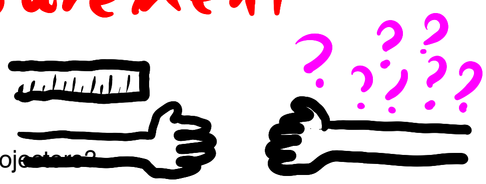
CMPT 476 Lecture 7

Partial measurement

What is a von neumann measurement?



What is the importance of rank in projectors?

Last lecture we discussed composite systems and in particular how to build & operate on them. However, to get something useful out of a quantum system we need to measure the result.

Going back to Alice and Bob, we saw that the joint state of their qubits is a unit vector in

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$$

We already know how to measure a state in $\mathbb{C}^4$. For instance, if the joint state is

pr(|01> 0

or Alice, the probability of getting the zero state is the sum of getting the 0 state as the first bit.

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Measuring produces "00" with probability $\frac{1}{2}$, and "11" with prob. $\frac{1}{2}$. However, this is a measurement on 2 distinct qubits. Even if they're physically close, how can we possibly measure 2 qubits simultaneously?

In this lecture we will discuss measurement more generally, and show that such a measurement can be described by local or partial measurements

(Partial measurement, informally)

Let $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ be a two qubit state (i.e. unit vector in $\mathbb{C}^2 \otimes \mathbb{C}^2$). Measuring the joint state produces:

$$
\begin{aligned}
&|00\rangle \quad \text{w/ prob. } |a|^2 \\
&|01\rangle \quad \text{w/ prob. } |b|^2 \\
&|10\rangle \quad \text{w/ prob. } |c|^2 \\
&|11\rangle \quad \text{w/ prob. } |d|^2
\end{aligned}
$$

The probability that we obtain 0 in the first bit is

$$|a|^2 + |b|^2$$

The ~~laws~~ postulates of Quantum mechanics state that this is the probability of measuring 0 in the first qubit even if only the first qubit is measured.

In this case, there is still uncertainty in the second qubit so the resulting state is not actually

$$a|00\rangle + b|01\rangle$$

This however is not a unit vector, so the actual state is the renormalized version

$$\frac{a|00\rangle + b|01\rangle}{\sqrt{|a|^2 + |b|^2}} = |0\rangle \otimes \left(\frac{a|0\rangle + b|1\rangle}{\sqrt{|a|^2 + |b|^2}} \right)$$

Aside

Observe that to normalize $|\psi\rangle$, divide by $\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle}$

E.x. $|\psi\rangle = \begin{bmatrix} 1 \\ i \end{bmatrix}$, $\langle\psi|\psi\rangle = 2$ hence not a unit vector.

$$|\psi'\rangle = \frac{|\psi\rangle}{\||\psi\rangle\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}, \quad \langle\psi'|\psi'\rangle = \frac{\langle\psi|\psi\rangle}{\||\psi\rangle\|^2} = \frac{\langle\psi|\psi\rangle}{\langle\psi|\psi\rangle} = 1$$

Lets verify that measuring the second qubit reproduces the probabilities of the joint measurement.

i.e. prob. $|0\rangle_A$ THEN $|0\rangle_B$ = prob. $|0\rangle_A|0\rangle_B$

prob. $|0\rangle_A$ THEN $|1\rangle_B$ = prob. $|0\rangle_A|1\rangle_B$

(we sometimes use this notation to denote the particular subsystem we're talking about.)

After measuring $|0\rangle_A$, we have the state

$$|0\rangle_A \otimes \left(\frac{a|0\rangle_B + b|1\rangle_B}{\sqrt{|a|^2+|b|^2}}\right)$$

Measuring qubit B produces

$$|0\rangle_B \text{ w/ prob } \left|\frac{a}{\sqrt{|a|^2+|b|^2}}\right|^2 = \frac{|a|^2}{|a|^2+|b|^2}$$

$$|1\rangle_B \text{ w/ prob } \left|\frac{a}{\sqrt{|a|^2+|b|^2}}\right|^2 = \frac{|b|^2}{|a|^2+|b|^2}$$

Now,

$$\text{prob. } |0\rangle_A \times \text{prob } |0\rangle_B = (|a|^2+|b|^2)\cdot\left(\frac{|a|^2}{|a|^2+|b|^2}\right) = |a|^2$$

$$\text{prob. } |0\rangle_A \times \text{prob } |1\rangle_B = (|a|^2+|b|^2)\cdot\left(\frac{|b|^2}{|a|^2+|b|^2}\right) = |b|^2$$

as expected :)

Intuition is the probability that we measure state $|i\rangle$ in a measurement of 1 qubit is the probability that we would get $|i\rangle$ for that qubit if we measured all qubits, and the resulting state is the normalized projection onto $|i\rangle$ for that qubit.

(Partial Von Neumann measurement)

Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{H}_A$ have basis $\{|e_i\rangle\}$.

Write $|\psi\rangle = \sum_i \alpha_i |e_i\rangle |\varphi_i\rangle$ where each $|\varphi_i\rangle$ is a unit vector.

Just a unit vector in H_B

A partial measurement on $\mathcal{H}_A$ w/ basis $\{|e_i\rangle\}$ produces result "i" with probability $|\alpha_i|^2$ and leaves the state as

$$|e_i\rangle|\varphi_i\rangle$$

Ex.

To reconcile this definition with our informal treatment, observe that

$$a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle = |0\rangle \otimes (a|0\rangle + b|1\rangle) + |1\rangle \otimes (c|0\rangle + d|1\rangle)$$

To make $a|0\rangle + b|1\rangle$ and $c|0\rangle + d|1\rangle$ unit vectors, we need to divide by $\sqrt{|a|^2+|b|^2}$ and $\sqrt{|c|^2+|d|^2}$, respectively:

$$|0\rangle \otimes \left(\frac{\sqrt{|a|^2+|b|^2}}{\sqrt{|a|^2+|b|^2}} \cdot (a|0\rangle + b|1\rangle)\right) + |1\rangle \otimes \left(\frac{\sqrt{|c|^2+|d|^2}}{\sqrt{|c|^2+|d|^2}} \cdot (c|0\rangle + d|1\rangle)\right)$$

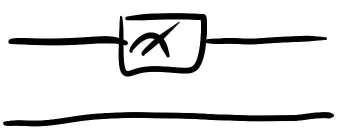
$$= \sqrt{|a|^2+|b|^2}\, |0\rangle \otimes \left(\frac{a|0\rangle + b|1\rangle}{\sqrt{|a|^2+|b|^2}}\right) + \sqrt{|c|^2+|d|^2}\, |1\rangle \otimes \left(\frac{c|0\rangle + d|1\rangle}{\sqrt{|c|^2+|d|^2}}\right)$$

Now we see that the probability of getting $|0\rangle_A$ with a partial Von Neumann measurement is

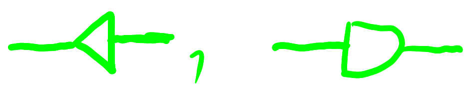
$$\left(\sqrt{|a|^2+|b|^2}\right)^2 = |a|^2 + |b|^2$$

Circuit notation

We use  to denote measurement in the computational basis. A partial measurement of the first qubit is then drawn



Other notations are



Ex.

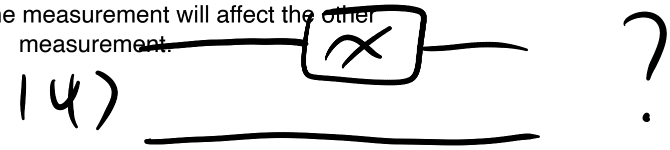
Back to Alice and Bob, we had initial state

Text

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$$

What happens if we measure Alice's qubit, i.e.

Even if you measure the two quibits far from each other,
the result from one measurement will affect the other
measurement.

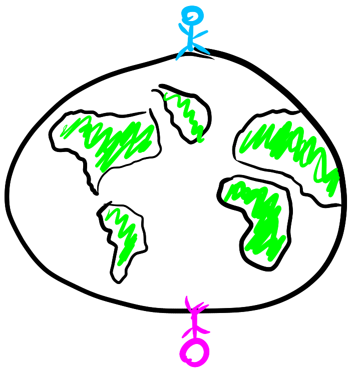


asure in, it will affect the probability that Bob obtains when measuring in the computational basis

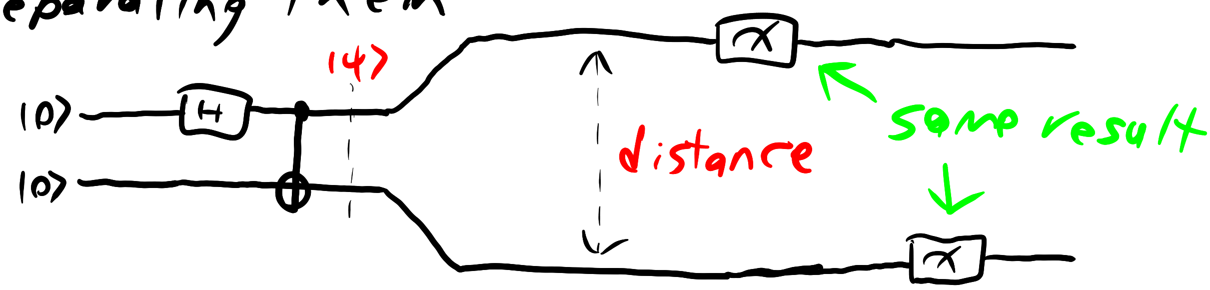
Well, each with probability $\frac{1}{2}$ we get:

- result 0 and state $|0\rangle_A|0\rangle_B$
- result 1 and state $|1\rangle_A|1\rangle_B$

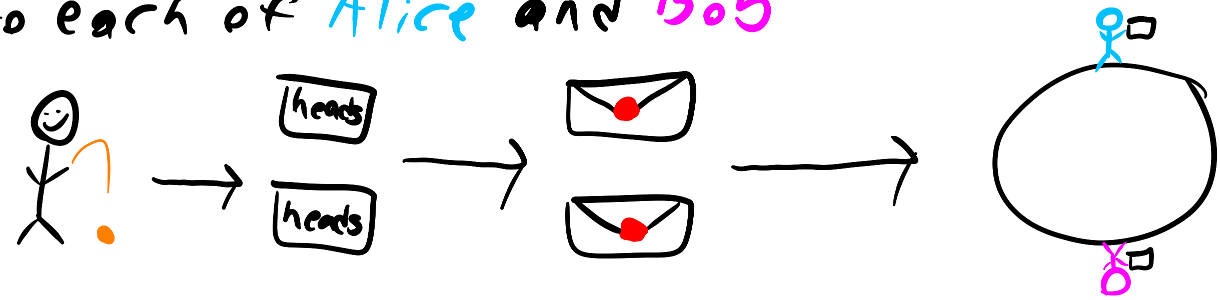
So measuring Alice's qubit collapses Bob's state even if they're physically distant



If you think it's an artificial consequence of our model, it's been experimentally demonstrated with thousands of KM's of distance. The key is to create the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$, called an EPR pair or Bell state by interacting two qubits then separating them



If there is no communication between Alice anb Bob, regardless of who does the measurement first, each get 0 or 1 with equal probability, though their results are always the same. This is perhaps not surprising: if I secretly flip a coin, then write the result on two pieces of paper, put them in envelopes and mail one to each of Alice and Bob



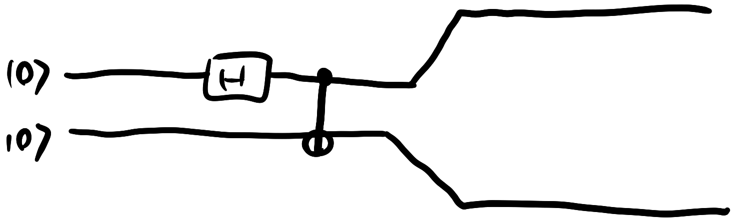
they would each see the same result, though it appears random to them. The coin here is called a hidden variable and while it explains this example, in 1964, John Bell showed that quantum mechanics can't be explained by any hidden variable model. We'll get to this soon...

But first the EPR paradox

(Einstein, Podolsky, Rosen)

(EPR paradox)

Suppose Alice and Bob prepare an EPR state and then take each qubit elsewhere:



Recall: $-\boxed{H}- = H = \frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$, CNOT gate $=$ CNOT $= \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\left(\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}\right)$$

$$=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

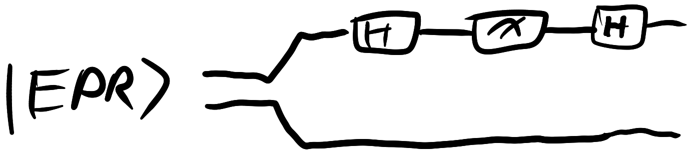
$$=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

$$=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

Note the difference between $H\otimes I$ and $I\otimes H$:

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1&0&0\\1&-1&0&0\\0&0&1&1\\0&0&1&-1\end{bmatrix}$$

Suppose Alice instead measures in the $\{|+\rangle, |-\rangle\}$ basis. This is the same thing as first applying H then measuring $\{|0\rangle, |1\rangle\}$ and applying H again.



What are the state & measurement statistics?

State before measurement is:

$$\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}}_{H \otimes I} \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{|EPR\rangle} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$$

Now Alice's measurement has the result:

- 0 and final state $\frac{\sqrt{2}}{2}\left(|00\rangle + |01\rangle\right) = |0\rangle|+\rangle$ (pr. $\frac{1}{2}$)
- 1 and final state $\frac{\sqrt{2}}{2}\left(|10\rangle - |11\rangle\right) = |1\rangle|-\rangle$ (pr. $\frac{1}{2}$)

Ask how the probability was calculated for this?

So Alice's choice of measurement basis changes Bob's state! $\{|0\rangle, |1\rangle\} \longrightarrow |0\rangle$ or $|1\rangle$

$\{|+\rangle, |-\rangle\} \longrightarrow |+\rangle$ or $|-\rangle$

Whether or not this is meaningful is up for debate, as no matter which basis Bob measures in, he'll get either result with ½ (global) probability

meas $\{|0\rangle, |1\rangle\}$

- $\xrightarrow{0.5} |0\rangle$
  - $\{|0\rangle, |1\rangle\} \xrightarrow{1} |0\rangle$
  - $\{|+\rangle, |-\rangle\} \xrightarrow{0.5} |+\rangle$, $\xrightarrow{0.5} |-\rangle$
- $\xrightarrow{0.5} |1\rangle$
  - $\{|+\rangle, |-\rangle\} \xrightarrow{0.5} |+\rangle$, $\xrightarrow{0.5} |-\rangle$
  - $\{|0\rangle, |1\rangle\} \xrightarrow{1} |1\rangle$

$Pr(|0\rangle) = 0.5$
$Pr(|1\rangle) = 0.5$
$Pr(|+\rangle) = 0.5$
$Pr(|-\rangle) = 0.5$

$\{|+\rangle, |-\rangle\}$

- $\xrightarrow{0.5} |+\rangle$
  - $\{|+\rangle, |-\rangle\} \xrightarrow{1} |+\rangle$
  - $\{|0\rangle, |1\rangle\} \xrightarrow{0.5} |0\rangle$, $\xrightarrow{0.5} |1\rangle$
- $\xrightarrow{0.5} |-\rangle$
  - $\{|0\rangle, |1\rangle\} \xrightarrow{0.5} |0\rangle$, $\xrightarrow{0.5} |1\rangle$
  - $\{|+\rangle, |-\rangle\} \xrightarrow{1} |-\rangle$

$Pr(|0\rangle) = 0.5$
$Pr(|1\rangle) = 0.5$
$Pr(|+\rangle) = 0.5$
$Pr(|-\rangle) = 0.5$

When we study mixed states we'll see that these states actually have the same representation.