CMPT 210: Probability and Computing

Lecture 9

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Recap

For events E and F, we wish to compute $\Pr[E|F]$, the probability of event E conditioned on F.

Approach 1: With conditioning, F can be interpreted as the *new sample space* such that for $\omega \notin F$, $\Pr[\omega|F] = 0$.

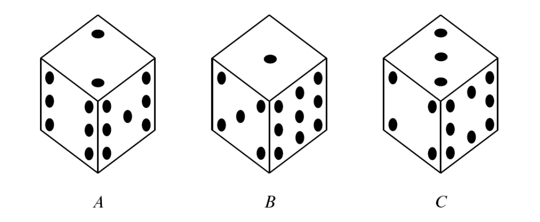
Approach 2: $\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$.

Multiplication Rule: For events $E_1, E_2, \ldots, E_n$,
$\Pr[E_1 \cap E_2 \ldots \cap E_n] = \Pr[E_1] \Pr[E_2|E_1] \Pr[E_3|E_1 \cap E_2] \ldots \Pr[E_n|E_1 \cap E_2 \cap \ldots E_{n-1}]$.

Tree Diagrams:

- Helpful in calculating probabilities in a sequential process (E.g. In the Monty Hall problem, the process is choose car location, choose door, reveal door).
- In a tree diagram, edge-weights correspond to conditional probabilities and leaf nodes correspond to outcomes.
- The probability of an outcome can be calculated by multiplying the relevant probabilities along a path.

Let us play a game with three strange dice shown in the figure. Each player selects one die and rolls it once. The player with the lower value pays the other player \$100. We can pick a die first, after which the other player can pick one of the other two.



Conditional Probability - Examples

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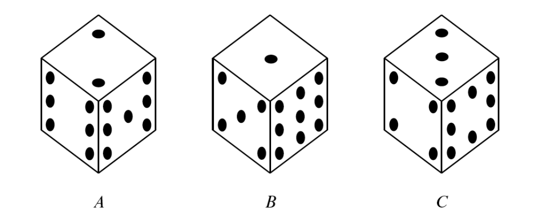

Q: Suppose we choose die B because it has a 9, and the other player selects die A. What is the probability that we will win?

Conditional Probability - Examples

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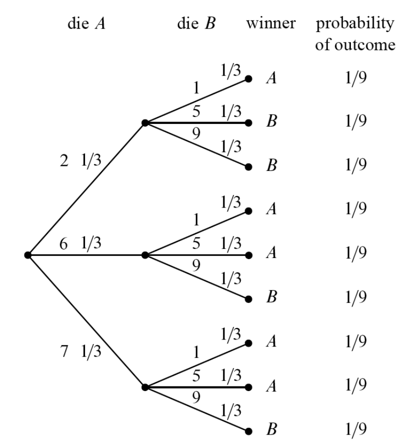

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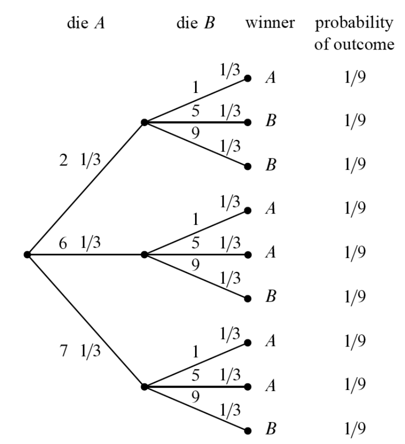

Identify Outcomes: Each leaf is an outcome and $\mathcal{S} = \{(2,1),(2,5),(2,9),(6,1),(6,5),(6,9),(7,1),(7,5),(7,9)\}$.

Identify Event: $E = \{(2,5),(2,9),(6,9),(7,9)\}$.

Compute probabilities: $\Pr[\text{Dice 1 is 6}] = \frac{1}{3}$.
$\Pr[(6,5)] = \Pr[\text{Dice 2 is 5} \ \cap \ \text{Dice 1 is 6}] = \Pr[\text{Dice 2 is 5} \mid \text{Dice 1 is 6}] \Pr[\text{Dice 1 is 6}] = \frac{1}{3}\frac{1}{3} = \frac{1}{9}$.

$\Pr[E] = \Pr[(2,5)] + \Pr[(2,9)] + \Pr[(6,9)] + \Pr[(7,9)] = \frac{4}{9}$.
Meaning that there is less than 50% chance of winning.

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Q: We get another chance – this time we know that die A is good (since we lost to it previously), we choose die A and the other player chooses die C. What is our probability of winning?

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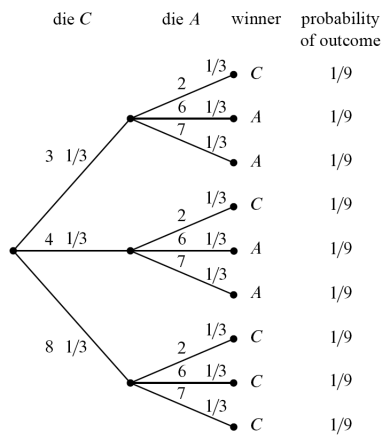

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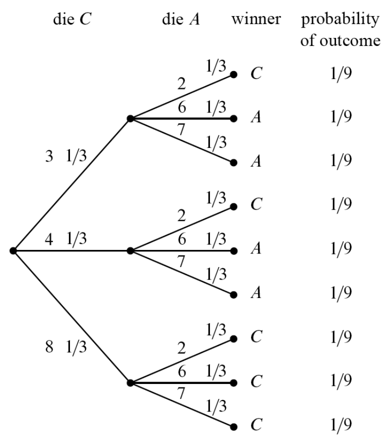

Now, $E = \{(3,6),(3,7),(4,6),(4,7)\}$ and hence $\Pr[E] = \frac{4}{9}$. Meaning that there is less than 50% chance of winning.

Conditional Probability - Examples

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- A beats B with probability $\frac{5}{9}$ (first game).
- C beats A with probability $\frac{5}{9}$ (second game).
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Since A will beat B more often than not, and B will beat C more often than not, it seems like A ought to beat C more often than not, that is, the "beats more often" relation ought to be transitive. But this intuitive idea is false: whatever die we pick, the second player can pick one of the others and be likely to win. So picking first is actually a disadvantage!

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This is the topic of some recent research and was covered in this article:
`https://www.quantamagazine.org/mathematicians-roll-dice-and-get-rock-paper-scissors-20230119/`

Conditional Probability - Examples

Q: A test for detecting cancer has the following accuracy – (i) If a person has cancer, there is a 10% chance that the test will say that the person does not have it. This is called a "false negative" and (ii) If a person does not have cancer, there is a 5% chance that the test will say that the person does have it. This is called a "false positive". For patients that have no family history of cancer, the incidence of cancer is 1%. Person X does not have any family history of cancer, but is detected to have cancer. What is the probability that the Person X does have cancer?

Conditional Probability - Examples

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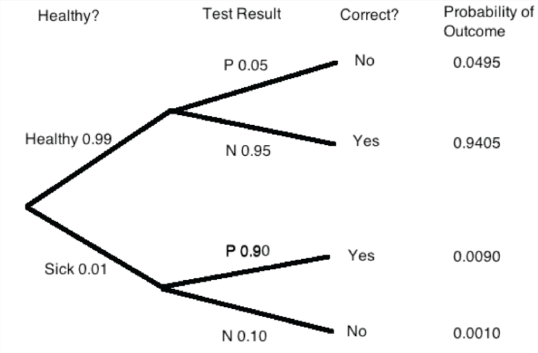
$\mathcal{S} = \{(Healthy, Positive), (Healthy, Negative), (Sick, Positive), (Sick, Negative)\}.$

A is the event that Person X has cancer. B is the event that the test is positive.



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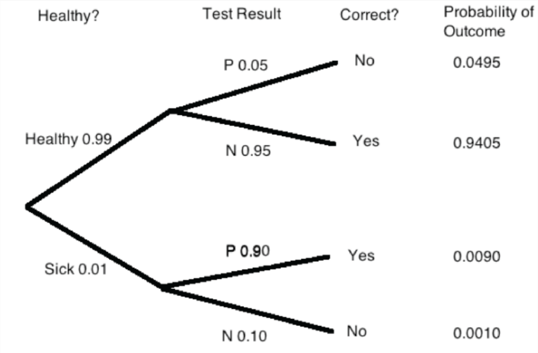

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{\Pr[\{(S,P)\}]}{\Pr[\{(S,P),(H,P)\}]} = \frac{0.0090}{0.0090+0.0495} \approx 15.4\%.$$

Questions?

Conditional Probability

Conditional probability for complement events: For events E, F, $\Pr[E^c|F] = 1 - \Pr[E|F]$.

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Conditional probability for complement events: For events E, F, $\Pr[E^c|F] = 1 - \Pr[E|F]$.

Proof: Since $E \cup E^c = \mathcal{S}$, for an event F such that $\Pr[F] \neq 0$,

$$\begin{aligned}
(E \cup E^c) \cap F &= \mathcal{S} \cap F = F \\
(E \cup E^c) \cap F &= (E \cap F) \cup (E^c \cap F) && \text{(Distributive Law)} \\
\implies \Pr[(E \cap F) \cup (E^c \cap F)] &= \Pr[(E \cup E^c) \cap F]
\end{aligned}$$

Since $E \cap F$ and $E^c \cap F$ are mutually exclusive events,

$$\begin{aligned}
\Pr[E \cap F] + \Pr[E^c \cap F] = \Pr[F] &\implies \frac{\Pr[E^c \cap F]}{\Pr[F]} = 1 - \frac{\Pr[E \cap F]}{\Pr[F]} \\
&\implies \Pr[E^c|F] = 1 - \Pr[E|F] && \text{(By def. of conditional probability)}
\end{aligned}$$

Bayes Rule

Bayes Rule: For events E and F if $\Pr[E] \neq 0$ and $\Pr[F] \neq 0$, then, $\Pr[F|E] = \frac{\Pr[E|F]\Pr[F]}{\Pr[E]}$.

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Proof: Using the formula for conditional probability,

$$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} \quad ; \quad \Pr[F|E] = \frac{\Pr[F \cap E]}{\Pr[E]}$$

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Allows us to compute $\Pr[F|E]$ using $\Pr[E|F]$. Later in the course, we will see an application of the Bayes rule to machine learning.

Law of Total Probability and Bayes rule

Law of Total Probability: For events E and F, $\Pr[E] = \Pr[E|F]\Pr[F] + \Pr[E|F^c]\Pr[F^c]$.

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Proof:

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\begin{aligned}
E &= (E \cap F) \cup (E \cap F^c) \\
\implies \Pr[E] &= \Pr[(E \cap F) \cup (E \cap F^c)] = \Pr[E \cap F] + \Pr[E \cap F^c] \\
&\hspace{10em} \text{(By union-rule for disjoint events)} \\
\Pr[E] &= \Pr[E|F]\Pr[F] + \Pr[E|F^c]\Pr[F^c] \qquad \text{(By definition of conditional probability)}
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Combining Bayes rule and Law of total probability

$$
\Pr[F|E] = \frac{\Pr[F \cap E]}{\Pr[E]} = \frac{\Pr[E|F]\ \Pr[F]}{\Pr[E]} \qquad \text{(By definition of conditional probability)}
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Questions?

Total Probability - Examples

Q: In answering a question on a multiple-choice test, a student either knows the answer or she guesses. Let p be the probability that she knows the answer and $1 - p$ the probability that she guesses. Assume that a student who guesses at the answer will be correct with probability $\frac{1}{m}$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that she answered it correctly?

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Let C be the event that the student answers the question correctly. Let K be the event that the student knows the answer. We wish to compute $\Pr[K|C]$.

We know that $\Pr[K] = p$ and $\Pr[C|K^c] = 1/m$, $\Pr[C|K] = 1$. Hence,
$\Pr[C] = \Pr[C|K]\Pr[K] + \Pr[C|K^c]\Pr[K^c] = (1)(p) + \frac{1}{m}(1-p)$.
$\Pr[K|C] = \frac{\Pr[C|K]\Pr[K]}{\Pr[C]} = \frac{mp}{1+(m-1)p}$.

Total Probability - Examples

Q: An insurance company believes that people can be divided into two classes — those that are accident prone and those that are not. Their statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability decreases to 0.2 for a non-accident-prone person. If we assume that 30% of the population is accident prone, what is the probability that a new policy holder will have an accident within a year of purchasing a policy?

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Let $A =$ event that a new policy holder will have an accident within a year of purchasing a policy. Let $B =$ event that the new policy holder is accident prone. We know that $\Pr[B] = 0.3$, $\Pr[A|B] = 0.4$, $\Pr[A|B^c] = 0.2$. By the law of total probability,
$\Pr[A] = \Pr[A|B]\Pr[B] + \Pr[A|B^c]\Pr[B^c] = (0.4)(0.3) + (0.2)(0.7) = 0.26.$

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Compute $\Pr[B|A] = \frac{\Pr[A|B]\ \Pr[B]}{\Pr[A]} = \frac{0.12}{0.26} = 0.4615$.

Total Probability - Examples

Q: Alice is taking a probability class and at the end of each week she can be either up-to-date or she may have fallen behind. If she is up-to-date in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.8 (or 0.2, respectively). If she is behind in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.6 (or 0.4, respectively). Alice is (by default) up-to-date when she starts the class. What is the probability that she is up-to-date after three weeks?

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Let U_i and B_i be the events that Alice is up-to-date or behind respectively after i weeks. Since Alice starts the class up-to-date, $\Pr[U_1] = 0.8$ and $\Pr[B_1] = 0.2$. We also know that $\Pr[U_2|U_1] = 0.8$, $\Pr[U_3|U_2] = 0.8$ and $\Pr[B_2|U_1] = 0.2$, $\Pr[B_3|U_2] = 0.2$. Similarly, $\Pr[U_2|B_1] = 0.6$, $\Pr[U_3|B_2] = 0.6$ and $\Pr[B_2|B_1] = 0.4$, $\Pr[B_3|B_2] = 0.4$.

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We wish to compute $\Pr[U_3]$. By the law of total probability,
$\Pr[U_3] = \Pr[U_3|U_2]\Pr[U_2] + \Pr[U_3|B_2]\Pr[B_2]$ and
$\Pr[U_2] = \Pr[U_2|U_1]\Pr[U_1] + \Pr[U_2|B_1]\Pr[B_1]$.

Hence, $\Pr[U_2] = (0.8)(0.8) + (0.6)(0.2) = 0.76$, and $\Pr[U_3] = (0.8)(0.76) + (0.6)(0.24) = 0.752$.