CMPT 210: Probability and Computing

Lecture 21

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Markov's Theorem

Markov's theorem formalizes the intuition on the last slide of the previous class, and can be stated as follows.

Markov's Theorem: If X is a non-negative random variable, then for all $x > 0$,

$$\Pr[X \geq x] \leq \frac{\mathbb{E}[X]}{x}.$$

We want to bound the probability of the r.v taking values beyond the mean.

c >= 1

If x = cE[X], Pr(X >= cE[X]) <= 1/c

This is a typical way to analyze randomized algorithms.

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Proof: Define $\mathcal{I}\{X \geq x\}$ to be the indicator r.v. for the event $[X \geq x]$. Then for all values of X, $x\mathcal{I}\{X \geq x\} \leq X$.

Case 1: I{X >= x} = 1, so we have that x <= (X = x)
Case 2: I{X < x} -> 0 <= X.

Since this is true, we can apply the expectation to both sides to obtain E[Ix{X >= x}] <= E[X]
We can pull the x out of the left hand side (it is a constant) to obtain
xE[I{X >= x}] <= E[X]
x * Pr(X >= x) <= E[X]
Pr(X >= x) <= E[X]/x

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$$\mathbb{E}[x\,\mathcal{I}\{X \geq x\}] \leq \mathbb{E}[X] \implies x\,\mathbb{E}[\mathcal{I}\{X \geq x\}] \leq \mathbb{E}[X] \implies x\Pr[X \geq x] \leq \mathbb{E}[X]$$
$$\implies \Pr[X \geq x] \leq \frac{\mathbb{E}[X]}{x}.$$

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$$\mathbb{E}[x\,\mathcal{I}\{X \geq x\}] \leq \mathbb{E}[X] \implies x\,\mathbb{E}[\mathcal{I}\{X \geq x\}] \leq \mathbb{E}[X] \implies x\Pr[X \geq x] \leq \mathbb{E}[X]$$
$$\implies \Pr[X \geq x] \leq \frac{\mathbb{E}[X]}{x}.$$

Since the above theorem holds for all $x > 0$, we can set $x = c\mathbb{E}[X]$ for $c \geq 1$. In this case, $\Pr[X \geq c\mathbb{E}[X]] \leq \frac{1}{c}$. Hence, the probability that X is "far" from the mean in terms of the multiplicative factor c is upper-bounded by $\frac{1}{c}$.

Markov's Theorem – Example

Q: Suppose there is a dinner party where n people check in their coats. The coats are mixed up during dinner, so that afterward each person receives a random coat. In particular, a person gets their own coat with probability $\frac{1}{n}$.

G is r.v equal to # of people that get their own coat.
E[G] = E[G1] + E[G2] + …. E[Gn] = 1
Since G takes on non-negative values, Pr(G >=5) <= E[G]/5 = 1/5
We are not making another assumption about G, so this bound is good.
Since we know nothing about the distribution, Markov's theorem is useful.

Markov's Theorem – Example

Q: Suppose there is a dinner party where n people check in their coats. The coats are mixed up during dinner, so that afterward each person receives a random coat. In particular, a person gets their own coat with probability $\frac{1}{n}$.

Recall that if G is the r.v. corresponding to the number of people that receive their own coat, then we used the linearity of expectation to derive that $\mathbb{E}[G] = 1$. Using Markov's Theorem,

$$\Pr[G \geq x] \leq \frac{\mathbb{E}[G]}{x} = \frac{1}{x}.$$

Hence, we can bound the probability that x people receive their own coat. For example, there is no better than 20% chance that more than 5 people get their own coat.

Markov's Theorem – Example

Q: If X is a non-negative r.v. such that $\mathbb{E}[X] = 150$, compute the probability that X is at least 200.

Pr(X >= 200) <= E[X]/200 = 3/4

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Markov's Theorem – Example

Q: If X is a non-negative r.v. such that $\mathbb{E}[X] = 150$, compute the probability that X is at least 200.

Q: If we are provided additional information that X can not take values less than 100 and $\mathbb{E}[X] = 150$, compute the probability that X is at least 200.

We need to shift the entire bound by 100.
Y = X - 100, since Y >= 0, we can apply Markov's theorem
Pr(X >= 200) = Pr(Y >= 100)
You can do algebraic operations inside the probabilities
Pr(X >= 200) = Pr(X - 100 >= 200 - 100) = Pr(Y >= 100) <= (E[X] - 100)/100 = 50/100 = 1/2

This a tighter bound than what we had without knowing that X is greater than 99.

Markov's Theorem – Example

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Q: If we are provided additional information that X can not take values less than 100 and $\mathbb{E}[X] = 150$, compute the probability that X is at least 200.

Define $Y := X - 100$. $\mathbb{E}[Y] = \mathbb{E}[X] - 100 = 50$ and Y is non-negative.

$$\Pr[X \geq 200] = \Pr[Y + 100 \geq 200] = \Pr[Y \geq 100] \leq \frac{\mathbb{E}[Y]}{100} = \frac{50}{100} = \frac{1}{2}$$

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$$\Pr[X \geq 200] = \Pr[Y + 100 \geq 200] = \Pr[Y \geq 100] \leq \frac{\mathbb{E}[Y]}{100} = \frac{50}{100} = \frac{1}{2}$$

Hence, if we have additional information (in the form of a lower-bound that a r.v. can not be smaller than some constant $b > 0$), we can use Markov's Theorem on the shifted r.v. (Y in our example) and obtain a tighter bound on the probability of deviation.

Chebyshev's Theorem

No constraint on X being positive or negative.

Chebyshev's Theorem: For a r.v. X and any constant $y > 0$,

$$\Pr[|X - \mathbb{E}[X]| \geq y] \leq \frac{\mathsf{Var}[X]}{y^2}.$$

The probability that the value of X - E[X] is <= E[x] - y or >= E[X] + y is bounded by Var[X]/y^2

Tells us that X takes on a value within E[x] + y and E[X] -y

If you want an additive bound, you want Chebysehv

If you want a multiplicative bound, you use Markov

Knowing the variance, Chebyshev's theorem provides a tighter bound.

Chebyshev's Theorem

Chebyshev's Theorem: For a r.v. X and any constant $y > 0$,

$$\Pr[|X - \mathbb{E}[X]| \geq y] \leq \frac{\mathrm{Var}[X]}{y^2}.$$

Proof: Use Markov's Theorem with some cleverly chosen function of X. Formally, for some function f such that $Y := f(X)$ is non-negative. Using Markov's Theorem for Y,

$$\Pr[f(X) \geq x] \leq \frac{\mathbb{E}[f(X)]}{x}$$

f(X) = |X - E[X]|^2
we need small x to be non-negative, so let x = y^2
Pr(|X - E[X]|^2 >= y^2] <= (E|X - E[X]|^2)/y^2
e add the absolute value since when we take the square, the absolute value stays
an modify the expression above to Pr(|X - E[X]| >= y] <= (E|X - E[X]|)^2/y^2 = Var[X]/y^2

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$$\Pr[f(X) \geq x] \leq \frac{\mathbb{E}[f(X)]}{x}$$

Choosing $f(X) = |X - \mathbb{E}[X]|^2$ and $x = y^2$ implies that $f(X)$ is non-negative and $x > 0$. Using Markov's Theorem,

$$\Pr[|X - \mathbb{E}[X]|^2 \geq y^2] \leq \frac{\mathbb{E}[|X - \mathbb{E}[X]|^2]}{y^2}$$

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$$\Pr[|X - \mathbb{E}[X]|^2 \geq y^2] \leq \frac{\mathbb{E}[|X - \mathbb{E}[X]|^2]}{y^2}$$

Note that $\Pr[|X - \mathbb{E}[X]|^2 \geq y^2] = \Pr[|X - \mathbb{E}[X]| \geq y]$, and hence,

$$\Pr[|X - \mathbb{E}[X]| \geq y] \leq \frac{\mathbb{E}[|X - \mathbb{E}[X]|^2]}{y^2} = \frac{\mathsf{Var}[X]}{y^2}$$

Chebyshev's Theorem

Chebyshev's Theorem bounds the probability that the random variable X is "far" away from the mean $\mathbb{E}[X]$ by an additive factor of x.

Let n = c sigma_x > 0

Pr(|X - E[X]| >= c * sigma_x) <= Var[X]/(c^2 sigma^2_x) = 1/c^2

The probability of the r.v deviating from the mean is extremely small.

Chebyshev's Theorem

Chebyshev's Theorem bounds the probability that the random variable X is "far" away from the mean $\mathbb{E}[X]$ by an additive factor of x.

If we set $x = c\sigma_X$ where σ_X is the standard deviation of X, then by Chebyshev's Theorem,

$$\Pr[(X \geq \mathbb{E}[X] + c\,\sigma_X) \cup (X \leq \mathbb{E}[X] - c\,\sigma_X)] = \Pr[|X - \mathbb{E}[X]| \geq c\sigma_X] \leq \frac{\mathrm{Var}[X]}{c^2\sigma_X^2} = \frac{1}{c^2}$$

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$$\begin{aligned} & \Pr[\mathbb{E}[X] - c\sigma_X < X < \mathbb{E}[X] + c\sigma_X] = \Pr[|X - \mathbb{E}[X]| \leq c\sigma_X] \\ \implies & \Pr[\mathbb{E}[X] - c\sigma_X < X < \mathbb{E}[X] + c\sigma_X] = 1 - \Pr[|X - \mathbb{E}[X]| \geq c\sigma_X] \geq 1 - \frac{1}{c^2}. \end{aligned}$$

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Chebyshev's Theorem is used to bound the probability that X is "concentrated" near its mean.

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Chebyshev's Theorem is used to bound the probability that X is "concentrated" near its mean.

Unlike Markov's Theorem, Chebyshev's Theorem does not require the r.v. to be non-negative, but requires knowledge of the variance.

Chebyshev's Theorem - Example

Q: If X is a non-negative r.v. such that $\mathbb{E}[X] = 100$ and $\sigma_X = 15$, compute the probability that X is at least 300.

Markov: bound is E[X] / 300 = 100/300 = 1/3

Chebyshev: Pr(|X - E[X]| >= X) <= var[X]/x^2
Pr(|X - 100| >= x] = Pr(X - 100 >= x U -X + 100 >= x]
= Pr(X >= 100 + x U X <= 100 - x]
Choose x = 200
Pr(X .>= 300 U X <= -100) = Pr(X >= 300)
This is true since X is non-negative.
Pr(|X - 100| >= 200] = Pr(X >= 300) <= Var[X]/200^2 = (15)^2/(200)^2 = 1/(178)

Chebyshev's bound is far smaller.

Chebyshev's Theorem - Example

Q: If X is a non-negative r.v. such that $\mathbb{E}[X] = 100$ and $\sigma_X = 15$, compute the probability that X is at least 300.

If we use Markov's Theorem, $\Pr[X \geq 300] \leq \frac{\mathbb{E}[X]}{300} = \frac{1}{3}$.

Chebyshev's Theorem - Example

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If we use Markov's Theorem, $\Pr[X \geq 300] \leq \frac{\mathbb{E}[X]}{300} = \frac{1}{3}$.

Note that $\Pr[|X - 100| \geq 200] = \Pr[X \leq -100 \cup X \geq 300] = \Pr[X \geq 300]$. Using Chebyshev's Theorem,

$$\Pr[X \geq 300] = \Pr[|X - 100| \geq 200] \leq \frac{\mathrm{Var}[X]}{(200)^2} = \frac{15^2}{200^2} \approx \frac{1}{178}.$$

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Hence, by exploiting the knowledge of the variance and using Chebyshev's inequality, we can obtain a tighter bound.

Chebyshev's Theorem - Example

Q: Consider a r.v. $X \sim \text{Bin}(20, 0.75)$. Plot the PDF$_X$, compute its mean and standard deviation and bound $\Pr[10 < X < 20]$.

Range of random variable is the domain of the PDF

Pr(X = x) = 20cx * (0.75)^x (1/4)^{20 - x}

E[X] = 15

Var[X] = np(1 - p) = 15 * 1/4 = 15/4

standard deviation is sqrt(15)/2

Pr(10 < x < 20] = 1 - Pr(X <=10 U X >= 20)

= 1 - Pr(lX - 15l >= 5)

Pr(lX - 15l >= 5) =< 3.75/25

lculated the event of the unwanted event occurring, we need to take the complement to bound the probability of the event we want to occur.

1 -

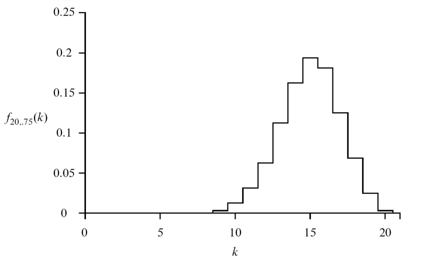
1 - Pr(lX - E[X]l >= 5) = 1 - 3.75/25

Review this slide.

Chebyshev's Theorem - Example

Q: Consider a r.v. $X \sim \text{Bin}(20, 0.75)$. Plot the PDF_X, compute its mean and standard deviation and bound $\Pr[10 < X < 20]$.

Range(X) = $\{0, 1, \ldots, 20\}$ and for $k \in$ Range(X), $f(k) = \binom{n}{k} p^k (1-p)^{n-k}$.



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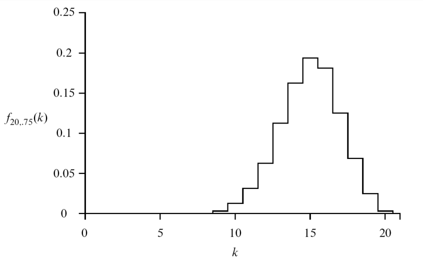

$\text{Range}(X) = \{0, 1, \ldots, 20\}$ and for $k \in \text{Range}(X)$,
$f(k) = \binom{n}{k} p^k (1-p)^{n-k}$.
$\mathbb{E}[X] = np = (20)(0.75) = 15$
$\text{Var}[X] = np(1-p) = 20(0.75)(0.25) = 3.75$ and hence
$\sigma_X = \sqrt{3.75} \approx 1.94$.

Chebyshev's Theorem - Example

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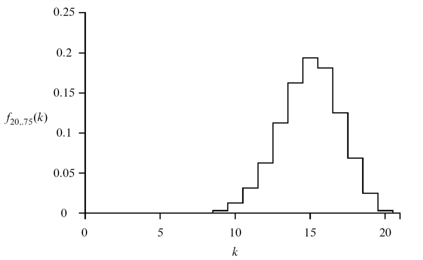

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\begin{aligned}
\Pr[10 < X < 20] &= 1 - \Pr[X \leq 10 \cup X \geq 20] \\
&= 1 - \Pr[|X - 15| \geq 5] \\
&= 1 - \Pr[|X - \mathbb{E}[X]| \geq 5] \\
&\geq 1 - \frac{\text{Var}[X]}{(5)^2} = 1 - \frac{3.75}{25} = 0.85.
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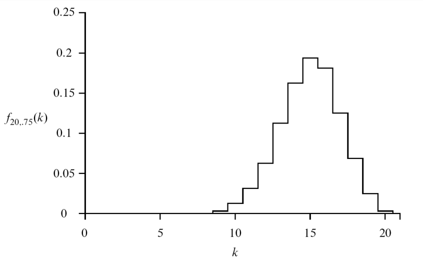

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&\geq 1 - \frac{\text{Var}[X]}{(5)^2} = 1 - \frac{3.75}{25} = 0.85.
\end{aligned}$$

Hence, the "probability mass" of X is "concentrated" around its mean.

Voter Poll

Q: Suppose there is an election between two candidates Donald Trump and Joe Biden, and we are hired by candidate Biden's election campaign to estimate his chances of winning the election. In particular, we want to estimate p, the fraction of voters favoring Biden before the election. We conduct a voter poll – selecting (typically calling) people uniformly at random (with replacement so that we can choose a person twice) and try to estimate p. What is the number of people we should poll to estimate p reasonably accurately and with reasonably high probability?

We want to create a collection of indicator variables which are mutually independent and which indicate whether the ith person voted Biden.
People that we call are identicallly distributed, with probability p of voting Biden.

If you call few people, then the approximation for p will be innaccurate.

Voter Poll

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Define X_i to be the indicator r.v. equal to 1 iff person i that we called favors Biden.

Assumption (1): The X_i r.v's are mutually independent since the people we poll are chosen randomly and we assume that their opinions do not affect each other.

Assumption (2): The people we call are identically distributed i.e. $X_i = 1$ with probability p.

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Assumption (2): The people we call are identically distributed i.e. $X_i = 1$ with probability p.

Suppose we poll n people and define $S_n := \sum_{i=1}^{n} X_i$ as the r.v. equal to the total number of people (amongst the ones we polled) that prefer Biden. $\frac{S_n}{n}$ is the *statistical estimate* of p.

Voter Poll

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Define X_i to be the indicator r.v. equal to 1 iff person i that we called favors Biden.

Assumption (1): The X_i r.v's are mutually independent since the people we poll are chosen randomly and we assume that their opinions do not affect each other.

S_n is binomially distributed.

Do we have a distribution for when we have more than two options?

This would be the multinomial distribution.

Assumption (2): The people we call are identically distributed i.e. $X_i = 1$ with probability p.

Suppose we poll n people and define $S_n := \sum_{i=1}^{n} X_i$ as the r.v. equal to the total number of people (amongst the ones we polled) that prefer Biden. $\frac{S_n}{n}$ is the *statistical estimate* of p.

Q: What is the distribution of S_n?

S_n ~ Bin(n, p)

Voter Poll

Delta is another parameter which tells you the lower bound of your probability. Delta is the probability of making an error.

We want to find for what n is our estimate for p accurate up to an error $\epsilon > 0$ and with probability $1 - \delta$ (for $\delta \in (0, 1)$). Formally, for what n is,

$$\Pr\left[\left|\frac{S_n}{n} - p\right| < \epsilon\right] \geq 1 - \delta$$

Delta is the probability of having the bad outcome occur.

The probability that the difference between the actual value and the estimate is less than epsilon.
If we take the complement of this expression, we obtain something we can manipulate using Chebyshev's theorem.

E[S_n/n] = 1/n * np = p

Voter Poll

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Since $S_n \sim \text{Bin}(n, p)$, $\mathbb{E}[S_n] = np$ and hence, $\mathbb{E}\left[\frac{S_n}{n}\right] = p$, meaning that our estimate is *unbiased* – in expectation, the estimate is equal to p. Hence, the above statement is equivalent to,

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Voter Poll

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$$\Pr\left[\left|\frac{S_n}{n} - \mathbb{E}\left[\frac{S_n}{n}\right]\right| < \epsilon\right] = 1 - \Pr\left[\left|\frac{S_n}{n} - \mathbb{E}\left[\frac{S_n}{n}\right]\right| \geq \epsilon\right] \geq 1 - \frac{\text{Var}[S_n/n]}{\epsilon^2}.$$

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Hence, the problem now is to find n such that,

$$1 - \frac{\mathrm{Var}[S_n/n]}{\epsilon^2} \geq 1 - \delta \implies \frac{\mathrm{Var}[S_n/n]}{\epsilon^2} < \delta$$

Voter Poll

Let us calculate the $\mathrm{Var}[S_n/n]$.

$$\begin{aligned}
\mathrm{Var}[S_n/n] &= \frac{1}{n^2}\mathrm{Var}[S_n] && \text{(Using the property of variance)} \\
&= \frac{1}{n^2}\, n\, p\,(1-p) = \frac{p\,(1-p)}{n} && \text{(Using the variance of the Binomial distribution)}
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Even if we are given epsilon and delta, we need p

Since p(1 - p) is maximized when p = 1/2, we can bound the value of p.

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Hence, if $n \geq \frac{1}{4\epsilon^2\delta}$, then $\Pr\left[\left|\frac{S_n}{n} - p\right| < \epsilon\right] \geq 1-\delta$ meaning that we have estimated p upto an error ϵ and this bound is true with high probability equal to $1-\delta$.

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For example, if $\epsilon = 0.01$ and $\delta = 0.01$ meaning that we want the bound to hold 99% of the time, then, we require $n \geq 250000$.

Pairwise Independent Sampling

Claim: Let $G_1, G_2, \ldots, G_n$ be pairwise independent random variables with the same mean μ and standard deviation σ. Define $S_n := \sum_{i=1}^{n} G_i$, then,

$$\Pr\left[\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right] \leq \frac{1}{n}\left(\frac{\sigma}{\epsilon}\right)^2.$$

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Proof: Let us compute $\mathbb{E}[S_n/n]$ and $\text{Var}[S_n/n]$.

$$\mathbb{E}[S_n] = \mathbb{E}\left[\sum_{i=1}^{n} G_i\right] = \sum_{i=1}^{n} \mathbb{E}[G_i] = n\mu \implies \mathbb{E}[S_n/n] = \frac{1}{n}\mathbb{E}[S_n] = \mu$$

(Using linearity of expectation)

$$\text{Var}[S_n] = \text{Var}\left[\sum_{i=1}^{n} G_i\right] = \sum_{i=1}^{n} \text{Var}[G_i] = n\sigma^2$$

(Using linearity of variance for pairwise independent r.v's)

$$\implies \text{Var}[S_n/n] = \frac{1}{n^2}\text{Var}[S_n] = \frac{\sigma^2}{n}$$

Pairwise Independent Sampling

Using Chebyshev's Theorem,

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Weak Law of Large Numbers: Let $G_1, G_2, \ldots, G_n$ be pairwise independent variables with the same mean μ and (finite) standard deviation σ. Define $X_n := \frac{\sum_{i=1}^{n} G_i}{n}$, then for every $\epsilon > 0$,

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Proof: Follows from the theorem on pairwise independent sampling since $\lim_{n\to\infty} \Pr[|X_n - \mu| \leq \epsilon] = \lim_{n\to\infty} \left[1 - \frac{\sigma^2}{n\epsilon^2}\right] = 1.$

Questions?