Chapter

3

Dimensional analysis in ecology

3.0 Dimensional analysis

Dimensional analysis is generally not part of the curriculum of ecologists, so that relatively few are conversant with this simple but remarkably powerful tool. Yet, applications of dimensional analysis are found in the ecological literature, where results clearly demonstrate the advantage of using this mathematical approach.

"Dimensional analysis treats the *general forms of equations* that describe natural phenomena" (Langhaar, 1951). The basic principles of this discipline were established by physicists (Fourier, 1822; Maxwell, 1871) and later applied by engineers to the very important area of small-scale modelling. Readers interested in the fundamentals and engineering applications of dimensional analysis should refer, for example, to Langhaar (1951), from which are taken several of the topics developed in the present chapter. Other useful references are Ipsen (1960), Huntley (1967), and Schneider (1994).

The use of dimensional analysis in ecology rests on the fact that a growing number of areas in ecological science use *equations*; for example, populations dynamics and ecological modelling. The study of equations is the very basis of dimensional analysis. This powerful approach can easily be used by ecologists, given the facts that it can be reduced to *a single theorem* (the Π theorem) and that many of its applications (Sections 3.1 and 3.2) only require a knowledge of elementary mathematics.

Dimensional analysis can resolve complex ecological problems in a simple and elegant manner. Readers should therefore not be surprised that ecological applications in the present chapter are of a rather high level, since the advantage of dimensional analysis lies precisely in its ability to handle complex problems. It follows that dimensional analysis is mainly useful in those cases where it would be difficult to resolve the ecological problem by conventional approaches.

3.1 Dimensions

All fields of science, including ecology, rest on a number of abstract entities such as the mass, length, time, temperature, speed, acceleration, radioactivity, concentration, energy or volume. These entities, which can be measured, are called *quantities*. Designing a *system of units* requires to: (1) arbitrarily choose a small number of *fundamental quantities*, on which a coherent and practical system can be constructed, and (2) arbitrarily assign, to each of these quantities, *base units* chosen as references for comparing measurements.

International System of Units

Various systems of units have been developed in the past, e.g. the British system and several versions of the metric system. The latter include the CGS metric system used by scientists (based on the centimetre, the gram and the second), the MKS (force) metric system used by engineers (based on the metre, the kilogram and the second, where the kilogram is the unit of *force*), and the MKS (mass) metric system (where the kilogram is the unit of *mass*). Since 1960, there is an internationally accepted version of the metric system, called the *International System of Units* (SI, from the French name *Système international d'unités*; see Plate 3.1, p. 142). The SI is based on seven *quantities*, to which are associated seven *base units* (Table 3.1; the mole was added to the SI in 1971 only). In addition to these seven base units, the SI recognizes two

Table 3.1 Base units of the International System of Units (Sl).

Fundamental quantity	Quantity symbol*	Dimension symbol	Base unit	Unit symbol
mass	m	[M]	kilogram	kg
length	l	[L]	metre†	m
time	t	[T]	second	s
electric current	I	[I]	ampere	A
thermodynamic temperature	T ‡	[θ]	kelvin‡	K
amount of substance	n	[N]	mole	mol
luminous intensity	I_v	[J]	candela	cd

* Quantity symbols are not part of the SI, and they are not unique.

† Spelled meter in the United States of America.

‡ In ecology, temperature is generally measured on the Celsius scale, where the unit is the *degree Celsius* (°C); the quantity symbol for temperatures expressed in °C is usually t. Note that the absolute temperature unit is the kelvin, *not degree kelvin*.

supplementary units, the radian (rad) and the steradian (sr), which measure planar and solid angles, respectively. All other units, called *derived units*, are combinations of the base and supplementary units. Some frequently used derived units have special names, e.g. volt, lux, joule, newton, ohm. It must be noted that: (1) *unit names* are written with small letters only, the sole exception being the degree Celsius; (2) *unit symbols* are written with small letters only, except the symbols of derived units that are surnames, whose first letter is a capital (e.g. Pa for pascal), and the litre (see Table 3.2, footnote). Unit symbols are *not* abbreviations, hence they are *never* followed by a dot.

Dimension

Table 3.2 shows that derived units are not only simple products of the fundamental units, but that they are often *powers* and *combinations of powers* of these units. Maxwell (1871) used symbols such as [M], [L], [T], and [θ] to represent the quantities mass, length, time and temperature (Table 3.1). The *dimensions* of the various quantities are *products of powers* of the symbols of fundamental quantities. Thus, the dimension of an area is $[L^2]$, of a volume $[L^3]$, of a speed $[LT^{-1}]$, and of an acceleration $[LT^{-2}]$. Table 3.2 gives the exponents of the dimensional form of the most frequently encountered quantities.

Since the various quantities are *products of powers,* going from one quantity to another is done simply by *adding* (or *subtracting) exponents* of the dimensions. For example, one calculates the dimensions of *heat conductivity* $W(mK)^{-1}$ by subtracting, from the dimension exponents of *power* W, the sum of the dimension exponents of *length* m and of *temperature* K:

$$[M^1L^2T^{-3}] / ([L^1] \times [\theta^1]) = [M^1L^{(2-1)}T^{-3}\theta^{-(1)}] = [M^1L^1T^{-3}\theta^{-1}]$$

The first three fundamental quantities (Table 3.1), mass [M], length [L], and time [T], are enough to describe any Newtonian mechanical system. Ecologists may require, in addition, temperature [θ], amount of substance [N], and luminous intensity [J]. Research in electromagnetism calls for electric current [I] and, in quantum mechanics, one uses the quantum state of the system [Ψ].

Four types of entities are recognized:

(1) *dimensional variables*, e.g. most of the quantities listed in Table 3.2;

(2) *dimensional constants*, for instance: the speed of light in vacuum $[LT^{-1}]$, $c = 2.998 \times 10^8$ m s^{-1}; the acceleration due to Earth's gravity at sea level $[LT^{-2}]$, $g = 9.807$ m s^{-2}; the number of elementary entities in a mole $N_A = 6.022 \times 10^{23}$ mol^{-1}, where N_A is the Avogadro number (note that the nature of the elementary entities in a mole must always be specified, e.g. mol C, mol photons);

(3) *dimensionless variables,* such as angles, relative density (Table 3.2), or dimensionless products which will be studied in following sections;

(4) *dimensionless constants,* e.g. π, e, 2, 7; it must be noted that exponents are, by definition, dimensionless constants.

Table 3.2 Dimensions, units, and names of quantities. Units follow the standards of the International System of Units (SI).

Quantity	[M]	[L]	[T]	[I]	[θ]	[N]	[J]	Units	Name*
mass	1	0	0	0	0	0	0	kg	kilogram
length	0	1	0	0	0	0	0	m	metre
time	0	0	1	0	0	0	0	s	second
electric current	0	0	0	1	0	0	0	A	ampere
temperature	0	0	0	0	1	0	0	K	kelvin
amount of substance	0	0	0	0	0	1	0	mol	mole
luminous intensity	0	0	0	0	0	0	1	cd	candela
absorbed dose	0	2	–2	0	0	0	0	$J\,kg^{-1} = Gy$	gray
acceleration (angular)	0	0	–2	0	0	0	0	$rad\,s^{-2}$	
acceleration (linear)	0	1	–2	0	0	0	0	$m\,s^{-2}$	
activity of radioactive source	0	0	–1	0	0	0	0	$s^{-1} = Bq$	becquerel
angle (planar)	0	0	0	0	0	0	0	rad	radian
angle (solid)	0	0	0	0	0	0	0	sr	steradian
angular momentum	1	2	–1	0	0	0	0	$kg\,m^2\,s^{-1}$	
angular velocity	0	0	–1	0	0	0	0	$rad\,s^{-1}$	
area	0	2	0	0	0	0	0	m^2	
compressibility	–1	1	2	0	0	0	0	Pa^{-1}	
concentration (molarity)	0	–3	0	0	0	1	0	$mol\,m^{-3}$	
current density	0	–2	0	1	0	0	0	$A\,m^{-2}$	
density (mass density)	1	–3	0	0	0	0	0	$kg\,m^{-3}$	
electric capacitance	–1	–2	4	2	0	0	0	$C\,V^{-1} = F$	farad
electric charge	0	0	1	1	0	0	0	$A\,s = C$	coulomb
electric conductance	–1	–2	3	2	0	0	0	$\Omega^{-1} = S$	siemens
electric field strength	1	1	–3	–1	0	0	0	$V\,m^{-1}$	
electric resistance	1	2	–3	–2	0	0	0	$V\,A^{-1} = \Omega$	ohm
electric potential	1	2	–3	–1	0	0	0	$W\,A^{-1} = V$	volt
energy	1	2	–2	0	0	0	0	$N\,m = J$	joule
force	1	1	–2	0	0	0	0	$kg\,m\,s^{-2} = N$	newton
frequency	0	0	–1	0	0	0	0	$s^{-1} = Hz$	hertz
heat capacity	1	2	–2	0	–1	0	0	$J\,K^{-1}$	
heat conductivity	1	1	–3	0	–1	0	0	$W(m\,K)^{-1}$	
heat flux density	1	0	–3	0	0	0	0	$W\,m^{-2}$	
illuminance	0	–2	0	0	0	0	1	$lm\,m^{-2} = lx$	lux
inductance	1	2	–2	–2	0	0	0	$Wb\,A^{-1} = H$	henry
light exposure	0	–2	1	0	0	0	1	lx s	
luminance	0	–2	0	0	0	0	1	$cd\,m^{-2}$	
luminous flux	0	0	0	0	0	0	1	$cd\,sr = lm$	lumen
magnetic field strength	0	–1	0	1	0	0	0	$A\,m^{-1}$	
magnetic flux	1	2	–2	–1	0	0	0	$V\,s = Wb$	weber
magnetic flux density	1	0	–2	–1	0	0	0	$Wb\,m^{-2} = T$	tesla
magnetic induction	1	0	–2	–1	0	0	0	$Wb\,m^{-2} = T$	tesla

* Only base units and special names of derived units are listed.

Table 3.2 Dimensions, units, and names of quantities (continued).

Quantity	[M]	[L]	[T]	[I]	[θ]	[N]	[J]	Units	Name
magnetic permeability	1	1	–2	–2	0	0	0	Ω s m^{-1}	
mass flow rate	1	0	–1	0	0	0	0	kg s^{-1}	
molality	–1	0	0	0	0	1	0	mol kg^{-1}	
molarity	0	–3	0	0	0	1	0	mol m^{-3}	
molar internal energy	1	2	–2	0	0	–1	0	J mol^{-1}	
molar mass	1	0	0	0	0	–1	0	kg mol^{-1}	
molar volume	0	3	0	0	0	–1	0	m^3 mol^{-1}	
moment of force	1	2	–2	0	0	0	0	N m	
moment of inertia	1	2	0	0	0	0	0	kg m^2	
momentum	1	1	–1	0	0	0	0	kg m s^{-1}	
period	0	0	1	0	0	0	0	s	
permittivity	–1	–3	4	2	0	0	0	F m^{-1}	
power	1	2	–3	0	0	0	0	J s^{-1} = W	watt
pressure	1	–1	–2	0	0	0	0	N m^{-2} = Pa	pascal
quantity of light	0	0	1	0	0	0	1	lm s	
radiant intensity	1	2	–3	0	0	0	0	W sr^{-1}	
relative density	0	0	0	0	0	0	0	(no unit)	
rotational frequency	0	0	–1	0	0	0	0	s^{-1}	
second moment of area	0	4	0	0	0	0	0	m^4	
specific heat capacity	0	2	–2	0	–1	0	0	J(kg K)$^{-1}$	
specific latent heat	0	2	–2	0	0	0	0	J kg^{-1}	
specific volume	–1	3	0	0	0	0	0	m^3 kg^{-1}	
speed	0	1	–1	0	0	0	0	m s^{-1}	
stress	1	–1	–2	0	0	0	0	N m^{-2} = Pa	pascal
surface tension	1	0	–2	0	0	0	0	N m^{-1}	
torque	1	2	–2	0	0	0	0	N m	
viscosity (dynamic)	1	–1	–1	0	0	0	0	Pa s	
viscosity (kinetic)	0	2	–1	0	0	0	0	m^2 s^{-1}	
volume [†]	0	3	0	0	0	0	0	m^3	
volume flow rate	0	3	–1	0	0	0	0	m^3 s^{-1}	
wavelength	0	1	0	0	0	0	0	m	
wave number	0	–1	0	0	0	0	0	m^{-1}	
work	1	2	–2	0	0	0	0	N m = J	joule

[†] The litre (spelt liter in the United States of America) is the *capacity* (vs. *cubic*) unit of volume. Its symbol (letter l) may be confused with digit one (1) in printed texts so that it was decided in 1979 that capital L could be used as well; 1 m^3 = 1000 L.

The very concept of *dimension* leads to immediate applications in physics and ecology. In physics, for example, one can easily demonstrate that the first derivative of distance with respect to time is a speed:

$$\text{dimensions of } \frac{dl}{dt}: \quad \left[\frac{\mathrm{L}}{\mathrm{T}}\right] = [\mathrm{LT}^{-1}], \text{ i.e. speed.}$$

Similarly, it can be shown that the second derivative is an acceleration:

$$\text{dimensions of } \frac{d^2 l}{dt^2} = \frac{d}{dt}\left(\frac{dl}{dt}\right): \quad \left[\frac{\mathrm{L}}{\mathrm{TT}}\right] = [\mathrm{LT}^{-2}], \text{ i.e. acceleration.}$$

Note that *italic*s are used for *quantity symbols* such as length (l), mass (m), time (t), area (A), and so on. This distinguishes them from *unit symbols* (roman type; Tables 3.1 and 3.2), and *dimension symbols* (roman capitals in brackets; Table 3.1).

Ecological application 3.1

Platt (1969) studied the efficiency of *primary (phytoplankton) production* in the *aquatic environment*. Primary production is generally determined at different depths in the water column, so that it is difficult to compare values observed under different conditions. The solution to this problem consists in finding a method to standardize the values, for example by transforming field estimates of *primary production* into values of *energy efficiency*. Such a transformation would eliminate the effect on production of solar irradiance at different locations and different depths. Primary production at a given depth $P(z)$ may be expressed in J m^{-3} s^{-1} [ML^{-1} T^{-3}], while irradiance at the same depth $E(z)$ is in J m^{-2} s^{-1} [MT^{-3}] (energy units).

The dimension of the ratio $P(z)/E(z)$, which defines the energy efficiency of primary production, is thus [L^{-1}]. Another property determined in the water column, which also has dimension [L^{-1}], is the *attenuation* of diffuse light as a function of depth. The *coefficient of diffuse light attenuation* (α) is defined as:

$$E(z_2) = E(z_1)\, e^{-\alpha(z_2 - z_1)}$$

where $E(z_2)$ and $E(z_1)$ are irradiances at depths z_2 and z_1, respectively. Given the fact that an exponent is, by definition, dimensionless, the dimension of α must be [L^{-1}] since that of depth z is [L].

Based on the dimensional similarity between efficiency and attenuation, and considering the physical aspects of light attenuation in the water column, Platt partitioned the attenuation coefficient (α) into physical (k_p) and biological (k_b) components, i.e. $\alpha = k_p + k_b$. The *biological attenuation coefficient* k_p may be used to estimate the attenuation of light caused by photosynthetic processes. In the same paper and in further publications by Platt & Subba Rao (1970) and Legendre (1971), it was shown that there exists a correlation in the marine environment between k_b and the concentration of chlorophyll a. The above papers used the calorie as unit of energy but, according to the SI standard, this unit should no longer be used. Coherency requires here that primary production be expressed in J m^{-3} s^{-1} and irradiance in J m^{-2} s^{-1} (or W m^{-2}).

This example illustrates how a simple reflection, based on dimensions, led to an interesting development in the field of ecology.

It is therefore useful to think in terms of *dimensions* when dealing with ecological equations that contain physical *quantities*. Even if this habit is worth cultivating, it would not however, in and of itself, justify an entire chapter in the present book. So, let us move forward in the study of dimensional analysis.

3.2 Fundamental principles and the Pi theorem

It was shown in the previous section that going from one quantity to another is generally done by multiplying or dividing quantities characterized by *different dimensions*. In contrast, additions and subtractions can only be performed on quantities having the *same dimensions* — hence the fundamental principle of *dimensional homogeneity*. Any equation of the general form

Dimensional homogeneity

$$a + b + c + \ldots = g + h + \ldots$$

is dimensionally homogeneous if and only if all quantities $a, b, c, \ldots g, h, \ldots$ have the *same dimensions*. This property applies to all equations of a *theoretical* nature, but it does not necessarily apply to those derived *empirically*. Readers must be aware that dimensional analysis only deals with dimensionally homogeneous equations. In animal ecology, for example, the basic equation for energy budgets is:

$$dW/dt = R - T \qquad \textbf{(3.1)}$$

where W is the mass of an animal, R its food ration, and T its metabolic expenditure rate (oxygen consumption). This equation, which describes growth dW/dt as a function of ration R and metabolic rate T, is dimensionally homogeneous. The rate of oxygen consumption T is expressed as mass per unit time, its dimensions thus being $[MT^{-1}]$, as those of food ration R. The dimensions of dW/dt are also clearly $[MT^{-1}]$. This same equation will be used in Ecological applications 3.2e and 3.3b, together with other ecological equations — all of which are dimensionally homogeneous.

In dimensional analysis, the correct identification of quantities to be included in a given equation is much more important than the exact form of the equation. Researchers using dimensional analysis must therefore have prior knowledge of the phenomenon under study, in order to identify the pertinent *dimensional variables* and *constants*. On the one hand, missing key quantities could lead to incomplete or incorrect results, or even to a deadlock. On the other hand, including unnecessary terms could overburden the solution needlessly. Hence, dimensional analysis cannot be conducted without first considering the ecological bases of the problem. A simple example, taken from hydrodynamics, will illustrate the dimensional method.

The question considered here relates to the work of many ecologists in aquatic environments, i.e. estimating the drag experienced by an object immersed in a current. Ecologists who moor current meters or other probes must consider the drag, lest the equipment might be carried away. To simplify the problem, one assumes that the immersed object is a smooth sphere and that the *velocity* of the current V is constant. The drag *force* F is then a function of: the *velocity* (V), the *diameter* of the sphere (D), the *density* of water (ρ), and its *dynamic viscosity* (η). The simplest equation relating these five quantities is:

$$F = f(V, D, \rho, \eta) \tag{3.2}$$

At first sight, nothing specifies the nature of the dependency of F on V, D, ρ, and η, except that such a dependency exists. Dimensional analysis allows one to find the form of the equation that relates F to the variables identified as governing the drag.

Dimensionless product

A number of variables are regularly encountered in hydrodynamics problems, i.e. F, V, L, ρ, η, to which one must also add g, the acceleration due to gravity. Some of these variables may be combined to form *dimensionless products*. Specialists of hydrodynamics have given names to some often-used dimensionless products:

$$\textit{Reynolds number}: Re = \frac{VL\rho}{\eta} = \frac{[\mathrm{LT^{-1}}]\,[\mathrm{L}]\,[\mathrm{ML^{-3}}]}{[\mathrm{ML^{-1}T^{-1}}]} = \frac{[\mathrm{ML^{-1}T^{-1}}]}{[\mathrm{ML^{-1}T^{-1}}]} = [1] \tag{3.3}$$

$$\textit{Newton number}: Ne = \frac{F}{\rho L^2 V^2} = \frac{[\mathrm{MLT^{-2}}]}{[\mathrm{ML^{-3}}]\,[\mathrm{L^2}]\,[\mathrm{L^2T^{-2}}]} = \frac{[\mathrm{MLT^{-2}}]}{[\mathrm{MLT^{-2}}]} = [1] \tag{3.4}$$

$$\textit{Froude number}: Fr = \frac{V^2}{Lg} = \frac{[\mathrm{L^2T^{-2}}]}{[\mathrm{L}]\,[\mathrm{T^{-2}}]} = \frac{[\mathrm{L^2T^{-2}}]}{[\mathrm{L^2T^{-2}}]} = [1] \tag{3.5}$$

Each of the above *products* is clearly *dimensionless*. It should also be noted that each product of this set is *independent* of the others, since each contains one exclusive variable, i.e. η for *Re*, F for *Ne*, and g for *Fr*. Finally, any other dimensionless product of these same variables would *inevitably* be a product of powers of dimensionless products from the above set. The three dimensionless products thus form a *complete set* of dimensionless products for variables F, V, L, ρ, η and g. It would obviously be possible to form other complete sets of dimensionless products using these same variables, by combining them differently.

The first important concept to remember is that of *dimensionless product*. This concept leads to the *sole* theorem of dimensional analysis, the Π theorem, which is also known as the Buckingham theorem.

Given the fundamental principle of dimensional homogeneity (see above), it follows that any equation that combines dimensionless products is dimensionally

homogeneous. Thus, a *sufficient* condition for an equation to be dimensionally homogeneous is that it could be reduced to an equation combining dimensionless products. Indeed, any equation that can be reduced to an equation made of dimensionless products is dimensionally homogeneous. Buckingham (1914) did show that this condition is not only *sufficient* but also *necessary*. This leads to the Π *(Pi) theorem* (the capital Greek letter Π is the mathematical symbol for product):

Π theorem

If an equation is dimensionally homogeneous, it can be reduced to a relationship among the members of a complete set of dimensionless products.

This theorem alone summarizes the whole theory of dimensional analysis.

The power of the Π theorem is illustrated by the solution of the drag problem, introduced above. Equation 3.2 is, by definition, dimensionally homogeneous:

$$F = f(V, D, \rho, \eta)$$

It may be rewritten as:

$$f(F, V, D, \rho, \eta) = 0 \qquad \textbf{(3.6)}$$

The complete set of dimensionless products of the five variables F, V, D, ρ, η contains two products, i.e. the Reynolds (*Re*) and Newton (*Ne*) numbers (D being a length, it is a quantity of type L). Hence, eq. 3.6 may be rewritten as a relation between the members of this complete set of dimensionless products (Π theorem):

$$Ne = f(Re)$$

$$\frac{F}{\rho V^2 D^2} = f(Re) \qquad \textbf{(3.7)}$$

In this equation, the function f is, for the time being, unknown, except that it depends on the sole dimensionless variable *Re*.

The projected area (A) of a sphere is:

$$A = \pi\,(D/2)^2 = (1/4)\,\pi\,D^2, \text{ so that } D^2 = 4A/\pi$$

which allows one to rewrite eq. 3.7 as:

$$\frac{F}{\rho V^2 \dfrac{4A}{\pi}} = f(Re)$$

$$\frac{F}{\rho V^2 A} = \frac{1}{2}\left(\frac{8}{\pi}\right) f(Re)$$

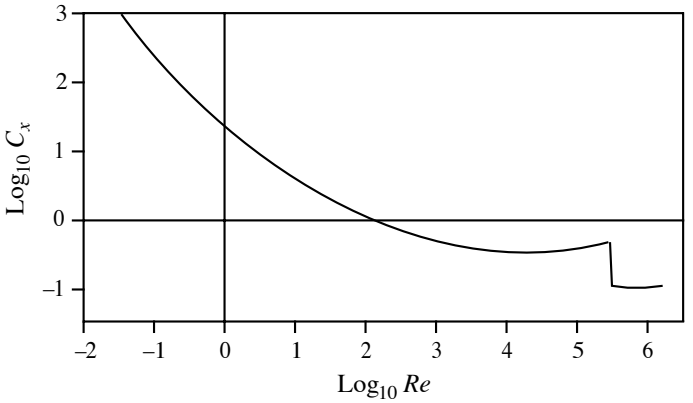

Figure 3.1 Drag coefficient on smooth spheres. Adapted from Eisner (1931).

In hydrodynamics, the term $(8/\pi)\,f\,(Re)$ is called the *drag coefficient* and is represented by $C_{x,}$ so that the drag exerted on a sphere is:

$$F = (1/2)\ C_x \rho V^2 A, \text{ where } C_x = (8/\pi)\,f\,(Re) \qquad \textbf{(3.8)}$$

Since C_x is a function of the sole dimensionless coefficient *Re,* the problem is resolved by determining, in the laboratory, the experimental curve of C_x as a function of *Re*. This curve will be valid for any density (ρ) or dynamic viscosity (η) of any fluid under consideration (the same curve can thus be used for water, air, etc.) and for objects of any size, or any flow speed. The curve may thus be determined by researchers under the most suitable conditions, i.e. choosing fluids and flow speeds that are most convenient for laboratory work. As a matter of fact, this curve is already known (Fig. 3.1).

Two important properties follow from the above example.

(1) First, data to build a *dimensionless graph* should be obtained under the most convenient conditions. For example, determining C_x for a sphere of diameter 3.48 m immersed in air at 14.4°C with a velocity of 15.24 m s^{-1} would be difficult and costly. In contrast, it would be much easier, in most laboratories, to determine C_x by using a sphere of diameter 0.61 m in water at 14.4°C with a speed of 5.79 m s^{-1}. In both cases, *Re* is the same so that the measured value of C_x is the same. This first property is the basis for *model testing* in engineering (Section 3.4), the sphere in air being here the *prototype* and that in water, the *model*.

Chart

(2) The dimensionless graph of Fig. 3.1 contains much more information than a set of *charts* depicting the function of the 4 variables. In a chart (Fig. 3.2), *a function of*

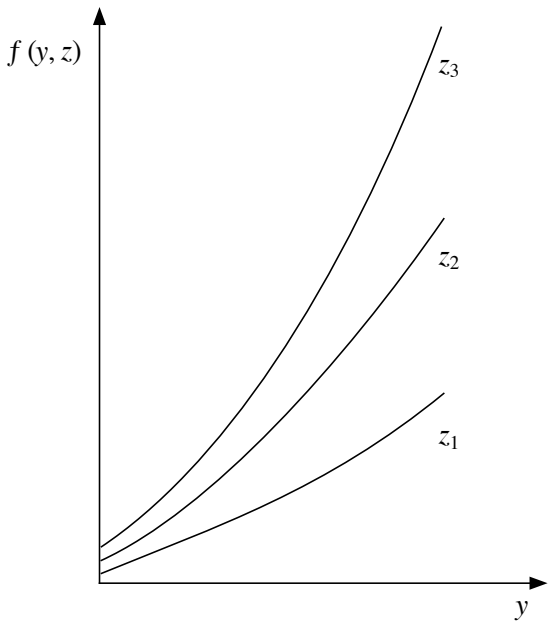

Figure 3.2 Chart representing a function of two variables. One curve is required for each value of the second variable ($z_1, z_2, z_3, \ldots$)

two variables is represented by a *family of curves*, one curve being required for each value of the second variable. A *function of three variables* would be represented by a *set of sets of charts*. Hence, for four variables and assuming that there were only five values measured per variable, a total of 625 experimental points would be required, i.e. five sets of five charts each. With 25 times fewer experimental points, one can easily obtain a dimensionless graph (e.g. Fig. 3.1) which is both more exact and much more convenient.

The above physical example illustrated the great simplicity and remarkable power of dimensional analysis. Let us now examine examples from ecology.

Ecological application 3.2a

This first example belongs to the disciplines of ecology and physiology, since it concerns the dimensions of animals and their muscular dynamics. Hill (1950) compared different cetaceans, as a set of similar animals which differ in size. All these cetaceans (porpoises, dolphins, and whales), with a 5000-fold mass range, travel at high speed (ca. 7.5 m s^{-1}) which they can maintain for a long time. Table 3.3 compares the two extreme cases of the common dolphin (*Delphinus delphis*) and the blue whale *(Balaenoptera musculus*).

Table 3.3 Body characteristics of two cetaceans.

	Common dolphin	Blue whale
Maximum length (m)	2.4	30
Maximum mass (10^3 kg)	0.14	150
Mass/length^3	0.01	0.006
Area/length^2	0.45	0.40

Since these two animals can maintain a cruising speed of ca. 7.5 m s^{-1} for long periods, one may assume that they are then in a physiological steady state. The question is: how is it possible for two species with such different sizes to cruise at the same speed?

To answer this question, one must first consider the drag (F) on a streamlined body moving in a fluid. The equation is similar to eq. 3.8, except that the drag coefficient C_x is replaced here by the *friction coefficient* C_f:

$$F = 0.5\, C_f \rho V^2 A$$

where ρ is the *density* of the fluid, V the *velocity* of the body, and A its *total surface area*. For *laminar* flow, $C_f \approx 1.33\, Re^{-1/2}$ whereas, for *turbulent* flow, $C_f \approx 0.455\, (\log_{10} Re)^{-2.58}$, Re being the *Reynolds number*. Low values of Re correspond to laminar flow, where resistance to motion is relatively weak, whereas high values of Re are associated with turbulent flow, which creates stronger resistance to motion. Normally, for a streamlined body, the flow is laminar over the front portion only and is turbulent towards the back.

The *power* developed by the muscles of moving cetaceans is calculated in three steps.

• Calculation of Re, for the animal under study:

$$Re \approx 7 \times 10^5\ (\text{s m}^{-2})\ VL, \text{ in sea water at } 5°\text{C}$$

• Calculation of drag (F):

$$F = 0.5\, C_f \rho V^2 A$$

C_f being computed from Re, using the equation for either laminar or turbulent flow.

• Calculation of power (P) developed during motion:

$$P = FV$$

For the purpose of the calculation, consider (1) a dolphin with a length of 2 m, weighing 80 kg, whose surface area is 1.75 m^2 and (2) a whale 25 m long, with a mass of 100 t and surface area of 250 m^2.

(1) The value of *Re* for a dolphin moving at 7.5 m s^{-1} is of the order of 10^7, which seems to indicate highly turbulent flow. In the case of *laminar* flow,

$$C_f = 1.33 \times (10^7)^{-1/2} = 4.2 \times 10^{-4}$$

and, for *turbulent* flow,

$$C_f = 0.455\ (\log_{10} 10^7)^{-2.58} = 3 \times 10^{-3}$$

The drag (F) corresponding to these two flow regimes is:

$$F \text{ (laminar)} = 0.5\ (4.2 \times 10^{-4})\ (1028 \text{ kg m}^{-3})\ (7.5 \text{ m s}^{-1})^2\ (1.75 \text{ m}^2) = 22 \text{ N}$$

$$F \text{ (turbulent)} = 0.5\ (3 \times 10^{-3})\ (1028 \text{ kg m}^{-3})\ (7.5 \text{ m s}^{-1})^2\ (1.75 \text{ m}^2) = 155 \text{ N}$$

The *power* ($P = F \times 7.5$ m s^{-1}) that a dolphin should develop, if its motion resulted in perfectly *laminar* flow, would be 165 W and, for *turbulent* flow, 1165 W. Since the size of a dolphin is of the same order as that of a man, it is reasonable to assume that the power it can develop under normal conditions is not higher than that of an athlete, i.e. a *maximum power* of 260 W. It follows that the flow must be laminar for the 9/10 front portion of the dolphin's body, with the rear 1/10 being perhaps turbulent. This conclusion is consistent with observations made in nature on dolphins. It is assumed that the absence of turbulence along the front part of the dolphin's body comes from the fact that the animal only uses its rear section for propulsion.

(2) The blue whale also swims at 7.5 m s^{-1}, its *Re* being ca. 12.5×10^7 which corresponds to a turbulent flow regime. A *laminar* flow would lead to a value

$$C_f = 1.33 \times (12.5 \text{ x } 10^7)^{-1/2} = 1.2 \times 10^{-4}$$

and a *turbulent* flow to

$$C_f = 0.455\ (\log_{10} 12.5 \times 10^7)^{-2.58} = 2.1 \text{ x } 10^{-3}$$

The corresponding drag (F) would be:

$$F \text{ (laminar)} = 0.5\ (1.2 \times 10^{-4})\ (1028 \text{ kg m}^{-3})\ (7.5 \text{ m s}^{-1})^2\ (250 \text{ m}^2) = 745 \text{ N}$$

$$F \text{ (turbulent)} = 0.5\ (2.1 \times 10^{-3})\ (1028 \text{ kg m}^{-3})\ (7.5 \text{ m s}^{-1})^2\ (250 \text{ m}^2) = 13 \text{ kN}.$$

The *power* a whale should develop, if its motion at 7.5 m s^{-1} was accompanied by *laminar* flow, would be 5.6 kW and, in the case of *turbulent* flow, 100 kW. The maximum power developed by a 80 kg dolphin was estimated to be 260 W so that, if the maximum power of an animal was proportional to its mass, a 10^5 kg whale should be able to develop 325 kW. One should, however, take into account the fact that the available energy depends on blood flow. Since cardiac rate is proportional to $(\text{mass})^{-0.27}$, the heart of a whale beats at a rate $(100/0.08)^{-0.27} \approx 1/7$ that of a dolphin. The *maximum power* of a whale is thus ca. 1/7 of 325 kW, i.e. 46.5 kW. This leads to the conclusion that laminar flow takes place along the 2/3 front portion of the animal and that only the 1/3 rear part can sustain turbulent flow.

Ecological application 3.2b

A second study, taken from the same paper as the previous application (Hill, 1950), deals with land animals. It has been observed that several terrestrial mammals run more or less at the same speed and jump approximately the same height, even if their sizes are very different. Table 3.4 gives some approximate maximal values. The question is to explain the similarities observed between the performances of animals with such different sizes.

Table 3.4 Performances (maximal values) of five mammals.

	Running speed (m s^{-1})	Height of jump (m)
Man	12	2
Horse	20	2
Greyhound (25 kg)	18	—
Hare	20	1.5
Deer	15	2.5

One of the explanations proposed by the author involves a relatively simple dimensional argument. The strength of tissues in the bodies of animals cannot be exceeded, during athletic performances, without great risk. For two differently sized animals, consider a pair of systems with lengths l_1 and l_2, respectively, carrying out similar movements within times t_1 and t_2, respectively. The stress at any point in these systems has dimensions $[ML^{-1}T^{-2}]$, which corresponds to the product of density $[ML^{-3}]$ with the square of speed $[L^2T^{-2}]$.

Assuming that the densities of systems are the same for the two species (i.e. $m_1 l_1^{-3} = m_2 l_2^{-3}$, which is reasonable, since the densities of bones, muscles, etc. are similar for all mammals), the stresses at corresponding points of the systems are in the ratio $(l_1^2 t_1^{-2}) : (l_2^2 t_2^{-2})$. If the two systems operate at speeds such that the stresses are the same at corresponding points, it follows that $(l_1 t_1^{-1}) = (l_2 t_2^{-1})$. In other words, the speed is the same at corresponding points of the two systems. It is therefore the strength of their tissues which would explain why athletic animals of very different sizes have the same upper limits for running speeds and jumping heights.

It is interesting to note that, over the years, the topic of maximal running speed of terrestrial mammals has been the subject of many papers, which considered at least four competing theories. These include the theory of geometric similarity, briefly explained in this example, and theories that predict an increase of maximum running speed with body mass. These are summarized in the introduction of a paper by Garland (1983), where maximum running speeds for 106 species of terrestrial mammals are analysed. The study led to several interesting conclusions, including that, even if maximal running speed is mass-independent within some mammalian orders, this is not the case when species from different orders are put together; there is then a tendency for running speed to increase with mass, up to an optimal mass of ca. 120 kg. This is quite paradoxical since, when considering mammals in general, limb bone proportions do scale consistently with geometric similarity. The author refers to Günther's (1975, p. 672) conclusion that "no single similarity criterion can provide a satisfactory quantitative explanation for every single function of an organism that can be submitted to dimensional analysis".

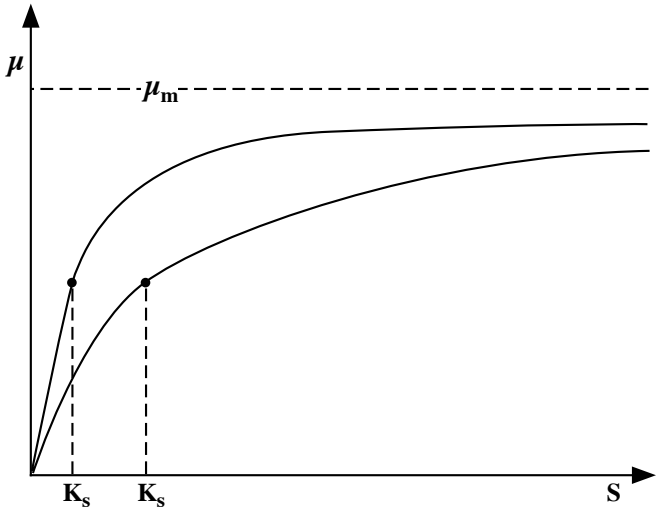

Figure 3.3 Illustration of the Michaelis-Menten equation, showing the role of parameter K_s. In the curve with higher K_s, μ approaches the asymptote μ_m more slowly than in the other curve.

Ecological application 3.2c

An example from aquatic ecology (Platt & Subba Rao, 1973) illustrates the use of dimensionless graphs. The dependence of phytoplankton growth on a given nutrient is often described by means of the Michaelis-Menten equation, borrowed from enzymology. In this equation, the *growth rate* (μ), with dimension $[T^{-1}]$, is a function of the *maximum specific* growth rate (μ_m), the *concentration* (S) of the nutrient, and the *concentration* (K_s) of nutrient at which the growth rate $\mu = 1/2\ \mu_m$:

$$\mu = \frac{1}{B}\frac{dB}{dt} = \frac{\mu_m S}{K_s + S}$$

$$[T^{-1}] = \frac{[1]}{[ML^{-3}]}\frac{[ML^{-3}]}{[T]} = \frac{[T^{-1}]\,[ML^{-3}]}{[ML^{-3}] + [ML^{-3}]}$$

where B is the concentration of phytoplankton *biomass*. This equation is that of a rectangular hyperbola, where K_s determines how fast the asymptote μ_m is approached. When K_s is high, μ approaches the asymptote μ_m slowly, which indicates a weak dependence of μ on S in the unsaturated part of the curve (Fig. 3.3).

In order to compare the effects of two different variables on phytoplankton growth, the authors defined a new entity $S_* = S/K_s$. Since this entity is dimensionless, the abscissa of the

graph $\mu(S_*)$ as a function of S_* is dimensionless; $\mu(S_*)$ stands for the specific growth rate, normalized to S_*. The Michaelis-Menten equation is thus rewritten as:

$$\mu(S_*) = \frac{\mu_m S_*}{(1 + S_*)}$$

Hence, the strength of the dependence of μ on S_* is:

$$\frac{d\mu(S_*)}{dS_*} = \frac{d}{dS_*}\left(\frac{\mu_m S_*}{1 + S_*}\right) = \frac{\mu_m}{(1 + S_*)^2}$$

Using this expression, it is possible to determine the relative strength of the dependence of μ on two different variables (i and j):

$$\xi(i,j) = \frac{d\mu(S^i_*)/dS^i_*}{d\mu(S^j_*)/dS^j_*} = \frac{\mu^i_m}{\mu^j_m}\left[\frac{(1 + S^j_*)^2}{(1 + S^i_*)^2}\right]$$

Under conditions that do not limit phytoplankton growth, the maximum specific growth rate is the same for the two variables, i.e. $\mu^i_m = \mu^j_m$. In such a case, the dependence of μ on the two variables becomes:

$$\xi(i,j) = (1 + S^j_*)^2 / (1 + S^i_*)^2$$

This dimensionless approach makes it possible to compare the effects of different variables on phytoplankton growth, regardless of the dimensions of these variables. Using the above equation, one could assess, for example, the relative importance of irradiance (µmol photons $m^{-2}s^{-1}$, also denoted µEinstein $m^{-2}s^{-1}$) [$NL^{-2}T^{-1}$] and of a nutrient [ML^{-3}] for phytoplankton growth.

Characteristic value

The method described here is actually of general interest in ecology, since it shows how to approach a problem involving several variables with no common measure. In all cases, it is recommended to transform the *dimensional* variables into *dimensionless ones*. The most obvious transformation, proposed by Platt & Subba Rao (1973), consists in dividing each variable by a *characteristic value*, which has the same dimensions as the variable itself. In the case of the Michaelis-Menten equation, the characteristic value is K_s, which has the same dimensions as S. This elegant and efficient approach is also used in parametric statistics, where variables are transformed through division by their standard deviations. For this and other transformations, see Section 1.5. The approach which consists in dividing an ecologically interesting variable by another variable with the same dimensions, so as to create a dimensionless variable, is known as "scaling" (e.g. in Schneider, 1994). Scaling analysis has been used, for example, in coral reef studies (Hatcher and Firth, 1985; Hatcher *et al*., 1987) and by Murray & Jumars (2002) to model steady-state diffusive uptake of nutrients by a spherical attached bacterium (study summarized by Legendre, 2004: 81-83).

The following example illustrates some basic characteristics of dimensional analysis. It also stresses a major weakness of the method, of which ecologists should be aware.

Ecological application 3.2d

The study discussed here (Kierstead & Slobodkin, 1953) did not use dimensional analysis, but it provides material to which the method may usefully be applied. The authors did develop their theory for phytoplankton, but it is general enough to be used with several other types of organisms. Given a water mass containing a growing population, which loses individuals (e.g. phytoplankton cells) by diffusion and regenerates itself by multiplication, the problem is to define the minimum size of the water mass below which the growth of the population is no longer possible.

The problem is simplified by assuming that: (1) the *diffusion* (D) of organisms remains constant within the water mass, but is very large outside where the population cannot maintain itself, and (2) the water mass is *one-dimensional* (long and narrow), so that the *concentration* (c) of organisms is a function of the *position* (x) along the axis of the water mass. The equation describing the growth of the population is thus:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + Kc$$

where K is the growth rate. On the right-hand side of the equation, the first term accounts for diffusion, while the second represents linear growth. A complicated algebraic solution led the authors to define a critical *length* (L_c) for the water mass, under which the population would decrease and above which it could increase:

$$L_c = \pi\sqrt{D/K}$$

It must be noted that this equation is analogous to that of the critical mass in a nuclear reactor. Associated with this critical length is a *characteristic time* (t) of the process, after which the critical length L_c becomes operative:

$$t = L_c^2/(8\pi^2 D)$$

The above results are those given in the paper of Kierstead and Slobodkin. The same problem is now approached by means of dimensional analysis, which will allow one to compare the *dimensional solution* of Platt (1981) to the algebraic solution of Kierstead and Slobodkin. In order to approach the question from a dimensional point of view, the dimensions of variables in the problem must first be specified:

x:	[L]	K:	$[T^{-1}]$
t:	[T]	D:	$[L^2T^{-1}]$

The only dimensions that are not immediately evident are those of D, but these can easily be found using the principle of dimensional homogeneity of theoretical equations.

The equation of Kierstead & Slobodkin involves three variables (c, t, x) and two constants (D, K). According to the general method developed in the previous ecological application, the variables are first *transformed* to dimensionless forms, through division by suitable *characteristic values*. *Dimensionless variables* C, T and X are defined using *characteristic values* c_*, t_* and x_*:

$$C = c/c_* \qquad T = t/t_* \qquad X = x/x_*$$

hence

$$c = C\,c_* \qquad t = T\,t_* \qquad x = X\,x_*$$

Substitution of these values in the equation gives:

$$\frac{c_* \partial C}{t_* \partial T} = D \frac{c_* \partial^2 C}{x_*^2 \partial X^2} + K c_* C$$

The next step is to make all terms in the equation dimensionless, by multiplying each one by x_*^2 and dividing it by D, after eliminating from all terms the common constant c_* :

$$\left[\frac{x_*^2}{D t_*}\right] \frac{\partial C}{\partial T} = \frac{\partial^2 C}{\partial X^2} + \left[\frac{K x_*^2}{D}\right] C$$

The resulting equation thus contains three *dimensionless variables* (C, T and X) and two *dimensionless products* (in brackets).

Since the dimensions of the two products are [1], these may be transformed to isolate the characteristic values x_* and t_* :

$$\text{since } \left[\frac{x_*^2}{D t_*}\right] = [1], \text{ it follows that } [\, t_* \,] = \left[\frac{x_*^2}{D}\right]$$

$$\text{since } \left[\frac{K x_*^2}{D}\right] = [1], \text{ it follows that } [\, x_*^2 \,] = \left[\frac{D}{K}\right] \text{ and thus } [\, x_* \,] = \left[\frac{D}{K}\right]^{1/2}$$

Using these relationships, the following proportionalities are obtained:

$$x_* \propto \sqrt{D/K} \text{ and } t_* \propto x_*^2 / D$$

Dimensional analysis thus easily led to the same results as those obtained by Kierstead and Slobodkin (1953), reported above, except for the constant factors π and $8\pi^2$. This same example will be reconsidered in the next section (Ecological application 3.3a), where the two dimensionless products will be calculated directly.

The above example illustrates the fact that *dimensional analysis cannot generate dimensionless constants*, which is a limit of the method that must be kept in mind. Thus, in order to take advantage of the power of dimensional analysis, one must give up some precision. It is obvious that such a simple method as dimensional analysis cannot produce the same detailed results as complex algebraic developments. As mentioned above (Section 3.0), dimensional analysis deals with *general forms* of equations. Yet, starting from simple concepts, one can progress quite far into complex problems, but the final solution is only partial. As noted by Langhaar (1951): "The generality of the method is both its strength and its weakness. With little effort, a partial solution to nearly any problem is obtained. On the other hand, a complete solution is not obtained."

Ecological application 3.2e

It often happens that ecologists must synthesize published data on a given subject, either as a starting point for new research, or to resolve a problem using existing knowledge, or else as a

basis for a new theoretical perspective. This is nowadays more necessary than ever, because of the explosion of ecological information. However, such syntheses are confronted to a real difficulty, which is the fact that available data are often very diversified, and must thus be unified before being used. Paloheimo & Dickie (1965) met this problem when they synthesized the mass of information available in the literature on the growth of fish as a function of food intake. As in the previous application, the authors did not themselves use dimensional analysis in their work. The dimensional solution discussed here is modified from Platt (1981).

The metabolism of fish may be described using the following relationship:

$$T = \alpha W^{\gamma}$$

where T is the rate of *oxygen consumption*, α specifies the level of *metabolic expenditure* per unit time, W is the *mass* of the fish, and γ specifies the rate of *change of metabolism* with body mass. Growth is expressed as a function of food ration (R), by means of the following equation:

$$\frac{dW}{dt} = R\left[e^{-(a+bR)}\right]$$

which shows that growth efficiency decreases by a constant fraction e^{-b} for each unit increase in the amount of food consumed per unit time. The value of R at maximum growth is determined, as usual, by setting the partial derivative equal to 0:

$$\frac{\partial}{\partial R}\left(\frac{dW}{dt}\right) = (1 - bR)\, e^{-(a+bR)} = 0$$

Growth is thus maximum when $bR = 1$.

The basic equation for the energy budget (eq. 3.1) is:

$$\frac{dW}{dt} = R - T$$

so that

$$T = R - \frac{dW}{dt}$$

Replacing, in this last equation, dW/dt by its expression in the second equation, above, and isolating R, one obtains:

$$T = R\left[1 - e^{-(a+br)}\right]$$

Then, replacing T by its expression in the first equation leads to:

$$\alpha W^{\gamma} = R\left[1 - e^{-(a+br)}\right]$$

which is a general equation for energy budgets. This equation may be used to calculate, for any fish of mass W, the ration R required to maintain a given metabolic level. Furthermore, with an increase in ration, the term $[1 - e^{-(a+bR)}]$ tends towards 1, which indicates that the metabolism then approaches R. In other words, growth decreases at high values of R.

Values for coefficient b and food intake found in the literature are quite variable. It was shown above that the product bR determines growth. Paloheimo & Dickie therefore suggested to standardize the relationship between growth and ration in terms of bR.

Since growth is maximum when $bR = 1$, the *ration* can be brought to a common measure by expressing it in units of $1/b$. On this new *scale*, the ration (r) is defined as:

$$r = bR$$

When growth is maximum, $bR = 1$, so that $R = 1/b$. Replacing, in the general equation for the energy budget, R by $1/b$ (and bR by 1) yields:

$$\alpha W^{\gamma} = 1/b\,[1 - e^{-(a+1)}]$$

so that

$$W = \left[\frac{1 - e^{-(a+1)}}{\alpha b}\right]^{1/\gamma}$$

from which it is concluded that the *mass* should be expressed in units of $(1/\alpha b)^{1/\gamma}$ in order to bring data from the literature to a common measure. On this new *scale*, the mass (w) is defined as:

$$w = (\alpha b)^{1/\gamma}\, W$$

so that

$$\frac{w^{\gamma}}{b} = \alpha W^{\gamma} = T$$

Using the scaled ration (r) and mass (w), the general equation for energy budgets may be rewritten as:

$$\frac{w^{\gamma}}{b} = \frac{r}{b}\,[1 - e^{-(a+r)}]$$

and finally

$$w^{\gamma} = r\,[1 - e^{-(a+r)}]$$

In this last equation, the use of r and w brings to a common measure the highly variable values of R and W, which are available in the literature for different species or for different groups within a given fish species.

These same results could have been obtained much more easily using dimensional analysis. As with all problems of the kind, it is essential, first of all, to identify the dimensions of variables involved in the problem. The first two equations are used to identify the dimensions of all variables in the study:

$$T = \alpha W^{\gamma}$$

$$[\mathrm{MT}^{-1}] = [\mathrm{M}^{(1-\gamma)}\mathrm{T}^{-1}]\,[\mathrm{M}^{\gamma}]$$

$$\frac{dW}{dt} = R\,[e^{-(a+bR)}]$$

$$[\mathrm{MT}^{-1}] = [\mathrm{MT}^{-1}]\,[1]^{\,[1]\,+\,[\mathrm{M}^{-1}\mathrm{T}]\,[\mathrm{MT}^{-1}]}$$

The dimensions of α, which were not immediately obvious, are determined using the principle of dimensional homogeneity (i.e. same dimensions on the two sides of the equation). The dimensions of a and b are also found by applying the principle of dimensional homogeneity, taking into account the fact that an exponent is by definition dimensionless.

The problem is then to define *characteristic values* (or, more appropriately, *scale factors*) so as to obtain dimensionless ration (r), mass (w), and time (τ). Obviously, these scale factors must contain the two dimensional parameters of the above equations, α and b.

Because the product bR is dimensionless, the scale factor r for ration is:

$$r = bR$$

The cases of w and τ require the calculation of unknown exponents. These are easily found by dimensional analysis. In order to do so, unknown exponents y and z are assigned to α and b, and these unknowns are solved using the principle of dimensional homogeneity:

Calculation of w:

$$[w] = [1] = [\alpha]^y\,[b]^z\,[W]$$

$$[W]^{-1} = [\alpha]^y\,[b]^z$$

$$[M^{-1}T^0] = [M^{(1-\gamma)}\,T^{-1}]^y\,[M^{-1}T]^z = [M^{y(1-\gamma)-z}\,T^{-y+z}]$$

so that $y(1-\gamma) - z = -1$

and $-y + z = 0$

hence $y = 1/\gamma = z$

Consequently, the scale factor w for the mass is:

$$w = (\alpha b)^{1/\gamma} W$$

Calculation of τ:

$$[\tau] = [1] = [\alpha]^y\,[b]^z\,[t]$$

$$[t]^{-1} = [\alpha]^y\,[b]^z$$

$$[M^0T^{-1}] = [M^{y(1-\gamma)-z}\,T^{-y+z}]$$

so that $y(1-\gamma) - z = 0$

and $-y + z = -1$

hence $y = 1/\gamma$ and $z = 1/\gamma - 1$

It follows that the scale factor τ for time is:

$$\tau = \alpha^{1/\gamma}\, b^{(1/\gamma - 1)} t$$

$$\tau = [(\alpha b)^{1/\gamma}/b]t$$

These scale factors can be used to compare highly diversified data. *Ration* is then expressed in units of $(1/b)$, *mass* in units of $(\alpha b)^{-1/\gamma}$, and *time* in units of $b/(\alpha b)^{-1/\gamma}$. With this approach, it is possible to conduct generalized studies on the food intake and growth of fish as a function of time.

Other applications of dimensionless products in ecology are found, for example, in Tranter & Smith (1968), Rubenstein & Koehl (1977), and Okubo (1987). The first application analyses the performance of plankton nets, the second explores the mechanisms of filter feeding by aquatic organisms, and the third examines various aspects of biofluid mechanics, including a general relationship between the Reynolds number (*Re*) and the sizes and swimming speeds of aquatic organisms from bacteria to whales. Platt (1981) provides other examples of application of dimensional analysis in

the field of biological oceanography. Legendre (2004, pp. 87-91) explains how the dimensional approach provided the main guideline to derive operational equations from a conceptual model on the fate of biogenic carbon in oceans. These equations were used by Beaugrand *et al*. (2010) to compute the effects of long-term changes in copepod biodiversity on carbon flows in the extratropical North Atlantic Ocean.

Ecological applications 3.2d and 3.2e showed that dimensional analysis may be a powerful tool in ecology. They do, however, leave potential users somewhat uncertain as to how personally apply this approach to new problems. The next section outlines a general method for solving problems of dimensional analysis, which will lead to more straightforward use of the method. It will be shown that it is not even necessary to know the basic *equations* pertaining to a problem, provided that all the pertinent *variables* are identified. The above last two examples will then be reconsidered as applications of the *systematic calculation of dimensionless products*.

3.3 The complete set of dimensionless products

As shown in the previous section, the resolution of problems using dimensional analysis involves two distinct steps: (1) the identification of variables pertinent to the phenomenon under study — these are derived from fundamental principles, for example of ecological nature — and (2) the computation of a complete set of dimensionless products. When the number of variables involved is small, complete sets of dimensionless products can be formed quite easily, as seen above. However, as the number of variables increases, this soon becomes unwieldy, so that one must proceed to a *systematic calculation of the complete set of dimensionless products*.

The physical example of the *drag on smooth spheres* (Section 3.2) will first be used to illustrate the principles of the calculation. The problem involved five variables (F, V, L, ρ, and η; see eq. 3.2), whose dimensions are written here in a *dimensional matrix*:

$$\begin{array}{c} \\ \mathrm{M} \\ \mathrm{L} \\ \mathrm{T} \end{array} \begin{array}{c} \begin{array}{ccccc} F & \eta & \rho & L & V \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -3 & 1 & 1 \\ -2 & -1 & 0 & 0 & -1 \end{bmatrix} \end{array} \qquad \textbf{(3.9)}$$

It must be kept in mind that the numbers in matrix 3.9 (i.e. dimensions) are *exponents*. The *dimensionless products* being sought are *products* of *powers* of variables in the matrix (columns). In each product, the exponents given to the variables must be such that the result is *dimensionless*.

In other words, the systematic calculation of dimensionless products consists in finding exponents x_1, x_2, x_3, x_4 and x_5 for variables F, η, ρ, L, and V, such that a product Π, of the general form

$$\Pi = F^{x_1}\eta^{x_2}\rho^{x_3}L^{x_4}V^{x_5}$$

be dimensionless. Taking into account the respective dimensions of the five variables, the general dimensions of Π are:

$$\Pi = [\mathrm{MLT^{-2}}]^{x_1}[\mathrm{ML^{-1}T^{-1}}]^{x_2}[\mathrm{ML^{-3}}]^{x_3}[\mathrm{L}]^{x_4}[\mathrm{LT^{-1}}]^{x_5}$$

$$\Pi = [\mathrm{M}^{(x_1+x_2+x_3)}\mathrm{L}^{(x_1-x_2-3x_3+x_4+x_5)}\mathrm{T}^{(-2x_1-x_2-x_5)}]$$

The exponents of dimensions [M], [L], and [T] carry exactly the same information as the dimensional matrix (eq. 3.9). These exponents could therefore have been written directly, using matrix notation:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -3 & 1 & 1 \\ -2 & -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad \textbf{(3.10)}$$

where the dimensional matrix is on the left-hand side.

Since the products Π are dimensionless, the exponent of each dimension [M], [L], and [T], respectively, must be *zero*. In follows that:

$$x_1 + x_2 + x_3 = 0$$

$$x_1 - x_2 - 3x_3 + x_4 + x_5 = 0$$

$$-2x_1 - x_2 - x_5 = 0$$

or, in matrix notation:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -3 & 1 & 1 \\ -2 & -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \mathbf{0} \qquad \textbf{(3.11)}$$

Calculation of dimensionless products Π is thus achieved by simultaneously solving three equations. However, the above system of equations is *indeterminate*, since there are only three equations for five unknowns. Arbitrary values must thus be assigned to two of the unknowns, for example x_1 and x_2. The general solution is then given in terms of x_1 and x_2. The steps are as follows:

(1) Matrix equation 3.11 is rewritten so as to isolate x_1 and x_2 together with the associated first two columns of the matrix. This operation simply involves transferring all terms in x_3, x_4 and x_5 to the right-hand side of the equation:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = - \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} \tag{3.12}$$

Note that there is now a negative sign in front of the matrix on the right-hand side. Matrix eq. 3.12 is identical to the algebraic form:

$$x_1 + x_2 = -x_3$$

$$x_1 - x_2 = 3x_3 - x_4 - x_5$$

$$-2x_1 - x_2 = x_5$$

(2) One then solves for the unknowns x_3, x_4 and x_5, using the general method of matrix inversion (Section 2.8):

$$-\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$-\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -2 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} \tag{3.13}$$

(3) The simplest approach consists in successively assigning the value 1 to each unknown while setting the other equal to 0, i.e. (1) $x_1 = 1$ and $x_2 = 0$ and (2) $x_1 = 0$ and $x_2 = 1$. It follows that the first two columns of the solution matrix are a *unit matrix*:

$$\begin{array}{c} F \;\; \eta \;\; \rho \;\; L \;\; V \\ x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; x_5 \\ \begin{matrix} \Pi_1 \\ \Pi_2 \end{matrix} \begin{bmatrix} 1 & 0 & -1 & -2 & -2 \\ 0 & 1 & -1 & -1 & -1 \end{bmatrix} \end{array} \qquad \textbf{(3.14)}$$

The dimensionless products of the *complete set* are therefore (as in Section 3.2):

$$\Pi_1 = \frac{F}{\rho L^2 V^2}$$, the *Newton number* (*Ne*; eq. 3.4)

$$\Pi_2 = \frac{\eta}{\rho L V}$$, the inverse of the *Reynolds number* ($1/Re$; eq. 3.3)

This example clearly shows that the systematic calculation of dimensionless products rests *solely* on recognizing the *variables* involved in the problem under consideration, without necessarily knowing the corresponding *equations*. The above solution, which was developed using a simple example, can be applied to all problems of dimensional analysis, since it has the following characteristics:

(1) Because the left-hand part of the solution matrix is an *identity matrix* (**I**), the dimensionless products Π are *independent* of one another. Indeed, given **I**, each product contains one variable which is not included in any other product, i.e. the first variable is only in Π_1, the second is only in Π_2, and so on.

(2) When partitioning the dimensional matrix, one must isolate *on the right-hand side* a matrix that can be *inverted*, i.e. a matrix whose determinant is non-zero.

(3) The *rank* (r) of the dimensional matrix is the order of the largest non-zero determinant it contains (Section 2.7). Therefore, it is always possible to isolate, *on the right-hand side*, a matrix of order r whose determinant is non-zero. The order r may however be lower than the number of rows in the dimensional matrix, as seen later.

(4) The *number* of *dimensionless products* in the *complete set* is equal to the number of variables isolated *on the left-hand side* of the dimensional matrix. It follows from item (3) that the number of dimensionless products is equal to the *total number of variables* minus the *rank of the dimensional matrix*. In the preceding example, the number of dimensionless products in the complete set was equal to the number of variables (5) minus the rank of the dimensional matrix (3), i.e. $5 - 3 = 2$ dimensionless products.

(5) When the last r columns of a dimensional matrix of order r do not lead to a non-zero determinant, the columns of the matrix must be rearranged so as to obtain a non-zero determinant.

Numerical example 1. An example will help understand the consequences of the above five characteristics on the general method for the systematic calculation of the complete set of dimensionless products. The dimensional matrix is as follows:

$$\begin{array}{c} \\ \mathrm{M} \\ \mathrm{L} \\ \mathrm{T} \end{array} \begin{array}{c} \begin{array}{ccccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 & V_7 \end{array} \\ \begin{bmatrix} 2 & 0 & 1 & 0 & -1 & -2 & 3 \\ 1 & 2 & 2 & 0 & 0 & 1 & -1 \\ 0 & 1 & 2 & 3 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

The rank (r) of this matrix is 3 (numerical example in Section 2.7), so that the number of dimensionless products of the complete set is equal to $7 - 3 = 4$. However, the determinant of the $r = 3$ last columns is zero:

$$\begin{vmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = 0$$

Calculating the complete set of dimensionless products thus requires a reorganization of the dimensional matrix by rearranging, for example, the columns as follows:

$$\begin{array}{c} \\ \mathrm{M} \\ \mathrm{L} \\ \mathrm{T} \end{array} \begin{array}{c} \begin{array}{ccccccc} V_1 & V_5 & V_7 & V_4 & V_2 & V_6 & V_3 \end{array} \\ \begin{bmatrix} 2 & -1 & 3 & 0 & 0 & -2 & 1 \\ 1 & 0 & -1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 3 & 1 & -1 & 2 \end{bmatrix} \end{array}$$

The solution then follows from the general method described above:

$$\begin{bmatrix} x_2 \\ x_6 \\ x_3 \end{bmatrix} = -\begin{bmatrix} 0 & -2 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_5 \\ x_7 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_6 \\ x_3 \end{bmatrix} = -\begin{bmatrix} 4 & 3 & -5 \\ -2 & -1 & 2 \\ -3 & -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_5 \\ x_7 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_6 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 & 9 & -9 & 15 \\ 5 & -4 & 5 & -6 \\ 8 & -7 & 7 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_5 \\ x_7 \\ x_4 \end{bmatrix}$$

$$\begin{array}{c} \\ \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} V_1 & V_5 & V_7 & V_4 & V_2 & V_6 & V_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & -11 & 5 & 8 \\ 0 & 1 & 0 & 0 & 9 & -4 & -7 \\ 0 & 0 & 1 & 0 & -9 & 5 & 7 \\ 0 & 0 & 0 & 1 & 15 & -6 & -12 \end{bmatrix} \end{array}$$

Numerical example 2. This example illustrates the case of a dimensional matrix whose *rank* is less than its number of rows. This matrix has already been considered in Section 2.7:

$$\begin{array}{c} \\ \mathrm{M} \\ \mathrm{L} \\ \mathrm{T} \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 2 & 1 & 3 & 4 \\ -1 & 6 & -3 & 0 \\ 1 & 20 & -3 & 8 \end{bmatrix} \end{array}$$

It was shown (Section 2.7) that the *rank* of this matrix is $r = 2$, so that it is not possible to find a combination of three columns that could be inverted. Any 3×3 submatrix would be *singular* (Section 2.8).

The solution consists in making the *number of rows* equal to the *rank*. This is done by eliminating any one row of the dimensional matrix, since the matrix has only two independent rows (Section 2.7). The number of dimensionless products in the complete set is thus equal to $4 - 2 = 2$.

$$\begin{array}{c} \\ \mathrm{M} \\ \mathrm{L} \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 2 & 1 & 3 & 4 \\ -1 & 6 & -3 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ \Pi_1 \\ \Pi_2 \end{array} \begin{array}{c} \\ \begin{bmatrix} 1 & 0 & -1/3 & -1/4 \\ 0 & 1 & 2 & -7/4 \end{bmatrix} \end{array}$$

It is possible to eliminate fractional exponents by multiplying each row of the solution matrix by its lowest common denominator:

$$\begin{array}{c} \\ \Pi_1 \\ \Pi_2 \end{array} \begin{array}{c} \\ \begin{bmatrix} 12 & 0 & -4 & -3 \\ 0 & 4 & 8 & -7 \end{bmatrix} \end{array}$$

Identical results would have been obtained if any other row of the dimensional matrix had been eliminated instead of row 3, since each of the three rows is a linear combination of the other two. This can easily be checked as exercise.

There now remains to discuss how to choose the ordering of variables in a dimensional matrix. This order determines the complete set of dimensionless products obtained from the calculation. The rules are as follows:

(1) The *dependent variable* is, of necessity, in the first column of the dimensional matrix, since it must be present in only one Π (the first dimensionless product is thus

called the *dependent dimensionless variable*). As a consequence, this first variable can be expressed as a function of all the others, which is the goal here. For example, in eq. 3.9, the *drag F* is in the first column of the dimensional matrix since it is clearly the *dependent variable*.

(2) The other variables are then arranged in decreasing order, based on their potential for experimental variation. Indeed, a maximum amount of information will result from experimentation if those variables with a wide range of experimental variability occur in a single Π.

(3) The initial ordering of variables must obviously be changed when the last *r columns* of the dimensional matrix have a zero determinant. However, one must then still comply as well as possible with the first two rules.

Two ecological applications, already discussed in Section 3.2, will now be treated using the systematic calculation of complete sets of dimensionless products.

Ecological application 3.3a

The first example reconsiders Ecological application 3.2d, devoted to the model of Kierstead & Slobodkin (1953). This model provided equations for the *critical size* of a growing phytoplankton patch and the *characteristic time* after which this critical size becomes operative.

The dimensional matrix of variables involved in the problem includes: *length x*, *time t*, *diffusion of cells D*, and *growth rate k*. The dependent variables being *x* and *t*, they are in the first two columns of the dimensional matrix:

$$\begin{array}{c} \\ \mathrm{L} \\ \mathrm{T} \end{array} \begin{array}{c} \begin{array}{cccc} x & t & D & k \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix} \end{array}$$

The rank of the dimensional matrix being 2, the number of dimensionless products is 4 – 2 = 2. These two products are found using the general method for calculating the complete set:

$$-\begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

$$\begin{array}{c} \\ \Pi_1 \\ \Pi_2 \end{array} \begin{array}{c} \begin{array}{cccc} x & t & D & k \end{array} \\ \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} x & t & D & k \end{array} \\ \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

$$\Pi_1 = kx^2/D \text{ and } \Pi_2 = tk$$

These two dimensionless products describe, as in Ecological application 3.2d, the *critical length x* and the *characteristic time t* as:

$$x \propto \sqrt{D/k} \text{ and } t \propto 1/k \propto x^2/D$$

Ecological application 3.3b

A second example provides an easy solution to the problem that confronted Paloheimo & Dickie (1965) concerning the synthesis of data on the growth of fish with respect to food intake. The question was discussed at length in Ecological application 3.2e, which led to three scale factors, for *food ration, mass,* and *time*. These scale factors were used by the authors to compare heterogeneous data from the ecological literature.

The solution is found directly, here, using the dimensional matrix of the six variables involved in the problem: *time t, mass W,* food *ration R,* rate of *oxygen consumption T,* rate of *metabolic expenditure* α, and coefficient b. The variables to be isolated being t, W, and R, they are in the first three columns of the dimensional matrix:

$$\begin{array}{c} \\ \mathrm{M} \\ \mathrm{T} \end{array} \begin{array}{c} \begin{array}{cccccc} t & W & R & T & \alpha & b \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & (1-\gamma) & -1 \\ 1 & 0 & -1 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

Since the *rank* of the dimensional matrix is $r = 2$, the number of dimensionless products is $6 - 2 = 4$. The four products are calculated by the method of the complete set:

$$-\begin{bmatrix} (1-\gamma) & -1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1/\gamma & 1/\gamma & 0 & 0 \\ [(1/\gamma)-1] & 1/\gamma & 1 & 1 \end{bmatrix}$$

$$\begin{array}{c} \\ \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \end{array} \begin{array}{c} \begin{array}{cccccc} t & W & R & T & \alpha & b \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1/\gamma & (1/\gamma)-1 \\ 0 & 1 & 0 & 0 & 1/\gamma & 1/\gamma \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

$$\Pi_1 = t\alpha^{1/\gamma} b^{(1/\gamma - 1)} = [(\alpha b)^{1/\gamma}/b]t$$

$$\Pi_2 = W\alpha^{1/\gamma} b^{1/\gamma} = (\alpha b)^{1/\gamma} W$$

$$\Pi_3 = Rb = bR$$

$$\Pi_4 = Tb = bT$$

The first three dimensionless products define the three scale factors already found in Ecological application 3.2e, i.e. Π_1 for *time,* Π_2 for *mass*, and Π_3 for *ration*. Π_4 defines a scale factor for *oxygen consumption*.

Direct calculations of complete sets of dimensionless products thus led to the same results as obtained before, but operations here were more straightforward than in Section 3.2.

It should not be necessary, after these examples, to dwell on the advantage of systematically calculating the complete set of dimensionless products. In addition to providing a rapid and elegant solution to problems of dimensional analysis, the above matrix method sets researchers on the right track when tackling a problem to be investigated using the dimensional tool. The success of a dimensional study depends

on: (1) adequate knowledge of the problem under study, so that *all* the pertinent variables are considered; and (2) clear ideas about which variables are functions of the others. It should be noted, as explained above, that the systematic calculation of the complete set of dimensionless products does not require prior knowledge of the fundamental equations. These, however, may be necessary to derive the dimensions of some complex variables. Dimensional analysis may be a powerful tool, provided that the ecological bases of the problem under consideration are thoroughly understood and that the objectives of the research are clearly stated.

3.4 Scale factors and models

Given the increased awareness in society for environmental problems, major engineering projects cannot be undertaken, in most countries, before their environmental impacts have been assessed. As a consequence, an increasing number of ecologists now work within multidisciplinary teams of consultants. At the planning stage, a powerful tool available to engineers, although very costly, is the small-scale *model*. *Tests* performed with such models help choose the most appropriate engineering solution. Actually, ecologists may encounter two types of model, i.e. mathematical and physical. *Mathematical models* are defined at the beginning of Section 10.3. *Physical models* are small-scale replica of the natural environment, to which changes can be made that reproduce those planned for the real situation. Tests with physical models (e.g. in wind tunnels or hydraulic flumes) are generally more costly to perform than mathematical simulations, so that the latter are becoming increasingly more popular than the former. *Physical models* are often based on dimensional analysis, so that it is this type of model that is considered here. It should be noted that physical models may originate from the empirical approach of engineers, which is distinct from the dimensional approach.

Physical model

In order to communicate with engineers conducting tests on small-scale models, ecologists must have some basic understanding of the principles governing model testing. In some cases, ecologists may even play a role in the study, when it is possible to integrate in the model variables of ecological significance (e.g. in a model of a harbour or estuary, such variables as salinity, sediment transport, etc.). Since small-scale models are based in part on dimensional analysis, their basic theory is thus relatively easy to understand. The actual testing, however, requires the specific knowledge and experience of model engineers. In addition to their possible involvement in applications of modelling to environmental impact studies, ecologists may at times use small-scale models to resolve problems of their own (e.g. studying the interactions between benthic organisms and sediment in a hydraulic flume). These various aspects are introduced here very briefly.

Prototype

In the vocabulary of physical modelling, the full-size system is called *prototype* and the small-size replica is called *model*. A model may be geometrically similar to the

Geometric similarity

prototype, or it may be distorted. In the case of *geometric similarity,* all parts of the model have the same shapes as the corresponding parts of the prototype. In certain cases, geometric similarity would lead to errors, so that one must use a *distorted model*. In such models, one or several scales may be distorted. For example, a geometrically similar model of an estuary could result in some excessively small water depths. With such depths, the flow in the model could become subject to surface tension, which would clearly be incorrect with respect to the real flow. In the model, the depth must therefore be relatively greater than in nature, hence a distorted model.

The physical example of the *drag on smooth spheres*, already discussed in Sections 3.2 and 3.3, is now used to introduce the basic principles of scaling and small-scale modelling. Equation 3.7 describes the *drag* (F) acting on a smooth sphere of *diameter* D, immersed in a stream with *velocity* V of a fluid with density ρ and dynamic viscosity η:

$$F = \rho V^2 D^2 f(Re) \tag{3.7}$$

$$F = \rho V^2 D^2 f\left(\frac{VD\rho}{\eta}\right)$$

In order to experimentally determine the drag, under convenient laboratory conditions (e.g. wind tunnel or hydraulic flume), it may be appropriate to use a geometrically similar model of the sphere. Quantities pertaining to the *model* are assigned *prime indices*. If the curve of the drag coefficient for smooth spheres was not known (Fig. 3.1), the estimation of F in the laboratory would require that the value of the *unknown function f* be the same for both the model and the prototype. In order to do so, the test engineer should make sure that the *Reynolds numbers* for the two systems are equal:

$$Re = Re'$$

$$\frac{VD\rho}{\eta} = \frac{V'D'\rho'}{\eta'} \tag{3.15}$$

Scale factor

A *scale factor* is defined as the ratio of the size of the model to that of the prototype. Scale factors are therefore *dimensionless numbers*. The scale factors (K) corresponding to eq. 3.15 are:

$$K_V = V'/V \qquad K_D = D'/D \qquad K_\rho = \rho'/\rho \qquad K_\eta = \eta'/\eta$$

These scales are used to rewrite eq. 3.15 as:

$$K_V K_D K_\rho = K_\eta \tag{3.16}$$

Because $Re = Re'$, the *scale factor of the unknown function f* is equal to unity:

$$K_{f(Re)} = 1 \qquad \textbf{(3.17)}$$

The ratio between the drag measured for the model and the real drag on the prototype is computed by combining eq. 3.7 with the above scale factors:

$$K_F = K_\rho K_V^2 K_D^2 K_{f(Re)}$$

Because of eq. 3.17, it follows that:

$$K_F = K_\rho K_V^2 K_D^2 \qquad \textbf{(3.18)}$$

Equation 3.16 is used to find the value of K_F:

$$K_V K_D K_\rho = K_\eta \qquad \textbf{(3.16)}$$

is squared $\quad K_V^2 K_D^2 K_\rho^2 = K_\eta^2$

from which $\quad K_V^2 K_D^2 K_\rho = K_\eta^2 / K_\rho$

and, given eq. 3.18 $\quad K_F = K_\eta^2 / K_\rho \qquad \textbf{(3.19)}$

Equation 3.19 leads to the following practical conclusions, for determining the drag on smooth spheres in the laboratory:

(1) If the model is tested using the *same fluid* as for the prototype, the *drag* measured during the test is the same as for the prototype. This follows from the fact that, if $K_\eta = 1$ and $K_\rho = 1$ (same fluid), K_F is equal to unity (eq. 3.19), hence $F' = F$.

(2) If testing is conducted using the *same fluid* as for the prototype, conservation of *Re* requires that the *velocity* for the model be greater than for the prototype (i.e. the model is smaller than the prototype). This follows from the fact that, when $K_\eta = 1$ and $K_\rho = 1$ (same fluid), $K_V K_D = 1$ (eq. 3.16); consequently any decrease in K_D must be compensated by a proportional increase in K_V.

(3) When it is more convenient to use *different fluids*, testing may be conducted while conserving *Re*. It has already been shown (Section 3.2) that, for example, going from a large-size prototype, in air, to a model 6 times smaller, in water, allows a reduction of the flow speed during the test by a factor of 3. The drag measured for the model would not, however, be necessarily the same as that of the prototype, since that force varies as a function of the ratio between the squares of the dynamic viscosities

(K_η^2) and the densities (K_ρ) of the two fluids (eq. 3.19). Knowing this ratio (K_F), it is easy to derive the drag for the model (F) from that measured during the test (F') since:

$$F = F'/K_F$$

In more complex cases, it is sometimes necessary to simultaneously conserve two or more dimensionless products that are incompatible. In such a situation, where a choice must be made between contradictory constraints, it rests on the test engineer to justify discrepancies in similarity and to apply theoretical corrections to compensate for them. Hence modelling, although derived from scientific concepts, becomes an art based on the experience of the researcher.

Similarity

A *general concept of similarity* follows from the previous discussion. In a Cartesian space, the *model* and the *prototype* are described by coordinates (x' y' z') and (x y z), respectively. Correspondence between the two systems is established by means of *scale factors* (K), which define *homologous* times as well as *homologous* points in the three dimensions of space:

$$t' = K_t t \qquad x' = K_x x \qquad y' = K_y y \qquad z' = K_z z$$

The *time scale factor* (K_t) would be used, for example, in the case of a flow where Δ'_t and Δ_t are the time intervals during which two homologous particles go through homologous parts of their respective trajectories. It would then be defined as

$$K_t = \Delta'_t/\Delta_t$$

Geometric similarity is defined as: $K_x = K_y = K_z = K_L$. In *distorted models*, a single length scale is usually modified, so that $K_x = K_y \neq K_z$. The ratio K_z/K_x is the *distortion factor*. It would be possible, using this same approach, to define characteristics of *kinematic similarity,* for similar motions, and of *dynamic similarity*, for systems subjected to homologous forces.

There are several types of similarity in addition to the geometric, dynamic and kinematic similarities. These include the *hydrodynamic*, *transport*, and *thermal similarities*. Readers interested in applications of dimensional analysis to the theory of biological similarity may refer to the review of Günther (1975), where the various types of physical similarity are briefly described.

Plate 3.1 The metre is the basis of the metric system, which was established during the French Revolution and became the *Système International d'Unités* (*SI*, International System of Units) in 1960. The metre was originally set as 10^{-7} of a quarter of the Earth meridional perimeter. In order to define the metre precisely, French astronomers Delambre and Méchain measured the meridian between Dunkerque and Barcelona between 1792 and 1799. The story of their work can be found in Guedj (1999, 2001). Copies of the standard metre engraved in marble were displayed at 16 locations in Paris to make the new measurement unit known and used by the people. The picture shows the last of these marble metres that is still at the site where it was originally placed, under the arcades of 36 rue de Vaugirard in Paris, across the street from the Palais du Luxembourg (seat of the French Senate), where it can be seen nowadays. Photo P. Legendre, 2002.