# CMPT 476 Lecture 9/10

## Mixed states and the Church of the higher Hilbert Space

Can we have an indefinite state vector?

If we have pure states, can we have impure states?


What is an input and output density operator?

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At this point we have all the basic ingredients of quantum computation, called the 4 **Postulates** of QM.

1. States are unit vectors in a Hilbert space $\mathcal{H}$
2. State evolution takes $|\psi\rangle$ to $U|\psi\rangle$ for some unitary operator $U$ on $\mathcal{H}$
3. Two systems with Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ have a combined state in $\mathcal{H}_A \otimes \mathcal{H}_B$

ity does not state that we can communicate faster than light

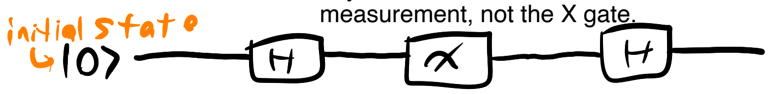
4. Measurement of $\mathcal{H}_A$ in basis $\{|e_i\rangle\}$ sends $\sum_i \alpha_i |e_i\rangle \otimes |\varphi_i\rangle$ to $|e_i\rangle \otimes |\varphi_i\rangle$ with probability $|\alpha_i|^2$

i phi

State in hilbert space H_B

Postulate #4 is troublesome because it takes us out of the land of linear algebra and requires a lot of case analysis and general annoyances. For example, suppose we have the circuit

measurement, not the X gate.




What is the resulting "state"? We have 2 cases:

$$|0\rangle \xrightarrow{H} |+\rangle \begin{cases} \xrightarrow{1/2} |0\rangle \xrightarrow{H} |+\rangle \\ \xrightarrow{1/2} |1\rangle \xrightarrow{H} |-\rangle \end{cases}$$

We do not need to preserve all the branches

What if we have $K$ nested measurements? We would have $2^K$ cases! There's got to be a better way!

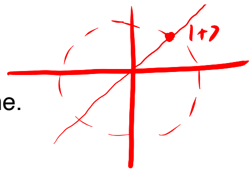
—— Mixed states have entered the chat ——

## (Projective measurements)

Before we talk about mixed states, let's talk more generally about measurements.

Recall that a **projector** on a Hilbert space is an operator $P$ such that $P^2 = P$, and can be viewed as projecting a state onto a linear subspace (e.g. a line)

$$\text{e.g. } \left( |+\rangle\langle+| \right)^2 = |+\rangle\langle+|+\rangle\langle+| = |+\rangle\langle+| \quad \Big\} \text{ projects onto the line}$$




Simplest subspace is one spanned by a line.

What is a direct sum?

Given a set of projectors $\{P_i\}$ satisfying

1. $\sum_i P_i = I$ (i.e. sums to the identity)

A collection of vector spaces which sum the whole space.

2. $P_i P_j = 0$ for all $i \neq j$ (i.e. projectors are orthogonal)

Then a **projective measurement** of state $|\psi\rangle$ with respect to $\{P_i\}$ produces result $i$ with probability

$$p(i) = \langle\psi| P_i |\psi\rangle$$

and leaves the state as

Final state is normalized projection

Where does this come from?

Divide by sqrt of p(i)

$$\frac{1}{\sqrt{p(i)}} \cdot P_i |\psi\rangle$$

What is a basis state?

## (Notation)

If pi has a rank of 1, then that means

A projective measurement is **complete** if each $P_i$ has rank 1 – that is, $\dim(\mathrm{im}(P_i)) = 1$, or $P_i|\psi\rangle$ is a **basis state**. Otherwise it is **incomplete** or **partial**

Ex. Computational basis measurement

Let $P_0 = |0\rangle\langle 0|$, $P_1 = |1\rangle\langle 1|$ — i.e. $P_i$ projects onto the $|i\rangle$ state. Then

1. $P_0 + P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

2. $P_0 P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Measuring the state $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$ wrt $\{P_0, P_1\}$:

Argument determining which projector is used

$$p(0) = \langle\psi|P_0|\psi\rangle$$
$$= \langle\psi|\cdot|0\rangle\langle 0|\cdot|\psi\rangle$$

sqrt(3)/2 * sqrt(3)/2

$$= \langle\psi|0\rangle\langle 0|\psi\rangle$$
$$= 3/4$$

Text

$$p(1) = \langle\psi|P_1|\psi\rangle$$
$$= \langle\psi|1\rangle\langle 1|\psi\rangle$$
$$= \frac{-i}{2}\cdot\frac{i}{2}$$
$$= 1/4$$

$$\frac{1}{\sqrt{p(0)}}P_0|\psi\rangle = \frac{2}{\sqrt{3}}|0\rangle\langle 0|\psi\rangle$$
$$= \frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2}|0\rangle$$
$$= |0\rangle$$

Normalized projection onto the |0> state

$$\frac{1}{\sqrt{p(1)}}P_1|\psi\rangle = 2|1\rangle\langle 1|\psi\rangle$$
$$= 2\cdot\frac{i}{2}|1\rangle$$
$$= i|1\rangle$$

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Ex. Partial measurement

Now let $P_0 = I \otimes |0\rangle\langle 0|$, $P_1 = I \otimes |1\rangle\langle 1|$.

1. $P_0 + P_1 = I \otimes |0\rangle\langle 0| + I \otimes |1\rangle\langle 1|$
$$= I \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)$$
$$= I \otimes I$$
$$= I_4$$

4x4 identity matrix

2. $P_0 P_1 = (I \otimes |0\rangle\langle 0|)(I \otimes |1\rangle\langle 1|)$
$$= I \otimes (|0\rangle\langle 0|\cdot|1\rangle\langle 1|)$$
$$= 0$$

Measuring $|\psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle)$ wrt $\{P_0, P_1\}$:

$$p(0) = \langle\psi|P_0|\psi\rangle$$

take the ket's and bra's with 0 in the second position

$$= \langle\psi|(I\otimes|0\rangle\langle 0|)|\psi\rangle$$

Only take basis vectors which have a zero in the second component

$$= \frac{1}{3}(\langle 00| + \langle 01| + \langle 10|)(I\otimes|0\rangle\langle 0|)(|00\rangle + |01\rangle + |10\rangle)$$

$$= \frac{1}{3}(\langle 00| + \langle 01| + \langle 10|)(|00\rangle + |10\rangle)$$

I operator I operator

How did we reach this step?

$$= \frac{1}{3}(\langle 00|00\rangle + \langle 10|10\rangle)$$

When you apply |0><0> onto |1>, we will get zero since 1 does not project onto the zero state and will return zero.

$$= \frac{2}{3}$$

$$\frac{1}{\sqrt{p(0)}}P_0|\psi\rangle = \frac{\sqrt{3}}{\sqrt{2}}\cdot\frac{1}{\sqrt{3}}(|00\rangle + |10\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

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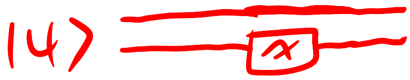
$$p(1) = \langle\psi|P_1|\psi\rangle = 1/3$$

$$\frac{1}{\sqrt{p(1)}}\cdot P_1|\psi\rangle = \sqrt{3}\cdot\frac{1}{\sqrt{3}}|01\rangle$$

$$= |01\rangle$$

Note: this is just the partial measurement





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Ex.

Important concept: Project onto more general subspaces

A common partial projective measurement is a parity measurement. Over $\mathbb{C}^2\otimes\mathbb{C}^2$, the parity 0 subspace is spanned by

What does parity tell us?

exclusive-OR, i.e. addition mod 2

$$|x\rangle|y\rangle \text{ s.t. } x\oplus y = 0$$

Even paruty subspace for C^2:
|00>, |11>
odd parity subspace:
|01>, |10>

likewise, the parity 1 subspace is spanned by

$$|x\rangle|y\rangle \text{ s.t. } x\oplus y = 1$$

Observe that $P_0 = |00\rangle\langle 00| + |11\rangle\langle 11|$ parity 0

$P_1 = |01\rangle\langle 01| + |10\rangle\langle 10|$ parity 1

project onto these spaces, respectively.

## (Observables)

Projective measurements are often described by physicists as measuring "an observable". Abstractly, an observable is something we can measure. Concretely, it's a Hermitian operator on the state space

Measuring an observable = doing a measurement
Obervable: has eigenvalues typically of 0, 1, -1
Important for hamiltonian states
Measuring the observable projects onto that eigenspaces

## (Hermitian operator)

An operator $T: \mathcal{H} \to \mathcal{H}$ is Hermitian iff

$$T^{\dagger} = T^{-1} = T \quad \text{(i.e. self-adjoint)}$$

**Ex**

Hermetian operators can be decomposed as the sum of the diagonals into their eigenbasis

The following are hermitian:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The following are not:

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}, \quad P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

## (Hermitian operators as measurements)

Hermitian operators, by something known as the spectral theorem* can be decomposed as a sum of projectors onto their eigenspaces:

Spectral theorem: if T is hermetian, we can write T as a diagonal operator

$$T = \sum_i \lambda_i |T_i\rangle\langle T_i|$$

Project onto eigenspace

Since their eigenspaces are disjoint and partition the Hilbert space (i.e. $\sum_i |T_i\rangle\langle T_i| = I$, $|T_i\rangle\langle T_i||T_j\rangle\langle T_j| = 0$) an observable defines a projective measurement onto its eigenspaces.

Is this an i or a J?

X gate can be composed of + observable - -observable

Normal operators?
Measuring X means you are measuring in the plus and minus basis

*we'll come back to this later

## Ex.

Observe that $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|.$

Then $|0\rangle$ is an eigenvector of $Z$ with eigenvalue $+1$ and $|1\rangle$ is an eigenvector with eigenvalue $-1$:

$$Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$

"Measuring $Z$" means measure $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, which projects a state $|\psi\rangle$ onto $Z$'s $+1$ eigenspace ($\mathrm{span}(|0\rangle)$) or $-1$ eigenspace ($\mathrm{span}(|1\rangle)$).

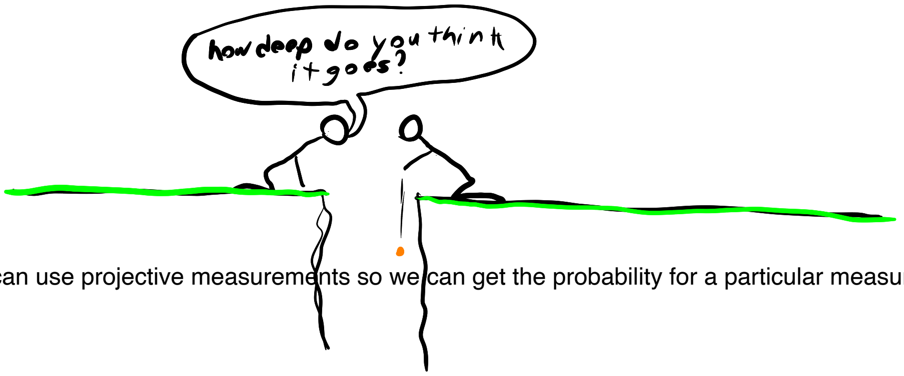
Likewise, physicists often talk about measuring $X$ or measuring in the $X$ basis to mean $\{|+\rangle, |-\rangle\}$, since

$$X|+\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle$$

$$X|-\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -|-\rangle$$

## (A final word on measurement)

We've only scratched the surface on measurement — there are other, even more general measurements. Projective measurements will suffice to capture all the cases we care about in this course.




We can use projective measurements so we can get the probability for a particular measurements

# (Pure & mixed states)

Complete projection: we project down into a specific basis state

Incomplete measurement: a partial measurement that projects onto a dimension higher than one

A unit vector $|\psi\rangle \in \mathcal{H}$ is called a pure state of $\mathcal{H}$, and represents a physical system in some definite (i.e. not probabilistic) superposition.

Pure state: uni vector in Hilbert space

After a measurement, we're left with a probability distribution over pure states, written as an ensemble

$$\{(|\psi_1\rangle, p_1), (|\psi_2\rangle, p_2), \ldots, (|\psi_k\rangle, p_k)\}$$

where $(|\psi_i\rangle, p_i)$ denotes that the system is in pure state $|\psi_i\rangle$ with probability $p_i$. The state of the system is

The mixed state representing this is (|0>, 1/2)(|1>, 1/2)

said to be mixed.

Applying the Hadamard gate to this produces |+> and |-> with probability 1/2

Is it probability 1/2 since the unitary matrix does not affect the length?

1/2 prob

$$|+\rangle \leftarrow |0\rangle$$

$$|+\rangle \leftarrow |1\rangle$$

1/2 prob

Ex.

Suppose Alice and Bob share an EPR pair

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

and Alice measures her qubit (in the comp. basis). The possibilities are:

- $|00\rangle$ with probability $1/2$
- $|11\rangle$ with probability $1/2$

We can describe their mixed state as the ensemble

$$\{(|00\rangle, 1/2), (|11\rangle, 1/2)\}$$

If Bob measures his qubit with 50% probability the joint state is $|00\rangle$, so his measurement returns $|0\rangle$ with 100% probability (!?)

When we measure

A measurement is not a unitary transformation, but we apply it to each state in the ensemble when we want to measure the probability of obtaining a specific state.

Note that applying a unitary transformation to a mixed state corresponds to applying it to each state in the ensemble:

$$U\{(|\psi_1\rangle, p_1), \dots, (|\psi_k\rangle, p_k)\} = \{(U|\psi_1\rangle, p_1), \dots, (U|\psi_k\rangle, p_k)\}$$

# (Density operators)

A more convenient representation of a mixed state is as a **density operator**. Given an ensemble

$$\{(|\psi_1\rangle, p_1), \dots, (|\psi_k\rangle, p_k)\}$$

(matrix)

on Hilbert space $\mathcal{H}$, the corresponding density operator is an operator $\rho$ (rho) on $\mathcal{H}$ defined by

Does this mean the probability of projecting onto phi_i?

The sum of the weighted projectors onto each of the pure state.
The state we project down to has probability p_i.

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Note that if we evolve the ensemble by $U$ to

$$\{(U|\psi_1\rangle, p_1), \dots, (U|\psi_k\rangle, p_k)\}$$

After

then the new density matrix is

$$\begin{aligned} \rho &= \sum_i p_i (U|\psi_i\rangle)(\langle\psi_i|U^\dagger) \\ &= U\left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right)U^\dagger \quad \text{(by linearity)} \\ &= U\rho U^\dagger \end{aligned}$$

So unitary evolution sends $\rho \longrightarrow U\rho U^\dagger$

Ex.

Calculate density operators for these ensembles:

For

We just need to sum the projectors over all of the pure states.

1. $\{(|+\rangle, 1)\} \longrightarrow \rho = |+\rangle\langle+| = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

2. $\{(|0\rangle, 1/2), (|1\rangle, 1/2)\} \longrightarrow \rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

$$= \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

o not have off diagonal elements, than we do not have quantum things happening

We have equal mixture of the two computational bases.

ing quantum happening in these matrices.

sity matrices show us how quantum a state is.

3. $H\{(|0\rangle, 1/2), (|1\rangle, 1/2)\} \longrightarrow \rho = H\left(\frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)H^{+}$

$$= \frac{1}{4}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

In all of the density operators above, observe that the entries along the diagonal sum to 1. This is a special property of density matrices, and comes from the fact that the entries on the diagonal are the probabilities of the outcomes of a measurement in the computational basis. To see this, we need another tool from linear algebra called the trace.

(Trace)

The trace of a matrix $A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & & \\ \vdots & & \ddots & \\ a_{n0} & & & a_{nn} \end{bmatrix}$ is

$$Tr(A) = \sum_{i=0}^{n} a_{ii}$$

In operator terms, the trace of $A$ on a Hilbert space $\mathcal{H}$ is

$$Tr(A) = \Sigma_i \langle e_i | A | e_i \rangle$$

for <u>any</u> orthonormal basis $\{|e_i\rangle\}$ of $\mathcal{H}$.

Sum up how much A projects on each of the components of the orthonormal basis?

## Ex.

$$Tr\left(X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 0, \quad Tr\left(H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) = 0$$

$$Tr\left(P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1, \quad Tr\left(CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\right) = 2$$

(Cyclicity of the trace)

Let $A$ and $B$ be $m \times n$ and $n \times m$ matrices, respectively. Then $AB$ and $BA$ are $m \times m$ and $n \times n$ matrices, respectively, and

$$Tr(AB) = Tr(BA)$$

## Ex.

The cyclicity property tells us that, for instance,

$$\langle \psi | \psi \rangle = Tr(\langle \psi | \psi \rangle) = Tr(|\psi\rangle\langle\psi|)$$

We can verify for $|+\rangle$:

$$\langle + | + \rangle = 1, \quad |+\rangle\langle +| = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad Tr(|+\rangle\langle +|) = \frac{1}{2} + \frac{1}{2} = 1$$

More generally, observe that

$$\langle \psi | \rho | \psi \rangle = Tr(\langle \psi | \rho | \psi \rangle) = Tr(|\psi\rangle\langle\psi| \rho)$$

Are we just switching around A, B, and C?

The previous observation connects measurement statistics to the trace of $\rho$. Recall that given a pure state $|\psi\rangle$, the probability of a measurement returning $|0\rangle$ is

$$p(0) = |\langle 0|\psi\rangle|^2 = \langle 0|\psi\rangle\langle\psi|0\rangle$$

As a density operator, the state $|\psi\rangle$ is represented as

If we sandwich the density operator between the two

$$\rho = |\psi\rangle\langle\psi|$$

so the probability of measuring $|0\rangle$ is

$$\langle 0|\rho|0\rangle = \mathrm{Tr}\,(|0\rangle\langle 0|\,\rho)$$

As $\langle i|\rho|i\rangle = \mathrm{Tr}\,(|i\rangle\langle i|\,\rho)$ are the entries along the diagonal, they must sum to 1. More generally, if $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, then the probability of measuring $|0\rangle$ is

$$\begin{aligned}\sum_i p_i\,(\langle 0|\psi_i\rangle\langle\psi_i|0\rangle) &= \langle 0|\Big(\sum_i p_i|\psi_i\rangle\langle\psi_i|\Big)|0\rangle \\ &= \langle 0|\rho|0\rangle \\ &= \mathrm{Tr}\,(|0\rangle\langle 0|\,\rho)\end{aligned}$$

(Projective measurements and density operators)

Let $\{P_i\}$ be a set of orthogonal projectors such that

$$\sum_i P_i = I$$

Then the projective measurement of a density matrix $\rho$ over $\{P_i\}$ produces result $i$ with probability

$$p(i) = \mathrm{Tr}\,(P_i\rho)$$

After we get result, we project down into the subspace spanned by p and we need to renormalize

and projects the system into the state

$$\frac{P_i\rho P_i}{\mathrm{Tr}(P_i\rho)}$$

$$\begin{tikzcd} & \dfrac{P_i\rho P_i}{\mathrm{Tr}(P_i\rho)} \\ \rho \arrow[ur, "i"] \arrow[dr, "j"'] & \\ & \dfrac{P_j\rho P_j}{\mathrm{Tr}(P_j\rho)} \end{tikzcd}$$

If we want to express the measurement result as a mixed state, then the measurement sends

$$\rho \longrightarrow \sum_i P_i\rho P_i \qquad\qquad \rho \xrightarrow[j]{i} P_i\rho P_i + P_j\rho P_j$$

q is a random probability

Ask about measuring mix states.
Ask about the result of a projective measurement on a density matrix

Ask about cyclicity of traces

Important property of density matrices:
Traces of density matrix is one since it sums up the probability over snapping to a specific basis.
tr(p) = sum <e_ilple_i>
sum of the probabilities of obtaining each of these results

(Density operators)

Density operators are useful! They

1. Contain all information about what is physically observable. That is two pure states $|\psi\rangle, |\varphi\rangle$ have the same measurement probabilities <u>over every basis</u> if and only if they have the same density matrix.

2. Simplify reasoning and calculations involving measurement — they can describe a mixture of <u>exponentially many</u> states with just a matrix of dimension $d \times d$, where $d = \dim(\mathcal{H})$

Plus, if you really dislike them you can turn the bras around to get a vector

$$\rho \simeq \Sigma_i \sqrt{p_i}\, |\psi_i\rangle |\psi_i\rangle$$

Now this seems like a silly thing to do, but is a simple example of what is colloquially known as going to the church of the larger Hilbert space. In particular, we can think of mixed states as pure states in a higher-dimensional Hilbert space, and operations like measurement as suitable operations (in particular versions, unitary) on the larger Hilbert space. We likely won't get to such techniques in this course, but they're quite useful in practice so it's good to be aware of. The map-state duality refers to a collection of similar ideas, for instance

$$T = \Sigma_{ij} T_{ij} |i\rangle\langle j| \simeq \Sigma_{ij} T_{ij} |i\rangle |j\rangle$$

In the end, it's all just different representations of the same linear algebra!

## (Reduced density operators)

Another useful application of density operators is to describe the local state of part of an entangled system. Consider the EPR pair

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$$

shared by Alice and Bob. While we can't describe Bob's qubit as a pure state separately from Alice's, we can describe Bob's density operator separately by "pretending" we measured Alice's qubit:

1/2 probability that Bob is in the zero state or in the one state

This measurement is destructive since we no longer have Alice's qubit

Ask professor why this is not a projective measurement?

$$|\psi\rangle_{AB} \xrightarrow{\text{measure } A} \left\{\left(\tfrac{1}{2}, |0\rangle_B\right), \left(\tfrac{1}{2}, |1\rangle_B\right)\right\}$$

e measures in computational basis, it sends bob's state to plus or minus

Bob's state becomes

$$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

There is no possibility to determine if Bob is in the 0 or plus state

What if Alice actually measured in the $\{|+\rangle, |-\rangle\}$ basis instead?

If we look at Bob's density matrix on his state,

$$(H\otimes I)|\psi\rangle_{AB} = \frac{1}{2}\left(|00\rangle + |10\rangle + |01\rangle - |11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|0\rangle|+\rangle + \frac{1}{\sqrt{2}}|1\rangle|-\rangle$$

|0>(|0> + |1>) + |1>(|0> - |1>)

measuring now gives Bob's density operator

Should this not be 1/4

Bob has a probability of 1/2 of being in the |+> state, and probability 1/2 of being in the |-> state.

$$\frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \quad \text{magic!}$$

In either basis Alice measured in, we have the same state of Bob's qubit.

This is called Bob's reduced density operator and is <u>independent</u> of the basis Alice hypothetically measures in

Partial trace is used to calculate the reduced density matrix.

sum: sum over the diagonal entries over one of the dimensions .

Formally, we calculate the reduced density operator by taking a partial trace (called tracing out a subsystem).

(Partial trace)

Bipartite hilbert space: a hilbert space that you can write as the tensor product of two hilbert spaces.

Let $\rho^{AB} = \sum_{ijlk} p_{ijlk} |e_i\rangle\langle e_j| \otimes |f_l\rangle\langle f_k|$ be a density operator on a composite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ with bases $\{|e_i\rangle\}$, $\{|f_l\rangle\}$ respectively. Then the

Orthonormal basis

partial trace over system $A$ is

$$\rho^B = Tr_A(\rho^{AB}) = \sum_{ijlk} p_{ijlk} \, Tr(|e_i\rangle\langle e_j|) \otimes |f_l\rangle\langle f_k|$$

0 if i != j, 1 otherwise

$$= \sum_{ijlk} p_{ijlk} \langle e_j | e_i \rangle |f_l\rangle\langle f_k|$$

he reduced density matrix of the other subsystems

Can write it as Sum over i

p_{ijlk} is the matrix entry

Ex.

The density matrix of Alice and Bob's EPR pair is

$$\rho^{AB} = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

we can write this as

$$\rho^{AB} = \frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$$

Tracing out Alice's system we get

$$Tr_A(\rho^{AB}) = \frac{1}{2}(\langle 0|0\rangle |0\rangle\langle 0| + \langle 0|1\rangle |0\rangle\langle 1| + \langle 1|0\rangle |1\rangle\langle 0| + \langle 1|1\rangle |1\rangle\langle 1|)$$

$$= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

as expected.

(Aside)

It's harder to work with partial traces of matrices, but it can be helpful to grok what's going on:

$$|0\rangle\langle 0|_A \qquad |0\rangle\langle 1|_A$$

Quadrant corresponding to Alice's 00 state

$$Tr_B \left( \left[ \begin{array}{cc|cc} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ \hline a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{array} \right] \right)$$

Quadrant corresponding to Alice's 01 state

$$|1\rangle\langle 0|_A \qquad |1\rangle\langle 1|_A$$

Quadrant corresponding to Alice's 10 state

Quadrant corresponding to Alice's 11 state

Sum along the diagonal of each of these matrices

$$\begin{bmatrix} Tr\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} & Tr\begin{bmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{bmatrix} \\ Tr\begin{bmatrix} a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} & Tr\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \end{bmatrix}$$

## (Local unitaries don't change a reduced density matrix)

Let $\rho$ be a density operator on a Hilbert Space $\mathcal{H}_A \otimes \mathcal{H}_B$ and $U$ be a unitary transformation on $\mathcal{H}_A$. Then

$$Tr_A((U \otimes I)\rho(U^\dagger \otimes I)) = Tr_A(\rho)$$

## Proof

Follows from the fact that $\{U|e_i\rangle\}$ is an orthonormal basis of $\mathcal{H}_A$ if $\{|e_i\rangle\}$ is an orthonormal basis. In particular,

$$Tr_A((U \otimes I)\rho(U^\dagger \otimes I)) = \sum_{ijlk} \rho_{ijlk} Tr(U|e_i\rangle\langle e_j|U^\dagger)|f_l\rangle\langle f_k|$$

$$= \sum_{ijlk} \rho_{ijlk} \langle e_j|U^\dagger U|e_i\rangle |f_l\rangle\langle f_k|$$

$$= Tr_A(\rho)$$

(No communication theorem)

The fact that we just proved is part of what is sometimes refered to as the no communication theorem. Namely, if Alice and Bob share an entangled state, nothing either does to their individual qubit can affect the other's reduced density matrix, and hence observable behaviour. Which is a good thing because if not, Alice and Bob could communicate faster than light and break relativity ☹. Next class we'll start to look at things entanglement does in fact break!

Let P^{AB} be a density operator on H_A X H_B, then for any unitary operator H_A -> H_A
Tr_A((U X I) p(U_dagger X I)) = Tr_A(P)
The reduced density matrix after applying the unitary is the same as if the unitary had not been applied
Nothing that Alice does to her qubit can affect the measurement results when Bob measures his qubit