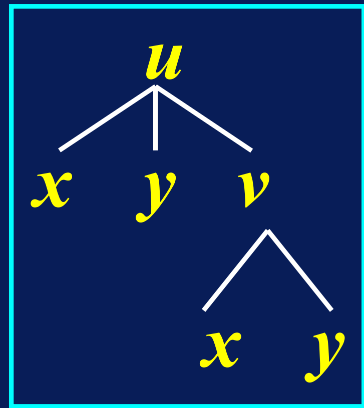
内容小结

1. 复合函数求导的链式法则

“连线相乘, 分线相加, 单路全导, 叉路偏导”

例如, $u = f(x, y, v), v = \varphi(x, y),$

$$\frac{\partial u}{\partial x} = f_1' + f_3' \cdot \varphi_1' ; \qquad \frac{\partial u}{\partial y} = f_2' + f_3' \cdot \varphi_2'$$



2. 一阶全微分形式不变性

对 $z = f(u, v)$, 不论 u , v 是自变量还是因变量,

$$\mathrm{d}z = f_u(u,v)\mathrm{d}u + f_v(u,v)\mathrm{d}v$$

思考题

1. 设 $z=\arctan\frac{x}{y}$, $x=u+v$, $y=u-v$, 求 $\frac{\partial z}{\partial v}$.

解

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$=\frac{1}{1+(\frac{x}{y})^2}\cdot\frac{1}{y}\cdot 1+\frac{1}{1+(\frac{x}{y})^2}\cdot(-\frac{x}{y^2})\cdot(-1)$$

$$=\frac{y+x}{x^2+y^2}=\frac{u}{u^2+v^2}$$

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2. 设 $u = f\left(\frac{x}{y}, \frac{y}{z}\right)$ 其中f可微,求u的一阶偏导数.

解

$$\frac{\partial u}{\partial x} = f_1' \cdot \frac{1}{y} = \frac{1}{y} f_1',$$

$$\frac{\partial u}{\partial y} = f_1' \cdot \left(-\frac{x}{y^2}\right) + f_2' \cdot \frac{1}{z} = -\frac{x}{y^2} f_1' + \frac{1}{z} f_2',$$

$$\frac{\partial u}{\partial z} = f_2' \cdot \left(-\frac{y}{z^2}\right) = -\frac{y}{z^2} f_2'.$$

3. 设 $z=f(u,x,y)$，$u=x\mathrm{e}^{y}$ 求 $\dfrac{\partial^2 z}{\partial x\partial y},\dfrac{\partial^2 z}{\partial x^2}$.

其中f具有连续的二阶偏导数 .

解

$$\frac{\partial z}{\partial x}=f_1'\cdot \mathrm{e}^{y}+f_2'$$

$$\frac{\partial^2 z}{\partial x\partial y}=\mathrm{e}^{y}f_1'+\mathrm{e}^{y}\cdot(f_{11}''\cdot x\mathrm{e}^{y}+f_{13}'')$$

$$+x\mathrm{e}^{y}f_{21}''+f_{23}''$$

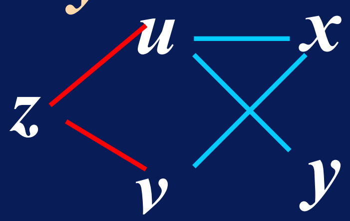
$$\frac{\partial^2 z}{\partial x^2}=\mathrm{e}^{y}\cdot(f_{11}''\cdot \mathrm{e}^{y}+f_{12}'')+f_{21}''\cdot \mathrm{e}^{y}+f_{22}''$$

$$=\mathrm{e}^{2y}f_{11}''+2\mathrm{e}^{y}f_{12}''+f_{22}''.$$

4. 设 $z = f(u,v), u = xy, v = e^x$,

求 $\dfrac{\partial^2 z}{\partial x \partial y}$.

解



$$z_x = f_1' \cdot y + f_2' \cdot e^x = y f_1' + e^x f_2',$$

$$z_{xy} = f_1' + y\, f_{11}'' \cdot x + e^x f_{21}'' \cdot x$$

$$= f_1' + xy f_{11}'' + x e^x f_{21}''.$$

5. 已知 $f(x,y)\Big|_{y=x^2}=1$， $f_1'(x,y)\Big|_{y=x^2}=2x$，求 $f_2'(x,y)\Big|_{y=x^2}$.

解 由 $f(x,x^2)=1$ 两边对 x 求导，得

$$f_1'(x,x^2)+f_2'(x,x^2)\cdot 2x=0$$

$$f_1'(x,x^2)=2x$$

$$f_2'(x,x^2)=-1$$

备用题

例1-1 设 $z = u^2 \ln v$，而 $u = \dfrac{x}{y}, v = 3x - 2y$，求 $\dfrac{\partial z}{\partial y}$.

解(方法1) 把 u， v代入，得到复合函数

$$z = \frac{x^2}{y^2}\ln(3x - 2y),$$

再利用多元函数求偏导　数的方法求 $\dfrac{\partial z}{\partial y}$:

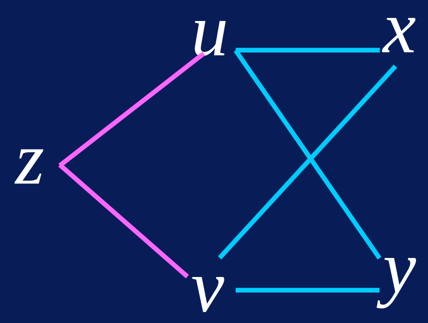
$$\frac{\partial z}{\partial y} = -\frac{2x^2}{y^3}\ln(3x - 2y) + \frac{x^2}{y^2}\cdot\frac{-2}{3x - 2y}$$

$$= -\frac{2x^2}{y^3}\ln(3x - 2y) - \frac{2x^2}{y^2}\cdot\frac{1}{3x - 2y}.$$

(方法2) 利用多元复合函数的求导法则:

$$z = u^2 \ln v$$

$$u = \frac{x}{y}, v = 3x - 2y$$



画出关系

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

写出公式

$$= 2u \ln v \cdot \left(-\frac{x}{y^2} \right) + u^2 \cdot \frac{1}{v} (-2)$$

求出各偏导数

$$= -\frac{2x^2}{y^3} \ln(3x - 2y) - \frac{2x^2}{y^3(3x - 2y)}.$$

将x，y代入

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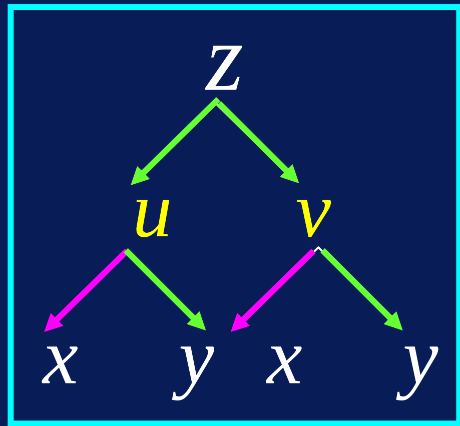
例1-2 设 $z = u^2 v - uv^2, u = x\sin y, v = x\cos y$,

求 $\dfrac{\partial z}{\partial x}$ 和 $\dfrac{\partial z}{\partial y}$.



解

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= (2uv - v^2)\sin y + (u^2 - 2uv)\cos y$$

$$= \frac{3x^2}{2}(\sin y - \cos y)\sin 2y$$

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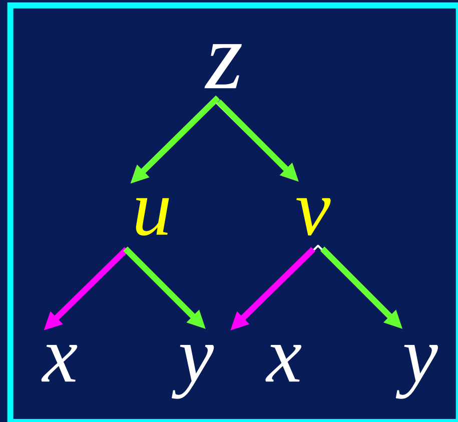
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= (2uv - v^2)\,x\cos y$$

$$+ (u^2 - 2uv)(-x\sin y)$$

$$= x^3(\sin y + \cos y)\left(\frac{3}{2}\sin 2y - 1\right).$$

$$z = u^2 v - uv^2,$$
$$u = x\sin y,$$
$$v = x\cos y,$$



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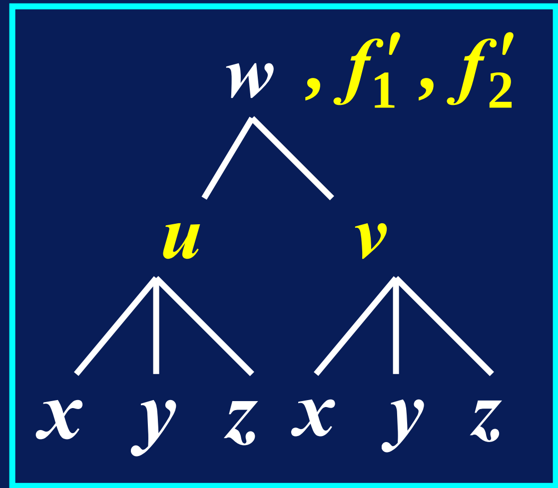
例1-3 设 $w=f(x+y+z,xyz)$, f 具有二阶连续偏导数, 求 $\dfrac{\partial w}{\partial x},\dfrac{\partial^2 w}{\partial x\partial z}$.



解 令 $u=x+y+z$, $v=xyz$, 则

$$w=f(u,v)$$

$$\frac{\partial w}{\partial x}=f_1'\cdot 1+f_2'\cdot yz$$

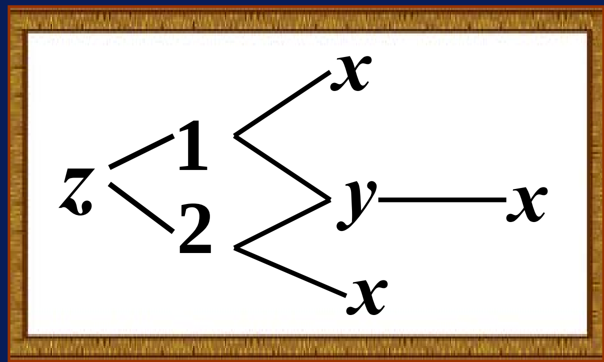
$$=f_1'(x+y+z,xyz)+\underline{yz\,f_2'(x+y+z,xyz)}$$

$$\frac{\partial^2 w}{\partial x\partial z}=f_{11}''\cdot 1+f_{12}''\cdot xy+y\,f_2'+yz[\,f_{21}''\cdot 1+f_{22}''\cdot xy]$$

$$=f_{11}''+y(x+z)f_{12}''+xy^2z\,f_{22}''+y\,f_2'$$

例3-1 设 $z=f(xy, x^2+y^2),\ y=\varphi(x),\ f$ 可微，

求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.



解 $\dfrac{\mathrm{d}z}{\mathrm{d}x}= f_1'\cdot\left(y+x\cdot\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$

$+f_2'\cdot\left(2x+2y\cdot\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$

$=[y+x\varphi'(x)]f_1'+2[x+\varphi(x)\cdot\varphi'(x)]f_2'$.

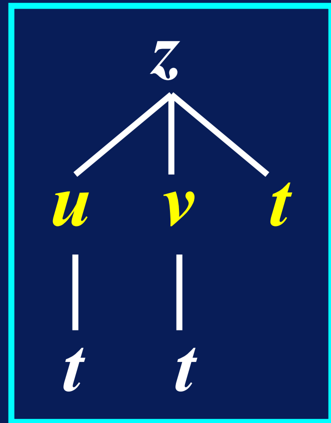
例3-2 设 $z = uv + \sin t, u = e^t, v = \cos t$, 求全导数 $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

解

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial t}$$

$$= v e^t - u\sin t + \cos t$$

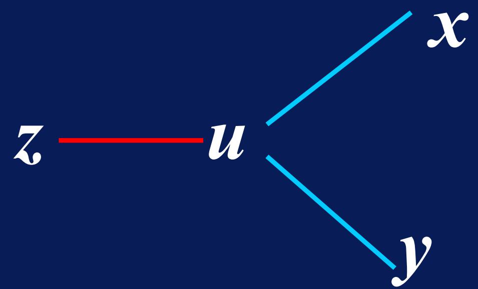
$$= e^t(\cos t - \sin t) + \cos t$$



例4-1 设$z = f(x+\varphi(y))$，其中f具有二阶连续偏导数，试证：$\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial^2 z}{\partial x^2}$.

证 令$u = x+\varphi(y)$，则$z = f(u), u = x+\varphi(y)$

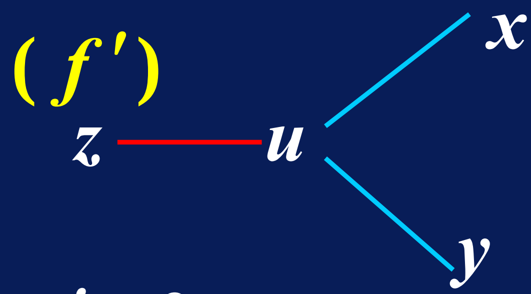
故



于是

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d}z}{\mathrm{d}u}\cdot\frac{\partial u}{\partial x} = f',$$

$$\frac{\partial z}{\partial y}=\frac{\mathrm{d}\,z}{\mathrm{d}\,u}\cdot\frac{\partial u}{\partial y}=f'\cdot\varphi',$$



$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}(f')=\frac{\mathrm{d}\,f'}{\mathrm{d}\,u}\cdot\frac{\partial u}{\partial y}=f''\cdot\varphi',$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}(f')=\frac{\mathrm{d}\,f'}{\mathrm{d}\,u}\cdot\frac{\partial u}{\partial x}=f'',$$

故 $\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial^2 z}{\partial x\partial y}=f'\cdot f''\cdot\varphi'=\dfrac{\partial z}{\partial y}\cdot\dfrac{\partial^2 z}{\partial x^2}.$

例4-2 设 $z=f(u)$, 方程 $u=\varphi(u)+\int_y^x p(t)\,dt$ 确定 u 是 x,y 的函数，其中 $f(u)$, $\varphi(u)$ 可微, $p(t),\varphi'(u)$ 连续, 且 $\varphi'(u)\neq 1$, 求 $p(y)\dfrac{\partial z}{\partial x}+p(x)\dfrac{\partial z}{\partial y}$.

解 $\dfrac{\partial z}{\partial x}=f'(u)\dfrac{\partial u}{\partial x}$, $\dfrac{\partial z}{\partial y}=f'(u)\dfrac{\partial u}{\partial y}$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\varphi'(u)\frac{\partial u}{\partial x}+p(x)\\ \frac{\partial u}{\partial y}&=\varphi'(u)\frac{\partial u}{\partial y}-p(y)\end{aligned}\right\}\Longrightarrow\left\{\begin{aligned}\frac{\partial u}{\partial x}&=\frac{p(x)}{1-\varphi'(u)}\\ \frac{\partial u}{\partial y}&=\frac{-p(y)}{1-\varphi'(u)}\end{aligned}\right.$$

$$\therefore\ p(y)\frac{\partial z}{\partial x}+p(x)\frac{\partial z}{\partial y}=f'(u)\left[p(y)\frac{\partial u}{\partial x}+p(x)\frac{\partial u}{\partial y}\right]=0$$

例4-3 设 $z=\frac{1}{x}f(xy)+y\varphi(x+y)$, f,φ具有连续导数，求 $\frac{\partial^2 z}{\partial x\partial y}$.

解 $$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}[\frac{1}{x}f(xy)]+\frac{\partial}{\partial x}[y\varphi(x+y)]$$

$$=[(-\frac{1}{x^2})f(xy)+\frac{1}{x}f'(xy)\cdot y]+y\varphi'(x+y)\cdot 1$$

$$=-\frac{1}{x^2}f(xy)+\frac{y}{x}f'(xy)+y\varphi'(x+y)$$

$$\frac{\partial z}{\partial x}=-\frac{1}{x^2}f(xy)+\frac{y}{x}f'(xy)+y\varphi'(x+y)$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

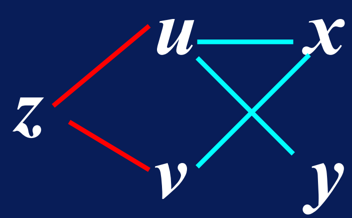
$$=\cancel{\left(-\frac{1}{x^2}\right)f'(xy)\cdot x}+\cancel{\frac{1}{x}f'(xy)}+\frac{y}{x}f''(xy)x$$

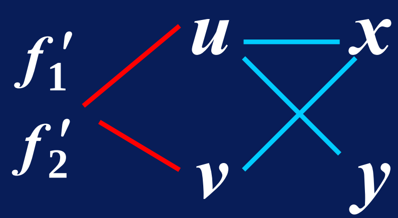
$$+[\varphi'(x+y)+y\varphi''(x+y)]$$

$$=yf''(xy)+\varphi'(x+y)+y\varphi''(x+y)$$

例5-1 设 $z = f(u,v), u = xy, v = e^x$，求 $\dfrac{\partial^2 z}{\partial x \partial y}$.

解





$$z_x = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \boxed{f_1'}\cdot\boxed{y} + \boxed{f_2'}\cdot e^x,$$

$$z_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_1'\cdot 1 + y\cdot f_{11}''\cdot x + e^x\cdot f_{21}''\cdot x$$

$$= f_{11}''\cdot xy + f_1' + f_{21}''\cdot xe^x.$$

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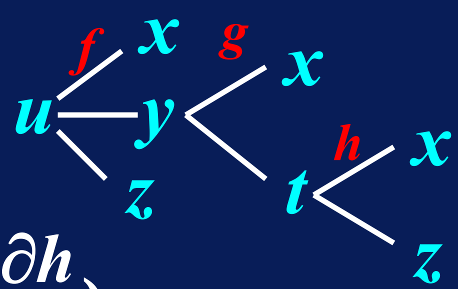
例6-1 设 $u = f(x, y, z),\ y = g(x, t), t = h(x, z)$

均可微，求 $\dfrac{\partial u}{\partial x}$ 及 $\dfrac{\partial u}{\partial z}$.

解

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x}$$



$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial t}\frac{\partial h}{\partial x}\right)$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial t}\frac{\partial h}{\partial x}$$

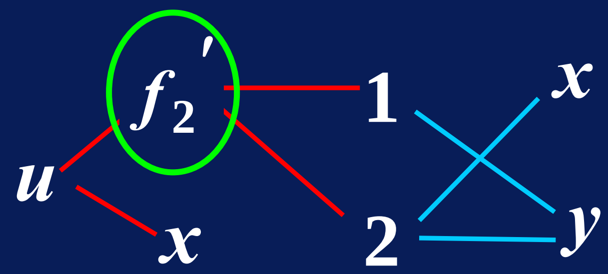
$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial t}\frac{\partial h}{\partial z}$$

例6-2 设 $u = xf(y,\frac{y}{x})$, f具有二阶偏导数，求 $\frac{\partial^2 u}{\partial x\partial y}$.

解

$$\frac{\partial u}{\partial x} = f(y,\frac{y}{x}) + xf_2'\cdot(-\frac{y}{x^2})$$

$$= f - \frac{y}{x}f_2'$$



$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial f}{\partial y} - \frac{1}{x}f_2' - \frac{y}{x}\frac{\partial f_2'}{\partial y}$$

$$= f_1' + f_2'\cdot\frac{1}{x} - \frac{1}{x}f_2' - \frac{y}{x}\left(f_{21}'' + f_{22}''\cdot\frac{1}{x}\right)$$

$$= f_1' - \frac{y}{x^2}\left(xf_{21}'' + f_{22}''\right).$$

例6-3 设函数 $z = f(x,y)$ 在点 $(1,1)$ 处可微，且

$$f(1,1)=1,\quad \left.\frac{\partial f}{\partial x}\right|_{(1,1)}=2,\quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)}=3,$$

$\varphi(x)=f(x,f(x,x))$，求 $\left.\dfrac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1}$. (2001考研)

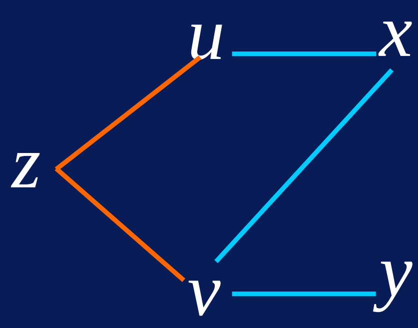
解 由题设 $\varphi(1)=f(1,f(1,1))=f(1,1)=1$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1}=3\varphi^2(x)\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=1}$$

$$=3\varphi^2(1)\Big[f_1'(x,f(x,x)) + f_2'(x,f(x,x))\big(f_1'(x,x)+f_2'(x,x)\big)\Big]\Big|_{x=1}$$

$$=3\cdot\left[2+3\cdot(2+3)\right]=51$$

例7-1 在自变量变换 $u = x, v = x^2 - y^2$ 下，求方程 $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = 0$ 的解 z.



解 将 z 看作由

$$z = z(u,v), u = x, v = x^2 - y^2$$

复合而成的复合函数，

则
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v},$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -2y\frac{\partial z}{\partial v}.$$

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代入原方程　，得

$$y\left(\frac{\partial z}{\partial u}+2x\frac{\partial z}{\partial v}\right)+x\left(-2y\frac{\partial z}{\partial v}\right)=0,$$

化简得 $\dfrac{\partial z}{\partial u}=0.$

这表明，函数 z不依赖于变量 u,只依赖于变量 v,

因此　　$z=f(v),$

其中f是任意可微的一元函数，

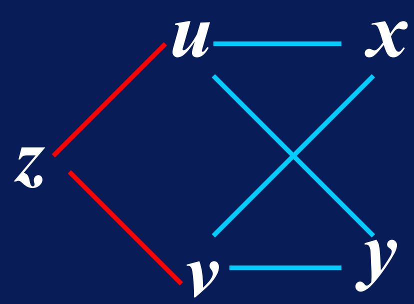
从而原方程的解为　$z=f(x^2-y^2).$

例7-2 设$u = x - 2\sqrt{y}, v = x + 2\sqrt{y}, (y > 0)$,变换

方程：$\dfrac{\partial^2 z}{\partial x^2} - y\dfrac{\partial^2 z}{\partial y^2} = \dfrac{1}{2}\dfrac{\partial z}{\partial y}$为$u$，$v$的方程

（其中所涉及的函数 z的二阶偏导数假定都连 续）.

解 z, u, v, x, y 的关系为



于是

$$z_x = z_u \cdot u_x + z_v v_x = z_u + z_v$$

$$z_{xx} = z_{uu} \cdot u_x + z_{uv} \cdot v_x + z_{vu} \cdot u_x + z_{vv} \cdot v_x$$

$$= z_{uu} + 2z_{uv} + z_{vv}$$

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$$z_y = z_u \cdot u_y + z_v \cdot v_y = -\frac{1}{\sqrt{y}} z_u + \frac{1}{\sqrt{y}} z_v$$

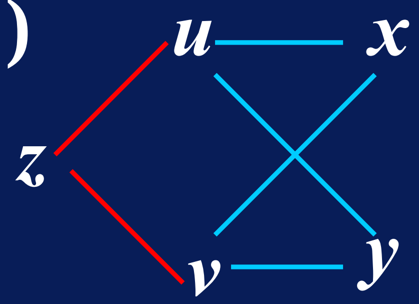
$$\boxed{\begin{array}{l} u = x - 2\sqrt{y} \\ v = x + 2\sqrt{y} \end{array}}$$

$$= \frac{1}{\sqrt{y}}(-z_u + z_v)$$

$$z_{yy} = -\frac{1}{2\sqrt{y^3}}(-z_u + z_v) + \frac{1}{\sqrt{y}}\left[-z_{uu} \cdot \left(-\frac{1}{\sqrt{y}}\right) - z_{uv} \cdot \frac{1}{\sqrt{y}}\right.$$

$$\left. + z_{vu} \cdot \left(-\frac{1}{\sqrt{y}}\right) + z_{vv} \cdot \frac{1}{\sqrt{y}}\right]$$

$$= -\frac{1}{2\sqrt{y^3}}(-z_u + z_v) + \frac{1}{y}(z_{uu} - 2z_{uv} + z_{vv})$$



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将z_{xx}, z_{yy}, z_y代入式：

$$\frac{\partial^2 z}{\partial x^2} - y\frac{\partial^2 z}{\partial y^2} = \frac{1}{2}\frac{\partial z}{\partial y}$$

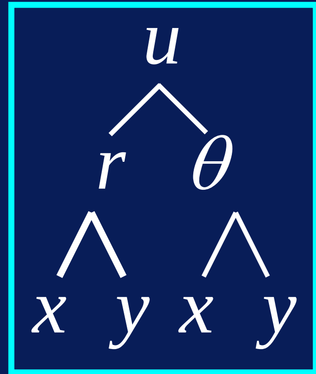
可得

$$4z_{uv} + \frac{1}{2\sqrt{y}}(-z_u + z_v) = \frac{1}{2\sqrt{y}}(-z_u + z_v)$$

化简得 $$z_{uv} = 0.$$

这是一个二阶双曲型偏微分方程的标准形式.

例7-3 设 $u=f(x,y)$二阶偏导数连续, 求下列表达式在极坐标系下的形式 (1) $\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2$, (2) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}$



解 已知$x=r\cos\theta$, $y=r\sin\theta$, 则

$$r=\sqrt{x^2+y^2},\ \theta=\arctan\frac{y}{x}$$

(1) $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\boxed{\frac{\partial r}{\partial x}=\frac{x}{r},\quad \frac{\partial \theta}{\partial x}=\frac{\frac{-y}{x^2}}{1+(\frac{y}{x})^2}=\frac{-y}{x^2+y^2}}$$

$$=\frac{\partial u}{\partial r}\frac{x}{r}-\frac{\partial u}{\partial \theta}\frac{y}{r^2}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$

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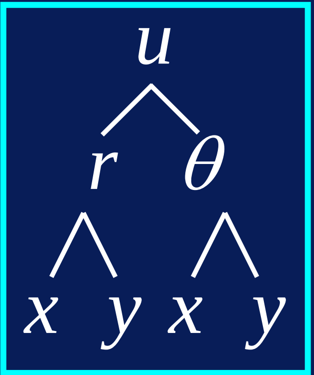
$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\boxed{\frac{\partial r}{\partial y}=\frac{y}{r},\quad \frac{\partial \theta}{\partial y}=\frac{\frac{1}{x}}{1+(\frac{y}{x})^2}=\frac{x}{x^2+y^2}}$$

$$=\frac{\partial u}{\partial r}\frac{y}{r}+\frac{\partial u}{\partial \theta}\frac{x}{r^2}$$

$$=\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$$



$$\therefore\quad (\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2=(\frac{\partial u}{\partial r})^2+\frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2$$

$$\frac{\partial u}{\partial x}=\boxed{\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}}$$

$\dfrac{\partial u}{\partial x}$

$r \quad \theta$

$x \quad y \quad x \quad y$

注意利用已有公式

(2) $$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial r}{\partial x}+\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \theta}{\partial x}$$

$$=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\cdot\cos\theta$$

$$+\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\cdot\left(-\frac{\sin\theta}{r}\right)$$

$$=\left(\frac{\partial^2 u}{\partial r^2}\cos\theta-\frac{\partial^2 u}{\partial\theta\partial r}\frac{\sin\theta}{r}+\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r^2}\right)\cos\theta$$

$$+\left(\frac{\partial^2 u}{\partial r\partial\theta}\cos\theta+\frac{\partial u}{\partial r}(-\sin\theta)-\frac{\partial^2 u}{\partial \theta^2}\frac{\sin\theta}{r}-\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right)\cdot\left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial r^2}\cos^2\theta-\cancel{2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}}+\frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2}$$

$$+\cancel{\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2}}+\frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$

同理可得

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}\sin^2\theta+\cancel{2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}}+\frac{\partial^2 u}{\partial\theta^2}\frac{\cos^2\theta}{r^2}$$

$$-\cancel{\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2}}+\frac{\partial u}{\partial r}\frac{\cos^2\theta}{r}$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$