CMPT 476 Lecture 17

More oracles!

Last class we discussed the black-box model and quantum query complexity with our first example of a truly quantum algorithm — Deutsch's algorithm. Today we continue with query algorithms, adding more complexity to our functions and the interference patterns leading to the desired answer. Just remember:

Quantum algorithms = Superposition, interference, & entanglement

1. Prepare superpositions of all x in {0, 1}^n
2. Phase the state by (-1)^(f(x))
3. Sum up all paths (values of x)

(Deutsch-Jozsa algorithm)

The next quantum algorithm we're going to see is a straightforward generalization of Deutsch's algorithm to the case when f takes n (rather than 1) inputs.

Let $f: \{0,1\}^n \longrightarrow \{0,1\}$. We say:

1. f is constant if $f(x) = f(y)$ $\forall x, y \in \{0,1\}^n$
2. f is balanced if $f(x) = 1$ for exactly half of the strings $x \in \{0,1\}^n$, and $f(x) = 0$ for the other half.

Deutsch=Josza's problem tries to find whether f is balanced or constant

We can make a probabilistic queries to get the right answer with a probability of 2/3

Deutsch-Josza's problem (DJ)

Input: a function $f: \{0,1\}^n \longrightarrow \{0,1\}$

Promise: f is either constant or balanced

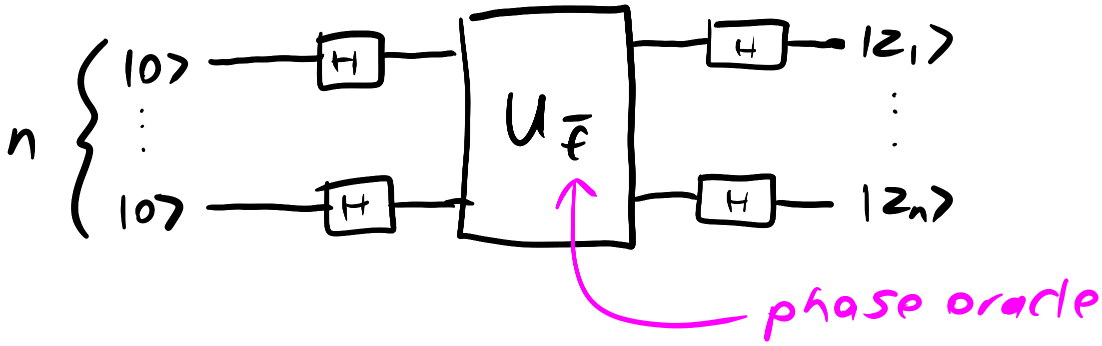
Goal: Determine whether f is constant or balanced

Fact: The classical query complexity is $2^{n-1}+1$

↳ Why? suppose the first 2^{n-1} queries (i.e. half the strings $x \in \{0,1\}^n$) give $f(x) = 0$. Then the other half of the strings could either all give 0 — hence f is constant — or could all give 1 — hence f is balanced.

Deutsch & Jozsa showed that the quantum query complexity of their problem is one!

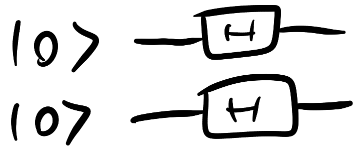
The Deutsch-Jozsa algorithm works analogously to Deutsch's algorithm, but with n qubits.



(Uniform superposition)

The first stage of the DJ algorithm is so common and important it deserves a separate analysis.

Consider the circuit



The state this circuit prepares is

$$
\begin{aligned}
(H|0\rangle)\otimes(H|0\rangle) &= \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\\
&= \frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+|11\rangle)\\
&= \frac{1}{2}\sum_{x\in\{0,1\}^2}|x\rangle
\end{aligned}
$$

This is a uniform superposition of $x\in\{0,1\}^2$.

In general,

$$
\begin{aligned}
H^{\otimes n}|0\rangle^{\otimes n} &= \underbrace{(H\otimes H\otimes\cdots\otimes H)}_{n\text{ times}}\underbrace{|0\rangle|0\rangle\cdots|0\rangle}_{n\text{ times}}\\
&= (H|0\rangle)^{\otimes n}\\
&= \left(\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\right)^{\otimes n}\\
&= \frac{1}{\sqrt{2}^n}\sum_{x\in\{0,1\}^n}|x\rangle
\end{aligned}
$$

Uniform superposition is obtained by applying Hadamard gate to n copies of zero

So, Deutsch-Jozsa first prepares the uniform superposition then uses $U_{\tilde{f}}|x\rangle = (-1)^{f(x)}|x\rangle$ to phase each string:

f tilde adds a phase to |X> state.

$$U_{\tilde{f}} H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2}^n} \sum_{x\in\{0,1\}} (-1)^{f(x)} |x\rangle$$

As in the Deutsch algorithm, the final $H^{\otimes n}$ is going to generate interference.

But how?

(Hadamard gate, abstractly)

Note that $H|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle)$, $x\in\{0,1\}$

We can write this more compactly as

the hadamard gate takes x and returns the + or - state, it encodes the relative phase

$$\begin{aligned} H|x\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle) \\ &= \frac{1}{\sqrt{2}} \sum_{z\in\{0,1\}} (-1)^{x\cdot z}|z\rangle \end{aligned}$$

Now what happens if we do this to an n-bit string?

$$\begin{aligned} H^{\otimes n}|x_1 x_2 \cdots x_n\rangle &= \left(\frac{1}{\sqrt{2}}\sum_{z_1} (-1)^{x_1\cdot z_1}|z_1\rangle\right) \otimes \cdots \otimes \left(\frac{1}{\sqrt{2}}\sum_{z_n} (-1)^{x_n\cdot z_n}|z_n\rangle\right) \\ &= \frac{1}{\sqrt{2}^n} \sum_{z_1\cdots z_n} (-1)^{x_1\cdot z_1 + \cdots + x_n z_n} |z_1\cdots z_n\rangle \\ &= \frac{1}{\sqrt{2}^n} \sum_{z\in\{0,1\}^n} (-1)^{x\cdot z}|z\rangle \end{aligned}$$

If it is even, we have a phase of 1. -1 otherwise.

← dot product over $\{0,1\}^n = \mathbb{Z}_2^n$

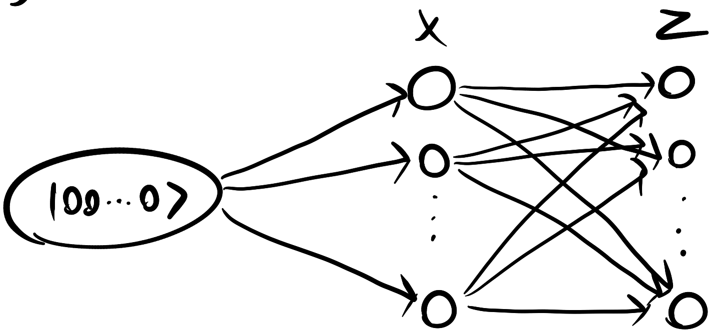
Note: we'll largely start using $\mathbb{Z}_2$ (integers mod 2) to refer to $\{0,1\}$ now. If $x,y\in\mathbb{Z}_2^n$, $x\cdot y = x_1y_1 \oplus \cdots \oplus x_ny_n$ $= x_1y_1 + \cdots + x_ny_n \bmod 2$

So, the final state in the DJ algorithm is

$$\begin{aligned} &H^{\otimes n}\left(\frac{1}{\sqrt{2}^n}\sum_{x\in\{0,1\}^n} (-1)^{f(x)}|x\rangle\right) \\ &= \frac{1}{\sqrt{2}^n}\sum_{x\in\{0,1\}^n} (-1)^{f(x)}\left(\frac{1}{\sqrt{2}^n}\sum_{z\in\{0,1\}^n} (-1)^{x\cdot z}|z\rangle\right) \\ &= \frac{1}{2^n}\sum_{x,z\in\{0,1\}^n} (-1)^{f(x)+x\cdot z}|z\rangle \end{aligned}$$

(Interference analysis)

The algorithm looks like this:



We need to figure out **which paths interfere.**

Consider a single z. The **amplitude** of this z is the sum over all paths leading to it:

$$\frac{1}{2^n}\sum_{x\in\{0,1\}}(-1)^{f(x)+x\cdot z}|z\rangle$$

Do I have 2^n possible values of x?

What is the amplitude of $z=00\cdots0$?

All zeros as amplitude state?

Case 1: f is constant

Then $\frac{1}{2^n}\sum_{x\in\{0,1\}}(-1)^{f(x)}|00\cdots0\rangle=\frac{1}{2^n}\sum_x(-1)^c|00\cdots0\rangle$

How are we obtaining this amplitude?

$$=\pm|00\cdots0\rangle$$

Case 2: f is balanced

2^(n-1) of both states differing by a phase difference since f is balanced?

The $\frac{1}{2^n}\sum_x(-1)^{f(x)}|00\cdots0\rangle=\frac{1}{2^n}\left(\sum_{x|f(x)=0}|00\cdots0\rangle+\sum_{x|f(x)=1}-|00\cdots0\rangle\right)$

$$=\frac{2^{n-1}}{2^n}|00\cdots0\rangle-\frac{2^{n-1}}{2^n}|00\cdots0\rangle$$

$$=0$$

So, if we measure at the end, if f is constant we get $|00\cdots0\rangle$ with 100% probability, and if f is balanced we got $|00\cdots0\rangle$ with 0% probability!

(Bernstein-Vazirani algorithm)

It is a way of framing the Deutsch-Jozsa algorithm to find out informations about n bits.

The Deutsch-Jozsa algorithm is not that impressive in reality, because we can solve the problem with 2/3 probability with 2 queries classically using a randomized algorithm. Bernstein & Vazirani came up with the next algorithm that gives a non-trivial speed-up over randomized algorithms too! Their algorithm is identical to Deutsch-Jozsa, but involves a specially-chosen promise on f.

Bernstein-Vazirani problem (BV)

Input: a function $f:\{0,1\}^n \rightarrow \{0,1\}$

Promise: $f(x) = s \cdot x \bmod 2 \quad \forall x \in \{0,1\}^n$ for some $s \in \{0,1\}^n$

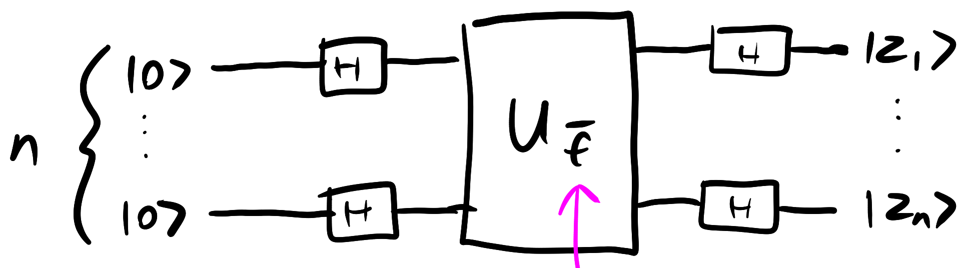
Goal: find the hidden string s

Fact

At least n since you can imagine the case where each query correctly gives you one of the bits of s?

The probabilistic query complexity of BV is at least n. Why? Because we need n bits of information and f only gives us 1 bit.

Bernstein & Vazirani's algorithm uses the exact same circuit as Deutsch & Jozsa's, but a different interference analysis



phase oracle

We only get constructive interference for the basis state zn.
Why do we not need to do an analysis of the other states?
All the amplitudes are in one state.

Final state: $\frac{1}{2^n}\sum_{x,z\in\{0,1\}^n} (-1)^{f(x)+x\cdot z}|z\rangle$

(Interference analysis)

The simple analysis is, just like Deutsch-Jozsa, to look at the amplitude of a well-chosen string. This time, we'll analyze interference when $z = s$.

$$\frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x) + x \cdot s} |s\rangle = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{s \cdot x + x \cdot s} |s\rangle$$

Ask about this.

$$= \frac{1}{2^n} \sum_{x \in \{0,1\}^n} |s\rangle$$

$$= |s\rangle$$

So measuring in the computational basis results in s with 100% probability!

Simple, right?