CMPT 476/981: Introduction to Quantum Algorithms
Assignment 2

Due **February 1st, 2024 at 11:59pm on coursys**
Complete individually and submit in PDF format.

Question 1 [3 points]: Optimal angles for the Zeno effect

In class we saw that we can *drag* a state from the $|0\rangle$ state to the $|1\rangle$ state by performing measurements in rotated bases. Given a basis $\mathcal{B} = \{|A\rangle, |B\rangle\}$ define the basis $\mathcal{B}$ *rotated* by an angle θ to be $\{\cos(\theta)|A\rangle + \sin(\theta)|B\rangle, -\sin(\theta)|A\rangle + \cos(\theta)|B\rangle\}$. Observe that this basis is in fact orthonormal via the identity $\cos^2(\theta) + \sin^2(\theta) = 1$.

1. Show that rotating the basis twice by θ is the same as rotating once by an angle of 2θ

   In the Piazza response, you said that it had to be an arbitrary p.

2. Calculate the angle θ and number of measurements needed to reach the $|1\rangle$ state with success probability p for some positive real number p close to 1.

   Not exactly p, but at least p

   Start in the 0 state, and then snap down to one of the possible basis vectors with a probability dependent on their distance

   Note: Assume that $\sin^2(x) = x^2$ when x is close to 0.

When x is close to 0, what does that mean? how close should it be?

Question 1 [2 points]: State discrimination

Using computational basis measurement, H gates, and *phase gates*

p = cos^2(theta)
sqrt(p) = cos(theta)
theta = cos^-1(sqrt(p))

$$P(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

where θ can be any real number, give a protocol to distinguish with 100% accuracy between the states

$$|\psi\rangle = \frac{1}{\sqrt{2}}(e^{i\pi/4}|0\rangle + |1\rangle), \qquad |\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i3\pi/4}|1\rangle)$$

Question 1 [4 points]: Pauli operators

Recall the definition of the I, X, Z, and Y gates:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

These are known as the *Pauli matrices* or gates.

1. Compute the matrices $X \otimes Z$ and $Z \otimes X$
2. Show that the non-identity Pauli matrices anti-commute: that is, $UV = -VU$ for every pair of X, Y, and Z matrices where $U \neq V$
3. Show that the Pauli matrices I, X, Z, Y are linearly independent
4. Show that the Pauli matrices form a basis for the space of 2×2 complex-valued matrices.

Question 1 [2 points]: Entanglement

Prove that the *controlled-Z* gate

$$CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is entangling. Do so by giving an explicit two-qubit (unentangled) state $|\psi\rangle \otimes |\phi\rangle$ and showing that $CZ(|\psi\rangle \otimes |\phi\rangle)$ is entangled.

Question 1 [2 points]: Partial measurement

Are we supposed to get separate probabilites for this?
Not the probability of obtaining one or the other?

Let

$$|\psi\rangle = \frac{i\sqrt{2}}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}\sqrt{2}}|01\rangle + \frac{\sqrt{2}}{2\sqrt{3}}|10\rangle.$$

Calculate the probabilities of measuring 0 or 1 in the first qubit, and the resulting normalized state vector in either case.

Question 1 [9 points]: Non-local games

In this question, we're going to examine another non-local game involving 3-parties, or 3 qubits. First let $|\psi\rangle = \frac{1}{2}(|000\rangle - |110\rangle - |011\rangle - |101\rangle)$

1. Give a 3-qubit circuit U consisting of X, H, and $CNOT$ gates such that

$$U\left(\frac{1}{\sqrt{2}}|000\rangle - \frac{1}{\sqrt{2}}|111\rangle\right) = |\psi\rangle.$$

2. Show that a partial measurement of any qubit in the $|\psi\rangle$ state leaves an entangled state in the remaining 2 qubits.

Factor out remaining qubit, and show that what is left cannot be written as the tensor product of two states

3. Compute the parity $a \oplus b \oplus c = a + b + c \mod 2$ of the measurement results if
   (a) All qubits are measured in the $\{0, 1\}$ basis.
   (b) Qubits 0 and 1 are measured in the $\{|+\rangle, |-\rangle\}$ basis and qubit 2 in the $\{0, 1\}$ basis.
   (c) Qubits 0 and 2 are measured in the $\{|+\rangle, |-\rangle\}$ basis and qubit 1 in the $\{0, 1\}$ basis.
   (d) Qubits 1 and 2 are measured in the $\{|+\rangle, |-\rangle\}$ basis and qubit 0 in the $\{0, 1\}$ basis.

For measurements in the l+>, l-> basis, you apply the hadamard gate to each qubit and then you
you compose it out of the 0 and 1 state in order to measure the parity.

Note: in the $\{|+\rangle, |-\rangle\}$ basis, we consider the result of measuring "+" to be 0 and the result of measuring "-" to be 1

4. Denote the measurement result of qubit i in the $\{0, 1\}$ basis by a_i, and in the $\{|+\rangle, |-\rangle\}$ basis by b_i. Is it possible that each a_i and b_i has a **pre-determined value** independent of which basis the other qubits are measured in? Give a convincing argument for your answer.

5. Give a quantum strategy (i.e. a strategy where involving a shared pre-entangled state) for a 3-player game where Alice, Bob, and Charlie are each given one bit x, y, and z respectively, and have to return a single bit a, b, c respectively such that $a \oplus b \oplus c = a \vee b \vee c$
Hint: use the state $|\psi\rangle$ as the initial shared state