

CMPT 476 Lecture 8

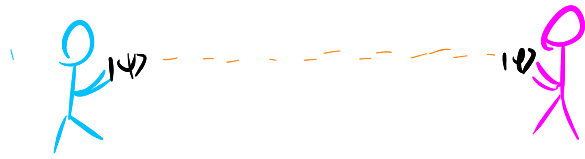
Bell's inequality and non-local games



Last class we discussed the **EPR paradox** which seemed to allow instantaneous communication between **Alice** and **Bob**. This led Einstein and others to believe that quantum theory was **incomplete** since it appeared to violate relativity. They posited that a complete picture of reality should be based on **local hidden variables**. Today, we look at **Bell's theorem** which famously showed that quantum mechanics **cannot** be predicted by any local hidden variable model.

(Local hidden variable models)

Let Alice and Bob share an entangled pair



(e.g. for concreteness, $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$)

A local hidden variable model asserts that for a set (or all?) possible single-qubit measurements M_i , the result of measuring M_i on Alice or Bob's qubit was pre-determined at the time of entanglement.

What does this mean?

Say we have 2 measurements M_x & M_z , each with 2 results, 0 or 1. Denote the result of measuring Alice or Bob's qubit $M_{\{x,z\}}(A)$ & $M_{\{x,z\}}(B)$. This means that the 4 values $M_x(A)$, $M_z(A)$, $M_x(B)$, $M_z(B)$ exist in the universe and are independent of who measures first and in which bases they measure.

(Bell's inequality)

$$\text{Let } A_0 = (-1)^{M_x(A)}, \quad A_1 = (-1)^{M_z(A)} \\ B_0 = (-1)^{M_x(B)}, \quad B_1 = (-1)^{M_z(B)}$$

Then (on average) \leftarrow needed to violate the inequality later

$$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$$

Pf

Note that $B_0 = \pm B_1$. If $B_0 = B_1$ then

$$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 = 2A_0 = \pm 2$$

Over n times, the average is hence

$$\frac{\sum_i A_0 B_0 + A_0 B_1 + A_1 B_0 + A_1 B_1}{n} \leq \frac{2n}{n} = 2$$

(Bell's theorem)

The inequality above can be **violated** with an entangled quantum state:

1. Start with the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

2. Let $M_x(A)$ be meas. in $\{|0\rangle, |1\rangle\}$

$M_z(A)$ be meas. in $\{|+\rangle, |-\rangle\}$ // $|0\rangle$

$M_x(B)$ be meas. in $\{\cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle, -\sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle\}$ // $|B\rangle$

$M_z(B)$ be meas. in $\{\cos\frac{\pi}{8}|0\rangle - \sin\frac{\pi}{8}|1\rangle, \sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle\}$ // $|B\rangle$

3. Now we calculate meas. statistics... **yuck**

$$|\langle 0 | \langle a_0 | \psi \rangle|^2 = |(\cos\frac{\pi}{8} \langle 00 | + \sin\frac{\pi}{8} \langle 01 |) \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)|^2$$

Am I measuring
psi in the basis

(computational basis) tensor-prod (rotated basis?)

$$= \frac{1}{2} \cos^2 \frac{\pi}{8}$$

only non-zero term

$$|\langle 0 | \langle a_1 | \psi \rangle|^2 = \frac{1}{2} \sin^2 \frac{\pi}{8}$$

$$|\langle 1 | \langle a_0 | \psi \rangle|^2 = \frac{1}{2} \sin^2 \frac{\pi}{8}$$

$$|\langle 1 | \langle a_1 | \psi \rangle|^2 = \frac{1}{2} \cos^2 \frac{\pi}{8}$$

$$\begin{aligned} \text{So } A_0 B_0 &= (-1)^0 (-1)^0 \frac{1}{2} \cos^2 \frac{\pi}{8} + (-1)^0 (-1)^1 \frac{1}{2} \sin^2 \frac{\pi}{8} + (-1)^1 (-1)^0 \frac{1}{2} \sin^2 \frac{\pi}{8} + (-1)^1 (-1)^1 \frac{1}{2} \cos^2 \frac{\pi}{8} \\ &= \cos^2 \frac{\pi}{8} - \sin^2 \frac{\pi}{8} \end{aligned}$$

$$= \frac{\sqrt{2}}{2}$$

(formally this is the expected value of $A_0 B_0$)

4. By a similar argument, the expected values are:

$$\mathbb{E}(A_0 B_1) = \frac{\sqrt{2}}{2}, \mathbb{E}(A_1 B_0) = \frac{\sqrt{2}}{2}, \mathbb{E}(A_1 B_1) = -\frac{\sqrt{2}}{2}$$

5. Now, what is the expected value of the sum?

$$\mathbb{E}(A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1)$$

$$= \mathbb{E}(A_0 B_0) + \mathbb{E}(A_0 B_1) + \mathbb{E}(A_1 B_0) - \mathbb{E}(A_1 B_1)$$

$$= 4 \cdot \frac{\sqrt{2}}{2}$$

$$= 2\sqrt{2}$$

$$\underline{\underline{> 2}}$$

(Non-local games)

A common theme in Quantum computer science is to phrase the above in a more computational way: as a **game** where **Alice** and **Bob** are not allowed to communicate directly, but share entanglement. It's helpful to see both perspectives, though non-local games fail* to capture the modern generalization of Bell's theorem to the **Kochen-Specker theorem** and notions of **contextuality**, which we lack the language to express at this point.

* Others may disagree with me here

(CHSH game (Clauser, Horne, Shimony & Holt))

The game is a co-operative one where Alice and Bob try to win against a referee. They are allowed to communicate before the game begins, but not while the game is on. Here are the rules:

1. The referee gives Alice and Bob each a bit a, b respectively in secret.
2. Without communicating, Alice and Bob each give the referee a bit back, x and y
3. Alice and Bob win if and only if

$$x \oplus y = a \cdot b$$

 exclusive OR, i.e. $x + y \bmod 2$

(The classical strategy)

$x = y = 0$. Observe that $x \oplus y = 0 = a \cdot b$ unless $a = b = 1$, so if a & b are random, win $\frac{3}{4}$ times



(Optimality for classical)

Note that any deterministic strategy, i.e. functions

$$f: a \mapsto x, \quad g: b \mapsto y$$

fails on at least one input (25% of the time), since

$$\sum_{a,b \in \{0,1\}} f(a) \oplus g(b) = 0 \bmod 2$$

$$\sum_{a,b \in \{0,1\}} a \cdot b = 1 \bmod 2$$

To see the previous fact we often arrange the outcomes in a table (called a **parity argument**)

a b		$f(a) \oplus g(b)$		$a \cdot b$
0	0	$f(0)$	$\oplus g(0)$	0
0	1	$f(0)$	$\oplus g(1)$	0
1	0	$f(1)$	$\oplus g(0)$	0
1	1	$f(1)$	$\oplus g(1)$	1

Summing the rows **mod 2** we see that $f(a)$ and $g(b)$ appear twice for each value of a & b , hence they cancel, since $x+x=0 \pmod 2$ for all $x \in \{0,1\}$

Since any **probabilistic** strategy is a weighted sum of deterministic ones, no such strategy can succeed more than

75% or **$\frac{3}{4}$** on average.

(Quantum strategy)

Pre-game: **Alice** and **Bob** prepare the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Alice: if $a=0$ then measure in the $\{|0\rangle, |1\rangle\}$ basis
 $a=1$ then measure in the $\{|+\rangle, |-\rangle\}$ basis

Bob: if $b=0$ then measure in $\{|0\rangle, |1\rangle\}$
 $b=1$ then measure $\{|b_0\rangle, |b_1\rangle\}$

Claim: **Alice** and **Bob** win **85%** of the time.

Proof: Suppose $a=b=0=a \cdot b$. Then Alice & Bob win if they measure **0&0** (i.e. $|0\rangle|0\rangle$) or **1&1** (i.e. $|1\rangle|1\rangle$). We computed these probabilities before:

$$\langle 0| \langle 0| \cdot |\psi\rangle + \langle 1| \langle 1| \cdot |\psi\rangle = \cos^2 \frac{\pi}{8} \approx 0.85$$

The other cases are similar

□

(Non-locality and Quantum computation)

The key take-away is that with **shared entanglement**, only **local** operations (i.e. operators & measurements only on individual qubits or subsystems) are needed to perform ^{some} tests which would **classically** require communication.

