Lie Algebras and Tensor Algebras

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1 Lie Algebras and Tensor Algebras

This section develops two fundamental algebraic structures which lie above vector spaces and associative algebras: Lie algebras, which capture infinitesimal symmetry, and tensor algebras, which generate the free associative algebra over a vector space. Both constructions play a central role in geometry, representation theory, and modern mathematical physics.

1.1 Lie Algebras

Definition 1 (Lie algebra)**.** *Let F be a field. A* Lie algebra *over F is a vector space $\mathfrak{g}$ together with a bilinear map*

$$[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

called the Lie bracket*, satisfying:*

1. ***Antisymmetry:***

$$[x, y] = -[y, x] \qquad \text{for all } x, y \in \mathfrak{g};$$

2. ***Jacobi identity:***

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \text{for all } x, y, z \in \mathfrak{g}.$$

Example 1. *Let A be an associative algebra over F. The commutator*

$$[x, y] = xy - yx$$

defines a Lie algebra structure on the underlying vector space of A.

Example 2. *The vector space of $n \times n$ matrices $\mathfrak{gl}_n(F)$ with the commutator bracket is a Lie algebra. The subspace of trace-zero matrices $\mathfrak{sl}_n(F)$ is also a Lie algebra.*

Structure Constants

Let $(e_1, \dots, e_n)$ be a basis of $\mathfrak{g}$. Then the bracket is determined by constants $c_{ij}^k \in F$ such that

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k.$$

The Jacobi identity is equivalent to the relations

$$\sum_{m=1}^{n} \left(c_{ij}^m c_{mk}^\ell + c_{jk}^m c_{mi}^\ell + c_{ki}^m c_{mj}^\ell \right) = 0 \qquad \text{for all } i, j, k, \ell.$$

1.2 Representations of Lie Algebras

Definition 2 (Representation)**.** *Let $\mathfrak{g}$ be a Lie algebra and V a vector space over F. A* representation *of $\mathfrak{g}$ on V is a linear map*

$$\rho : \mathfrak{g} \to \mathrm{End}(V)$$

such that

$$\rho([x, y]) = [\, \rho(x), \rho(y) \,] = \rho(x)\rho(y) - \rho(y)\rho(x).$$

Example 3. *The* adjoint representation *is defined by*

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \qquad \mathrm{ad}(x)(y) = [x, y].$$

1.3 Tensor Algebras

Definition 3 (Tensor algebra)**.** *Let V be a vector space over F. The* tensor algebra *of V is the graded vector space*

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n},$$

with $V^{\otimes 0} = F$, equipped with the associative multiplication

$$(v_1 \otimes \cdots \otimes v_m) \; * \; (w_1 \otimes \cdots \otimes w_n) \; = \; v_1 \otimes \cdots \otimes v_m \otimes w_1 \otimes \cdots \otimes w_n.$$

Remark 1. *$T(V)$ is the* free associative algebra *generated by V. That is, for any associative algebra A and linear map $\varphi : V \to A$, there exists a unique algebra homomorphism*

$$\widetilde{\varphi} : T(V) \to A$$

extending φ.

1.4 Exterior and Symmetric Algebras

The tensor algebra contains two fundamental quotient algebras:

Definition 4 (Exterior algebra)**.** *The* exterior algebra $\Lambda(V)$ *is the quotient of* $T(V)$ *by the ideal generated by*

$$v \otimes v \qquad (v \in V).$$

The induced product is denoted $\wedge$ *and is antisymmetric.*

Definition 5 (Symmetric algebra)**.** *The* symmetric algebra $S(V)$ *is the quotient of* $T(V)$ *by the ideal generated by*

$$v \otimes w - w \otimes v.$$

It is the free commutative algebra generated by V*.*

Remark 2. *These constructions underlie multilinear algebra, differential forms, and polynomial algebras. They also serve as foundations for Clifford algebras and spin geometry.*

1.5 The Universal Enveloping Algebra

Definition 6 (Universal enveloping algebra)**.** *For a Lie algebra* $\mathfrak{g}$*, the* universal enveloping algebra $U(\mathfrak{g})$ *is the quotient of the tensor algebra* $T(\mathfrak{g})$ *by the ideal generated by the relations*

$$x \otimes y - y \otimes x - [x, y].$$

Remark 3. $U(\mathfrak{g})$ *provides the bridge between Lie algebras and associative algebras, and plays a central role in representation theory.*

1.6 Interpretation in Kleis

The structures above naturally admit the following Kleis specifications:

```
structure LieAlgebra(g) over Field(F) {
  operation bracket : g × g → g
  axiom antisymmetry:
    (x y : g). bracket(x,y) = - bracket(y,x)
  axiom jacobi:
    (x y z : g).
     bracket(x, bracket(y,z))
```

```
    + bracket(y, bracket(z,x))
    + bracket(z, bracket(x,y)) = 0
}

structure TensorAlgebra(T) over VectorSpace(V) {
  operation tensor : V × V → V V
  operation multiply : T × T → T
  axiom associativity:
     (a b c : T). multiply(multiply(a,b),c)
                 = multiply(a,multiply(b,c))
}
```

These structures provide the mathematical foundations for differential operators, representation theory, and multilinear constructions within the Kleis language.