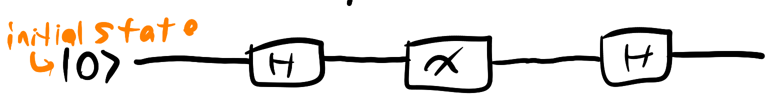
CMPT 476 Lecture 9/10

Mixed states and the Church of the higher Hilbert Space

At this point we have all the basic ingredients of quantum computation, called the 4 **Postulates** of QM.

1. States are unit vectors in a Hilbert space $\mathcal{H}$
2. State evolution takes $|\psi\rangle$ to $U|\psi\rangle$ for some unitary operator U on $\mathcal{H}$
3. Two systems with Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ have a combined state in $\mathcal{H}_A \otimes \mathcal{H}_B$
4. Measurement of $\mathcal{H}_A$ in basis $\{|e_i\rangle\}$ sends $\sum_i \alpha_i |e_i\rangle \otimes |\varphi_i\rangle$ to $|e_i\rangle \otimes |\varphi_i\rangle$ with probability $|\alpha_i|^2$

Postulate #4 is troublesome because it takes us out of the land of linear algebra and requires a lot of case analysis and general annoyances. For example, suppose we have the circuit



What is the resulting "state"? We have 2 cases:

$$|0\rangle \xrightarrow{H} |+\rangle \begin{cases} \xrightarrow{1/2} |0\rangle \xrightarrow{H} |+\rangle \\ \xrightarrow{1/2} |1\rangle \xrightarrow{H} |-\rangle \end{cases}$$

What if we have k nested measurements? We would have 2^k cases! There's got to be a better way!

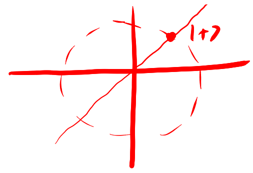
Mixed states have entered the chat

(Projective measurements)

Before we talk about mixed states, let's talk more generally about measurements.

Recall that a **projector** on a Hilbert space is an operator P such that $P^2 = P$, and can be viewed as projecting a state onto a linear subspace (e.g. a line)

$$\text{e.g. } \left(|+\rangle\langle +|\right)^2 = |+\rangle\langle +|+\rangle\langle +| = |+\rangle\langle +| \quad \Big\} \text{ projects onto the line}$$



Given a set of projectors $\{P_i\}$ satisfying

1. $\sum_i P_i = I$ (i.e. sums to the identity)
2. $P_i P_j = 0$ for all $i \neq j$ (i.e. projectors are orthogonal)

Then a **projective measurement** of state $|\psi\rangle$ with respect to $\{P_i\}$ produces result i with probability

$$p(i) = \langle \psi | P_i | \psi \rangle$$

and leaves the state as

$$\frac{1}{\sqrt{p(i)}} \cdot P_i |\psi\rangle$$

(Notation)

A projective measurement is **complete** if each P_i has rank 1 – that is, $\dim(\mathrm{im}(P_i)) = 1$, or $P_i|\psi\rangle$ is a basis state. Otherwise it is **incomplete** or **partial**

Ex. Computational basis measurement

Let $P_0 = |0\rangle\langle 0|$, $P_1 = |1\rangle\langle 1|$ — i.e. P_i projects onto the $|i\rangle$ state. Then

1. $P_0 + P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$
2. $P_0 P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Measuring the state $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$ wrt $\{P_0, P_1\}$:

$$\begin{aligned} p(0) &= \langle\psi|P_0|\psi\rangle \\ &= \langle\psi|\cdot|0\rangle\langle 0|\cdot|\psi\rangle \\ &= \langle\psi|0\rangle\langle 0|\psi\rangle \\ &= 3/4 \end{aligned}$$

$$\begin{aligned} \frac{1}{\sqrt{p(0)}} P_0|\psi\rangle &= \frac{2}{\sqrt{3}}|0\rangle\langle 0|\psi\rangle \\ &= \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2}|0\rangle \\ &= |0\rangle \end{aligned}$$

$$\begin{aligned} p(1) &= \langle\psi|P_1|\psi\rangle \\ &= \langle\psi|1\rangle\langle 1|\psi\rangle \\ &= \frac{-i}{2} \cdot \frac{i}{2} \\ &= 1/4 \end{aligned}$$

$$\begin{aligned} \frac{1}{\sqrt{p(1)}} P_1|\psi\rangle &= 2\,|1\rangle\langle 1|\psi\rangle \\ &= 2 \cdot \frac{i}{2}|1\rangle \\ &= i|1\rangle \end{aligned}$$

Ex. Partial measurement

Now let $P_0 = I \otimes |0\rangle\langle 0|$, $P_1 = I \otimes |1\rangle\langle 1|$.

1. $$\begin{aligned} P_0 + P_1 &= I \otimes |0\rangle\langle 0| + I \otimes |1\rangle\langle 1| \\ &= I \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\ &= I \otimes I \\ &= I_4 \end{aligned}$$
   4x4 identity matrix
2. $$\begin{aligned} P_0 P_1 &= (I \otimes |0\rangle\langle 0|)(I \otimes |1\rangle\langle 1|) \\ &= I \otimes (|0\rangle\langle 0| \cdot |1\rangle\langle 1|) \\ &= 0 \end{aligned}$$

Measuring $|\psi\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle\right)$ wrt $\{P_0, P_1\}$:

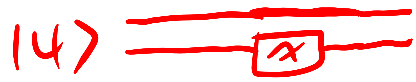
$$
\begin{aligned}
p(0) &= \langle\psi|P_0|\psi\rangle \\
&= \langle\psi|(I\otimes|0\rangle\langle 0|)|\psi\rangle \\
&= \frac{1}{3}\left(\langle 00| + \langle 01| + \langle 10|\right)\underbrace{(I\otimes|0\rangle\langle 0|)}_{\text{take the ket's and bra's with 0 in the second position}}\left(|00\rangle + |01\rangle + |10\rangle\right) \\
&= \frac{1}{3}\left(\langle 00| + \langle 01| + \langle 10|\right)\left(|00\rangle + |10\rangle\right) \\
&= \frac{1}{3}\left(\langle 00|00\rangle + \langle 10|10\rangle\right) \\
&= \frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\sqrt{p(0)}} P_0|\psi\rangle &= \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{1}{\sqrt{3}}\left(|00\rangle + |10\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)
\end{aligned}
$$

$$p(1) = \langle\psi|P_1|\psi\rangle = 1/3$$

$$
\begin{aligned}
\frac{1}{\sqrt{p(1)}} \cdot P_1|\psi\rangle &= \sqrt{3} \cdot \frac{1}{\sqrt{3}}|01\rangle \\
&= |01\rangle
\end{aligned}
$$

Note: this is just the partial measurement



Ex.

A common partial projective measurement is a parity measurement. Over $\mathbb{C}^2\otimes\mathbb{C}^2$, the parity 0 subspace is spanned by

$$|x\rangle|y\rangle \text{ s.t. } \underbrace{x\oplus y}_{\text{exclusive-OR, i.e. addition mod 2}} = 0$$

likewise, the parity 1 subspace is spanned by

$$|x\rangle|y\rangle \text{ s.t. } x\oplus y = 1$$

Observe that
$$P_0 = \overbrace{|00\rangle\langle 00| + |11\rangle\langle 11|}^{\text{parity 0}}$$
$$P_1 = \overbrace{|01\rangle\langle 01| + |10\rangle\langle 10|}^{\text{parity 1}}$$

project onto these spaces, respectively.

(Observables)

Projective measurements are often described by physicists as measuring "an observable". Abstractly, an observable is something we can measure. Concretely, it's a Hermitian operator on the state space.

(Hermitian operator)

An operator $T: \mathcal{H} \to \mathcal{H}$ is Hermitian iff

$$T^{+} = T^{-1} = T \quad \text{(i.e. self-adjoint)}$$

Ex.

The following are hermitian:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The following are not:

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}, \quad P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(Hermitian operators as measurements)

Hermitian operators, by something known as the spectral theorem* can be decomposed as a sum of projectors onto their eigenspaces:

$$T = \sum_i \lambda_i |T_i\rangle\langle T_i|$$

Since their eigenspaces are disjoint and partition the Hilbert space (i.e. $\sum_i |T_i\rangle\langle T_i| = I$, $|T_i\rangle\langle T_i||T_j\rangle\langle T_j| = 0$) an observable defines a projective measurement onto its eigenspaces.

*we'll come back to this later

Ex.

Observe that $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$.

Then $|0\rangle$ is an eigenvector of Z with eigenvalue $+1$ and $|1\rangle$ is an eigenvector with eigenvalue -1:

$$Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$

"Measuring Z" means measure $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, which projects a state $|\psi\rangle$ onto Z's $+1$ eigenspace ($\mathrm{span}(|0\rangle)$) or -1 eigenspace ($\mathrm{span}(|1\rangle)$).

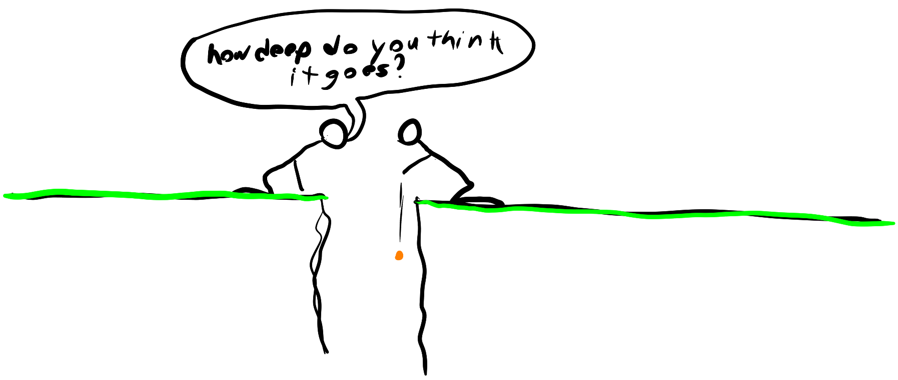
Likewise, physicists often talk about measuring X or measuring in the X basis to mean $\{|+\rangle, |-\rangle\}$, since

$$X|+\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle$$

$$X|-\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -|-\rangle$$

(A final word on measurement)

We've only scratched the surface on measurement — there are other, even more general measurements. Projective measurements will suffice to capture all the cases we care about in this course.



(Pure & mixed states)

A unit vector $|\psi\rangle \in \mathcal{H}$ is called a pure state of $\mathcal{H}$, and represents a physical system in some definite (i.e. not probabilistic) superposition.

After a measurement, we're left with a probability distribution over pure states, written as an ensemble

$$\{(|\psi_1\rangle, p_1), (|\psi_2\rangle, p_2), \ldots, (|\psi_k\rangle, p_k)\}$$

where $(|\psi_i\rangle, p_i)$ denotes that the system is in pure state $|\psi_i\rangle$ with probability p_i. The state of the system is said to be mixed.

Ex.

Suppose Alice and Bob share an EPR pair

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

and Alice measures her qubit (in the comp. basis). The possibilities are:

- $|00\rangle$ with probability $1/2$
- $|11\rangle$ with probability $1/2$

We can describe their mixed state as the ensemble

$$\{(|00\rangle, 1/2), (|11\rangle, 1/2)\}$$

If Bob measures his qubit with 50% probability the joint state is $|00\rangle$, so his measurement returns $|0\rangle$ with 100% probability (!?)

Note that applying a unitary transformation to a mixed state corresponds to applying it to each state in the ensemble:

$$U\{(|\psi_1\rangle, p_1), \dots, (|\psi_k\rangle, p_k)\} = \{(U|\psi_1\rangle, p_1), \dots, (U|\psi_k\rangle, p_k)\}$$

(Density operators)

Contains the information needed to apply unitaries or measurements
size = dXd for d = dim(H)

A more convenient representation of a mixed state is as a density operator. Given an ensemble

$$\{(|\psi_1\rangle, p_1), \dots, (|\psi_k\rangle, p_k)\}$$

(matrix)

on Hilbert space $\mathcal{H}$, the corresponding density operator is an operator ρ (rho) on $\mathcal{H}$ defined by

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Note that if we evolve the ensemble by U to

$$\{(U|\psi_1\rangle, p_1), \dots, (U|\psi_k\rangle, p_k)\}$$

then the new density matrix is

$$\begin{aligned}\rho &= \sum_i p_i (U|\psi_i\rangle)(\langle\psi_i|U^\dagger) \\ &= U\left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right)U^\dagger \quad \text{(by linearity)} \\ &= U\rho U^\dagger\end{aligned}$$

So unitary evolution sends $\rho \longrightarrow U\rho U^\dagger$

Ex.

Calculate density operators for these ensembles:

1. $\{(|+\rangle, 1)\} \longrightarrow \rho = |+\rangle\langle+| = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

2. $\{(|0\rangle, 1/2), (|1\rangle, 1/2)\} \longrightarrow \rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

$$= \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

3. $H\{(|0\rangle, 1/2), (|1\rangle, 1/2)\} \longrightarrow \rho = H\left(\frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)H^{+}$

$$= \frac{1}{4}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

In all of the density operators above, observe that the entries along the diagonal sum to 1. This is a special property of density matrices, and comes from the fact that the entries on the diagonal are the probabilities of the outcomes of a measurement in the computational basis. To see this, we need another tool from linear algebra called the trace.

(Trace)

The trace of a matrix $A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & & \\ \vdots & & \ddots & \\ a_{n0} & & & a_{nn} \end{bmatrix}$ is

$$Tr(A) = \sum_{i=0}^{n} a_{ii}$$

In operator terms, the trace of A on a Hilbert space $\mathcal{H}$ is

$$Tr(A) = \sum_i \langle e_i | A | e_i \rangle$$

for <u>any</u> orthonormal basis $\{|e_i\rangle\}$ of $\mathcal{H}$.

<u>Ex.</u>

$$Tr\left(X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 0, \quad Tr\left(H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) = 0$$

$$Tr\left(P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1, \quad Tr\left(CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\right) = 2$$

(Cyclicity of the trace)

Let A and B be $m \times n$ and $n \times m$ matrices, respectively. Then AB and BA are $m \times m$ and $n \times n$ matrices, respectively, and

$$Tr(AB) = Tr(BA)$$

<u>Ex.</u>

The cyclicity property tells us that, for instance,

$$\langle \psi | \psi \rangle = Tr(\langle \psi | \psi \rangle) = Tr(|\psi\rangle\langle\psi|)$$

We can verify for $|+\rangle$:

$$\langle + | + \rangle = 1, \quad |+\rangle\langle +| = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad Tr(|+\rangle\langle+|) = \frac{1}{2} + \frac{1}{2} = 1$$

More generally, observe that

$$\langle \psi | \rho | \psi \rangle = Tr(\langle \psi | \rho | \psi \rangle) = Tr(|\psi\rangle\langle\psi| \rho)$$

The previous observation connects measurement statistics to the trace of ρ. Recall that given a pure state $|\psi\rangle$, the probability of a measurement returning $|0\rangle$ is

$$p(0) = |\langle 0|\psi\rangle|^2 = \langle 0|\psi\rangle\langle\psi|0\rangle$$

As a density operator, the state $|\psi\rangle$ is represented as

$$\rho = |\psi\rangle\langle\psi|$$

so the probability of measuring $|0\rangle$ is

$$\langle 0|\rho|0\rangle = \mathrm{Tr}\,(|0\rangle\langle 0|\,\rho)$$

As $\langle i|\rho|i\rangle = \mathrm{Tr}\,(|i\rangle\langle i|\,\rho)$ are the entries along the diagonal, they must sum to 1. More generally, if $\rho = \Sigma_i\, p_i |\psi_i\rangle\langle\psi_i|$, then the probability of measuring $|0\rangle$ is

$$\begin{aligned}\Sigma_i\, p_i\,(\langle 0|\psi_i\rangle\langle\psi_i|0\rangle) &= \langle 0|\left(\Sigma_i\, p_i |\psi_i\rangle\langle\psi_i|\right)|0\rangle \\ &= \langle 0|\rho|0\rangle \\ &= \mathrm{Tr}\,(|0\rangle\langle 0|\,\rho)\end{aligned}$$

(Projective measurements and density operators)

Let $\{P_i\}$ be a set of orthogonal projectors such that

$$\Sigma_i\, P_i = I$$

Then the projective measurement of a density matrix ρ over $\{P_i\}$ produces result i with probability

$$p(i) = \mathrm{Tr}\,(P_i\,\rho)$$

and projects the system into the state

$$\frac{P_i \rho P_i}{\mathrm{Tr}(P_i \rho)}$$

$$\begin{tikzcd} & \dfrac{P_i\rho P_i}{\mathrm{Tr}(P_i\rho)} \\ \rho \arrow[ur, "i"] \arrow[dr, "j"'] & \\ & \dfrac{P_j\rho P_j}{\mathrm{Tr}(P_j\rho)} \end{tikzcd}$$

If we want to express the measurement result as a mixed state, then the measurement sends

$$\rho \longrightarrow \Sigma_i\, P_i \rho P_i \qquad \rho \xrightarrow[j]{i} P_i\rho P_i + P_j \rho P_j$$

(Density operators)

Density operators are useful! They

1. Contain all information about what is physically observable. That is two pure states $|\psi\rangle, |\varphi\rangle$ have the same measurement probabilities over every basis if and only if they have the same density matrix.

2. Simplify reasoning and calculations involving measurement — they can describe a mixture of exponentially many states with just a matrix of dimension $d \times d$, where $d = \dim(\mathcal{H})$

Plus, if you really dislike them you can turn the bras around to get a vector

$$\rho \simeq \sum_i \sqrt{p_i}\, |\psi_i\rangle|\psi_i\rangle$$

Now this seems like a silly thing to do, but is a simple example of what is colloquially known as going to the church of the larger Hilbert space. In particular, we can think of mixed states as pure states in a higher-dimensional Hilbert space, and operations like measurement as suitable operations (in particular versions, unitary) on the larger Hilbert space. We likely won't get to such techniques in this course, but they're quite useful in practice so it's good to be aware of. The map-state duality refers to a collection of similar ideas, for instance

$$T = \sum_{ij} T_{ij} |i\rangle\langle j| \simeq \sum_{ij} T_{ij} |i\rangle|j\rangle$$

In the end, it's all just different representations of the same linear algebra!

(Reduced density operators)

Another useful application of density operators is to describe the local state of part of an entangled system. Consider the EPR pair

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$$

Shared by Alice and Bob. While we can't describe Bob's qubit as a pure state separately from Alice's, we can describe Bob's density operator separately by "pretending" we measured Alice's qubit:

$$|\psi\rangle_{AB} \xrightarrow{\text{measure } A} \left\{\left(\tfrac{1}{2}, |0\rangle_B\right), \left(\tfrac{1}{2}, |1\rangle_B\right)\right\}$$

$$\|$$

$$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

What if Alice actually measured in the $\{|+\rangle, |-\rangle\}$ basis instead?

$$(H\otimes I)|\psi\rangle_{AB} = \frac{1}{2}\left(|00\rangle + |10\rangle + |01\rangle - |11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|0\rangle|+\rangle + \frac{1}{\sqrt{2}}|1\rangle|-\rangle$$

measuring now gives Bob's density operator

$$\frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \frac{1}{4}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \quad \text{magic!}$$

This is called Bob's reduced density operator and is <u>independent</u> of the basis Alice hypothetically measures in

Formally, we calculate the reduced density operator by taking a partial trace (called tracing out a subsystem).

(Partial trace)

Let $\rho^{AB} = \sum_{ijlk} p_{ijlk} |e_i\rangle\langle e_j| \otimes |f_l\rangle\langle f_k|$ be a density operator on a composite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ with bases $\{|e_i\rangle\}$, $\{|f_l\rangle\}$ respectively. Then the partial trace over system A is

$$\rho^{B} = \mathrm{Tr}_A(\rho^{AB}) = \sum_{ijlk} p_{ijlk} \mathrm{Tr}(|e_i\rangle\langle e_j|) \otimes |f_l\rangle\langle f_k|$$
$$= \sum_{ijlk} p_{ijlk} \langle e_j|e_i\rangle |f_l\rangle\langle f_k|$$

Ex.

The density matrix of Alice and Bob's EPR pair is

$$\rho^{AB} = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

we can write this as

$$\rho^{AB} = \frac{1}{2}\left(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|\right)$$

Tracing out Alice's system we get

$$\mathrm{Tr}_A(\rho^{AB}) = \frac{1}{2}\left(\langle 0|0\rangle |0\rangle\langle 0| + \langle 0|1\rangle |0\rangle\langle 1| + \langle 1|0\rangle |1\rangle\langle 0| + \langle 1|1\rangle |1\rangle\langle 1|\right)$$
$$= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

as expected.

(Aside)

It's harder to work with partial traces of matrices, but it can be helpful to grok what's going on:

$$\operatorname{Tr}_B\left(\begin{array}{c} |0\rangle\langle 0|_A \qquad |0\rangle\langle 1|_A \\ \left[\begin{array}{cc|cc} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ \hline a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{array}\right] \\ |1\rangle\langle 0|_A \qquad |1\rangle\langle 1|_A \end{array}\right) = \begin{bmatrix} \operatorname{Tr}\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} & \operatorname{Tr}\begin{bmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{bmatrix} \\ \operatorname{Tr}\begin{bmatrix} a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} & \operatorname{Tr}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \end{bmatrix}$$

(Local unitaries don't change a reduced density matrix)

Let ρ be a density operator on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and U be a unitary transformation on $\mathcal{H}_A$. Then

$$\operatorname{Tr}_A\big((U\otimes I)\rho(U^\dagger \otimes I)\big) = \operatorname{Tr}_A(\rho)$$

Proof

Follows from the fact that $\{U|e_i\rangle\}$ is an orthonormal basis of $\mathcal{H}_A$ if $\{|e_i\rangle\}$ is an orthonormal basis. In particular,

$$\begin{aligned} \operatorname{Tr}_A\big((U\otimes I)\rho(U^\dagger\otimes I)\big) &= \textstyle\sum_{ij\ell k} \rho_{ij\ell k} \operatorname{Tr}\big(U|e_i\rangle\langle e_j|U^\dagger\big)\,|f_\ell\rangle\langle f_k| \\ &= \textstyle\sum_{ij\ell k} \rho_{ij\ell k} \langle e_j|U^\dagger U|e_i\rangle\,|f_\ell\rangle\langle f_k| \\ &= \operatorname{Tr}_A(\rho) \end{aligned}$$

(No communication theorem)

The fact that we just proved is part of what is sometimes refered to as the no communication theorem. Namely, if Alice and Bob share an entangled state, nothing either does to their individual qubit can affect the other's reduced density matrix, and hence observable behaviour. Which is a good thing because if not, Alice and Bob could communicate faster than light and break relativity ☹. Next class we'll start to look at things entanglement does in fact break!