

Even Higher Algebraic Structures

Modules, Algebras, Group Actions, and Representations

Kleis Language Project

1 Higher Algebraic Structures: Modules, Algebras, Group Actions, and Representations

This section develops the next layer of abstract algebra above groups, rings, fields, and vector spaces. The structures introduced here provide the algebraic foundation for linear algebra, representation theory, and many constructions appearing in the semantic foundations of Kleis.

1.1 Modules Over Rings

Definition 1 (R -module). *Let R be a ring. An R -module is an abelian group $(M, +)$ together with a scalar multiplication*

$$\cdot : R \times M \rightarrow M$$

such that, for all $r, s \in R$ and $x, y \in M$,

$$\begin{aligned} r \cdot (x + y) &= r \cdot x + r \cdot y, \\ (r + s) \cdot x &= r \cdot x + s \cdot x, \\ (rs) \cdot x &= r \cdot (s \cdot x), \\ 1 \cdot x &= x, \end{aligned}$$

where 1 denotes the multiplicative identity of R .

Remark 1. *Modules generalize vector spaces by allowing the scalars to come from an arbitrary ring rather than a field. When R is a field, an R -module is precisely a vector space over R .*

1.2 Algebras Over a Ring or Field

Definition 2 (Algebra over a ring). *Let R be a ring. An R -algebra is an R -module $(A, +, \cdot)$ together with a bilinear multiplication*

$$* : A \times A \rightarrow A$$

*such that $(A, *, 1)$ is a ring and scalar multiplication in A is compatible with multiplication:*

$$r \cdot (a * b) = (r \cdot a) * b = a * (r \cdot b).$$

Example 1.

1. The polynomial ring $R[x]$ is an R -algebra.
2. The matrix ring $M_n(R)$ is an R -algebra.
3. Over a field F , any associative algebra (Lie, Clifford, tensor, exterior, symmetric) is an F -algebra.

1.3 Group Actions

Definition 3 (Group action). Let G be a group and X a set. A (left) action of G on X is a function

$$\alpha : G \times X \rightarrow X \quad \text{denoted } g \cdot x,$$

such that, for all $g, h \in G$ and $x \in X$,

$$\begin{aligned} e \cdot x &= x, \\ (gh) \cdot x &= g \cdot (h \cdot x), \end{aligned}$$

where e is the identity of G .

Remark 2. A group action is a homomorphism $G \rightarrow \text{Bij}(X)$, the group of all bijections of X .

1.4 Representation Theory

Definition 4 (Linear representation). Let G be a group and F a field. A representation of G on a vector space V over F is a group homomorphism

$$\rho : G \rightarrow \text{GL}(V),$$

where $\text{GL}(V)$ denotes the group of invertible linear transformations of V .

Example 2.

1. The trivial representation: $\rho(g) = \text{id}$.
2. The permutation representation of S_n on \mathbb{R}^n .
3. The adjoint representation of a Lie group on its Lie algebra.

Remark 3. Representations allow groups to be studied through the linear algebra of vector spaces. Much of modern algebra, geometry, and quantum physics is based on the interaction between groups and their representations.

1.5 Structural Overview

The higher algebraic hierarchy extends the basic tower:

$$\text{Magma} \subset \text{Semigroup} \subset \text{Monoid} \subset \text{Group} \subset \text{Ring} \subset \text{Field}.$$

Above these, one obtains:

$$\text{Field} \implies \text{Vector Space}, \quad \text{Ring} \implies \text{Module}, \quad \text{Module} \implies R\text{-Algebra}.$$

Group actions and group representations provide the fundamental means by which algebraic symmetry is expressed within linear structures.

1.6 Interpretation in Kleis

The structures introduced above admit direct specification in Kleis. For instance:

```
structure Module(M) over Ring(R) {
    operation (+) : M × M → M
    operation (·) : R × M → M
}

structure Algebra(A) over Ring(R) {
    structure module : Module(A) over Ring(R)
    operation (*) : A × A → A
    axiom bilinear:
        (r : R, x y : A).
        r · (x * y) = (r · x) * y = x * (r · y)
}

structure Representation(G, V) over Field(F) {
    operation act : G × V → V
    axiom compatibility:
        (g h : G, v : V).
        act(g · h, v) = act(g, act(h, v))
}
```