

# CMPT 476/981: Introduction to Quantum Algorithms

## Assignment 1

Due **January 18, 2024 at 11:59pm on coursys**  
Complete individually and submit in PDF format.

### Question 1 [3 points]: A universal classical gate

The *NAND* gate is a classical gate with the following truth table:

$x$	$y$	$NAND(x, y)$
0	0	1
0	1	1
1	1	0

1. fanout original bit and apply NAND gates  
2. AND gate: Apply NAND, apply fanout and apply NAND again.  
OR gate: Apply FANOUT to both bits, and then for each pair apply NAND, and then apply NAND to the remainder  
FANOUT gate: Apply FANOUT to the bit

1. Show that the NOT gate can be implemented with NAND gates and FANOUT. You may draw a circuit or simply give the algebraic expression.
2. Show that the gate set  $\{NAND, FANOUT\}$  is universal for classical computation by giving implementations of each gate in the universal gate set  $\{AND, OR, NOT, FANOUT\}$ .

### Question 2 [6 points]: Dirac notation

Let  $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{i}{\sqrt{3}}|1\rangle + \frac{-1}{\sqrt{3}}|2\rangle$ ,  $|\phi\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{-i}{\sqrt{2}}|2\rangle$  be two states of a **qutrit** (i.e. a three-level or three-dimensional system).

1. Give the explicit column vectors of  $|\psi\rangle$  and  $|\phi\rangle$   $v = 1/\sqrt{3}(3) [1 \ 0 \ 0] + i/\sqrt{3}(3)[0 \ 1 \ 0] -1/\sqrt{3}(3)[0 \ 0 \ 1]$
2. Calculate the following:  $\phi = 1/\sqrt{2}(2)[0 \ 1 \ 0] -1/\sqrt{2}(2)[0 \ 0 \ 1]$ 
  - $\langle\psi|\psi\rangle$
  - $\langle\phi|\phi\rangle$
  - $\langle\psi|\phi\rangle$
  - $|\psi\rangle\langle\phi|$
  - $|\psi\rangle \otimes |\phi\rangle$
3. Is the vector  $|\psi\rangle + |\phi\rangle$  a unit vector? If not, normalize it to get a unit vector.

Probability of receiving 1 or 0 is both 1/2

### Question 3 [4 points]: Gates and measurement

Suppose we have a qubit initially in the state  $\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle$  for some  $\theta \in \mathbb{R}$ .

1. Calculate the probabilities of receiving result “0” or “1” if the qubit is measured.
2. Recall the definition of the Hadamard gate, which has the vectors  $|+\rangle$  and  $|-\rangle$  as its columns:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

If we first apply the Hadamard gate to the initial state  $\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle$  and then measure, what are the probabilities of receiving the “0” and “1” results as an function of  $\theta$ ?

Note: this is the same thing as measuring the initial state in the  $|+\rangle, |-\rangle$  basis.

### Question 4 [5 points]: Eigenvectors

Recall that an *eigenvector* of a matrix  $A$  is a vector  $|v\rangle$  such that  $A|v\rangle = \lambda|v\rangle$  for some scalar *eigenvalue*  $\lambda$ .

1. Let  $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ . Find two **unit** vectors  $|+_Y\rangle, |-_Y\rangle$  such that

$$\begin{aligned} Y|+_Y\rangle &= |+_Y\rangle \\ Y|-_Y\rangle &= -|-_Y\rangle \end{aligned}$$

2. Let  $U$  be the 2 by 2 matrix with columns  $|+_Y\rangle$  and  $|-_Y\rangle$ . Is  $U$  unitary?
3. Calculate  $U^\dagger Y U$ .