CMPT 210: Probability and Computing

Lecture 3

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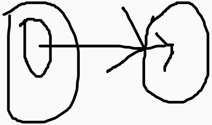
Recap - Counting

Product Rule: For sets $A_1, A_2 \ldots, A_m$, $|A_1 \times A_2 \times \ldots \times A_m| = \prod_{i=1}^{m} |A_i|$ (E.g: Selecting one course each from every subject.)

Sum rule: If $A_1, A_2 \ldots A_m$ are disjoint sets, then, $|A_1 \cup A_2 \cup \ldots \cup A_m| = \sum_{i=1}^{m} |A_i|$ (E.g Number of rainy, snowy or hot days in the year).

Generalized product rule: If S is the set of length k sequences such that the first entry can be selected in n_1 ways, after the first entry is chosen, the second one can be chosen in n_2 ways, and so on, then $|S| = n_1 \times n_2 \times \ldots n_k$. (E.g Number of ways n people can be arranged in a line = $n!$)

Division rule: $f : A \to B$ is a k-to-1 function, then, $|A| = k|B|$. (E.g. For arranging people around a round table, f : seatings $\to$ arrangements is an n-to-1 function).



Counting subsets (Combinations)

Q: How many size-k subsets of a size-n set are there?

Example: How many ways can we select 5 books from 100?

If n = 5

Let us form a permutation of the n elements, and pick the first k elements to form the subset. Every size k subset can be generated this way. There are $n!$ total such permutations.

The order of the first k elements in the permutation does not matter in forming the subset, and neither does the order of the remaining $n-k$ elements.

There are (n-k)! many ways to rearrange the remaining elements

The first k elements can be ordered in $k!$ ways and the remaining $n-k$ elements can be ordered in $(n-k)!$ ways. Using the product rule, $k! \times (n-k)!$ permutations map to the same size k subset.

When you count number of permutations, you have to include all the different orders.

Since it is a set, the order of the elements do not matter.

Hence, the function f : permutations $\rightarrow$ size k subsets is a $k! \times (n-k)!$-to-1 function. By the division rule, $|\text{permutations}| = k! \times (n-k)!\,|\text{size } k \text{ subsets}|$. Hence, the total number of size k subsets $= \frac{n!}{k! \times (n-k)!}$.

k = 3

n **choose** $k = \binom{n}{k} = \frac{n!}{k! \times (n-k)!}$.

If on an exam, you can leave the answer unexpanded

a b c d e

Select top k elements

Counting subsets (Combinations)

nck = n!/k!(n-k)!
nc(n-k) = n!/(n-k)!k!

Q: Prove that $\binom{n}{k} = \binom{n}{n-k}$ - both algebraically (using the formula for $\binom{n}{k}$) and combinatorially (without using the formula)

Q: Which is bigger? $\binom{8}{4}$ vs $\binom{8}{5}$?

Combinatorial proof:
nck is the same as choosing k things from a box of n items.
nc(n-k) is the same as choosing (n-k) things to throw away, showing you all the ways to select n things

8c4 = 8!/4!4!
8c5 = 8!/5!3!

Counting subsets – Example

Q: How many m-bit binary sequences contain exactly k ones?

10001 -> {1, 5}
01010 <- = {2, 4}

Consider set $A = \{1, \ldots, m\}$ and selecting S, a subset of size k. For example, say $m = 10, k = 3$ and $S = \{3, 7, 10\}$. S records the positions of the 1's, and can mapped to the sequence 001 0001 001. Similarly, every m-bit sequence with exactly k ones can be mapped to a subset S of size k. Hence, there is a bijection:
f : m-bit sequence with exactly k ones $\to$ subsets of size k from size m-set, and
$|m\text{-bit sequence with exactly } k \text{ ones}| = |\text{subsets of size } k| = \binom{m}{k}$.

Q: Suppose we want to buy 10 donuts. There are 5 donut varieties – chocolate, lemon-filled, sugar, glazed, plain. What is the number of ways to select the 10 donuts?

Recall that the number of ways of selecting 10 donuts with 5 varieties = number of 14-bit sequences with exactly 4 ones $= \binom{14}{4} = 1001$.

(n + k - 1)c(k - 1)

Q: What is the number of ways of choosing n things with k varieties?

Think of how you can map a question to a donut or bit string question

Counting subsets – Example

Maybe: count all the number of
sequences and exclude the sequences which contain less than k ones?
The sum from k to n (assuming k is less or equal to n) is (n k)
This will get you all the sequences

This is the complement of the set below

Q: What is the number of n-bit binary sequences with at least k ones?

+

Q: What is the number of n-bit binary sequences with less than k ones?

||

Q: What is the total number of n-bit binary sequences?

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

$$\sum_{i=0}^{k-1} \binom{n}{i}$$

3: 2^n
2: sigma_0_k-1(nck)

$$\sum_{i=k}^{n} \binom{n}{i}$$

Set you need to count is all the subsets of a bit string with k ones, k + 1 ones, k + 2, ... n ones

Binomial Theorem

For all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,

Sum of exponents of a and b must sum to n. Cannot be greater than n

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Example: If $a = b = 1$, then $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ (result from previous slide).

If $n = 2$, then $(a+b)^2 = \binom{2}{0} a^2 + \binom{2}{1} ab + \binom{2}{2} b^2 = a^2 + 2ab + b^2$.

Q: What is the coefficient of the terms with ab^3 and a^2b^3 in $(a+b)^4$? .

Q: For $a, b > 0$, what is the coefficient of $a^{2n-7}b^7$ and $a^{2n-8}b^8$ in $(a+b)^{2n} + (a-b)^{2n}$?

1: 4c0 * a^4 + 4c1 * a^3b + 4c2 * a^2b^2 + 4c3 * ab^3 + 4c4 * b^4
Coefficient of ab^3 is 4
coefficient of a^2b^3 is 0
2:
part a:
I am looking for the eighth term in the sequence.
I can sum the values from each sequence and then factor out the coefficient
For a^(2n - 7)b^7:
8th element in (a + b)^(2n) = (2n)c7
8th element in (a - b b)^(2n) = -(2n)c7
Adding coefficients, then it will sum to 0 since (2n)c7 -(2n)c7 = 0

9th element in (a + b)^(2n) = (2n)c8
9th element in (a - b b)^(2n) = (2n)c8

Coefficient is 2 * (2n)2c8

Counting Practice

Q: A fair die (with numbers $\{1, 2, 3, 4, 5, 6\}$ is rolled 6 times in succession.

- How many rolls will have no 6?
- How many rolls will have each number once?
- How many rolls will have 6 come up exactly once?
- How many rolls will have 6 come up exactly k times (for $k \leq 6$)?

1: 5^6 many ways to have rolls with no six
2: 6! many ways of rolling each number once
3: 6 * 5^5 many ways of rolling six once
4: 6^k * 5^(n - k) many ways of rolling a six k times

Counting Practice

Q: How many 5 digit numbers are there which contain at least one zero? Note that a number is different from a string, i.e. 01234 is not a 5-digit number and is hence not allowed.

Conditions: numbers cannot start with zero.

Strategy:
count all the numbers which do not contain zeroes and subtract it from all the ways you can make a number.

9 * 10 * 10 * 10 * 10 many ways of making a number
9 * 9 * 9 * 9 * 9 many ways of making a number without zeroes.
There are 9*10^4 - 9^5 many ways of making a number with at least one zero.

Counting Practice

Q: How many non-negative integer solutions $(x_1, x_2, x_3 \geq 0)$ are there to the following equation:

$$x_1 + x_2 + x_3 = 40$$

I have to find all the ways of dividing 40 ones among three variables.
There are 42c2 many non-negative integer solutions.

Questions?