第二节

多元函数的偏导数

- 一、 偏导数的概念
- 二、偏导数的计算
- 三、偏导数的几何意义
- 四、高阶偏导数

一、偏导数的概念

1.引例 弦线的振动问题.

研究弦在点 x_0 处的振动速度与加速度， 就是将振幅 $u(x,t)$ 中的 x 固定于 x_0 处，求 $u(x_0,t)$ 关于 t 的一阶导数与二阶导数.

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2. 定义8.6 设函数 $z = f(x, y)$ 在点 $P_0(x_0, y_0)$ 的某邻域 $U(P_0)$ 内有定义．若当固定 y 在 y_0, $z = f(x, y_0)$ 在 $x = x_0$ 处的导数存在，即

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

存在，则称此极限为 $z = f(x, y)$ 在点 (x_0, y_0) 处 对 x 的**偏导数**，记为

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)};\quad \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)};\quad \left.z_x\right|_{(x_0, y_0)};\quad f_x(x_0, y_0).$$

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注 $f_x(x_0, y_0)$

$$= \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x} = \frac{\mathrm{d}}{\mathrm{d}x} f(x, y_0)\Big|_{x=x_0}$$

同样可定义 函数 $f(x, y)$ 在点(x_0, y_0) 对 y 的偏导数

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y}$$

$$= \frac{\mathrm{d}}{\mathrm{d}y} f(x_0, y)\Big|_{y=y_0}$$

记为 $\left.\frac{\partial z}{\partial y}\right|_{(x_0, y_0)}$； $\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)}$； $z_y\big|_{(x_0, y_0)}$； $f_y(x_0, y_0)$.

注　1° 偏导函数

若函数 $z = f(x, y)$ 在域 D 内每一点 (x, y) 处对 x 的（或 y）偏导数都存在，称该偏导数为 $z = f(x, y)$ 对自变量 x (或 y) 的**偏导函数**，也简称为**偏导数**，记为

$$\frac{\partial z}{\partial x},\ \frac{\partial f}{\partial x},\ z_x\ ,\ f_x(x, y),\ f_1'(x, y)$$

$$\frac{\partial z}{\partial y},\ \frac{\partial f}{\partial y},\ z_y\ ,\ f_y(x, y),\ f_2'(x, y)$$

由此可知： $f_x(x_0, y_0) = f_x(x, y)\big|_{(x_0, y_0)}$

$$f_y(x_0, y_0) = f_y(x, y)\big|_{(x_0, y_0)}$$

2° 偏导数的概念可以推广到二元以上函数

例如：三元函数 $u = f(x, y, z)$ 在点(x, y, z) 处对 x 的偏导数定义为

$$f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$f_y(x, y, z) = ?$$

$$f_z(x, y, z) = ?$$ (请自己写出)

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3° 可(偏)导

若 $z=f(x,y)$在点(x_0,y_0)处的两个偏导数

$f_x(x_0,y_0), f_y(x_0,y_0)$均存在，则称 $f(x,y)$

在点(x_0,y_0)处可(偏)导.

4° 偏导数 $\dfrac{\partial z}{\partial x}$是一个整体记号，不能拆分

$$\frac{\partial z}{\partial x} \neq \partial z / \partial x$$

例1 一定量理想气体的状态方程 $pV = RT$

(R 为常数)，求证: $\dfrac{\partial p}{\partial V}\cdot\dfrac{\partial V}{\partial T}\cdot\dfrac{\partial T}{\partial p} = -1$

证 $p = \dfrac{RT}{V}$, $\dfrac{\partial p}{\partial V} = -\dfrac{RT}{V^2}$

$V = \dfrac{RT}{p}$, $\dfrac{\partial V}{\partial T} = \dfrac{R}{p}$

$T = \dfrac{pV}{R}$, $\dfrac{\partial T}{\partial p} = \dfrac{V}{R}$

$$\therefore \frac{\partial p}{\partial V}\cdot\frac{\partial V}{\partial T}\cdot\frac{\partial T}{\partial p} = -\frac{RT}{pV} = -1$$

说明: 此例表明, 偏导数记号是一个整体记号, 不能看作分子与分母的商！

5° 若 $f(x, y_0)$为分段函数，分段点为 x_0,

则求 $f_x(x_0, y_0)$时，须用偏导数定义 .

3. 偏导数存在与连续的关系

对于一元函数：可导 $\underset{\not\Longleftarrow}{\Longrightarrow}$ 连续

对于多元函数：可（偏）导 $\underset{\not\Longleftarrow}{\not\Longrightarrow}$ 连续

例2 证明：函数 $z=\sqrt{x^2+y^2}$ 在 $(0,0)$ 点连续，但两个偏导数均不存在.

证 $\forall \varepsilon>0$, 取 $\delta \le \varepsilon$,

则当 $\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}<\delta$ 时,

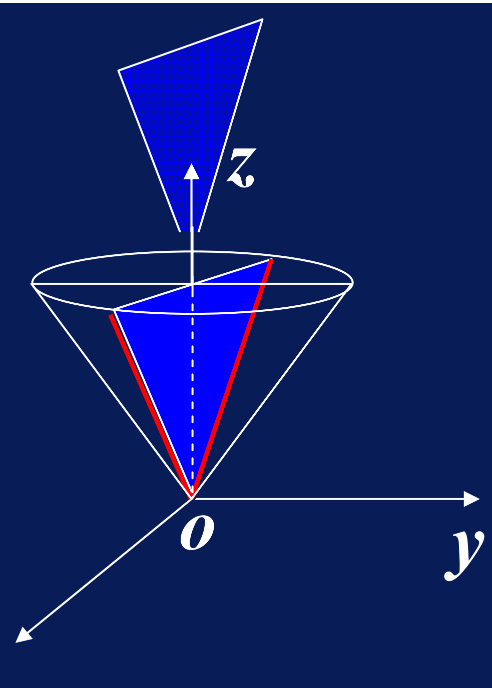
便有 $\left|\sqrt{x^2+y^2}-\sqrt{0^2+0^2}\right|$

$$=\sqrt{x^2+y^2}<\delta\le\varepsilon,$$

故函数在点 $(0,0)$ 处连续.

但 $\lim\limits_{x \to 0} \dfrac{f(x,0) - f(0,0)}{x}$

$$= \lim_{x \to 0} \frac{\sqrt{x^2} - 0}{x} = \lim_{x \to 0} \frac{|x|}{x}$$



此极限不存在，故函数在 $(0,0)$点处关于自变量x的偏导数不存在．

同理,关于自变量y的偏导数也不存在．

注 对于二元函数：可偏导 $\not\longleftarrow$ 连续

例3 设 $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$,

求 $f_x(0,0)$, $f_y(0,0)$, 并讨论$f(x,y)$在(0,0)处的连续性.

解(方法1) $f_x(0,0)=\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x}$

$$=\lim_{x\to 0}\frac{\dfrac{x\cdot 0}{\sqrt{x^2+0}}-0}{x}=0$$

同理可求得 $f_y(0,0)=0.$

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$$\because \lim_{\substack{x\to 0\\ y=kx\to 0}} f(x,y)=\lim_{\substack{x\to 0\\ y=kx\to 0}}\frac{xy}{x^2+y^2}$$

$$=\lim_{x\to 0}\frac{x\cdot kx}{x^2+(kx)^2}=\frac{k}{1+k^2}$$

其值随 k 的不同而变化，

$$\therefore \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{x^2+y^2}\text{不存在.}$$

从而 $f(x,y)$ 在点$(0,0)$并不连续!

(方法2) $z = f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0\quad, & x^2+y^2 = 0 \end{cases}$

$\because \quad f(x,0) \equiv 0, \quad f(0,y) \equiv 0,$

$$\therefore \quad f_x(0,0) = \frac{\mathrm{d}}{\mathrm{d}x} f(x,0)\Big|_{x=0} = 0$$

$$f_y(0,0) = \frac{\mathrm{d}}{\mathrm{d}y} f(0,y)\Big|_{y=0} = 0$$

注 对于二元函数：可偏导 $\not\longrightarrow$ 连续

二、偏导数的计算

由偏导数的定义可知，偏导数的计算可归结为一元函数的导数计算.

$$f_x(x_0,y_0)=\frac{\mathrm{d}}{\mathrm{d}x}f(x,y_0)\Big|_{x=x_0}$$

$$f_y(x_0,y_0)=\frac{\mathrm{d}}{\mathrm{d}y}f(x_0,y)\Big|_{y=y_0}$$

求某个具体的点处的偏导数时方便

例4 求 $z = x^2 + 3xy + y^2$ 在点(1 , 2) 处的偏导数.

解(方法1) 先求后代

$$\frac{\partial z}{\partial x} = 2x + 3y, \quad \frac{\partial z}{\partial y} = 3x + 2y$$

$$\therefore \quad \left.\frac{\partial z}{\partial x}\right|_{(1,2)} = 2\cdot 1 + 3\cdot 2 = 8,$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = 3\cdot 1 + 2\cdot 2 = 7$$

(方法2) 先代后求

$$z\big|_{y=2} = x^2 + 6x + 4$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,\,2)} = \left.\frac{\mathrm{d}\,z(x,2)}{\mathrm{d}\,x}\right|_{x=1} = \frac{\mathrm{d}}{\mathrm{d}\,x}(x^2 + 6x + 4)\Big|_{x=1}$$

$$= (2x+6)\Big|_{x=1} = 8$$

$$z\big|_{x=1} = 1 + 3y + y^2$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,\,2)} = \left.\frac{\mathrm{d}\,z(1,y)}{\mathrm{d}\,y}\right|_{y=2} = \frac{\mathrm{d}}{\mathrm{d}\,y}(1 + 3y + y^2)\Big|_{y=2}$$

$$= (3+2y)\Big|_{y=2} = 7$$

例5 设 $z = x^y \ (x > 0, \text{且} \ x \neq 1)$，求证

$$\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = 2z$$

证 $\because \dfrac{\partial z}{\partial x} = yx^{y-1}, \qquad \dfrac{\partial z}{\partial y} = x^y \ln x$

$$\therefore \frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = \frac{x}{y} yx^{y-1} + \frac{1}{\ln x} x^y \ln x$$

$$= x^y + x^y = 2z$$

例6 设$f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$

求$f(x,y)$的偏导数.

解 当（x,y）$\neq(0,0)$时， 有

$$f_x(x,y)=\frac{2xy(x^4+y^2)-x^2y\cdot 4x^3}{(x^4+y^2)^2}$$

$$=\frac{2xy(y^2-x^4)}{(x^4+y^2)^2}$$

$$f_y(x,y) = \frac{x^2(x^4+y^2) - x^2 y \cdot 2y}{(x^4+y^2)^2}$$

$$= \frac{x^2(x^4-y^2)}{(x^4+y^2)^2}.$$

当$(x,y)=(0,0)$时，由偏导数的定义得

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x,0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0,$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0}{\Delta y} = 0.$$

于是

$$f_x(x,y)=\begin{cases}\dfrac{2xy(y^2-x^4)}{(x^4+y^2)^2}, & x^2+y^2\neq 0,\\[2ex] 0, & x^2+y^2=0,\end{cases}$$

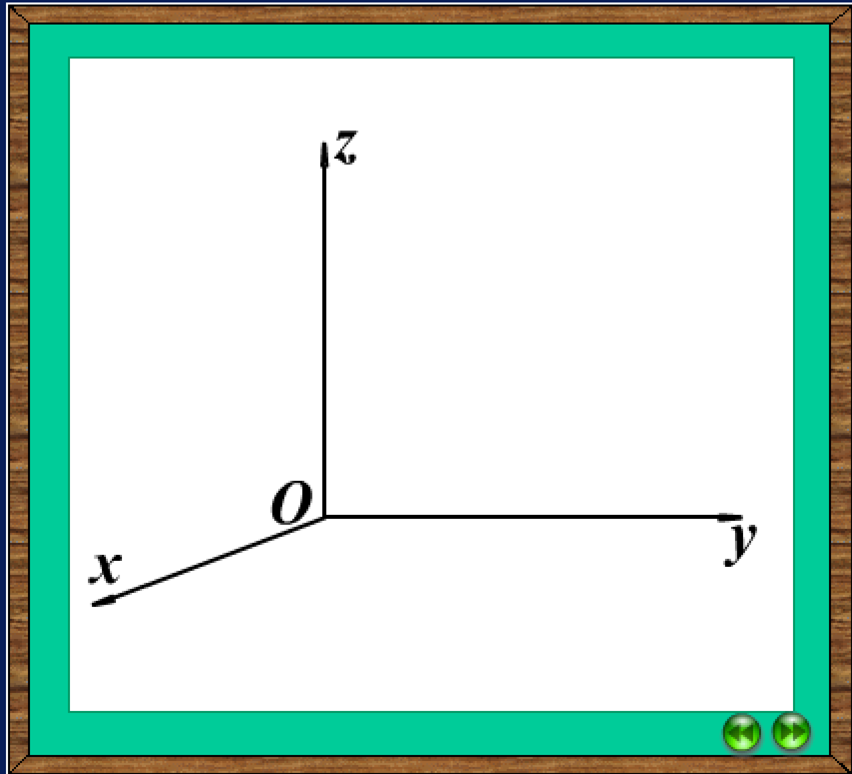
$$f_y(x,y)=\begin{cases}\dfrac{x^2(x^4-y^2)}{(x^4+y^2)^2}, & x^2+y^2\neq 0,\\[2ex] 0, & x^2+y^2=0.\end{cases}$$

三、偏导数的几何意义

设 $M_0(x_0, y_0, f(x_0, y_0))$ 为曲面 $z = f(x, y)$ 上一点,

$$\tan\alpha = \left.\frac{\mathrm{d}\, f(x, y_0)}{\mathrm{d}\, x}\right|_{x=x_0}$$

$$= f_x(x_0, y_0)$$



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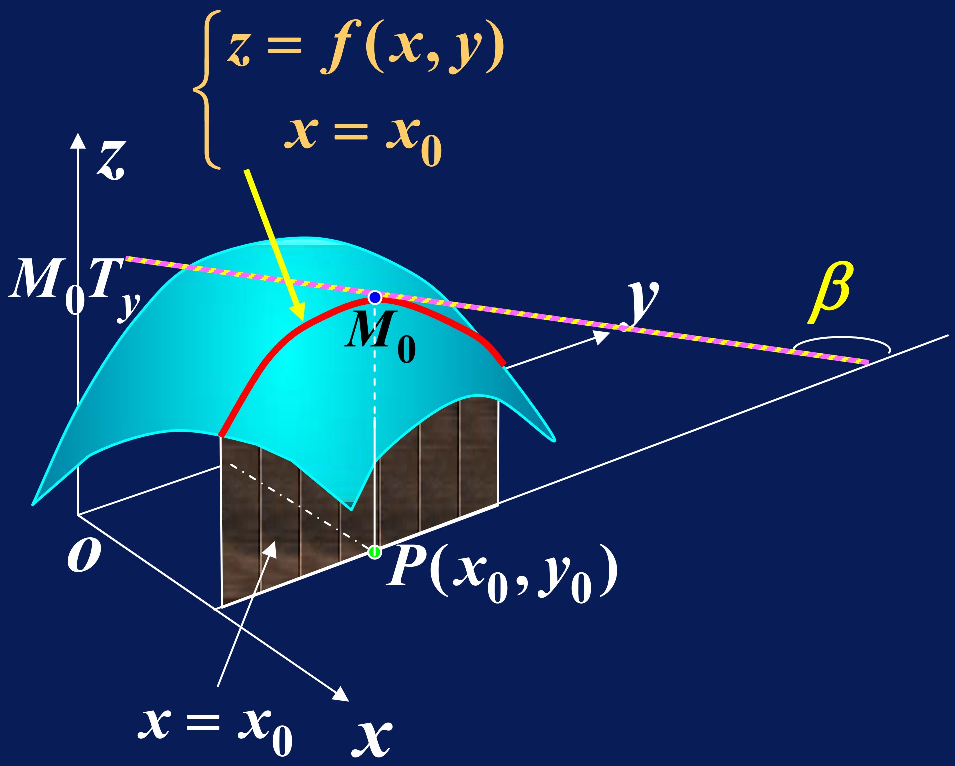
$$\tan\beta = f_y(x_0, y_0) = \frac{\mathrm{d}}{\mathrm{d}y} f(x_0, y)\Big|_{y=y_0}$$

是曲线 $\begin{cases} z = f(x, y) \\ x = x_0 \end{cases}$

在点 M_0 处的切线 M_0T_y

对 y 轴的斜率.



例7 求曲线 $\begin{cases} z = \sqrt{1 + x^2 + y^2}, \\ x = 1, \end{cases}$

在点 $(1,1,\sqrt{3})$ 处的切线与 y 轴正向的夹角 β

解　根据偏导数的几何意义　　,有

$$\tan\beta = \left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=1}} = \left.\frac{2y}{2\sqrt{1+x^2+y^2}}\right|_{\substack{x=1\\y=1}}$$

$$= \frac{1}{\sqrt{3}} \qquad \text{故} \quad \beta = \frac{\pi}{6}$$

四、高阶偏导数

设 $z = f(x, y)$在域 D 内存在连续的偏导数

$$\frac{\partial z}{\partial x} = f_x(x, y), \qquad \frac{\partial z}{\partial y} = f_y(x, y)$$

若这两个偏导数仍存在偏导数，则称它们是

$z = f(x, y)$的二阶偏导数．按求导顺序不同,

有下列四个二阶偏导数:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y); \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x, y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x, y); \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y)$$

类似可以定义更高阶的偏导数.

例如，$z = f(x, y)$ 关于 x 的三阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial x^3}$$

$z = f(x, y)$ 关于 x 的 $n-1$ 阶偏导数，再关于

y 的一阶偏导数为

$$\frac{\partial}{\partial y}\left(\frac{\partial^{n-1} z}{\partial x^{n-1}}\right) = \frac{\partial^n z}{\partial x^{n-1}\partial y}$$

例8 求函数 $z=e^{x+2y}$ 的二阶偏导数及 $\dfrac{\partial^3 z}{\partial y\partial x^2}$.

解 $\dfrac{\partial z}{\partial x}=e^{x+2y}$ $\quad \dfrac{\partial z}{\partial y}=2e^{x+2y}$ $\quad \dfrac{\partial^2 z}{\partial x^2}=e^{x+2y}$

$$\frac{\partial^2 z}{\partial x\partial y}=2e^{x+2y}\quad \frac{\partial^2 z}{\partial y\partial x}=2e^{x+2y}\quad \frac{\partial^2 z}{\partial y^2}=4e^{x+2y}$$

$$\frac{\partial^3 z}{\partial y\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y\partial x}\right)=2e^{x+2y}$$

注 此处 $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial^2 z}{\partial y\partial x}$,

但这一结论并不总是成立.

问题：二阶混合偏导数一定都相等吗？不一定！

例如：
$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_x(x,y)=\begin{cases} y\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0\,, & x^2+y^2=0 \end{cases}$$

$$f_y(x,y)=\begin{cases} x\dfrac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_x(x,y)=\begin{cases} y\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_y(x,y)=\begin{cases} x\dfrac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_{xy}(0,0)=\lim_{y\to 0}\frac{f_x(0,y)-f_x(0,0)}{y}=\lim_{y\to 0}\frac{-y}{y}=-1$$

$$f_{yx}(0,0)=\lim_{x\to 0}\frac{f_y(x,0)-f_y(0,0)}{x}=\lim_{x\to 0}\frac{x}{x}=1$$

二者不等

问题： 具备怎样的条件，混合偏导数 相等？

定理 若 $f_{xy}(x,y)$ 和 $f_{yx}(x,y)$ 都在点 (x_0, y_0) 连续, 则

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$ (证明略)

本定理对 n 元函数的高阶混合偏导数也成立.

例如, 对三元函数 $u = f(x, y, z)$, 当三阶混合偏导数

在点 (x, y, z) 连续时, 有

$$f_{xyz}(x,y,z) = f_{yzx}(x,y,z) = f_{zxy}(x,y,z)$$

$$= f_{xzy}(x,y,z) = f_{yxz}(x,y,z) = f_{zyx}(x,y,z)$$

例9 证明函数 $u=\frac{1}{r}, r=\sqrt{x^2+y^2+z^2}$ 满足拉普拉斯

方程 $\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$

偏微分方程

证 $\frac{\partial u}{\partial x}=-\frac{1}{r^2}\frac{\partial r}{\partial x}=-\frac{1}{r^2}\cdot\frac{x}{r}=-\frac{x}{r^3}$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{1}{r^3}+\frac{3x}{r^4}\cdot\frac{\partial r}{\partial x}=-\frac{1}{r^3}+\frac{3x^2}{r^5}$$

利用对称性，有 $\frac{\partial^2 u}{\partial y^2}=-\frac{1}{r^3}+\frac{3y^2}{r^5}, \frac{\partial^2 u}{\partial z^2}=-\frac{1}{r^3}+\frac{3z^2}{r^5}$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{3}{r^3}+\frac{3\underbrace{\boxed{(x^2+y^2+z^2)}}_{=r^2}}{r^5}=0$$

内容小结

1. 偏导数的概念及有关结论

- 定义; 记号; 几何意义
- 函数在一点偏导数存在 $\not\Longrightarrow$ 函数在此点连续
- 混合偏导数连续 $\Longrightarrow$ 与求导顺序无关

2. 偏导数的计算方法

- 求一点处偏导数的方法 $\begin{cases}\text{先代后求} \\ \text{先求后代} \\ \text{利用定义}\end{cases}$
- 求高阶偏导数的方法 —— 逐次求导法

(与求导顺序无关时, 应选择方便的求导顺序)

备用题

例2-1 设$z = f(x, y) = \sqrt{|xy|}$, 求$f_x(0, 0)$, $f_y(0, 0)$.

解 $f_x(0,0) = \lim\limits_{x\to 0} \dfrac{f(x,0) - f(0,0)}{x}$

$$= \lim_{x\to 0} \frac{\sqrt{|x\cdot 0|} - 0}{x} = 0$$

同理可求得 $f_y(0,0) = 0.$

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例5-1 求函数 $u = \mathrm{e}^{xy}\cos yz$ 的偏导数.

解 $u_x = y\mathrm{e}^{xy}\cos yz,$

$$u_y = (\cos yz)\mathrm{e}^{xy}x + \mathrm{e}^{xy}(-\sin yz)z$$

$$= \mathrm{e}^{xy}(x\cos yz - z\sin yz),$$

$$u_z = \mathrm{e}^{xy}(-\sin yz)\,y = -y\mathrm{e}^{xy}\sin yz$$

例5-2 设$z=\arcsin\frac{x}{\sqrt{x^2+y^2}}$，求$\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$.

解
$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\right)_x' \quad (y\neq 0)$$

$$=\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{y^2}{\sqrt{(x^2+y^2)^3}} \quad (\sqrt{y^2}=|y|)$$

$$=\frac{|y|}{x^2+y^2}.$$

$$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-\dfrac{x^2}{x^2+y^2}}}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\right)_y'$$

$$=\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{(-xy)}{\sqrt{(x^2+y^2)^3}}$$

$$=-\frac{x}{x^2+y^2}\operatorname{sgn}\frac{1}{y}\qquad(y\neq 0)$$

例6-1 设$f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$

求$f_x(x,y)$及$f_y(x,y)$.

解: 当$x^2+y^2\neq 0$时,

$$f_x(x,y)=\frac{\partial}{\partial x}\left(\frac{x^2y}{x^2+y^2}\right)=\frac{2xy(x^2+y^2)-x^2y\cdot 2x}{(x^2+y^2)^2}$$

$$=\frac{2xy^3}{(x^2+y^2)^2}$$

$$f_y(x,y)=\frac{\partial}{\partial y}\left(\frac{x^2 y}{x^2+y^2}\right)=\frac{x^2(x^2+y^2)-x^2 y\cdot 2y}{(x^2+y^2)^2}$$

$$=\frac{x^2(x^2-y^2)}{(x^2+y^2)^2}$$

在（0,0）点, 注意到 $f(x,0)=\begin{cases}0, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}=0$

故有 $f_x(0,0)=\left.\frac{\mathrm{d}}{\mathrm{d}x}f(x,0)\right|_{x=0}=0$

同理 $f_y(0,0)=\left.\frac{\mathrm{d}}{\mathrm{d}y}f(0,y)\right|_{y=0}=0.$

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$$\therefore f_x(x,y)=\begin{cases}\dfrac{2xy^3}{(x^2+y^2)^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$$

$$f_y(x,y)=\begin{cases}\dfrac{x^2(x^2-y^2)}{(x^2+y^2)^2}, & (x^2+y^2)\neq 0,\\ 0, & (x^2+y^2)\neq 0.\end{cases}$$

例8-1 求下列函数的一阶和二阶偏导数

(1) $z=\ln(x+y^2)$;

解 $\dfrac{\partial z}{\partial x}=\dfrac{1}{x+y^2}$, $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{-1}{(x+y^2)^2}$,

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{-2y}{(x+y^2)^2},$$

$$\frac{\partial z}{\partial y}=\frac{2y}{x+y^2},\quad \frac{\partial^2 z}{\partial y^2}=\frac{2\cdot(x+y^2)-2y\cdot 2y}{(x+y^2)^2}=\frac{2(x-y^2)}{(x+y^2)^2}.$$

(2) $z = x^y$. 由例5知

$$\frac{\partial z}{\partial x} = yx^{y-1}, \qquad \frac{\partial z}{\partial y} = x^y \ln x,$$

$$\frac{\partial^2 z}{\partial x^2} = y(y-1)x^{y-2}, \qquad \frac{\partial^2 z}{\partial y^2} = x^y \ln^2 x.$$

$$\frac{\partial^2 z}{\partial x \partial y} = x^{y-1} + yx^{y-1} \ln x,$$

例8-2 求下列函数的偏导数

(1) $F(x,y)=\int_{3y}^{2x}\frac{\sin t}{t}\mathrm{d}t$，求$F_x, F_y$;

(2) $F(x,y)=x\int_{y}^{x}\mathrm{e}^{-3t^2}\mathrm{d}t$，求$\frac{\partial^2 F}{\partial x\partial y}$.

解 (1) $F_x=2\cdot\frac{\sin 2x}{2x}=\frac{\sin 2x}{x}$.

$$F_y=-\frac{\sin 3y}{3y}\cdot 3=-\frac{\sin 3y}{y}.$$

(2) $F_x=\int_{y}^{x}\mathrm{e}^{-3t^2}\mathrm{d}t+x\cdot\mathrm{e}^{-3x^2}$

$$F_{xy}=-\mathrm{e}^{-3y^2}.$$

例9-1 验证函数$u(x,y)=\ln\sqrt{x^2+y^2}$满足二维拉普拉斯方程

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

解 $\because \quad u=\ln\sqrt{x^2+y^2}=\frac{1}{2}\ln(x^2+y^2),$

$$\therefore \quad \frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}, \qquad \frac{\partial u}{\partial y}=\frac{y}{x^2+y^2},$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}=\frac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2},$$

$$\frac{\partial u}{\partial y}=\frac{y}{x^2+y^2},$$

$$\therefore\quad \frac{\partial^2 u}{\partial x^2}=\frac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2},$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{(x^2+y^2)-y\cdot 2y}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}.$$

$$\therefore\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{x^2-y^2}{(x^2+y^2)^2}=0.$$