CMPT 476 Lecture 18

Simon's algorithm

So far the algorithms we've seen have been pretty trivial. On the one hand, it seems the "problems" were reverse-engineered based on the interference patterns generated from the Hadamard transform. On the other hand, we didn't even really get a non-trivial advantage over classical algorithms on those problems.

Today we'll look at the first non-trivial quantum algorithm which gives an **exponential speed-up** in the black-box model over classical computations, albeit for another **synthetic** problem.

(Simon's algorithm)

While the Bernstein-Vazirani algorithm gives a **query complexity speed-up** compared to probabilistic algorithms, in the end it's only **linear** since we have an $O(n)$ classical algorithm

$$f(100\cdots0) = s_1$$
$$f(010\cdots0) = s_2$$
$$f(001\cdots0) = s_3$$
$$\vdots$$
$$f(000\cdots1) = s_n$$

Moreover, BV itself takes $O(n)$ total work so it's not a "real" speed-up. In 1994, Simon set out to show that there could be no "real" speed-up and instead found the first "truly" exponential speed-up.

not really...

Simon's Problem

Input: A function $f: \{0,1\}^n \longrightarrow \{0,1\}^n$

Promise: $\exists s \in \{0,1\}^n$ such that $s \neq 0$ and $f(x) = f(y)$ if & only if $x = y$ or $x = y \oplus s$

bitwise xor of Y and S

Goal: Find s.

Simon's problem is **artificial** (like all problems we've seen so far), but it's closely related to and based on the hardness of finding **collisions**

$$x, y \text{ such that } f(x) = f(y)$$

Cryptographic hash functions like **SHA** are based on the hardness of finding collisions too.

First: how do we solve Simon classically?

Ex.

Suppose we have $f: \{0,1\}^3 \to \{0,1\}^3$ with the truth table

x	f(x)
000	101
001	011
010	111
011	010
100	011
101	111
110	011
111	101

Note - the values of f don't matter only collisions

If we sample $f(001) = 011$ and $f(110) = 011$, then we've found $x = 001$ & $y = 110$ such that $f(x) = f(y)$. Since $f(x) = f(y)$ iff $x = y \oplus s$, then

$$s = x \oplus y = 001 \oplus 110 = 111$$

If however $f(x) \neq f(y)$, we've learned nothing about s. We know there are collisions since $s \neq 0$ So, the question is how many queries do we need to find one with high probability?

We need to use at least sqrt(2^n) queries to have 50% chance of finding s.

Fact

Any probabilistic algorithm solving Simon's problem with probability $\frac{1}{2}$ must use $\Omega(\sqrt{2^n})$ queries.

f can be any function as long as it satisfies the conditions.

(at least)

This number comes from the birthday paradox:

The probability one pair out of 23 people share a birthday is > 50%

In general, to find a collision out of N uniformly distributed possibilities with >50% probability you need

Simon created an algorithm to sample the orthogonal subspace of s

$\approx \sqrt{N}$ samples.

(A quantum algorithm)

Simon's algorithm is our first one which is meaningfully different from Deutsch's. It uses interference on collisions

$$|x\rangle, |y = x \oplus s\rangle \text{ where } f(x) = f(y)$$

to repeatedly **sample** vectors orthogonal to s.

We first recall some facts from linear algebra.

Note that $\{0,1\}^n = \mathbb{Z}_2^n$ is a vector space.

(Orthogonal subspace)

Given a subspace S of a vector space V, denote by $S^\perp$ the **orthogonal complement** of S in V.

$$S^\perp = \{v \in V \mid s \cdot v = 0 \ \forall s \in S\}$$

The orthogonal complement of S is a subspace of V, and

$$\dim(V) = \dim(S) + \dim(S^\perp)$$

Rank-nullity theorem

(Finding a complement)

If we do not know s, how do we know s orthogonal?

Let $s \in \mathbb{Z}_2^n$. Then $S^\perp = (\text{span}\{s\})^\perp$. Suppose we can randomly sample from $S^\perp$ — how could we find s?

We can prepare an equally weighted superposition of the vectors in s orthogonal without knowing what s is.

Idea:

Take many samples $x_1 \cdots x_k$ and solve the linear system for $s = s_1 \cdots s_n$

How does this help us find s?

$$\begin{bmatrix} x_1^T \\ \vdots \\ x_k^T \end{bmatrix} \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} x_1 \cdot s \\ \vdots \\ x_k \cdot s \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

if s is in this space, do we not have a dimension of 2^n?

How do we know that the dimension of s_orthogonal is n - 1?

How many x's do we need? Well if they're **linearly independent** then $n-1$ since

$$\dim(s^\perp) = \dim(\mathbb{Z}_2^n) - \dim(\text{span}\{s\})$$
$$= n - 1$$

Professor: the same vector is not included in the perpendicular subspace.

We have a single vector s, so the basis for s is s itself.

Linear algebra in finite fields

Dot product in z_{2}^{n} is (xor x1y1 z2y2

(Simon's algorithm, high-level)

At a high-level, Simon's algorithm works like this

1. Set $A = []$
2. while rank$(A) < n-1$ do:
    3. Prepare a uniform superposition $\frac{1}{\sqrt{|S^\perp|}} \sum_{z \in S^\perp} |z\rangle$
    4. Measure to get $z_i \in S^\perp$

    } quantum part

    5. Append row z_i to A
6. Solve $As = 0$ for s

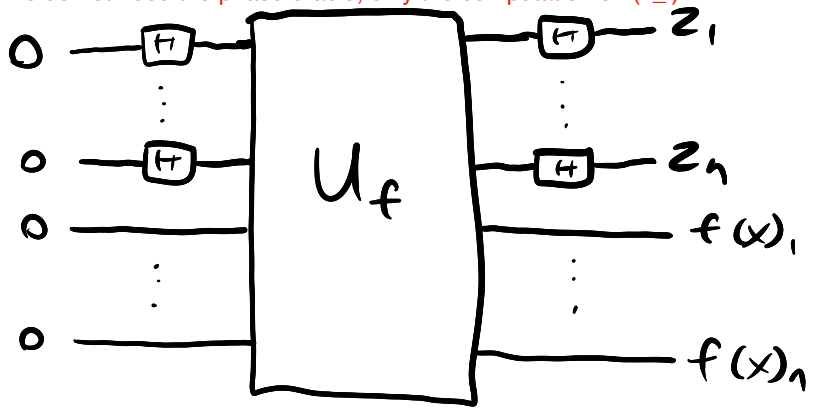
(The quantum part)

The genius of Simon's algorithm is in observing that interference can be used to prepare the state

$$\frac{1}{\sqrt{|S^\perp|}} \sum_{z \in S^\perp} |z\rangle$$

The idea is to generate a uniform superposition of $x \in \{0,1\}^n$, then pair up collisions with a call to f.

We will query the function on every z_i
We do not need the phase oracle, only the computation of f(z_i)



Here we use a **state oracle** $U_f |x\rangle|0\rangle = |x\rangle|f(x)\rangle$, so the state after U_f is

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle|f(x)\rangle$$

Now, suppose we measure the second register to get one particular value $w = f(x)$. The resulting state will be a (normalized) sum over all $y \in \{0,1\}^n$ such that $f(y) = w$. Observe that there are exactly two strings $x, x \oplus s$ such that $f(x) = f(y)$. This step helps you find the collisions.

You are finding all the x's such that f(x) = w

Why? suppose $\exists y, z \in \{0,1\}^n$ s.t. $f(x) = f(y) = f(z)$.
Then $x = s \oplus y = s \oplus z \Rightarrow y \oplus z = 0 \Rightarrow y = z$
Likewise, since $s \neq 0$, $x \oplus s \neq x$.

Ask professor to clarify about this.

So after measuring $f(x)$ the state is

$$\frac{1}{\sqrt{2}}\left(|x\rangle + |x \oplus s\rangle\right)|f(x)\rangle, \quad x \in \{0,1\}^n$$

Now what is the effect of the Hadamard gates? Since $H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} (-1)^{x \cdot z} |z\rangle$, we have

Applying the n hadamard gates: We get an equal weight superposition over all bit strings with a phase of -1 to the dot product of x and z

$$\left(H^{\otimes n} \frac{1}{\sqrt{2}}\left(|x\rangle + |x \oplus s\rangle\right)\right)|f(x)\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2^n}} \sum_z (-1)^{x \cdot z}|z\rangle + \frac{1}{\sqrt{2^n}} \sum_z (-1)^{(x \oplus s) \cdot z}|z\rangle\right)|f(x)\rangle$$

$$= \frac{1}{\sqrt{2^{n+1}}} \sum_z \left[(-1)^{x \cdot z} + (-1)^{(x \oplus s) \cdot z}\right]|z\rangle|f(x)\rangle$$

Text

Now which $|z\rangle$'s have non-zero amplitude? If $x \cdot z \neq (x \oplus s) \cdot z$, then

If we distribute the dot product, we obtain
xz != xz XOR sz
The only way we can get these values not to match is if z is not in the orthogonal subspace.

$$(-1)^{x \cdot z} + (-1)^{(x \oplus s) \cdot z} = 0$$

But $(x \oplus s) \cdot z = x \cdot z \oplus s \cdot z$, so $x \cdot z = (x \oplus s) \cdot z$ implies

$$s \cdot z = 0, \text{ or } z \in s^{\perp}!$$

So, if we measure the first register, we get some $z \in s^{\perp}$ as required. Or, in our original language the final quantum state from which we sample is

$$\frac{1}{\sqrt{2^{n-1}}} \sum_{z \in s^{\perp}} (-1)^{x \cdot z} |z\rangle|f(x)\rangle$$

(The role of measuring $|f(x)\rangle$)

One thing to note is that the measurement of $|f(x)\rangle$ is actually irrelevant — it only serves to simplify the analysis. If we don't measure $f(x)$, the only difference in the final state is that it is a superposition over all possible x's:

$$N \sum_{x \in \{0,1\}^n} \sum_{z \in S^\perp} (-1)^{x \cdot z} |z\rangle|f(x)\rangle$$

↑ normalization factor

Measuring the first register still gives us some $z \in S^\perp$ since for any x, y where $f(x) = f(y)$,

$$|z\rangle|f(x)\rangle \text{ and } |z\rangle|f(y)\rangle$$

have the same phase, since $x \cdot z = (x \oplus s) \cdot z = y \cdot z$.

(Complexity analysis)

Since Simon's algorithm involves a loop where we repeatedly sample from $S^\perp$ until we have a basis, we need to know how many times we need to repeat. This part makes it a probabilistic runtime because we could technically get the sequence of samples

$$z, z, z, z, z, \ldots \quad \text{(i.e. same string each time)}$$

ve n - 2 basis vectors, the size of the subpace is 2^(n - 2)

The size of the orthogonal complex

In reality, each time we sample, we get something not in the span of the previous samples with high probability. To see why, think about the extremal cases: when we've seen 1 sample, or $n-2$ samples.

- 1 sample z: $|\text{span}\{z\}| = 1$,
  $|\mathbb{Z}_2^n - \text{span}\{z\}| = 2^n - 1$

Why would nobody implement this?

span

- $n-2$ samples $z_1, \ldots, z_{n-2}$: $|\text{span}\{z_i\}| \leq 2^{n-2}$
  $|\mathbb{Z} - \text{span}\{z_i\}| \geq 2^n - 2^{n-2}$
  $= 3 \cdot 2^{n-2}$

Ask professor what the error with this was.

The probability

So even when we already have $n-2$ l.i. ind. samples, we get the last one with $3/4$ probability!

Fact the expected number of samples needed is n

Expected number: what you get after n results.

This gives Simon's algorithm an expected quantum query complexity of $O(n)$ with $O(n^3)$ additional classical work to solve the linear system.

Ask about the n^3 work.

Alternatively, we could just sample $m \in O(n)$ times and if we haven't found a basis yet just give up. In practice, $m \approx n+1$ is enough to give a high probability of success. This is a common idea in probabilistic algorithms, where a **deterministic algorithm with a probabilistic runtime** can be turned into a **probabilistic algorithm with a deterministic runtime**.

(Recap of algorithms so far)

So far we've seen a series of algorithms with increasing separation between their **classical** and **quantum** query complexities. To summarize,

Problem	Classical query complexity	Probabilistic query complexity	Quantum query complexity
Deutsch	2	2	1
Deutsch-Jozsa	$O(2^n)$	2	1
Bernstein-Vazirani	$O(n)$	$O(n)$	1
Simon	$O(2^n)$	$O(2^n)$	$O(n)$

For the classical and probabilistic query complexity, it should be sqrt(2^n)

The query complexity does not assume anything on how it is implemented.

However, this is just the **query** complexity — we need to ask whether such a speed-up is **meaningful** in practice.

(Is query complexity meaningful?)

It depends — in particular on whether we can learn the property more efficiently by looking at the implementation of f. In polynomial interpolation, say we have a sub-routine in python which implements f. If this sub-routine is something like

Ask what the professor means about peeking into the function.

power operator

```
def f(x):
    return a_0 + a_1*x + ... + a_d*(x**d)
```

then we can get the answer in O(1) by opening up the black-box. On the other hand, f may be implemented by some other means — e.g. some financial algorithm — and we want to learn a property which is not apparent from the implementation.

Simon's algorithm is a period finding transformation over z^n_2 using the fourier transform.

Since we don't have true black boxes in quantum computation, the true complexity is relative to the implementation of the oracle, and to give a speed-up over classical computing it must give a speed-up over any classical algorithm given the same or a comparable implementation of f. That rules out a quantum speed-up for the algorithms we've seen over any implementation that explicitly uses the hidden string. Other options for the function in Simon's algorithm for example are

If we implemented Simon's algorithm using a matrix of size n - 1, we could solve for finding s in polynomial time.

$$f(x) = Ax, \quad \text{rank}(A) = n-1$$

In this case, the best classical algorithm runs in time $O(n^3)$ by gaussian elimination, so there can be no (non-trivial) quantum speed-up.

Next we'll start to look at algorithms which offer true speed-ups, starting with an algorithm inspired by Simon's...

SHOR!

If we only have entanglement between sets of qubits of size k (k fixed), we have a polynomia

Ask professor about this.

If we do not use interference, then our algorithm becomes probabilitstic. Why?

e have n states in our superposition, we need to apply a few one qubit states. Polynomial run time.

The ingredients for a quantum speed up:

1. We need a large superposition of states
2. We need interference
3. We need a highly entangled state (highly entangled: a large number of qubits entangled)

e 11 wires, none of which are entanged, then we can represent it an a tensor product of two vectors

l and we have the qubits in subsets at most of size m, then every qubit subset has a size of 2^m, which is polynomial in n.

ubset is in c^(2^m), which means it has a size 2^m. We only need to store n/m * 2^m amplitudes.