A Sequence with Doubling Ratio and Full-Density Subset Sums

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Abstract

We construct a nondecreasing sequence of integers $A = \{a_1 \le a_2 \le \cdots\}$ with

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$$

such that for every cofinite subsequence $A' \subseteq A$, the finite subset-sum set

$$P(A') = \left\{ \sum_{x\in B} x : B \subseteq A' \text{ finite} \right\}$$

has asymptotic density 1 in $\mathbb{N}$.

1 Problem

Does there exist a nondecreasing sequence A of integers with

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$$

such that $P(A')$ has density 1 for every cofinite subsequence $A' \subseteq A$?

2 Construction by Blocks

2.1 Block lengths

Define

$$k_n := 4 + \lceil \log_2 \log_2(n+16) \rceil \qquad (n \ge 0).$$

Then $k_n \to \infty$ slowly and $2^{k_n} \asymp \log n$ as $n \to \infty$.

2.2 Block scales

Let $M_0 := 10$ and define recursively

$$M_{n+1} := \left\lfloor \left(2^{k_n} - \tfrac{3}{2}\right) M_n \right\rfloor .$$

Since $M_n \geq 10$, we have

$$(2^{k_n} - 2)M_n < M_{n+1} < (2^{k_n} - 1)M_n. \tag{1}$$

2.3 Block entries

Block n consists of

$$M_n,\ 2M_n,\ 4M_n,\ \ldots,\ 2^{k_n-2}M_n,\ (2^{k_n-1} - 1)M_n + 1.$$

Concatenating all blocks yields a nondecreasing sequence A.

3 Limit of Consecutive Ratios

Inside a block, consecutive ratios equal 2 except at the final step:

$$\frac{(2^{k_n-1} - 1)M_n + 1}{2^{k_n-2}M_n} = 2 - \frac{1}{2^{k_n-2}} + \frac{1}{2^{k_n-2}M_n} = 2 + o(1).$$

Across block boundaries,

$$\frac{M_{n+1}}{(2^{k_n-1} - 1)M_n + 1} = 2 + O(2^{-k_n}) + O(M_n^{-1}).$$

As $k_n, M_n \to \infty$, all ratios tend to 2, hence

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2.$$

4 Density for Cofinite Subsequences

Let A' be cofinite in A. Then there exists n_0 such that all blocks with $n \geq n_0$ are contained in A'. Set $T := A_{\geq n_0}$.

4.1 Digit sets

Fix $n \geq n_0$ and write $k = k_n$. Define

$$c_n := (2^{k-1} - 1)M_n.$$

Set

$$B_n = \left\{ jM_n \mid 0 \leq j \leq 2^{k-1} - 1 \right\} \cup \left\{ jM_n + 1 \mid 2^{k-1} \leq j \leq 2^k - 2 \right\}.$$

Then $|B_n| = 2^k - 1$, $c_n \in B_n$, $c_n + 1 \notin B_n$, and $B_n \subset [0, M_{n+1})$.

Lemma 1 (Greedy covering)**.** *For every integer x with $0 \leq x \leq M_{n+1}$, there exists $b \in B_n$ such that*

$$0 \leq x - b \leq M_n.$$

4.2 Greedy expansion

Fix $N \geq n_0$. For each $m \leq M_{N+1}$ define recursively

$$r_{N+1} := m, \qquad b_n := \max\{b \in B_n : b \leq r_{n+1}\}, \qquad r_n := r_{n+1} - b_n,$$

for $n = N, \dots, n_0$. Then

$$m = \sum_{n=n_0}^{N} b_n + d, \qquad 0 \leq d \leq M_{n_0}. \tag{2}$$

4.3 Correction lemma

Lemma 2. *If at least M_{n_0} indices satisfy $b_n = c_n$, then $m \in P(T)$.*

Proof. Since each block contains both unshifted multiples jM_n and shifted ones $jM_n + 1$, the choice $b_n = c_n$ gives access to a $+1$ adjustment. With at least M_{n_0} such indices, the remainder $d \leq M_{n_0}$ can be absorbed by switching the block sum c_n to the last block element $c_n + 1$ in at most d positions. $\square$

4.4 Counting exceptions

Let E_N denote the number of integers $m \leq M_{N+1}$ not in $P(T)$. We give a detailed bound showing

$$\frac{E_N}{M_{N+1}} \longrightarrow 0 \quad (N \to \infty).$$

Fix $N \geq n_0$ and set

$$s_n := |B_n| = 2^{k_n} - 1, \qquad a_n := s_n - 1 = 2^{k_n} - 2.$$

For each $m \leq M_{N+1}$ the greedy expansion produces a unique vector

$$(b_{n_0}, \dots, b_N) \in B_{n_0} \times \cdots \times B_N$$

and a remainder $d \in [0, M_{n_0}]$, with

$$m = \sum_{n=n_0}^{N} b_n + d.$$

Thus the map $m \mapsto (b_{n_0}, \dots, b_N, d)$ is injective. By the correction lemma, $m \notin P(T)$ implies fewer than M_{n_0} indices with $b_n = c_n$. Therefore

$$E_N \leq (M_{n_0} + 1) \cdot \#\{(b_{n_0}, \dots, b_N) : \#\{n : b_n = c_n\} < M_{n_0}\}. \tag{3}$$

For a fixed subset $I \subset \{n_0, \dots, N\}$ of size j, the number of vectors with $b_n = c_n$ exactly for $n \in I$ is

$$\prod_{n \notin I} a_n,$$

since there are a_n non-special choices at $n \notin I$. Summing over all I with $|I| = j$ and then over $j \le M_{n_0} - 1$ gives

$$E_N \le (M_{n_0} + 1) \left(\prod_{n=n_0}^{N} a_n \right) \sum_{j=0}^{M_{n_0}-1} e_j \left(\frac{1}{a_{n_0}}, \dots, \frac{1}{a_N} \right), \tag{4}$$

where e_j is the jth elementary symmetric sum. Using the bound

$$e_j(x_1, \dots, x_r) \le \frac{(x_1 + \dots + x_r)^j}{j!},$$

we obtain

$$E_N \le (M_{n_0} + 1) \left(\prod_{n=n_0}^{N} a_n \right) \sum_{j=0}^{M_{n_0}-1} \frac{S_N^j}{j!}, \qquad S_N := \sum_{n=n_0}^{N} \frac{1}{a_n}.$$

On the other hand, the recurrence $M_{n+1} = \lfloor (a_n + \frac{1}{2}) M_n \rfloor$ implies $M_{n+1} \ge (a_n + \frac{1}{2}) M_n - 1$. Writing $M_{n+1} \ge (a_n + \frac{1}{2}) M_n \left(1 - \frac{1}{(a_n + \frac{1}{2}) M_n} \right)$ and using $k_n \ge 6$ (hence $a_n \ge 62$ and $M_{n+1} \ge a_n M_n$), we get $M_n \ge 10 \cdot 62^n$ and $\sum_n \frac{1}{(a_n + \frac{1}{2}) M_n} < \infty$. Therefore the product $\prod_n \left(1 - \frac{1}{(a_n + \frac{1}{2}) M_n} \right)$ is bounded below by a constant $c > 0$, and hence

$$M_{N+1} \ge c \cdot M_{n_0} \prod_{n=n_0}^{N} \left(a_n + \tfrac{1}{2} \right).$$

Hence

$$\frac{E_N}{M_{N+1}} \le \frac{C_{n_0}}{c} \left(\prod_{n=n_0}^{N} \frac{a_n}{a_n + \frac{1}{2}} \right) \left(\sum_{j=0}^{M_{n_0}-1} \frac{S_N^j}{j!} \right). \tag{5}$$

where $C_{n_0} := (M_{n_0} + 1)/M_{n_0}$. Since

$$\log \left(\frac{a_n}{a_n + \frac{1}{2}} \right) = -\log \left(1 + \frac{1}{2a_n} \right) \le -\frac{1}{4a_n}$$

for all n (using $a_n \ge 2^{k_n} - 2 \ge 62$), we have

$$\prod_{n=n_0}^{N} \frac{a_n}{a_n + \frac{1}{2}} \le \exp \left(-\frac{1}{4} S_N \right).$$

Because $2^{k_n} \asymp \log n$, we have $a_n \asymp \log n$ and therefore

$$S_N = \sum_{n=n_0}^{N} \frac{1}{a_n} \asymp \sum_{n=n_0}^{N} \frac{1}{\log n} \to \infty.$$

The exponential decay in (5) dominates the fixed polynomial factor $\sum_{j=0}^{M_{n_0}-1} S_N^j / j!$, so

$$\frac{E_N}{M_{N+1}} \to 0,$$

and $P(T)$ has asymptotic density 1.

5 Conclusion

The constructed sequence A satisfies

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2,$$

and for every cofinite subsequence $A' \subseteq A$, the set $P(A')$ has density 1. Hence the answer to the original problem is *yes*.

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