A lifting line solver

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1 Reference

Lifting-line theory - Wikipedia

2 Lifting line theory summary

This section summarizes the main ideas and equations of lifting line theory:

2.1 Lift of an airfoil

The lift produced by an infinitesimal span airfoil L_y in steady state is given by the Kutta–Joukowski theorem:

$$\frac{dL}{dy} = L_y = \rho V \Gamma(y, \alpha) \tag{1}$$

Where Γ represents the circulation around the airfoil. If the airfoil is at an angle α, the lift is given by:

$$\Gamma = VC(y, \alpha) = V 2\pi c(y) \sin(\alpha(y)) \tag{2}$$

Where $c(y)$ is the local chord length and $\alpha(y)$ is the local angle of attack considering any wing twist.

The lift can be integrated over the span to obtain the total lift.

2.2 Lifiting line

Because the circulation is not constant along the wingspan, the variation in circulation needs to be compensated by the shedding of a vortex. The shed vortices instead induce a vertical velocity over the wingspan.

This induced velocity can be computed by:

$$w(y) = \int_{-b/2}^{b/2} \frac{d\Gamma(\tilde{y})}{4\pi\,(y - \tilde{y})} \tag{3}$$

Note the integrand is singular at $y = \tilde{y}$, so it is understood in terms of the Cauchy principal value.

The circulation at a point can be extended to account for the effect of the induced velocity:

$$\Gamma(y) = V 2\pi c(y) \left(\alpha(y) + \frac{w(y)}{V} \right) \tag{4}$$

$$= 2\pi c(y) \left(V\alpha(y) + \int_{-b/2}^{b/2} \frac{d\Gamma(\tilde{y})}{4\pi\,(y - \tilde{y})} \right) \tag{5}$$

And the total lift and drag can be computed as:

$$L = \rho V \int_{-b/2}^{b/2} \Gamma(y, \alpha) dy \tag{6}$$

$$D = \rho \int_{-b/2}^{b/2} \Gamma(y, \alpha) w(y) dy \tag{7}$$

Note: wikipedia states the induced drag includes $\alpha(y)$ instead of $w(y)$, however this could lead to induced drag simply due to the angle of attack in the case of an infinite wing.

3 Solution

The solution process consists of find the circulation profile for a given wing geometry, i.e. $c(y)$ and $\alpha(y)$.

In order to compute the circulation profile, we can discretize the problem and solve it numerically.

3.1 Discretization scheme

Since we are considering an integral in the Cauchy singular value sense, we will discretize the integral using a rectangle rule.

The rectangle rule value is done as follows:

$$F(x) = \int_a^b f(x)dx \approx \sum_{i=0}^{N} f(x_i)\Delta_i x_i \tag{8}$$

Where Δ_i is the difference operator which can be expressed as:

$$\Delta_i f(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2} \tag{9}$$

Considering that at the limits we will instead use:

$$\Delta_0 f(x_0) = \frac{f(x_1) - f(x_0)}{2} \tag{10}$$

$$\Delta_N f(x_N) = \frac{f(x_N) - f(x_{N-1})}{2} \tag{11}$$

This rule let us compute the Cauchy value by simply excluding the integrand at the singularity point.

3.1.1 Example of the integration of a Cauchy singular value

We could check the accuracy of this scheme by integrating:

$$F(x) = \int_{-1}^{2} \frac{dx}{x} \tag{12}$$

The function is odd about the singularity at $x = 0$, which is the use case for the Cauchy singular value, so we can split the integration into two parts and play with the limits. Whose analytic solution is:

$$\begin{align}
F(x) &= \int_{-1}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{2} \frac{dx}{x} = \tag{13}\\
&= \int_{1}^{\epsilon} \frac{dx}{x} + \int_{\epsilon}^{2} \frac{dx}{x} = \tag{14}\\
&= \ln(x)|_1^{\epsilon} + \ln(x)|_{\epsilon}^{2} = \tag{15}\\
&= \ln(2) - \ln(1) = \ln(2) \tag{16}
\end{align}$$

TODO: check the discretization scheme against this problem.

3.2 Discretization of the circulation

Instead of solving the equation directly, we can discretize the problem and solve it numerically:

$$\Gamma(y_i) = 2\pi c(y_i) \left(V\alpha(y_i) + \frac{1}{4\pi} \sum_{j=1}^{N} \frac{\Delta_j \Gamma(y_j)}{(y_i - y_j)} \right) \tag{17}$$

Where Δ_j is the difference operator defined above.

3.3 Solver

The will implement an iterative solver using considering the error and a step size λ, such that:

$$\tilde{\Gamma}(y_i) = 2\pi c(y_i) \left(V\alpha(y_i) + \frac{1}{4\pi} \sum_{j=1}^{N} \frac{\Delta_j \Gamma(y_j)}{(y_i - y_j)} \right) \tag{18}$$

And the next value in the iteration will be:

$$\Gamma(y_i) = \Gamma(y_i) + \lambda \left(\tilde{\Gamma}(y_i) - \Gamma(y_i) \right) \tag{19}$$

In the hope that it converges :)