THE ASTROPHYSICAL JOURNAL, 217: 425–433, 1977 October 15

THE SIZE DISTRIBUTION OF INTERSTELLAR GRAINS

JOHN S. MATHIS, WILLIAM RUMPL, AND KENNETH H. NORDSIECK
Washburn Observatory, University of Wisconsin–Madison
Received 1977 January 24; accepted 1977 April 11

ABSTRACT

The observed interstellar extinction over the wavelength range $0.11\ \mu m < \lambda < 1\ \mu m$ was fitted with a very general particle size distribution of uncoated graphite, enstatite, olivine, silicon carbide, iron, and magnetite. Combinations of these materials, up to three at a time, were considered. The cosmic abundances of the various constituents were taken into account as constraints on the possible distributions of particle sizes.

Excellent fits to the interstellar extinction, including the narrowness of the λ2160 feature, proved possible. Graphite was a necessary component of any good mixture, but it could be used with any of the other materials. The particle size distributions are roughly power law in nature, with an exponent of about −3.3 to −3.6. The size range for graphite is about 0.005 μm to about 1 μm. The size distribution for the other materials is also approximately power law in nature, with the same exponent, but there is a narrower range of sizes: about 0.025–0.25 μm, depending on the material. The number of large particles is not well determined, because they are gray. Similarly, the number of small particles is not well determined because they are in the Rayleigh limit. This power-law distribution is drastically different from an Oort–van de Hulst distribution, which is much more slowly varying for small particles but drops much faster for particles larger than average.

The extinction was also fitted with spherical graphite particles plus cylinders of each of the other materials. Linear and circular polarizations were then determined for the cylinders on the assumption of Davis-Greenstein alignment. The extinction was quite satisfactory, but the linear polarization reached a maximum in the ultraviolet (about 1600 Å). This is because the mixture contains many small particles. If the small particles are not elongated or aligned, the wavelength dependence of the polarization can be fitted, but the larger particles which are aligned do not provide enough polarization per magnitude of extinction. However, a fit to polarization and extinction can be achieved if the material responsible for the polarization contributes only a small part of the extinction but consists of fairly large particles and is very well aligned. Dielectric particles with coatings could also provide the polarization.

Subject headings: interstellar: matter — polarization — ultraviolet: spectra

I. INTRODUCTION

The wavelength dependence of the interstellar extinction is remarkably uniform over the sky (Bless and Savage 1972; York *et al.* 1973; Nandy *et al.* 1976). The wavelength dependence of the linear polarization is much more variable, but is a simple function of (λ/λ_{max}), where λ_{max} is the wavelength of maximum polarization (generally about 5500 Å). In only a few directions, such as the Orion or ρ Oph regions, does the extinction differ markedly from the "standard" law. In these regions the linear polarization is also peculiar (Serkowski, Mathewson, and Ford 1976). The circular polarization (Martin 1974, 1975) has been observed for comparatively few objects. Its most interesting property is a change in signs, occurring at about λ_{max}.

There have been dozens of papers throughout the years on interpreting the extinction of dust in terms of various constituents. Most of these have necessarily dealt with the spectral region accessible from the ground. This paper will discuss fitting the extinction data from $\lambda = 1\ \mu m$ to $\lambda = 0.11\ \mu m$.

We will concentrate on the extinction rather than the polarization data for two reasons. First, the observed extinction, if normalized, is well observed and uniform, while polarization is much more variable. Second, extinction can be calculated with much more confidence than can the linear polarization, which in turn is more predictable than the circular polarization. We discuss our method of fitting the extinction in § II and the results of such fitting in § III. In § IV we discuss polarization rather briefly.

II. TECHNIQUES OF FITTING EXTINCTION

We wish to find a suitable size distribution of particles of various materials: graphite; silicon carbide, SiC; enstatite, (Fe, Mg)SiO_3; olivine, $(Fe, Mg)_2SiO_4$; iron; and magnetite, Fe_3O_4. Other materials such as magnesium carbonates (Gillett, Forrest, and Merrill 1973) could be added, but we feel that most features of the problem will emerge by considering the six substances already mentioned. If the complex indices of refraction of each material are known, the extinction cross section of a sphere or an infinite cylinder of a given size are readily determined (Greenberg 1968). The net extinction is then an integral over the size distribution $n(a)$ of each material, where $n(a)da$ is the number of particles in the interval $(a, a + da)$. There is no reliable theoretical way of predicting $n(a)$; and we will assume a very general distribution, with many variable parameters, and minimize the squared deviations of the total predicted extinction from the observed extinction.

We consider the size distribution of each material to be a linear combination of simple size distributions which we refer to as "bins." We divide the size interval from 0 to 1 μm into discrete sizes a_j. The jth bin includes a uniform distribution of sizes in the interval a_{j-1} to a_j. We then have, for the extinction of the jth bin at the ith wavelength for the mth material

$$C_{mji} = \int_{a_{j-1}}^{a_j} \hat{n}_j(a) C_m(a, \lambda_i) da \,,$$

$$\hat{n}_j(a) = 1\ \mu\text{m}^{-1} \quad (a_{j-1} < a < a_j) \,, \tag{1}$$

$$\hat{n}_j(a) = 0 \,, \quad \text{otherwise} \,,$$

where $C_m(a, \lambda)$ is the extinction cross section of the material m, at wavelength λ_i, appropriate to the shape (sphere or cylinder) we are considering. The bin sizes were chosen to completely cover the range of sizes $0 < a < 1$ μm, in 18 steps, such that $a_j \approx 1.5 a_{j-1}$, $a_1 = 0.0025$ μm, and $a_0 = 0$. Several slightly different choices of a_j were made to study effects of the choices on the final distribution. The observed interstellar extinction is compared to a predicted extinction, which is a linear combination of the extinctions of all bins of each material we wish to consider. The linear combination is chosen to achieve the best fit, in a certain sense to be defined below. The extinction predicted for the mixture at the wavelength λ_i is

$$C_{\text{pred},i} = \sum_m \sum_j X_{mj} C_{mji} \,, \tag{2}$$

where the coefficients X_{mj} have been chosen to minimize a quantity χ^2 defined as follows:

$$\chi^2 = \sum_i (C_{\text{pred},i} - C_{\text{obs},i})^2 / (\sigma_i C_{\text{obs},i})^2 \,. \tag{3}$$

The quantities σ_i have the formal role of relative errors associated with the observed extinction at λ_i, but were in fact made artificially small at some wavelengths in order to force a fit of the calculated extinction there, as will be discussed below.

There are many more parameters to be chosen, X_{mj}, than there are extinction points to be fitted, $C_{\text{obs},i}$. The underdetermination is only apparent, because the choice of the X_{mj} is subject to constraints. Clearly X_{mj} must be nonnegative,

$$X_{mj} > 0 \,, \tag{4a}$$

and, very importantly, the number of the various atoms used by the materials cannot exceed the cosmic abundances of each element. If the element E has an atomic abundance relative to hydrogen of A_E, and the molecule of constituent m contains N_{Em} atoms of E, then

$$A_E \geq \sum_m N_{Em} \sum_j X_{mj} V_j \rho_m (\mu_m m_{\text{H}})^{-1} \,, \tag{4b}$$

where ρ_m is the density of the material, μ_m is the molecular weight of molecule m, m_{H} is the mass of the hydrogen atom, and V_j is the total material volume associated with the jth bin:

$$V_j = \int_{a_{j-1}}^{a_j} \hat{n}_j(a) v(a) da \,. \tag{5}$$

Here $v(a) = 4\pi a^3/3$ for spheres and πa^2 for cylinders (the extinction is calculated per unit length for the latter).

The abundances of C, Mg, Si, and Fe, relative to 10^6 H atoms, were taken from Cameron (1973) to be 370, 33, 31, and 26, respectively. Oxygen is so abundant that it cannot be locked up in uncoated grains. The column density of H atoms per magnitude of color excess E_{B-V} was taken from Jenkins and Savage (1974) to be 7.5×10^{21} atoms mag^{-1} cm^{-2}. $C_{\text{obs},i}$ was normalized to the number of optical depths of extinction per H atom which gives an extinction difference between λ4350 and λ5500 of $(1.086 \times 7.5 \times 10^{21})^{-1} = 1.22 \times 10^{-22}$. The wavelength dependence of $C_{\text{obs},i}$ was taken from two sources: Code *et al.* (1976), which will be referred to as the OAO extinction, and from Nandy *et al.* (1975), which will be called the TD-1 extinction. They are in fact quite similar.

Table 1 gives the 16 wavelengths used, and the values of the uncertainty parameters, σ_i, associated with each wavelength. The σ_i's are smaller than the differences between the OAO and TD-1 extinctions and should not be interpreted as measures of the observational errors on $C_{\text{obs},i}$. Instead, they were chosen to be artificially small (0.001) at the wavelengths of 2000, 2160, 2300, and 2500 Å in order to force the computed extinction to provide an excellent fit to the width and position of the λ2160 bump, both of which are very well determined (Savage 1975). This procedure of normalizing the predicted values to the bump by means

TABLE 1

ADOPTED WAVELENGTHS AND WEIGHTS*

λ (Å)	σ_i*	λ (Å)	σ_i*
1100.........	0.15	2500.........	0.001
1300.........	0.15	3000.........	0.03
1430.........	0.10	3460.........	0.01
1670.........	0.02	4350.........	0.01
1800.........	0.005	4861.........	0.01
2000.........	0.001	5555.........	0.01
2160.........	0.001	6563.........	0.01
2300.........	0.001	10000.........	0.01

* For definition, see discussion after eq. (3).

of small σ's of course tends to make χ^2 artificially large; hence a fit is acceptable if $\chi^2 < 100$, say, instead of the usual criterion of requiring χ^2 to be not much larger than the number of wavelength points fitted. The rather large values of σ_i associated with 1300 Å and 1100 Å arise from the fact that the extinction does vary among stars at these wavelengths.

The technique for minimizing χ^2, subject to linear constraints like the inequalities (4), is a fairly well-known process known as quadratic programming. It is described briefly in Faber (1972). The actual algorithm used in this paper is described in Cottle (1968) and Cottle and Danztig (1968).

In order to calculate the C_{mij} from equation (1), one needs the indices of refraction of each substance at each wavelength. For graphite, the index is different for light with electric vector parallel to the basal plane than for light polarized with the electric vector perpendicular to the basal plane. Constants for the perpendicular direction were taken from Taft and Philipp (1965), Tosatti and Bassani (1970), and others. They will be discussed further in the next section. The extinction cross sections computed with these constants we will call $C_{\mathrm{Gr,per}}$. The optical constants used when the electric vector is parallel to the basal plane were taken from Tosatti and Bassani (1970). The extinction cross section computed with these values will be called $C_{\mathrm{Gr,par}}$. For the extinction of a graphite particle with incident unpolarized radiation, we used $(2C_{\mathrm{Gr,per}} + C_{\mathrm{Gr,par}})/3$, which is appropriate for small-sized ($a \ll \lambda/2\pi$) particles (Greenberg 1966). This relation becomes increasingly inexact as the size is increased, but to our knowledge no extinction cross sections are available for finite-sized spheroids of anisotropic material. In view of the necessarily approximate nature of our calculated graphite extinction, we feel that consideration of such effects as temperature dependence of optical constants is not warranted.

The other materials all have isotropic optical properties, although of course the shape of the cylinders renders their extinction dependent upon the polarization of the incident radiation. For Si C we used the constants from Thibault (1944) and Philipp and Taft (1960). For enstatite, we used those of Huffman and Stapp (1973); for iron, those of Wickramasinghe and Nandy (1971).

III. RESULTS OF FITTING EXTINCTION

In this section we consider the results of fitting the extinction with combinations of spherical particles of the various materials mentioned earlier. Cylinders are discussed in § IV. Let us denote graphite by the symbol C; olivine, by Ol; enstatite, En; and magnetite, Mag. We will refer to a mixture of graphite and olivine particles as (C + Ol) with similar notation for other mixtures.

Because high accuracy is required for wavelengths near the λ2200 bump, how well we fit the extinction depends very strongly upon the optical constants we adopt for wavelengths near the bump. The constants of graphite are especially important, particularly for radiation with the electric vector perpendicular to the basal plane of the crystal. The constants of Tosatti and Bassani (1970; hereafter TB) allow quite acceptable fits when graphite is used with another material. However, if we use exactly the optical constants of Taft and Philipp (1965; hereafter TP), we find no reasonable fit for graphite plus any one other material; $\chi^2 > 200$ for all mixtures. Such a large value of χ^2 corresponds to a substantial difference between the predicted and observed extinctions. The problems of fitting with the TP constants disappear if they are slightly changed at two of the wavelengths at which they were experimentally measured. We obtain a good fit if we increase the imaginary part of the index of refraction by 10% (from 1.71 to 1.90) at the measurement wavelength of 2254 Å, and 8% (from 2.08 to 2.24) at 2384 Å. Of course these changes in measured values are reflected, by interpolation, into changes at our wavelengths in the range 2000–2500 Å. The changes in the TP values are well within the experimental uncertainties of about 20% at most wavelengths as shown by the disagreements of TP, TB, Greenaway *et al.* (1969), Carter *et al.* (1965), Zeppenfeld (1968), and Klucker, Skibowski, and Steinman (1974). Near the bump, where the constants are changing rapidly, the agreement among the various determinations is only to within about 50%. For the remainder of this paper, when we use "graphite," we mean the TB constants, or, almost equivalently, the TP constants with the modifications described above.

The uncertainties in the optical constants of course impose a limit on the absolute significance of any conclusions we (or anyone else) can make by using calculated cross sections of particles, especially since the substances which constitute real interstellar grains are by no means pure. Furthermore, their crystal structure has possibly been damaged by either radiation or cosmic rays. Our approximation to the cross section of graphite grains (by combining cross sections of spheres using the perpendicular and parallel optical constants) imposes similar limits. In this paper, we only hope to show that with reasonable optical constants some mixtures can reproduce the observed extinction law very well, and are therefore rendered plausible while others cannot reproduce it. This test for plausibility is a quite restrictive one which only a very limited number of mixtures survive. We will find some general properties shared by all plausible mixtures which we consider. Of course we cannot rule out models with coated grains by our calculations.

Attempts to fit the extinction were made using each individual substance, all combinations of two substances, and many of the combinations of three substances. Graphite alone gave $\chi^2 = 1200$; each of the other materials alone gave $\chi^2 = 3 \times 10^5$. Most of the problems of fitting arise from the bump. While it is true as Huffman and Stapp (1971) suggested that silicates can qualitatively reproduce the bump, they fail badly in our program because of our insistence on a fit over a long-wavelength baseline and because of the abundance constraints. Even with the abundance constraints removed, silicates did no better than $\chi^2 = 675$ for enstatite. This is better than for graphite alone, but still not acceptable. The best enstatite mixture used 3.2 times as much silicon as is available cosmically. As Ney, Strecker, and Gehrz (1973) have pointed out, the presence of a strong silicate emission feature at 10 μm in the Orion Nebula coupled with the fact that θ Ori shows the smallest known λ2160 bump suggests on observational grounds that the bump is not caused by silicates. In addition, Day, Steyer, and Huffman (1974) have shown that the strength of the 10 μm emission feature suggests that silicates are not the major producer of extinction in the visual.

Some mixtures achieved excellent fits, with $\chi^2 \approx 10$. All of these mixtures contained graphite used in conjunction with another material. The (C + Ol), (C + En), and (C + SiC) combinations gave quite satisfactory results with χ^2 increasing slightly in the order given. However, (C + Mag) and (C + Fe) are not excluded by our results, with $\chi^2 = 44$ and 69, respectively. Thus we cannot make any statement about which materials must be used in conjunction with graphite.

The values of χ^2 for any three-component mixture are no larger than those for any pair of the components, since an allowed solution for the three components is to have $X_{mj} = 0$ for all values of j for any one component. Sometimes the three-component mix gave a dramatic reduction in χ^2. Using the TB constants for graphite, the (C + En) mixture gave $\chi^2 = 22$, while (C + En + Ol) gave $\chi^2 = 5$. However, the fit is quite acceptable for the binary mixture. The extinction in the tertiary mixture was always caused mostly by C, followed by the material which makes the best binary mixture with C (Ol, En, SiC, in that order), followed by the third component. For example, at 5500 Å in a (C + Ol + SiC) mixture the relative contributions to the extinction are 0.65, 0.27, and 0.082, respectively. The optical properties (albedo and phase function) agreed with those of (C + Ol) to two significant figures. For this reason, only binary mixtures will be discussed in this paper, although almost all possible combinations were actually investigated.

The size distributions obtained were all generally of the same character, with $n(a)$ decreasing rapidly with a. Figure 1 shows a plot of the size distribution for the (C + Ol) mixture which is typical of other mixtures as well. There are two important characteristics of all acceptable distributions. First, there is a wide range in sizes of particles (roughly between 0.005 and 0.25 μm for graphite and 0.025 and 0.25 μm for the other material). Second, there is a rapid decrease of number with size. There are gaps in the distribution, as expected when the number used in each bin is independent of that used in adjacent bins. However, if we connect the nonzero values in Figure 1, we obtain approximately a power law, with $n(a) \propto a^{-q}$, with q in the range $3.3 < q < 3.6$ for the various substances. As Figure 1 shows, there can be rather substantial deviations from this power-law distribution at some sizes, but the trend of $n(a)$ over the large interval in a is clearly power law in nature. This is a general

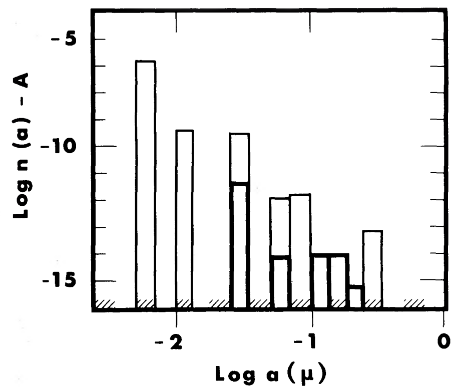

FIG. 1.—The size distribution function for a graphite–olivine mixture, OAO extinction, using single independent bins, spherical particles. The size limits of each bin are cross-hatched along the horizontal axis. *Abscissa:* logarithm of $n(a)$, the number of particles per micron sized for a column density of one H atom cm^{-2}, displaced downward by an amount A for clarity. *Light boxes:* graphite, with $A = 0$. *Heavy boxes:* olivine, with $A = 2$.

feature of all our acceptable distributions for all substances.

Although it is not clear from Figure 1 alone, we find from considering all acceptable distributions that the distribution for both the graphite and the other material followed approximately the same power law with a wider spread in sizes for the graphite.

The numbers of very small particles and the size of the smallest particles are not well determined by our procedure. The reason is that for particles in the Rayleigh limit, with $a \ll \lambda/2\pi$, the extinction per gram of material is independent of the particle sizes and therefore cannot be used to determine the size distribution. The smallest particles selected by minimizing χ^2 are in the Rayleigh limit for all but the smallest wavelengths, and hence the small-size cutoff is poorly determined.

The largest particles chosen varied between 0.25 and 1 μm for various mixtures. The fact that our distributions have 0.25 μm or larger particles means that rather gray extinction is needed, since these particles are near the geometrical limit except for the longest wavelengths we considered. Still larger particles could be present but would use more mass to provide the same extinction.

The gaps occurring whenever independent bins are used are, of course, unphysical. We therefore ensured that there would be a monotonically decreasing size distribution by considering bins which contain particles distributed uniformly in the range $0 < a < a_j$, instead of $a_{j-1} < a < a_j$. We determined extinction cross sections C'_{mji} such that

$$C'_{mji} = \sum_{k<j} C_{mki} = \int_0^{a_j} \hat{n}'(a) C_m(a, \lambda_i) da \,,$$

$$\hat{n}'(a) = 1\ \mu\text{m}^{-1}, \quad 0 < a \le a_j \,, \qquad (6)$$

$$\hat{n}'(a) = 0 \,, \quad \text{otherwise} \,.$$

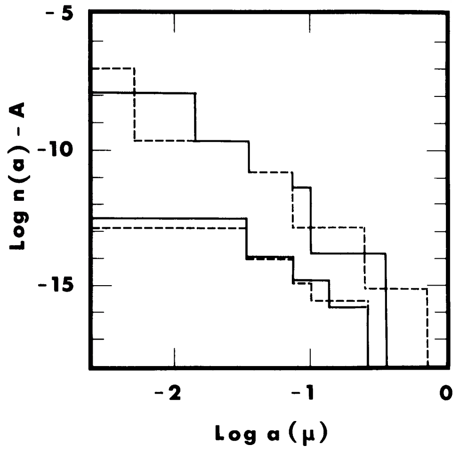

FIG. 2.—Same as Fig. 1, using bins which are uniform in particle sizes from zero to a microns (the "monotonically decreasing distribution"). *Solid curves:* OAO extinction; *dashed curves:* TD-1 extinction. *Upper lines:* graphite, with $A = 0$. *Lower lines:* olivine, with $A = 3$.

Clearly, any linear combination of $\hat{n}'(a)$ with non-negative coefficients must be monotonically decreasing with a. Coefficients X'_j are obtained from minimizing χ^2 in just the standard way, which C'_{mji} used in place of C_{mji}. We will refer to the resulting distribution as a "monotonically decreasing distribution." Of course, this procedure forces the solution to contain particles of sizes which are not of the optimum sizes since independent bins allow the greatest freedom of choice. For the (C + Ol) mixture shown in Figure 1, χ^2 was increased by about 40% for monotonically decreasing bins. This increase is quite acceptable.

Figure 2 shows log $n(a)$, plotted against log a for the (C + Ol) mixture with monotonically decreasing sizes. Figure 3 shows the same plot for (C + SiC). Each distribution extends formally to zero sizes, but in fact the particles smaller than half the minimum size chosen provide little extinction and use little material and hence can be excluded. With this in mind, we see that an approximately power-law distribution is required for the monotonically decreasing as well as for the single bin distributions. Again, the substance other than graphite shows a smaller range of sizes than the graphite does, but shows about the same exponent.

Figure 4 shows the calculated extinction from the (C + Ol) mixture of Figure 2 and the contribution to that extinction from the graphite. Graphite is the major contributor to the extinction at all wavelengths. The same is true for the other acceptable mixtures.

According to our models, one or more of the heavy elements (Si, Mg, or Fe) contained in our materials is completely locked up in the grains, naturally into the material used with the graphite. Sometimes, but not always, carbon was also used up. However, lowering the allowed abundance of C relative to H from 3.7×10^{-4} to 2.4×10^{-4} made little difference, but further lowering it to 1.8×10^{-4} increased χ^2 drastically. Thus the abundance of C above about 2×10^{-4} might be observed in the ISM. The depletions observed by *Copernicus* (Spitzer and Jenkins 1975) show mild depletion of C and severe depletions of Mg, Si, and Fe as predicted by our models.

It has been suggested (Gilra 1972) that the narrowness and position of the λ2200 feature implies that graphite particles, if they produce the feature, must be near the Rayleigh limit (i.e., $a \ll 0.04\ \mu$m), uncoated with ices, and spherical. However, in our mixtures the extinction at λ2160 tends to be quite well distributed among the various sizes with roughly half contributed by particles with $a > 0.04\ \mu$m. We need small graphite particles to provide only a narrow peak in extinction at 2160 Å superposed upon a broader feature from the larger particles. It follows from Gilra's work that *our small graphite particles must be uncoated.*

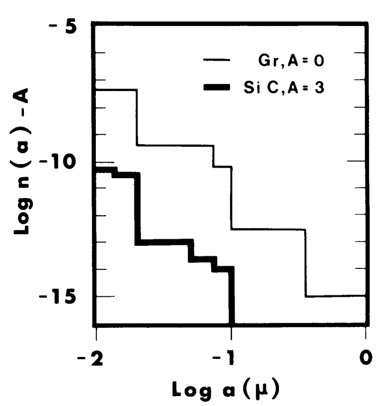

FIG. 3.—Same as Fig. 2, except for graphite–SiC mixture, with only OAO extinction shown. $A = 0$ for graphite, $A = 3$ for SiC.

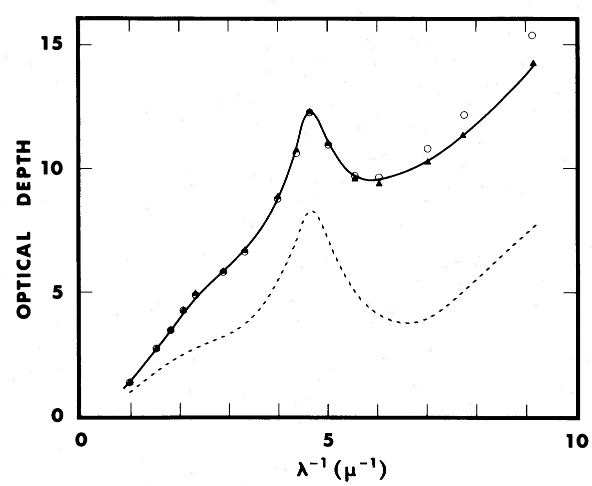

FIG. 4.—Optical depths, for column densities of 10^{22} H atoms cm^{-2}, versus inverse wavelength. *Solid line:* observed by OAO. *Triangles:* the extinction of (C + Ol) mixture of Fig. 2. *Dashed line:* the contribution of graphite to the extinction. *Dots:* a mixture of graphite and olivine, $n(a) \propto a^{-3.5}$, 0.005 μm $< a < 0.25\ \mu$m, forced to fit at the maximum of the "bump" at 4.6 μm^{-1}.

We therefore suggest that the approximate uniformity of the interstellar extinction law over the entire sky arises from having the same broad distribution of sizes established, rather than from needing any one or two special sizes be abundant everywhere. The details of the distribution at very large particles (which provide gray extinction) or small particles (in the Rayleigh limit) are not important. It seems reasonably likely that a uniformity in the distribution function in the middle sizes ($0.01 < a < 0.1\ \mu$m) could be established stochastically, such as by collisions soon after particles are formed in a stellar atmosphere, or in the process of formation, so that the uniformity is simply the reflection of a most probable distribution.

The approximately power-law distribution found by minimizing χ^2 leads naturally to the question: What would the extinction of an exactly power-law distribution be? Figure 4, *dots*, shows the extinction of a (C + Ol) mixture with $\log n(a) = K_C - 3.5 \log (a/1\ \mu\text{m})$ for graphite and $\log n(a) = K_{Ol} - 3.5 \log (a/1\ \mu\text{m})$ for olivine ($0.005\ \mu\text{m} < a < 0.25\ \mu$m). The constants, for $n(a)$ in units of particles per H atom per micron, are $K_C = -15.24$ and $K_{Ol} = -15.21$. All the available silicon was assumed to reside in the olivine together with enough C in the graphite to make the extinction agree with the observed extinction at $\lambda = 2160$ Å. Hence the amount of graphite is the only adjusted parameter in the distribution; the exponent and size limits were optimized (but of course the general results of the χ^2-fitting went into guessing reasonable values for them). The fit is quite satisfactory.

The observed extinction law has been fitted many times in the past by adjusting parameters of the Oort–van de Hulst (1946) distribution or an approximation to it. Such distributions have a fairly flat $n(a)$ out to some value of a, say $\bar{a}$, and then a fairly abrupt drop for $a > \bar{a}$. In fact, the Oort–van de Hulst distribution would not look very different, on the scale of Figure 2, from a single rectangular bin used in constructing our monotonically decreasing mixtures. Hence our monotonically decreasing distributions are roughly composed of superpositions of six or eight Oort–van de Hulst distributions. Similarly, sometimes Gaussian distributions have been used; they correspond roughly to one of our single bins because they decrease so rapidly for both small and large sizes.

Some tests were made to determine how much the derived $n(a)$ depends upon the values of the a_j limits adopted for the various bins. We have tried changing the size limits on our bins in various ways and found similar results. Hence we are confident that the nature of $n(a)$ is not influenced by our choices of a_j.

For each mixture we calculate the scattering and phase properties as well as the extinction. We express the scattering by means of the albedo, ϖ, and the phase function through g, the average value of the cosine of the scattering angle. Witt and Lillie (1973) and Lillie and Witt (1976) determined ϖ and g from OAO measurements of the diffuse galactic light. Morgan, Nandy, and Thompson (1976) made similar determinations using TD-1 data. Figure 5 shows the newer determination of ϖ and the predicted values for

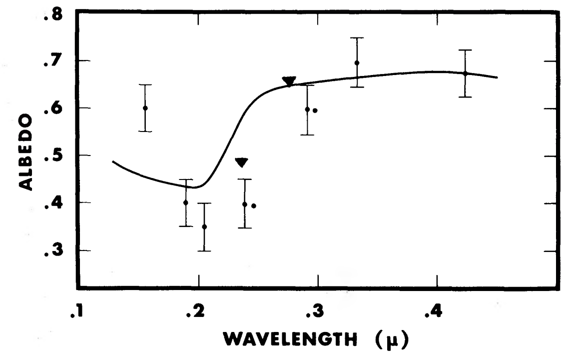

FIG. 5.—The albedo versus wavelength. *Solid curve:* the (C + Ol) mixture of Fig. 2. *Dots:* observational values of Lillie and Witt (1976). *Triangles:* values from Morgan *et al.* (1976), with $g = 0.75$.

the (C + Ol), monotonically decreasing mixture of Figure 2. The predicted values agree fairly well with the observed ones, both predictions and observations having a minimum of ϖ at about 2000 Å. The minimum is caused by graphite absorption and is a feature common to all of our acceptable mixtures. The values of the phase parameter g are all about 0.5–0.6 for all wavelengths.

Another plausibility check on our mixtures is the ratio of the strength of the 10 μm absorption feature, usually attributed to silicates, to the visual extinction (Martin 1975). Let τ_{10} be the optical depth at a maximum of the 10 μm absorption feature. The ratio A_V/τ_{10} has been reliably determined for only one star: VI Cyg No. 12 (Rieke 1974). The value given there is 24 ± 4. To produce this value, the silicates in our models would have to have an absorption strength of 10^4 cm^{-1}, which is about equal to what is actually observed for both terrestrial and lunar silicate rocks, but a factor of 6 less than the measured value for pure olivine. Thus, to have all or most of the non-graphite extinction in the form of silicates is consistent with the strength of the 10 μm feature.

We have neglected coatings, presumably of H_2O, NH_3, and CN_4, in this study. There are several reasons why this is reasonable. One is that it is important to see if the extinction can be well fitted without them. As mentioned above, the narrowness and position of the 2160 Å bump require that there be no coatings on our very small graphite particles. The uniformity of the normal extinction suggests that the particle size distribution does not vary greatly from region to region, which is more difficult to explain if one has to have uniformity in coatings as well as cores. Unfortunately, the determinations of the depletions of oxygen and nitrogen in the interstellar medium (Spitzer and Jenkins 1975) depend on rather saturated lines and are therefore consistent with either very little or quite extensive coatings. We suspect that the grains in very dense regions, such as the ρ Oph cloud (Carrasco, Strom, and Strom 1973), appear larger (or at least a larger λ_{max} of polarization) because they are coated, as one might expect in a dense region of lower interstellar radiation density and of temperature lower than normal. The 3.07 μm H_2O absorption band is in fact observed in such dense clouds as the Orion Becklin-Neugebauer object (Gillett and Forrest 1973) or a peculiar cloud in Cygnus (Merrill and Soifer 1974). Our assumption that there are no coatings is most likely to break down for the large particles (whose temperature fluctuations are small, and whose temperatures are low) where our graphite cross sections are most likely to be incorrect anyway.

We have two reservations about our models other than those already mentioned. We now briefly discuss each in turn.

We have some doubts about the use of the classical Mie theory which arise from the very careful experiments of Huffman (1975) on graphite smoke. He measures both the particle sizes and the extinction of the smoke directly. He finds good agreement for $\lambda >$ 1700 Å but increasingly poor agreement for shorter wavelengths, in the sense that the experimental values become much larger than the Mie calculations. The cause of this discrepancy is still unknown. If the cross section of graphite below 1700 Å is in fact higher than the Mie calculations predict, we would require fewer small particles than the present models or a larger lower limit to the size distribution. One possibility, suggested by D. R. Huffman (private communication), is that the smoke consists of "glassy" carbon (Hagemann, Gudat, and Kunz 1974) which has a λ2200 Å feature similar in shape to that of graphite, but much weaker. We found that the glassy carbon cannot be substituted in place of graphite for fitting the interstellar extinction because its λ2200 feature is so weak that one needs about 3 times as much carbon as is cosmically available.

We have also ignored the effects of damage to the crystals of the material from either photons or cosmic rays. Drapatz and Michel (1976) have discussed a radiation-damage model of the bump and conclude that such a model is attractive but requires a large concentration of crystal defects.

IV. LINEAR AND CIRCULAR POLARIZATION

Observations of polarization provide further information about the nature of interstellar particles. However, polarization imposes less direct constraints than does extinction; it depends upon the difference of cross sections in two orthogonal directions while extinction is quite independent of particle shape (Greenberg and Hong 1975). We thus must consider particle shapes for polarization. Furthermore the particles must be aligned, and the alignment mechanism is by no means certain.

We have considered mixtures of graphite spheres plus cylinders of the other materials. We determine the size distribution completely on the basis of fitting the extinction. After the size distribution is determined we investigate the polarization properties of the mixture.

We assume the cylinders are perfectly aligned by the Davis-Greenstein (1951) mechanism, so that they are rotating in the plane perpendicular to the magnetic field. In order to calculate the maximum linear polarization possible, we assume that the field is perpendicular to the line of sight. The extinction and linear polarization are calculated for each bin by the expressions in Greenberg (1968); the circular polarization was discussed by Martin (1972), and in this paper we follow his sign convention. We ignored the polarization but not the extinction of graphite partly because we could see no sensible method of calculating the polarization of anisotropic flakes with tensor constants by combinations of infinite cylinders having uniform indices of refraction. This approach is close to assuming that graphite flakes are randomly oriented because they are difficult to align (Greenberg 1969; Purcell 1969). Furthermore, Greenberg (1969) has shown that for *small* aligned graphite particles, the wavelength dependence of polarization should be that of the extinction, contrary to observations. However, one cannot say what the polarization of large aligned graphite flakes would be.

We found that mixtures with cylinders provided acceptable fits for the extinction, although the χ^2 was larger than it was for spheres. As with spheres, there were no reasonable fits to the extinction without graphite in the mixture.

Our standard bin sizes, each covering a range of 1.5 in size, are too coarse for a really careful study for polarization; linear and (especially) circular polarization are fairly rapidly varying functions of $x = 2\pi a/\lambda$ (Martin 1972). Having each bin cover a range of particle sizes of a factor of 1.5 precludes discussing the wavelength of maximum linear polarization to within the same factor, so that only two of our individual bins have their maxima of polarization in the visible range. Since the extinction is fitted without regard to polarization, one or both of these bins might be missing from the best mixture. However, some general conclusions may be drawn from our coarse analysis.

For brevity, we will refer to the fraction of linear polarization as Q, after the Stokes parameter (although the polarization is actually Q/I in the standard notation). Similarly, the fraction of circular polarization will be called V.

The observations of Q peak somewhere in the visible portion of the spectrum. In our models, Q peaked at about 1600 Å, or about 0.4 as long a wavelength as it should. The circular polarization V is observed to be positive at long wavelengths, to go through zero at about the wavelength of the maximum linear polarization, and to become negative at short wavelengths (Martin 1974). Our models had the same behavior as expected for dielectrics (but, as mentioned, the wrong wavelength of maximum Q). We predict the maximum Q at too short a wavelength because Q and V are strongly influenced by particles which are so small that their extinction in the visible is not important. Our models required these particles for the ultraviolet extinction.

The reason why small particles can be important for Q and V while being unimportant for visual extinction is that Q and V are proportional to the difference in cross sections for two orthogonal directions while extinction is proportional to the sum. The cross sections in orthogonal directions almost cancel for large particles (i.e., in the geometrical limit) while their magnitude (and hence sum) continues to increase. Hence their polarization, relative to their extinction, is small. For small particles, one cross section is much larger than the other, and the polarization is about the same as the extinction.

Martin (1974) fits both Q and V very well with a dielectric material whose size distribution (approximately Oort–van de Hulst) was determined by fitting the extinction throughout the *visible* region. We also find excellent agreement of predicted and observed Q and V for several mixtures *so long as we confine our attempt to fit the extinction to* $\lambda > 3000$ Å. Our difficulties in fitting the polarization arise solely from our fitting the entire wavelength range of extinction.

One might bring about agreement of model and polarization observations if the offending small particles are all spherical or randomly oriented while the large particles responsible for making Q peak in the visual are aligned and elongated. However, in order to have Q always peak in the visual, we need to have the variations in the sizes of the aligned particles for various stars be roughly the range of variation of $\lambda_{\max}$ which is less than a factor of 1.5. We can imagine a size distribution being this uniform, but such narrow variations in some size-dependent alignment mechanism is rather difficult to imagine. Also, having the small particles fail to contribute to the polarization makes it even more difficult to explain the amount of polarization per optical depth of extinction, which seems to be difficult for all models including ours to account for.

The largest observed value of visual polarization per extinction optical depth, which we will call Q/E, is about 0.06. Our mixes without Fe or magnetite, assuming *perfect* Davis-Greenstein alignment, have $Q/E = 0.15$, of which 80% is contributed by particles so small that their polarization maximum is in the ultraviolet ($\lambda < 3000$ Å). Hence one could not obtain enough visual polarization per magnitude of extinction, even with perfect alignment, if the small particles did not contribute.

The metallic substances, Fe and magnetite, have $Q/E = 0.3$ for the mix, if all particles are aligned. However, the small particles contribute over 90% of the polarization so that there is again not enough visual polarization if small particles are not aligned. We agree with the suggestion (Shapiro 1975) that well-aligned, large flakes of magnetite could provide the polarization. The high alignment of flakes could perhaps be provided by the "pinwheel" mechanism (Purcell 1975). Variations in $\lambda_{\max}$ over the sky could reflect changes in particle sizes with especially large particles (with large $\lambda_{\max}$) in dense regions.

On the other hand, it is entirely possible that the polarization is provided by grains which have coatings. Since we have been able to fit the extinction well with uncoated graphite and any other uncoated grains, sphere or cylinder, we expect no difficulty with fitting it with graphite plus coated grains of almost any sort. The coated grains possibly can be more easily aligned in the magnetic field than uncoated ones. The larger $\lambda_{\max}$ in denser clouds and the somewhat grayer extinction are then easily explained by assuming that coatings in dense clouds are thicker than normal.

V. SUMMARY

1. We have been able to fit the interstellar extinction from infrared (1 μm) to ultraviolet wavelengths. The strength and width of the $\lambda 2160$ feature are very well reproduced.
2. A variety of mixtures are acceptable. *All include graphite particles*, which are assumed to be spherical and uncoated. The graphite is the main contributor to the $\lambda 2160$ feature. Probably the small particles of graphite must be uncoated if the feature is to be kept narrow. Uncoated silicates, silicon carbide, and metallic particles may be used with the graphite. Presumably these substances could also be coated since the extinction they supply does not need any very narrow features.
3. The required particle-size distribution is quite broad, with approximately a power-law distribution: $n(a) \propto a^{-3.5}$. This type of distribution was found for all substances. The sizes for graphite vary from about 0.005 μm to about 1 μm. This distribution provides that particles of a variety of sizes contribute to the extinction at any wavelength. The uniformity of the extinction over the sky is explained by having the exponent in power-law distribution rather uniform. The exponent may be the result of a stochastic process in forming the grains and therefore represents the most probable distribution in some sense. The wavelength of the extinction bump near $\lambda 2160$ is set by small particles of graphite.
4. The wavelength of maximum polarization of our models occurs at too short a wavelength (in the UV) if graphite is not aligned and if all the nongraphite particles are aligned. However, the polarization can be explained by either having large particles of magnetite aligned by the Davis-Greenstein mechanism or by having aligned coated particles of silicates or other dielectric material.

This work was completed while one of the authors (J. S. M.) was a US Senior Scientist Awardee of the Alexander von Humboldt Foundation of the Federal Republic of Germany. I appreciate the hospitality of the Foundation and of the Max-Planck-Institut für Radioastronomie, Bonn, Federal Republic of Germany.

It is a pleasure to acknowledge partial support of this work by the US National Science Foundation. Computer time was donated by the Space Astronomy Laboratory, University of Wisconsin-Madison. Useful conversations were held with many people, including Drs. A. D. Code, D. P. Gilra, D. F. Huffman, N. Panagia, P. G. Mezger, K. W. Michel, and B. D. Savage, and especially L. F. Smith.

REFERENCES

Bless, R. C., and Savage, B. D. 1972, *Ap. J.*, **171**, 293.
Cameron, A. G. W. 1973, *Space Sci. Rev.*, **15**, 122.
Carrasco, L., Strom, S. E., and Strom, K. M. 1973, *Ap. J.*, **182**, 95.
Carter, J. G., Heubner, R. H., Hamm, R. N., and Birkhoff, R. D. 1965, *Phy. Rev.*, **137A**, 639.
Code, A. D., Davis, J., Bless, R. C., and Hanbury Brown, R. 1976, *Ap. J.*, **203**, 417.
Cottle, R. W. 1968, in *Mathematics of the Decision Sciences*, Vol. **1**, p. 144.
Cottle, R. W., and Dantzig, G. B. 1968, *Linear Algebra and Its Applications* (NY: Elsevier), p. 103.
Davis, L., and Greenstein, J. L. 1951, *Ap. J.*, **114**, 206.
Day, K. L., Steyer, T. R., and Huffman, D. R. 1974, *Ap. J.*, **191**, 145.
Drapatz, S., and Michel, K. W. 1976, *Mitt. Astr. Ges.*, **40**, 187.
Faber, S. M. 1972, *Astr. Ap.*, **20**, 361.
Gillett, F. C., and Forrest, W. J. 1973, *Ap. J.*, **179**, 483.
Gillett, F. C., Forrest, W. J., and Merrill, K. M. 1973, *Ap. J.*, **183**, 87.
Gilra, D. P. 1972, in *The Scientific Results of OAO-2*, ed. A. D. Code, NASA SP310, p. 295.
Greenaway, D. L., Harbeke, G., Bassani, F., and Tosatti, E. 1969, *Phys. Rev.*, **178**, 1340.
Greenberg, J. M. 1966, *Ap. J.*, **145**, 57.
———. 1968, in *Nebulae and Interstellar Matter*, ed. B. M. Middlehurst and L. H. Aller (Chicago: University of Chicago Press), p. 221.
———. 1969, *Physica*, **41**, 67.
Greenberg, J. M., and Hong, S. S. 1974, *IAU Symposium No. 60, Galactic Radio Astronomy*, ed. F. J. Kerr and S. C. Simonson (Dordrecht: Reidel), p. 155.
Hagemann, J. J., Gudat, W., and Kunz, C. 1974, *Deutsches Elektronen-Synchrotron DESY*, SR-7417.
Huffman, D. R. 1975, *Astr. Space Sci.*, **34**, 175.
Huffman, D. R., and Stapp, J. L. 1971, *Nature Phys. Sci.*, **229**, 45.
Jenkins, E. B., and Savage, B. D. 1974, *Ap. J.*, **187**, 243.
Klucker, R., Skibowski, M., and Steinman, W. 1974, *Phys. Status Solidi* (*B*), **65**, 703.
Lillie, C. F., and Witt, A. N. 1976, *Ap. J.*, **208**, 64.
Martin, P. G. 1972, *M.N.R.A.S.*, **159**, 179.
———. 1974, *Ap. J.*, **187**, 461.
———. 1975, *Ap. J.*, **202**, 389.
Merrill, K. M., and Soifer, B. T. 1974, *Ap. J.* (*Letters*), **189**, L27.
Morgan, D. H., Nandy, K., and Thompson, G. I. 1976, *M.N.R.A.S.*, **177**, 531.
Nandy, K., Thompson, G. I., Jamar, C., Monfils, A., and Wilson, R. 1975, *Astr. Ap.*, **44**, 195.
Ney, E., Strecker, D., and Gehrz, R. 1973, *Ap. J.*, **180**, 809.
Oort, J. H., and van de Hulst, H. C. 1946, *Bull. Astr. Inst. Netherlands*, **10**, 187.
Philipp, H. R., and Taft, E. A. 1960, in *Conference on Silicon Carbide* (Boston: Pergamon), p. 366.
Purcell, E. M. 1969, *Ap. J.*, **158**, 433.
———. 1975, in *Dusty Universe Symposium*, ed. G. B. Field and A. G. W. Cameron (New York: Neale Watson), p. 155.
Rieke, G. H. 1974, *Ap. J.* (*Letters*), **193**, L81.
Savage, B. D. 1975, *Ap. J.*, **199**, 92.
Serkowski, K., Mathewson, D. S., and Ford, V. C. 1975, *Ap. J.*, **196**, 261.
Shapiro, P. R. 1975, *Ap. J.*, **201**, 151.
Spitzer, L., and Jenkins, E. B. 1975, *Ann. Rev. Astr. Ap.*, **13**, 133.
Taft, E. A., and Philipp, H. R. 1965, *Phys. Rev.*, **138A**, 197 (TP).
Thibault, N. W. 1944, *Amer. Mineral.*, **29**, 327.
Tosatti, E., and Bassani, F. 1970, *Nuovo Cimento*, **65B**, 161 (TB).
Wickramasinghe, N. C., and Nandy, K. 1971, *M.N.R.A.S.*, **153**, 205.
Witt, A. N., and Lillie, C. F. 1973, *Astr. Ap.*, **25**, 397.
York, D. G., Drake, J. F., Jenkins, E. B., Morton, D. C., Rogerson, J. B., and Spitzer, L. 1973, *Ap. J.* (*Letters*), **182**, L1.
Zeppenfeld, K. 1968, *Zs. f. Physik*, **211**, 391.

John S. Mathis, Kenneth H. Nordsieck, and William Rumpl: Washburn Observatory, University of Wisconsin-Madison, 475 N. Charter Street, Madison, WI 53706