Final Homework

This homework is due to June 11th 2018

1. Consider the one-dimensional, transient (i.e. time-dependent) heat conduction equation without heat generating sources

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right),$$

where ρ is density, c_p heat capacity, k thermal conductivity, T temperature, x distance, and t time. If the thermal conductivity, density and heat capacity are constant over the model domain, the equation can be simplified to

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

where

$$\kappa = \frac{k}{\rho c_p}$$

is the thermal diffusivity (a common value for rocks is $\kappa = 10^{-6} m^2/s$). We are interested in the temperature evolution versus time, $T(x,t)$, given an initial temperature distribution Figure.1. An example would be the intrusion of a basaltic dike in cooler country rocks. The country rock has a temperature of $300°C$ and the dike a total width $W = 5m$, with a magma temperature of $1200°C$. In addition we assume that the temperature far away from the dike center (at $|L/2|$ where $L = 100m$) remains at a constant temperature, $300°C$.

Let the $T_d(t)$ be maximum temperature of dike when time is t.

- Plot $T_d(t)$ graph by using natural cubic spline interpolation.
- Find the time t_1 such $T_d(t_1) = 600°C$ by using bisection, Newton's method or secant method.

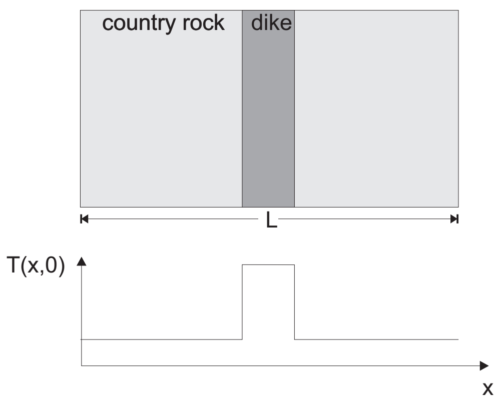

Figure 1: Setup of the thermal cooling model considered here

2. The electric field is related to the charge density by the divergence relationship

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

where E is electric field, ρ charge density, and ϵ_0 is permittivity. The electric field is related to the electric potential by a gradient relationship,

$$E = -\nabla\Phi.$$

Therefore the potential is related to the charge density by Poisson's equation

$$\nabla \cdot \nabla\Phi = \Delta\Phi = -\frac{\rho}{\epsilon_0}.$$

In a charge-free region of space, this becomes Laplace's equation,

$$\Delta\Phi = 0.$$

The walls of a rectangular cylinder are constrained in potential as shown in 2. The walls at $x = 0, a$ and $y = 0$ have zero potential, while those at $y = b$ have the potential distributions V, respectively. The region inside the cylinder is free space.

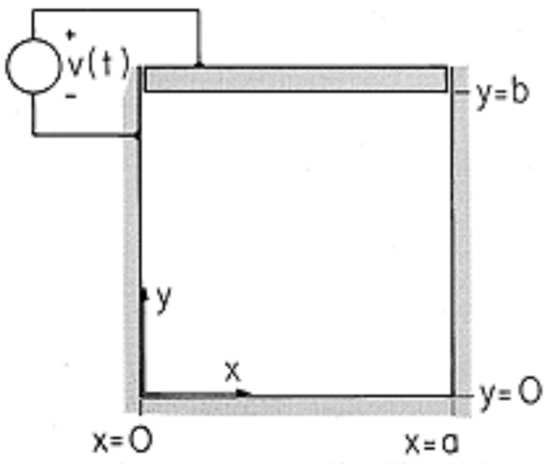

Figure 2: Setup of the rectangular cylinder.

Assume that $a = b = 1$ and $V = 1$. Determine the potential in the free region inside the cylinder.

- Plot the potential $\Phi(x, y)$ and error, $|\Phi(x, y) - \Phi_e(x, y)|$, where the exact potential distribution is

$$\Phi_e = \sum_{n=1,\text{odd}}^{\infty} \frac{4V}{n\pi} \frac{\sinh \frac{n\pi}{a} y}{\sinh \frac{n\pi b}{a}} \sin \frac{n\pi x}{a}.$$

  and $0 < x, y < 1$.
- Plot h-error graph in log-log scale with $h = 1/10, 1/20, 1/40, 1/80$, where the error is

$$e = |\Phi(0.5, 0.8) - \Phi_e(0.5, 0.8)|.$$

- Find convergence rate by using least squares method.