CMPT 210: Probability and Computing

Lecture 6

Sharan Vaswani

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Recap

Sample (outcome) space $\mathcal{S}$: Nonempty (countable) set of possible outcomes. Example: When we threw one dice, the sample space is $\{1, 2, 3, 4, 5, 6\}$.

Outcome $\omega \in \mathcal{S}$: Possible "thing" that can happen. Example: When we threw one dice, a possible outcome is $\omega = 1$.

Event E: Any subset of the sample space. Example: When we threw one dice, a possible event is $E = \{6\}$ (first example) or $E = \{3, 6\}$ (second example).

Probability function on a sample space $\mathcal{S}$ is a total function $\Pr : \mathcal{S} \to [0, 1]$. For any $\omega \in \mathcal{S}$,

$$0 \leq Pr[\omega] \leq 1 \quad ; \quad \sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1 \quad ; \quad \Pr[E] = \sum_{\omega \in E} \Pr[\omega]$$

Union: For mutually exclusive events $E_1, E_2, \ldots, E_n$,
$\Pr[E_1 \cup E_2 \cup \ldots E_n] = \Pr[E_1] + \Pr[E_2] + \ldots + \Pr[E_n]$.

Probability Rules

Complement rule: $\Pr[E] = 1 - \Pr[E^c]$.

Proof: Recall that $E \cap E^c = \{\}$ and $E \cup E^c = \mathcal{S}$. Since E and E^c are disjoint,

$$\Pr[E \cup E^c] = \Pr[E] + \Pr[E^c] \implies \Pr[S] = \Pr[E] + \Pr[E^c] \implies \Pr[E^c] \quad = 1 - \Pr[E].$$

Inclusion-Exclusion rule: For any two events E, F, $\Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$.

Know how to do this proof

May be on midterm, but it is unlikely

Proof:

$$\Pr[E \cup F] = \sum_{\omega \in \{E \cup F\}} \Pr[\omega] = \sum_{\omega \in \{E - F\}} \Pr[\omega] + \sum_{\omega \in \{F - E\}} \Pr[\omega] + \sum_{\omega \in \{E \cap F\}} \Pr[\omega]$$

(Since disjoint)

$$= \left[\sum_{\omega \in \{E - F\}} \Pr[\omega] + \sum_{\omega \in \{E \cap F\}} \Pr[\omega]\right] + \left[\sum_{\omega \in \{F - E\}} \Pr[\omega] + \sum_{\omega \in \{E \cap F\}} \Pr[\omega]\right] - \sum_{\omega \in \{E \cap F\}} \Pr[\omega]$$

$$= \sum_{\omega \in E} \Pr[\omega] + \sum_{\omega \in F} \Pr[\omega] - \sum_{\omega \in \{E \cap F\}} \Pr[\omega] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$$

Probability Rules

Used in machine learning algorithms

All probabilities are non-negative

Union Bound: For any two events E, F, $\Pr[E \cup F] \leq \Pr[E] + \Pr[F]$.

Proof: By the inclusion-exclusion rule, $\Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$. Since probabilities are non-negative, $\Pr[E \cap F] \geq 0$ and hence, $\Pr[E \cup F] \leq \Pr[E] + \Pr[F]$.

Union Bound: For any events $E_1, E_2, E_3, \ldots E_n$,

$$\Pr[E_1 \cup E_2 \cup E_3 \ldots \cup E_n] \leq \sum_{i=1}^{n} \Pr[E_i]$$

Use inclusion of

The events contained in B that are not within A have a non-zero probability of occuring

so pr(B - A) > 0

Monotonicity rule: For events A and B, if $A \subset B$, then $\Pr[A] < \Pr[B]$.

B = A U (B - A)

Proof:

pr(B) = pr(A) + pr(B - A)

$$\Pr[A] = \sum_{\omega \in A} \Pr[\omega] = \sum_{\omega \in B} \Pr[\omega] - \sum_{\omega \in \{B - A\}} \Pr[\omega] \implies \Pr[A] < \Pr[B]$$

(Since probabilities are non-negative.)

Uniform Probability Spaces

Events can have different probabilites, but outcomes must have different probabilities

Definition: A probability space is uniform if $\Pr[\omega]$ is the same for every outcome $\omega \in \mathcal{S}$.

Since $\sum_{\omega \in \mathcal{S}} Pr[\omega] = 1 \implies \Pr[\omega] = \frac{1}{|\mathcal{S}|}$ for all $\omega \in \mathcal{S}$.

Example: For a standard dice, $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$, $\Pr[\{1\}] = \Pr[\{2\}] = \ldots = \Pr[\{6\}] = 1/6$.

$\Pr[E] = \sum_{\omega \in E} Pr[\omega] = |E|\, Pr[\omega] = \frac{|E|}{|\mathcal{S}|}$.

Example: For a standard dice, if $E = \{3, 6\}$, then, $\Pr[E] = \frac{|E|}{|\mathcal{S}|} = \frac{2}{6} = 1/3$.

Hence, for uniform probability spaces, computing the probability is equivalent to counting the outcomes we "care" about.

For uniform probability space, calculating probabilities is equivalent to counting size of space.

Back to throwing dice

e the probability of getting an odd number.

p(1) = p(3) = p(5) = p

ability of getting an even number is 2p

pr(2) = pr(4) = pr(6) = 2p

p probabilites of all outcomes must equal one.

9p = 1

p = 1/9

2p = 2/9

Q: Suppose we have a loaded (not "standard") dice such that the probability of getting an even number is twice that of getting an odd number (all even numbers are equally likely, and so are the odd numbers). What is the probability of getting a 6?

Let p be the probability of getting an odd number. Probability of getting an even number $= 2p$.

$\sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1 \implies 3p + 3(2p) = 1 \implies p = \frac{1}{9}$. Hence, probability of getting an odd number $= \frac{1}{9}$. Probability of getting a 6 $=$ Probability of getting an even number $= \frac{2}{9}$.

Q: What is the probability that we get either a 3 or a 6? 3/9 = 1/3

Q: What is the probability that we get a prime number

1 is not prime.

P(get 2, 3, 5) =

+ 2/9 + 1/9 + 1/9 =

4/9

Probability Examples

Q: Suppose we select a card at random from a standard deck of 52 cards. What is the probability of getting:

- A spade 13/ 52 = 1/4
- A spade facecard 3/52 =
- A black card 26/52 = 1/2
- The queen of hearts 1/52
- An ace 4/52 = 1/13

Probability Examples

Q: A class consists of 6 men and 4 women. An exam is given and the students are ranked according to their performance. Assuming that no two students obtain the same scores and all rankings are considered equally likely, what is the probability that women receive the top 4 scores?

Uniform probability space

Change it into rearranging string of 1111 000000

In general, let the number of men be m and let the number of women be w.

Number of possible rankings = Number of all permutations = $(m+w)!$.

S = set of all possible rankings

E -> set of the rankings where the four women get the top scores

The event of interest is that where the women achieve the top scores.

pr(E) = |E|/|s| since it is a uniform probability space

In a possible ranking, let's fix the top w slots for women. The w women can be arranged in $w!$ ways. And the m men can be arranged in $m!$ ways. Hence, total number of rankings where women receive the top scores = $m! \quad w!$.

Since all rankings are equally likely, probability that women receive the top w scores = $\frac{m!w!}{(m+w)!}$.
In this case, since $m = 6$ and $w = 4$, probability that women receive the top 4 scores = $\frac{6!\,4!}{10!}$.

Probability Examples

S -> all rankings

E -> women receive the top t scores

|S| = (m + w)!

Q: A class consists of m men and w women. An exam is given and the students are ranked according to their performance. Assuming that no two students obtain the same scores and all rankings are considered equally likely, what is probability that women receive the top t ($t \leq w$) scores?

We must select t amount of women from the set to receive the top rankings

Anyone else can be arranged in any way

Number of ways to select the t women that have top scores $= \binom{w}{t}$. The top t women can be arranged in $t!$ ways. The number of remaining students is equal to $m + w - t$. These can be arranged in $(m + w - t)!$ ways. Hence, total number of rankings where women receive the top t scores $= \binom{w}{t} \quad (m + w - t)! \quad t!$.

As before, the total number of rankings $= (m + w)!$. Since all rankings are equally likely, the probability that women receive the top t scores $= \frac{\binom{w}{t}\,(m+w-t)!\,t!}{(m+w)!} = \frac{w!\,(m+w-t)!}{(w-t)!(m+w)!}$

If t = w, then we get the answer on the previous slide selection made randomly => uniform probability space

E = set of all committees of size five which has 3 cs and two math students

S = set of all committees of size 5

Q: A committee of size 5 is to be selected from a group of 6 CS and 9 Math students (no double majors allowed). If the selection is made randomly, what is the probability that the committee consists of 3 CS and 2 Math students?

Number of possible ways of selecting the committee $= |\mathcal{S}| = \binom{15}{5}$.

The event of interest (E) requires choosing 3 CS and 2 Math students. Number of ways we can select the CS students $= \binom{6}{3}$. Similarly, number of ways we can select the Math students $= \binom{9}{2}$.

Can leave answer as this.

Hence, $|E| = \binom{6}{3}\binom{9}{2} \implies \Pr[E] = \frac{|E|}{|\mathcal{S}|} = \frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}}$.

On exam, you must define was S and E is.

ecause for every selection of the math students, you have a selection of the cs students

Probability Examples

S: ways to select k items

Each item is equally likely

E: selections of k items that has alpha.

pr(E) = |E|/|S|

Q: From a set of n items a random sample of size k is to be selected. What is the probability a given item (α) will be among the k selected items?

Number of ways of choosing the sample $= \binom{n}{k}$.

If we want a particular item in the sample, number of ways of choosing the other items $= \binom{n-1}{k-1}$.

Hence, probability that a given item will be among the k selected $= \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$.

Probability Examples

E -> event that alpha is one of the k selected

F -> event that B is selected

Q: From a set of n items a random sample of size k is to be selected. Given two items of interest: α and β, what is the probability that (i) both α and β will be among the k selected (ii) at least one of α or β will be among the k selected (iii) neither α nor β will be among the k selected?

(i) If we want both α and β to be in the sample, number of ways of choosing the other items $= \binom{n-2}{k-2}$. Hence, probability that both α and β will be in the sample $= \frac{\binom{n-2}{k-2}}{\binom{n}{k}} = \frac{k(k-1)}{n(n-1)}$.

(ii) Let A be the event that item α is in the selection. $\Pr[A] = \frac{k}{n}$. Similarly B be the event that item β is in the selection. $\Pr[B] = \frac{k}{n}$. We want to compute $\Pr[A \cup B]$. By the union-rule, $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$. Hence, probability that either α or β will be among the k selected items $= \frac{2k}{n} - \frac{k(k-1)}{n(n-1)}$.

pr(ii) +
pr(iii) = 1

(iii) If we want neither α nor β to be in the sample, number of ways of choosing the items $= \binom{n-2}{k}$. Hence, probability that neither α nor β will be in the sample $= \frac{\binom{n-2}{k}}{\binom{n}{k}} = \frac{(n-k)(n-k-1)}{n(n-1)}$.

Calculate pr(E U F) = pr(E) + pr(F) - pr(EF)

Another way is to take the complement of pr(alpha and beta not in selection)

Probability - Examples

All permutations are equally likely
S -> all possible permuations
E -> all possible permutations where the third letter is B

Q: Let us consider random permutations of the letters (i) ABBA (ii) ABBA'. What is the probability that the third letter is B?

A' is different from A.

a: 3!/2! ways of rearranging remaining letters.
There is 4!/2!2! ways to rearrange all the lltters.
The probability is 3/6 = 1/2

Assuming a' is unique: there are 4!/2! ways to rearrange the letters
here are 3! ways to rearrange the letters such that the third letter is B
the probability is 3! / 4! 2!

Questions?

Birthday Paradox

When using pidgeon hole principle, define the function and say what is a and b

75

Q: There are [redacted] students in the class. What is the probability that two students have their birthdays in the same week? 1 by the pidgeon hole principle

Q: In this class, what is the probability that two students share the same birthday? Assume that (i) each student is equally likely to be born on any day of the year, (ii) no leap years and (iii) student birthdays are independent of each other.

Let n be the number of students, and let d be the number of days in the year. Let's order the students according to their ID. A birthday sequence is (11 Feb, 23 April, 31 August, ...). First let's count the number of possible birthday sequences.

The first student's birthday can be one of d days. Similarly, the second student's birthday can be one of d days, and so on. By the product rule, the total number of birthday sequences $=$ $d \times d \times \ldots = d^n$.

Let n -> students.

d -> # of days

S -> set of all possible birthday sequences

E -> set of all birthday sequences where two students have the same birthday

sequence example: (11 Feb, 23 March ...)

|S| = 365^75

Birthday Paradox

The event of interest is that two students share the same birthday. Let us compute the probability of the event that NO two students share the same birthday, and then use the complement rule.

The first birthday can be chosen in d ways, the second in $d-1$ ways, and so on. By the generalized product rule, the number of birthday sequences such that no birthday is shared $=$ $d \times (d-1) \times (d-2) \times \ldots (d-(n-1))$.

Hence, the probability that no two students share the same birthday

$$= \frac{\text{the number of birthday sequences such that no birthday is shared}}{\text{total number of birthday sequences}} = \frac{d \times (d-1) \times (d-2) \times \ldots (d-(n-1))}{d^n}$$

Text
Text

$$= \left(1 - \frac{0}{d}\right) \times \left(1 - \frac{1}{d}\right) \ldots \left(1 - \frac{n-1}{d}\right) \leq \exp(-0/d) \times \exp(-1/d) \ldots \exp(-(n-1)/d)$$

$$(\text{for } x > 0,\ 1 - x \leq \exp(-x))$$

$$= \exp\left(\frac{-0}{d} + \frac{-1}{d} + \ldots \frac{-(n-1)}{d}\right) = \exp\left(-\frac{n(n-1))}{2d}\right)$$

Text

Birthday Paradox

Probability that two students share a birthday $\geq 1 - \exp\left(-\frac{n(n-1))}{2d}\right)$. Let's plot for $d = 365$.

As n increases, the value of the expression goes closer to 1.

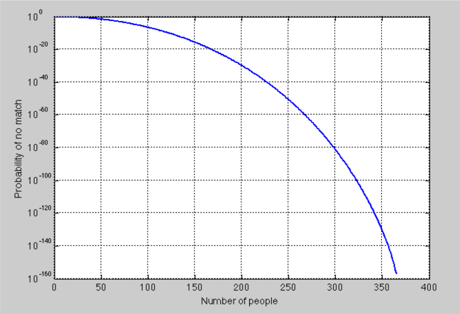

Figure 1: Plotting $\exp\left(-\frac{n(n-\mathbf{1}))}{\mathbf{2}d}\right)$ for $d = 365$

In our class, there is $> 96.4\%$ that two students have the same birthday!

Birthday Principle

If there are n pigeons and d pigeonholes, then the probability that two pigeons occupy the same hole is $\geq 1 - \exp\left(-\frac{n(n-1))}{2d}\right)$

For $n = \lceil\sqrt{2d}\rceil$, probability that two pigeons occupy the same hole is about $1 - \frac{1}{e} \approx 0.632$.

Example: If we are randomly throwing $\lceil\sqrt{2d}\rceil$ balls into d bins, then the probability that two balls land in the same bin is around 0.632.

Later in the course, we will see applications of this principle to load balancing.

Questions?