***Black Path*: Game Analysis**

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MATH-321: Introduction to Game Theory

Introduction

Black Path (originally just named *Black*) is a two-player topological, combinatorial game invented in 1960 by William "Larry" Black as an MIT undergraduate. *Black Path* was later introduced to the public in 1963 by Matin Gardner in his *Scientific American* column, "Mathematical Games" [MG1, MG2].

Black Path is a topological game, which means that players construct paths that twist and turn across the board or field. Larry Black and his friends drew upon their studies of other topological games—namely *Bridg-It* and *Hex*—to ultimately develop *Black Path* [MG1]. In *Bridg-It* and *Hex*, the players are also connecting lines from one edge of the board to another. However, in *Black Path* the goal is to avoid "falling off" the edge of the board, whereas in the other two games, the goal is to be the first to connect the edges.

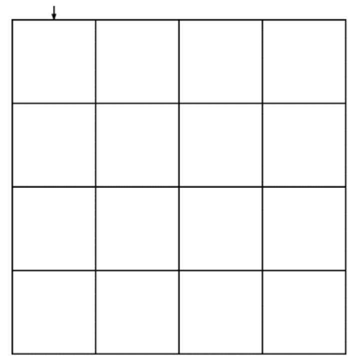
Black Path is interesting to study because it requires a different kind of strategy than *Bridg-It* and *Hex* to avoid the edges rather than to aim for them. It is also interesting to see that the placement of tiles does not only affect where the path goes, but the unused path segments can be either a detriment or an advantage when considering future plays.

In terms of modern-day games, *Black Path* has inspired the game *Tsuro*. *Tsuro* is a multi-player board game with similar rules, except each player choses a pattern to play each move from a private hand of cards drawn at random from a deck. This makes *Black Path* fun to analyze because some winning principles of the combinatorial game could potentially provide some insight into the probabilistic version that is *Tsuro*.

Finally, this analysis of *Black Path* will be modifying the role of the starting arrow slightly. In the originally published game, the starting arrow is fixed, pointing down, above the top-left square on the board. In the version of the game analyzed here, the starting arrow is part of the initial configuration of the board, along with the dimensions of the board. This adjustment was made to allow for more general analysis of the game than those that have been done prior.

Setup and Gameplay

<u>The Board</u>

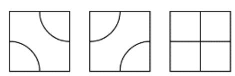


The game is played on a rectangular board that is divided into squares. The board may be of any dimensions. If no move has been played on a square, then the square is blank. Otherwise, the square will contain one of the legal patterns. Further, one of the border edges is marked with a starting arrow. This arrow is continued by the path composed of the patterned squares.

To the right is an example of an empty board with the top-left square designated as the starting position.

<u>Player Movement</u>

Each of the legal patterns consists of two lines connecting unique edges. As such, there are three legal patterns available to both players, namely the two quarter-circle Truchet tiles and one cross.

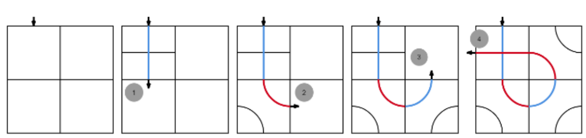
Here are illustrations of the legal patterns: 

The current player must extend the current path by playing one of the legal tiles onto the square indicated by the arrow.

<u>End Game Conditions</u>

The current player loses if they play a pattern that causes the path to run off any edge of the board.

<u>Sample Game</u>

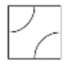
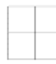


Game Analysis

<u>Black Path Notation</u>

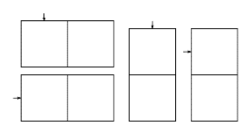
For the following analysis, the following notation will be used.

- Let r refer to the number of rows in the game board.
- Let c refer to the number of columns in the game board.
- Let $d \in \{N, E, W, S\}$ refer to one of the cardinal directions (North, East, West, South).

- Let $S(i,j)$ refer to the square in row i and column j with $S(1,1)$ referring to the top-left square.
- Let $E(i,j,d)$ refer to the edge on $S(i,j)$ in direction d.
- Let $B(r,c,E(i,j,d))$ refer to the starting board of size $r \times c$ with the starting arrow on $E(i,j,d)$.
- Let $P(NE,SW)$ refer to the pattern: 
- Let $P(NW,SE)$ refer to the pattern: 
- Let $P(NS,EW)$ refer to the pattern: 

<u>Simple Game Positions</u>

Firstly, consider the simplest starting board: a single square. Of course, no matter where the starting arrow is placed, the first player is forced into playing on the only square, and the second player wins immediately.



Now, consider the starting boards consisting of exactly two squares: $\{B\big(1,2,E(1,1,N)\big), B\big(1,2,E(1,1,E)\big), B\big(2,1,E(1,1,N)\big), B\big(2,1,E(1,1,E)\big)\}$. No matter the board, the first player's only rational move is to extend the path from the first square to the second. They may do this by playing the appropriate Truchet tile when the starting arrow is at $E(1,1,N)$ or by playing $P(NS,EW)$ when the starting arrow is at $E(1,1,E)$. Now, on the second player's turn, the game has been reduced to the single square game, and the second player loses.

The argument can be extended for the starting boards consisting of exactly three squares. For a rectangular board to have three squares, it must be either 1×3 or 3×1.

If the starting arrow is on a corner square, then the first player has the same rational options as in the two-square games. This means that after the first player's turn, the three-square game has been reduced to the two-square game, making it a second-player win. In fact, of the legal patterns, the cross is the only rational move for the second player from this position.

However, if the starting arrow is on the middle square, then the first player is given a choice in the direction to go. For the three-square game, either direction will force a loss for player two because there is no possibility for paths to cross back on a single line of squares. This means that $B\big(1,3,E(1,2,-)\big) \cong B\big(1,2,E(1,1,-)\big)$ and $B\big(3,1,E(2,1,-)\big) \cong B\big(3,1,E(1,1,-)\big)$.

From this pattern, it is clear to see that the outcome of a $1 \times c$ or $r \times 1$ board is dependent on the number squares in the path as well as the location of the starting arrow. If the starting arrow is not in one of the corners, then it functionally turns an even-length path into an odd-length path. So,

$$B(r, 1, -) \in \begin{cases} \mathcal{N} \text{ if } r \text{ is even and starting arrow is in corner} \\ \mathcal{N} \text{ if } r \text{ is odd and starting arrow is in middle} \\ \mathcal{P} \text{ otherwise} \end{cases}$$

$$B(1, c, -) \in \begin{cases} \mathcal{N} \text{ if } c \text{ is even and starting arrow is in corner} \\ \mathcal{N} \text{ if } c \text{ is odd and starting arrow is in middle} \\ \mathcal{P} \text{ otherwise} \end{cases}$$

Further, it can be further concluded that, apart from the first move and potentially the last move, the only rational tile to play when playing on a single row or column is the cross tile, $P(NS, EW)$.

Rectangular Board Positions: Fixed Starting Arrow

The following are descriptions and expansions of the winning strategies presented in *Winning Ways for your Mathematical Plays* [WW]. Note that as in the original game, the starting arrow is fixed at $E(1,1,N)$. Let the starting board be $B(r, c, E(1,1,N))$.

When the starting board has at least one even dimension—and therefore has an even number of squares—the first player has a winning strategy using the *One Hand Tied Principle*. The first player should partition the board into dominoes (1×2 or 2×1 rectangles). This is trivial to do. If there are an even number of rows, then they can partition the board into all vertical dominoes. Otherwise, they can partition the board into all horizontal dominoes.

Then, the first player proceeds with playing as though they are playing multiple, disconnected two-square games. Since it has been shown in the previous section that the two-square game is always a first player win, the even-square game is also a first player win.

When the starting board has no even dimension—and therefore has an odd number of squares— and the second player is able to partition the board into dominoes, the second player can steal the above *One Hand Tied* strategy. It now remains to be shown when the second player is able to produce such a partitioning.

Because the starting square is the top-left corner, the first player's move splits that row/column into even paths, which can be easily partitioned. Then, the rectangle composed of the remaining rows/columns has at least one even dimension, so they can use the same partitioning method as the first player on an even-square board.

Therefore,

$$B(r, c, E(1,1,N)) \in \begin{cases} \mathcal{N} \text{ if one of } r, c \text{ is even} \\ \mathcal{P} \text{ if both } r, c \text{ are odd} \end{cases}$$

Rectangular Board Positions: General Case

Now, consider the general rectangular *Black Path* game where the starting arrow may start on any outside edge of the board. To the best of the author's knowledge, the following expansions made to the analysis by Berlekamp et. al [WW] are new and original.

The first observation is that *Black Path* essentially reduces to a mutilated chessboard domino tiling problem. Namely, the winner of the game is the player who is able to produce a domino tiling of the board. Because the game board is a grid of squares, a simple application of Gomory's Theorem will be sufficient to solve any *Black Path* game.

According to Gomory's Theorem, if one black square and one white square are removed from anywhere on an 8×8 chessboard, then the remaining board can be covered with 31 dominoes [MCP]. A visual demonstration of Gomory's Theorem can be found on the Wolfram Demonstrations Project[1]. Otherwise, if two squares of the same color are removed, then a domino tiling of the board is impossible.

Now, apply a chessboard coloring to the starting *Black Path* game board, with the top-left square colored white. It can be seen that as long as there are an even number of squares, then there is a domino tiling. Therefore, a starting board with an even number of squares is a first player win via the strategy introduced in the previous section.

If there are an odd number of squares, then the game outcome depends on the color of the starting square. Notice that because the starting square is white, there will be one extra white square on a starting board with an odd number of squares. Then, if the starting square is white, the second player

[1] https://demonstrations.wolfram.com/GomorysTheorem/

starts with an equal number of black and white squares and has a winning strategy. Otherwise, the starting square is black, and the second player starts with two extra white squares. Neither the first now the second player are able to produce a domino tiling, and there is no clear winning strategy.

Conclusion

It is interesting to see how *Black Path* reduces to chessboard coloring and domino tiling. The final outcomes for the general game of *Black Path* are as follows:

$$B(r, c, E(i, j, -)) \in \begin{cases} \mathcal{N} \text{ if one of } r, c \text{ is even} \\ \mathcal{P} \text{ if both } r, c \text{ are odd and } S(i, j) \text{ is white} \\ ? \text{ if both } r, c \text{ are odd and } S(i, j) \text{ is black} \end{cases}$$

While Gomory's Theorem provides a simple proof of a winning strategy for 75% of the starting configurations, the case of an odd board with a black starting square is still ambiguous.

Further analysis of this game might find inspiration in Berlekamp's solution to an original version of *Black Path* as published in [MG2, p. 153-154]. In his analysis, Berlekamp introduces the concept of a "split domino" composed of two same-colored squares connected by previously placed tiles. By utilizing the "split domino", the second player wins the original game even with two extra squares. However, Berlekamp's solution cannot directly be applied to the version of *Black Path* discussed here. The simplest counterexample is the 3-square path with a middle (black) starting square. This game is clearly a first player win, and the second player has no chance to take advantage of their "split domino".

Additional analysis of *Black Path* may also look at other variations of the game—such as playing with hexagonal tiles, or using a different set of legal tiles.

Source Code

A simple MATLAB implementation of the Black Path game can be found on the author's GitHub[2].

[2] https://github.com/eaw8044/black-path-game

References

[MCP] Watkins, John J. “Chapter One: Introduction.” *Across the Board the Mathematics of Chessboard Problems*, Princeton University Press, Princeton, NJ, 2004, pp. 1–24.

[MG1] Gardner, Martin. “MATHEMATICAL GAMES.” *Scientific American*, vol. 209, no. 4, Scientific American, a division of Nature America, Inc., 1963, pp. 124–32, http://www.jstor.org/stable/24936325.

[MG2] Gardner, Martin. “MATHEMATICAL GAMES.” *Scientific American*, vol. 209, no. 5, Scientific American, a division of Nature America, Inc., 1963, pp. 144–58, http://www.jstor.org/stable/24935942.

[WW] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. *Winning Ways for your Mathematical Plays*. A K Peters, Ltd, Natick, MA, second edition, 2001. First edition published in 1982 by Academic Press.