# Partial identification via conditional linear programs

## with an application to learning individualized treatment rules

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## Overview

Problem: estimate expectation of unknown linear functions

$$\theta = \mathbb{E}[\langle \boldsymbol{c}(X), \boldsymbol{p}^*(X) \rangle]$$

see also [1, 2]

- Know $J$ constraints on $K$ variables $\boldsymbol{p}(x)$: $Ap^*(x) = b(x)$
- $\boldsymbol{b}(x)$ and $\boldsymbol{c}(x)$ are identifiable

For each value of covariates $x$, have pair of conditional LPs

$$\theta_L(x) = \min_{A\boldsymbol{p}=\boldsymbol{b}(x)} \langle \boldsymbol{c}(x), \boldsymbol{p} \rangle \text{ and } \theta_U(x) = \max_{A\boldsymbol{p}=\boldsymbol{b}(x)} \langle \boldsymbol{c}(x), \boldsymbol{p} \rangle$$

Sharp, covariate-assisted bounds: $\mathbb{E}[\theta_L(X)] \leq \theta \leq \mathbb{E}[\theta_U(X)]$

This work: estimation of (regularized) bounds + policy learning

## Application: collective utility functions

Setting: RCT or Obs. Study (i.e. ignorability + overlap)

- Covariates $X$, binary decision $D$
- Discrete potential utilities $U(0), U(1)$; $\tau = U(1) - U(0)$
- Stochastic treatment policy $\pi(X)$

Individual expected utility under $\pi$ as a stochastic intervention

$$U(\pi(X)) = U(0) + \pi(X) \times \tau$$

Collective utility functions to aggregate individual utilities [3]

$$V^\lambda(\pi) = \frac{1}{\lambda}\mathbb{E}[(U(\pi(X))^\lambda - 1)]$$

| Egalitarian rule | 0 | Arithmetic mean utility | $\infty$ |
|---|---|---|---|
| $-\infty$ | Geometric mean utility | 1 | Inequality-seeking |

Nash cooperative bargaining value function

$$V^N(\pi) = \mathbb{E}[\{\tau > 0\} \log \pi(X) + \{\tau < 0\} \log(1 - \pi(X))]$$

Log-loss for positive treatment effect

- Unconstrained opt: $\pi(x) = P(\tau > 0 \mid X = x, \tau \neq 0)$
- Bargaining + rationality conditions + affine invariance [4]

Goal: minimize regret relative to oracle w/knowledge of POs

- Oracle: $\pi^o = (\tau \geq 0)$; Regret $R(\pi) = V(\pi^o) - V(\pi)$
- Requires knowledge of joint distribution of $U(0), U(1)$
- Margins $A\boldsymbol{p}(x) = \boldsymbol{b}(x)$; regret for PO pairs $\boldsymbol{c}(\pi(x))$

Find minimax regret policy $\pi^* \in \text{argmin}_{\pi \in \Pi} \max V(\pi^o) - V(\pi)$


[1] Vira Semenova. Aggregated Intersection Bounds and Aggregated Minimax Values, 2024. URL http://arxiv.org/abs/2303.00982. arXiv:2303.00982.

[2] Wenlong Ji, Lihua Lei, and Asher Spector. Model-Agnostic Covariate-Assisted Inference on Partially Identified Causal Effects, November 2024. URL http://arxiv.org/abs/2310.08115. arXiv:2310.08115.

[3] Hervé Moulin. *Axioms of Cooperative Decision Making*. Econometric Society Monographs. Cambridge University Press, Cambridge, 1988. ISBN 978-0-521-36055-5. doi: 10.1017/CCOL0521360552. URL https://www.cambridge.org/core/product/481FCBCDD15F3CCEE2FE381E7BF17B3D.

[4] Mamoru Kaneko and Kenjiro Nakamura. The Nash Social Welfare Function. *Econometrica*, 47(2):423–435, 1979. ISSN 0012-9682. doi: 10.2307/1914191. URL https://www.jstor.org/stable/1914191.

[5] Jonathan Weed. An explicit analysis of the entropic penalty in linear programming. In *Proceedings of the 31st Conference On Learning Theory*, pages 1841–1855. PMLR, July 2018. URL https://proceedings.mlr.press/v75/weed18a.html. ISSN: 2640-3498.

[6] Amy Finkelstein, Sarah Taubman, Bill Wright, Mira Bernstein, Jonathan Gruber, Joseph P. Newhouse, Heidi Allen, Katherine Baicker, and Oregon Health Study Group. The Oregon Health Insurance Experiment: Evidence from the First Year. *The Quarterly Journal of Economics*, 127(3):1057–1106, August 2012. ISSN 0033-5533. doi: 10.1093/qje/qjs020. URL https://doi.org/10.1093/qje/qjs020.


## De-biased estimation of bounds from conditional linear programs

### Plugin basic feasible solutions

Solution in terms of optimal basis $\mathcal{B}_U^*(x) = \{i_1, \ldots, i_J\}$

$$\boldsymbol{p}_U(x) = A^{-1}_{\mathcal{B}_U^*(x)}\boldsymbol{b}(x) \Rightarrow \theta_U = \mathbb{E}\left[\langle \boldsymbol{c}(x), A^{-1}_{\mathcal{B}_U^*(x)}\boldsymbol{b}(x) \rangle\right]$$

Plugin optimal basis feasible solution:

$$\widehat{B}_U(x) \in \text{argmax}_{B \in \mathcal{B}} \langle \hat{\boldsymbol{c}}(x), A_B^{-1}\hat{\boldsymbol{b}}(x) \rangle$$

- Directly read off simplex algorithm (avg polynomial time)

De-biased estimator w/plugin basic feasible solution:

$$\hat{\theta}_U = \hat{\mathbb{E}}\left[\langle \hat{\boldsymbol{c}}(X) + \hat{\varphi}_c, \hat{\boldsymbol{p}}_U(X) \rangle + \langle \hat{\boldsymbol{c}}(X), A^{-1}_{\widehat{B}_U(X)}\hat{\varphi}_b \rangle\right]$$

Margin condition: $P(\text{best sol'n - 2nd best sol'n} \leq t) \leq t^\alpha$

$$\left|\mathbb{E}[\hat{\theta}_U - \theta_U]\right| \lesssim \left(\|\hat{\boldsymbol{b}} - \boldsymbol{b}\|_\infty + \|\hat{\boldsymbol{c}} - \boldsymbol{c}\|_\infty\right)^{1+\alpha} + o_p(n^{-1/2})$$

### Entropic regularization

Entropic-regularized solution:

$$\boldsymbol{p}_U^\eta(x) = \underset{A\boldsymbol{p}=\boldsymbol{b}(x)}{\text{argmax}} \langle \boldsymbol{c}(X), \boldsymbol{p} \rangle + \eta^{-1}\text{Entropy}(\boldsymbol{p})$$

Plugin solution in terms of dual variables

$$\hat{p}_U^\eta(x) = \exp\left(-A'\lambda(\hat{\boldsymbol{b}}(x), \hat{\boldsymbol{c}}(x)) + \eta\hat{\boldsymbol{c}}(x)\right)$$

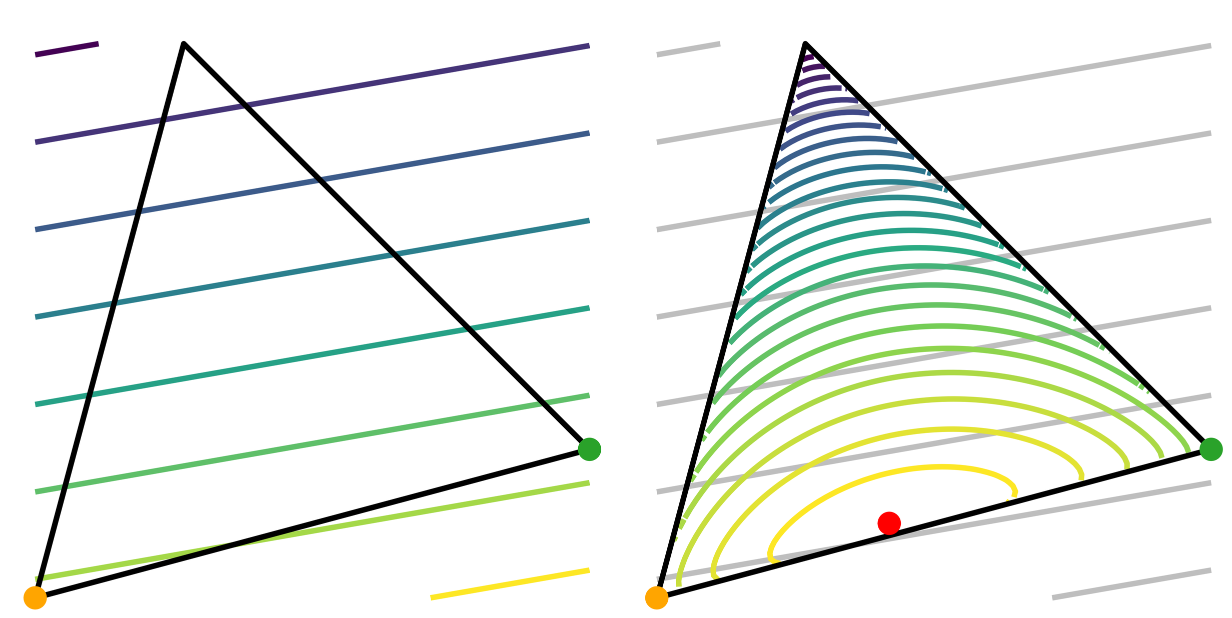
- Strongly convex, unconstrained, fast w/Sinkhorn algo

De-biased estimator w/entropic regularized solution:

$$\hat{\theta}_U^\eta = \hat{\mathbb{E}}[\langle \hat{\boldsymbol{c}}(X) + \hat{\varphi}_c, \hat{\boldsymbol{p}}_U^\eta(X) \rangle + \langle \hat{\boldsymbol{c}}(X), \nabla_b \hat{\boldsymbol{p}}_U^\eta(X)\hat{\varphi}_b + \nabla_c \hat{\boldsymbol{p}}_U^\eta(X)\hat{\varphi}_c \rangle]$$

If regularization penalty $\frac{1}{\eta}$ is small enough[5] relative to margin

$$\left|\mathbb{E}\left[\hat{\theta}_U^\eta - \theta_U\right]\right| \lesssim e^{-\eta} + o_p(n^{-1/2})$$



## Learning minimax regret policies

Unregularized minimax regret policy:

$$\hat{\pi} = \underset{\pi \in \Pi}{\text{argmin}}\, \hat{\mathbb{E}}\left[\langle \boldsymbol{c}(\pi(X)), A^{-1}_{\widehat{B}_U(X)}(\hat{\boldsymbol{b}}(X) + \hat{\varphi}_b) \rangle\right]$$

Excess regret $\lesssim$ Complexity($\Pi$) + $\|\hat{\boldsymbol{b}} - \boldsymbol{b}\|_\infty^{1+\alpha}$

Regularized minimax regret policy:

$$\hat{\pi}^\eta = \underset{\pi \in \Pi}{\text{argmin}}\, \hat{\mathbb{E}}[\langle \boldsymbol{c}(\pi(X)), \hat{\boldsymbol{p}}_U^\eta(X) + \nabla_b \hat{\boldsymbol{p}}_U^\eta(X)\hat{\varphi}_b \rangle]$$

Excess regret $\lesssim$ Complexity($\Pi$) + regularization bias

## Empirical Illustration: Oregon Health Insurance Experiment

Lottery for Medicaid enrollment [6]

- $X$: socioeconomic + health characteristics
- $D$: Medicaid offer; $U(d)$: -(# of ED visits)

Overall ITT: Medicaid offer ↑ ED visits by 12%

Q How suboptimal was random assignment?

Q What other targeting rules minimize maximum regret?