CMPT 476 Lecture 15

Reversible computation

Now that we have a well-defined model of what a quantum computation should be, we can ask whether we can do some things faster on a quantum computer. But first we need to narrow down the relationship of quantum and classical computing.

There are different models of computation, but they end up being equivalent.

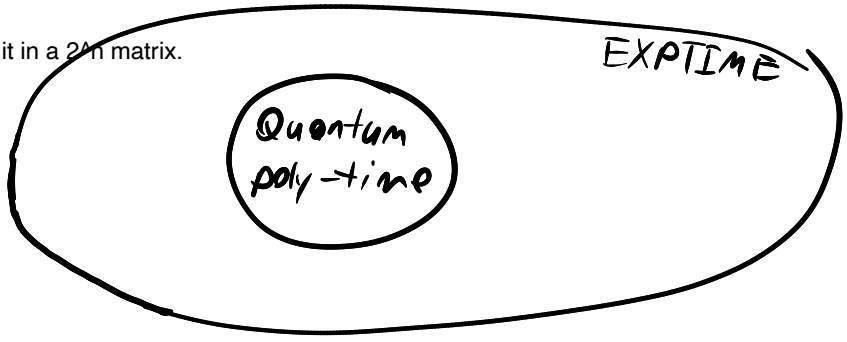
First question:

Can a classical (probabilistic) computer do everything a quantum one can?

with all states initialized to the zero state and some approximately universal gate set is applied,

Yes! Given a quantum circuit with n qubits and m gates we can simulate it classically in time $O(2^n m)$ by explicit matrix-vector multiplication and probabilistic choices for measurements (or density matrices).

ve n qubits, we can represent it in a 2^n matrix.



Can a classical computer do everything a quantum computer can?
Yes. We can represent gates and states as matrices and vectors respectively

We would be using a pseudo random generator to perform the measurement.
We can also do the density matrix, which is better since we are given the output distribution
Time commplexity: O(2^n * m), since for n qubits we can simulate it as a 2^n vector.

Second question: Can a quantum computer do everything a classical computer can?

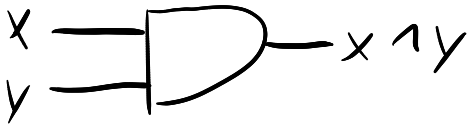
essor about your answer that every probabilistic computation can simulate a deterministic computation.

Yes! But it's a bit complicated...

Remember: Classical computation can be simulated by {NOT, XOR, AND, FANOUT}
FANOUT can be simulated by CNOT
XOR can be applied by CNOT
NOT can be applied by X gate.

(Invertibility)

Recall the classical AND gate



The AND gate has no inverse since

$00, 01, 10 \longmapsto 0$

Since quantum gates U are invertible ($U^{-1} = U^\dagger$) and hence quantum circuits (without measurement) are invertible, we can not implement AND directly!

Second question, redux

can we do classical computation invertibly?

(Reversible computation)

Researchers looked at this question independently of quantum computing, motivated by reducing energy use of computers. In the 60's, Landauer showed that erasing one bit of information dissipates

T is temperature system is in.

What is K_B and T?

$K_B T \ln 2$ Joules (Landauer's Principle)

Boltman constant

of energy. This led to a model of reversible computing where information, and hence energy, is conserved in a computation.

Measurements are irreversible

How could we compute conservatively?

The obvious answer is whenever you compute a gate, save the inputs:

We inject energy by adding a bit, and remove energy by taking away a bit in the inverse

map the ancilla to 0, we have to dissapate some energy to measure it.

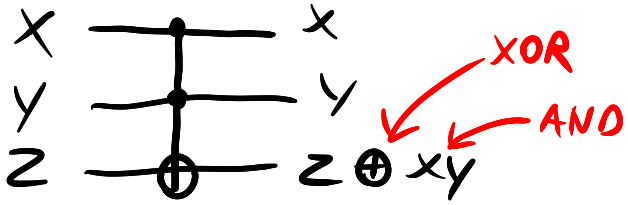
This is invertible in the sense that we can retrieve the values of x & y from the output, but it isn't conservative — we add energy to the system by adding a bit! Conversely, if we "invert" the computation, we erase the bit of information $x \wedge y$.

For energy to be truly conserved (and computation truly invertible/quantum) we need to account for the initial state of the bit which will store the result $x \wedge y$.

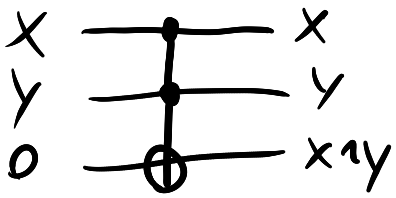
(The Toffoli gate)

The Toffoli gate is a 3-bit classical gate proposed by Toffoli in 1980 as a reversible AND.

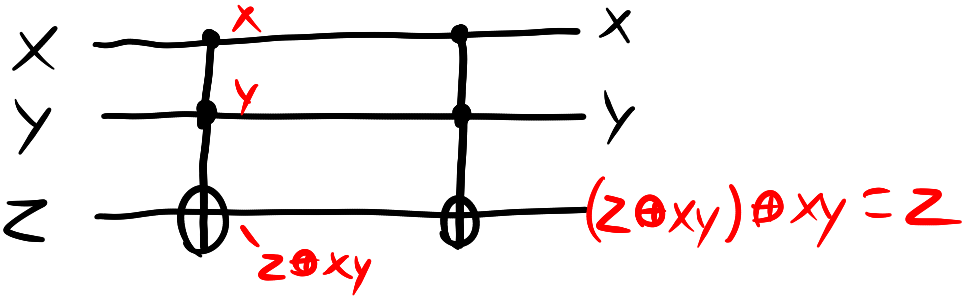
The toffoli gate allows you to implement a reversible and.



If $z=0$, then $z \oplus xy = xy = x \wedge y$, so we can reversibly AND two bits into an ancilla which we previously initialized in the 0 state



As a function of 3 bits, the Toffoli gate is also invertible (and is in fact self-inverse)



As a matrix,

$$\text{Toffoli} = \begin{array}{c} \begin{matrix} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{matrix}$$

Note that the Toffoli gate is a controlled-CNOT:

$$\text{Toffoli}: |0\rangle|y\rangle|z\rangle \longmapsto |0\rangle|y\rangle|z\rangle$$
$$|1\rangle|y\rangle|z\rangle \longmapsto |1\rangle|y\rangle|z \oplus y\rangle = |1\rangle(\text{CNOT}|y\rangle|z\rangle)$$

(Universality of Toffoli (and reversible computing))

We can then find an invertible function that represents f^-1

Any function $f: \{0,1\}^n \longrightarrow \{0,1\}^m$ can be implemented as a circuit over {Toffoli} with sufficiently many 0/1-initialized ancillas and some left over garbage.

Any function from n to m bits is invertible

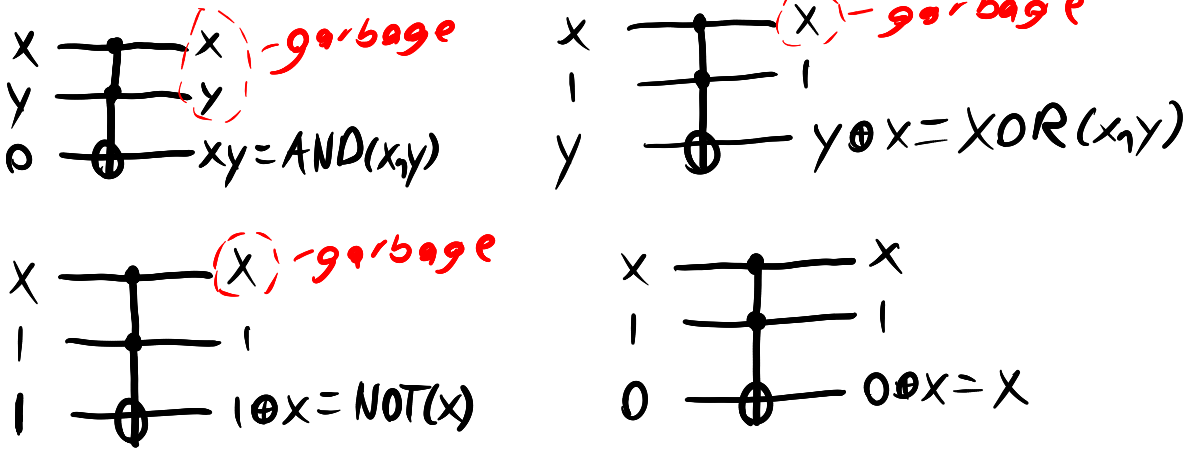
0 suffices if you add NOT which is reversible... NOT(0)=1

The function is its self inverse

Proof Sketch

To prove this, we can give toffoli circuits which simulate the classical gates.

Recall that {AND, XOR, NOT, FANOUT} are univ. We can simulate each gate with ancillas & garbage:



Hence, write f as a circuit over {AND, XOR, NOT, FANOUT} and replace each gate with the circuit above.

How much space does this use?

Since every circuit above uses 1-2 ancillas and leaves some garbage which can't be re-used, we simulate a circuit having T gates and S space with T gates but $Q(T+S)$ space!

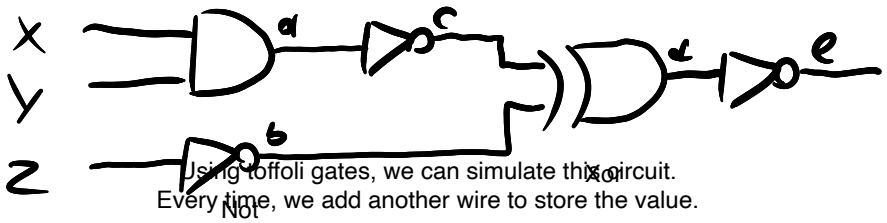
Moreover, all of this garbage can't be used again until the computer is "reset" somehow.

(In a quantum context, the garbage is also entangled with the state, so leaving it laying around can have unintended consequences)

Can you isolate the garbage?

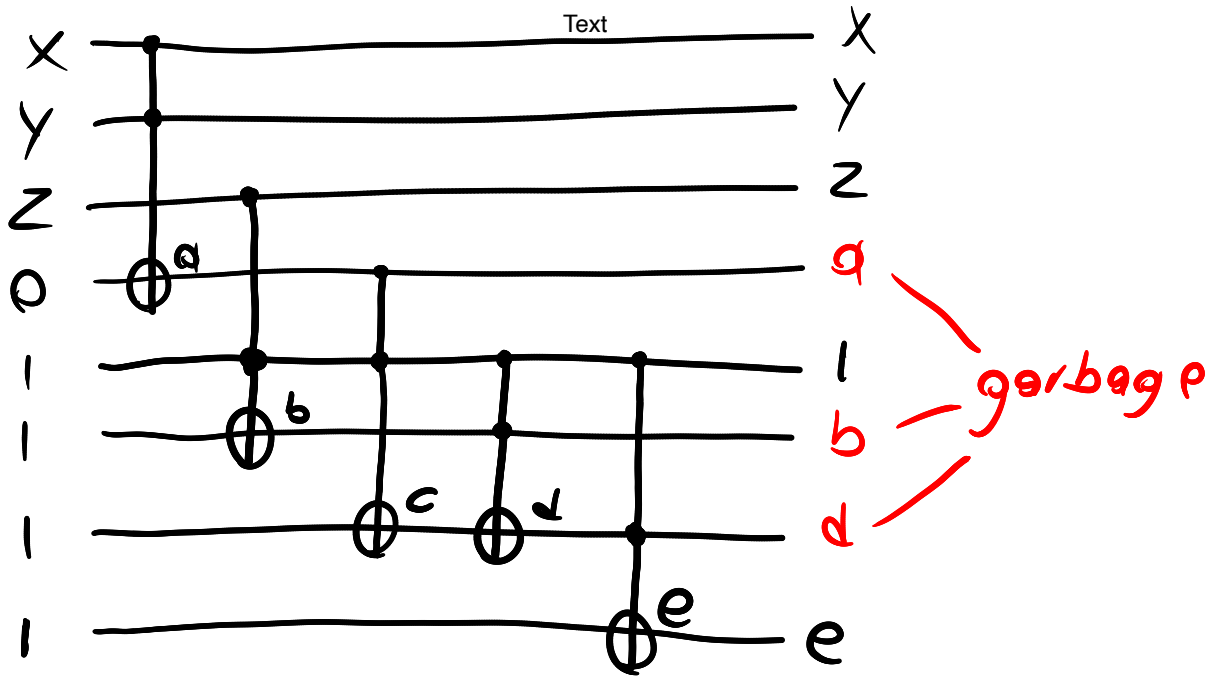
Ex.

To implement the classical circuit



Using toffoli gates, we can simulate this circuit.
Every time, we add another wire to store the value.

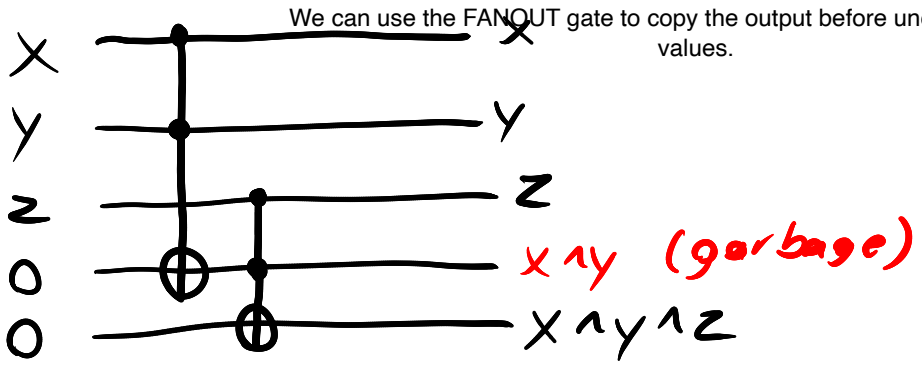
We use a Toffoli to compute each intermediate or temporary value (a, b, c, d, e)



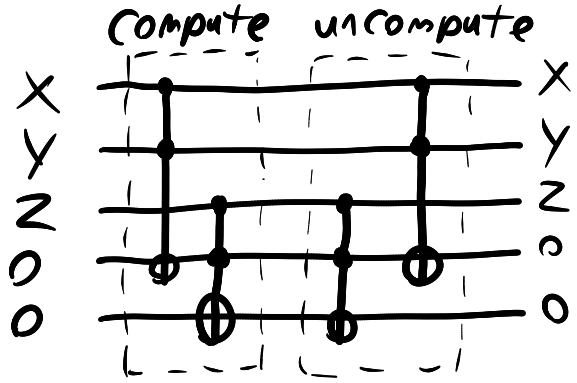
Issue:
e size of the input is so large that the algorithm will use exponential space.
Solution:
get the value you want and uncompute the garbage values.

(Uncomputation)

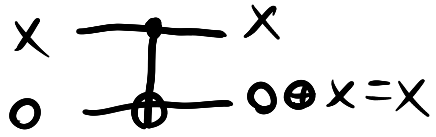
We can reversibly re-claim space by uncomputing garbage. E.g. to compute $x \wedge y \wedge z$



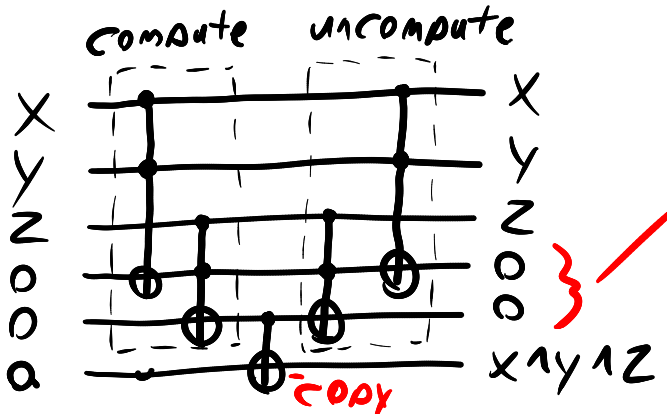
Running the circuit in reverse would return the ancilla storing $x \wedge y$ to 0 (called uncomputing) but it would also uncompute $x \wedge y \wedge z$.



To uncompute temporary values (i.e. garbage) we can copy any values we wish to save and then run in reverse. Recall that the CNOT gate is reversible and can copy a bit.



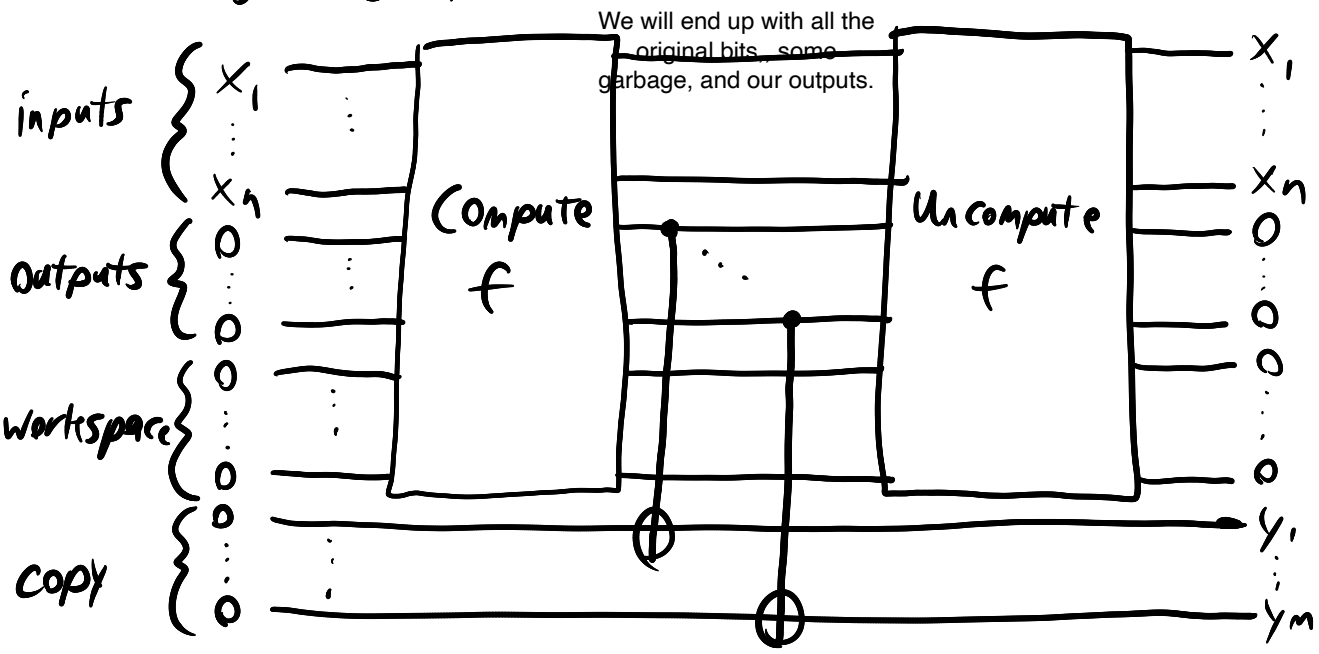
So



all temporary space reclaimed!

(Bennett's trick)

To reversibly compute a function $f: \{0,1\}^n \to \{0,1\}^m$ with no garbage, compute f, copy, then uncompute:



We use the CNOT gates to copy over the outputs.

If the copy register itself has initial state $|a\rangle = |a_1 a_2 \cdots a_m\rangle$, then the above circuit implements

$$|x\rangle|00\ldots0\rangle|a\rangle \longmapsto |x\rangle|00\ldots0\rangle|a \oplus f(x)\rangle$$

(Reversibility)

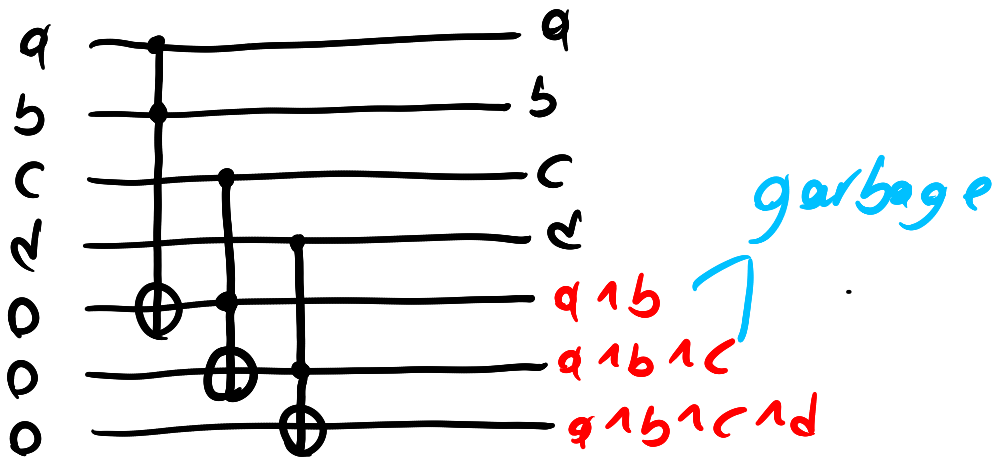
For any classical function $f: \{0,1\}^n \to \{0,1\}^m$,

$$|x\rangle|a\rangle \longmapsto |x\rangle|a \oplus f(x)\rangle$$

is an invertible (in fact, self inverse) transformation.

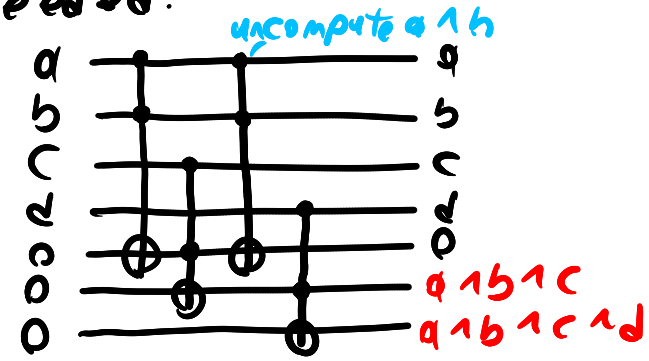
Ex.

Let $f(a,b,c,d) = a \wedge b \wedge c \wedge d$. A circuit for f is



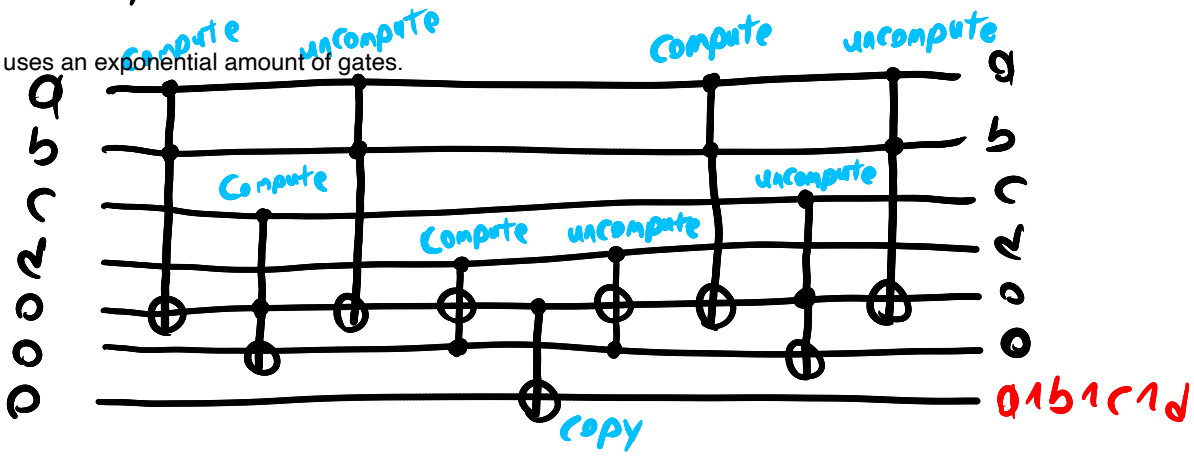
We do not need (And A B) since we can use

We could reclaim all space at the end of computation, in which case we would use 8 bits total and 7 gates. We could instead add an intermediate cleanup state to reclaim space once $a \wedge b$ is no longer needed:



In this case we can reuse the first ancilla to compute the result. However, to uncompute $a \wedge b \wedge c$ we actually need to recompute $a \wedge b$!

ing each of the values. It uses an exponential amount of gates.



The result is only 7 bits but 9 gates.

(Bennett 1989)

An irreversible circuit with T gates and space S can be simulated by a reversible one with

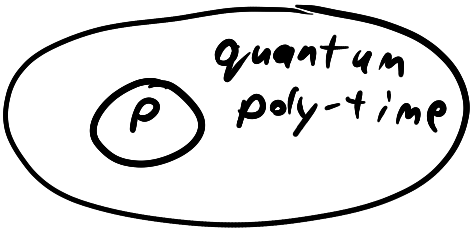
- Time $O(T^{1+\varepsilon})$ & space $O(S \log T)$
- or Time $O(T)$ & space $O(ST^{\varepsilon})$

for any $\varepsilon > 0$. Epsilon is any positive constant.

These result from different choices of when to uncompute temporary values and are called pebble games or pebbling strategies.

(Reversibility & quantum computing)

Since reversible computations are a subset of quantum computations, this tells us that quantum computers can simulate classical ones in polynomial time and space.



In general, the power of quantum computing is believed to lie somewhere between P & NP