CMPT 210: Probability and Computing

Lecture 8

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Recap - Conditional Probability

For events E and F, we wish to compute $\Pr[E|F]$, the probability of event E conditioned on F.

Approach 1: With conditioning, F can be interpreted as the *new sample space* such that for $\omega \notin F$, $\Pr[\omega|F] = 0$.

Example: For computing Pr(we get a 6|the outcome is even), the new sample space is $F = \{2, 4, 6\}$ and the resulting probability space is uniform. $\Pr[\{\text{even number}\}] = \frac{1}{3}$ and $\Pr[\{\text{odd number}\}] = 0$.

Approach 2: $\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$.

Example: $E \cap F = \{6\}$. $\Pr[E \cap F] = \frac{1}{6}$. $\Pr[F] = \Pr[\{2\}] + \Pr[\{4\}] + \Pr[\{6\}] = \frac{1}{2}$. Hence, $\frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/6}{1/2} = \frac{1}{3}$.

Conditional Probability - Generalization to multiple events

Multiplication Rule: For events E_1, E_2, E_3, $\Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_1] \Pr[E_2|E_1] \Pr[E_3|E_1 \cap E_2]$.

Union: inclusion-exclusion

Find the probability that the sequence of events occur

Proof:

interesection: conditional probability

rhs: pr(E1) * pr(E2|E1) * pr

$$\Pr[E_1] \Pr[E_2|E_1] \Pr[E_3|E_1 \cap E_2] = \Pr[E_1] \frac{\Pr[E_2 \cap E_1]}{\Pr[E_1]} \frac{\Pr[E_1 \cap E_2 \cap E_3]}{\Pr[E_1 \cap E_2]} = \Pr[E_1 \cap E_2 \cap E_3]$$

We can order the events to compute $\Pr[E_1 \cap E_2 \cap E_3]$ more easily. For example,

$$\Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_2] \Pr[E_3|E_2] \Pr[E_1|E_2 \cap E_3]$$

Order of the events does not matter. You only need the events to occur sequentially

Can extend this to n events i.e. in general,

$$\Pr[E_1 \cap E_2 \ldots \cap E_n] = \Pr[E_1] \Pr[E_2|E_1] \Pr[E_3|E_1 \cap E_2] \ldots \Pr[E_n|E_1 \cap E_2 \cap \ldots E_{n-1}]$$

Conditional Probability - Examples

Q: The organization that Jones works for is running a father–son dinner for those employees having at least one son. Each of these employees is invited to attend along with his youngest son. If Jones is known to have two children, what is the conditional probability that they are both boys given that he is invited to the dinner? Assume that the sample space S is given by $S = \{(b,b),(b,g),(g,b),(g,g)\}$ and all outcomes are equally likely. For instance, (b,g) means that the younger child is a boy and the older child is a girl.

Uniform probability space

Scenario we are conditioning on

The event that we care about is Jones has both boys. Hence, $E = \{(b,b)\}$.

E: the probability that both of Hone's children are male

Additional information that we are conditioning on is that Jones is invited to the dinner meaning that he has at least one son. Hence, $F = \{(b,b),(b,g),(g,b)\}$.

F: at least one child is male

Hence, $E \cap F = \{(b,b)\}$, $\Pr[E \cap F] = \frac{|E \cap F|}{|\mathcal{S}|} = \frac{1}{4}$. $\Pr[F] = \frac{|F|}{|\mathcal{S}|} = \frac{3}{4}$.

$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/4}{3/4} = \frac{1}{3}$.

Conditional Probability - Examples

Conditional probability

Q. Ms. Perez figures that there is a 30 percent chance that her company will set up a branch office in Phoenix. If it does, she is 60 percent certain that she will be made manager of this new operation. What is the probability that there will be a branch in Phoenix and Perez will be its office manager?

intersect

E = Perez will be a branch office manager; F = her company will set up a branch office in Phoenix; $E \cap F$ = Perez will be an office manager in the Phoenix branch.

From the question, we know that $\Pr[F] = 0.3$, $\Pr[E|F] = 0.6$. Hence,
$\Pr[E \cap F] = \Pr[E]\,\Pr[E|F] = 0.3 \times 0.6 = 0.18$.

Conditional Probability Examples

Q: Suppose we have a bowl containing 6 white and 5 black balls. We randomly draw a ball. What is the probability that we draw a black ball?

Sampling without replacement

Q: We randomly draw two balls, one after the other (without putting the first back). What is the probability that we (i) draw a black ball followed by a white ball (ii) draw a white ball followed by a black ball (iii) we get one black ball and one white ball (iv) both black (v) both white?

B1 = Draw black first, W1 = Draw white first. B2 = Black second, W2 = White second.

(i) $\Pr[B1] = \frac{5}{11}$. $\Pr[W2|B1] = \frac{6}{10}$. Hence, $\Pr[B1 \cap W2] = \Pr[B1]\Pr[W2|B1] = \frac{30}{110}$.

(ii) $\Pr[W1] = \frac{6}{11}$. $\Pr[B2|W1] = \frac{5}{10}$. Hence, $\Pr[W1 \cap B2] = \Pr[W1]\Pr[B2|W1] = \frac{30}{110}$.

(iii) $G = (B1 \cap W2) \cup (W1 \cap B2)$. Events $B1 \cap W2$ and $B2 \cap W1$ are mutually exclusive. By the union rule for mutually exclusive events, $\Pr[G] = \Pr[B1 \cap W2] + \Pr[W1 \cap B2] = \frac{60}{110}$.

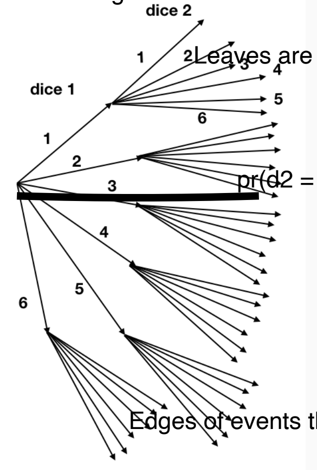
(iv) $\Pr[B1 \cap B2] = \Pr[B1]\Pr[B2|B1] = \frac{20}{110}$.

(v) $\Pr[W1 \cap W2] = \Pr[W1]\Pr[W2|W1] = \frac{30}{110}$.

Questions?

Q: Suppose we throw two standard dice one after the other. What is the probability that we get two 6's in a row?

Tree diagram consists of things we are doing sequentially



Leaves are our outcomes

pr(d2 = 3 |d1 = 3)

edge weights represent probabilities

Independent events do not affect each other

Identify Outcomes: Each leaf is an outcome and $\mathcal{S} = \{(1,1),(1,2),(1,3),\ldots(6,6)\}$.

Text
Text

Identify Event: $E = \{(6,6)\}$.

Compute probabilities: $\Pr[\text{Dice 1 is 6}] = \frac{1}{6}$.
$\Pr[(6,3)] = \Pr[\text{Dice 2 is 3} \cap \text{ Dice 1 is 6}] = \Pr[\text{Dice 2 is 3} \mid \text{Dice 1 is 6}]\ \Pr[\text{Dice 1 is 6}] = \frac{1}{6}\frac{1}{6} = \frac{1}{36}$.

$\Pr[E] = \Pr[\text{dice 1 is 6 } \cap \text{dice 2 is 6 }] = \frac{1}{36}$.

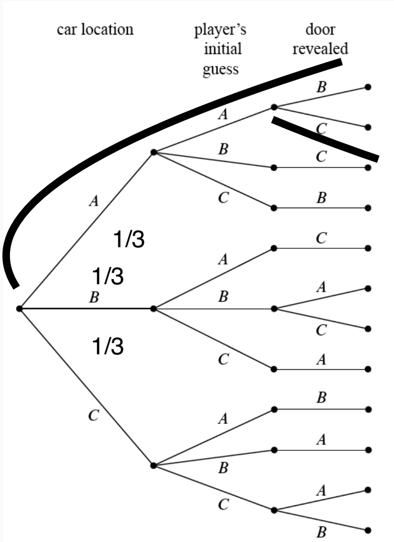
Edges of events that follow from another event are conditional probabilities

Monty Hall Problem

Q: Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say A, and the host, who knows what's behind the doors, opens another door, say C, which has a goat. He says to you, "Do you want to pick door B?" Is it to your advantage to switch your choice of doors?

- The car is equally likely to be hidden behind each of the three doors.
- The player is equally likely to pick each of the three doors, regardless of the car's location.
- After the player picks a door, the host must open a different door with a goat behind it and offer the player the choice of staying with the original door or switching.
- If the host has a choice of which door to open, then he is equally likely to select each of them.

Tree Diagram for the Monty Hall Problem - Identify Outcomes



Car

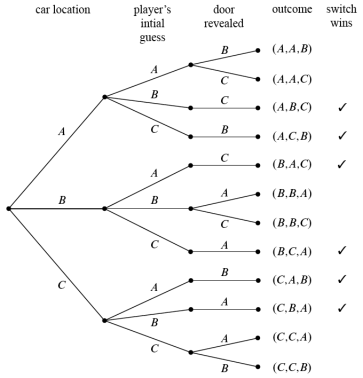
Since the player picks uniformly, the guess will always have a probability of 1/3

$\mathcal{S} = \{(A, A, B), (A, A, C), (A, B, C), (A, C, B), \ldots\}.$
$E_1 =$ Prize is behind door C $=$
$\{(C, A, B), (C, B, A), (C, C, A), (C, C, B)\}$

pr(host will open door B|(A, A))
(A, A) = car location is A, player's guess is A

All the probabilities that occur after conditioning have to sum to 1.

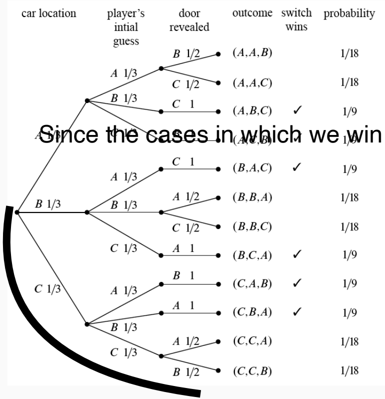
Tree Diagram for the Monty Hall Problem - Identify Event



$E =$ Switching wins $=$
$\{(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A)\}$

$\Pr[(A, A)] = \Pr[\text{Car is at A } \cap \text{ Player picks A }] =$
$\Pr[\text{Player picks A} \mid \text{Car is at A}] \Pr[\text{Car is at A}] = \frac{1}{3}\frac{1}{3} = \frac{1}{9}.$

$\Pr[(A, A, B)] = \Pr[\text{Door B is revealed } \cap \text{ AA}] =$
$\Pr[\text{Door B is revealed} \mid \text{AA}] \Pr[\text{AA}] = \frac{1}{2}\frac{1}{9} = \frac{1}{18}.$



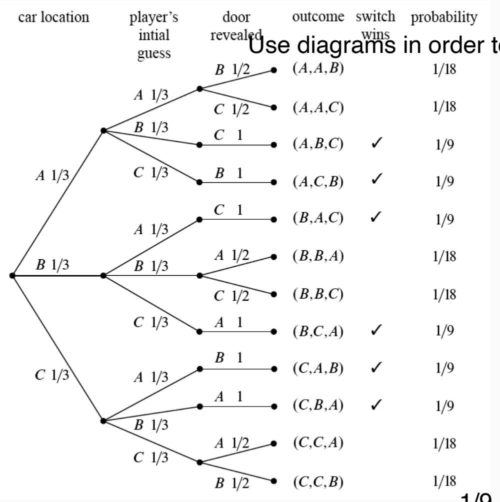
Since the cases in which we win are mutually disjoint, then we can sum up the probabilities.

$$\Pr[E] = \Pr[(A,B,C)] + \Pr[(A,C,B)] + \Pr[(B,A,C)] + \Pr[(B,C,A)] + \Pr[(C,A,B)] + \Pr[(C,B,A)] = \frac{1}{9} \times 6 = \frac{2}{3}.$$

If you switch in the scenario CCB, then you will lose

Monty Hall Problem and Conditional Probability

Use diagrams in order to avoid any errors from intuition



Q: What is the probability of winning by switching, if we pick door A and door B is opened.

$$\Pr[\text{win by switching} \mid \text{pick A and door B is opened}] = \frac{\Pr[\text{win by switching} \cap \text{pick A and door B is opened}]}{\Pr[\text{pick A and door B is opened}]}$$

$$= \frac{\Pr[(C,A,B)]}{\Pr[\{(A,A,B),(C,A,B)\}]} = \frac{1/9}{1/9+1/18} = \frac{2}{3}.$$

Q: Compute

$\Pr[\text{win by switching} \mid \text{pick A and door C is opened}]$?

1/9 1/9 + 1/18

1/9 * 18/3 = 2/3

With tree diagrams, it is easy to identify conditional probability

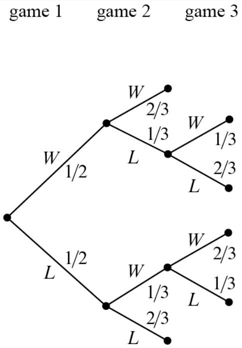
Questions?

Conditional Probability - Examples

If you have two edges coming out of a node, make sure they add up to one.

Q: In a best-of-three series, the local hockey team wins the first game with probability $\frac{1}{2}$. In subsequent games, their probability of winning is determined by the outcome of the previous game. If the team won the previous game, then they are invigorated by victory and win the current game with probability $\frac{2}{3}$. If they lost the previous game, then they are demoralized by defeat and win the current game with probability only $\frac{1}{3}$. What is the probability that the local team wins the series, given that they win the first game? Note that the series is over as soon as a team wins two games.

Conditional Probability - Examples



outcome	event A: win the series	event B: win game 1	outcome probability
WW	✓	✓	1/3
WLW	✓	✓	1/18
WLL		✓	1/9
LWW	✓		1/9
LWL			1/18
LL			1/3

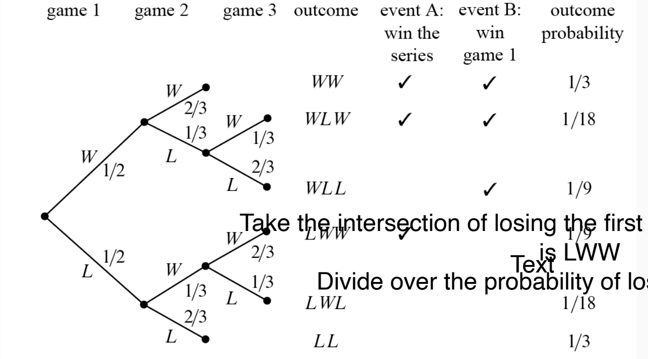
In this case, the probability of winning the next game is influenced by the probability of winning the last game

Sample space: $\mathcal{S} = \{(W, W), (W, L, W), (W, L, L), (L, W, W), (L, W, L), (L, L)\}$.

Events: $T = \{(W, W), (W, L, W), (L, W, W)\}$, $F = \{(W, W), (W, L, W), (W, L, L)\}$.

$$\Pr[T|F] = \frac{\Pr[T \cap F]}{\Pr[F]} = \frac{\Pr[\{(W, W), (W, L, W)\}]}{\Pr[\{(W, W), (W, L, W), (W, L, L)\}]} = \frac{1/3 + 1/18}{1/3 + 1/18 + 1/9} = \frac{7}{9}$$

Conditional Probability - Examples



Take the intersection of losing the first game and winning, which is LWW

Text

Divide over the probability of losing the first game

Q: What is the probability that the team wins the series if they lose Game 1?

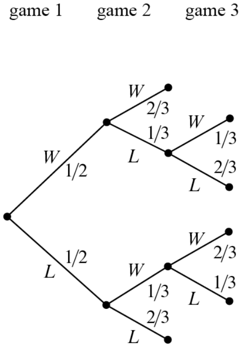
Q: What is the probability that the team wins the series?

The probability that a team wins is 1/2

Q: What is the probability that the series goes to Game 3?

1/3 : do 1 - p(series ends in two games) = 1 -1/3 -1/3 = 1/3

Conditional Probability - Examples



outcome	event A: win the series	event B: win game 1	outcome probability
WW	✓	✓	1/3
WLW	✓	✓	1/18
WLL		✓	1/9
LWW	✓		1/9
LWL			1/18
LL			1/3

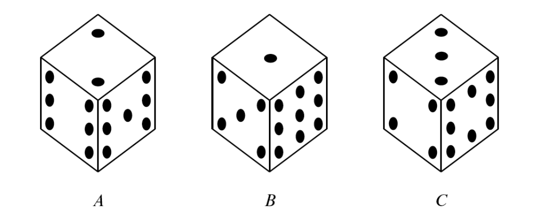
Q: What is the probability that the team won their first game given that they won the series?

Recall that $T = \{(W,W),(W,L,W),(L,W,W)\}$, $F = \{(W,W),(W,L,W),(W,L,L)\}$. We wish to compute $\Pr[F|T] = \frac{\Pr[F \cap T]}{\Pr[T]} = \frac{\Pr[\{(W,W),(W,L,W)\}]}{\Pr[\{(W,W),(W,L,W),(L,W,W)\}]} = \frac{1/3+1/18}{1/3+1/18+1/9} = \frac{7}{9}$.

Remember: this is not a uniform probability space

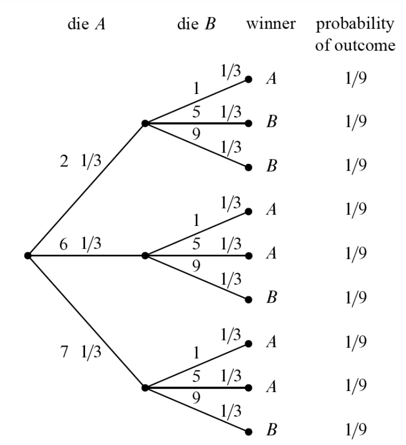
Conditional Probability - Examples

Let us play a game with three strange dice shown in the figure. Each player selects one die and rolls it once. The player with the lower value pays the other player \$100. We can pick a die first, after which the other player can pick one of the other two.



Q: Suppose we choose die B because it has a 9, and the other player selects die A. What is the probability that we will win?

Conditional Probability - Examples



Identify Outcomes: Each leaf is an outcome and $\mathcal{S} = \{(2,1),(2,5),(2,9),(6,1),(6,5),(6,9),(7,1),(7,5),(7,9)\}$.

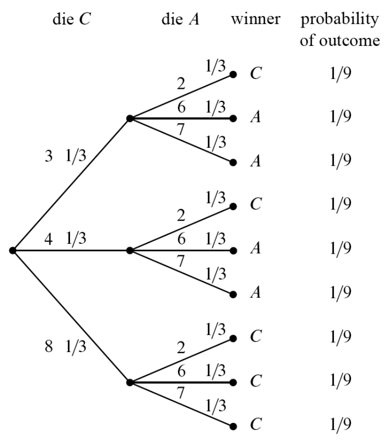
Identify Event: $E = \{(2,5),(2,9),(6,9),(7,9)\}$.

Compute probabilities: $\Pr[\text{Dice 1 is 6}] = \frac{1}{3}$.
$\Pr[(6,5)] = \Pr[\text{Dice 2 is 5} \ \cap \ \text{Dice 1 is 6}] = \Pr[\text{Dice 2 is 5} \mid \text{Dice 1 is 6}] \Pr[\text{Dice 1 is 6}] = \frac{1}{3}\frac{1}{3} = \frac{1}{9}$.

$\Pr[E] = \Pr[(2,5)] + \Pr[(2,9)] + \Pr[(6,9)] + \Pr[(7,9)] = \frac{4}{9}$.
Meaning that there is less than 50% chance of winning.

Conditional Probability - Examples

Q: We get another chance – this time we know that die A is good (since we lost to it previously), we choose die A and the other player chooses die C. What is our probability of winning?



Now, $E = \{(3,6),(3,7),(4,6),(4,7)\}$ and hence $\Pr[E] = \frac{4}{9}$. Meaning that there is less than 50% chance of winning.

Conditional Probability - Examples

We get yet another chance, and this time we choose die C, because we reason that die A is better than B, and C is better than A.

We can construct a similar tree diagram to show that the probability that we win is again $\frac{4}{9}$.

- A beats B with probability $\frac{5}{9}$ (first game).
- C beats A with probability $\frac{5}{9}$ (second game).
- B beats C with probability $\frac{5}{9}$ (third game).

Since A will beat B more often than not, and B will beat C more often than not, it seems like A ought to beat C more often than not, that is, the "beats more often" relation ought to be transitive. But this intuitive idea is false: whatever die we pick, the second player can pick one of the others and be likely to win. So picking first is actually a disadvantage!

This is the topic of some recent research and was covered in this article:
`https://www.quantamagazine.org/mathematicians-roll-dice-and-get-rock-paper-scissors-20230119/`