CHAPTER 1

FUNDAMENTAL PARTICLE VIBRATION THEORY

The production of sound always involves some vibrating source. Such a source is often of irregular shape, and rarely do all parts of the vibrating surface move as a unit. It is the very complexity of the vibration of a sound source that makes it necessary to consider first the simplest vibrating body, the *particle*. The motion of actual sources may approximate that of a particle, particularly at low frequencies. Whenever this approximation may not be made, the vibrating surface may be broken up into smaller areas, infinitesimal if desired, the sum effect of which is equivalent to that of the total surface area of the actual source. The mathematics of this summation may be extremely complicated, but approximations will often lead to useful results.

1–1 Simple harmonic motion of a particle. Simple harmonic motion originates, in mechanics, because of the existence of some kind of unbalanced elastic force. With such a force, Newton's second law becomes, for a particle of mass m, free to move along the x-axis,

$$m\ddot{x} = -Kx. \tag{1–1}$$

In the expression on the right for the force, K is called the elastic constant, and the negative sign indicates that the restoring force always acts towards the origin. Equation (1–1) may also be written

$$\ddot{x} = -\omega^2 x, \tag{1–2}$$

where $\omega^2 = K/m$. This differential equation completely defines the type of motion and from it all other properties of simple harmonic motion may be obtained. By integrating Eq. (1–2) twice, the displacement equation may be shown to be of the form

$$x = x_m \cos(\omega t + \alpha), \tag{1–3}$$

where x_m is the amplitude of the motion and α is called the phase angle. The quantities x_m and α are essentially constants of integration, whose values depend upon the mathematical boundary conditions. They may easily be determined, for instance, if one knows the value of x and of the velocity, $\dot{x}$, at either the time $t = 0$, or at any other specific value of the time. Whether the cosine or the sine function appears in Eq. (1–3) is dependent upon these boundary conditions. If, for instance, α turns out to be $\pm\pi/2$, Eq. (1–3) may be written in the sine form. The angular frequency, ω, is equal to $2\pi f$, where f is the repetition rate in cycles per unit time.

Besides the displacement equation, two similar equations for the velocity, $\dot{x}$, and the acceleration, $\ddot{x}$, are important:

$$\dot{x} = -\omega x_m \sin(\omega t + \alpha), \tag{1-4}$$
$$\ddot{x} = -\omega^2 x_m \cos(\omega t + \alpha). \tag{1-5}$$

These are obtained by a simple differentiation of Eq. (1-3). All three equations can also be obtained by considering the projection, on a diameter of a circle, of the motion of a particle moving around the circle with a constant speed, as is usually shown in elementary physics. The phase relationship is apparent from Eqs. (1-3), (1-4), and (1-5). The velocity and displacement bear a 90° relationship, while acceleration and displacement are 180° apart. The 90° relationship which always results from differentiating a sine or cosine function will be an important feature of our discussion of sound waves in air, as will be seen later.

1-2 Energy in SHM. In sound, we are always dealing with the vibration of material bodies, or media having the property of mass, and since the particle being considered is moving, it will, in general, have a kinetic energy equal to $\frac{1}{2}m(\dot{x})^2$. This energy varies with the velocity, being zero at the ends of the motion, where $x = x_m$, and a maximum when the particle is passing through the position $x = 0$. Since no dissipative force is being considered, the total energy of the system must remain constant. Therefore when the kinetic energy decreases, as the particle approaches $x = x_m$, the potential energy must increase. Clearly, the maximum potential energy must equal the maximum kinetic energy. The maximum potential energy, $(E_p)_m = \int_0^{x_m} Kx\,dx = \frac{1}{2}Kx_m^2$. It is easy to show that this energy is equal to the maximum kinetic energy, $(E_k)_m$, possessed by the particle when it is moving through the central position. For, if $\dot{x}_m$ is the maximum velocity,

$$(E_p)_m = \frac{1}{2}\,Kx_m^2 = \frac{1}{2}\,K\frac{(\dot{x}_m)^2}{\omega^2} = \frac{1}{2}\,m(\dot{x}_m)^2 = (E_k)_m. \tag{1-6}$$

At positions other than the central one and the extreme end points, the energy is partly kinetic and partly potential. The total energy of the system may obviously be taken as either the maximum potential energy or the maximum kinetic energy. Using the latter,

$$E_{\text{total}} = \tfrac{1}{2}m(\dot{x}_m)^2 = \tfrac{1}{2}m\omega^2 x_m^2 = \tfrac{1}{2}m(4\pi^2)f^2 x_m^2. \tag{1-7}$$

It is interesting to note that for particles of equal mass executing simple harmonic motions of the same energy but of different frequencies, the amplitudes must be inversely proportional to the frequency. The paper cone of a radio loudspeaker, fed with the same energy at a variety of frequencies, will have imperceptible amplitudes at the high audible frequencies, whereas at low frequencies, visible amplitudes of as much as a millimeter or two may easily occur.

1-3 Combinations of SHM's along the same straight line. The combination of several collinear simple harmonic vibrations may be discussed either analytically or, more conveniently, by use of the graphical method commonly employed in a.c. circuit theory. This method is fundamentally based on the rectilinear projection of uniform circular motion, so often used in elementary physics to introduce SHM. In Fig. 1-1 the length of the vector represents the amplitude of the motion, x_m. The vector is conventionally assumed to rotate counterclockwise at the angular rate, ω (in radians per second). It is clear that the expression for the instantaneous projection of this vector, i.e., $x_m \cos(\omega t + \alpha)$, where α is the starting angle at $t = 0$, is identical with the displacement equation for SHM, Eq. (1-3).

Suppose, now, that we wish to represent the simultaneous execution, by a particle, of several SHM's along x, of differing amplitude, frequency, and phase angle. Each of these separate motions may be represented as the projection of an appropriate rotating vector. The simplest case to consider is when the frequencies are the same. The total displacement of the particle is

$$x_r = x_1 + x_2 + \cdots + x_n,$$

where x_1, x_2, etc., represent the separate displacements. Since all angular frequencies are the same, the relative angles between the different amplitude vectors are maintained at all times. Therefore it is possible at any time, such as at time $t = 0$, to sum up vectorially the several amplitude vectors and to consider the total motion, x, to be simply the projection of this resultant upon the x-axis. In Fig. 1-2 two amplitude vectors $(x_m)_1$ and $(x_m)_2$ are drawn for the time $t = 0$. The magnitude of the resultant

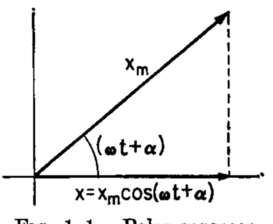


FIG. 1-1. Polar representation of SHM.

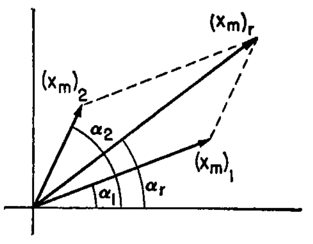


FIG. 1-2. Amplitude summation for two SHM's of the same frequency.

vector, $(x_m)_r$, may be obtained most simply by finding its x- and y-components, as is done in mechanics with force vectors:

$$(x_m)_r = \sqrt{[\Sigma(x_m)_x]^2 + [\Sigma(x_m)_y]^2}. \tag{1-8}$$

Also,

$$\tan \alpha_r = \frac{\Sigma(x_m)_y}{\Sigma(x_m)_x},$$

where $\Sigma(x_m)_x$ and $\Sigma(x_m)_y$ are the sums of the x- and y-components of the separate amplitude vectors at the time $t = 0$. The total motion, x_r, may then be written:

$$x_r = (x_m)_r \cos(\omega t + \alpha_r). \tag{1-9}$$

It is seen that such a combination of SHM's is always equivalent to a single pure SHM. This is a fact of fundamental practical importance in the production of music. In the first violin section of an orchestra, for instance, while at a given instant all violins are presumably playing at the same frequency and with approximately the same amplitudes, the relative phases are quite randomly related. Since these relative phases undoubtedly are shifting continuously due to slight frequency variations, the phase of the sum effect at the ear is also changing. As we shall see later, the ear ordinarily is insensitive to phase effects in music, and in the case of the violinists, only a single note of the common approximate frequency is heard.

This vector method of summing up SHM's of the same frequency but of differing phase will prove very useful in Chapter 4 in the consideration of sound diffraction.

Example. Reduce the following two collinear SHM's to a single equivalent vibration, finding the amplitude and the phase angle.

$$x_1 = 5 \cos(\omega t + 65°),$$
$$x_2 = 7 \cos(\omega t + 30°).$$

The two amplitude vectors are located at the time $t = 0$, as in Fig. 1-3. Making use of the cosine law, the resultant amplitude, $(x_m)_r$, may be found directly:

$$(x_m)_r = \sqrt{(5)^2 + (7)^2 + 2(5)(7)\cos 35°} = 11.4.$$

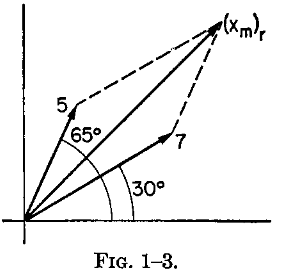


FIG. 1-3.

Or, using the x- and y-components:

$$\Sigma(x_m)_x = 5 \cos 65° + 7 \cos 30° = 8.18,$$

$$\Sigma(x_m)_y = 5 \sin 65° + 7 \sin 30° = 8.03,$$

$$(x_m)_r = \sqrt{[\Sigma(x_m)_x]^2 + [\Sigma(x_m)_y]^2} = \sqrt{(8.18)^2 + (8.03)^2} = 11.4.$$

The phase angle of the resultant vibration is arctan $\Sigma(x_m)_y/\Sigma(x_m)_x$ = arctan (0.982) = 44° 30′. Therefore the equation for x_r is

$$x_r = 11.4 \cos (\omega t + 44° 30').$$

1-4 Two collinear SHM's whose frequencies differ by a small amount. Beats. The phenomenon of beats, in sound, is a familiar one. As it is commonly observed, it is the slow, audible "throbbing," or variation in intensity, associated with two sounds of nearly the same frequency which alternately reinforce and partially or completely cancel each other. In Fig. 1-4a are shown two SHM's of slightly different frequency, the ordinate being the displacement and the abscissa, time. In the presence of two such sound waves, a layer of air (equivalent to the particle under discussion) will execute a motion which is the graphical sum of the two separate motions. In Fig. 1-4b is drawn the graphical sum of the curves of Fig. 1-4a. The periodic variation in amplitude, in the case of the sum curve, is to be expected, in view of the effects observed aurally.

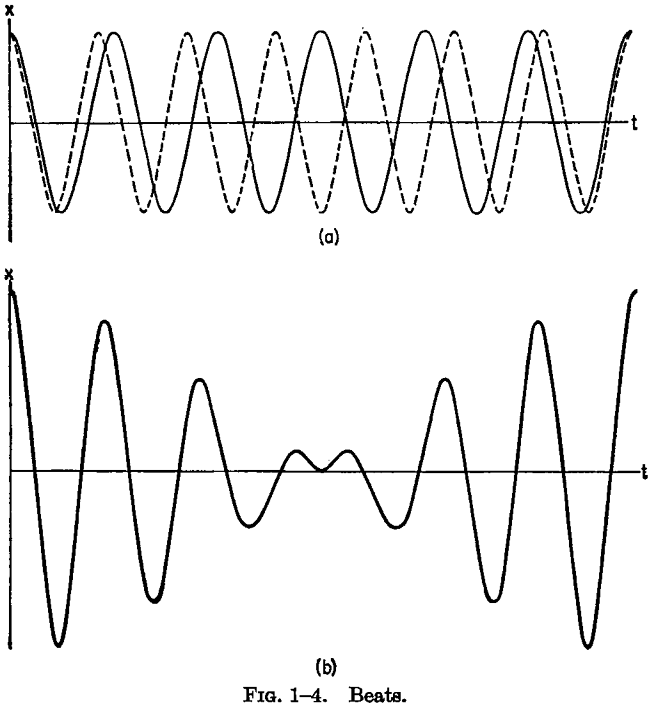


FIG. 1-4. Beats.

The two separate SHM'S may be represented analytically as

$$x_1 = (x_m)_1 \cos [\omega t + \alpha_1] \tag{1-10}$$

and

$$x_2 = (x_m)_2 \cos [(\omega + \Delta\omega)t + \alpha_2], \tag{1-11}$$

where $\Delta\omega$ is small compared with ω. Equation (1-11) may be rewritten:

$$x_2 = (x_m)_2 \cos \{\omega t + [(\Delta\omega)t + \alpha_2]\} \tag{1-12}$$

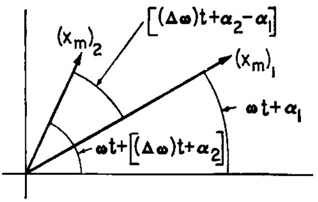


FIG. 1-5.

Both x_1 and x_2 may be thought of as projections of rotating vectors, as discussed in Section 1-3. Since the two amplitude vectors, $(x_m)_1$ and $(x_m)_2$, rotate with nearly the same angular velocity, the term $[(\Delta\omega)t + \alpha_2]$ in Eq. (1-12) may be considered as a slowly changing phase angle. When the two vectors are in the positions shown in Fig. 1-5, the resultant vector, $(x_m)_r$, may be computed by means of the cosine law:

$$(x_m)_r = \sqrt{(x_m)_1^2 + (x_m)_2^2 + 2(x_m)_1(x_m)_2 \cos [(\Delta\omega)t + \alpha_2 - \alpha_1]} \tag{1-13}$$

The magnitude of $(x_m)_r$ will slowly change, as time goes on, due to the variation of the cosine function in Eq. (1-13) with the time. The maximum and minimum values of $(x_m)_r$ will occur when the cosine function is equal to $+1$ and -1, respectively. The corresponding values for $(x_m)_r$ will be $(x_m)_1 + (x_m)_2$ and $(x_m)_1 - (x_m)_2$, assuming $(x_m)_1 > (x_m)_2$. The frequency, f_b, of this cyclic change in $(x_m)_r$ is plainly $\Delta\omega/2\pi$. Since $\Delta\omega$ is the difference between the angular frequencies for the two vibrations, ω_1 and ω_2, f_b will equal the difference between the vibration rates, f_1 and f_2.

If f_1 is nearly equal to f_2, what has been said in the preceding paragraph regarding the variations in $(x_m)_r$ will closely describe the variation in the amplitude of the motion along x, which is the *projection* of $(x_m)_r$. It is the projection of $(x_m)_r$, of course, which represents the instantaneous sum of x_1 and x_2, and which directly describes the beat phenomenon. The maximum value of $x = x_1 + x_2$ will vary periodically, at a frequency very close to the beat frequency, f_b, between limits which are very nearly $(x_m)_1 + (x_m)_2$ and $(x_m)_1 - (x_m)_2$. These are not exact statements, since in general $(x_m)_1$ and $(x_m)_2$ will not become coincident when in the horizontal position. However, since the two amplitude vectors are rotating with nearly the same angular velocity, it is clear that at whatever angle to the x-axis coincidence occurs, the two vectors will have only slight relative displacement by the time they *do* reach the horizontal, and the above statements are, for all practical purposes, valid. If $\Delta\omega$ is quite large compared with ω (not the case in ordinary sound beats) this method of interpretation has little meaning.

1–5 Mathematical vs audible beats. There is an interesting distinction between what might be called "mathematical" and "audible" beats. It can be shown that unless the two angular frequencies ω_1 and ω_2 are commensurate, that is, unless the ω's and therefore the two actual frequencies f_1 and f_2 bear a whole number ratio, the sum motion will *never* repeat exactly. Therefore no recurring beat phenomenon, in the strict mathematical sense, will exist. In addition, if the whole number relation *does* exist, each separate vibration must execute some integral number of cycles before a repetition of the sum motion can occur. To take numerical examples, suppose the two frequencies are 406 and 404 cycles/sec, respectively. The two vibrations will be in phase twice each second. This can readily be seen by reducing the frequency ratio to the smallest whole number ratio, i.e., $\frac{203}{202}$. If the two frequencies start in phase, after $\frac{1}{2}$ second, when they have executed 203 and 202 cycles respectively, they will be in phase again. The two beats each second obtained in this way would indicate that the beat frequency is always $f_1 - f_2$. This, however, is not invariably so, for if the two frequencies were 407 and 404 the difference frequency would indicate three beats a second, whereas $\frac{407}{404}$ being already the smallest whole number ratio, there is a mathematical repetition only *once* a second.

The above statements can be easily checked by consideration of the rotating vector example.

The *audible* effect of beats contains none of the subtleties discussed above. If two sources initially emit sound waves of the same frequency and then one frequency is gradually raised, the beat effect begins to occur smoothly and continuously, with no gaps occurring at discrete frequencies. This is because the ear is sensitive only to the envelope of the sum function, as in Fig. 1–4b, and an absence of an exact mathematical repetition *within* the envelope goes unnoticed.

When the simple difference frequency becomes greater than about ten per second, the alternation in intensity is no longer observed and, instead, one receives the impression of a steady sound which is either harmonious or discordant, depending on the frequency interval. (This will be discussed later in Chapter 9 in connection with consonance and dissonance.) With ordinary sound intensities a real difference frequency is never observed, that is, a third musical note is never evident. (With very large sound intensities, it is another matter. See Chapter 9.) This is not surprising, since there are really only two SHM's involved. The true beat effect is merely the alternation in intensity of what appears to be one frequency.

1–6 Combinations of more than two SHM's of different frequencies. From the discussion just concluded, a mixture of frequencies not bearing a whole number relation is equivalent to no repetitive steady state vibration. This situation is not often encountered in problems in sound — at least one does not usually attempt to analyze problems in which it *does* arise. Most musical instruments, fortunately, vibrate in such a way as to give rise to a "fundamental" tone and "overtones," all of which bear whole number ratios to one another, and consequently the over-all vibration is a repeating function. There is a theorem, due to Fourier, so powerful in its ability to analyze such a repeating function into its separate component frequencies that it deserves considerable attention in any discussion of vibration and sound.

1-7 Fourier's theorem. Stated briefly, this theorem asserts that any single-valued periodic and continuous function may be expressed as a summation of simple harmonic terms, finite or infinite in number (depending on the form of the function), whose frequencies are integral multiples of the repetition rate of the given function.* The restrictions that the function be single-valued and continuous are easily met in the case of the vibrations of material bodies, and the theorem is therefore of the greatest use in acoustics.

The most useful analytic expression for the harmonic series for periodic functions of the time is as follows:

$$x = f(t) = A_0 + A_1 \sin \omega t + A_2 \sin 2\omega t + \cdots + A_n \sin (n\omega t) + \cdots + B_1 \cos \omega t + B_2 \cos 2\omega t + \cdots + B_n \cos (n\omega t) + \cdots, \tag{1-14}$$

where the A's and B's are constants, to be determined.

Every term in this series may not always be present, depending on the nature of the function to be expanded. This will be made clear presently by an illustrative example. The presence or absence of a term will be known when one determines the constants A_0, A_n, and B_n. Formulas for this determination are obtained quite easily.

1-8 Determination of the Fourier coefficients. The constant term, A_0, is obtained by multiplying both sides of Eq. (1-14) by dt and then integrating over the time $t = T$, where T is the period ($T = 2\pi/\omega$) of the first term of *lowest* frequency. With this integration, all sine and cosine terms will disappear, since the area under any integral number of sine or cosine cycles is zero. Only the constant term will remain, and solving for A_0,

$$A_0 = \frac{1}{T} \int_0^T x \, dt. \tag{1-15}$$

* There are a number of additional mathematical restrictions placed upon the form of the function. The theorem fully applies to all functions encountered in problems in acoustics.

To evaluate A_0 it is necessary, of course, to have the expression for x as a function of time.

To obtain a typical coefficient, A_n, for the sine series, both sides of Eq. (1–14) are multiplied by $\sin (n\omega t)\, dt$ and again integrated from $t = 0$ to $t = T$. On the right-hand side, all but one of the integrations will involve products of the type $\sin (n\omega t) \sin (n'\omega t)\, dt$, where n and n' are different integers. Since

$$\sin (n\omega t) \sin (n'\omega t) = \frac{\cos [(n - n')\omega t] - \cos [(n + n')\omega t]}{2},$$

and since the integration will always be over an integral number of cycles, the result of all integrations on the right-hand side of Eq. (1–14) will be zero, except in the case where $n = n'$. For this latter case, the integration becomes

$$A_n \int_0^T \sin^2 (n\omega t)\, dt = A_n \frac{T}{2}.$$

Therefore, integration of both sides of Eq. (1–14) yields

$$\int_0^T x \sin (n\omega t)\, dt = A_n \frac{T}{2}.$$

Solving for A_n, we obtain

$$A_n = \frac{2}{T} \int_0^T x \sin (n\omega t)\, dt. \qquad (1\text{–}16)$$

In a similar way, by multiplying each term in (1–14) by $\cos (n\omega t)\, dt$ and integrating, term by term, from $t = 0$ to $t = T$, one may obtain the expression for B_n, the coefficient of a typical cosine term in the series:

$$B_n = \frac{2}{T} \int_0^T x \cos (n\omega t)\, dt. \qquad (1\text{–}17)$$

Whether or not the integrations represented by Eqs. (1–15), (1–16), and (1–17) are feasible will, of course, depend on the nature and complexity of the function, $x = f(t)$, to be expanded. In addition, while the harmonic series can be shown always to be convergent, so that the coefficients A_n and B_n become progressively smaller as the frequency of the term rises, this rate of convergence may be slow in the case of certain functions. In these cases, it may be necessary to include a large number of harmonic terms in order to achieve a reasonably good equivalence to the original function. In problems in sound the convergence is frequently fairly rapid. In addition, to the average ear, the over-all effect due to a complex sound vibration is often only slightly modified if the very high harmonics are removed or ignored.

In a function which exactly represents the combination of a finite number of pure sine or cosine variations, the series obtained by analysis of the sum function will contain a finite, not an infinite, number of terms. Analysis, for instance, of the vibration effect known as beats will yield only the two frequencies present. Similarly, the complex sound constituting the sum of three pure musical notes will analyze into those three frequencies alone.

Example. To illustrate the application of the formulas developed above for the series coefficients, an analysis of the function represented graphically by the so-called "saw-toothed" wave will suffice. This function, shown graphically in Fig. 1–6, may be defined analytically as

$$f(t) = b\left(\frac{1}{2} - \frac{t}{T}\right)$$

for the time interval $t = 0$ to $t = T$. After this time the function repeats with a fundamental period, T ($1/T$ is then the frequency of the first sine or cosine term). Then

$$A_0 = \frac{1}{T}\int_0^T x\,dt = \frac{b}{T}\int_0^T \left(\frac{1}{2} - \frac{t}{T}\right) dt = 0.$$

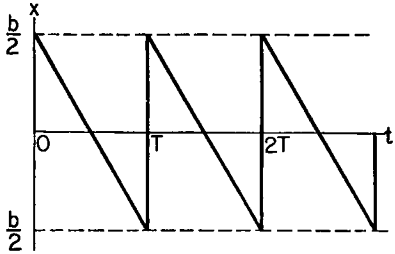


FIG. 1–6. Graph of saw-tooth wave.

It should be noted that A_0 is here zero because of the complete symmetry of the graph about the time axis. Wherever this symmetry is lacking, the constant term will not be zero.

The coefficient of a typical sine term becomes, in this problem,

$$A_n = \frac{2b}{T}\int_0^T \left(\frac{1}{2} - \frac{t}{T}\right) \sin\,(n\omega t)\,dt = \frac{2b}{n\pi}.$$

The amplitudes of the successive terms are then

$$\frac{2b}{\pi}, \frac{2b}{2\pi} \cdots \frac{2b}{n\pi}.$$

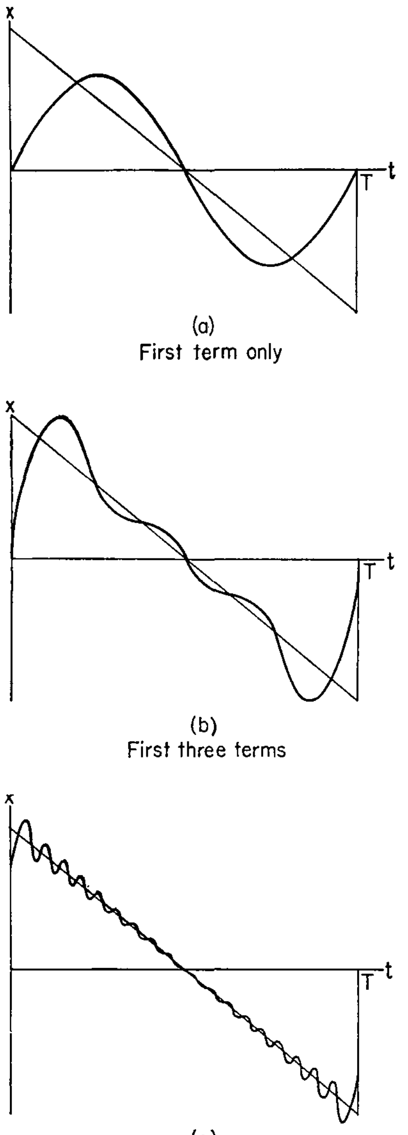
The cosine series is, in this problem, completely absent, since

$$B_n = \frac{2b}{T}\int_0^T \left(\frac{1}{2} - \frac{t}{T}\right) \cos\,(n\omega t)\,dt = 0,$$

regardless of the value of n. The complete series equivalent to the saw-tooth wave is therefore

$$x = f(t) = \frac{2b}{\pi}\left(\sin\omega t + \frac{1}{2}\sin 2\omega t + \cdots + \frac{1}{n}\sin\,(n\omega t) + \cdots\right).$$

1–9 Even and odd functions. In general, the absence of all the sine terms, or of all the cosine terms, depends on whether the original repeating function is "even" or "odd." An even function is one such that $f(t) =$



(a)
First term only

(b)
First three terms

(c)
First twenty terms

FIG. 1–7. The effect of including additional terms in the Fourier series.

$f(-t)$. An odd function, on the other hand, is one where $f(t) = -f(-t)$. If the saw-tooth graph of Fig. 1–6 is repeated to the left of the origin, where t is negative, it may be verified that in this problem the conditions for an odd function are satisfied. Therefore the equivalent series contains only the sine terms. For the function to be even there must plainly be a mirror symmetry around the y-axis. This symmetry obtains, for instance, in the case of a simple cosine curve, there being, in this case, no sine terms. In the interests of saving computing labor, it will pay to first classify the given function as either even or odd. There are many functions, of course, which are neither even nor odd, in which case there will be both sine and cosine terms.

1–10 Convergence. It is clear in the problem just discussed that the harmonic terms become of smaller and smaller amplitude as the frequency rises. The complete infinite series must be considered for a complete equivalence. In Fig. 1–7 one can see the approach to the saw-tooth wave as more and more terms are added. The precision desired determines how far the computation is carried out. In general, it is near points of abrupt changes of slope that the "fit" is poorest, when using a finite number of terms.

The example above will suffice to show the general method of computing the Fourier coefficients. Other practice problems of a similar nature will be found at the end of the chapter.

1–11 Application of the Fourier analysis to empirical functions. Because one starts the analysis already knowing the analytic expression for the function, it may appear that problems of the above type are very artificial. Experimentally, the motions of vibrating bodies and the vibrations of air itself are usually picked up by electrical or electromagnetic means and are studied by means of a recording galvanometer or an oscilloscope, and we therefore have a graph to analyze, not an analytic function. From the principles of the Fourier analysis just discussed, graphical methods may be developed whereby, through the use of selected ordinates, the amplitude of the various harmonic terms may be determined with any desired precision. (This material may be found in many texts on electrical engineering.) In recent years many so-called harmonic analyzers have been built which, by mechanical or electronic means or a combination of both, perform the desired analysis with great saving of labor and with the highest precision. In Chapter 10 there is described an acoustical equivalent to the optical diffraction grating that may be used to determine very quickly the approximate harmonic content in any complex sound.

1–12 Damped vibrations of a particle. So far no force other than an elastic restraining force has been assumed to act upon the particle (or upon the sound source treated as equivalent to a particle). No such mechanical system exists in nature (at least in the large scale or macroscopic world!), since some sort of friction or dissipative force is always present. It will be assumed that the dissipative force acting upon the particle is in the nature of fluid friction and is of the form $F = -r\dot{x}$. The constant r is the force per unit velocity. The negative sign is necessary to show that the force is always opposite in direction to the velocity. In general, fluid friction is a function of the velocity raised to some power. The first power is used here as a first approximation. If the velocity is not too great, this approximation is reasonably good and, in addition, the use of the first power greatly simplifies the differential equation.

For a particle moving under the action of an elastic force and also of a viscous force of the above type, Newton's second law may be written

$$m\ddot{x} = -Kx - r\dot{x}$$

or, after transposing all terms to the left,

$$m\ddot{x} + r\dot{x} + Kx = 0. \tag{1–18}$$

This linear differential equation arises many times in different branches of physics. The student of electricity, for instance, will encounter an equation of exactly this form when he studies the transient behavior of an L–R–C circuit. This analogy will be discussed in more detail in Chapter 5.

The solution to Eq. (1–18) may be obtained in a number of ways. In Chapter 5, when the use of complex quantities is introduced, a more general method of solving this and related equations will be discussed. At this point, a change of variable will yield results more quickly. Let

$$x = y\epsilon^{-bt}, \tag{1–19}$$

where b is an arbitrary constant. Differentiating Eq. (1–19) and substituting in Eq. (1–18), we obtain

$$\ddot{y} + \left(\frac{r}{m} - 2b\right)\dot{y} + \left(\frac{K}{m} + b^2 - \frac{r}{m}b\right)y = 0. \tag{1–20}$$

In this new equation in y, the constants m, r, and K are fixed by the nature of the system being considered, but the constant b which appears first in the change of variable equation, Eq. (1–19), may be selected quite arbitrarily. If b is chosen equal to $r/2m$, the second differential term in Eq. (1–20) will vanish and the whole equation will take the much simpler form

$$\ddot{y} + (\omega_u^2 - b^2)y = 0, \tag{1–21}$$

where ω_u^2 has been substituted for K/m. The values of y which are solutions to Eq. (1–21) can be obtained quite simply. Then, according to Eq. (1–19), x, the actual particle displacement, may be obtained by simply multiplying the value of y by ϵ^{-bt}.

There are three important types of solutions to Eq. (1–21), whose form depends on the values of the system parameters, m, r, and K.

1–13 Case I. $\omega_u^2 < b^2 \left(\text{or } \dfrac{K}{m} < \dfrac{r^2}{4m^2}\right)$. **Large frictional force.** When the system constants are such that ω_u^2 is less than b^2, the algebraic sign of the coefficient of y in Eq. (1–21) is negative. The solution to the differential equation can then readily be shown to be

$$y = A_1\epsilon^{\sqrt{b^2-\omega_u^2}\,t} + A_2\epsilon^{-\sqrt{b^2-\omega_u^2}\,t}, \tag{1–22}$$

A_1 and A_2 being integration constants. Therefore, using Eq. (1–19), we find that

$$x = A_1\epsilon^{-(b-\sqrt{b^2-\omega_u^2})\,t} + A_2\epsilon^{-(b+\sqrt{b^2-\omega_u^2})\,t}. \tag{1–23}$$

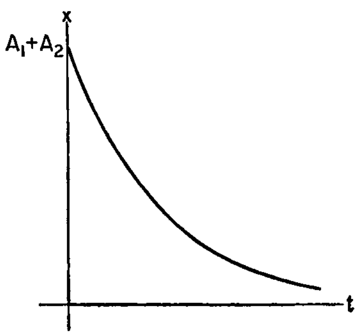


FIG. 1–8. Graph of Eq. (1–23).

The values of the integration constants A_1 and A_2 may be determined if the initial or other time conditions of the problem are known. Since $b > \sqrt{b^2 - \omega_u^2}$, both exponents are intrinsically negative, and the particle, once displaced, will always return to the position $x = 0$ asymptotically with time. The rate of this approach to zero will depend on the values of ω_u and b. The graph in Fig. 1-8 shows the subsequent motion after $t = 0$, in the case where both A_1 and A_2 are positive. Two initial conditions, such as displacement and velocity at the time $t = 0$, may be used to determine the constants A_1 and A_2.

1-14 Case II. $\omega_u^2 > b^2 \left(\text{or } \dfrac{K}{m} > \dfrac{r^2}{4m^2}\right)$. **Small frictional force.** In this case the coefficient of y in Eq. (1-21) changes sign, i.e., $(\omega_u^2 - b^2)$ is positive, and the equation is readily recognized as in the form for SHM. Clearly, then, the solution is

$$y = y_m \cos(\omega' t + \alpha),$$

and therefore

$$x = y_m \epsilon^{-bt} \cos(\omega' t + \alpha). \tag{1-24}$$

where

$$\omega' = \sqrt{\omega_u^2 - b^2}.$$

Equation (1-24) describes a *damped* harmonic motion, whose effective amplitude, $x_m = y_m \epsilon^{-bt}$, dies out exponentially with the time. The initial amplitude and phase angle are, respectively, y_m and α. The constant $b(= r/2m)$ determines the time rate of damping. The envelope of the curve represented by Eq. (1-24) is, effectively, the exponential curve $x = y_m \epsilon^{-bt}$, as shown in Fig. 1-9.

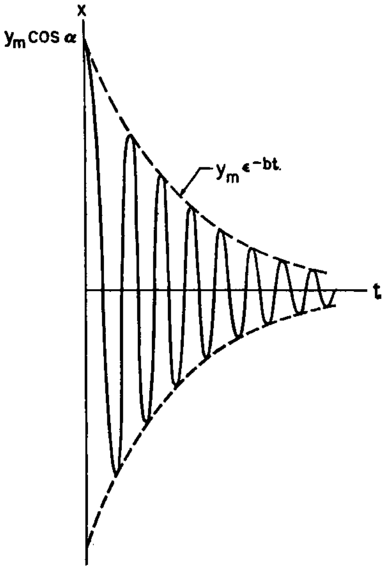


FIG. 1-9.

With no damping, i.e., when $b = 0$, the frequency of the motion is $\omega_u/2\pi$. Where damping exists, the natural frequency is always lowered, since the frequency is $\omega'/2\pi$ and ω' is always less than ω_u. In fact, as the value of b is increased (say, by keeping m constant and increasing the frictional coefficient, r), the oscillation frequency approaches zero as b approaches ω_u. Practical sound sources are usually so lightly damped that the damping factor, b, does not greatly affect the frequency. This is especially true because of the quadratic relation between ω', ω_u, and b.

The length of time required for an oscillation to die out is of practical importance in sound. The time for x to become zero is, of course, infinite from the mathematical point of view, but in the case of sound waves an amplitude below a certain minimum will be inaudible to the ear, and some quantitative measure of the rate at which the amplitude diminishes is desirable. A commonly used quantity for this measure is the modulus of decay, $1/b$, often called the *time constant*. This is the time for the amplitude of the cosine function in Eq. (1-24) to drop to the fraction $1/\epsilon$ of its initial value. Since $b = r/2m$, it will be seen that a large frictional coefficient, r, and a small mass, m, will make the time constant small. A small time constant implies a rapid rate of decay. It will be seen in Chapter 7 that the moduli of decay of the different harmonic frequencies generated by musical instruments are of considerable importance in determining the quality of the sound produced.

1-15 Case III. $\omega_u^2 = b^2\left(\text{or } \dfrac{K}{m} = \dfrac{r^2}{4m^2}\right)$. Critical damping.

This is a case of more importance in scientific instrument design than in the behavior of sound sources. When $\omega_u^2 = b^2$, Eq. (1-21) becomes simply

$$\ddot{y} = 0. \tag{1-25}$$

The solution to this equation is a straight line, of the form

$$y = A_1t + A_2,$$

where A_1 and A_2 are again constants of integration. The expression for x then becomes

$$x = \epsilon^{-bt}(A_1t + A_2). \tag{1-26}$$

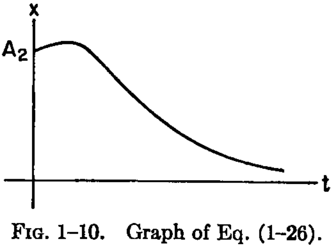


FIG. 1-10. Graph of Eq. (1-26).

Plotted, this equation does not look greatly different from the solution for the case where b^2 is greater than ω_u^2 (friction large). Figure 1-10 represents the plot of Eq. (1-26) for the case where A_1 is large and where both A_1 and A_2 are positive (this is, of course, not necessarily so). If A_1 is numerically large, x will increase at first, but eventually the exponential coefficient will bring about a reversal of slope and x will approach zero as time progresses. Rarely are actual sound sources so critically damped.

Example. An example will show how the physical constants of the vibrating system are used and also the technique for the evaluation of the constants of integration.

A particle of mass 3 gm is subject to an elastic force of 27 dyne-cm^{-1} and a damping force of 6 dyne-cm^{-1}-sec. It is displaced a distance of $+1.0$ cm from its equilibrium position and released. It is required to determine whether or not the motion is oscillatory and, if so, to find its period; also the complete equation for x as a function of time is to be obtained, with the numerical values of the amplitude x_m and the phase angle α.

From the data given, $\omega_u = \sqrt{K/m} = 3$ sec^{-1} and $b = r/2m = 1.0$ sec^{-1}. Since $\omega_u > b$, the solution is oscillatory, of the form:

$$x = y_m \epsilon^{-bt} \cos\left(\sqrt{\omega_u^2 - b^2}\, t + \alpha\right).$$

The period is $2\pi/\sqrt{\omega_u^2 - b^2} = 2.23$ sec. To find the initial amplitude y_m and the phase angle α (the integration constants), the initial position and velocity may be used. Differentiating x, we obtain

$$\dot{x} = -y_m \epsilon^{-bt}\left[\sqrt{\omega_u^2 - b^2} \sin\left(\sqrt{\omega_u^2 - b^2}\, t + \alpha\right) + b \cos\left(\sqrt{\omega_u^2 - b^2}\, t + \alpha\right)\right].$$

Setting $t = 0$ and inserting the values $x = 1.0$ and $\dot{x} = 0$, two equations may be obtained for the determination of α and y_m, that is:

$$\tan \alpha = -\frac{b}{\sqrt{\omega_u^2 - b^2}}$$

and

$$y_m = \frac{1.0}{\cos \alpha}.$$

Solving for α and y_m, we obtain

$$\tan \alpha = -0.353; \qquad \alpha = -19°\,30'; \qquad y_m = 1.06 \text{ cm}.$$

Therefore the complete expression for x is

$$x = 1.06\epsilon^{-1.0t} \cos(2.82t - 19°\,30').$$

1-16 Forced vibrations. All sound sources are set into vibration by some external source of energy, capable of supplying some kind of periodic force. Sometimes the mechanism of this energy transfer is quite complicated, as, for instance, in the excitation of a violin string or in the sounding of an organ pipe. A simpler example to consider is the setting into motion of a pendulum by the application of an external force of a periodic nature.

In practice, the periodic driving force is rarely a simple harmonic variation of a single frequency. The cone of a radio loudspeaker which is reproducing music, for instance, is being driven by a variable force equivalent to a mixture of periodic forces of assorted frequencies. If, however, we can discover how the particle will behave under the action of a driving force of one particular frequency, we are ready, by means of the superposition principle, to understand its motion when there are many frequencies.

1–17 The differential equation. Let the instantaneous driving force be represented by $F = F_m \cos \omega t$, where, as before, $\omega = 2\pi f$. Writing Newton's second law for a particle subject to an elastic and a damping force, we obtain

$$m\ddot{x} + r\dot{x} + Kx = F_m \cos \omega t. \tag{1–27}$$

The general solution to this equation is made up of two parts, mathematically speaking. The first part is the *complementary function*, which is the solution to Eq. (1–27), with the right-hand side set equal to zero. Since this modified differential equation is exactly the one just discussed under the heading of "damped vibrations of a particle," it is clear that the complementary function may actually take one of three forms, depending on the factors m, r, and K.

The complete solution to Eq. (1–27) must contain, besides the complementary function, a second part, which constitutes a *particular solution* to Eq. (1–27), with the right-hand side $\neq 0$. This particular solution must satisfy the complete differential equation for all values of the time t. It will be remembered that in the presence of damping all solutions to the simpler differential equation (where the right-hand side of (1–27) is set equal to zero) are of a form such that x approaches zero with the passage of time. This part of the general solution to Eq. (1–27) (i.e., the complementary function) is therefore called the *transient* part. With physical vibrations it can usually be neglected after a short time. The remaining part of the solution, the *particular* or *steady state solution* referred to, will then be the only significant part, for later times. It is this important steady state solution that we shall now consider.

1–18 The steady state solution for forced vibrations. To obtain the steady state part of the solution to Eq. (1–27), it is most convenient to compare the differential equation with an exactly similar one arising in electricity. If an emf, varying in a simple harmonic manner, is impressed upon a series circuit with inductance L, resistance R, and capacitance C, we may write the equation

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = E_m \cos \omega t, \tag{1–28}$$

where E_m is the maximum value of the impressed emf and q is the charge on the capacitor at any instant. Since for an electrical circuit the current i is equal to dq/dt, we may write Eq. (1–28) in terms of the current:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\, dt = E_m \cos \omega t. \tag{1–29}$$

A comparison of Eq. (1–27) with (1–28) will show the mathematical form to be identical. In addition, there is an equation for the mechanical system in terms of the *velocity* $\dot{x}$, which is the exact counterpart of Eq. (1–29) for the current i, $= \dot{q}$. This means that if the electrical equations have been solved, the equations for the mechanical system have also been solved. Writing down the solutions to the electrical equations, we have only to replace the electrical parameters with those of the particle system and to insert the variable x instead of the variable q to obtain the solutions to the mechanical equations.

The steady state solution to Eq. (1–29), the electrical equation in terms of the current, is the ordinary expression for the instantaneous value of the alternating current in an L–R–C circuit, familiar to most students of elementary electricity. The expression for this current, i, is

$$i = \frac{E_m}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} \cos(\omega t - \alpha), \tag{1–30}$$

where

$$\tan \alpha = \frac{\omega L - \dfrac{1}{\omega C}}{R}. \tag{1–31}$$

In this equation the angle α represents the phase relationship between the impressed potential and the current. Not so familiar is the expression for the charge q. The equation for q may be easily obtained by integrating Eq. (1–30) with respect to time. (Note that the constant of integration must be zero, since there is no d.c. component to the impressed potential.)

$$q = \frac{E_m}{\omega\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} \sin(\omega t - \alpha). \tag{1–32}$$

It will be remembered that the expression in the denominator of Eq. (1–30) is called the total electrical *impedance* of the circuit, while the collection of terms, $\left(\omega L - \dfrac{1}{\omega C}\right)$, is called the circuit *reactance*, commonly represented by the symbol X.

We can now write the analogous equations for the mechanical system, where the displacement x replaces charge, and the velocity $\dot{x}$ replaces current:

$$x = \frac{F_m}{\omega\sqrt{r^2 + \left(\omega m - \dfrac{K}{\omega}\right)^2}} \sin(\omega t - \alpha) \tag{1–33}$$

and

$$\dot{x} = \frac{F_m}{\sqrt{r^2 + \left(\omega m - \frac{K}{\omega}\right)^2}} \cos(\omega t - \alpha). \tag{1-34}$$

So exact is the parallel between the mechanical and the electrical problem that it is common to use for the mechanical system such expressions as "mechanical impedance," "mechanical resistance," and "mechanical reactance." (This use of the concept of impedance, as applied to a particle or its equivalent, is not to be confused with the idea of "radiation impedance," to be introduced in Chapter 5. This latter concept is used only in connection with wave propagation and involves a quite different use of the word impedance.) Note that in the comparison of the mechanical and the electrical parameters, r is analogous to R, m to L, and $1/K$ to C. $1/K$ is called the "compliance" of the system, since it is the reciprocal of K, the elastic "stiffness" constant. More will be said about the use of analogies in Chapter 5.

1-19 Velocity and displacement resonance. In the electrical equation, (1-30), so-called series resonance occurs when the current is in phase with the applied potential or, from (1-31), when the reactance is zero, i.e., $\omega L = 1/\omega C$. Under these conditions, since the impedance is a minimum, the value of the current, I_m, will be a maximum, and so will the "root mean square" current, I_{rms}. For the mechanical system, this means that the criterion for *velocity* resonance is that $\omega m = K/\omega$. If this condition is brought about by the variation in the angular driving frequency ω, other parameters remaining constant, $\dot{x}_m$ will then be a maximum. This corresponds to the maximum current observed in the circuit.

Of more interest in the mechanical than in the electrical problem is another kind of resonance, *displacement* resonance. Again considering ω as the variable, this resonance may be said to occur when the amplitude of x, i.e., x_m, in Eq. (1-33) is a maximum. Since ω appears outside the radical in the denominator, as well as inside, it is necessary to differentiate with respect to ω the coefficient of the sine expression on the right and set the result equal to zero, in order to determine the exact criterion for resonance. The necessary condition may be stated as follows:

$$\omega^2 = \omega_u^2 - 2b^2, \tag{1-35}$$

where ω_u^2, as earlier, $= K/m$ and $b = r/2m$.

It should be noted that if the frictional coefficient, r, is small, so that $2b^2$ is much less than ω_u^2, then the condition for amplitude resonance is very nearly that $\omega^2 = \omega_u^2$. Since $\omega_u^2 = K/m$, this condition is seen to be identical with that for velocity resonance. It is worth noting that with low damping the frequency f at which amplitude resonance occurs is identical with the natural frequency of vibration of the particle under the action of an elastic force only, i.e., $f = \omega/2\pi = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$. (With velocity resonance this is *always* true, regardless of the degree of the damping.) When the damping is large, however, so that the term $2b^2$ in Eq. (1–35) becomes important, the frequency for amplitude resonance is lowered. Indeed, if the damping is so large that $2b^2$ is greater than ω_u^2, there is then no true resonance at all, since in Eq. (1–35) ω is then imaginary.

In Fig. 1–11 are shown a number of curves for different degrees of damping, each curve being a plot of the amplitude x_m against the angular frequency of the driving force, ω. With low damping it is seen that resonance virtually occurs when $\omega = \omega_u$. When the damping is increased, the position of the maximum shifts to the left. Curve 4 represents the transition case such that with any increased damping, no true maximum occurs.

1–20 The amplitude at resonance. It is clear from Fig. 1–11 that the maximum value of x_m at resonance is a function of the degree of damping. The exact value of this maximum ordinate, $(x_m)_{res}$, is determined by inserting the condition given by Eq. (1–35) into the expression

$$(x_m)_{res} = \frac{F_m}{\omega\sqrt{r^2 + \left(\omega m - \frac{K}{\omega}\right)^2}}. \tag{1–36}$$

A simpler, approximate expression for $(x_m)_{res}$ may be readily obtained if the damping is low (usually the case in acoustics). In this case the condition for amplitude resonance is practically that for velocity resonance, i.e., that the mechanical reactance $X = (\omega m - K/\omega) = 0$. We then have

$$(x_m)_{res} = \frac{F_m}{\omega r} = \frac{F_m}{\omega_u r}. \tag{1–37}$$

For low damping, it is seen that the amplitude at resonance is inversely proportional to the frictional coefficient, r, becoming very large as the

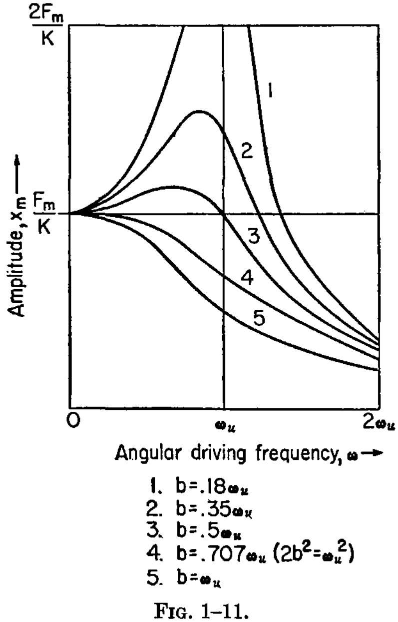


FIG. 1–11.

damping factor approaches zero. As in the case of electrical resonance, it is near resonance that the amplitude is affected most markedly by the value of the dissipative element. Well off resonance, it is the mechanical reactance, $\left(\omega m - \frac{K}{\omega}\right)$, that mainly determines the amplitude.

1-21 Phase relationships. In general, varied phase relations will obtain between particle displacement and driving force, and between particle velocity and driving force. In the latter case, the phase angle relationship should be familiar from alternating current circuit theory; at velocity resonance, $\dot{x}$ and F are in phase. At frequencies above resonance the effect of the mechanical mass reactance, ωm, predominates; $\dot{x}$ lags F by a greater and greater angle, approaching $\pi/2$ for values of ωm large compared with K/ω and r. Below resonance, the angle is a lead, since it is the term K/ω, containing the compliance, that is important, and the angle approaches $\pi/2$ for large values of K/ω. In the case of the displacement x the angles are different, since the displacement is 90° out of phase with the velocity. Figure 1-12 is a graph of the phase angle between the displacement x and the driving force, $-(\alpha + \pi/2)$, plotted against angular driving frequency for various values of b. When $\omega = \omega_u$, regardless of the value of b, the phase angle is $\pi/2$ and is a lag. At very low frequencies the angle approaches zero; for very high frequencies the lag approaches π. For low damping, where b is small, the phase angle shifts rather abruptly as the driving frequency is varied from a little below the value $\omega_u/2\pi$ to a little above. With greater damping the change is more gradual.

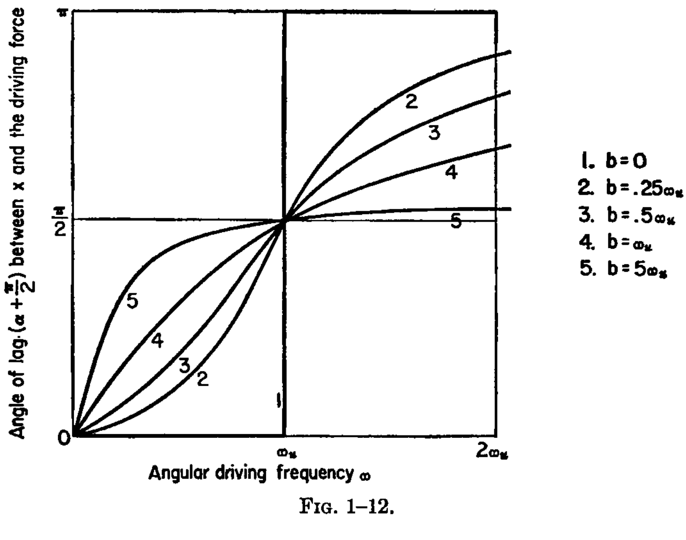


FIG. 1-12.

A simple demonstration of the above phase relationships can be set up as in Fig. 1–13. A heavy and a light plumb bob, M and m respectively, are suspended from a somewhat flexible common support, such as a horizontally stretched string. The two pendulum lengths are adjusted to be slightly different. If the heavy pendulum is set swinging, the lighter one will soon begin to oscillate also, due to the coupling at the support. The amplitude of this induced motion will alternately build up and die down as energy flows back and forth between the two pendulums in this coupled system (the heavier one, having the larger mass and energy, will not be appreciably affected). During the peaks of the induced oscillations, the above phase relations may be clearly seen. The driven system here is one of very low damping. Therefore if the heavier pendulum is longer and the driving frequency consequently lower than that of the driven system, the phase angle will be almost zero. On the other hand, if the heavier pendulum is shortened so that the driving frequency is higher than that of the driven system, the two pendulums will be almost 180° out of phase, being at opposite ends of their motions at the same time. When the two pendulums are of the same length ($\omega = \omega_u$), the 90° relationship can also be clearly seen.

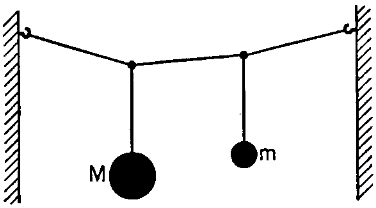


FIG. 1–13. Forced oscillations.

1–22 Energy transfer in forced oscillations. Unless the two pendulums in the above experiment are of nearly the same length, very little energy will be transferred. This is in line with common experience and can readily be shown with equations. The instantaneous power delivered to the particle system is $F\dot{x}$. This is the analog of electrical power, ei. In both the electrical and the mechanical case it is the time average of this product over a large number of cycles which constitutes the real power delivered. In the electrical case the time average of the product $ei(= E_m \cos \omega t\, I_m \cos (\omega t - \alpha))$ becomes $E_{rms} I_{rms} \cos \alpha = I^2_{rms} R$, where E_{rms} and I_{rms} are root mean square values. The angle α is the angle between current and applied potential, and R is the circuit resistance. Analogously, for the particle, since F and $\dot{x}$ are periodic functions of the time, just as are e and i, real average mechanical power may be written $F_{rms}\dot{x}_{rms} \cos \alpha$ or $(\dot{x}_{rms})^2 r$. The expression $(\dot{x}_{rms})^2 r$ shows that with a system having fixed damping characteristics the power delivered will be a maximum whenever the velocity, $\dot{x}_{rms}$, is a maximum. $\dot{x}_{rms}$ is itself a function of r and at resonance $= F_{rms}/r$. Therefore real power, Ω_{res}, at resonance may be written

$$\Omega_{res} = (\dot{x}_{rms})^2_{res} r = \frac{(F_{rms})^2}{r}. \tag{1-38}$$

(Compare with the electrical equivalent, $(E^2_{rms})/R$.) The lower the damping, obviously, the greater will be the delivered power.

Example. A particle has a mass of 2 gm. It is free to vibrate under the action of an elastic force of 128 dyne-cm^{-1} and a damping force of 8 dyne-cm^{-1}-sec. A periodically varying outside force of maximum value 256 dynes is applied to the particle. It is required to find the frequency $(f_{res})_d$ for displacement resonance and also the frequency $(f_{res})_v$ for velocity resonance, and the approximate amplitude at displacement resonance.

In this case $\omega_u^2 = K/m = 64$ sec^{-2} and $b^2 = r^2/4m^2 = 4$ sec^{-2}. For displacement resonance, $\omega^2 = \omega_u^2 - 2b^2$ Therefore the required frequency is

$$(f_{res})_d = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\omega_u^2 - 2b^2} = 1.19 \text{ sec}^{-1}.$$

For velocity resonance, $\omega m = K/\omega$, or

$$(f_{res})_v = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = 1.27 \text{ sec}^{-1}.$$

Since $2b^2$ is considerably less than ω_u^2, we may use the approximate expression for the amplitude at displacement resonance:

$$(x_m)_{res} \cong \frac{F_m}{\omega r} = 4.0 \text{ cm}.$$

1-23 Some applications of the theory of forced vibrations. From the graphs of Fig. 1-11 several useful conclusions can be drawn. If we are interested in transferring the maximum energy at a *single* frequency to a system capable of vibration, it is obvious from the graphs and from the previous discussion of energy that the damping factor of the system should be as small as possible and that the driving frequency should be near the natural frequency of the system. The crystal vibrators used in the production of ultrasonic waves are good examples of low-damped systems. In addition, the smaller the damping factor, the longer the persistence of any sound energy set up after the driving force has ceased. The vibrations of musical instruments persist for an appreciable time after energy ceases to be supplied. There are two kinds of damping involved in the decay of these vibrations. First, there are the internal frictions set up within the sound source (string, bar, or plate, as the case may be). This type of friction is undesirable from an energy point of view, as it results in the degeneration of vibrational energy into thermal energy. The second kind of damping is due to the presence of the surrounding air, and constitutes the only mechanism by which sound energy is radiated into space. For high radiation efficiency this type of damping should be large compared with the damping due to internal friction. In Chapter 5 this aspect of vibration will be discussed more fully in connection with "radiation resistance."

Ever since the advent of the phonograph and the radio set there has been a need for a source of sound reproduction which is capable of vibration at *all* audible frequencies, with no partiality to any one such frequency. For many different reasons the ideal source has not been found; some of the difficulties will be discussed later. A study of the curves of Fig. 1-11 will suggest one possible solution to the problem. By designing the system (treated as a particle) so as to have rather high damping, and by placing the resonant frequency above audibility, we may obtain a virtually aperiodic response to a driving force over a wide, useful frequency range.

In Fig. 1-11, Curve 3 shows this approximately aperiodic property for values of $\omega < \omega_u$. Unfortunately, in order for the system to have this type of response, the damping must be quite large. If, in a radio loudspeaker, the damping could be mainly that due to the air load, this would be all to the good, for the sound radiating efficiency would then be high. Unless the area of the vibrating source is impossibly large, as will be shown, the damping due to the air is likely to be much smaller than is necessary to approach critical damping. The required damping must then be obtained by artificially increasing the internal losses, which will result in very low over-all sound efficiency. Fortunately for efficiency, it is actually undesirable for such a sound source to have strictly aperiodic properties. Sound sources are usually poor radiators at low frequencies, for reasons not connected with their own intrinsic vibration properties. By reducing the damping well below the critical and placing the loudspeaker resonance near the lower end of the audible spectrum, the increased amplitude near resonance will make the output more uniform.

There is another interesting use that may be made of the phase angle graphs of Fig. 1-12. In general, the motion of a particle undergoing forced oscillations, with or without damping, will lag the driving force by some small time which will depend, in a rather complicated way, upon the driving frequency. This means that when a series of frequencies of particular relative phases are impressed upon the particle (constituting a complex forced vibration), the resulting particle motion will not be a complete replica of the variation in the driving force because of the assorted phase lags. If, however, referring to Fig. 1-12, a damping factor is so chosen as to make the phase angle approximately linear with driving frequency (such as with $b \cong .75\omega$), the original phase relationship will be maintained.

This can be seen from Eqs. (1-33) and (1-34). If $\alpha \propto \omega$ or $\alpha = A\omega$, then the angle on the right-hand side may be written $[\omega(t - A)]$, showing what amounts to a simple shift of the time axis, all frequencies being shifted together by the amount A. In actual practice, it is usually unnecessary to worry about phase shifts in sound, since the ear, at least for stimuli of the usual steady state type, is unaware of the phase relations in a complex sound wave. This may not be true, however, in the case of short-duration transients.

1-24 The importance of the transient response. A word may be said here about the transient response of a system equivalent to a particle, undergoing forced vibrations. The transient part of the solution to Eq. (1-27), while of short duration, may have considerable effect on the quality of a musical instrument and in some cases may distort or even mask the desired steady state frequency. The difference in the quality of a violin during the rapid playing of scales as compared with the sound of long, sustained notes is quite apparent. It is only in the latter case that the transient vibrations have had time to die out. The characteristic sound of a drum is due entirely to a transient, the driving force being of very short duration. Consider again the radio loudspeaker, whose purpose it is to transform into sound all driving frequencies applied to it. Whenever a new driving force is applied, there may be an important transient response amounting to some 10 to 20 vibrations or more at the natural frequency of the diaphragm, this frequency having nothing whatever to do with the driving frequency. As a result, all sounds which are abruptly cut off appear to have a "tail" or "hangover." Short duration sounds, like those originating from the drum, appear to have about the same monotonous frequency (i.e., that of the *speaker* resonance). These effects are minimized by an increase in the damping factor, but for most present-day radio reproducers this damping is not sufficient to overcome these effects.

1-25 Superposition of SHM's mutually perpendicular. This interesting case is important mainly because of the modern use of the oscilloscope in the study and measurement of sound. In this instrument, vertical and horizontal motions are imparted to an electron beam by means of vertical and horizontal field forces. If these forces vary sinusoidally with the time, the luminous spot on the screen will execute the motions to be described. The curves traced are called *Lissajous' figures*. Lissajous himself obtained these figures originally by observing the rectilinear vibrations of a particle while sighting through a microscope, itself mounted upon the prong of a tuning fork, free to oscillate at right angles to the particle motion.

There is almost no limit to the variety of the curves that may be obtained, depending on the amplitudes, frequencies, and relative phases of the two motions. If the frequencies are the same, but the amplitudes and phase angles are different, the equations for the vertical and horizontal motions may be written:

$$x = x_m \cos (\omega t + \alpha_1) \tag{1-39}$$

and

$$y = y_m \cos (\omega t + \alpha_2). \tag{1-40}$$

If the time is eliminated between these two equations, the following equation is obtained:

$$\frac{x^2}{x_m^2} + \frac{y^2}{y_m^2} - \frac{2xy}{x_m y_m} \cos (\alpha_1 - \alpha_2) - \sin^2 (\alpha_1 - \alpha_2) = 0. \tag{1-41}$$

This represents an ellipse whose eccentricity and inclination depend upon the phase relations and the amplitudes. If the relative phase angle $(\alpha_1 - \alpha_2)$ happens to be $\pi/2$, the principal axes of the ellipse are vertical and horizontal, since the term containing the product xy is absent. If, in addition, $x_m = y_m$, the ellipse becomes a circle.

If the relative phase angle is zero, the equation degenerates into two identical straight lines, given by

$$y = \frac{y_m}{x_m} x. \tag{1-42}$$

These curves can be made the basis of an exceedingly sensitive test for frequency measurement. If the vertical motion is of unknown frequency and if the frequency of the horizontal motion can be controlled with a calibrated variable frequency electrical oscillator, it is only necessary to adjust the oscillator until a stationary ellipse, circle, or straight line appears, and then read off the unknown frequency.

If the frequencies of the vertical and horizontal motions are not the same, no steady pattern will appear upon an oscilloscope unless the two frequencies bear a whole number relationship, as indicated earlier in con-

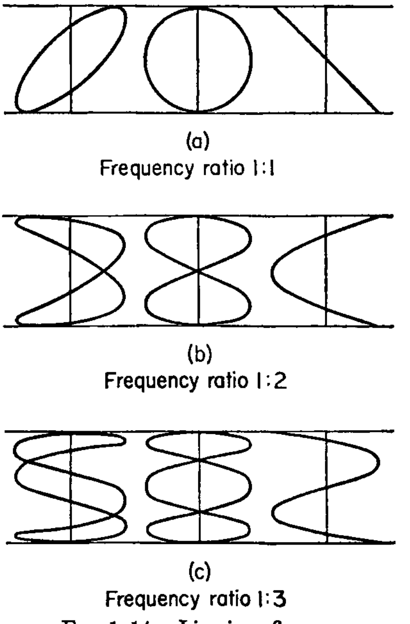


FIG. 1-14. Lissajous figures.

nection with beats. The steady patterns are all closed curves representing higher degree equations. A few of the simpler ones are illustrated in Fig. 1–14. Some of the patterns may be used, practically, to determine the ratio of a known to an unknown frequency, provided that the whole number ratio of the two frequencies does not involve integers which are too large. In this latter case, the patterns are too crowded to interpret easily.

PROBLEMS

1. Using a single pair of rectangular axes, draw three graphs to represent, for simple harmonic motion, the displacement x, the velocity $\dot{x}$, and the acceleration $\ddot{x}$, each as a function of the time. Besides showing the relative phases, indicate the maximum values of the three variables in terms of the proper constants.

2. (a) For simple harmonic motion, find the displacement x as a function of the time, by integrating the equation $m\ddot{x} = -Kx$. (b) Show that the period of the motion is given by $T = 2\pi\sqrt{m/K}$.

3. (a) Find the displacement x as a function of the time, if the differential equation for the motion is $m\ddot{x} = +Ax$, where A is a constant. Assume that the initial velocity is not zero, but has some value v_0. (Why is this necessary?) (b) Is the resulting motion periodic? Give a physical description of the motion.

4. A perfectly elastic ball is bouncing on a rigid floor. If the constant height to which it rebounds is h, find the period of the motion. Is the motion simple harmonic?

5. Two collinear harmonic motions of the same frequency have amplitudes of 2 cm and 3 cm respectively, and corresponding phase angles of $+10°$ and $+30°$. Find by the "method of components" used in mechanics (a) the amplitude, and (b) the phase angle of the sum vibration.

6. Two collinear simple harmonic motions are given by

$$x_1 = (x_m)_1 \cos (2\pi ft + \alpha_1)$$

and

$$x_2 = (x_m)_2 \cos (2\pi ft + \alpha_2).$$

By expanding the cosines of the sums of angles and adding, show that the resultant displacement x so obtained is equivalent to that obtained by the purely vector method.

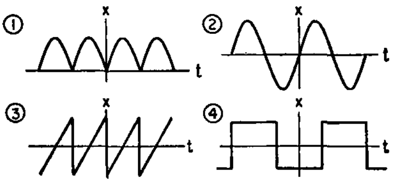


Fig. 1–15.

7. Two collinear simple harmonic motions have frequencies of 1024 and 1021 cycles-sec^{-1} respectively. (a) What is the number of "mathematical" beats per second? Of audible beats? (b) Answer the same questions if the two frequencies are 1024 and 1022 cycles-sec^{-1}.

8. (a) Which of the graphs of Fig. 1–15 represent even functions and which represent odd functions? (b) In which cases will a Fourier expansion involve a constant term?

9. Find the first few terms of the Fourier series equivalent to the square wave specified by $x = a$, from $t = 0$ to $t = T/2$, and $x = -a$, from $t = T/2$ to $t = T$.

10. Show graphically how close to the square wave is the sum of the first three periodic terms in the solution to problem 10.

11. The current in a circuit with a half-wave rectifier is given by $i = I_m \sin (2\pi ft)$ from $t = 0$ to $t = T/2$, and $i = 0$ from $t = T/2$ to $t = T$. Find the first few terms of the equivalent Fourier series.

12. A triangular wave is represented by the analytical expressions $x = 2at/T$ from $t = 0$ to $t = T/2$, and $x = 2a\ (1 - t/T)$ from $t = T/2$ to $t = T$. Find the first few terms of the Fourier expansion.

13. A telephone receiver diaphragm is considered as a particle of mass 1.0 gm. When displaced a distance 1.0 mm from its equilibrium position, the elastic restoring force is 10^6 dynes. The frictional force opposing its motion is 4.0×10^3 dynes per unit velocity (in cm-sec^{-1}). (a) If the diaphragm is displaced and then released, will its subsequent motion be oscillatory or not? (b) Find its natural frequency both with and without the presence of the damping force.

14. The diaphragm in problem 13 is driven by a force $F = 10^5 \cos (2\pi ft)$ dynes. (a) Plot a curve of velocity amplitude vs the driving frequency, from $f = 0$ to values of f beyond the resonance frequency. (b) Compute the frequency for displacement resonance, and compare with the frequency for velocity resonance.

15. It is desired to halve the free-oscillation resonance frequency (with damping) of the diaphragm of problem 13. If this is to be done by changing the mass alone, what will the new mass be?