The Exploration of Harmonic Motion and The Journey to Nonlinear Mechanics

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Contents

Abstract

The exploration and study of Harmonic Motion can give us an insight into how different physical systems work and knowledge about our world. Before we dive into this fascinating type of motion we ask that you have our Jupyter Notebook open as you follow along. The link to this can be found above. It will be very helpful for you to understand how the motion looks and acts. It will also help you grasp how the different conditions affect a system. We have also made a YouTube Playlist with different video simulations in case you cannot access the notebook or do not want too. We begin our story at the simplest problem, the problem that gives rise to the phenomenon known as Simple Harmonic Motion. We will then begin to delve into the wondrous world of Pendulums. We will first explore the Simple Pendulum, in all its glory before moving onto the Damped Driven Pendulum, the successor of the Simple Pendulum. And then we will find ourselves faced with the most fiercest Pendulum we have seen, the Double Pendulum. We will explore the chaos that the Double Pendulum brings to the universe and try our best to analyze, learn and understand the motion that these beautiful systems create.

1 Introduction

1.1 Hooke's Law

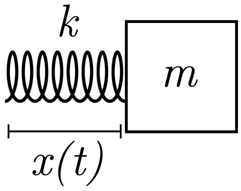
Hooke's Law is very simple: it states the Restoring Force of a spring is proportional to the distance from the equilibrium position.

$$F_{Spring} = -kx$$

Where k is the spring constant, and x is the distance from the equilibrium position.

1.2 Friction-less Spring

Consider the following mass on the end of a spring confined to move in the x-direction on a friction-less surface:



Newtons 2^{nd} Law yields:

$$F_{net} = m\ddot{x}(t) = -kx(t)$$

giving us the following equation of motion,

$$\ddot{x}(t) = -\frac{k}{m}x(t) \tag{1}$$

The equation above is sometimes refered to as "The world's simplest second order DE".[1] This represents the fact that the spring is a restoring force, always pointing in the opposite direction of the displacement. To solve it explicitly we can choose

$$\dot{x}(0) = 0 \qquad x(0) = x_0$$

We have from class/assignments

$$\omega = \sqrt{\frac{k}{m}}$$

$$x(t) = A\cos\omega t + B\sin\omega t$$

$$x'(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

Solving for constants A and B,

$$x(0) = x_0 = A$$

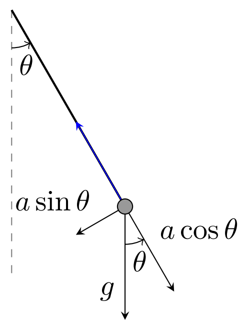
$$x'(0) = 0 = B\omega\ ;\ B = 0$$

$$x(t) = x_0\cos\omega t \tag{2}$$

Section 1) in the Jupyter Notebook depicts a simulation of the Mass on Spring System. Or you can view the **Mass On Spring Video Simulation**

2 Simple Pendulum

Consider the following mass m, subjected to earths gravitational field, attached to a mass-less string of fixed length l.



[1] John Wainwright and Joe West. *Introduction to Differential Equations.* 1st ed. Pearson, 2017.

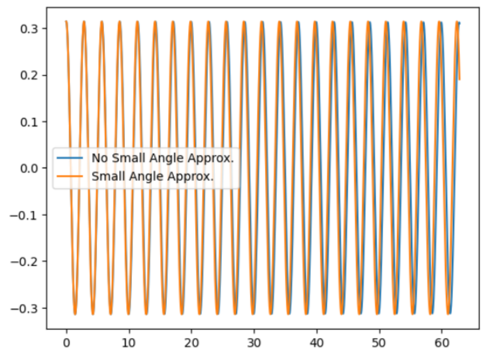
As we saw in class, we can use define a Lagrangian for the system and use the Euler Lagrange equations to derive the following equation of motion.

$$\ddot{\phi} + \frac{g}{l} sin\phi = 0 \tag{3}$$

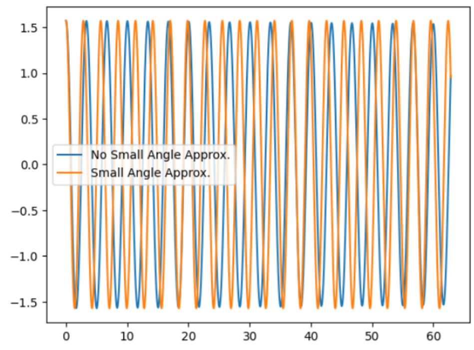
Letting $u = \dot{\phi}$:

$$\begin{cases} \dot{\phi} = u \\ \dot{u} = -\frac{g}{l} sin\phi \end{cases} \tag{4}$$

Which is a system of ODEs and can be be solved numerically using a Runge-Kutta (RK) method. Refer to the Jupyter notebook for plots of the functions and for simulations. Or you can view the **Simple Pendulum Simulation Video**



Theta Vs Time of a simple pendulum with the small angle approximation and a simple pendulum without the small angle approximation, both starting at a small angle.



Theta Vs Time of a simple pendulum with the small angle approximation and a simple pendulum without the small angle approximation, both starting at a larger angle.

This shows that the small angle approximation cannot be used at larger angles, but is very accurate for small angles.

2.1 Nonlinearity and Chaos

For a system to exhibit chaos its equations of motion **must** be non-linear, however having a non-linear equation **does not** mean there is chaos. Equation (1) is an example of a linear equation and (3) of a non-linear one. Almost all of the linear equations in mechanics are able to be solved analytically, but almost none of the non-linear ones are. Often when linear problems had to be addressed, they were solved in terms of linear approximations and the rich complexities in non-linear systems went unrecognized.[2]

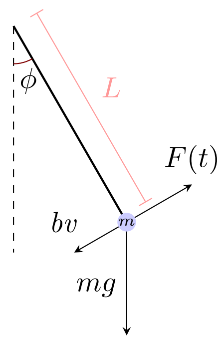
French mathematician Poincaré (1854-1912) was the first to study the phenomenon of chaos, but his work went largely unnoticed, overshadowed by other physics discoveries in the 20th century. With the development of computers to aid in solutions to nonlinear systems, chaos has become a popular phenomenon to study.It turns out these chaotic systems are pretty special, and throughout our paper, we hope to convince you they are worth looking into.

For instance, the equation of motion for the non-linear simple pendulum (3) does not exhibit chaos. It turns out that non-linearity is not a guarantee for chaos. For a system to exhibit chaos it must be non linear and "sufficiently complicated." As we will see in the following sections, the Damped Driven Pendulum can be "sufficiently complicated" with a "sufficiently complicated" driving force and the Double Pendulum is "sufficiently complicated" with many conditions, and thus exhibit chaos.

3 Damped Driven Pendulum

Problem

Consider the same mass and string as before but now with additional driving and damping forces as shown below.



[2] J.R. Taylor. *Classical Mechanics.* University Science Books, 2005.

3.1 Equation of Motion

The equation of motion is simply $\Gamma = I\ddot{\phi}$, where $I = mL^2$ is the moment of inertia and Γ is the net torque about the pivot. The damping force has magnitude of bv and we will denote the driving force by F(t). Analysing the torques and using Newton's 2nd law we get the following equation of motion

$$mL^2\ddot{\phi} = -bL^2\dot{\phi} - mgLsin\phi + LF(t) \tag{5}$$

and we shall assume that the driving force is sinusoidal and of the form

$$F(t) = F_0 \ cos(\omega t)$$

where F_0 is the *drive amplitude* and ω the **drive frequency**. Subbing into (5) and rearranging, we get

$$\ddot{\phi} + \frac{b}{m}\dot{\phi} + \frac{g}{L}sin\phi = \frac{F}{mL}cos(\omega t).$$

We will now rename $\frac{b}{m}$ as 2β as we have previously done in the course,

$$\frac{b}{m} = 2\beta$$

where β is called the **damping constant**. Similarly the coefficient $\frac{g}{L}$ is just ω_0^2,

$$\frac{g}{L} = \omega_0^2$$

where ω_0 is the **natural frequency** of the pendulum. Finally, we notice that the coefficients F_0/mL and ω_0^2 are of the same dimension. It is convenient to rewrite the former as $F_0/mL = \gamma\omega_0^2$. Introducing a dimensionless parameter of the form,

$$\gamma = \frac{F_0}{mL\omega_0^2} = \frac{F_0}{mg},$$

which we will call the **drive strength** and which is simply the ratio of drive amplitude F_0 to weight mg. γ is a dimensionless parameter that measures the strength of the driving force. If $\gamma \geq 1$ the driving force is strong enough to give rise to larger amplitudes. Making all the substitutions we arrive at the final form of our equation

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 sin\phi = \gamma\omega_0^2 cos\omega t \tag{6}$$

We will proceed in a similar fashion as we did for the Simple Pendulum case letting $u = \dot{\phi}$,

$$\begin{cases} \dot{\phi} = u \\ \dot{u} = \gamma\omega_0^2 cos\omega t - 2\beta u - \omega_0^2 sin\phi \end{cases} \tag{7}$$

which once again we can solve numerically using a Runge Kutta method. We will now consider the solutions of (6) for different values of γ using our numerical plots to guide our analysis.

3.2 Linear approximation

If we choose our initial angle near the equilibrium position i.e. ($\phi_0 \approx 0$) and set the drive strength to be small, $\gamma << 1$ we expect for ϕ to remain small at all times. Therefore we can approximate $sin\phi \approx \phi$ and eq (7) becomes,

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2\phi = \gamma\omega_0^2 cos\omega t \tag{8}$$

Some may recognize this as the linear oscillator, it is explored in detail in Taylor(2005, Section 5.5). The equation of motion has transient terms that depend on the initial conditions but quickly die out and become irrelevant. Thus, after the pendulum released from rest the behavior of the pendulum briefly depends on the initial conditions, but rapidly the transients die out and the motion approaches a unique "*attractor*". The pendulum then oscillates with the same frequency of the driving force:

$$\phi(t) = Acos(\omega t \ - \delta) \tag{9}$$

If we include another term in the Taylor expansion ,i.e $sin\phi \approx \phi - \frac{1}{6}\phi^3$ and sub into the exact equation of motion (7), we get the approximation

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2(\phi - \frac{1}{6}\phi^3) = \gamma\omega_0^2 cos\omega t \tag{10}$$

Using the identity

$$cos^3 x = \frac{1}{4}(cos3x + 3cosx)$$

one can show that eq (8) will have vanishing transient terms dependent on initial conditions just as linear , but will have an attractor of the form

$$\phi(t) = Acos(\omega t \ - \delta) + Bcos3(\omega t - \delta) + ... + C_i cosn(\omega t - \delta) \tag{11}$$

for $n > 3 \in \mathbb{Z}$ Notice these terms all have frequencies $n\omega$ for various integers, n. In fact, any term oscillating with frequency equal to an integer multiple of ω is called a harmonic of the drive frequency. The nth harmonic has frequency $n\omega$ and period $\tau_n = \frac{2\pi}{m\omega} = \frac{\tau}{n}$ where $\tau = \frac{2\pi}{\omega}$ Note that in one drive perios, the nth harmonic repeats n times and thus will have cycled back to its originial value. Therefore (11) which is made up of various harmonics, will hace the same period as the driving force. Turns out this not so linear approximation is valid for driving strength $\gamma \leq 1$. We will soon see that increasing the driving strength to slightly greater than $\gamma = 1$ gives rise to some dramatically different behaviour.

3.3 Chaos in the DDP

As we continue to increase γ we see that the attractor will no longer have the same period as the driving frequency but now will become an integer multiple of it. In particular, we can observe that there are certain values of γ for which the period of the driving force actually doubles. In the late 1970, physicsist Mitchell Feigenbaum showed that many system exhibit similar phenomena, and all cascading systems show the same geometric acceleration. The intervals (drive strength in our case) satisfy

$$(\gamma_{n+1} - \gamma_n) \approx \frac{1}{\delta}(\gamma_n - \gamma_{n-1}) \tag{12}$$

where

$$\delta = 4.6692016$$

is called the **Feigenbaum Number**. Since period doubling is such a widespread occurence and δ has the same value for so many different systems, this phenomenon is ofter regarded as **universal**. Equation 12 implies that the intervals between successive γ_i terms approaches zero rapidly, therefore γ_i approaches a finite limit γ_c,

$$\gamma_n \to \gamma_c \quad (asn \to \infty)$$

Therefore our sequence of thresholds γ_i satisfies

$$\gamma_1 < \gamma_2 < \cdots < \gamma_i < \cdots < \gamma_c$$

For our DDP, the limit γ_c is found to be

$$\gamma_c = 1.0829$$

We will see that beyond the critical value ϕ_c, chaos sets in, and this period doubling phenomenon is called a **route to chaos**. However, it is not the only route, and systems are observed to exhibit chaos without first going through a period-doubling cascades. The route to chaos is **not** unique.

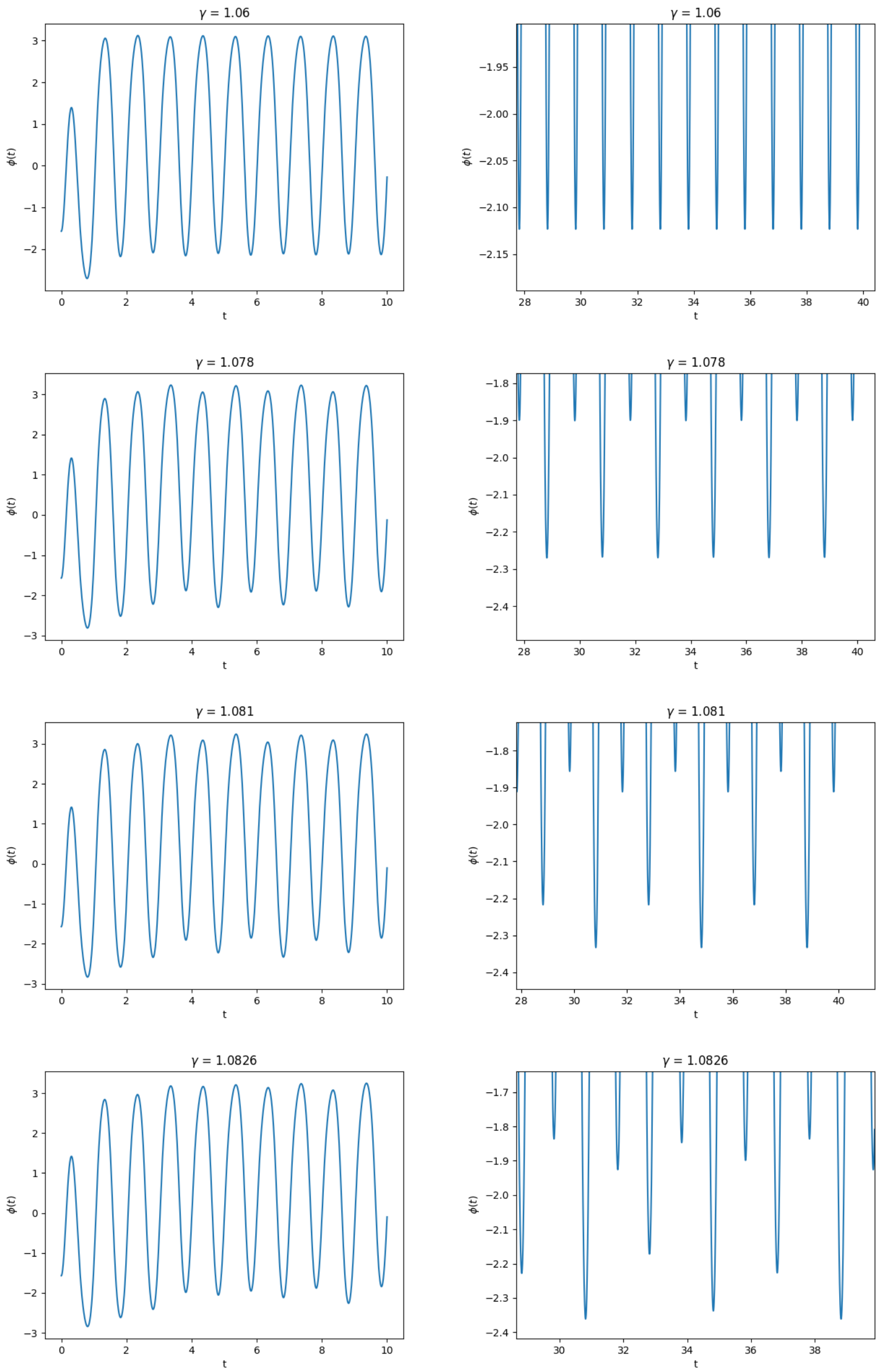


Figure 1: demonstration of period doubling cascade

3.4 Sensitivity to Initial Conditions

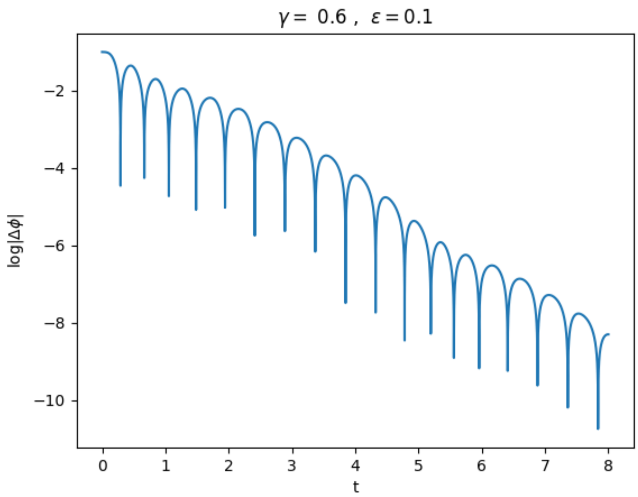
Let's consider two indentical DDP's with slightly different initial conditions, i.e,

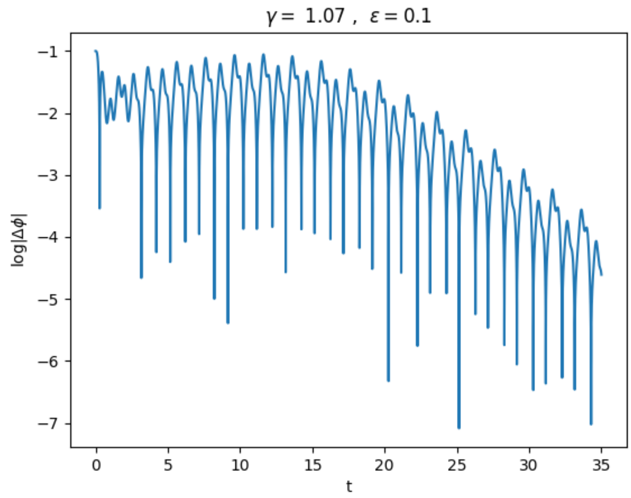
$$\phi_1(0) = \phi_2(0) + \epsilon$$

As time evolves, we ask the following question, do the motions of the two pendulums remain constant? Do they converge to one another? Or do they diverge from each other? Let us consider the difference between our two solutions, which we denote by $\Delta\phi(t)$

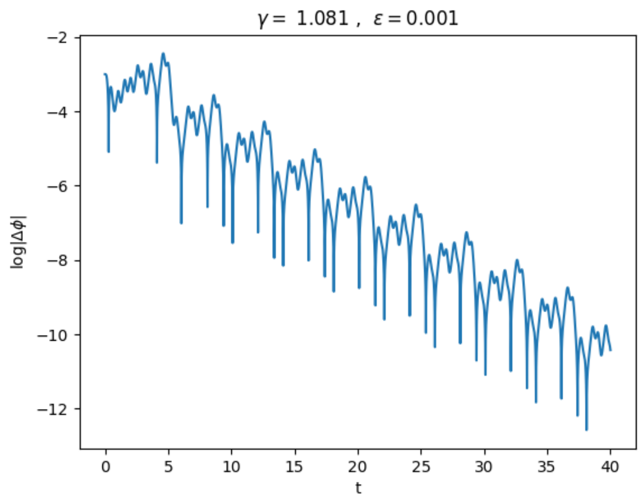
$$\Delta\phi(t) = \phi_2(t) - \phi_1(t)$$

if we plot $\Delta\phi(t)$ against t for various values of $\gamma < \gamma_c$ what we will see will be something similar to an exponential decay (oscillations around an exponential decay). Therefore differences in initial conditions converge to 0. For $\gamma > \gamma_c$ we see exponential increase, so our differences in initial conditions diverge. This is more clear if we instead plot the log base 10 of $\Delta\phi$.

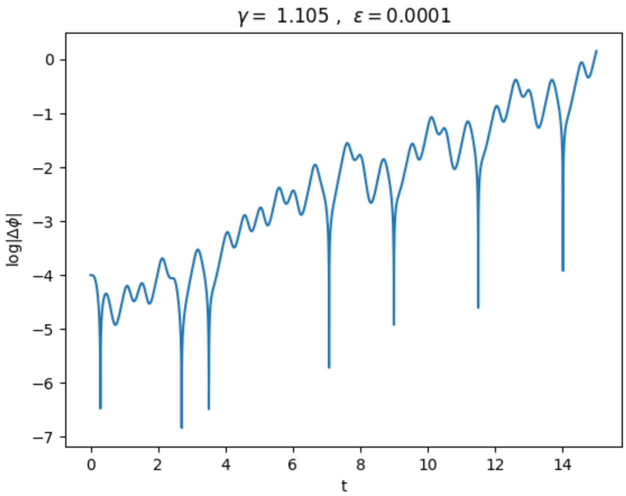




$\gamma = 1.07$, $\varepsilon = 0.1$
$\log|\Delta\phi|$
t
−1
−2
−3
−4
−5
−6
−7
0
5
10
15
20
25
30
35



$\gamma = 1.081$, $\varepsilon = 0.001$
$\log|\Delta\phi|$
t
−2
−4
−6
−8
−10
−12
0
5
10
15
20
25
30
35
40



$log_{10}|\Delta\phi|$ vs t for different values of γ

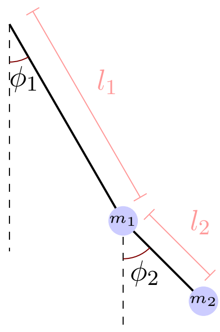
The Liapunov Exponent
We can generalize the previous analysis with

$$|\Delta\phi(t)| \sim Ke^{\lambda t}$$

with λ being often referred to as the Lyapunov exponent. when λ is positive we have a chance of observing sensitivity to initial conditions, a symptom of chaotic motion.

You can view the Jupyter Notebook to play around with different types of driving forces to see what resulting motion occurs or you can view the **Damped Driven Pendulum Simulation Video**

4 Double Pendulum



Let us now explore the Double Pendulum. Consider the system above. Deriving the equations of motion for this system using Newtonian mechanics is very difficult, so we choose to use Lagrangian mechanics instead. As we have seen in class and on assignments, the Lagrangian is given by

$$\begin{aligned}\mathcal{L} = \frac{1}{2}(m_1+m_2)l_1(l_1\dot{\phi_1}^2 + 2gcos\phi_1)\\ + \frac{1}{2}m_2l_2(l_2\dot{\phi_2}^2 + 2l_1cos(\phi_1-\phi_2)\dot{\phi_1}\dot{\phi_2} + 2gcos\phi_2)\end{aligned} \tag{13}$$

Hence the Euler-Lagrange equations get written as

ϕ_1:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \phi_1} =& - m_2l_1l_2\dot{\phi_1}\dot{\phi_2}sin(\phi_1-\phi_2) - (m_1+m_2)gl_1sin\phi_1\\ \frac{\partial \mathcal{L}}{\partial \dot{\phi_1}} =&(m_1+m_2)l_1^2\dot{\phi_1} + m_2l_1l_2cos(\phi_1-\phi_2)\dot{\phi_2}\\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi_1}}\right) =&(m_1+m_2)l_1^2\ddot{\phi_1} + m_2l_1l_2(cos(\phi_1-\phi_2)\ddot{\phi_2} - sin(\phi_1-\phi_2)\dot{\phi_1}\dot{\phi_2} - \dot{\phi}_2^2)\end{aligned}$$

ϕ_2:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \phi_2} =& - m_2l_2(l_1sin(\phi_1-\phi_2)\dot{\phi_1}\dot{\phi_2} - 2gsin\phi_2)\\ \frac{\partial \mathcal{L}}{\partial \dot{\phi_2}} =&m_2l_2(l_2\dot{\phi_2} + l_1cos(\phi_1-\phi_2)\dot{\phi_1})\\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi_2}}\right) =&m_2l_2(l_2\ddot{\phi_2} + l_1cos(\phi_1-\phi_2)\ddot{\phi_1} - l_1sin(\phi_1-\phi_2)(\dot{\phi_1}^2 - \dot{\phi_1}\dot{\phi_2}))\end{aligned}$$

So we get the following two simplified equations of motion,

$$\ddot{\phi_1} + \frac{l_2}{l_1}\frac{m_2}{m_1+m_2}cos(\phi_1-\phi_2)\ddot{\phi_2} = -\frac{l_2}{l_1}\frac{m_2}{m_1+m_2}sin(\phi_1-\phi_2)\dot{\phi}_2^2 - \frac{g}{l_1}sin\phi_1 \tag{14}$$

$$\ddot{\phi_2} + \frac{l_1}{l_2}cos(\phi_1-\phi_2)\ddot{\phi_1} = \frac{l_1}{l_2}sin(\phi_1-\phi_2)\dot{\phi_1}^2 - \frac{g}{l_2}sin\phi_2 \tag{15}$$

if we let $\alpha = \frac{l_2}{l_1}$ be the ratio of our two lengths, $\mu_1 = \frac{m_2}{m_1+m_2}$ be a form of the reduced mass, and let $\omega_{01}^2 = \frac{g}{l_1} = \frac{g\alpha}{l_2}$ and $\omega_{02}^2 = \frac{g}{l_2} = \frac{g}{l_1\alpha}$ be the corresponding natural frequencies. Also note that $\omega_{01}^2 = \omega_{02}^2\alpha$ so we will let $\omega_0 = \omega_{01}$ and thus $\omega_{02} = \frac{\omega_0}{\alpha}$ now subbing in our new constants,

$$\ddot{\phi}_1 + \alpha\mu_1 cos(\phi_1 - \phi_2)\ddot{\phi}_2 = -\alpha\mu_1 sin(\phi_1 - \phi_2)\dot{\phi}_2^2 - \omega_0 sin\phi_1 \tag{16}$$

$$\ddot{\phi}_2 + \frac{1}{\alpha}cos(\phi_1 - \phi_2)\ddot{\phi}_1 = \frac{1}{\alpha}sin(\phi_1 - \phi_2)\dot{\phi}_1^{\,2} - \frac{\omega_0}{\alpha}sin\phi_2 \tag{17}$$

now just as we have done before, we will let $\dot{\phi}_1 = u$ and $\dot{\phi}_2 = v$ and make the substitutions into (16) and (17) and apply the Runge-Kutta method on the resulting system of equations to get our solutions.

What is most striking about these equations is just how much more complex the motion of the pendulum is, now that we have added only one extra mass. This is a chaotic system, which can be defined as any system which is highly sensitive to small changes in initial conditions. For us, this manifests as a an infinitesimal change in the initial position of the pendulum leading to vastly different motion shortly after.

A popular way to think about chaos is the so called "butterfly effect", where a butterfly flaps its wings, and starts a causal chain that eventually leads to a tornado. You can also think about this mathematically as a system with deterministic equations of motion, but such high sensitivity to the initial conditions that meaningfully accurate predictions can never be computed. Interestingly, the study of chaos is a relatively new field that has formed over the last few decades.

The system can be viewed with our Jupyter Notebook or by viewing the **Double Pendulum Simulation Video**

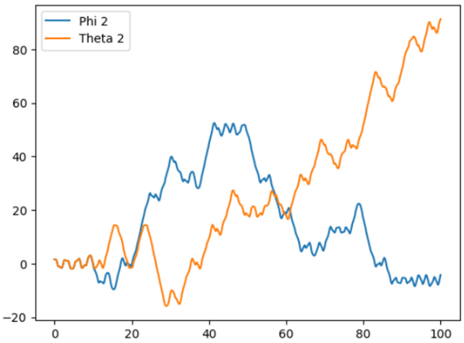
Note that in the sections below, we have introduced a second pendulum, whose angles we will describe with θ instead of ϕ.

4.1 Chaos in the Double Pendulum

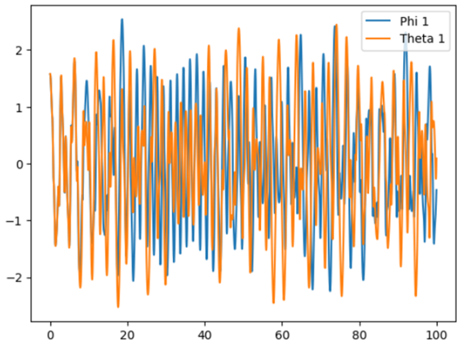
In this section we will closely dissect the motion of 2 pendulums, that are offset slightly by plotting both ϕ_1 and θ_1, as well as ϕ_2 and θ_2, both with respect to time. In our simulation, both pendulums start 0.01 radians apart from each other. This demonstrates just how sensitive this system is to its initial conditions.

It is clear from the graphs below, that there is a certain period of time, for which a reasonably accurate, and possibly even periodic, prediction of the motion could be made. This is the period of time in which the two curves are coincident, meaning if we are certain of our initial conditions to 0.01 radians, it will take around 10 seconds for the system to become unpredictable. We will call this period of time the interval of **predictability**, and the rest the **chaotic** interval.

Second Angles vs Time (Fig. 1)



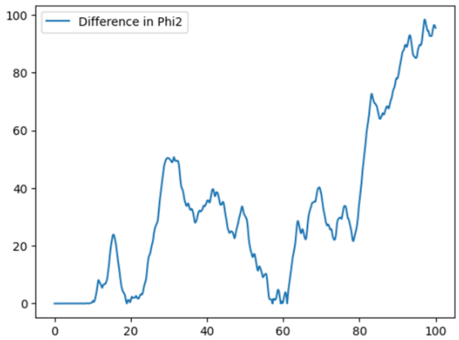
First Angles vs Time (Fig. 2)



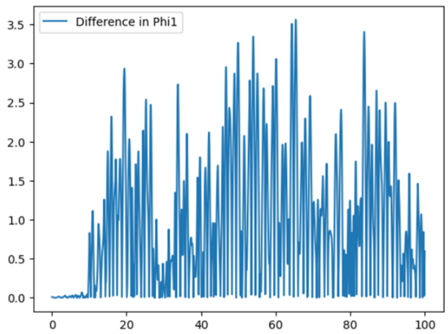
Angle over Time

Based on our data, which can be seen in the notebook and above, we could argue that the unpredictability can be better described based on the second angle of the pendulum, which has a tendency to all of a sudden switch directions prior to the chaotic interval. This can be seen in the graphs of the angles of either pendulum with respect to time below.

Difference in Second Angle (Fig. 3)



Difference in First Angle (Fig. 4)



Angle over Time

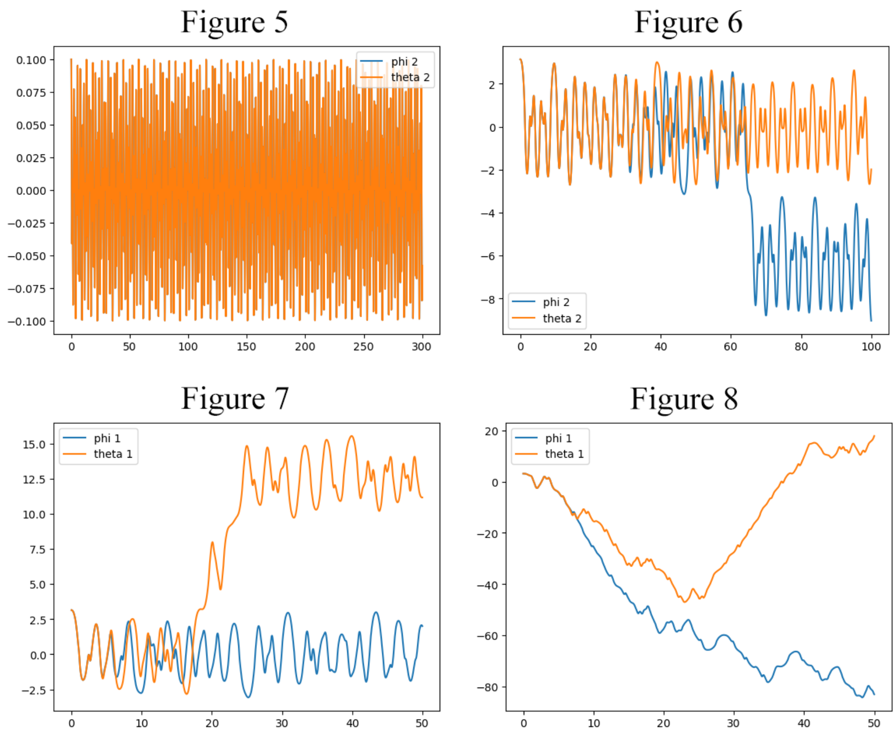
This is also a useful way to think, because it seems to much more clearly define the length of the predictable interval. The advantage of this, is in finding some small angle ϵ such that,

$$\epsilon > |\phi_2 - \theta_2| \text{ on the interval of predictability.}$$

There does seem to be a problem with this approach, however. If there is not enough energy in the system, the second mass will never be able to swing around the first one. In order to quantify this energy, we must look at the equilibrium solutions, which can be found by setting both angles in our DE constant. We get the following:

$$sin(\phi_1) = sin(\phi_2) = 0$$

Which implies that we have 4 equilibrium solutions, specifically any combination of $\phi_1 = 0, \pi$ and $\phi_2 = 0, \pi$. We can investigate the stability of these solutions by simulating a very small deviation in either angle from equilibrium.



Angle over Time

Figure 5 (IC: $\phi_1 = \phi_2 = 0$):
This shows the pendulum oscillating around $\phi_1 = \phi_2 = 0$. This is the only stable equilibrium.

Figure 6 (IC: $\phi_1 = 0, \phi_2 = \pi$):
This shows the **second** pendulum eventually making a full rotation. This is seen in the plot as a large spike in the angle, followed by oscillation around a new, shifted angle (note that the **first** pendulum never makes any such rotation). This is an unstable equilibrium.

Figure 7 (IC: $\phi_1 = \pi, \phi_2 = 0$):
This shows the **first** pendulum eventually making a full rotation. This is an

unstable equilibrium.

Figure 8 (IC: $\phi_1 = \phi_2 = \pi$):
This shows the **first** pendulum making many full rotations, never really oscillating around any one angle. This is an unstable equilibrium.

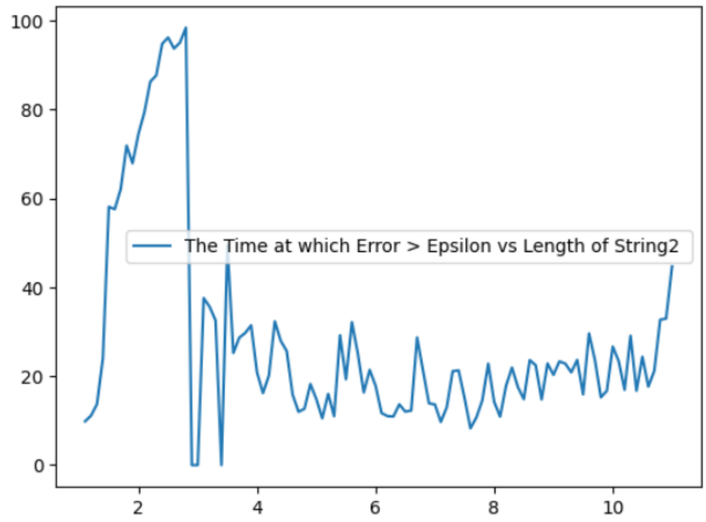
Clearly there is very different behaviour based on these 4 types of initial conditions. The most unique, however, is illustrated in figure 1, where no chaos is displayed, even after 5 minutes. This seems to imply that the smaller the initial angles (hence the lesser the total energy), the less chaotic the system.

4.2 Chaos and the Conditions of the System

(note that in this section, variable may not be typeset. This is because we are using the names of the variables in the included notebook instead.)

How do the masses, lengths and starting angles affect the outcome of the system? Does increasing or decreasing the mass change the point in time where our system becomes chaotic? Note that when we are changing the different conditions below, we leave all the other conditions equal to 1 unit, and the initial angles = $\frac{\pi}{2}$ radians (unless we are changing the angle, in which case one angle is set to $\frac{\pi}{2}$ and the other 0).

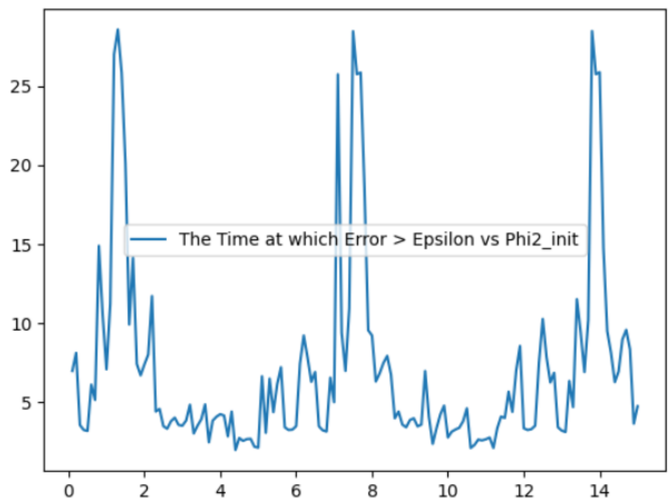
As we saw above we can define an ϵ to be a bound on our error function. (Our Error function is simply the difference between ϕ_2 and θ_2)And when our error function is greater then this ϵ we will store that point in time to plot against different conditions. The first Error $> \epsilon$ vs Condition we look at will be Error $> \epsilon$ vs Length of String 2.



The point in time at which Difference/Error > Epsilon vs Length String 2

As you can see, as we increase the length of the 2nd string to roughly about 2.5 units we reach a peak, where it takes 100 seconds before the 2 pendulums diverge past our error bound. After that it dips and then slowly starts to increase as the length increases.

Now we will take a look at Error > ϵ vs The initial Phi2 angle. We can make a prediction that the graph should be "roughly periodic", meaning every 2π units, we should see a repeat.

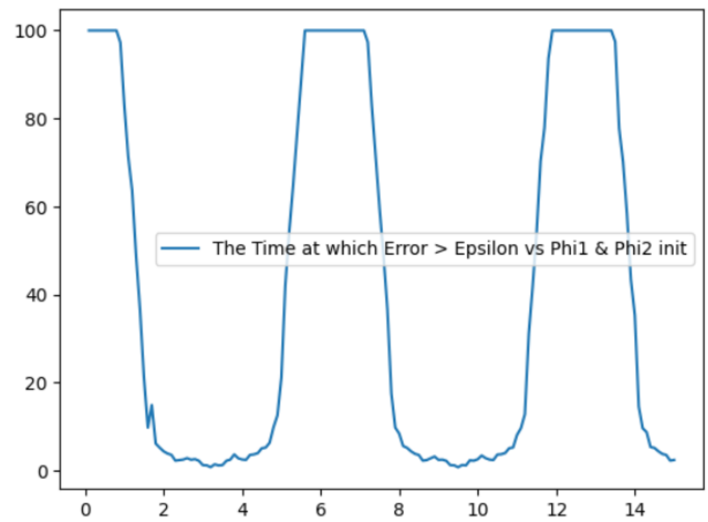


The point in time at which Difference/Error > Epsilon vs Phi2

As we can see from the graph above, there seem to be repeating peaks which is what we would expect as we go around the circle. However the placement of the peaks are very interesting. We would expect these peaks to be close to $2n\pi$ units apart, where n is an integer, as that is when we can apply the small angle approximation and the system becomes linear. However they actually appear roughly around $\pi/2 + 2n\pi$ units apart.

Our assumption on why this happens is, if you recall, we have the other angle set to be $\pi/2$ Radians thus the 2 angles are equal to each other at that point. To see if this assumption is right or not we will now explore what happens if we set both angles equal to each other and slowly increase that angle. Recall that each pendulum is described by ϕ and θ, where 1 and 2 denote the mass the angle is with respect to. In this system $\Phi_1(0) = \Phi_2(0)$ and we are increasing both angles equally.

We should see that at every $\pi/2$ release, (every $\pi/2 + 2n\pi$ units), the graph reads 25 seconds as those conditions would line up with our previous graph. We should also see peaks every $2n\pi$ units on this graph as that is when we would be able to apply the small sin approximation to BOTH angles.



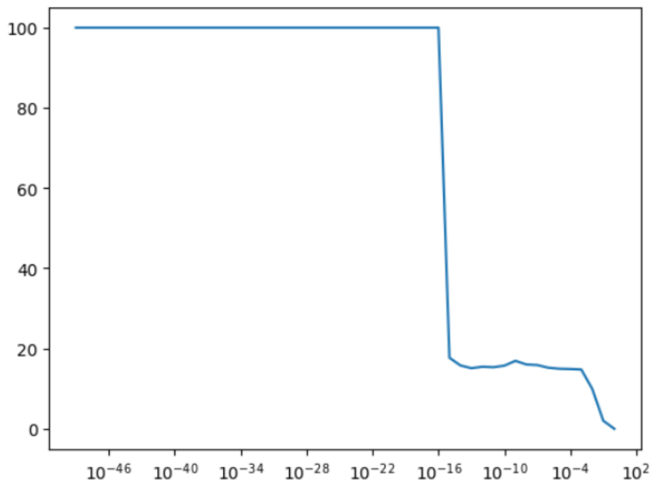
The point in time which Difference/Error > Epsilon vs Both Initial Angles

As we can see from the graph above our assumption was correct. There are repeating peaks every 2π units that would look more like asymptotes if we could look at an infinite amount of time. This also implies our system is very stable at smaller angles of release.

The straight lines appear because we cannot calculate that far ahead in time with the computers that are available to us, (The program already takes about 10 minutes to run), so we just set it equal to the max time. That is why there are straight lines on the peaks.

This graph reiterates that the smaller the initial angles are, the system becomes less chaotic and more predictable, which makes a lot of sense as, again, that is when we can apply the small angle approximation to make our system linear.

We will now explore what happens as we increase the difference between ϕ_1 and θ_1, and ϕ_2 and θ_2. This might be hard to understand/explain, but if you recall, we always set the 2 pendulums (described by ϕ and θ) to be 0.01 Radians apart, Now we are starting them even closer together and seeing what happens as we increase that difference. So at the very end of the graph, ϕ will start 1 Radian apart from θ (Note that $\phi_1 = \phi_2$ in this system.)



The point in time at which Difference/Error > Epsilon vs The Difference in Both Angles

This tells us that when the "error" or difference between the two angles is on the order of 10^{-16} we get divergence/chaos in the system after 20 seconds of evolution. It would be interesting to see how this graph changes if we were able to look past more then 100 seconds.

5 Conclusion

In conclusion, we explored the many different types of Harmonic Motion, went face to face with the Chaotic Double Pendulum and survived using computational methods and virtual simulations to study the behaviour of chaotic systems. We saw that the period of time in which a chaotic system can be accurately predicted is measurable using our error function, and we explored how different parameters on the double pendulum affect this predictability. We saw that in order for a chaotic system to be accurately predicted, it requires a more accurate measurement than could ever realistically be made. This is what makes chaotic systems so intriguing, as the smallest difference in initial conditions can change the entire motion of the object(s). The study of chaotic systems can almost be connected to the chaos of life, in how one action in your life could change the entire trajectory of your future and that we can never truly know what comes next.

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