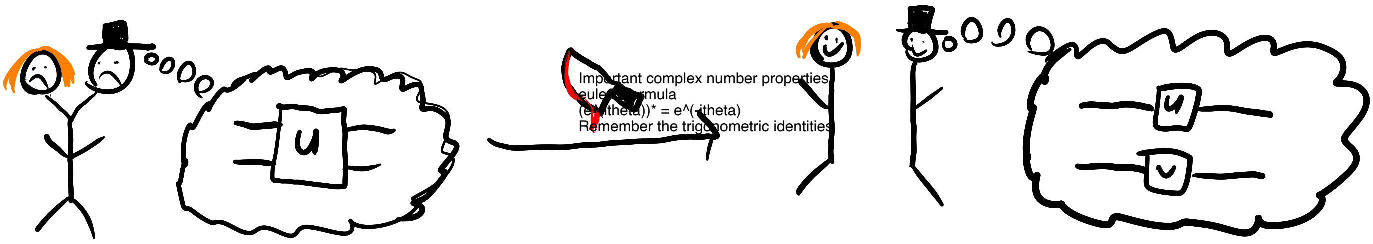
CMPT 476 Lecture 6

A few qubits more

Review complex numbers



Important complex number properties
euler formula
(e^(itheta))* = e^(-itheta)
Remember the trigonometric identities

What do we know about QM so far?

- States $|\psi\rangle = \sum_i a_i |i\rangle = \begin{bmatrix} a_0 \\ \vdots \\ a_{d-1} \end{bmatrix} \in \mathbb{C}^d$ s.t. $\sum |a_i|^2 = 1$

- Measurement sends $|\psi\rangle$ to $|i\rangle$ with prob. $|a_i|^2$

- Gates send $|\psi\rangle \longrightarrow U|\psi\rangle$ where $U \in \mathcal{L}(\mathbb{C}^d)$ is unitary ($U^\dagger = U^{-1} \iff U^\dagger U = UU^\dagger = I$) and has matrix $\sum_{ij} u_{ij} |i\rangle\langle j|$, or

$$\begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0d-1} \\ u_{10} & u_{11} & & \\ \vdots & & \ddots & \\ u_{d-10} & & & u_{d-1d-1} \end{bmatrix}$$

An example of a unitary on $\mathbb{C}^4$ is the CNOT gate from lecture 2

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Today we will learn how to build composite systems like $\mathbb{C}^4$ from two qubits (i.e. $\mathbb{C}^2$), how to operate on them, and some implications.

How does a 4-dimensional system arise? Could be

- An electron in one of 4 orbitals (is this even possible?)
- A photon with a polarization ($\updownarrow$ or $\leftrightarrow$) and path ($\uparrow$ or $\rightarrow$)
- Two physically separate qubits

Each of the above has 4 physical states, but drastically different dynamics. The last one in particular can only evolve according to local operations on either qubit. We shall see that with entanglement, local operations have surprising power.

First, suppose Alice's qubit has state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and Bob's has state $|\psi\rangle = \gamma|0\rangle + \delta|1\rangle$. If both Alice and Bob measure their qubits, what is the distribution of outcomes?

Assuming the quibits are not entangled.

- $|0\rangle$ and $|0\rangle \longrightarrow |\alpha|^2 \cdot |\gamma|^2$
- $|0\rangle$ and $|1\rangle \longrightarrow |\alpha|^2 \cdot |\delta|^2$
- $|1\rangle$ and $|0\rangle \longrightarrow |\beta|^2 \cdot |\gamma|^2$
- $|1\rangle$ and $|1\rangle \longrightarrow |\beta|^2 \cdot |\delta|^2$

We saw in lecture 2 that this is the joint prob. distribution

$$\underbrace{\begin{bmatrix} |\alpha|^2 \\ |\beta|^2 \end{bmatrix} \otimes \begin{bmatrix} |\gamma|^2 \\ |\delta|^2 \end{bmatrix}}_{\text{tensor product}} = \begin{bmatrix} |\alpha|^2|\gamma|^2 \\ |\alpha|^2|\delta|^2 \\ |\beta|^2|\gamma|^2 \\ |\beta|^2|\delta|^2 \end{bmatrix}$$

Because the quibits are isolated, there is no interference.
If the quibits are entangled, can we have interference?

Thinking of quantum amplitudes as "probabilities" we can wonder if

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \otimes \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{bmatrix}$$

is a sensible representation of the 4-dimensional state. Indeed,

We know that \gamma^2+\delta^2 is one, so the entire expression evaluates to one.

$$|\alpha\beta|^2 = |\alpha|^2|\beta|^2 \Rightarrow |\alpha\gamma|^2 + |\alpha\delta|^2 + |\beta\gamma|^2 + |\beta\delta|^2$$
$$= |\alpha|^2(|\gamma|^2 + |\delta|^2) + |\beta|^2(|\gamma|^2 + |\delta|^2)$$
$$= 1$$

(State of a composite system, pt 1)

Given two systems in states $|\psi\rangle$ and $|\varphi\rangle$, their joint state is $|\psi\rangle \otimes |\varphi\rangle$

Assume Alice starts in the state |+> and bob starts in |0>

Going back to Alice and Bob, what if they brought their qubits together (i.e. $|\psi\rangle \otimes |\varphi\rangle$) and then applied a CNOT to the joint state?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{bmatrix} = \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\delta \\ \beta\gamma \end{bmatrix}$$

Suppose $\alpha = \frac{1}{\sqrt{2}}$, $\beta = \frac{1}{\sqrt{2}}$, $\gamma = 1$, $\delta = 0$. Then

Compose the tensor product out of a linear combination of the |0> and |1> states and then apply the CNOT gate.

$$\begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\delta \\ \beta\gamma \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \neq \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$$

(Why?)

Recall that in a probabilistic setting we said that Alice & Bob's states are correlated if the joint state $|\psi\rangle \neq \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix}$. In Quantum mechanics we say their states are entangled. Since the above postulate says nothing about composite systems with entangled states, we need to generalize it a bit using more linear algebra.

(Tensor products)

Let V, W be two Vector spaces (e.g. Hilbert spaces)
The **tensor product** $V \otimes W$ is a vector space such that:

1. $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$ Important.
2. $v \otimes w \in V \otimes W \quad \forall v \in V, w \in W$
3. $(v+u) \otimes w = v \otimes w + u \otimes w$
4. $v \otimes (u+w) = v \otimes u + v \otimes w$
5. $(cv) \otimes w = v \otimes (cw) = c(v \otimes w)$, c scalar

$\otimes$ is bilinear

Linear in either argument

The definition above describes tensor products abstractly (and is in fact missing a crucial abstract property). Since we work in finite dimensions it's easier to simply give a basis for $V \otimes W$

(Basis of $V \otimes W$)

Let V & W be V.S. with orthonormal bases $\{|e_i\rangle\}$, $\{|f_j\rangle\}$ respectively. Then $V \otimes W$ has an orthonormal basis

$$\{ |e_i\rangle \otimes |f_j\rangle = |e_i\rangle|f_j\rangle \}$$

Ex.

$\mathbb{C}^2 \otimes \mathbb{C}^2$ has basis $\{|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle\}$

If $|\psi\rangle = i|1\rangle$ and $|\varphi\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$, then

$$|\psi\rangle \otimes |\varphi\rangle = i|1\rangle \otimes \left(\frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\right)$$
$$= \frac{i}{\sqrt{5}}|1\rangle|0\rangle + \frac{2i}{\sqrt{5}}|1\rangle|1\rangle$$

|0>|0> = tensor-product(|0>, |0>)

(Multi-qubit quantum computing & notation)

$\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^4$ look very similar...

	$\mathbb{C}^2 \otimes \mathbb{C}^2$	$\mathbb{C}^4$
dim	$2 \times 2 = 4$	4
basis	$\{\lvert 0\rangle\lvert 0\rangle, \lvert 0\rangle\lvert 1\rangle, \lvert 1\rangle\lvert 0\rangle, \lvert 1\rangle\lvert 1\rangle\}$	$\{\lvert 0\rangle, \lvert 1\rangle, \lvert 2\rangle, \lvert 3\rangle\}$ $\Downarrow$ Binary expansion $\{\lvert 00\rangle, \lvert 01\rangle, \lvert 10\rangle, \lvert 11\rangle\}$

We say $\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$ via the isomorphism $\lvert i\rangle\lvert j\rangle \longleftrightarrow \lvert ij\rangle$. In practice we view $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$ and write the following equivalenty:

$$e_i = \lvert i\rangle = \lvert \underbrace{i_1 i_0}_{\text{binary exp.}}\rangle = \lvert i_1\rangle\lvert i_0\rangle = \lvert i_1\rangle \otimes \lvert i_0\rangle$$

More generally, for n qubits,

$$\overbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}^{n \text{ times}} \simeq \mathbb{C}^{2^n}$$

$$\lvert i\rangle = \lvert i_{n-1} \cdots i_0\rangle = \lvert i_{n-1}\rangle \cdots \lvert i_0\rangle = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{ith entry}$$

And for n qudits of dimension d,

$$\mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d \simeq \mathbb{C}^{d^n}$$

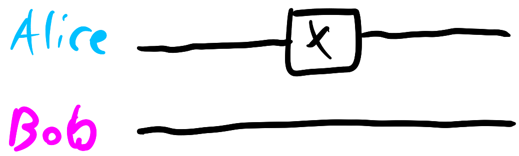
(More notation)

To make it even more confusing, depending on the context the following are the same

$$\lvert\psi\rangle \otimes \lvert\varphi\rangle, \quad \lvert\psi\rangle\lvert\varphi\rangle, \quad \lvert\psi, \varphi\rangle$$

(local operations)

Back to our example, suppose Alice wants to apply a NOT/X gate to her qubit. Diagrammatically,



We haven't used circuit diagrams much yet because we only had 1 qubit. They'll factor in more now. Note that

$$-\boxed{A}-\boxed{B}- = BA$$

since time flows left to right

If their states are not entangled we might view this circuit as sending

$$\underbrace{|\psi\rangle}_{\text{Alice}} \otimes \underbrace{|\varphi\rangle}_{\text{Bob}} \longrightarrow (X|\psi\rangle) \otimes |\varphi\rangle$$

If they are entangled, we need a way of describing "X on Alice's qubit" as a linear operator over $\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$.

State of n qubits = superposition of n-bit strings

superposition of n-bit strings != n bits in superposition

ition, we have two amplitudes for each bit (one to measure the one and zero state), so

for n bits we have 2^n amplitudes

For a superposition of n-bit strings,

(Tensor product of operators)

Let $A: V \to V$, $B: W \to W$ be linear operators on V & W. Then $A \otimes B : V \otimes W \longrightarrow V \otimes W$ is the linear operator defined by

$$(A \otimes B)|v\rangle \otimes |w\rangle = (A|v\rangle) \otimes (B|w\rangle)$$

Ask Matt to clarify on why the transformation must be uniquely defined

As a matrix, this corresponds to

Barf! ^

$$\begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \otimes \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} = \begin{bmatrix} A_{00}\begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} & A_{01}\begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} \\ A_{10}\begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} & A_{11}\begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} \end{bmatrix}$$

of two states, it represented the joint probability of two states.

does not have the significance of a joint probability.

Ex.

Recall that $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Compute:

1. $X \otimes I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$
2. $I \otimes X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
3. $Z \otimes Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
4. $(X \otimes I)(|0\rangle \otimes |1\rangle) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |1\rangle \otimes |1\rangle$

(Properties of tensor products)

1. $s(A \otimes B) = sA \otimes B = A \otimes sB$
2. $(A+B) \otimes C = A \otimes C + B \otimes C$ (left distributive)
3. $A \otimes (B+C) = A \otimes B + A \otimes C$ (right distributive)
4. $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ (associative)
5. $(A \otimes B)^{+} = A^{+} \otimes B^{+}$ (+-distributivity)
6. $(A \otimes B)(C \otimes D) = AC \otimes BD$ (bi-functor)

dagger, not plus.

Ex.

$(|1\rangle \otimes |0\rangle)^{+}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$

I thought the tensor products of 1 and 0 would cancel out.

$= (\langle 1| \otimes \langle 0|)(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$ (+-dist)

$= (\langle 1| \otimes \langle 0|)(|0\rangle \otimes |1\rangle) - (\langle 1| \otimes \langle 0|)(|1\rangle \otimes |0\rangle)$ (dist.)

$= (\langle 1|0\rangle \otimes \langle 0|1\rangle) - (\langle 1|1\rangle \otimes \langle 0|0\rangle)$ (bi-func)

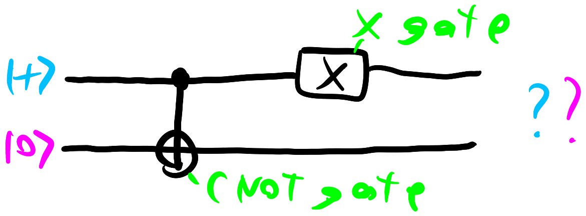
$= -1$ (surprised?)

(Back to our friends)

We now have all the ingredients to compute the final state of our Alice and Bob example. To recap:

1. Alice starts in state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$, Bob in state $|0\rangle$
2. Alice and Bob jointly apply CNOT
3. Alice separates her qubit and applies X

We can write this as a Circuit



Recalling that $CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, the final state is

$$\left(\overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{\boxed{X}}\right) \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}^{\oplus} \left(\overbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{|+\rangle = \ \ |0\rangle =}\right)$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

Alt. calculation

CNOT: $|00\rangle \longmapsto |00\rangle$
$|01\rangle \longmapsto |01\rangle$
$|10\rangle \longmapsto |11\rangle$
$|11\rangle \longmapsto |10\rangle$

X: $|0\rangle \longmapsto |1\rangle$
$|1\rangle \longmapsto |0\rangle$

I thought a linear combination of operators would act on the corresponding vectors. Why is it that the X gate is applied on both vectors?

So

$$(X \otimes I)\,CNOT\left(\tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|10\rangle\right)$$
$$= (X \otimes I)\left(\tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle\right)$$
$$= \tfrac{1}{\sqrt{2}}|10\rangle + \tfrac{1}{\sqrt{2}}|01\rangle$$

Dirac is best :)

(Is quantum computing stochastic?)

Measurement statistics in the Alice and Bob example are identical whether we measure first and proceed stochastically, taking

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \xrightarrow{\text{measure}} \frac{1}{2}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

We might wonder if all QC can be modeled by stochastic matrices. The answer lies in the hadamard gate

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

which has the measurement statistics of a coin flip:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow{\text{measure}} \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

However, unlike a classical coin flip

$$\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

flipping the coin again gets us back to the original state:

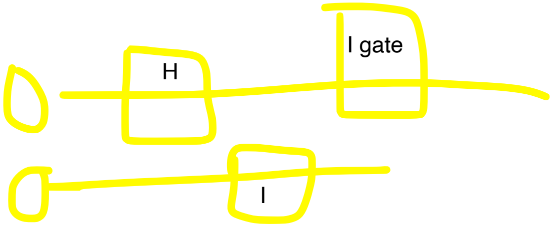
$$H\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) = |0\rangle \xrightarrow{\text{measure}} \begin{bmatrix}1\\0\end{bmatrix}$$

Can we have a negative amplitude?
Whenever we saw amplitudes, they always had a non-negative magnitude by taking their absolute value.
What does negative mean here?
What does positive amplitude mean?

In contrast,

$$\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

The reason is negative amplitudes giving rise to interference. We will examine this and other quantum effects in the coming weeks.



When you do (H tensor-prod I), you are applying the hadamard gate to the first quibit.

(I tensor-prod X) CNOT(H tensor-prod I)

Easier way:

apply each computation on the basis vectors

EX:

(I tensor-pod X)CNOT(H tensor-prod I)I00>

If we measure a four dimensional system in the computational basis where the quibits are separated,

There is no way to assure that measuring a four dimensional system can be done at the same time by multiple people.