CMPT 210: Probability and Computing

Lecture 17

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Recap

Expectation/mean of a random variable R is denoted by $\mathbb{E}[R]$ and "summarizes" its distribution. Formally, $\mathbb{E}[R] := \sum_{\omega \in \mathcal{S}} \Pr[\omega]\, R[\omega]$ Generalization of the mean.

Alternate definition of expectation: $\mathbb{E}[R] = \sum_{x \in \text{Range}(R)} x \Pr[R = x]$.

Linearity of Expectation: For n random variables $R_1, R_2, \ldots, R_n$ and constants $a_1, a_2, \ldots, a_n$, $\mathbb{E}\left[\sum_{i=1}^{n} a_i R_i\right] = \sum_{i=1}^{n} a_i\, \mathbb{E}[R_i]$.

Conditional Expectation: For random variable R, the expected value of R conditioned on an event A is given by:

$$\mathbb{E}[R|A] = \sum_{x \in \text{Range}(R)} x \Pr[R = x|A]$$

Law of Total Expectation

If R is a random variable $\mathcal{S} \to V$ and events $A_1, A_2, \ldots A_n$ form a partition of the sample space i.e. for all i, j, $A_i \cap A_j = \emptyset$ and $A_1 \cup A_2 \cup \ldots \cup A_n = \mathcal{S}$, then,

$$\mathbb{E}[R] = \sum_i \mathbb{E}[R|A_i] \Pr[A_i] .$$

Analogous to Pr(A) = sum_Pr(F I A_i] * Pr(A_i)

E(R) = sum(x * Pr(R = x)

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$$\mathbb{E}[R] = \sum_i \mathbb{E}[R|A_i] \Pr[A_i] \,.$$

Proof :

$$\mathbb{E}[R] = \sum_{x \in \text{Range}(R)} x \Pr[R = x] = \sum_{x \in \text{Range}(R)} x \sum_i \Pr[R = x|A_i] \Pr[A_i]$$

(Law of total probability)

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$$= \sum_i \Pr[A_i] \sum_{x \in \text{Range}(R)} x \Pr[R = x | A_i]$$

$$\implies \mathbb{E}[R] = \sum_i \Pr[A_i] \, \mathbb{E}[R|A_i].$$

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Conditional Expectation - Examples

Q: Suppose that 49.6% of the people in the world are male and the rest female. If the expected height of a randomly chosen male is 5 feet 11 inches, while the expected height of a randomly chosen female is 5 feet 5 inches, what is the expected height of a randomly chosen person?

et H be the random variable equal to height of person

You condition on events.

M = event person is male

F is event person is female

E(H) = Pr(M)E(H I M) + Pr(F)E(H I F)

= 0.496 * 71 + 0.504 * 65

=

Split up sample space into cases.

Q: Suppose that 49.6% of the people in the world are male and the rest female. If the expected height of a randomly chosen male is 5 feet 11 inches, while the expected height of a randomly chosen female is 5 feet 5 inches, what is the expected height of a randomly chosen person?

Define H to be the random variable equal to the height (in feet) of a randomly chosen person. Define M to be the event that the person is male and F the event that the person is female.
We wish to compute $\mathbb{E}[H]$ and we know that $\mathbb{E}[H|M] = 5 + \frac{11}{12}$ and $\mathbb{E}[H|F] = 5 + \frac{5}{12}$.
$\Pr[M] = 0.496$ and $\Pr[F] = 1 - 0.496 = 0.504$.
Hence, $\mathbb{E}[H] = \mathbb{E}[H|M]\Pr[M] + \mathbb{E}[H|F]\Pr[F] = \frac{71}{12}(0.496) + \frac{65}{12}(0.504)$.

Questions?

Randomized Quick Select

Given an array A of n distinct numbers, return the k^{th} smallest element in A for $k \in [1, n]$.

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Algorithm Randomized Quick Select

```
function QuickSelect(A, k)
If Length(A) = 1, return A[1].
Select p ∈ A uniformly at random.
Construct sets Left := {x ∈ A | x < p} and Right := {x ∈ A | x > p}.
r = |Left| + 1 {Element p is the r^th smallest element in A.}
if k = r then
    return p
else if k < r then
    QuickSelect(Left, k)
else
    QuickSelect(Right, k − r)
end if
```

Everything in left is smaller than p, everything in right is greater than p.

Given A = {4, 3, 7, 2}
Using p = 3
Left = {2}
Right = {4, 7}}
|Left| = 1, r = 1 + 1 = 2
If k = 2, then you have found it
If k = 3, then the element we want is in Right

You do k - r as the new value for k since you have discarded the r values on the right hand side.

If $A = \{2, 7, 0, 1, 3\}$ and we wish to find the 2^{nd} smallest element meaning that $k = 2$. According to the algorithm, $p \sim \text{Uniform}(A)$. Say $p = 3$.

We do not need the collection to be sorted.

Randomized Quick Select

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Then after step 1, $\text{Left} = \{2, 0, 1\}$ and $\text{Right} = \{7\}$. $r := |\text{Left}| + 1 = 3 + 1 = 4$. Since $r > k$, we recurse on the left-hand side by calling the algorithm on $\{2, 0, 1\}$ with $k = 2$.

Since r > k, we want to search the left side.

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$p \sim \text{Uniform}(\{2, 0, 1\})$. Say $p = 1$. After step 2, Left $= \{0\}$ and Right $= \{2\}$. $r := |\text{Left}| + 1 = 1 + 1 = 2$. Since $r = k$, we terminate the recursion and return $p = 1$ as the second-smallest element in A.

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$r := |\text{Left}| + 1 = 1 + 1 = 2$. Since $r = k$, we terminate the recursion and return $p = 1$ as the second-smallest element in A.

Q: Run the algorithm if $p = 0$ in the first step?

Right = {x l x > 0}
Left = {}
If k = 1, you would return 0 since r = 0 + 1
If k = 2, then you would need to search in Right

Randomized Quick Select

If $A = \{2, 7, 0, 1, 3\}$ and we wish to find the 2^{nd} smallest element meaning that $k = 2$. According to the algorithm, $p \sim \text{Uniform}(A)$. Say $p = 3$.

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$p \sim \text{Uniform}(\{2, 0, 1\})$. Say $p = 1$. After step 2, Left $= \{0\}$ and Right $= \{2\}$.
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Q: Run the algorithm if $p = 1$ in the first step?

Randomized Quick Select – Analysis

O(nlog(n)) + O(k)

O(k) comes from traversing the list to the kth element

Alternate way: Sort the elements in A and return the k^{th} element in the sorted list. Uses $O(n\log(n))$ comparisons.

Randomized Quick Select – Analysis

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Q: Can Randomized Quick Select do better – what is the maximum number of comparisons required by Randomized Quick Select in the worst-case?

O(n^2) in worse case.

Every time we choose the pivot, it is the smallest one.

A {2, 3, 0, 1, 7} k = n

P = 0 -> Left= {} , right = {2, 3, 1, 7}

p = 1 -> Left = {}, right = {2, 3, 7}

First step: n - 1 comparisons

Second step: n - 2 comparisons

third step: n - 3 comparisons

Randomized Quick Select – Analysis

Look at the worst case and then take a look at the average.
It could be that the worst case rarely happens.

Alternate way: Sort the elements in A and return the k^{th} element in the sorted list. Uses $O(n \log(n))$ comparisons.

Q: Can Randomized Quick Select do better – what is the maximum number of comparisons required by Randomized Quick Select in the worst-case?

In the worst case, Randomized Quick Select is worse than the naive strategy of sorting and returning the k^{th} element. What about the average (over the pivot selection) case?

Expectation

Randomized Quick Select – Analysis

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Claim: For any array A with n distinct elements, and for any $k \in [n]$, Randomized Quick Select performs fewer than $8n$ comparisons in expectation.

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Text

Claim: For any array A with n distinct elements, and for any $k \in [n]$, Randomized Quick Select performs fewer than $8n$ comparisons in expectation.

In order to prove this claim, we will need to prove the following lemma.

Randomized Quick Select – Analysis

Lemma: The child sub-problem's array (either Left or Right) after the partitioning (in Line 4 of the algorithm) has expected size smaller than $\frac{7n}{8}$.

In every step on average, the size of the left and right children will be less than 7n/8

Randomized Quick Select – Analysis

Lemma: The child sub-problem's array (either Left or Right) after the partitioning (in Line 4 of the algorithm) has expected size smaller than $\frac{7n}{8}$.

Proof: Define a "good" event $\mathcal{E}$ that the randomly chosen pivot splits the array roughly in half.

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Proof: Define a "good" event $\mathcal{E}$ that the randomly chosen pivot splits the array roughly in half.

r is the position of the pivot

Formally, if n is the length of the array, then $\mathcal{E}$ is the event that $r \in \left(\frac{n}{4}, \frac{3n}{4}\right]$ (for simplicity, let us assume that n is divisible by 4.) Since p is chosen uniformly at random, $\Pr[\mathcal{E}] = \frac{3n/4 - n/4}{n} = \frac{1}{2}$.

Since we only care about the elements from (n/4, 3n/4], we want to get rid of n/4 elements from the beginning.

We have a divisor of n since there are n possible places to select.

If F = {1, 2, 3,4 ,5 , 6, 7, 8}, E = {3, 4, 5, 6}
Pr(E) = 4/8 = 1/2

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Recall that $|\text{Left}| = r - 1$ and $|\text{Right}| = n - r$. Hence if event $\mathcal{E}$ happens, then $|\text{Left}| < \frac{3n}{4}$ and $|\text{Right}| < \frac{3n}{4}$. Hence, $|\text{Child}| < \frac{3n}{4}$. If event $\mathcal{E}$ does not happen, in the worst-case, $|\text{Child}| < n$.

In one scenario, Left can contain all the elements in (n/4, 3n/4), which has less elements than

n/4, than the greatest size of Right is < n - n/4 = 3n/4

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|Child| < 3n/4 only in the best scenario.

$$\begin{aligned}\mathbb{E}[|\text{Child}|] &= \mathbb{E}[|\text{Child}| \mid \mathcal{E}] \Pr[\mathcal{E}] + \mathbb{E}[|\text{Child}| \mid \mathcal{E}^c] \Pr[\mathcal{E}^c] \\ &< \frac{3n}{4}\frac{1}{2} + (n)\frac{1}{2} = \frac{7n}{8}.\end{aligned}$$

We have n since the size of the child is always less than n.

Hence on average, the size of the child sub-problem is smaller than $\frac{7n}{8}$, proving the lemma.

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In order to upper-bound the total number of comparisons, we use the Lemma with an induction on n. Recall that we need to prove that Randomized Quick Select requires fewer than $8n$ comparisons in expectation.

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Base case: If $n = 1$, then we require $0 < 8(1)$ comparisons. Hence the base case is satisfied.

Line 2 of the algorithm

Randomized Quick Select – Analysis

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Base case: If $n = 1$, then we require $0 < 8(1)$ comparisons. Hence the base case is satisfied.

Inductive Step: Assume that for all $m < n$,
$\mathbb{E}$[Total number of comparisons for size m array] $< 8\,m$.

Inductive hypothesis. Strong induction.

n -1 comes from the number of comparisons in the first iteration

E((n - 1) + total number of comparisons for child sub-problem)
= (n - 1) + E(# of comparisons for the child sub-problems)
< (n - 1) + 8(E(|child|) (by inductive hypothesis)
< n - 1 + 8(7/8n) = 8n - 1 < 8n

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$$\begin{aligned} &\mathbb{E}[\text{Total number of comparisons for size } n \text{ array}] \\ &= \mathbb{E}[(n-1) + \text{Total number of comparisons in child sub-problem}] \end{aligned}$$

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&= (n-1) + \mathbb{E}[\text{Total number of comparisons in child sub-problem}] \quad \text{(Linearity of expectation)}
\end{aligned}$$

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&= (n-1) + \mathbb{E}[\text{Total number of comparisons in child sub-problem}] && \text{(Linearity of expectation)} \\
&< (n-1) + 8\,\mathbb{E}[|\text{Child}|] && \text{(Induction hypothesis)}
\end{aligned}$$

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&< (n-1) + 8\,\mathbb{E}[|\text{Child}|] \quad \text{(Induction hypothesis)}\\
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\end{aligned}
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Hence, for any $k \in [n]$, on average, Randomized Quick Select requires fewer than $8n$ comparisons, even though it might require $O(n^2)$ comparisons in the worst-case.

Questions?

Independence of random variables

We define two random variables R_1 and R_2 to be independent if for *all* $x_1 \in \text{Range}(R_1)$ and $x_2 \in \text{Range}(R_2)$, events $[R_1 = x_1]$ and $[R_2 = x_2]$ are independent. More formally, we require,

$$\Pr[(R_1 = x_1) \cap (R_2 = x_2)] = \Pr[(R_1 = x_1)] \Pr[(R_2 = x_2)]$$

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$$\Pr[(R_1 = x_1) \cap (R_2 = x_2)] = \Pr[(R_1 = x_1)] \Pr[(R_2 = x_2)]$$

Q: Suppose we toss three independent, unbiased coins. Let C be r.v. equal to the number of heads that appear and M be the r.v. that is equal to 1 if all the coins match. Are random variables C and M independent?

Range(c) = {0, 1, 2, 3}

Range(M) = {0, 1}

If we want to show independence, we need to show that every pair of events are independent.

We need to show that the eight pair of events are independent.

If you can find a single counterexample, you do not need to prove any other cases.

Pr(M = 1) = 2/8 = 1/4

Only two cases where all three coins match.

Pr(C = 3) = 1/8, Pr(M = 1) = 1/4

Pr(c= 3 and m = 1) = 1/8 != 1/8 * 1/4

Therefore, C and M are not independent.

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$\text{Range}(C) = \{0, 1, 2, 3\}$ and $\text{Range}(M) = \{0, 1\}$. $\Pr[C = 3] = \frac{1}{8}$ and $\Pr[M = 1] = \frac{1}{4}$. $\Pr[(C = 3) \cap (M = 1)] = \frac{1}{8} \neq \frac{1}{32} = \Pr[C = 3] \Pr[M = 1]$. Hence, C and M are not independent.

Independence - Examples

Range(H_1) = {0, 1}

Q: If H_1 is the indicator r.v. equal to one if the first toss is a heads, are H_1 and M independent?

We have four possible events to look at.
Pr(H_1) = 1/2
Pr(M = 1) = 1/4
Pr(M = 0) = 3/4
Pr(h1 = 0 and M = 1) = Pr({TTT}) = 1/8 = Pr(H_1 = 0) & Pr(M = 1)
Pr(H1 = 1 and M = 0) = 3/8 = Pr(H1 = 1) = 1/2 * Pr(M = 0) = 1/2 * 3/4

H1 = 1 and M = 0 = ({HTT}, {HTH}, {HHT})

Independence - Examples

Q: If H_1 is the indicator r.v. equal to one if the first toss is a heads, are H_1 and M independent?
$\Pr[H_1 = 1] = \Pr[H_1 = 0] = \frac{1}{2}$, $\Pr[M = 1] = \frac{1}{4}$, $\Pr[M = 0] = \frac{3}{4}$.
$\Pr[H_1 = 0 \cap M = 1] = \Pr[\{TTT\}] = \frac{1}{8} = \Pr[H_1 = 0] \Pr[M = 1]$.
$\Pr[H_1 = 1 \cap M = 1] = \Pr[\{HHH\}] = \frac{1}{8} = \Pr[H_1 = 1] \Pr[M = 1]$.
$\Pr[H_1 = 0 \cap M = 0] = \Pr[\{THH, THT, TTH\}] = \frac{3}{8} = \Pr[H_1 = 0] \Pr[M = 0]$.
$\Pr[H_1 = 1 \cap M = 0] = \Pr[\{HHT, HTH, HTT\}] = \frac{3}{8} = \Pr[H_1 = 1] \Pr[M = 0]$.
Hence, H_1 and M are independent.

Independence of random variables

Q: If R_1 and R_2 are not independent, is $\mathbb{E}[R_1 + R_2] = \mathbb{E}[R_1] + \mathbb{E}[R_2]$?

Yes.

Use the proof of linearity of expectation.

linearity of expectation holds regardless of whether the random variables are independent or not.

Independence of random variables

Q: If R_1 and R_2 are not independent, is $\mathbb{E}[R_1 + R_2] = \mathbb{E}[R_1] + \mathbb{E}[R_2]$?

Yes! Recall the derivation of the linearity of expectation. We never assumed that R_1 and R_2 are independent for the proof and the linearity of expectation holds regardless of whether the random variables are independent.

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Q: If R_1 and R_2 are independent, is $\mathbb{E}[R_1 R_2] = \mathbb{E}[R_1]\,\mathbb{E}[R_2]$?

Range(R_1) = {1, 2}
Range(R_2) = {3, 4}
Range(R_1R_2) = {3, 4, 6, 8}

E(R1R2) = sum_all x in range(R_1R_2) x * Pr(R_1R_2 = x)

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Does not hold if they are dependent.

Q: If R_1 and R_2 are independent, is $\mathbb{E}[R_1 R_2] = \mathbb{E}[R_1]\,\mathbb{E}[R_2]$? Yes!

$$\mathbb{E}[R_1 R_2] = \sum_{x \in \text{Range}(R_1 R_2)} x \Pr[R_1 R_2 = x] = \sum_{r_1 \in \text{Range}(R_1), r_2 \in \text{Range}(R_2)} r_1 r_2 \Pr[R_1 = r_1 \cap R_2 = r_2]$$

$$(x = r_1\, r_2)$$

Q: If R_1 and R_2 are not independent, is $\mathbb{E}[R_1 + R_2] = \mathbb{E}[R_1] + \mathbb{E}[R_2]$?

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Revise this proof.

Q: If R_1 and R_2 are independent, is $\mathbb{E}[R_1 R_2] = \mathbb{E}[R_1]\,\mathbb{E}[R_2]$? Yes!

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$$= \sum_{r_1 \in \text{Range}(R_1)} \sum_{r_2 \in \text{Range}(R_2)} r_1 r_2 \Pr[R_1 = r_1 \cap R_2 = r_2] \qquad \text{(Splitting the sum)}$$

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$$\begin{aligned}
\mathbb{E}[R_1 R_2] &= \sum_{x \in \text{Range}(R_1 R_2)} x \Pr[R_1 R_2 = x] = \sum_{r_1 \in \text{Range}(R_1), r_2 \in \text{Range}(R_2)} r_1 r_2 \Pr[R_1 = r_1 \cap R_2 = r_2] && (x = r_1\, r_2) \\
&= \sum_{r_1 \in \text{Range}(R_1)} \sum_{r_2 \in \text{Range}(R_2)} r_1 r_2 \Pr[R_1 = r_1 \cap R_2 = r_2] && \text{(Splitting the sum)} \\
&= \sum_{r_1 \in \text{Range}(R_1)} \sum_{r_2 \in \text{Range}(R_2)} r_1 r_2 \Pr[R_1 = r_1] \Pr[R_2 = r_2] && \text{(Independence)} \\
&= \sum_{r_1 \in \text{Range}(R_1)} r_1 \Pr[R_1 = r_1] \sum_{r_2 \in \text{Range}(R_2)} r_2 \Pr[R_2 = r_2] = \mathbb{E}[R_1]\mathbb{E}[R_2]
\end{aligned}$$

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Independence of random variables

Alternate definition of independence – two random variables R_1 and R_2 are independent if for *all* $x_1 \in \text{Range}(R_1)$ and $x_2 \in \text{Range}(R_2)$,

$$\Pr[(R_1 = x_1) | (R_2 = x_2)] = \Pr[(R_1 = x_1)]$$
$$\Pr[(R_2 = x_2) | (R_1 = x_1)] = \Pr[(R_2 = x_2)]$$

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Similar to events, random variables $R_1, R_2, \ldots, R_n$ are mutually independent if for all $x_1, x_2, \ldots, x_n$, events $[R_1 = x_1], [R_2 = x_2], \ldots [R_n = x_n]$ are mutually independent.

Mutual Independence of events: A set of events is said to be mutually independent if the probability of each event in the set is the same no matter which of the events has occurred. For events E_1, E_2 and E_3 to be mutually independent, all the following equalities should hold:

$$\Pr[E_1 \cap E_2] = \Pr[E_1]\Pr[E_2] \quad \Pr[E_1 \cap E_3] = \Pr[E_1]\Pr[E_3]$$
$$\Pr[E_2 \cap E_3] = \Pr[E_2]\Pr[E_3] \quad \Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_1]\Pr[E_2]\Pr[E_3].$$

Alternatively, (i) $\forall i$ and $j \neq i$, $\Pr[E_i|E_j] = \Pr[E_i]$ and (ii) $\forall i$ and $j, k \neq i$, $\Pr[E_i|E_j \cap E_k] = \Pr[E_i]$.

eneralize this for n events, is this the sum from 2 to n of (n c n)

Expectation/Independence - Examples

Q: Suppose there is a dinner party where n people check in their coats. The coats are mixed up during dinner, so that afterward each person receives a random coat. In particular, a person gets their own coat with probability $\frac{1}{n}$. What is the expected number of people who get their own coat?

Let G be the number of people who get back their coats.

G_i is the event that person i gets their coat back.
G = G_1 + G_2 + G_3 + G_4 + ….. G_n
E(G) = E(G_1 + G_2 + G_3 + G_4 + ….. G_n)
= 1/n * n = 1.

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Let G be the number of people that get their own coat. We wish to compute $\mathbb{E}[G]$. Define G_i to be the indicator r.v. that person i gets their own coat. Observe that $G = G_1 + G_2 + \ldots + G_n$ and by linearity of expectation $\mathbb{E}[G] = \mathbb{E}[G_1] + \mathbb{E}[G_2] + \ldots + \mathbb{E}[G_n]$. For each i, $\mathbb{E}[G_i] = \Pr[G_i] = \frac{1}{n}$. Hence, $\mathbb{E}[G] = 1$ meaning that on average one person will correctly receive their coat.

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Q: If G_i is the indicator r.v. that person i gets their own coat, are the random variables $G_1, G_2, \ldots G_n$ mutually independent?

No. Since if $G_1 = G_2 = \ldots G_{n-1} = 1$, then,
$\Pr[G_n = 1 | (G_1 = 1 \cap G_2 = 1 \cap \ldots \cap G_{n-1} = 1)] = 1 \neq \frac{1}{n} = \Pr[G_n = 1]$. Conditioning on $(G_1, G_2, \ldots, G_{n-1})$ changes $\Pr[G_n]$, and hence the r.v's are not independent. Notice that we have used the linearity of expectation even though these r.v's are not mutually independent.

Questions?

Joint distributions

For a given experiment, we are often interested not only in the PDFs of individual random variables but also in the relationships between two or more random variables. For example, we might be interested in the mean time of failure and its connection with different number of components in the system.

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If $\mathrm{Range}[X] = \{x_1, x_2, \ldots x_n\}$, $\mathrm{Range}[Y] = \{y_1, y_2, \ldots y_n\}$, then for $x \in \mathrm{Range}(X)$,
$[X = x] = [X = x \cap y = y_1] \cup [X = x \cap y = y_2] \cup \ldots \cup [X = x \cap y = y_n]$
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$\implies \Pr[X = x] = \Pr[X = x \cap y = y_1] + \Pr[X = x \cap y = y_2] + \ldots + \Pr[X = x \cap y = y_n]$.

$\implies \mathrm{PDF}_X[x] = \sum_i \mathrm{PDF}_{X,Y}[x, y_i]$.
Hence, we can obtain the distribution for each r.v. from the joint distribution by "marginalizing" over the other r.v's.

Joint distributions - Examples

Q: Suppose that 3 batteries are randomly chosen from a group of 3 new, 4 used but still working, and 5 defective batteries. If the batteries are distinct and we let X and Y denote, respectively, the number of new and used but still working batteries that are chosen, completely specify $\text{PDF}_{X,Y}$.

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For $i \in [3], j \in [3]$, $\mathrm{PDF}_{X,Y}[i,j] = \Pr[X = i \cap Y = j | X + Y \leq 3] = \frac{\binom{3}{i}\binom{4}{j}\binom{5}{3-i-j}}{\binom{12}{3}}$.

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$\text{PDF}_{X,Y}[0,0] = \frac{\binom{5}{3}}{\binom{12}{3}} = 10/220$, $\text{PDF}_{X,Y}[1,2] = \frac{\binom{3}{1}\binom{4}{2}\binom{5}{2}}{\binom{12}{3}} = 18/220$.

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Table 4.1 $P\{X = i, Y = j\}$.

i \ j	0	1	2	3	Row Sum $= P\{X = i\}$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
Column Sums $= P\{Y = j\}$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	

Questions?