CMPT 210: Probability and Computing

Lecture 22

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Recap

Tail inequalities bound the probability that the r.v. takes a value much different from its mean.

Markov's Theorem: If X is a non-negative random variable, then for all $x > 0$, $\Pr[X \geq x] \leq \frac{\mathbb{E}[X]}{x}$.

Chebyshev's Theorem: For a r.v. X and all $x > 0$, $\Pr[|X - \mathbb{E}[X]| \geq x] \leq \frac{\mathrm{Var}[X]}{x^2}$.

Markov's theorem measures a multiplicative difference
Chebyshev's theorem measures from an additive difference.

Pairwise Independent Sampling

Claim: Let $G_1, G_2, \ldots, G_n$ be pairwise independent random variables with the same mean μ and standard deviation σ. Define $S_n := \sum_{i=1}^{n} G_i$, then,

$$\Pr\left[\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right] \leq \frac{1}{n}\left(\frac{\sigma}{\epsilon}\right)^2.$$

This is true for all epsilon.

mu is the true mean.
S_n /n is the average of the estimate

Epsilon measures the probability of obtaining an error.

In the voter poll problem, the G_i are the indication r.v for the people polled.

No distribution assumption on S_n.

As epsilon decreases, the probability of the bad event occurring increases.
As sigma increases, the probability of the bad event occurring.
Epsilon is fixed, and n is modified to allow you to obtain the probability you want.

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Since we can replace mu by E[S_n /n], we can replace it and apply Chebyshev's theorem.

Proof: Let us compute $\mathbb{E}[S_n/n]$ and $\mathrm{Var}[S_n/n]$.

$$\mathbb{E}[S_n] = \mathbb{E}\left[\sum_{i=1}^{n} G_i\right] = \sum_{i=1}^{n} \mathbb{E}[G_i] = n\mu \implies \mathbb{E}[S_n/n] = \frac{1}{n}\mathbb{E}[S_n] = \mu$$

(Using linearity of expectation)

$$\mathrm{Var}[S_n] = \mathrm{Var}\left[\sum_{i=1}^{n} G_i\right] = \sum_{i=1}^{n} \mathrm{Var}[G_i] = n\sigma^2$$

(Using linearity of variance for pairwise independent r.v's)

$$\implies \mathrm{Var}[S_n/n] = \frac{1}{n^2}\mathrm{Var}[S_n] = \frac{\sigma^2}{n}$$

As n increases, then the variance decreases.

Pairwise Independent Sampling

Using Chebyshev's Theorem,

$$\Pr\left[\left|\frac{S_n}{n} - \mathbb{E}\left[\frac{S_n}{n}\right]\right| \geq \epsilon\right] = \Pr\left[\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right] \leq \frac{\mathrm{Var}[S_n/n]}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$$

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Hence, for arbitrary pairwise independent r.v's, if n increases, the probability of deviation from the mean μ decreases.

Weak Law of Large Numbers: Let $G_1, G_2, \ldots, G_n$ be pairwise independent variables with the same mean μ and (finite) standard deviation σ. Define $X_n := \frac{\sum_{i=1}^{n} G_i}{n}$, then for every $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr[|X_n - \mu| \leq \epsilon] = 1.$$

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If the standard deviation is not finite, then the numerator in is infinite.

$$\lim_{n\to\infty} \Pr[|X_n - \mu| \leq \epsilon] = 1.$$

Proof: Follows from the theorem on pairwise independent sampling since $\lim_{n\to\infty} \Pr[|X_n - \mu| \leq \epsilon] = \lim_{n\to\infty} \left[1 - \frac{\sigma^2}{n\epsilon^2}\right] = 1$.

Questions?

Sums of Random Variables

If we know that the r.v X is (i) non-negative and (ii) $\mathbb{E}[X]$, we can use Markov's Theorem to bound the probability of deviation from the mean.

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Markov and Chebyshev theorems hold for a generic r.v

Chernoff holds for a sum of r.v's

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We need more than pairwise independence.

Chernoff Bound: Let $T_1, T_2, \ldots, T_n$ be mutually independent r.v's such that $0 \le T_i \le 1$ for all i. If $T := \sum_{i=1}^{n} T_i$, for all $c \ge 1$ and $\beta(c) := c\ln(c) - c + 1$,

$$\Pr[T \ge c\mathbb{E}[T]] \le \exp(-\beta(c)\,\mathbb{E}[T])$$

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$$\Pr[T \geq c\mathbb{E}[T]] \leq \exp(-\beta(c)\,\mathbb{E}[T])$$

If $T_i \sim \mathrm{Ber}(p)$ and are mutually independent, then $T_i \in \{0, 1\}$ and we can use the Chernoff bound to bound the deviation from the mean for $T \sim \mathrm{Bin}(n, p)$. In general, if $T_i \in [0, 1]$, the Chernoff Bound can be used even if the T_i's have different distributions!

Chernoff Bound – Binomial Distribution

Q: Bound the probability that the number of heads that come up in 1000 independent tosses of a fair coin exceeds the expectation by 20% or more.

n = 1000
p = 1/2

Pr(T > 1.2E[T]) <= e^{-(1.2 ln(1.2) - 1.2 + 1)500}

T_i is the indicator r.v that we get a heads on attempt i

For each T_i, T_i can take on either 0 or 1.
Each T_i is mutually independent.

c = 1.2
E[T] = 500 = 1000 * 1/2 = np. T ~ Bin(n, p)
beta(c) = 1.2(ln(c)) - 1.2 + 1 = 1.2ln(c) - 0.2

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Let T_i be the indicator r.v. for the event that coin i comes up heads, and let T denote the total number of heads. Hence, $T = \sum_{i=1}^{1000} T_i$. For all i, $T_i \in \{0, 1\}$ and are mutually independent r.v's. Hence, we can use the Chernoff Bound.

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We want to compute the probability that the number of heads is larger than the expectation by 20% meaning that $c = 1.2$ for the Chernoff Bound. Computing $\beta(c) = c\ln(c) - c + 1 \approx 0.0187$. Since the coin is fair, $\mathbb{E}[T] = 1000\,\frac{1}{2} = 500$. Plugging into the Chernoff Bound,

$\Pr[T \geq c\mathbb{E}[T]] \leq \exp(-\beta(c)\mathbb{E}[T]) \implies \Pr[T \geq 1.2\,\mathbb{E}[T]] \leq \exp(-(0.0187)\,(500)) \approx 0.0000834.$

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The probability that T - E[T] >= (c - 1)E[T] is less than Pr[|T - E[T]| >= (c - 1)E[T]] since the last expression considers the values outside the range on both sides. We do not consider only the range of values greater than what we are looking for.

Since the coin is fair, $\mathbb{E}[T] = 1000\,\frac{1}{2} = 500$. Plugging into the Chernoff Bound,

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Comparing this to using Chebyshev's inequality,

This term acts as x in chebyshev's theorem.

$$\Pr[T \geq c\mathbb{E}[T]] = \Pr[T - \mathbb{E}[T] \geq (c-1)\mathbb{E}[T]] \leq \Pr[|T - \mathbb{E}[T]| \geq (c-1)\,\mathbb{E}[T]]$$

$$\leq \frac{\mathrm{Var}[T]}{(c-1)^2\,(\mathbb{E}[T])^2} = \frac{1000\,\frac{1}{4}}{(1.2-1)^2(500^2)} = \frac{250}{0.2^2\,500^2} = \frac{250}{10000} = 0.025.$$

Chernoff Bound – Lottery Game

Q: Pick-4 is a lottery game in which you pay \$1 to pick a 4-digit number between 0000 and 9999. If your number comes up in a random drawing, then you win \$5,000. Your chance of winning is 1 in 10000. If 10 million people play, then the expected number of winners is 1000. When there are 1000 winners, the lottery keeps \$5 million of the \$10 million paid for tickets. The lottery operator's nightmare is that the number of winners is much greater – especially at the point where more than 2000 win and the lottery must pay out more than it received. What is the probability that will happen? (Assume that the players' picks and the winning number are random, independent and uniform)

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Let T_i be an indicator for the event that player i wins. Then $T := \sum_{i=1}^{n} T_i$ is the total number of winners. Using the independence assumptions, we can conclude that T_i are independent, as required by the Chernoff bound.

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We wish to compute $\Pr[T \geq 2000] = \Pr[T \geq 2\mathbb{E}[T]]$. Hence $c = 2$ and $\beta(c) \approx 0.386$. By the Chernoff bound,

$$\Pr[T \geq 2\mathbb{E}[T]] \leq \exp(-\beta(c)\mathbb{E}[T]) = \exp(-(0.386)\,1000) < \exp(-386) \approx 10^{-168}$$

Comparing the Bounds

For r.v's $T_1, T_2, \ldots T_n$, if $T_i \in \{0,1\}$ and $\Pr[T_i = 1] = p_i$. Define $T := \sum_{i=1}^{n} T_i$. By linearity of expectation, $\mathbb{E}[T] = \sum_{i=1}^{n} p_i$. For $c \geq 1$,

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$$\implies \Pr[T - \mathbb{E}[T] \geq (c-1)\mathbb{E}[T]] \leq \frac{\mathrm{Var}[T]}{(c-1)^2\,(\mathbb{E}[T])^2)} \qquad (x = (c-1)\,\mathbb{E}[T])$$

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If the T_i's are pairwise independent, by linearity of variance, $\mathrm{Var}[T] = \sum_{i=1}^{n} p_i\,(1-p_i)$. Hence, $\Pr[T \geq c\mathbb{E}[T]] \leq \frac{\sum_{i=1}^{n} p_i\,(1-p_i)}{(c-1)^2 \left(\sum_{i=1}^{n} p_i\right)^2}$. If for all i, $p_i = 1/2$, then, $\Pr[T \geq c\mathbb{E}[T]] \leq \frac{1}{(c-1)^2\,n}$.

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Chernoff Bound: If T_i' are mutually independent, then, $\Pr[T \geq c\mathbb{E}[T]] \leq \exp(-\beta(c)\,\mathbb{E}[T]) = \exp\left(-(c\ln(c) - c + 1)\,\left(\sum_{i=1}^{n} p_i\right)\right)$. If for all i, $p_i = 1/2$, $\Pr[T \geq c\mathbb{E}[T]] \leq \exp\left(-\frac{n\,(c\ln(c)-c+1)}{2}\right)$.

Questions?