A note on the harmonic sum of cycle lengths in graphs with kn edges

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Abstract

Let G be a graph on n vertices with $m := |E(G)| = kn$ edges (so $k = m/n$), and let $a_1 < a_2 < \cdots$ be the distinct lengths of cycles appearing in G. Define the harmonic sum of cycle lengths by $L(G) := \sum_i 1/a_i$. Assuming an exact extremal statement (quoted in a survey of Montgomery) that for large d the complete bipartite graph $K_{d,n-d}$ minimises $L(G)$ among graphs with at least $d(n-d)$ edges, we give a short, fully explicit proof that $L(G) \gg \log k$ (indeed $L(G) \geq \frac{1}{2}\log k - O(1)$).

1 Definitions and the Erdős–Hajnal harmonic-sum parameter

Throughout, graphs are finite and simple.

Definition 1 (Cycle-length set and harmonic sum)**.** Let G be a graph. Define the *cycle-length set*

$$\mathcal{C}(G) := \{\ell \in \mathbb{N} : G \text{ contains a cycle of length } \ell\}.$$

Define the *harmonic sum of cycle lengths*

$$L(G) := \sum_{\ell \in \mathcal{C}(G)} \frac{1}{\ell}.$$

Equivalently, if $a_1 < a_2 < \cdots$ lists the distinct elements of $\mathcal{C}(G)$, then $L(G) = \sum_i 1/a_i$.

This parameter was proposed by Erdős and Hajnal as a measure of how "dense" the set of cycle lengths is. A classical theorem of Gyárfás–Komlós–Szemerédi (1984) shows that $L(G) \geq c \log \delta(G)$ for graphs of sufficiently large minimum degree $\delta(G)$, and yields corresponding statements for density/average-degree assumptions. See [2].

In this note we do *not* reprove the Gyárfás–Komlós–Szemerédi argument. Instead, we show that once one assumes the (claimed) *exact* extremal result identifying the minimiser of $L(G)$ at a given edge threshold, the lower bound $L(G) \gg \log k$ becomes an immediate computation.

2 The extremal hypothesis

The following statement is quoted (as "forthcoming work") in Montgomery's survey on cycles and expansion [1]. We take it as a black-box assumption.

Assumption 2 (Exact extremal minimiser at the $d(n-d)$ threshold)**.** *There exists an integer d_0 and a function $n_0 : \mathbb{N} \to \mathbb{N}$ such that for every integer $d \geq d_0$ and every $n \geq n_0(d)$, every n-vertex graph H with*

$$|E(H)| \geq d(n-d)$$

satisfies

$$L(H) \;\geq\; L\big(K_{d,n-d}\big),$$

with equality attained uniquely by $H \cong K_{d,n-d}$.

Remark 3. This is precisely the "exact extremal result corresponding to Theorem 2.3" described in [1], i.e. the minimiser is the complete *bipartite* graph with parts of sizes d and $n-d$.

3 Cycle lengths in complete bipartite graphs

We now determine $\mathcal{C}(K_{d,n-d})$ and hence compute $L(K_{d,n-d})$.

Lemma 4 (Cycle-length spectrum of $K_{d,n-d}$)**.** *Let $2 \leq d \leq n-d$. Then*

$$\mathcal{C}\big(K_{d,n-d}\big) = \{4, 6, 8, \dots, 2d\}.$$

Proof. First, $K_{d,n-d}$ is bipartite, so every cycle alternates between the two parts and therefore has even length. Thus $\mathcal{C}(K_{d,n-d}) \subset 2\mathbb{N}$.

If a cycle has length $2t$, it uses exactly t vertices in each part. Since the smaller part has size d, necessarily $t \leq d$, so $2t \leq 2d$. Hence no cycle length exceeds $2d$.

Conversely, fix any integer t with $2 \leq t \leq d$. Choose distinct vertices $x_1, \dots, x_t$ in the d-vertex part and distinct vertices $y_1, \dots, y_t$ in the $(n-d)$-vertex part. Since the graph is complete bipartite, all edges $x_i y_j$ exist, and in particular

$$x_1 y_1 x_2 y_2 \cdots x_t y_t x_1$$

is a cycle of length $2t$. This shows every even length $2t$ with $2 \leq t \leq d$ occurs. Therefore $\mathcal{C}(K_{d,n-d}) = \{4, 6, \dots, 2d\}$. □

Corollary 5 (Harmonic sum of cycle lengths in $K_{d,n-d}$)**.** *For $2 \leq d \leq n-d$,*

$$L\big(K_{d,n-d}\big) = \sum_{t=2}^{d} \frac{1}{2t} = \frac{1}{2}\left(H_d - 1\right),$$

where $H_d := \sum_{t=1}^{d} \frac{1}{t}$ is the dth harmonic number.

Proof. By Lemma 4,

$$L\big(K_{d,n-d}\big) = \sum_{\ell \in \{4,6,\dots,2d\}} \frac{1}{\ell} = \sum_{t=2}^{d} \frac{1}{2t} = \frac{1}{2}\sum_{t=2}^{d} \frac{1}{t} = \frac{1}{2}(H_d - 1).$$

□

4 A standard lower bound for harmonic numbers

Lemma 6 (Integral lower bound)**.** *For every integer $d \geq 1$,*

$$H_d \;\geq\; \log(d+1).$$

Proof. Since $x \mapsto 1/x$ is decreasing on $(0, \infty)$, for each integer $t \geq 1$ and all $x \in [t, t+1]$ we have $1/x \leq 1/t$. Hence

$$\int_t^{t+1} \frac{dx}{x} \;\leq\; \int_t^{t+1} \frac{dx}{t} = \frac{1}{t}.$$

Summing from $t = 1$ to d yields

$$\int_1^{d+1} \frac{dx}{x} = \sum_{t=1}^{d} \int_t^{t+1} \frac{dx}{x} \leq \sum_{t=1}^{d} \frac{1}{t} = H_d.$$

The integral equals $\log(d+1)$, giving $H_d \geq \log(d+1)$. □

5 Main implication: $L(G) \gg \log k$ from the extremal minimiser

We now prove the desired lower bound for graphs with kn edges.

Theorem 7. *Assume Assumption 2. Let G be a graph on n vertices with $m := |E(G)|$ edges, and set $k := m/n$. Let $a_1 < a_2 < \cdots$ be the distinct cycle lengths in G. Then there exists k_0 (depending only on d_0 from Assumption 2) such that whenever $k \geq k_0$ and $n \geq n_0(\lfloor k \rfloor)$,*

$$\sum_i \frac{1}{a_i} = L(G) \;\geq\; \frac{1}{2}\log k - O(1),$$

and in particular $L(G) \gg \log k$ as $k \to \infty$.

Proof. If k is bounded (say $1 \leq k \leq 2$), then $\log k = O(1)$ and the statement $L(G) \gg \log k$ is trivial after adjusting constants. Hence we assume $k \geq 2$.

Write $m := |E(G)|$ and $k = m/n$. Since $m \leq \binom{n}{2}$, we have

$$k \leq \frac{n-1}{2}.$$

In particular $d := \lfloor k \rfloor \le (n-1)/2 < n/2$, so $d \le n-d$ and the standing hypothesis of Lemma 4 is satisfied.

Let $d := \lfloor k \rfloor$. Then $d \le k$ and $d \ge 1$. We claim that

$$|E(G)| \ge d(n-d).$$

Indeed,

$$|E(G)| = m = kn \;\ge\; dn,$$

and since $n-d \le n$ we have $d(n-d) \le dn$. Thus $|E(G)| \ge d(n-d)$.

If furthermore $d \ge d_0$ and $n \ge n_0(d)$ (in particular, $n \ge n_0(\lfloor k \rfloor)$), Assumption 2 implies

$$L(G) \;\ge\; L\big(K_{d,n-d}\big).$$

By Corollary 5 and Lemma 6,

$$L\big(K_{d,n-d}\big) = \frac{1}{2}(H_d - 1) \;\ge\; \frac{1}{2}(\log(d+1) - 1).$$

Since $k \ge 2$ gives $\lfloor k \rfloor \ge k/2$, we have $d+1 \ge k/2$ and so

$$\log(d+1) \;\ge\; \log\Big(\frac{k}{2}\Big) = \log k - \log 2.$$

Therefore, for $k \ge \max\{2, d_0\}$,

$$L(G) \;\ge\; \frac{1}{2}(\log(d+1) - 1) \;\ge\; \frac{1}{2}(\log k - \log 2 - 1) = \frac{1}{2}\log k - \frac{1}{2}(\log 2 + 1).$$

This is of the form $\frac{1}{2}\log k - O(1)$, and it implies $L(G) \gg \log k$. $\square$

Remark 8 (Sharpness up to the constant 1/2). For the extremal graph itself,

$$L\big(K_{d,n-d}\big) = \sum_{t=2}^{d} \frac{1}{2t} = \frac{1}{2}(H_d - 1) \sim \frac{1}{2}\log d.$$

Thus, even with the best possible argument, one cannot replace the coefficient 1/2 in front of $\log k$ by any larger absolute constant in general (up to lower-order terms).

Acknowledgement of context

The original lower bound $L(G) \gg \log d$ (for average degree/minimum degree parameters) was proved by Gyárfás–Komlós–Szemerédi in [2]. Montgomery's survey [1] discusses later improvements and states the exact extremal characterisation (Assumption 2) as forthcoming work; assuming that characterisation, the $\Omega(\log k)$ bound for the edge-density parameter follows directly from evaluating $L(K_{d,n-d})$.

References

[1] R. Montgomery, *Cycles and expansion in graphs*, EMS Magazine 138 (2025), DOI: 10.4171/MAG/287.

[2] A. Gyárfás, J. Komlós, and E. Szemerédi, *On the distribution of cycle lengths in graphs*, J. Graph Theory 8 (1984), 441–462.