Algebraic Foundations

Magma, Semigroup, and Monoid

Kleis Language Project

1 Algebraic Foundations: Magma, Semigroup, and Monoid

This appendix recalls the universal–algebraic hierarchy underlying many of the core abstractions in Kleis. The presentation follows the style of Bourbaki and classical algebraic specification.

1.1 Magma

Definition 1 (Magma)**.** *A* magma *is a pair* $(M, \star)$ *consisting of a set* M *together with a binary operation*

$$\star : M \times M \to M.$$

No algebraic axioms are required: in particular, associativity and identity need not hold.

Example 1. *Let* $M = \{1, 2, 3\}$ *and define* $x \star y = x$ *for all* $x, y \in M$*. Then* $(M, \star)$ *is a magma which is neither associative nor admits an identity element.*

1.2 Semigroup

Definition 2 (Semigroup)**.** *A* semigroup *is a magma* $(S, \star)$ *whose operation is associative:*

$$(x \star y) \star z = x \star (y \star z) \qquad \text{for all } x, y, z \in S.$$

Remark 1. *A semigroup may fail to have a neutral element. Thus the usual notion of a* fold *over an empty list is not available for arbitrary semigroups.*

1.3 Monoid

Definition 3 (Monoid)**.** *A* monoid *is a semigroup* $(M, \cdot)$ *together with a distinguished element* $e \in M$*, called the* identity*, such that*

$$e \cdot x = x \qquad \text{and} \qquad x \cdot e = x \qquad \text{for all } x \in M.$$

Thus a monoid is a triple $(M, \cdot, e)$ *satisfying associativity and both left and right identity laws.*

Example 2 (Classical monoids)**.**

1. $(\mathbb{N}, +, 0)$ *is a monoid.*

2. $(\textit{Strings}, \textit{concatenation}, \varepsilon)$ *is a monoid, where* ε *is the empty string.*

3. *For any set* X*, the endofunctions* X^X *with composition form a monoid; the identity element is* id_X.

1.4 Relationship Between the Structures

Every monoid is a semigroup, and every semigroup is a magma. The converse implications do not hold. In summary:

$$\text{Monoid} \subset \text{Semigroup} \subset \text{Magma}.$$

1.5 Categorical Interpretation

Proposition 1. *A monoid is precisely a category with a single object.*

Proof. Let $\mathcal{C}$ be a category with one object $*$. Its morphisms $\mathrm{Hom}(*, *)$ carry a natural associative composition, and the identity morphism id_* is a unit. Conversely, given a monoid $(M, \cdot, e)$, construct a category with one object whose endomorphisms are the elements of M and whose composition is $\cdot$. □

1.6 Interpretation in Kleis

The algebraic hierarchy of Kleis mirrors the universal–algebraic definitions above. In Kleis, the corresponding specifications are:

```
structure Magma(M) {
  operation (•) : M × M → M
}

structure Semigroup(S) extends Magma(S) {
  axiom associativity:
    (x y z : S). (x • y) • z = x • (y • z)
}

structure Monoid(M) extends Semigroup(M) {
  element e : M
  axiom left_identity:  (x : M). e • x = x
  axiom right_identity: (x : M). x • e = x
}
```

Thus the standard fold and accumulation operators become available precisely at the level of monoids, in agreement with both algebraic and category-theoretic semantics.