CMPT 210: Probability and Computing

Lecture 7

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Recap

Sample (outcome) space $\mathcal{S}$: Nonempty (countable) set of possible outcomes.

Outcome $\omega \in \mathcal{S}$: Possible "thing" that can happen.

Event E: Any subset of the sample space. An event E "happens" if the outcome ω (from some process) is in set E i.e. if $\omega \in E$.

Probability function on a sample space $\mathcal{S}$ is a total function $\Pr : \mathcal{S} \to [0,1]$. For any $\omega \in \mathcal{S}$,

$$0 \leq Pr[\omega] \leq 1 \quad ; \quad \sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1 \quad ; \quad \Pr[E] = \sum_{\omega \in E} \Pr[\omega]$$

Different pairs, but there can be distinct pairs

Complement rule: $\Pr[E] = 1 - \Pr[E^c]$

Inclusion-Exclusion rule: For any two events E, F, $\Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$.

Uniform probability space: A probability space is said to be uniform if $\Pr[\omega]$ is the same for every outcome $\omega \in \mathcal{S}$. In this case, $\Pr[E] = \frac{|E|}{|\mathcal{S}|}$.

Birthday Principle

If there are n pigeons and d pigeonholes, then the probability that two pigeons occupy the same hole is $\geq 1 - \exp\left(-\frac{n(n-1))}{2d}\right)$

For $n = \lceil\sqrt{2d}\rceil$, probability that two pigeons occupy the same hole is about $1 - \frac{1}{e} \approx 0.632$.

Example: If we are randomly throwing $\lceil\sqrt{2d}\rceil$ balls into d bins, then the probability that two balls land in the same bin is around 0.632.

Later in the course, we will see applications of this principle to load balancing.

Conditioning on?

Conditioning is revising probabilities based on partial information (an event).

Q: Suppose we throw a "standard" dice, what is the probability of getting a 6 $\boxed{\text{if}}$ we are told that the outcome is even?

Additional information

Conditioning

Know what event we are conditioning on

We wish to compute Pr[we get a 6|the outcome is even] or Probability of getting a 6 *given* that the outcome is even or Probability of a 6 *conditioned on the event* that the outcome is even.

Creation of a probability space does not imply that it is an uniform probability space.

You can only divide by the sample space iff you have a uniform probability space

Sample space: $S = \{1, 2, 3, 4, 5, 6\}$, Event: $E = \{6\}$. Additional information: Event $F = \{2, 4, 6\}$ has happened. With conditioning on F, *new sample space* $S' = F = \{2, 4, 6\}$. Since each outcome in $S' = \{2, 4, 6\}$ is equally likely, the new probability space is still a uniform probability space. Hence, conditioned on the event that the outcome is even, $\sum_{\omega \in S'} \Pr[\omega] = 1$ and $\Pr[\{\text{even number}\}] = \frac{1}{3}$ and $\Pr[\{\text{odd number}\}] = 0$. Hence, $\Pr[\{6\}] = \frac{1}{3}$.

Q: What is the probability of getting either a 3 or 6 if we are told that the outcome is even? $E = \{3, 6\}$, $F = \{2, 4, 6\}$. With conditioning, the new sample space $S' = \{2, 4, 6\}$. By the same reasoning as above, $\Pr[\{6\}] = \frac{1}{3}$ and $\Pr[\{3\}] = 0$. Hence, $\Pr[E] = \Pr[\{3\}] + \Pr[\{6\}] = \frac{1}{3}$.

F' = {5, 6}. pr(6lF') = 1/3

Text

Conditional Probability

Q: Suppose we throw two standard dice one after the other. What is the probability that the sum of the dice is 6?

Recall the sample space consists of tuples, i.e. $S = \{(1,1),(1,2),(1,3),\dots,(6,6)\}$. The event E consists of outcomes such that the sum of the dice is 6. $E = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$. Since all outcomes are equally likely, this is a uniform probability space, and $\Pr[E] = \frac{|E|}{|S|} = \frac{5}{36}$.

Q: Suppose I tell you that the first dice came up 4. Given this information, what is the probability that the sum of the dice is 6?

Let F be the event that the first dice came up 4. We wish to compute $\Pr[E|F]$, the probability that the sum of the dice is 6 conditioned on the event that the first dice came up 4.

Uniform probability space

With conditioning, the new sample space $S' = F = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$. For this new probability space, $E = \{(4,2)\}$. Since each outcome in S' is equally likely, $\Pr[E|F] = \frac{|E|}{|S'|} = \frac{1}{6}$.

Conditional Probability

Conditional Probability Rule: For two events E and F, $\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$, where $\Pr[F] \neq 0$.

When E = {2}, P(E|F) = 1/3

When E = {2, 4}, P(E|F) = 2/3

When E = {2, 4, 6}, P(E|F) = 1

Proof: By conditioning on F, the only outcomes we care about are in F i.e. for $\omega \notin F$, $\Pr[\omega|F] = 0$.

Since we want to compute the probability that event E happens, we care about the outcomes that are in E. Hence, the outcomes we care about lie in both E *and* F, meaning that $\omega \in E \cap F$.

$\implies \Pr[E|F] \propto \sum_{\omega \in (E \cap F)} \Pr[\omega]$. By definition of proportionality, for some constant $c > 0$, $\Pr[E|F] = c \sum_{\omega \in (E \cap F)} \Pr[\omega]$.

If it is proportional, what is the relation? as pr(E intersect F) increases, pr(E | F) increases

C is unknown

Is it because we divide by pr(F) that we need to include the constant?

We know that $\Pr[F|F] = 1$ (probability of event F given that F has happened). Hence, $\Pr[F|F] = 1 = c \sum_{\omega \in F} \Pr[\omega] \implies c = \frac{1}{\sum_{\omega \in F} \Pr[\omega]}$.

F intersect F = F

Ask professor about this relationship

Substituting the value of c,

$$\Pr[E|F] = \frac{\sum_{\omega \in (E \cap F)} \Pr[\omega]}{\sum_{\omega \in F} \Pr[\omega]} = \frac{\Pr[E \cap F]}{\Pr[F]}, \text{where } \Pr[F] \neq 0.$$

This formula gives an alternate way to compute conditional probabilities.

Back to throwing dice

Q: Suppose we throw a "standard" dice, what is the probability of getting a 6 if we are told that the outcome is even?

Sample space: $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ and $\Pr[1] = \Pr[2] = \ldots = \Pr[6] = \frac{1}{6}$.
Event: $E = \{6\}$. We are conditioning on $F = \{2, 4, 6\}$.
$\Pr[\text{we get a 6}|\text{the outcome is even}] = \Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$.
$E \cap F = \{6\}$. $\Pr[E \cap F] = \frac{1}{6}$. $\Pr[F] = \Pr[\{2\}] + \Pr[\{4\}] + \Pr[\{6\}] = \frac{1}{2}$.
Hence, $\frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/6}{1/2} = \frac{1}{3}$.

Q: What is the probability of getting either a 3 or 6 if we are told that the outcome is even?

Q: Suppose we throw two standard dice one after the other. What is the probability that the sum of the dice is 6 given that the first dice came up 4?

ii: pr(outcome is even) = 1/2
pr(getting 3 or 6l outcome is even) = 1/3
iii: pr(first dice was four) = 1/6
pr(sum to six l first dice was four) = 1/6

Conditional Probability - Examples

Q: Suppose we select a card at random from a standard deck of 52 cards. What is the probability of getting:

- A spade conditioned on the event that I picked the red color
- A spade facecard conditioned on the event that I picked the black color
- A black card conditioned on the event that I picked a spade facecard
- The queen of hearts given that I picked a queen
- An ace given that I picked a spade

E = spade facecard
F = black card
E intersect F = all spades (jack, queen, king)

i: p(spade | red was picked) = 0
p(spade facecard | black was picked) = 3/26
iii: p(black card | spade facecard) = 1
iiii: p(queen of hearts | queen was picked) = 1/4
iiiii: p(ace | spade was picked) = 1/13

Conditional Probability - Examples

Q: The organization that Jones works for is running a father–son dinner for those employees having at least one son. Each of these employees is invited to attend along with his youngest son. If Jones is known to have two children, what is the conditional probability that they are both boys given that he is invited to the dinner? Assume that the sample space S is given by $S = \{(b,b),(b,g),(g,b),(g,g)\}$ and all outcomes are equally likely. For instance, (b,g) means that the younger child is a boy and the older child is a girl.

The event that we care about is Jones has both boys. Hence, $E = \{(b,b)\}$.

Additional information that we are conditioning on is that Jones is invited to the dinner meaning that he has at least one son. Hence, $F = \{(b,b),(b,g),(g,b)\}$.

Hence, $E \cap F = \{(b,b)\}$, $\Pr[E \cap F] = \frac{|E \cap F|}{|\mathcal{S}|} = \frac{1}{4}$. $\Pr[F] = \frac{|F|}{|\mathcal{S}|} = \frac{3}{4}$.

$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/4}{3/4} = \frac{1}{3}$.

Conditional Probability - Examples

Q: Ms. Perez figures that there is a 30 percent chance that her company will set up a branch office in Phoenix. If it does, she is 60 percent certain that she will be made manager of this new operation. What is the probability that there will be a branch in Phoenix and Perez will be its office manager?

Text

E = Perez will be a branch office manager; F = her company will set up a branch office in Phoenix; $E \cap F$ = Perez will be an office manager in the Phoenix branch.

From the question, we know that $\Pr[F] = 0.3$, $\Pr[E|F] = 0.6$. Hence,
$\Pr[E \cap F] = \Pr[E] \Pr[E|F] = 0.3 \times 0.6 = 0.18$.

Conditional Probability Examples

Q: Suppose we have a bowl containing 6 white and 5 black balls. We randomly draw a ball. What is the probability that we draw a black ball? 5/11

Q: We randomly draw two balls, one after the other (without putting the first back). What is the probability that we (i) draw a black ball followed by a white ball (ii) draw a white ball followed by a black ball (iii) we get one black ball and one white ball (iv) both black (v) both white?

B1 = Draw black first, W1 = Draw white first. B2 = Black second, W2 = White second.

(i) $\Pr[B1] = \frac{5}{11}$. $\Pr[W2|B1] = \frac{6}{10}$. Hence, $\Pr[B1 \cap W2] = \Pr[B1] \Pr[W2|B1] = \frac{30}{110}$.

(ii) $\Pr[W1] = \frac{6}{11}$. $\Pr[B2|W1] = \frac{5}{10}$. Hence, $\Pr[W1 \cap B2] = \Pr[W1] \Pr[B2|W1] = \frac{30}{110}$.

(iii) $G = (B1 \cap W2) \cup (W1 \cap B2)$. Events $B1 \cap W2$ and $B2 \cap W1$ are mutually exclusive. By the union rule for mutually exclusive events, $\Pr[G] = \Pr[B1 \cap W2] + \Pr[W1 \cap B2] = \frac{60}{110}$.

(iv) $\Pr[B1 \cap B2] = \Pr[B1] \Pr[B2|B1] = \frac{20}{110}$.

(v) $\Pr[W1 \cap W2] = \Pr[W1] \Pr[W2|W1] = \frac{30}{110}$.

Questions?