CORRIGENDUM TO "LOCALLY DEFINABLE GROUPS IN O-MINIMAL STRUCTURES"

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In [3], the second author develops the theory of locally definable groups. In particular, using the notion of connectedness given by Definition 3.7 in [3], he studies the connected locally definable subgroups of a locally definable group obtaining similar results as in the definable case (see Section 3.1 in [3]). However, we show that this definition of connectedness is not correct for the category of locally definable groups. Moreover, we show that with this definition some results in [3] are false.

Counterexample (to Proposition 3.9 in [3]). Let R be an $\aleph_1$-saturated real closed field and consider the definable sets $Z_n = (-n, -\frac{1}{n}) \cup (\frac{1}{n}, n)$ for $n \in \mathbb{N}$, $n > 1$. Then $\mathcal{Z} = \bigcup_{n>1} Z_n$ is a connected locally definable group with the multiplicative operation of R. Intuitively, we see that $\mathcal{Z}$ is the disjoint union of $\bigcup_{n>1}(\frac{1}{n}, n)$ and $\bigcup_{n>1}(-n, -\frac{1}{n})$, but neither of these sets is definable. Consider also the locally definable subgroup $\mathcal{H} = \bigcup_{n>1}(\frac{1}{n}, n)$ of $\mathcal{Z}$. Both $\mathcal{Z}$ and $\mathcal{H}$ are compatible, connected, normal and $dim(\mathcal{Z}) = dim(\mathcal{H})$, which is a contradiction with Proposition 3.9 in [3].

The flaw in the proof of Proposition 3.9 in [3] comes from the following incorrect statement: given a locally definable group $\mathcal{Z}$, a connected compatible locally definable normal subgroup of $\mathcal{Z}$ with the same dimension contains all connected locally definable subgroups of $\mathcal{Z}$.

Inspired by the theory of locally semi-algebraic spaces from [1], Definition 3.7 must be replaced by the following.

Definition. *Let $\mathcal{Z}$ be a locally definable group. We say that a set $Z \subset \mathcal{Z}$ is **connected** if there is no subset $U \subset \mathcal{Z}$ such that (i) the intersection of U with every definable subset of $\mathcal{Z}$ is definable and (ii) $U \cap \mathcal{Z}$ is a non-empty proper subset of $\mathcal{Z}$ which is closed and open in the topology induced on Z by $\mathcal{Z}$.*

Observe that every definable subset of a locally definable group satisfies condition (i) above and that therefore the new notion of connectedness is stronger than Definition 3.7 in [3]. Also note that, using the notation of

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[2], clearly a subset of a locally definable group satisfying (i) above is a compatible locally definable subset (see Definition 2.8 in [2]).

With this new definition, we can prove a correct version of Proposition 3.9 in [3] just following its proof.

Proposition 1. *Let $\mathcal{Z}$ be a locally definable group. Then there is a unique connected compatible locally definable subgroup $\mathcal{Z}^0$ of $\mathcal{Z}$ with dimension dim $\mathcal{Z}$. Moreover, the following hold:*
(i) $\mathcal{Z}^0$ contains all connected locally definable subgroups of $\mathcal{Z}$;
(ii) $\mathcal{Z}^0$ is the smallest compatible locally definable subgroup of $\mathcal{Z}$ such that $(\mathcal{Z} : \mathcal{Z}^0) < \aleph_1$, and
(iii) $\mathcal{Z}^0$ is normal.

Proposition 3.9 in [3] (compare with Proposition 2.18 in [2]) is used extensively in both papers [3] and [2]. Therefore in both papers the definition of connectedness must be replace by the new one, in which case all the results remain true.

In spite of Definition 3.7 in [3] is not correct for the category of locally definable groups, it still makes sense and therefore we will call it "weakly connected". The next proposition gives us a relation between the notions of connectedness and weakly connectedness.

Proposition 2. *Let $\mathcal{Z}$ be a locally definable group which is not weakly connected. Then $\mathcal{Z}^0$ (that of Proposition 1) is definable and contains all weakly connected locally definable subgroups of $\mathcal{Z}$.*

Proof. Let U be a definable subset of $\mathcal{Z}$ such that U is both open and closed. By Corollary 3.6 in [3] there exist a locally definable subset $\{z_r : r \in S\}$ of $\mathcal{Z}$ such that $\mathcal{Z} = \bigcup_{r \in S} z_r \mathcal{Z}^0$. Since $\mathcal{Z}^0$ is connected it is easy to prove that if $z_r \mathcal{Z}^0 \cap U \neq \emptyset$ then $z_r \mathcal{Z}^0 \subset U$. Therefore $U = \bigcup_{r \in S'} z_r \mathcal{Z}^0$ for some $S' \subset S$. Since U is definable, by saturation we deduce that S' is finite and $z_r \mathcal{Z}^0$ is definable for each $r \in S'$. Hence $\mathcal{Z}^0$ is definable and it is closed and open. Suppose $\mathcal{H}$ is a weakly connected locally definable subgroup of $\mathcal{Z}$. Since $\mathcal{Z}^0$ is definable, open, closed and $\mathcal{H} \cap \mathcal{Z}^0 \neq \emptyset$ we deduce that $\mathcal{H} \cap \mathcal{Z}^0 = \mathcal{H}$, i.e, $\mathcal{H}$ is contained in $\mathcal{Z}^0$. $\square$

Corollary. *Let $\mathcal{Z}$ be a locally definable group. Then there exists a unique weakly connected locally definable subgroup of $\mathcal{Z}$ which contains all weakly connected locally definable subgroups of $\mathcal{Z}$. Moreover, this weakly connected locally definable subgroup equals $\mathcal{Z}$ or $\mathcal{Z}^0$ (so in particular is compatible).*

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