Appendix A. Bourbaki–Style Foundations of Kleis

A.0. Introduction

The present appendix establishes, in a form adhering to the structural method, the foundational principles upon which the Kleis language is constructed. Kleis is intended to serve as a formal system for the description of mathematical structures, their relations, and the operations defined upon them. Rather than adopting a computational viewpoint, we adopt here the classical framework of *structures* as introduced in the *Éléments de mathématique*.

Throughout, we assume the existence of a background universe $\mathscr{U}$ sufficiently large to contain all sets considered. No distinction is made between sets and classes except where explicitly noted. All constructions are understood to take place within $\mathscr{U}$.

A.1. Species of Structures

A *species of structure* in the sense of Bourbaki is determined by:

(i) a finite family of underlying sets,

(ii) a finite family of operations of finite arity,

(iii) a finite family of relations of finite arity,

(iv) a finite family of axioms expressed as identities or implications.

In Kleis, such a species corresponds precisely to a declaration of the form:

```
structure S(X1, ..., Xn) { ... }
```

The sets $X_1, \ldots, X_n$ constitute the underlying sets of the structure; the operations and relations correspond to the `operation` and `element` declarations; the axioms correspond to the `axiom` declarations.

Thus, a Kleis structure is a *formal species of structure* in Bourbaki's sense.

A.2. Morphisms of Species and the `extends` Relation

Given two species of structures $\mathcal{S}$ and $\mathcal{T}$, a *morphism of species* is any mapping which respects:

- the number and positions of underlying sets,
- the signature of operations and relations,
- the axioms imposed.

The Kleis construct:

```
structure T(...) extends S(...)
```

represents precisely such a morphism of species in which:

- each operation of $\mathcal{S}$ is preserved in $\mathcal{T}$,
- each axiom of $\mathcal{S}$ holds in $\mathcal{T}$,
- additional operations and axioms may be added.

Thus, `extends` corresponds to the inclusion of one species of structure into another.

A.3. Structures Dependent on Other Structures and the `over` Clause

In Bourbaki's formalism, certain structures are defined relative to a previously chosen structure. For example, a vector space is defined only after a field has been designated.

Kleis formalizes this by the construct:

```
structure V(X) over Field(F) { ... }
```

This expresses that V is a species whose definition presupposes the choice of a structure of type `Field`. Formally, this corresponds to the notion of a *parametrized species of structure*, or, in categorical language, to a *fibred* family of structures indexed by the models of the parameter species.

A.4. Realizations of Structures and the `implements` Clause

A *structure of species* $\mathcal{S}$ in the sense of Bourbaki is given by:

- a family of sets,
- interpretations of each operation symbol as a genuine function,
- interpretations of relations as subsets,
- verification that all axioms hold.

In Kleis:

```
implements Field() { ... }
```

declares that the species `Field` is realized on the set $\mathbb{R}$. The operations and elements given in the implementation constitute the interpretation of the signature, and the axioms are asserted to hold in this interpretation.

Thus `implements` corresponds exactly to the Bourbaki notion of a *structure of a given species.*

A.5. Conditions on Realizations and the `where` Clause

Many mathematical constructions impose side conditions on underlying sets or operations (e.g., commutativity, order-completeness). In Bourbaki's methodology, these appear as *subspecies of structures*.

The Kleis construct:

```
implements R(T) where C(T)
```

specifies that the realization of R on T is valid only in cases where T satisfies an auxiliary species C.

Thus, `where` produces a *subspecies* of the original species, in the Bourbaki sense.

A.6. Nested Structures and Composite Species

A Bourbaki structure may itself contain other structures whose underlying set is identical. For example, a ring consists of an additive commutative group and a multiplicative monoid defined on the same set.

In Kleis:

```
structure Ring(R) {
    structure additive : AbelianGroup(R)
    structure multiplicative : Monoid(R)
}
```

This corresponds to a composite species of structure in Bourbaki's terminology: the ring structure is obtained by amalgamating two subspecies defined on the same underlying set and adding compatibility axioms.

Thus, nested Kleis structures represent the Bourbaki notion of a structure *with internal substructures of species.*

A.7. Algebraic Data Types and Inductive Species

An inductive Kleis data type:

```
data Option(T) = None | Some(T)
```

corresponds to the initial model of a finitary algebraic species. In Bourbaki's terminology, this is a *free structure* on the given signature.

Given a functor:

$$F(X) = 1 + T \times X,$$

the type defined above is the initial object of the category of F-algebras.

A.8. Axioms as Structural Relations

A Kleis axiom:

```
axiom associativity:
    (x y z : M). (x • y) • z = x • (y • z)
```

corresponds to an identity constraint on the operations of the species. In Bourbaki's setting, these axioms complete the specification of the structure by imposing structural relations on the operations.

A.9. Functoriality and Polymorphism

Many Kleis expressions, particularly polymorphic type constructors, correspond to functors between categories of structures.

For a type operator such as:

```
Matrix(m,n,T)
```

we obtain a mapping:

$$\mathsf{Matrix}_{m,n} : \mathscr{U} \to \mathscr{U},$$

which, when restricted to appropriate categories of structures, becomes a covariant functor.

Polymorphic Kleis functions correspond to families of functions natural with respect to such functorial actions.

A.10. Summary

In this appendix, we have identified the principal constructs of the Kleis language with the corresponding notions in Bourbaki's theory of structures:

Kleis Construct	**Bourbaki Concept**
`structure`	species of structure
`extends`	morphism of species / species inclusion
`over`	parametrized species
`implements`	structure of a given species
`where`	subspecies via conditions
nested structures	composite species / associated structures
algebraic data types	free or inductive structures
polymorphism	functorial dependence

Thus, Kleis may be regarded as a formal, executable language for defining and manipulating Bourbaki-style mathematical structures.