CMPT 210: Probability & Computing
Assignment 1

Total marks: 200

Due: Via Coursys at 11.59 pm, Tuesday, 30 January
Late Submission: 11.59 pm, Thursday, 1 February

Instructions on how to solve, write, and submit your assignment

Solutions: Solutions to assignments must be your own. Use sample exercises from the lectures to learn methods and approaches you can use. In some cases expect that you will need to make a substantial effort to solve a problem. Discussing and collaborating with other people is okay, as long as you produce your own solution. For instance, even when two people try to solve the problem together and extensively discuss possible solutions, if they write down that solution independently, it is acceptable.

We treat any kind of academic dishonesty very seriously. For the SFU policy on academic dishonesty see the part of University Policy S 10.01 relevant for us:

> e. Cheating in assignments, projects, examinations or other forms of evaluation by:
> i. using, or attempting to use, another student's answers;
> ii. providing answers to other students;
> iii. failing to take reasonable measures to protect answers from use by other students; or
> iv. in the case of students who study together, submitting identical or virtually identical assignments for evaluation unless permitted by the course Instructor or supervisor.
>
> University Policy S 10.01
> Code of Academic Integrity and Good Conduct
> 4.1.2 Forms of Academic Dishonesty

Note that this policy treats copying and allowing to copy equally. Should anyone be caught submitting a work too similar to someone else's work, or a source found on the web, a record of the violation will permanently stay in their student file.

Writing the solution: There is no strict prescribed way to present your solution. However, make sure that your solution can be understood by another person without any help on from you. The onus to present your solution in a clear understandable way is on you. Any unclear steps or arguments will be considered incorrect. If you use some method, or result presented in this course you do not need to explain it. However, if you would like to use a result or approach from elsewhere, please, give a reference and explain what you are doing in more details.

Submission: The assignment needs to be submitted via Coursys. For some flexibility, each student is allowed 1 late-submission and can submit the assignment on the Thursday after.

(1) **[30 marks] Set Theory**

- Prove that
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$
using both (i) Venn diagrams [5 marks] and (ii) the distributive law for binary literals. [10 marks]

- Using that $|A \cup B| = |A| + |B| - |A \cap B|$ and the laws of set theory, prove that $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$. [15 marks]

(2) **[20 marks]** Set A has r elements and set B has n elements.

- What is the number of possible total functions $f : A \to B$? [3 marks]
- If $r > n$, what is the number of injective functions $f : A \to B$? [5 marks]
- If $r \leq n$, what is the number of injective functions $f : A \to B$? [5 marks]
- If $r = n$, what is the number of bijective functions $f : A \to B$? [5 marks]
- If $r < n$, what is the number of bijective functions $f : A \to B$? [2 marks]

(3) **[20 marks]** We have 5 identical black balls and 1 ball each of 5 colors {red, green, blue, yellow, violet}. In how many different ways can we choose 5 balls?

(4) **[15 marks]** A license plate consists of either:

- 3 upper-case letters followed by 3 digits (standard plate)
- 5 upper-case letters (vanity plate)
- 2 characters – each of which is either an upper-case letter or number (big shot plate)

If P is the set of all possible plates, and $L = \{A, B, \dots, Z\}$ is the set of upper-case letters, and $D = \{0, 1, \dots, 9\}$ is the set of digits,

- Express P in terms of L, D using the set union $\cup$ and product $\times$ operations. [10 marks]
- Using the sum rule and the product rule, compute $|P|$. [5 marks]

(5) **[15 marks]** In Lecture 4, we used a combinatorial technique to prove Pascal's identity:
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$
Use the definitions of $\binom{n}{k}$ and $n!$ to prove Pascal's identity algebraically.

(6) **[10 marks]** In how many ways can 12 identical mathematics books be distributed among the students Anna, Beth, Candy, and Daniel?

(7) **[10 marks]** Next week, I'm going to get really fit! On day 1, I'll exercise for 5 minutes. On each subsequent day, I'll exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be (5, 6, 9, 9, 9, 11, 12). How many such sequences are possible?

(8) **[20 marks]** How many positive integers less than or equal to 2023 are divisible by 3, 5 or 7?

(9) **[10 marks]** SFU ID numbers are 9 digit numbers that start with 3. How many students do we need in this class such that there are at least two students with the same sum of their SFU ID digits?

(10) **[20 marks]** Prove that the number of different ways $2n$ students can be paired up is equal to $\frac{(2n)!}{2^n\, n!}$

(11) **[30 marks]** Prove that

$$n\, 2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}$$

using two methods:

- **Combinatorially**: For this, let S be the number of length-n strings that can be composed using a's, b's and must contain exactly one c. For example, if $n = 4$, then, *aacb*, *abbc* are valid strings. On the other hand, *aabb*, *acac* are not valid. Count $|S|$ in two ways to prove the above identity. [20 marks]
- **Algebraically**: Apply the Binomial Theorem to $(1 + x)^n$ and take derivatives. [10 marks]