

CMPT 210: Probability and Computing

Lecture 17

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Expectation/mean of a random variable R is denoted by $\mathbb{E}[R]$ and “summarizes” its distribution. Formally, $\mathbb{E}[R] := \sum_{\omega \in \mathcal{S}} \Pr[\omega] R[\omega]$

Alternate definition of expectation: $\mathbb{E}[R] = \sum_{x \in \text{Range}(R)} x \Pr[R = x]$.

Linearity of Expectation: For n random variables R_1, R_2, \dots, R_n and constants a_1, a_2, \dots, a_n , $\mathbb{E}[\sum_{i=1}^n a_i R_i] = \sum_{i=1}^n a_i \mathbb{E}[R_i]$.

Conditional Expectation: For random variable R , the expected value of R conditioned on an event A is given by:

$$\mathbb{E}[R|A] = \sum_{x \in \text{Range}(R)} x \Pr[R = x|A]$$

Law of Total Expectation

If R is a random variable $\mathcal{S} \rightarrow V$ and events A_1, A_2, \dots, A_n form a partition of the sample space i.e. for all i, j , $A_i \cap A_j = \emptyset$ and $A_1 \cup A_2 \cup \dots \cup A_n = \mathcal{S}$, then,

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Proof:

$$\mathbb{E}[R] = \sum_{x \in \text{Range}(R)} x \Pr[R = x] = \sum_{x \in \text{Range}(R)} x \sum_i \Pr[R = x|A_i] \Pr[A_i]$$

(Law of total probability)

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Conditional Expectation - Examples

Q: Suppose that 49.6% of the people in the world are male and the rest female. If the expected height of a randomly chosen male is 5 feet 11 inches, while the expected height of a randomly chosen female is 5 feet 5 inches, what is the expected height of a randomly chosen person?

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Define H to be the random variable equal to the height (in feet) of a randomly chosen person. Define M to be the event that the person is male and F the event that the person is female.

We wish to compute $\mathbb{E}[H]$ and we know that $\mathbb{E}[H|M] = 5 + \frac{11}{12}$ and $\mathbb{E}[H|F] = 5 + \frac{5}{12}$.

$\Pr[M] = 0.496$ and $\Pr[F] = 1 - 0.496 = 0.504$.

Hence, $\mathbb{E}[H] = \mathbb{E}[H|M] \Pr[M] + \mathbb{E}[H|F] \Pr[F] = \frac{71}{12}(0.496) + \frac{65}{12}(0.504)$.

Questions?

Randomized Quick Select

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Algorithm Randomized Quick Select

```
1: function QuickSelect( $A, k$ )
2:   If  $\text{Length}(A) = 1$ , return  $A[1]$ .
3:   Select  $p \in A$  uniformly at random.
4:   Construct sets  $\text{Left} := \{x \in A \mid x < p\}$  and  $\text{Right} := \{x \in A \mid x > p\}$ .
5:    $r = |\text{Left}| + 1$  {Element  $p$  is the  $r^{\text{th}}$  smallest element in  $A$ .}
6:   if  $k = r$  then
7:     return  $p$ 
8:   else if  $k < r$  then
9:     QuickSelect( $\text{Left}, k$ )
10:  else
11:    QuickSelect( $\text{Right}, k - r$ )
12:  end if
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If $A = \{2, 7, 0, 1, 3\}$ and we wish to find the 2^{nd} smallest element meaning that $k = 2$.
According to the algorithm, $p \sim \text{Uniform}(A)$. Say $p = 3$.

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Then after step 1, $\text{Left} = \{2, 0, 1\}$ and $\text{Right} = \{7\}$. $r := |\text{Left}| + 1 = 3 + 1 = 4$. Since $r > k$, we recurse on the left-hand side by calling the algorithm on $\{2, 0, 1\}$ with $k = 2$.

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$p \sim \text{Uniform}(\{2, 0, 1\})$. Say $p = 1$. After step 2, $\text{Left} = \{0\}$ and $\text{Right} = \{2\}$.

$r := |\text{Left}| + 1 = 1 + 1 = 2$. Since $r = k$, we terminate the recursion and return $p = 1$ as the second-smallest element in A .

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Claim: For any array A with n distinct elements, and for any $k \in [n]$, Randomized Quick Select performs fewer than $8n$ comparisons in expectation.

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Claim: For any array A with n distinct elements, and for any $k \in [n]$, Randomized Quick Select performs fewer than $8n$ comparisons in expectation.

In order to prove this claim, we will need to prove the following lemma.

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Proof: Define a “good” event \mathcal{E} that the randomly chosen pivot splits the array roughly in half.

Formally, if n is the length of the array, then \mathcal{E} is the event that $r \in (\frac{n}{4}, \frac{3n}{4}]$ (for simplicity, let us assume that n is divisible by 4.) Since p is chosen uniformly at random, $\Pr[\mathcal{E}] = \frac{3n/4 - n/4}{n} = \frac{1}{2}$.

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Recall that $|\text{Left}| = r - 1$ and $|\text{Right}| = n - r$. Hence if event \mathcal{E} happens, then $|\text{Left}| < \frac{3n}{4}$ and $|\text{Right}| < \frac{3n}{4}$. Hence, $|\text{Child}| < \frac{3n}{4}$. If event \mathcal{E} does not happen, in the worst-case, $|\text{Child}| < n$.

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By using the law of total expectation,

$$\begin{aligned}\mathbb{E}[|\text{Child}|] &= \mathbb{E}[|\text{Child}| \mid \mathcal{E}] \Pr[\mathcal{E}] + \mathbb{E}[|\text{Child}| \mid \mathcal{E}^c] \Pr[\mathcal{E}^c] \\ &< \frac{3n}{4} \frac{1}{2} + (n) \frac{1}{2} = \frac{7n}{8}.\end{aligned}$$

Hence on average, the size of the child sub-problem is smaller than $\frac{7n}{8}$, proving the lemma.

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Hence, for any $k \in [n]$, on average, Randomized Quick Select requires fewer than $8n$ comparisons, even though it might require $O(n^2)$ comparisons in the worst-case.

Questions?

Independence of random variables

We define two random variables R_1 and R_2 to be independent if for *all* $x_1 \in \text{Range}(R_1)$ and $x_2 \in \text{Range}(R_2)$, events $[R_1 = x_1]$ and $[R_2 = x_2]$ are independent. More formally, we require,

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Q: Suppose we toss three independent, unbiased coins. Let C be r.v. equal to the number of heads that appear and M be the r.v. that is equal to 1 if all the coins match. Are random variables C and M independent?

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$\text{Range}(C) = \{0, 1, 2, 3\}$ and $\text{Range}(M) = \{0, 1\}$. $\Pr[C = 3] = \frac{1}{8}$ and $\Pr[M = 1] = \frac{1}{4}$.
 $\Pr[(C = 3) \cap (M = 1)] = \frac{1}{8} \neq \frac{1}{32} = \Pr[C = 3] \Pr[M = 1]$. Hence, C and M are not independent.

Q: If H_1 is the indicator r.v. equal to one if the first toss is a heads, are H_1 and M independent?

Independence - Examples

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$$\Pr[H_1 = 1] = \Pr[H_1 = 0] = \frac{1}{2}, \Pr[M = 1] = \frac{1}{4}, \Pr[M = 0] = \frac{3}{4}.$$

$$\Pr[H_1 = 0 \cap M = 1] = \Pr[\{TTT\}] = \frac{1}{8} = \Pr[H_1 = 0] \Pr[M = 1].$$

$$\Pr[H_1 = 1 \cap M = 1] = \Pr[\{HHH\}] = \frac{1}{8} = \Pr[H_1 = 1] \Pr[M = 1].$$

$$\Pr[H_1 = 0 \cap M = 0] = \Pr[\{THH, THT, TTH\}] = \frac{3}{8} = \Pr[H_1 = 0] \Pr[M = 0].$$

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Hence, H_1 and M are independent.

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$$\mathbb{E}[R_1 R_2] = \sum_{x \in \text{Range}(R_1 R_2)} x \Pr[R_1 R_2 = x] = \sum_{r_1 \in \text{Range}(R_1), r_2 \in \text{Range}(R_2)} r_1 r_2 \Pr[R_1 = r_1 \cap R_2 = r_2] \\ (x = r_1 r_2)$$

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Independence of random variables

Alternate definition of independence – two random variables R_1 and R_2 are independent if for *all* $x_1 \in \text{Range}(R_1)$ and $x_2 \in \text{Range}(R_2)$,

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Similar to events, random variables R_1, R_2, \dots, R_n are mutually independent if for all x_1, x_2, \dots, x_n , events $[R_1 = x_1], [R_2 = x_2], \dots [R_n = x_n]$ are mutually independent.

Mutual Independence of events: A set of events is said to be mutually independent if the probability of each event in the set is the same no matter which of the events has occurred. For events E_1, E_2 and E_3 to be mutually independent, all the following equalities should hold:

$$\Pr[E_1 \cap E_2] = \Pr[E_1] \Pr[E_2] \quad \Pr[E_1 \cap E_3] = \Pr[E_1] \Pr[E_3]$$

$$\Pr[E_2 \cap E_3] = \Pr[E_2] \Pr[E_3] \quad \Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_1] \Pr[E_2] \Pr[E_3].$$

Alternatively, (i) $\forall i$ and $j \neq i$, $\Pr[E_i|E_j] = \Pr[E_i]$ and (ii) $\forall i$ and $j, k \neq i$, $\Pr[E_i|E_j \cap E_k] = \Pr[E_i]$.

Expectation/Independence - Examples

Q: Suppose there is a dinner party where n people check in their coats. The coats are mixed up during dinner, so that afterward each person receives a random coat. In particular, a person gets their own coat with probability $\frac{1}{n}$. What is the expected number of people who get their own coat?

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Let G be the number of people that get their own coat. We wish to compute $\mathbb{E}[G]$. Define G_i to be the indicator r.v. that person i gets their own coat. Observe that $G = G_1 + G_2 + \dots + G_n$ and by linearity of expectation $\mathbb{E}[G] = \mathbb{E}[G_1] + \mathbb{E}[G_2] + \dots + \mathbb{E}[G_n]$. For each i , $\mathbb{E}[G_i] = \Pr[G_i] = \frac{1}{n}$. Hence, $\mathbb{E}[G] = 1$ meaning that on average one person will correctly receive their coat.

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No. Since if $G_1 = G_2 = \dots = G_{n-1} = 1$, then,

$\Pr[G_n = 1 | (G_1 = 1 \cap G_2 = 1 \cap \dots \cap G_{n-1} = 1)] = 1 \neq \frac{1}{n} = \Pr[G_n = 1]$. Conditioning on $(G_1, G_2, \dots, G_{n-1})$ changes $\Pr[G_n]$, and hence the r.v.'s are not independent. Notice that we have used the linearity of expectation even though these r.v.'s are not mutually independent.

Questions?

Joint distributions

For a given experiment, we are often interested not only in the PDFs of individual random variables but also in the relationships between two or more random variables. For example, we might be interested in the mean time of failure and its connection with different number of components in the system.

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If $\text{Range}[X] = \{x_1, x_2, \dots, x_n\}$, $\text{Range}[Y] = \{y_1, y_2, \dots, y_n\}$, then for $x \in \text{Range}(X)$,

$$[X = x] = [X = x \cap y = y_1] \cup [X = x \cap y = y_2] \cup \dots \cup [X = x \cap y = y_n]$$

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$$\implies \text{PDF}_X[x] = \sum_i \text{PDF}_{X,Y}[x, y_i].$$

Hence, we can obtain the distribution for each r.v. from the joint distribution by “marginalizing” over the other r.v's.

Joint distributions - Examples

Q: Suppose that 3 batteries are randomly chosen from a group of 3 new, 4 used but still working, and 5 defective batteries. If the batteries are distinct and we let X and Y denote, respectively, the number of new and used but still working batteries that are chosen, completely specify $\text{PDF}_{X,Y}$.

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$$\text{For } i \in [3], j \in [3], \text{PDF}_{X,Y}[i,j] = \Pr[X = i \cap Y = j | X + Y \leq 3] = \frac{\binom{3}{i} \binom{4}{j} \binom{5}{3-i-j}}{\binom{12}{3}}.$$

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$$\text{PDF}_{X,Y}[0,0] = \frac{\binom{5}{3}}{\binom{12}{3}} = 10/220, \text{PDF}_{X,Y}[1,2] = \frac{\binom{3}{1} \binom{4}{2} \binom{5}{2}}{\binom{12}{3}} = 18/220.$$

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Table 4.1 $P\{X = i, Y = j\}$.

$i \backslash j$	0	1	2	3	Row Sum $= P\{X = i\}$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
Column Sums = $P\{Y = j\}$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	

Questions?