CHAPTER 2

PLANE WAVES IN AIR

2–1 Introduction. The displacement of a single particle held in an elastic suspension gives rise to a simple harmonic vibration around a fixed point. On the other hand, the displacement of a portion of an extended medium, having the properties of distributed mass and elasticity, always results in *waves*, traveling out from the disturbed region. It is the mass or inertial property of the medium which keeps the propagation speed from becoming infinite, and in general the greater the specific mass (i.e., the density), the lower this speed will be. Conversely, the "stiffer" the medium, the greater will be the unbalanced force upon a portion of the medium adjacent to the disturbed region and the greater the resulting acceleration. A "stiff" medium will therefore make for a high propagation speed.

For the study of plane waves the general physical approach will be as follows. We will assume a deformable medium, having both elastic and inertial properties. A particular type of deformation will be assumed to exist at a certain location in space, at a particular time. Then, in view of the physical properties of the medium, we shall see that in the region being considered the degree of deformation changes with time and, in addition, new deformations appear in adjacent regions. This may be stated quite simply in terms of the rates of change of the degree of deformation with both position and time, i.e., in the form of a differential equation. It is the solution to this differential equation that describes completely the wave phenomena.

The student of electricity and magnetism should compare this procedure with the method of demonstrating the necessity, under certain conditions, for the existence of electromagnetic waves. In the case of electromagnetic waves we start, not with the mechanical properties of a material medium, but with Coulomb's law and the principle of electromagnetic induction. Two "fields," the electric and the magnetic, are assumed to be locally distorted, in the sense that they have local values differing from those existing in the surrounding regions. Just as the mechanical deformations in air then change with time and position, so one can also express the rates of change of the two fields with time and space coordinates. It is not strange that both the differential equation and the integral equation obtained in this manner are similar in many respects to the equations for sound waves.

As a preliminary to a more formal description of wave phenomena, let us clearly state the properties assumed for the medium. We describe it as a continuous, isotropic medium, of uniform density and having the property of perfect elasticity. As long as sound sources and receivers have dimensions large compared with the mean spacing between molecules, air is, to all intents and purposes, continuous. (More will be said about the molecular point of view of sound propagation in Chapter 6.) The assumption that air is perfectly elastic deserves additional attention. Sound attenuation actually *does* occur, due to the presence of dissipative factors. As sound energy is projected through limited regions of the air, viscous stresses in the nature of shear appear near the lateral boundaries of the disturbance, and tend to dissipate the energy associated with the wave motion. Away from the boundaries these effects are of negligible importance. A second possible means of wave energy dissipation is by the process of heat flow between adjacent regions of compression and rarefaction. Such a heat flow, because of its irreversible nature, would result in a constant degradation of the wave-motion energy into the energy of uncoordinated thermal motions. (This decrease in wave amplitude is not to be confused with the operation of the ordinary inverse square law in the case of spherical waves, where the same total wave energy simply spreads into a larger and larger volume.) The heat conductivity of gases is low, and therefore over the audible spectrum the deformation process can be described quite accurately as adiabatic, and heat flow is not a significant dissipative factor. More will be said on this matter in Chapter 6.

One other assumption will be made in the course of setting up the differential equations for waves, namely, that the disturbance in the normal mass distribution for the medium will always remain small. This is true in any ordinary sound wave and this assumption will greatly simplify the mathematics. Moreover, as a consequence of this assumption, there will appear certain important physical features of wave propagation characteristic of small amplitude waves only.

A few definitions will be of use in setting up the wave equations.

2–2 Dilatation and condensation. Let V_0 be the volume occupied by any fixed mass of air with no wave disturbance present. Similarly, ρ_0 is the density of the air under the same conditions. Then, if there is some small deformation of the medium, so that V_0 is increased by a small amount v, and ρ_0 is changed similarly by a small amount ρ_d, we may state that

$$\left.\begin{aligned}\text{Dilatation} &= \delta = v/V_0 \\ \text{and} \qquad \text{Condensation} &= s = \rho_d/\rho_0.\end{aligned}\right\} \qquad (2\text{–}1)$$

These dimensionless ratios describe the instantaneous fractional change in volume and density at a point in a field of sound. They are small, but not truly differential quantities, and they vary in value both with position in space and with time. In the manner of physics, dilatation and condensation are sometimes treated as true differentials when their magnitudes are small compared with other variables.

In view of Eqs. (2-1), we may write for the volume V of the chosen mass of air and for the density ρ in the presence of the distortion in the air,

$$V = V_0(1 + \delta)$$

and (2-2)

$$\rho = \rho_0(1 + s).$$

Therefore

$$(1 + \delta)(1 + s) = 1, \tag{2-3}$$

since ρV is constant, for constant mass.

If s and δ are small,

$$s \cong -\delta, \tag{2-4}$$

neglecting the product, $s\delta$, in comparison with s or δ. The values of s and δ rarely exceed 10^{-3} for ordinary sound waves, so that the error in this assumption is negligible. For the large amplitude waves which accompany explosions, the simple relation of Eq. (2-4) can no longer be assumed and the exact expression of Eq. (2-3) must be used. This greatly complicates the mathematics, as will be seen in Chapter 6.

2-3 Bulk modulus. One other definition, from elasticity, will be useful. For an elastic, isotropic medium, the bulk modulus is

$$\mathcal{B} = -V\frac{dP}{dV}, \tag{2-5}$$

where P and V represent the pressure and volume respectively of a given mass. With this definition the constant $\mathcal{B}$ is always positive, since the volume will decrease when the pressure increases and vice versa.

For a perfect gas there are two such moduli, the adiabatic modulus, $\mathcal{B}_a$, and the isothermal modulus, $\mathcal{B}_i$. The ratio $\mathcal{B}_a/\mathcal{B}_i = \gamma$, where γ is the ratio of the specific heat of the gas at constant pressure to the specific heat at constant volume ($= 1.4$ for air). Since the variations involved in sound propagation in air are closely adiabatic in nature, we will be concerned only with $\mathcal{B}_a$. Used without a subscript, $\mathcal{B}$ will be assumed, therefore, to be $\mathcal{B}_a$.

The relationship between pressure and volume for a gas is not a linear one, so that in general the value of $\mathcal{B}$ does not remain constant, for a given mass of gas, when the total pressure and volume are varied. However, for sound propagated in the ordinary open air, any variations due to changes in atmospheric conditions and also due to the presence of the sound wave itself are quite small. Therefore there is little error in assuming $\mathcal{B}$ constant. (For normal open air conditions, $\mathcal{B}$ is of the order of 1.4×10^6 dynes/cm², or 1.4×10^5 newtons/m².)

While Eq. (2–5) is the precise definition of $\mathcal{B}$, we may substitute small finite changes in pressure and volume for the differential changes without introducing any serious error. Let p and v be such small variations in the total pressure P and the volume V, due to the presence of the sound wave. If P_0 and V_0 are the normal undisturbed values for a given mass of gas, we may write

$$\mathcal{B} = -\frac{p}{v/V_0}$$

or

$$p = -\mathcal{B}\delta = \mathcal{B}s. \tag{2–6}$$

This is a most useful relation between the small "excess pressure" p and the condensation s. At any one position in space, both quantities will vary periodically as the wave passes by, and since they are linearly related through the bulk modulus, they will always be in phase.

2–4 Significant variables in the field of sound. The state of the air through which sound waves are traveling may be discussed in terms of any one of several physical variables. We have defined and related three such variables, the dilatation δ, the condensation s, and the excess pressure p. In setting up the wave equation in the next section, we shall introduce a fourth important variable, the "particle displacement" ξ, together with its time and space derivatives. The three quantities already discussed are related by quite simple equations and are also simply related to ξ. It is therefore equally correct to describe the wave as a traveling variation in the pressure, the condensation, the dilatation, or the particle displacement.

In modern experimental acoustics, microphones of an electrical type are used almost exclusively to detect sound waves. Such microphones respond primarily to the pressure variable in the wave. In addition, the "particle velocity" $\dot{\xi}$, which represents the time derivative of the particle displacement, presently to be introduced, will be a particularly convenient variable when we come to the use of electrical analogs. In our later discussion we shall make more use of "sound pressure" and "particle velocity" than of any other of the variables so far introduced. These other quantities will, however, be useful in setting up the wave equations and in addition they are important to a complete understanding of the physical nature of a longitudinal wave.

2–5 The differential equation for plane waves. The problem of the one-dimensional wave, where the deformations in the medium are a function of one cartesian space coordinate, is the simplest to analyze. Such a wave is called *plane* because conditions are uniform over the cartesian plane specified by the one space coordinate. Most sound waves are not plane, but at a considerable distance from sound sources of ordinary size and of any shape the curvature of the wave front is small, and the wave-front shape, for all practical purposes, becomes plane. At nearer points, where this cannot be assumed, we must make use of the more complicated three-dimensional equations developed in the next chapter.

In the following sections we shall make free use of partial derivatives. The integral equations in this chapter will often involve three or more variables, the two independent ones being the space coordinate x and the time t. The important physical parameters in the field of sound, p, ξ, $\dot{\xi}$, s, etc., will each be a function of both x and t. When we write $\partial\xi/\partial x$, we shall be assuming that time, t, is held constant, whereas when $\partial\xi/\partial t$ is used, it is understood that x is held constant.

Let the air be deformed, at a given instant of time, along the x-direction only (Fig. 2–1). Assume a layer of air, originally of thickness dx and of unit cross section, to be displaced along x in such a way that the face originally at x has moved a distance ξ, and the face at $x + dx$ has moved a distance $\xi + d\xi$. The increased thickness of the layer of air, due to the deformation, is plainly $d\xi$. Since $d\xi$ can be written as $\dfrac{\partial\xi}{\partial x}dx$, we can then evaluate the dilatation, at this instant, for the layer of unit area:

$$\delta = \frac{v}{V_0} = \frac{\frac{\partial\xi}{\partial x}\,dx}{dx} = \frac{\partial\xi}{\partial x}. \tag{2–7}$$

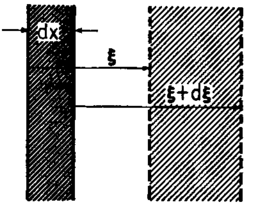

FIG. 2–1.

Due to the deformation of the medium, the pressures on the two faces of the layer will now be slightly different by a differential amount, dP. Assuming a positive increment in pressure with increasing x, the net force is to the left and therefore negative. This net force along the x-axis is

$$-P_{x+dx} + P_x = -dP = -d(P_0 + p) = -dp. \tag{2–8}$$

Therefore, writing Newton's second law for the matter within the layer, we have

$$-dp = \rho_0\, dx\, \frac{\partial^2\xi}{\partial t^2}, \tag{2–9}$$

where ρ_0 is the normal undisturbed density of the air. (This equation neglects the second order difference between the acceleration of the face displaced by an amount ξ, and that of the face displaced by an amount $\xi + d\xi$.) The acceleration is expressed as a partial derivative in recognition of the fact that ξ is a function of both x and t. Since the small change in the excess pressure dp is $\dfrac{\partial p}{\partial x}\,dx$, Eq. (2–9) may be written

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial x} = \frac{\partial^2 \xi}{\partial t^2}. \tag{2-10}$$

This form of the wave equation involves four variables. By making use of Eq. (2–6), the number may be reduced to three. Differentiating (2–6) partially with respect to x, time being assumed constant, we obtain

$$\frac{\partial p}{\partial x} = -\mathcal{B}\frac{\partial \delta}{\partial x} = -\mathcal{B}\frac{\partial^2 \xi}{\partial x^2}. \tag{2-11}$$

Therefore Eq. (2–10) may be written

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\mathcal{B}}{\rho_0}\frac{\partial^2 \xi}{\partial x^2}$$

or, letting $\mathcal{B}/\rho_0$ equal c^2,

$$\frac{\partial^2 \xi}{\partial t^2} = c^2\frac{\partial^2 \xi}{\partial x^2}. \tag{2-12}$$

This is the most common form for the differential equation for plane waves, where ξ is the dependent variable and x and t are the independent variables. Equation (2–12) uniquely relates rates of change of ξ with respect to position and rates of change of ξ with respect to time. Before we discuss the solution of Eq. (2–12) and how it implies wave production, we should make clear the meaning of the variable ξ as applied to air.

2–6 Physical significance of the particle displacement, ξ. A gas like air is not, of course, made up of molecules having any fixed mean position in the medium, like the atoms of a solid. Even without the presence of a wave, gas molecules are in constant motion, with average velocities far in excess of any velocities associated with the wave motion (see Chapter 6). However, from a statistical point of view, a fluid, either gas or liquid, may be treated much as a solid because, when in the undisturbed state, molecules leaving a certain region as a result of their random motions are replaced by an exactly equivalent number of molecules, having exactly the same properties, thus keeping the macroscopic properties of the medium the same. Similarly, during the vibration cycle associated with the wave motion, the fact that a continuously changing group of molecules is involved rather than a fixed set is of no moment, so long as the average properties of the aggregate remain the same. In view of this equivalence, it is quite proper to speak of "particle" displacements, velocities, and accelerations for a fluid with much the same meaning as for a solid.

2–7 Solution of the wave equation. The most general solution of the differential equation, (2–12), can be shown to be of the form

$$\xi = f(x \pm ct), \tag{2-13}$$

where $c = \sqrt{\mathcal{B}/\rho_0}$. The exact nature of the function f is determined by the boundary conditions peculiar to a specific problem, in particular by the nature and behavior of the sound source. There is no mathematical restriction that the function be periodic, although in practical sound production this is usually the case.

If the reader has never encountered an equation of the type given by (2–13), he will see nothing in it to indicate a wave. A closer scrutiny of the term $(x \pm ct)$, however, will reveal its wave implications. Let us assume the negative sign for the term ct. At some specific time t_1 and at some specific position x_1, ξ will have some particular value ξ_1. If a small increment of time is added to the time t_1, so that t becomes $t_1 + \Delta t$, there will then be a slightly greater value of $x = x_1 + \Delta x$ such that the total value of $(x - ct)$ will remain the same (i.e., such that $(x_1 - ct_1) = [(x_1 + \Delta x) - c(t_1 + \Delta t)]$). Therefore ξ will still have the value ξ_1. Putting it as simply as possible, after a short time Δt, the same value of ξ will recur at a point a little farther along in the $+x$ direction. This is equivalent to describing a traveling disturbance, where the whole graphical representation of Eq. (2–13) moves along the x-axis from left to right. Figure 2–2 will help to clarify this interpretation of Eq. (2–13).

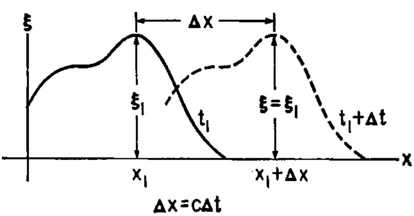

FIG. 2–2.

A similar consideration will show that with a positive sign in front of the term ct, the disturbance will move in the direction of $-x$. Whether the motion is in the positive or the negative direction, for identical values of the argument on the right of Eq. (2–13), and therefore for identical values of ξ, Δx must equal $c\,\Delta t$. The velocity of travel is therefore $\Delta x/\Delta t = c$. Since the quantity c, which is equal to $\sqrt{\mathcal{B}/\rho_0}$, is almost a constant, the velocity is independent of the nature of the function f. It should also be clear, from the above analysis, that for the small amplitude disturbances here considered, no change in graphical "shape" will occur during the propagation. If this were not true, the whole character of musical sound and speech would vary with the distance between the source and the observer!

Using the adiabatic bulk modulus and the density for dry air at normal atmospheric pressure and 0° C, the velocity of sound c becomes very nearly 33,100 cm/sec, or 331 m/sec. This is in close agreement with experiment. In Chapter 6 we shall consider some of the reasons for variations in this figure.

2–8 Disturbances of a periodic nature. The word *disturbance* is purposely used here instead of "wave" because *wave* generally implies a recurring pattern or, mathematically, a repeating function, and there are no such restrictions on the solution of Eq. (2–12). Much of the sound associated with music is transient in nature, with no true steady state frequencies. The transient component of the air disturbance travels with the same speed as does the regularly periodic portion, and plays an important part in determining the over-all effect on the hearer. For simplicity, most of our attention will be directed to disturbances of a steady state nature, originating from sustained vibrations at the source.

The following periodic expression for ξ satisfies the differential equation for plane waves:

$$\xi = \xi_m \cos \frac{2\pi}{\lambda} (x \pm ct), \tag{2–14}$$

since ξ is a function of $(x \pm ct)$. The quantities ξ_m and λ are constants. Equation (2–14) may also be written

$$\xi = \xi_m \cos \left[\pm \frac{2\pi}{\lambda} (ct \pm x) \right] = \xi_m \cos \frac{2\pi}{\lambda} (ct \pm x). \tag{2–15}$$

The student may check directly that any function $f(ct \pm x)$ is a solution of the differential equation, as well as $f(x \pm ct)$. Written either way, the use of the negative sign signifies a disturbance traveling in the $+x$ direction. We shall, for the most part, use the argument $(ct \pm x)$, since this form leads to the interpretation of phase in the conventional manner.

In physical problems, the solution of a differential equation must not only satisfy this equation but must also fit the boundary conditions. Suppose that the source of the plane waves being considered is one side of a rigid vibrating plate, the motion of every point of whose surface may be described by the equation

$$Q = Q_m \cos 2\pi ft, \tag{2–16}$$

Q being the instantaneous displacement of the surface of the plate. Such a source is often called an "acoustic piston." The air adjacent to the vibrating surface of the source must have a motion identical with that of the source itself. Let x in the wave equation (2–15) be measured from the source position. Then, provided that the constant $\lambda = c/f$, and if $\xi_m = Q_m$, it is seen that Eq. (2–15) becomes identical with (2–16) at $x = 0$. Thus the form of Eq. (2–15) is correct to fit this particular boundary condition.

2–9 The wavelength. The relation $\lambda = c/f$ will suggest that this constant is the *wavelength*, or the distance between adjacent crests in the traveling disturbance. That this interpretation is correct will be evident if Eq. (2–15) is rewritten as

$$\xi = \xi_m \cos 2\pi \left(\frac{c}{\lambda} t \pm \frac{x}{\lambda} \right). \tag{2-17}$$

Assuming time to be held constant, Eq. (2-17) becomes a relation between two variables only, ξ and x, and represents a sort of "frozen" picture of the various air layer displacements at a given instant. It is then seen that there is a spatial repetition of a given value of ξ every time x changes by an amount λ. This is the ordinary idea of wavelength. With x held constant, Eq. (2-17) becomes a relation between the two variables ξ and t. It then describes, as a function of time, and while the wave passes by, the vibration of a particular layer of air around its equilibrium position. The frequency of this vibration will be c/λ. In either case, the plot of ξ vs x or of ξ vs t is a sinusoid, whose position along the x- or t-axis, as the case may be, is determined by the particular value of x or t that is chosen.

2-10 Graphical representation. In the ordinary graphs of ξ vs t, the particle displacement ξ is plotted vertically, along the y-direction. It must be emphasized that since the wave is longitudinal, the actual physical direction of the displacement of a layer of air is parallel to the x-axis. In this connection it will be recalled that in setting up the original differential equation, a positive value of ξ was measured to the *right* of the equilibrium position, i.e., in the direction of $+x$. In Fig. 2-3 the graph is placed below the physical picture in order to clarify these relationships. The dashed lines represent the equilibrium positions for selected layers of air.

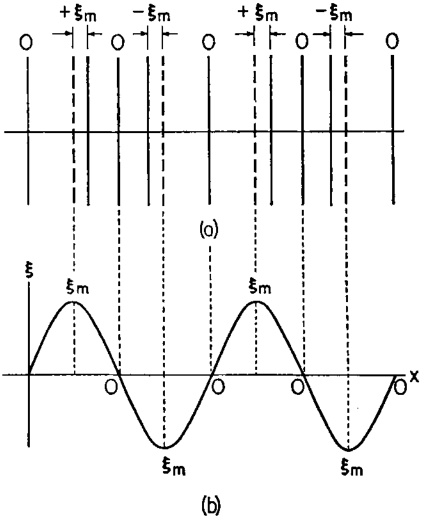

FIG. 2-3. Graphical representation of $\xi = f(x)$, time being held constant. Dashed lines in part (a) represent central position for each vibrating layer of air.

2-11 Waves containing more than one frequency component. If the vibrating source is simultaneously executing a number of simple harmonic motions of different frequencies, each of these motions will contribute a separate component displacement of the air. The total value of ξ, by the superposition principle, is the sum of these contributions. In general, then, we may write, for a wave traveling in the $+x$ direction,

$$\xi = (\xi_m)_1 \cos\left[\frac{2\pi}{\lambda_1}(ct - x) + \alpha_1\right] + (\xi_m)_2 \cos\left[\frac{2\pi}{\lambda_2}(ct - x) + \alpha_2\right] + \cdots + (\xi_m)_n \cos\left[\frac{2\pi}{\lambda_n}(ct - x) + \alpha_n\right], \quad (2\text{–}18)$$

where each of the λ's is associated with one of the particular frequencies present. Note that it is necessary to introduce a separate phase angle α into each component of the wave expression, the values of these angles being determined by the exact behavior of the source, where $x = 0$. The reader should recognize that the right-hand side of Eq. (2–17) is somewhat similar, in terms of the wave equation, to the harmonic series of a Fourier analysis, as discussed in Chapter 1.

As was brought out in Section 2–7, the speed of wave propagation, c, for small amplitude waves, is dependent only on the elastic and inertial properties of the medium. The various frequencies present in a complex wave, as represented by Eq. (2–18), all travel with the same speed. This means that the phenomenon of dispersion, so important in light, is almost nonexistent in sound. This is to be expected, in view of the purely mechanical nature of longitudinal waves. At very high audible frequencies and in the ultrasonic region, anomalous effects do occur (see Chapter 6), but not for the ordinary audible range. There is one interesting special case where dispersion *does* occur in air with ordinary sound frequencies and intensities, i.e., in the propagation of waves along an exponential horn. This will be mentioned again when this type of horn is discussed in Chapter 5.

2–12 Alternate forms for the steady state solution to the wave equation. By means of the relations given in Sections 2–2 and 2–3, it is possible to rewrite Eq. (2–15) in terms of any of the various field parameters. The results for a wave traveling in the $+x$ direction are summarized below, along with the original equation in terms of ξ.

$$\begin{aligned}
&\text{(a)}\quad \xi = \xi_m \cos\frac{2\pi}{\lambda}(ct - x),\\
&\text{(b)}\quad \dot{\xi} = -2\pi\frac{c}{\lambda}\xi_m \sin\frac{2\pi}{\lambda}(ct - x),\\
&\text{(c)}\quad \delta = \frac{\partial \xi}{\partial x} = \frac{2\pi}{\lambda}\xi_m \sin\frac{2\pi}{\lambda}(ct - x), \qquad (2\text{–}19)\\
&\text{(d)}\quad s = -\frac{\partial \xi}{\partial x} = -\frac{2\pi}{\lambda}\xi_m \sin\frac{2\pi}{\lambda}(ct - x),\\
&\text{(e)}\quad p = \mathcal{B}s = -\mathcal{B}\frac{\partial \xi}{\partial x} = -\mathcal{B}\frac{2\pi}{\lambda}\xi_m \sin\frac{2\pi}{\lambda}(ct - x).
\end{aligned}$$

(The coefficient on the right of Eq. (2–19b) could also have been written as $2\pi f\xi_m$.)

It should be pointed out that the determination of any one of the group of field properties in a plane wave in free space uniquely determines all the others, since they are all simply related. This fact is important in connection with experimental acoustical measurements in the path of plane waves. A "pressure" detector may be used to measure indirectly all other important quantities as well.

2–13 Phase relationships. The phase relationships which appear in the above set of equations are worth some attention. The particle velocity $\dot{\xi}$, the condensation s, and the excess pressure p are all in phase. This means that the density and the pressure are a maximum when a layer of air is moving through its *central* position (where $\dot{\xi}$ is a maximum), *not* when it is at the extreme ends of its motions, as one might expect. The dilatation is, of course, 180° out of phase with the condensation and the excess pressure. The quantities $\dot{\xi}$; δ, s, and p are all 90° out of phase with the displacement.

The algebraic signs of these quantities introduce some subtleties in the phase relationships. In the first place, the variables ξ and $\dot{\xi}$ in Eqs. (2–19) represent *vector* quantities. As assumed in Section 2–5, $+\xi$ is in the $+x$ direction and $-\xi$ is in the $-x$ direction. This applies also to the particle velocity, $\dot{\xi}$. The variables δ and s may also be plus or minus, but since they represent *scalar* quantities, the algebraic sign simply indicates whether the volume or density change is an increase or a decrease. The excess pressure, p, is also a scalar, in this sense. This difference in the interpretation of sign for the vector and for the scalar equations in Eqs. (2–19) must be kept clearly in mind in connection with reflection phenomena.

For a wave traveling in the $-x$ direction, where $(ct - x)$ is replaced by $(ct + x)$, there will be a positive sign on the right-hand side of Eqs. (2–19d) and (2–19e), after the differentiation. No change of sign will take place, however, in Eq. (2–19b). This means that there will now be a 180° phase relationship between $\dot{\xi}$ and either s or p. When the particle velocity is to the right (positive), the density and total pressure will be *less* than the normal ρ_0 and P_0 respectively.

The vector interpretation of a positive or negative ξ or $\dot{\xi}$ is unaffected by the direction of wave propagation, since their sign is tied up with the sign convention associated with a fixed x-axis.

Example. A large flat plate is radiating plane sound waves from one side only. The amplitude of its motion is 0.01 mm and the frequency of vibration is 1000 cycles-sec^{-1}. For any point in the path of the waves, find the maximum values of ξ, $\dot{\xi}$, s, δ, and p.

Assume the velocity of sound c to be 331 m/sec and the bulk modulus, $\mathcal{B}$, to be 1.4×10^6 dynes-cm^{-2}. The amplitude of motion in the air is the same as that of the source. Therefore

$$\begin{aligned}
\xi_m &= 10^{-3}\ \text{cm},\\
\dot{\xi}_m &= 2\pi f \xi_m = 2\pi(10^3)(10^{-3}) = 6.28\ \text{cm-sec}^{-1},\\
s_m = \delta_m &= \frac{2\pi}{\lambda}\,\xi_m = \frac{2\pi}{33.1}\,10^{-3} = 1.9 \times 10^{-4},\\
p &= \mathcal{B}s = (1.4 \times 10^6)(1.9 \times 10^{-4}) = 2.66 \times 10^2\ \text{dynes-cm}^{-2}.
\end{aligned}$$

These values are typical of sounds of high intensity.

2–14 Energy in the wave. For the simple harmonic motion of a mass particle, the energy was seen to be, on the average, half kinetic and half potential. One would therefore suspect that in a sound wave the energy is also so divided. While this turns out to be the case, there are certain features of energy storage which are peculiar to longitudinal waves and deserve some discussion.

2–15 Kinetic energy. Consider a longitudinal wave of sinusoidal form, progressing in the $+x$ direction. For a thin layer of air, of thickness dx and of unit cross section, moving with a velocity $\dot{\xi}$, the instantaneous kinetic energy is

$$dE_k = \frac{1}{2}\rho_0(\dot{\xi})^2\,dx. \tag{2–20}$$

The average kinetic energy density e_k in the medium may be obtained by integrating this expression with respect to x over an integral number of wavelengths $n\lambda$ (keeping the time constant), and then dividing the result by the volume this total energy occupies:

$$e_k = \frac{\frac{1}{2}\,\rho_0(\dot{\xi}_m)^2 \displaystyle\int_0^{n\lambda} \sin^2 \frac{2\pi}{\lambda}\,(ct - x)dx}{n\lambda} = \frac{1}{4}\,\rho_0(\dot{\xi}_m)^2 = \frac{1}{4}\,\rho_0(4\pi^2 f^2)\xi_m^2. \tag{2–21}$$

It will be noted that the result is identical with what one would expect for the time average of the kinetic energy of a particle whose mass is the mass per unit volume of the gas.

2–16 Potential energy. To obtain the corresponding *potential energy* density, we must consider the properties of a perfect gas. In Section 2–3 it was indicated that the volume and pressure changes that occur in air during the passage of longitudinal waves of audible frequencies are nearly adiabatic in character. The graph of Fig. 2–4 represents a small portion of a PV diagram for an adiabatic variation, using a fixed mass of gas. Let us suppose a volume V_0 to be reduced to a slightly smaller volume, V, the decrease being called v. As a result, the pressure will rise slightly from P_0 to P, the increase being called p. The relations between these quantities are shown in the graph. Assuming the curve to be straight for small changes, the work ΔW done upon the gas during the volume change, or the energy ΔE_p, stored potentially within the gas, will be the area under the curve between V and V_0:

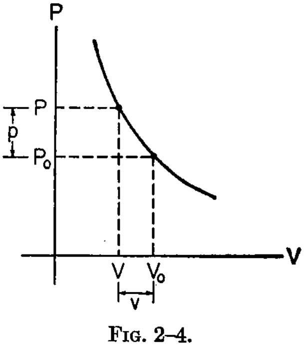

FIG. 2-4.

$$\Delta W = \Delta E_p = (P_0 + \frac{1}{2}p)v = P_0 v + \frac{1}{2}pv. \tag{2-22}$$

For simple harmonic variations around mean pressures and volumes, the v's and p's are alternately plus and minus. The average value of the first term on the right-hand side, $P_0 v$, over any integral number of cycles is therefore zero. In the second term, however, when v changes sign, so does p. The product sign is therefore always the same, showing potential energy to be stored in the gas whether there is a compression or a rarefaction. It is interesting to see that while the air is, of course, always "compressed" in the absolute sense, *total* pressures never changing sign, the medium nevertheless acts just like an unstressed spring which is alternately compressed and stretched.

The second term on the right-hand side of Eq. (2-22), $\frac{1}{2}pv$, which alone contributes to the average potential energy stored in the medium, may be written in terms of the other field parameters by means of the following transformations.

Since

$$p = -\mathcal{B}\delta \quad \text{and} \quad v = V_0\delta, \tag{2-23}$$

therefore

$$\frac{1}{2}pv = \frac{1}{2}\mathcal{B}\delta^2 V_0. \tag{2-24}$$

The minus sign of the first relation of Eq. (2-23) may be ignored, since a positive sign for p and a negative sign for δ both signify work done *on* the gas and therefore an increase in the stored potential energy. For a thin layer of air of unit cross section in the path of the wave, the potential energy becomes

$$dE_p = \frac{1}{2}\mathcal{B}\,\delta^2\,dx. \tag{2-25}$$

By partially differentiating the right-hand side of Eq. (2–13), first with respect to x and then with respect to t, we see that

$$\frac{\partial \xi}{\partial x} = \pm \frac{1}{c}\frac{\partial \xi}{\partial t}. \tag{2–26}$$

Inserting this value of $\partial\xi/\partial x(= \delta)$ into Eq. (2–25), we obtain

$$dE_p = \frac{1}{2}\frac{\mathcal{B}}{c^2}(\dot{\xi})^2\,dx = \frac{1}{2}\rho_0(\dot{\xi})^2\,dx. \tag{2–27}$$

This last expression is identical with Eq. (2–20), representing the instantaneous *kinetic* energy of a thin layer of air. It therefore follows that the *average potential energy density* in a region containing an integral number of wavelengths will also be $\frac{1}{4}\rho_0(\dot{\xi}_m)^2$.

2–17 Total energy density in the wave. The total average energy density in the wave will be the sum of the kinetic and the potential energies, or

$$e_{\text{total}} = \frac{1}{4}\rho_0(\dot{\xi}_m)^2 + \frac{1}{4}\rho_0(\dot{\xi}_m)^2 = \frac{1}{2}\rho_0(\dot{\xi}_m)^2. \tag{2–28}$$

One of the interesting things about the energy in a wave disturbance, not readily foreseen, is that the kinetic and the potential energies move along together in *identical regions*. Since the instantaneous energies can both be written in terms of the instantaneous particle velocity, they are each a maximum at the same position in space and also at other places and times are zero together. This is, of course, not true for the vibration of a single isolated mass particle. A plot of the total energy distribution in space for the wave, Fig. 2–5, shows a sort of pseudoquantum nature, regions of large total energy alternating with regions of little or no energy. In the case of traveling transverse waves on a stretched string, the above remarks do not apply. For these waves, as will be seen in Chapter 7, the kinetic energy maxima and the potential energy maxima are not coincident.

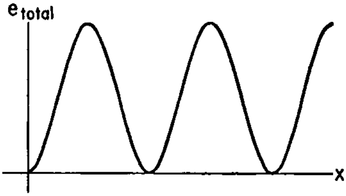

FIG. 2–5. Graph of energy distribution along x at a given instant of time.

2–18 Sound intensity. This important measure of sound wave amplitudes is defined as the energy flow across an area, per unit area and per unit time. This energy will plainly be equal to that contained in a column of unit cross section and of length c, the velocity of sound. Therefore the intensity I is the product of the total energy density, derived above, and the velocity c.

There are several different ways of writing the expression for the sound wave intensity, in view of the interrelations between all the important parameters. Several useful forms, whose validity are given below.

$$\text{(a)}\ \frac{1}{2}\rho_0 c(\dot{\xi}_m)^2. \qquad \text{(d)}\ p_{\text{rms}}\dot{\xi}_{\text{rms}}.$$

$$\text{(b)}\ \frac{1}{2}\rho_0 c\, 4\pi^2 f^2(\xi_m)^2. \qquad \text{(e)}\ \frac{p^2_{\text{rms}}}{\rho_0 c}. \qquad (2\text{–}29)$$

$$\text{(c)}\ \rho_0 c(\dot{\xi}_{\text{rms}})^2.$$

The form given in Eq. (2–29c) is algebraically similar to the expression for electrical power, RI^2, where $\rho_0 c$ replaces R and $\dot{\xi}$ takes the place of I. Extensive use of this analogy will be made in Chapter 5.

2–19 Units of intensity. Using cgs units, intensity is measured in ergs-cm^{-2}-sec^{-1}. For ordinary audible waves, intensities range from about 10^{-9} to about 10^{+3} cgs units. In mks units this corresponds to a range of from 10^{-12} to 1.0 joule-meter^{-2}-sec^{-1} These numbers are an indication of how small are the energies associated with sound. The total energy coming from the throats of a crowd at a football game, in response to some spectacular play on the field, might perhaps be enough to heat a cup of coffee! Even the great crescendos of a large symphony orchestra involve very little sound wave energy. All of this is a tribute to the sensitivity of the human ear.

2–20 The decibel. For two sounds of intensities I_1 and I_2, one is said to be of a greater intensity than the other by a number of *decibels* (db), where

$$\text{Intensity difference in db} = 10 \log_{10} \frac{I_1}{I_2}. \qquad (2\text{–}30)$$

The decibel is therefore not an absolute, but a comparative measure of intensity and is consequently a pure number. Without the factor 10, the comparison is in *bels*, a unit too large for most practical purposes.

It is because of the sensitivity of the ear, and the range of intensities to which it responds, that the decibel scale has been devised. The scale is based on the well-known observation that the human sensory response to a given increase in an objective stimulus is approximately proportional to the ratio of the increase in stimulus to the stimulus already present. To give a concrete example, the ear is capable of detecting a very small increase in sound intensity when the background intensity is low; with a great deal of background noise, a much larger increase of intensity is necessary to give to the ear the same sensations.

If h represents the sensation delivered to the brain, and g the objective stimulus, the proportion may be expressed mathematically as

$$h \propto \frac{\Delta g}{g}. \tag{2-31}$$

If this statement is essentially correct, as seems to be the case for sensations of sight, pain, etc., as well as for hearing, then with greater changes in stimuli Eq. (2–31) may be integrated between definite limits, to obtain

$$h_1 - h_2 = \log \frac{g_1}{g_2}. \tag{2-32}$$

It is upon this equation that the decibel scale is based.

The range of audible intensities mentioned above, 10^{+3} to 10^{-9} ergs-cm^{-2}-sec^{-1}, may be converted into a decibel comparison by inserting the ratio $10^3/10^{-9} = 10^{12}$ for I_1/I_2 in Eq. (2–30). There is then seen to be an approximate range of 120 db between very weak and very intense sounds (ranging from the so-called threshold of hearing to the threshold of feeling). Apart from the nature of the ear response, the numerical convenience of this compressed scale is obvious. The decibel is also a convenient sized unit to use because any intensity difference of the order of one decibel may usually be ignored as far as the ear is concerned. The average ear is unable, even under ideal laboratory conditions, to tell that two sounds differ in intensity when their difference, measured in power per unit area, is less than about 10%, and under ordinary listening conditions the difference must be much greater.

There is an important consequence of the ear's rather crude ability to differentiate among varying sound intensities. It was pointed out earlier that it is difficult to express the physics of actual sound problems in terms of precise mathematics, and often even more difficult to solve these approximate equations. Fortunately, a discrepancy of 10–15% between theory and experiment, as far as intensity is concerned, is of no significance to the ear. This is a great comfort to the designer of practical acoustical equipment.

2–21 Intensity "level"; pressure "level." In recent years there has been devised an absolute intensity scale, known as the *intensity level*. This scale is based on the arbitrary selection of a low reference intensity, I_0, with which other intensities are compared. The value generally used for I_0 for plane waves is 10^{-16} watt-cm^{-2} = 10^{-9} erg-cm^{-2}-sec^{-1}, an intensity which corresponds approximately to the average threshold of hearing, or the weakest sound which can be heard. With I_0 specified, the intensity level in decibels of some sound of intensity I is then computed by replacing the ratio I_1/I_2 in Eq. (2–30) by I/I_0. Using the reference intensity given above, the intensity level in a noisy machine shop might be 100 db.

Since virtually all modern sound detectors respond directly to the variations of *pressure* in the wave disturbance, rather than to the intensity itself, the so-called *pressure level* is considered more fundamental than the intensity level. By Eq. (2–29e) the intensity in a plane wave is seen to be proportional to p^2. Therefore we may write Eq. (2–30) in the form

$$\text{Intensity difference in db} = 10 \log_{10} \frac{p^2}{p_0^2} = 20 \log_{10} \frac{p}{p_0}. \tag{2–33}$$

The rms value of p_0, the standard reference pressure, is commonly taken to be 0.0002 dyne-cm^{-2}. (This corresponds closely to the pressure in a wave whose intensity is the reference one given above, 10^{-16} watt-cm^{-2}.) The pressure level in a plane wave is therefore, by Eq. (2–33), 20 times the logarithm to the base 10 of the ratio of the pressure in the wave to the reference pressure, p_0.

Neither intensity level nor pressure level is the same as *loudness level*, which is a measure of subjective response that will be defined in Chapter 9.

PROBLEMS

1. Consider a plane wave traveling in the $+x$ direction. Using the same pair of rectangular axes, plot as a function of the time, for a fixed value of x, the particle displacement ξ, the particle velocity $\dot{\xi}$, the dilatation δ, and the excess pressure p. Besides showing the relative phases, indicate also the maximum values of each variable in terms of c, λ, etc.

2. Repeat problem 1 for a wave moving in the $-x$ direction.

3. Two plane waves are traveling along the x-axis. The particle displacements due to the separate waves are given by

$$\xi_1 = \xi_m \cos \frac{2\pi}{\lambda} (ct - x)$$

and

$$\xi_2 = -\xi_m \cos \frac{2\pi}{\lambda} (ct + x).$$

For the two waves, at the position $x = 0$, find the relative phase of (a) ξ_1 compared with ξ_2, (b) $\dot{\xi}_1$ compared with $\dot{\xi}_2$, and (c) p_1 compared with p_2.

4. For a certain plane wave traveling through air, the maximum value of the excess or acoustic pressure is 0.1 dyne-cm^{-2}. If the frequency is 1000 cycles-sec^{-1}, and the density of air is 1.29×10^{-3} gm-cm^{-3}, find (a) the maximum particle displacement, (b) the maximum particle velocity, (c) the maximum condensation, and (d) the maximum dilatation.

5. For the wave in problem 4, find (a) the average kinetic energy per unit volume, (b) the average potential energy per unit volume, and (c) the wave intensity, all in cgs units. (d) Also determine these values expressed in mks units.

6. A plane wave in air has an intensity of 40 erg-cm^{-2}-sec^{-1} and a frequency, f. A second wave, also in air, has an intensity of 10 erg-cm^{-2}-sec^{-1}, and a frequency $f/2$. For the two waves, find the ratio of (a) the maximum particle displacements, (b) the maximum particle velocities, and (c) the maximum acoustic pressures.

7. A plane wave in air and a plane wave in hydrogen have the same intensity and are of the same frequency. Find for the two waves the relative values of (a) the maximum particle displacements, (b) the maximum particle velocities, and (c) the maximum acoustic pressures. The density of hydrogen is 9×10^{-5} gm-cm^{-3} and the value of c is 1270 m-sec^{-1}.

8. The intensity level in a longitudinal wave in air is 20 db. Assuming a reference intensity of 10^{-9} erg-cm^{-2}-sec^{-1}, find the absolute intensity in the wave in cgs units. Assuming the rms reference pressure to be 2×10^{-4} dyne-cm^{-2}, find the rms acoustic pressure for the *pressure* level of 20 db.

9. Sound wave B has an intensity 10 db greater than wave A. Wave C has an intensity 10 db greater than wave B. (a) What is the intensity of wave C relative to wave A, in db? (b) Find the absolute ratio of the intensity of B to A, C to B, and C to A.

10. The pressure level in a sound wave in air is 30 db. (a) Find the absolute value of the acoustic pressure in the wave, in cgs units. (b) Also find the maximum value of the particle velocity. (Use the standard reference pressure given in problem 8.)