Assn 1 → out tonight
→ due next Tuesday

CMPT 210: Probability and Computing

Lecture 5

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January 23, 2024

{1, 2, ... 6} ; each number → equally likely

// → "loaded" (opposite)

Q: Suppose we throw a standard dice. What is the probability that the number that comes up is 6?

1/6

Introduction to Probability - Throwing dice

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[uniform prob. space]

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Probability of getting a 6 $= \frac{\text{Number of ways in which the thing we care about happens}}{\text{Total number of ways in which something can happen}} = \frac{1}{6}$.

Q: Suppose we throw a standard dice. What is the probability that we get either a 3 or a 6?

possible things → {1, 2, ... 6}

(outcome)

things we care about = 3 or a 6

(event)

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$E = \{3, 6\}$

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$$\text{Probability of getting either a 3 or a 6} = \frac{\text{Number of ways in which the event we care about happens}}{\text{Total number of outcomes}} = \frac{\boxed{2}}{6}.$$

$= 1/3$

Q: Suppose we throw two standard dice one after the other. What is the probability that we get two 6's in a row?

What are the $\boxed{\text{possible outcomes}}$ that can happen? The first dice comes up one of the numbers in $1, 2, 3, 4, 5, 6$, the second dice comes up one of the numbers in $1, 2, 3, 4, 5, 6$.

If we consider both dice together, what are the possible outcomes – first dice is 1, second dice is 1; first is 1, second is 2, and so on. Let us write this compactly. The space of outcomes is $\{(1, 1), (1, 2), (1, 3), \ldots, (6, 6)\}$.

tuples

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$(1,2) \rightarrow (2,1)$

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Probability of getting two 6's in a row $= \frac{\text{Number of ways in which the event we care about happens}}{|\text{outcome space}|} = \frac{1}{36}$.

Probability Basics

Sample (outcome) space $\boxed{\mathcal{S}}$: Nonempty (countable) set of possible outcomes. *Example*: When we threw one dice, the sample space is $\{1, 2, 3, 4, 5, 6\}$. When we threw two die, the sample space is $\{(1, 1), (1, 2), (1, 3), \ldots\} = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$ (using the relation between sets and sequences).

Cartesian Products

The sample space is not necessarily numbers. *Example*: If we are randomly choosing colors from the rainbow, then $\mathcal{S} = \{\text{violet, indigo, blue, green, yellow, orange, red}\}$.

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several symbol for outcome

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Outcome $\boxed{\omega} \in \mathcal{S}$: Possible "thing" that can happen. *Example*: When we threw one dice, a possible outcome is $\omega = 1$. For the rainbow example, the color "red" is a possible outcome.

Sample space → set

outcomes → elements

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Subset → event.

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Outcome $\omega \in \mathcal{S}$: Possible "thing" that can happen. *Example*: When we threw one dice, a possible outcome is $\omega = 1$. For the rainbow example, the color "red" is a possible outcome.

Event E: Any subset of the sample space. *Example*: When we threw one dice, a possible event is $E = \{6\}$ (first example) or $E = \{3, 6\}$ (second example). When we threw two die, a possible event is $E = \{(6, 6)\}$.

tuple

throwing the dice

set

An event E "happens" if the outcome ω (from some process) is in set E i.e. if $\omega \in E$.

$E = \{6\}$

Union of events

Since the event E is a set, all the set theory we learned is useful!

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Suppose E, F are two events in $\mathcal{S}$. Define the union $E \cup F$ to consist of outcomes that are either in E or F (this is just the definition of the union of two sets). Formally,

logical OR

$\omega \in S$

$$G = E \cup F = \{\omega | \omega \in E \text{ OR } \omega \in F\}.$$

→ at least one of

Another way to interpret this is to say event G occurs if either event E or event F occurs. E or F happens.

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Example: We considered the case where we threw one dice and cared about getting *either* 3 or 6. In this case, event G happens if we get either 3 or 6. Formally, $E = \{3\}$, $F = \{6\}$, $G = E \cup F = \{3, 6\}$. And $\boxed{G}$ occurs when the number that shows up is either 3 or 6.

Prob of $G = E \cup F$

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$S = \{1, \dots 6\}$

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Can define union between more than two events in the same way we defined union between more than two sets. $\boxed{G} = E_1 \cup E_2 \cup \dots E_n$. G happens when *at least one* of the events E_i happen.

$i = 1 \dots n.$

Intersection of events

Suppose E, F are two events in $\mathcal{S}$. Define the intersection $E \cap F$ to consist of outcomes that are in both E and F (this is just the definition of the intersection of two sets). Formally,

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$\boxed{E} = \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$, F is the event we get a 6 for the second dice.
$\boxed{F} = \{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6)\}$, $G = E \cap F = \{(6,6)\}$. G happens when both E and F happen i.e. the first dice has a 6 and the second dice has 6.

$E, F \subset S$; $S = \{(1,1),(1,2) \ldots (6,6)\}$; $F = \{(1,6);(2,6), \ldots (6,6)\}$

$\boxed{E} = \{(6,1),(6,2),(6,3),(6,4) \ldots (6,6)\}$ $\quad E \cap F = \{(6,6)\}$

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Can define intersection between more than two events in the same way we defined intersection between more than two sets. $\boxed{G} = E_1 \cap E_2 \cap \ldots E_n$. G happens when *all* of the events E_i happen.

$i = 1 \cdots n$

Mutually exclusive and complement events

Mutually exclusive events: If E and F are two events such that $E \cap F = \{\}$, then events E and F are mutually exclusive.

Example: We threw one dice and want to get both 3 *and* 6. This is not possible. Formally, $E = \{6\}$, $F = \{3\}$ and $E \cap F = \{\}$, hence, events E and F are mutually exclusive.

↳ both events cannot happen at the same time.

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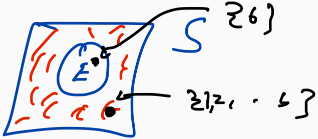
Complement of an event: If E is an event, then its complement $\boxed{E^c}$ is defined such that $E \cap E^c = \{\}$ and $E \cup E^c = \mathcal{S}$. Event E^c will occur if and only if event E does not occur.

Example: We threw one dice and want to get a 6 i.e. we define $E = \{6\}$. $E^c = \{1, 2, 3, 4, 5\}$.

Two complement events are mutually exclusive, but two mutually exclusive events need not be the complements of each other. *Example*: $E = \{6\}$ and $F = \{3\}$ are mutually exclusive, but not complements.

mutual exclusivity $\not\Rightarrow$ complement

$\Leftarrow$



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Subset: If $E \subset F$, then if E happens F will happen. *Example*: When we throw one dice, if $E = \{3\}$ and $F = \{1, 2, 3\}$ i.e. E is the event that we get 3 and F is the event that we can either $1, 2, 3$. Clearly, if E happens, F will happen.

Axioms of Probability

$\omega \in S \rightarrow$ is assigned prob

Probability $\boxed{\textbf{function}}$ on a sample space $\mathcal{S}$ is a $\boxed{\text{total function}}$ $\Pr : \mathcal{S} \to [0, 1]$.

domain   co-domain

For any $\omega \in \mathcal{S}$, $0 \leq Pr[\omega] \leq 1$; $\sum_{\boxed{\omega \in \mathcal{S}}} \Pr[\omega] = 1$

Probability space: The outcome space $\mathcal{S}$ together with the probability function.

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Recall that we can define functions on sets. In this case, for an event E, $\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$.

$E = \{1, 2\}$

$\Pr[E] = \Pr[1] + \Pr[2]$

$\omega \in E$

$S = \{1, \ldots 6\}$

$\boxed{\omega} = 1 \ldots$

$= 2 \ldots 6$

$\omega \in S$

$\omega \to$ element of S

Axioms of Probability

Probability function on a sample space $\mathcal{S}$ is a total function $\Pr : \mathcal{S} \to [0, 1]$. $E_1 = \{3\}$

For any $\omega \in \mathcal{S}$, $0 \leq Pr[\omega] \leq 1$; $\sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1$ $E_2 = \{4\}$

Probability space: The outcome space $\mathcal{S}$ together with the probability function. $E_1 \cup E_2 = \{3, 4\}$

Recall that we can define functions on sets. In this case, for an event E, $\boxed{\Pr[E] = \sum_{\omega \in E} \Pr[\omega]}$.

Union: For mutually exclusive events $E_1, E_2, \ldots, E_n$ (sets $E_1, E_2, \ldots, E_n$ are disjoint),
$\boxed{\Pr[E_1 \cup E_2 \cup \ldots E_n]} = \Pr[E_1] + \Pr[E_2] + \ldots + \Pr[E_n]$.

$= \sum_{\boxed{\omega} \in \{E_1 \cup \cdots E_n\}} \Pr[\omega]$ → def of calculating the prob of an event

$= \boxed{\sum_{\omega \in E_1} \Pr[\omega]} + \boxed{\sum_{\omega \in E_2} \Pr[\omega]} + \ldots \boxed{\sum_{\omega \in E_n} \Pr[\omega]}$

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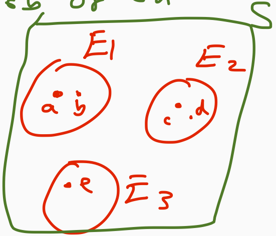

Axioms of Probability

Probability function on a sample space $\mathcal{S}$ is a total function $\Pr : \mathcal{S} \to [0, 1]$.

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Recall that we can define functions on sets. In this case, for an event E, $\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$.

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Proof:

$$\Pr[E_1 \cup E_2 \cup \ldots E_n] = \sum_{\omega \in \{\boldsymbol{E_1} \cup \boldsymbol{E_2} \cup \ldots E_n\}} \Pr[\omega]$$

Since E_i's are disjoint, any ω can only be in one of $E_1, E_2, \ldots E_n$

$$= \sum_{\omega \in \boldsymbol{E_1}} \Pr[\omega] + \sum_{\omega \in \boldsymbol{E_2}} \Pr[\omega] + \ldots + \sum_{\omega \in E_n} \Pr[\omega] = \Pr[E_1] + \Pr[E_2] + \ldots + \Pr[E_n].$$

Back to throwing dice

each outcome is equally likely.

(formal)

Possible outcomes

Q: Suppose we throw a standard dice. What is the probability that the number that comes up is 6?

$S = \{1, 2, 3, 4, 5, 6\}$

$\boxed{1 \geq Pr[1] \geq 0}$;

$$\sum_{w \in S} Pr[w] = 1$$

$Pr[1] + Pr[2] + \dots Pr[6] = 1$

$Pr[1] = Pr[2] = \dots Pr[6]$

$\Rightarrow Pr[6] = 1/6.$

Back to throwing dice

Q: Suppose we throw a standard dice. What is the probability that the number that comes up is 6?

$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$. Since the dice is "standard", each outcome is equally likely, i.e. $\Pr[1] = \Pr[2] = \ldots = \Pr[6]$.

Since $\Pr[\mathcal{S}] = 1 \implies \sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1 \implies \Pr[1] + \Pr[2] + \ldots \Pr[6] = 1 \implies \Pr[6] = \frac{1}{6}$.

Back to throwing dice

$G = E \cup F$.

Q: Suppose we throw a standard dice. What is the probability that we get either a 3 or a 6?

$E = \{3\}$

$F = \{6\}$

$E \cap F = \phi$

$\Rightarrow E, F$ are disjoint

$E \cup F = \{3, 6\}$

$\Pr[E \cup F] = \Pr[E] + \Pr[F]$

$= \Pr[3] + \Pr[6]$

$= 1/6 + 1/6$

$= \underline{1/3}$

$G = \{3, 6\}$

$\Pr[G] = \sum_{\omega \in G} \Pr[\omega] = \Pr[3] + \Pr[6]$

$= 1/3.$

Back to throwing dice

Q: Suppose we throw a standard dice. What is the probability that we get either a 3 or a 6?

$E = \{3\}$, $F = \{6\}$, $G = \{3, 6\}$. Since $E \cap F = \{\}$, E and F are mutually exclusive events, implying that $\Pr[G] = \Pr[E] + \Pr[F] = \Pr[\{3\}] + \Pr[\{6\}] = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

Hence, probability of getting either a 3 or a 6 is equal to $\frac{1}{3}$.

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Hence, probability of getting either a 3 or a 6 is equal to $\frac{1}{3}$.

Q: Compute the probability of getting either 1, 2 or 3. $\longrightarrow$

Q: Compute the probability of getting an even number.

Q: Compute the probability of getting either $\{$1, 2, 3, 4, 5, 6$\}$

$S = \{1 \ldots 6\}$

$E_1 = \{1\}$

$E_2 = \{2\}$

$E_3 = \{3\}$

$= \Pr[E_1] + \Pr[E_2] + \Pr[E_3]$

$\Rightarrow \Pr[E_1 \cup E_2 \cup E_3]$

$E_1 = \{2\}$

$E_2 = \{4\}$

$E_3 = \{6\}$

$1/2$

$|S| = 1$

$= S$

$\Pr[S] = 1$

Probability Rules

Complement rule: $\Pr[E] = 1 - \Pr[E^c]$.

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Proof: Recall that $E \cap E^c = \{\}$ and $E \cup E^c = \mathcal{S}$. Since E and E^c are disjoint,

$$\Pr[E \cup E^c] = \Pr[E] + \Pr[E^c] \implies \Pr[S] = \Pr[E] + \Pr[E^c] \implies \Pr[E^c] \quad = 1 - \Pr[E].$$

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Inclusion-Exclusion rule: For any two events E, F, $\Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$.

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Proof:

$$\Pr[E \cup F] = \sum_{\omega \in \{E \cup F\}} \Pr[\omega] = \sum_{\omega \in \{E - F\}} \Pr[\omega] + \sum_{\omega \in \{F - E\}} \Pr[\omega] + \sum_{\omega \in \{E \cap F\}} \Pr[\omega]$$

(Since disjoint)

Probability Rules

Complement rule: $\Pr[E] = 1 - \Pr[E^c]$.

Proof: Recall that $E \cap E^c = \{\}$ and $E \cup E^c = \mathcal{S}$. Since E and E^c are disjoint,

$$\Pr[E \cup E^c] = \Pr[E] + \Pr[E^c] \implies \Pr[S] = \Pr[E] + \Pr[E^c] \implies \Pr[E^c] \quad = 1 - \Pr[E].$$

Inclusion-Exclusion rule: For any two events E, F, $\Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$.

Proof:

$$\Pr[E \cup F] = \sum_{\omega \in \{E \cup F\}} \Pr[\omega] = \sum_{\omega \in \{E - F\}} \Pr[\omega] + \sum_{\omega \in \{F - E\}} \Pr[\omega] + \sum_{\omega \in \{E \cap F\}} \Pr[\omega]$$

(Since disjoint)

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Proof:

$$\Pr[A] = \sum_{\omega \in A} \Pr[\omega] = \sum_{\omega \in B} \Pr[\omega] - \sum_{\omega \in \{B - A\}} \Pr[\omega] \implies \Pr[A] < \Pr[B]$$

(Since probabilities are non-negative.)