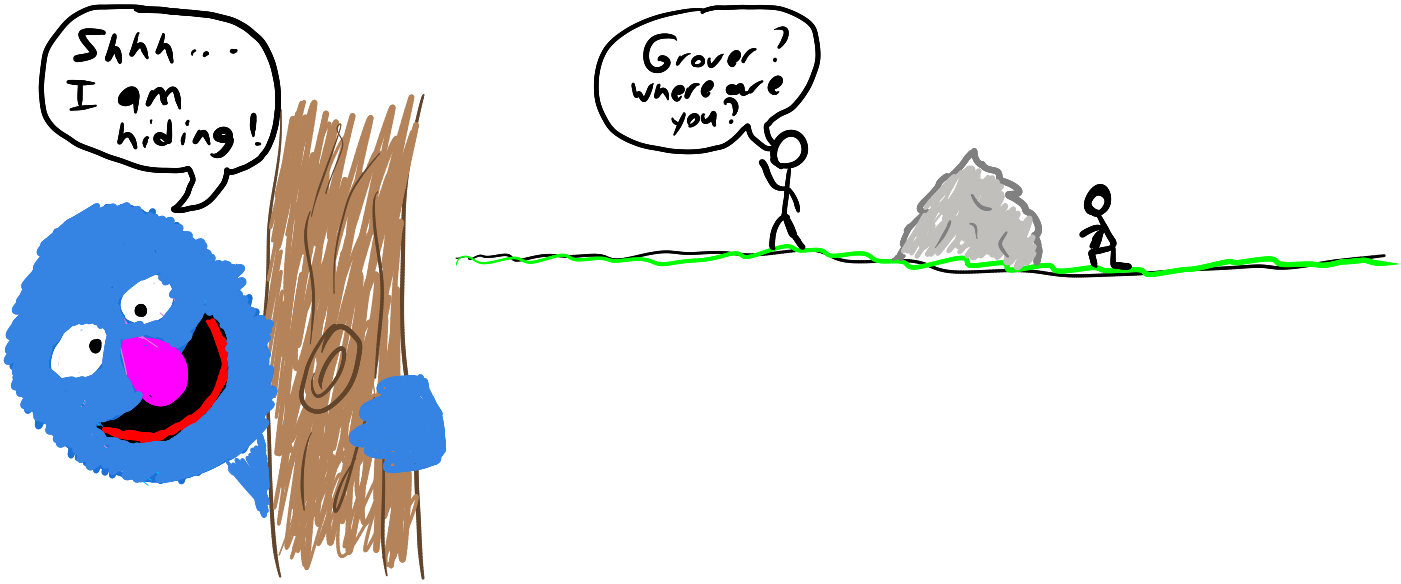
CMPT 476 Lecture 25

Grover's Search



Finally we come to the last of the classic quantum algorithms - Grover's 1996 search algorithm. This algorithm is intrinsically different from Fourier-based algorithms, and encompasses the other main approach to quantum speed-ups: Amplitude amplification. In contrast to Fourier-based algorithms, the quantum speed-up due to Grover's algorithm is only polynomial, and so it won't itself move intractable problems to the tractable pile, but the algorithmic components are used in many other algorithms, and theoretically a polynomial speed-up may still be useful in some practical contexts.

The polynomial speedups brought by Grover search are counteracted by the doubt of how practical it would be to implement it.

HHL uses Grover's search? Ask professor about this.

(Black-box searching)

The problem that Grover's algorithm solves is the black-box or unstructured search problem.

Unstructured search problem

input: a function $f : \{0,1\}^n \longrightarrow \{0,1\}$

goal: find $x \in \{0,1\}^n$ such that $f(x) = 1$

We rely on continued applications of f to find the solution.

We call this an unstructured search problem because it amounts to brute force searching — trying all possible values $x \in \{0,1\}^n$ until a solution to $f(x) = 1$ is found.

Ex.

The SAT problem can be phrased as an unstructured search. Given a propositional formula φ, let $[\![\varphi]\!] : \{0,1\}^n \longrightarrow \{0,1\}$ be the function that evaluates φ on some assignment to its set of n variables. The SAT problem reduces to finding some x such that $[\![\varphi]\!]x = 1$.

Many problems can be phrased as or solved by unstructured search:

- Collision finding
- Hash function inversion
- NP-complete decision problems
- Combinatorial optimization problems
- Unordered databases — Finding one element y in an unordered array

  f(x) = y < n and coll[x] = y,
- etc. — Finding a vertex cover of size k.

  You pass in the vertex and see if it is of size k.

(Classical complexity of unstructured search)

Informally, if there are many solutions, we can find one with decently high probability using just a few queries on a classical compater. So instead, imagine the worst-case scenario: f has exactly one solution.

Worst-case query complexity

Let $f:\{0,1\}^n \to \{0,1\}$ have exactly one solution $f(x)=1$. Then $O(2^n)$ queries are needed classically to find the solution with at least $\frac{1}{2}$ probability.

You must make 2^{n - 1} queries to have a 1/2 probability of obtaining the correct answer.

Intuitively, each query has a $\frac{1}{2^n}$ probability of being the unique solution, assuming f is arbitrary, so we need to check at least $\frac{2^n}{2} = 2^{n-1}$ of the possible inputs to find the solution with $1/2$ probability.

Quantum query complexity for unstructured search is O(sqrt(2^n))

(Grover's quantum algorithm)

At first glance, unstructured search seems like a prime candidate for quantum computation:

1. Prepare $\frac{1}{\sqrt{2^n}}\sum_x |x\rangle|0\rangle$
2. Compute $\frac{1}{\sqrt{2^n}}\sum_x |x\rangle|f(x)\rangle$
3. ???

You could post select, measuring the second register until you get a state with the answer you want.

4. Profit!

If we tried to find $f(x)=1$ by measuring $|f(x)\rangle$ in step 3. we would find such an x with probability just $\left|\frac{1}{\sqrt{2^n}}\right|^2 = \frac{1}{2^n}$ since

$$\frac{1}{\sqrt{2^n}} \sum_x |x\rangle|f(x)\rangle = \frac{1}{\sqrt{2^n}} \sum_{x|f(x)=0} |x\rangle|0\rangle + \frac{1}{\sqrt{2^n}} \sum_{x|f(x)=1} |x\rangle|1\rangle$$

$$= \frac{\sqrt{2^n-1}}{\sqrt{2^n}} |\psi_{f(x)=0}\rangle|0\rangle + \frac{1}{\sqrt{2^n}}|x\rangle|1\rangle$$

What we instead need to do is **amplify** the amplitude of the **correct state** $|x\rangle|1\rangle$.

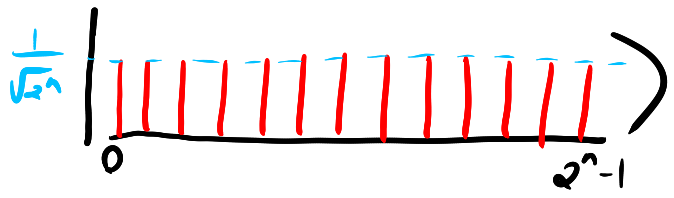
Grover showed that we can do this by switching to the **phase oracle** and **inverting about the mean**.

(Inversion about the mean)

Ask professor to clarify about this.

Suppose we prepare the equal weight superposition of bit strings $\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$.

We can visualize this state as 2^n equal, positive real numbers



What happens if we apply the phase oracle

$$U_f : |x\rangle \longmapsto (-1)^{f(x)}|x\rangle$$

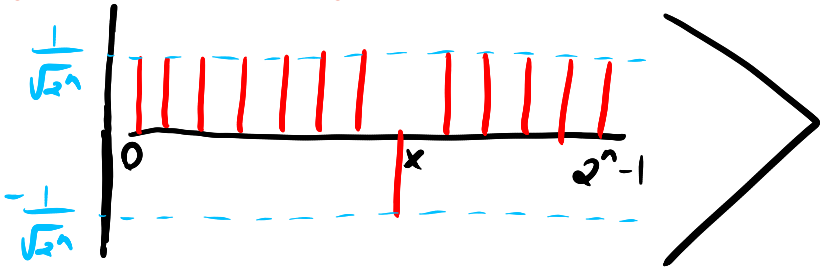
to this state, where $f(x)=1$ for exactly one x?

The state becomes

$$\frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{x|f(x)=0} |x\rangle - \frac{1}{\sqrt{2^n}} \sum_{x|f(x)=1} |x\rangle$$

Which we can visualize as

By swapping the phase of x, we are pulling the mean of the amplitudes down.

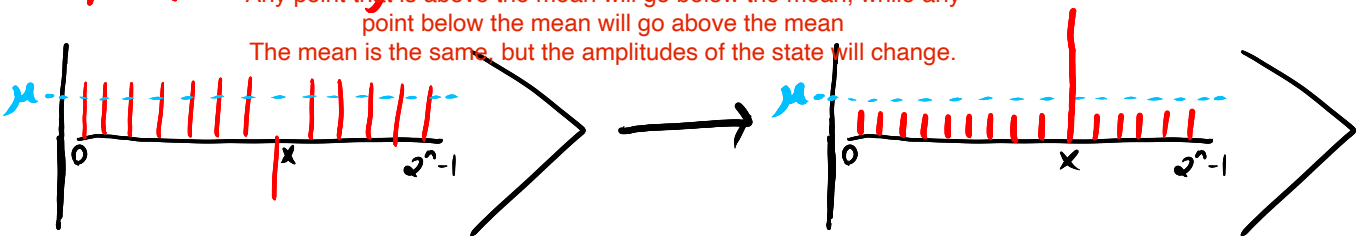


Now, what is the average or mean μ of the amplitudes?

$$\mu = \frac{1}{\sqrt{2^n}} \left(\frac{2^n - 1 - 1}{2^n} \right) = \frac{1}{\sqrt{2^n}} \left(1 - \frac{1}{2^{n-1}} \right) \approx \frac{1}{\sqrt{2^n}}$$

The term inverting about the mean means reflecting about the mean line, i.e.

Any point that is above the mean will go below the mean, while any point below the mean will go above the mean
The mean is the same, but the amplitudes of the state will change.



By reflecting about the mean, we add twice the mean of the amplitude to the desired amplitude.

Now the amplitude of x is much bigger!

Mathematically, we send α to α' such that $\mu - \alpha = -(\mu - \alpha')$, so $\alpha' = 2\mu - \alpha$ and the amplitude of x is hence

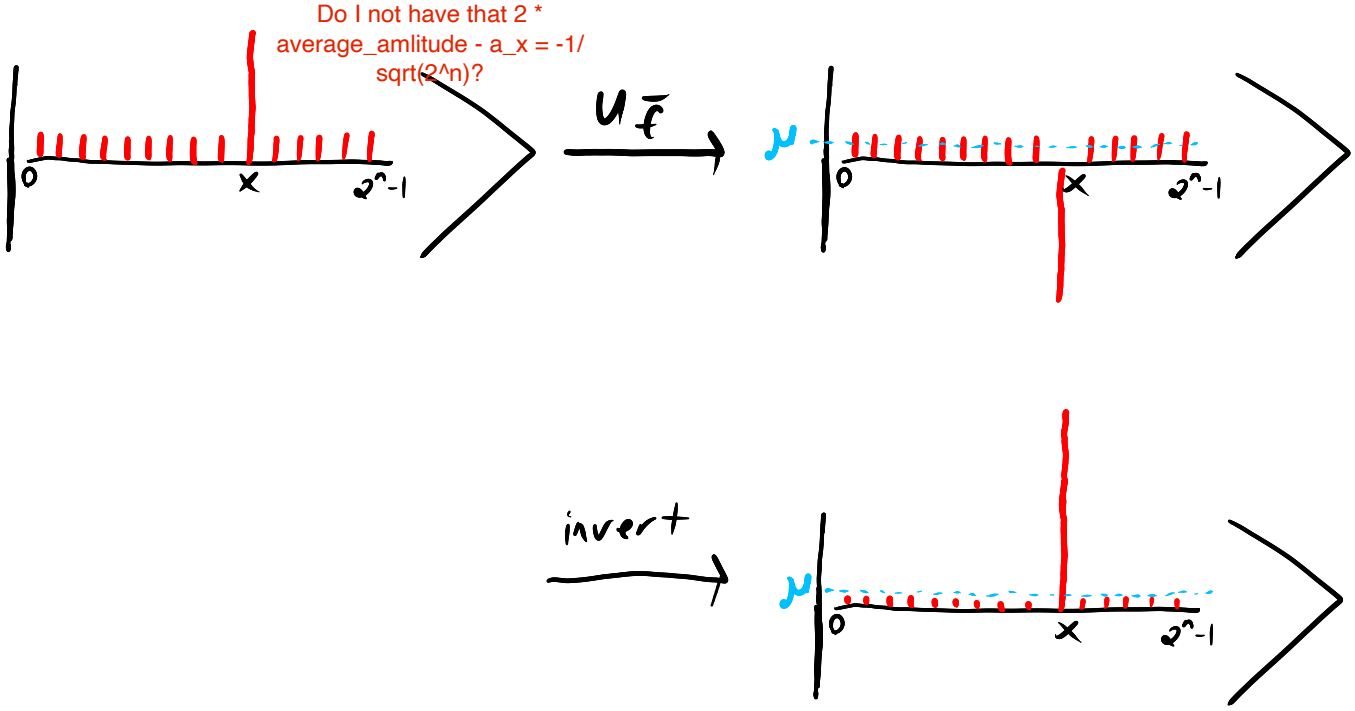
alpha_x is the amplitude for any x.

$$\approx \frac{2}{\sqrt{2^n}} - \left(\frac{-1}{\sqrt{2^n}} \right) = \frac{3}{\sqrt{2^n}}$$

after inversion.

Now what happens if we repeat this process?

Ask TA what happens when a_x = 3/sqrt(2^n). Do I not have that 2 * average_amlitude - a_x = -1/sqrt(2^n)?



This is basically Grover's algorithm. We do still need to figure out how we might invert about the mean however.

Since the mean is a function of 1/sqrt(2^n), we need to add until we amplify the amplitude of the desired state. Ask the professor about needing O(sqrt(2^n)) applications.

(Inversion about the mean)

The inversion about the mean subroutine, like the QFT in Shor's algorithm, is the heart of Grover's search algorithm. Specifically, observe that

This operator reflects over the mean, sends a superposition of x to a superposition of x with a new amplitude.

$$U_{diff} : \sum_x \alpha_x |x\rangle \longmapsto \sum_x (2\mu - \alpha_x)|x\rangle$$

inverts about the mean, where $\mu = \frac{1}{2^n}\sum_x \alpha_x$.

U_{diff} is called the Grover diffusion operator and can be verified to be unitary, notably since it is self-inverse with the mean remaining invariant.

So how can we implement it?

(Implementing Grover's diffusion operator)

To implement U_{diff}, it will be helpful to use our intuition of it as a reflection. First, what state(s) does U_{diff} fix? (i.e. $U_{diff}|\psi\rangle = |\psi\rangle$). Well, if $\alpha_x = \mu$ for all x ($|\psi\rangle$ is a uniform superposition $\frac{1}{\sqrt{2^n}}\sum_x |x\rangle$), then

Why do we have $2^{n} - 1$ states which have a mean zero? If we remove the zero bitstring and we use the fact that xy = 1 for half of the x's if y != 0, sum over all xs [(-1)^{xy}|y>] has a mean of zero.

$$U_{diff}\left(\frac{1}{\sqrt{2^n}}\sum_x |x\rangle\right) = \sum_x \left(2\mu - \frac{1}{\sqrt{2^n}}\right)|x\rangle$$

$$= \sum_x \frac{1}{\sqrt{2^n}}|x\rangle \quad \text{since } \mu = \frac{1}{\sqrt{2^n}}$$

If the average is 0, then U_diff sends |x> to -|x>

$$= \frac{1}{\sqrt{2^n}}\sum_x |x\rangle$$

The states are in the plane of reflection? Ask professor to clarify about this.

So U_{diff} fixes the uniform superposition. Now, what state(s) does U_{diff} reflect (i.e. $U_{diff}|\psi\rangle = -|\psi\rangle$).

We know such a state must be orthogonal to the uniform superposition $|s\rangle = \frac{1}{\sqrt{2^n}}\sum_x |x\rangle$, as it is a -1 eigenvector. What states does $|s\rangle^{\perp}$ contain?

$$\frac{1}{\sqrt{2^n}}\sum_z (-1)^{y\cdot z}|z\rangle \in |s\rangle^{\perp}$$

for any $y \neq 00\cdots0$ because $y\cdot z = 1$ for exactly half the values of z, hence

$$\left(\frac{1}{\sqrt{2^n}}\sum_z (-1)^{y\cdot z}\langle z|\right)\left(\frac{1}{\sqrt{2^n}}\sum_x |x\rangle\right) = \frac{1}{2^n}\sum_z (-1)^{y\cdot z}\langle z|z\rangle$$

$$= 0$$

Now, what does U_{diff} do to those vectors? Well, since $\sum_z (-1)^{y\cdot z} = 0$, their mean is 0, hence

$$U_{diff}|\psi\rangle = -|\psi\rangle$$

Finally, noting that all $\frac{1}{\sqrt{2^n}} \sum_z (-1)^{y \cdot z} |z\rangle$ are linearly independent (and in fact equal to $H^{\otimes n}|y\rangle$) and there are $2^n - 1$ such orthogonal vectors, they must span the entire subspace orthogonal to $|s\rangle$.

So

U_{diff} is a reflection along the line $|\psi\rangle$ which we can write as

We can view $|s\rangle\langle s|$ as a projector onto the hilbert space spanned by $|s\rangle$.
Ask the professor about the comment the other student noticed that it was a projection onto an even weight superposition basis.

$$U_{diff} = 2|s\rangle\langle s| - I$$

Alternatively, we see that U_{diff} has $+1$ eigenspace $\{|s\rangle\}$ and -1 eigenspace $\mathbb{C}^{2^n} - \{|s\rangle\}$, so by the spectral theorem

I because it represents the resolution of the identity,
the subtraction to represent the rest of the subspace without s?

Projector of s

$$U_{diff} = |s\rangle\langle s| - (I - |s\rangle\langle s|)$$
$$= 2|s\rangle\langle s| - I$$

(A concrete circuit)

To devise a circuit for U_{diff}, note that

$$H^{\otimes n}|0\rangle = |s\rangle$$

So,

$$U_{diff} = 2H^{\otimes n}|0\rangle\langle 0|H^{\otimes n} - I$$
$$= H^{\otimes n}(2|0\rangle\langle 0| - I)H^{\otimes n}$$

Now, $2|0\rangle\langle 0| - I$ sends $|0\rangle \longmapsto -|0\rangle$
$|x\rangle \longmapsto |x\rangle \quad \forall x \neq 0$

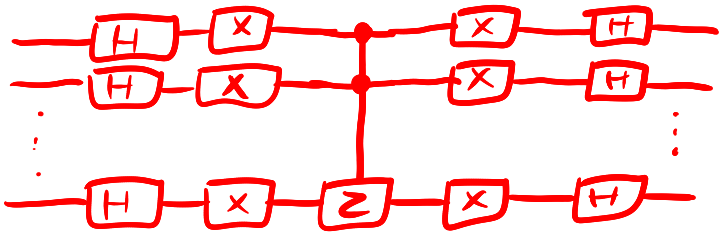
Ask TA how this sends |0> to -|0>.
It seems to send |0> to |0> and |x> to -|x>

This is exactly the n-1 controlled Z gate with 0 & 1 swapped! That is,

$$2|0\rangle\langle 0| - I = X^{\otimes n}(2|1\rangle\langle 1| - I)X^{\otimes n}$$
$$= X^{\otimes n}(C^{n-1}Z)X^{\otimes n}$$

Ask TA to explain this part.

So, we can implement U_{diff} with the circuit

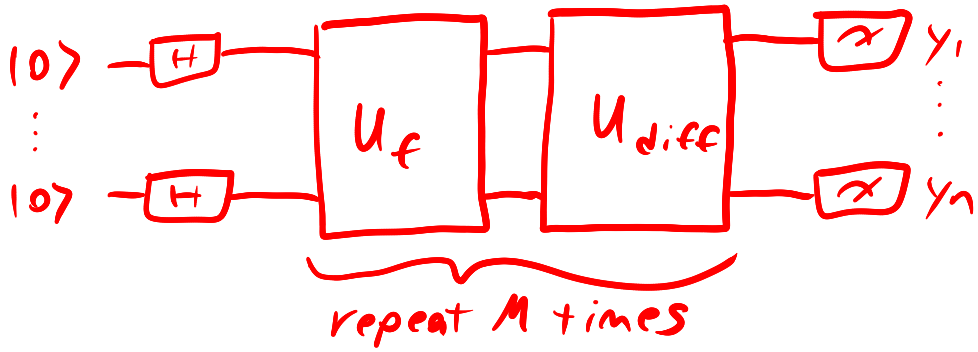


(Grover's search algorithm)

Given a classical function $f: \{0,1\}^n \to \{0,1\}$, Grover's algorithm proceeds as follows:

1. Prepare $|s\rangle = H^{\otimes n}|0\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$
2. For $i = 1$ to M
3. Apply $U_{diff} U_f$
4. Measure to get $|y\rangle$, $y \in \{0,1\}^n$

As a circuit,



We still need to figure out a value of M, but for now let's just say $M \approx \sqrt{2^n}/2$ since each iteration adds $2\mu \approx \frac{2}{\sqrt{2^n}}$ amplitude to the good state. We'll do the full analysis next class for the generalized version — Amplitude Amplification