CMPT 210: Probability and Computing

Lecture 19

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March 21, 2024

Recap

Variance: Standard way to measure the deviation from the mean. For r.v. X, $\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in \text{Range}(X)} (x - \mu)^2 \Pr[X = x]$, where $\mu := \mathbb{E}[X]$.

Alternate Definition: $\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

If $X \sim \text{Ber}(p)$, $\text{Var}[X] = p(1 - p)$.

If $X \sim \text{Uniform}(\{v_1, v_2, \ldots v_n\})$, $\text{Var}[X] = \frac{[v_1^2 + v_2^2 + \ldots v_n^2]}{n} - \left(\frac{[v_1 + v_2 + \ldots v_n]}{n}\right)^2$.

Variance - Examples

Q: If $R \sim \text{Geo}(p)$, calculate $\text{Var}[R]$.

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Recall that for a coin s.t. $\Pr[\text{heads}] = p$, R is the r.v. equal to the number of coin tosses we need to get the first heads. Let A be the event that we get a heads in the first toss. Using the law of total expectation,

$$\mathbb{E}[R^2] = \mathbb{E}[R^2|A]\Pr[A] + \mathbb{E}[R^2|A^c]\Pr[A^c]$$

$\mathbb{E}[R^2|A] = 1$ ($R^2 = 1$ if we get a heads in the first coin toss) and $\Pr[A] = p$. Hence,

$$\mathbb{E}[R^2] = (1)\,(p) + \mathbb{E}[R^2|A^c]\,(1-p) \quad ; \quad \mathbb{E}[R^2|A^c] = \sum_{k=1} k^2 \Pr[R = k|A^c]$$

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$\mathbb{E}[R^2|A] = 1$ ($R^2 = 1$ if we get a heads in the first coin toss) and $\Pr[A] = p$. Hence,

k-2 since the probability of the first attempt
failing is not considered.

$$\mathbb{E}[R^2] = (1)(p) + \mathbb{E}[R^2|A^c](1-p) \quad ; \quad \mathbb{E}[R^2|A^c] = \sum_{k=1} k^2 \Pr[R = k|A^c]$$

Note that $\Pr[R = k|A^c] = \Pr[R = k|\text{ first toss is a tails}] = (1-p)^{k-2}p = \Pr[R = k-1]$

$$\implies \mathbb{E}[R^2|A^c] = \sum_{k=1} k^2 \Pr[R = k-1] = \sum_{t=0} (t+1)^2 \Pr[R = t] \qquad (t := k-1)$$

g(R) = (R + 1)^2
E[g(R)] = sum(g(R) * pr(R = x))

Variance - Examples

Continuing from the previous slide,

For the t = 0 term, it always has zero probability of occurring.

$$\mathbb{E}[R^2|A^c] = \sum_{t=0}(t+1)^2 \Pr[R=t] = \sum_{t=0} t^2 \Pr[R=t] + 2\sum_{t=0} t \Pr[R=t] + \sum_{t=0} \Pr[R=t]$$

$$= \sum_{t=1} t^2 \Pr[R=t] + 2\sum_{t=1} t \Pr[R=t] + \sum_{t=1} \Pr[R=t] = \mathbb{E}[R^2] + 2\mathbb{E}[R] + 1$$

Variance - Examples

Continuing from the previous slide,

$$\begin{aligned}\mathbb{E}[R^2|A^c] &= \sum_{t=0}(t+1)^2 \Pr[R=t] = \sum_{t=0} t^2 \Pr[R=t] + 2\sum_{t=0} t \Pr[R=t] + \sum_{t=0} \Pr[R=t] \\ &= \sum_{t=1} t^2 \Pr[R=t] + 2\sum_{t=1} t \Pr[R=t] + \sum_{t=1} \Pr[R=t] = \mathbb{E}[R^2] + 2\mathbb{E}[R] + 1\end{aligned}$$

Putting everything together,

$$\mathbb{E}[R^2] = (1)\,(p) + (\mathbb{E}[R^2] + 2\mathbb{E}[R] + 1])\,(1-p) \implies p\,\mathbb{E}[R^2] = p + 2(1-p)\mathbb{E}[R] + (1-p)\mathbb{E}[1]$$

$$\implies p\,\mathbb{E}[R^2] = p + \frac{2(1-p)}{p} + (1-p) \qquad (\mathbb{E}[R] = \tfrac{1}{p},\ \mathbb{E}[1] = 1)$$

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$$\implies \mathbb{E}[R^2] = \frac{2(1-p)}{p^2} + \frac{1}{p} \implies \mathbb{E}[R^2] = \frac{2-p}{p^2}$$

$$\implies \mathrm{Var}[R] = \mathbb{E}[R^2] - (\mathbb{E}[R])^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

As p -> 1, Var[R] -> 0
As p -> 0. Var[R] -> infinity

Variance - Examples

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$$\begin{aligned}\mathbb{E}[R^2|A^c] &= \sum_{t=0}(t+1)^2\Pr[R=t] = \sum_{t=0} t^2\Pr[R=t] + 2\sum_{t=0} t\ \Pr[R=t] + \sum_{t=0}\Pr[R=t] \\ &= \sum_{t=1} t^2\Pr[R=t] + 2\sum_{t=1} t\ \Pr[R=t] + \sum_{t=1}\Pr[R=t] = \mathbb{E}[R^2] + 2\mathbb{E}[R] + 1\end{aligned}$$

Putting everything together,

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Standard Deviation

Standard Deviation: For r.v. X, the standard deviation in X is defined as:

$$\sigma_X := \sqrt{\mathrm{Var}[X]} = \sqrt{\mathbb{E}[X^2] - (\mathbb{E}[X])^2}$$

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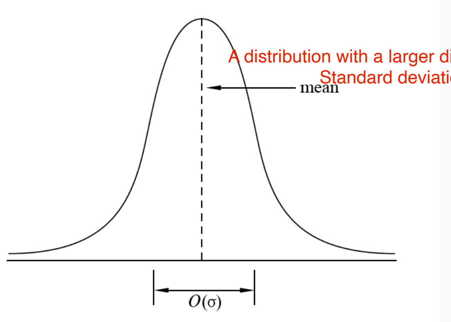

A distribution with a larger difference in the middle has a larger standard deviation.
Standard deviation has the same units as the expectations.

Standard deviation for a "bell"-shaped distribution indicates how wide the "main part" of the distribution is.

Properties of Variance

Adding a b only shifts the distribution, but does not change the distance to the mean.

Q: For constants a, b and r.v. R, $\text{Var}[a\,R + b] = a^2\text{Var}[R]$.

E[(aR + b)^2] - E^2(aR + b)
E[a^2R^2 + aRb + aRb b^2]
= a^2E[R^2] + 2abE[R] + E[b^2]

E^2[aR + b] = (aE[R] + b)^2 = a^2E[R] + 2baE[R] + b^2
Difference of the two expressions:
a^2(E[R^2] - E[R])

Properties of Variance

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Proof:

$$
\begin{aligned}
\text{Var}[aR + b] &= \mathbb{E}[(aR+b)^2] - (\mathbb{E}[aR+b])^2 = \mathbb{E}[a^2 R^2 + 2abR + b^2] - (\mathbb{E}[aR] + \mathbb{E}[b])^2 \\
&= (a^2\mathbb{E}[R^2] + 2ab\mathbb{E}[R] + b^2) - (a\mathbb{E}[R] + b)^2 \\
&= (a^2\mathbb{E}[R^2] + 2ab\mathbb{E}[R] + b^2) - (a^2(\mathbb{E}[R])^2 + 2ab\mathbb{E}[R] + b^2) \\
&= a^2 \left[\mathbb{E}[R^2] - (\mathbb{E}[R])^2\right] \\
\implies \text{Var}[aR + b] &= a^2 \text{Var}[R]
\end{aligned}
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&= (a^2\mathbb{E}[R^2] + 2ab\mathbb{E}[R] + b^2) - (a\mathbb{E}[R] + b)^2 \\
&= (a^2\mathbb{E}[R^2] + 2ab\mathbb{E}[R] + b^2) - (a^2(\mathbb{E}[R])^2 + 2ab\mathbb{E}[R] + b^2) \\
&= a^2\left[\mathbb{E}[R^2] - (\mathbb{E}[R])^2\right]
\end{aligned}
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$$\implies \text{Var}[aR+b] = a^2\text{Var}[R]$$

Similarly, for the standard deviation,

$$\sigma_{aR+b} = \sqrt{\text{Var}[aR+b]} = \sqrt{a^2\text{Var}[R]} = |a|\,\sigma_R$$

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&= (a^2\mathbb{E}[R^2] + 2ab\mathbb{E}[R] + b^2) - (a\mathbb{E}[R] + b)^2 \\
&= (a^2\mathbb{E}[R^2] + 2ab\mathbb{E}[R] + b^2) - (a^2(\mathbb{E}[R])^2 + 2ab\mathbb{E}[R] + b^2) \\
&= a^2\left[\mathbb{E}[R^2] - (\mathbb{E}[R])^2\right]
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Similarly, for the standard deviation,

$$\sigma_{aR+b} = \sqrt{\mathrm{Var}[aR+b]} = \sqrt{a^2\mathrm{Var}[R]} = |a|\,\sigma_R$$

Note the difference from the property of expectation,

$$\mathbb{E}[aR+b] = a\mathbb{E}[R] + b$$

Properties of Variance

Recall that for r.v's R and S, $\mathbb{E}[R+S] = \mathbb{E}[R] + \mathbb{E}[S]$. In general, such a property is not true for the variance, i.e. variance of a sum is not necessarily equal to the sum of the variances.

Var[R + S] = E[(R + S)^2] - E^2[R + S]

If the r.v's R and S are *independent*, $\mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S]$.

(R + S)^2 = R^2 + S^2 + 2RS

E[(R + S)^2] = E[R^2] + E[S^2] + 2E[R]E[S]

E[R + S] = E[R] + E[S]

E^2[R + S] = E^2[R] + E^2[S] + 2E[R]E[S]

Difference of both terms:

E[R^2] - E^2[R] + E[S^2] - E^2[S] = Var[R] + Var[S]

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Proof:

$$\begin{aligned}\text{Var}[R+S] &= \mathbb{E}[(R+S)^2] - (\mathbb{E}[R+S])^2 = \mathbb{E}[R^2 + S^2 + 2RS] - (\mathbb{E}[R] + \mathbb{E}[S])^2 \\ &= \mathbb{E}[R^2 + S^2 + 2RS] - [(\mathbb{E}[R])^2 + (\mathbb{E}[S])^2 + 2\mathbb{E}[R]\,\mathbb{E}[S]]\end{aligned}$$

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&= \mathbb{E}[R^2 + S^2 + 2RS] - [(\mathbb{E}[R])^2 + (\mathbb{E}[S])^2 + 2\mathbb{E}[R]\,\mathbb{E}[S]] \\
&= [\mathbb{E}[R^2] - (\mathbb{E}[R])^2] + [\mathbb{E}[S^2] - (\mathbb{E}[S])^2] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]) \\
&= \text{Var}[R] + \text{Var}[S] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S])
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\end{aligned}$$

Recall that if r.v. are independent, $\mathbb{E}[RS] = \mathbb{E}[R]\,\mathbb{E}[S]$,

$$\implies \text{Var}[R+S] = \text{Var}[R] + \text{Var}[S]$$

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&= \text{Var}[R] + \text{Var}[S] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S])
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Properties of Variance

Pairwise Independence: Random variables $R_1, R_2, R_3, \ldots R_n$ are *pairwise* independent if for any pair R_i and R_j, for $x \in \text{Range}(R_i)$ and $y \in \text{Range}(R_j)$, events $\Pr[R_i = x]$ and $\Pr[R_j = y]$ are pairwise independent implying that $\Pr[(R_i = x) \cap (R_j = y)] = \Pr[R_i = x] \Pr[R_j = y]$.

Pairwise independence does not imply mutual independence, but the converse is true.

Mutual independence are for any subset of events.

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We can prove that for any pair of pairwise independent r.v's, R_i and R_j, $\mathbb{E}[R_i R_j] = \mathbb{E}[R_i] \, \mathbb{E}[R_j]$.

Similar to E[R1R2] = Pr(R1)Pr(R2)

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For pairwise independent random variables $R_1, R_2, R_3, \ldots R_n$, $\text{Var}[\sum_{i=1}^n R_i] = \sum_{i=1}^n \text{Var}[R_i]$.

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For pairwise independent random variables $R_1, R_2, R_3, \ldots R_n$, $\text{Var}[\sum_{i=1}^{n} R_i] = \sum_{i=1}^{n} \text{Var}[R_i]$.

$$\textit{Proof}: \text{Var}[R_1 + R_2 + \ldots R_n] = \mathbb{E}[(R_1 + R_2 + \ldots R_n)^2] - (\mathbb{E}[R_1 + R_2 + \ldots R_n])^2$$

This is an instance of the multinomial theorem.

$$= \sum_{i=1}^{n} [\mathbb{E}[R_i^2] - (\mathbb{E}[R_i])^2] + 2 \sum_{i,j \mid 1 \le i < j \le n} [\mathbb{E}[R_i R_j] - \mathbb{E}[R_i]\,\mathbb{E}[R_j]]$$

If the r.v's are independent, this term is zero.

$$\implies \text{Var}[R_1 + R_2 + \ldots R_n] = \sum_{i=1}^{n} \text{Var}[R_i] \qquad \text{(Since the r.v's are pairwise independent)}$$

Properties of Variance

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We can prove that for any pair of pairwise independent r.v's, R_i and R_j, $\mathbb{E}[R_i R_j] = \mathbb{E}[R_i]\,\mathbb{E}[R_j]$.

For pairwise independent random variables $R_1, R_2, R_3, \ldots R_n$, $\text{Var}[\sum_{i=1}^n R_i] = \sum_{i=1}^n \text{Var}[R_i]$.

$$\textit{Proof}: \text{Var}[R_1 + R_2 + \ldots R_n] = \mathbb{E}[(R_1 + R_2 + \ldots R_n)^2] - (\mathbb{E}[R_1 + R_2 + \ldots R_n])^2$$

$$= \sum_{i=1}^n [\mathbb{E}[R_i^2] - (\mathbb{E}[R_i])^2] + 2 \sum_{i,j|1 \leq i < j \leq n} [\mathbb{E}[R_i R_j] - \mathbb{E}[R_i]\,\mathbb{E}[R_j]]$$

$$\implies \text{Var}[R_1 + R_2 + \ldots R_n] = \sum_{i=1}^n \text{Var}[R_i] \qquad \text{(Since the r.v's are pairwise independent)}$$

Importantly, we do not require the r.v's to be mutually independent. Mutual independence $\implies$ pairwise independence, but pairwise independence $\not\Rightarrow$ mutual independence.

Variance - Examples

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Define R_i to be the indicator random variable that we get a heads in toss i of the coin. Recall that R is the random variable equal to the number of heads in n tosses.

Each of the R_i's are mutually independent

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Define R_i to be the indicator random variable that we get a heads in toss i of the coin. Recall that R is the random variable equal to the number of heads in n tosses.

Hence,

$$R = R_1 + R_2 + \ldots + R_n \implies \text{Var}[R] = \text{Var}[R_1 + R_2 + \ldots + R_n]$$

Since $R_1, R_2, \ldots, R_n$ are mutually independent indicator random variables,

$$\text{Var}[R] = \text{Var}[R_1] + \text{Var}[R_2] + \ldots + \text{Var}[R_n]$$

Since the variance of an indicator (Bernoulli) r.v. is $p(1 - p)$,

$$\text{Var}[R] = n\, p\, (1 - p).$$

Questions?

Matching Birthdays

Q: In a class of n students, what is the probability that two students share the same birthday? Assume that (i) each student is equally likely to be born on any day of the year, (ii) no leap years and (iii) student birthdays are independent of each other.

Pr(matching birthday) = 1 - Perm(365, n)/(365)^n

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$$\Pr[\text{two students share the same birthday}] = 1 - \frac{d \times (d-1) \times (d-2) \times \ldots (d-(n-1))}{d^n}$$

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Q: On average, how many pairs of students have matching birthdays?

Define M to be the number of pairs of students with matching birthdays. For a fixed ordering of the students, let $X_{i,j}$ be the indicator r.v. corresponding to the event $E_{i,j}$ that the birthdays of students i and j match. Hence,

$$M = \sum_{i,j|1 \leq i < j \leq n} X_{i,j} \implies \mathbb{E}[M] = \mathbb{E}[\sum_{i,j|1 \leq i < j \leq n} X_{i,j}] = \sum_{i,j|1 \leq i < j \leq n} \mathbb{E}[X_{i,j}] = \sum_{i,j|1 \leq i < j \leq n} \Pr[E_{i,j}]$$

(Linearity of expectation)

Matching Birthdays

For a pair of students i, j, let B_i be the r.v. equal to the day of student i's birthday. Range(B_i) = {1,2,. . . , d}. For all $k \in [d]$, $\Pr[B_i = k] = 1/d$ (each student is equally likely to be born on any day of the year).

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$$E_{i,j} = (B_i = 1 \cap B_j = 1) \cup (B_i = 2 \cap B_j = 2) \cup \dots$$

$$\implies \Pr[E_{i,j}] = \sum_{k=1}^{d} \Pr[B_i = k \cap B_j = k] = \sum_{k=1}^{d} \Pr[B_i = k] \Pr[B_j = k] = \sum_{k=1}^{d} \frac{1}{d^2} = \frac{1}{d}$$

(student birthdays are independent of each other)

$$\implies \mathbb{E}[M] = \sum_{i,j|1 \le i < j \le n} \Pr[E_{i,j}] = \frac{1}{d} \sum_{i,j|1 \le i < j \le n} (1) = \frac{1}{d}[(n-1) + (n-2) + \dots + 1] = \frac{n(n-1)}{2d}$$

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The number of pairs when starting with the first student is n - 1 since there are n - 1 students to pick from.

The number of pairs when starting with the second student is n - 2 since there are n - 2 students to pick from (we do not count the first student again and will do that outside).

(student birthdays are independent of each other)

$$\implies \mathbb{E}[M] = \sum_{i,j|1 \le i < j \le n} \Pr[E_{i,j}] = \frac{1}{d} \sum_{i,j|1 \le i < j \le n} (1) = \frac{1}{d}[(n-1) + (n-2) + \ldots + 1] = \frac{n(n-1)}{2d}$$

Since we have nc2 pairs, we divide by a factor of two.

75 instead of 41

Hence, in our class of 75 students, on average, there are $\frac{(21)(41)}{365} = 7.60$ students with matching birthdays.

Matching Birthdays

Q: Are the $X_{i,j}$ r.v's mutually independent?

No. If conditioning on Xik =1 and xkj = 1, then Xij must be 1

Matching Birthdays

Q: Are the $X_{i,j}$ r.v's mutually independent?

No, because if $X_{i,j} = 1$ and $X_{j,k} = 1$, then,
$\Pr[X_{i,k} = 1 | X_{j,k} = 1 \cap X_{i,j} = 1] = 1 \neq \frac{1}{d} = \Pr[X_{i,k} = 1]$.

n the additional information fixes the probability, so the r.v's are not mutually independent.

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Q: Are the $X_{i,j}$ pairwise independent?

Yes, because for all i, j and i', j' (where $i \neq i'$), $\Pr[X_{i,j} = 1 | X_{i',j'} = 1] = \Pr[X_{i,j} = 1]$ because if students i' and j' have matching birthdays, it does not tell us anything about whether i and j have matching birthdays.

Matching Birthdays

Q: If M is the random variable equal to the number of pairs of students with matching birthdays, calculate $\mathrm{Var}[M]$.

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$$\mathrm{Var}[M] = \mathrm{Var}[\sum_{i,j|1\leq i<j\leq n} X_{i,j}]$$

Since $X_{i,j}$ are pairwise independent, the variance of the sum is equal to the sum of the variance.

$$\implies \mathrm{Var}[M] = \sum_{i,j|1\leq i<j\leq n} \mathrm{Var}[X_{i,j}] = \sum_{i,j|1\leq i<j\leq n} \frac{1}{d}\left(1-\frac{1}{d}\right) = \frac{1}{d}\left(1-\frac{1}{d}\right)\frac{n\,(n-1)}{2}$$

(Since $X_{i,j}$ is an indicator (Bernoulli) r.v. and $\Pr[X_{i,j}=1]=\frac{1}{d}$)

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(Since $X_{i,j}$ is an indicator (Bernoulli) r.v. and $\Pr[X_{i,j}=1]=\frac{1}{d}$)

Hence, in our class of 75 students, the standard deviation for the matching birthdays is equal to $\sqrt{\frac{(37)\,(75)}{365}\,\frac{364}{365}} \approx 2.75$.

Questions?

Covariance

For two random variables R and S, the covariance between R and S is defined as:

$$\mathrm{Cov}[R, S] := \mathbb{E}[(R - \mathbb{E}[R])\,(S - \mathbb{E}[S])] = \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]$$

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$$\begin{aligned}
\mathrm{Cov}[R,S] &= \mathbb{E}[(R-\mathbb{E}[R])\,(S-\mathbb{E}[S])] \\
&= \mathbb{E}\left[RS - R\,\mathbb{E}[S] - S\,\mathbb{E}[R] + \mathbb{E}[R]\,\mathbb{E}[S]\right] \\
&= \mathbb{E}[RS] - \mathbb{E}[R\,\mathbb{E}[S]] - \mathbb{E}[S\,\mathbb{E}[R]] + \mathbb{E}[R]\,\mathbb{E}[S] \\
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The covariance between two r.v's is symmetric i.e. $\text{Cov}[R,S] = \text{Cov}[S,R]$.

Covariance

For two arbitrary (not necessarily independent) r.v's, R and S,

$$\mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S] + 2\,\mathrm{Cov}[R,S]$$

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Recall from Slide 6, where we showed that,

$$\mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]) = \mathrm{Var}[R] + \mathrm{Var}[S] + 2\,\mathrm{Cov}[R,S].$$

If R and S are independent, $\mathrm{Cov}[R,S] = 0$ and we recover the formula for the sum of independent variables.

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For $R = S$, $\mathrm{Var}[R+R] = \mathrm{Var}[R] + \mathrm{Var}[R] + 2\mathrm{Cov}[R,R] = \mathrm{Var}[R] + \mathrm{Var}[R] + 2\mathrm{Var}[R] = 4\mathrm{Var}[R]$ which is consistent with our previous formula that $\mathrm{Var}[2R] = 2^2\mathrm{Var}[R]$.

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Generalization to multiple random variables $R_1, R_2, \ldots R_n$ (Recall from Slide 7):

$$\mathrm{Var}\left[\sum_{i=1}^{n} R_i\right] = \sum_{i=1}^{n} \mathrm{Var}[R_i] + 2 \sum_{1 \le i < j \le n} \mathrm{Cov}[R_i, R_j]$$

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$$\implies \mathbb{E}[X] = \Pr[A]\,;\, \mathbb{E}[Y] = \Pr[B];\, \mathbb{E}[XY] = \Pr[A \cap B]$$

$$\implies \text{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = \Pr[A \cap B] - \Pr[A]\Pr[B]$$

If $\text{Cov}[X, Y] > 0 \implies \Pr[A \cap B] > \Pr[A]\Pr[B]$. Hence,

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} > \frac{\Pr[A]\Pr[B]}{\Pr[B]} = \Pr[A]$$

If $Cov[X, Y] > 0$, it implies that $\Pr[A|B] > \Pr[A]$ and hence, the probability that event A happens increases if B is going to happen/has happened. Similarly, if $Cov[X, Y] < 0$, $\Pr[A|B] < \Pr[A]$. In this case, if B happens, then the probability of event A decreases.

Correlation

The correlation between two r.v's R_1 and R_2 is defined as:

$$\mathrm{Corr}[R_1, R_2] = \frac{\mathrm{Cov}[R_1, R_2]}{\sqrt{\mathrm{Var}[R_1]\,\mathrm{Var}[R_2]}}$$

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If $R_1 = R_2 = R$, then, $\text{Corr}[R, R] = \frac{\text{Cov}[R,R]}{\sqrt{\text{Var}[R]\,\text{Var}[R]}} = \frac{\text{Var}[R]}{\text{Var}[R]} = 1$.

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If R_1 and R_2 are independent, $\mathrm{Cov}[R_1, R_2] = 0$ and $\mathrm{Corr}[R_1, R_2] = 0$.

If $R_1 = -R_2 = R$, then,

$$\begin{aligned}\mathrm{Corr}[R, -R] &= \frac{\mathrm{Cov}[R, -R]}{\sqrt{\mathrm{Var}[R]\,\mathrm{Var}[-R]}} = \frac{\mathrm{Cov}[R, -R]}{\sqrt{\mathrm{Var}[R]\,(-1)^2\mathrm{Var}[R]}} = \frac{\mathrm{Cov}[R, -R]}{\mathrm{Var}[R]} \\ &= \frac{\mathbb{E}[-R^2] - \mathbb{E}[R]\,\mathbb{E}[-R]}{\mathrm{Var}[R]} = \frac{-\mathbb{E}[R^2] + \mathbb{E}[R]\,\mathbb{E}[R]}{\mathrm{Var}[R]} = \frac{-\mathrm{Var}[R]}{\mathrm{Var}[R]} = -1\end{aligned}$$

Questions?