Notes 06: Rational Expressions and Equations

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1 Fractions, but with Variables

As the title of this section suggests, there is nothing here that you don't already know how to work with or at least haven't seen before. At this point, you *should* be able to solve any problem you see here - in principle. In practice, these problems are entirely new and it's unlikely that you've seen anything exactly like them. For example, the problem below is pretty straightforward if you know how to evaluate fractions with division and how to factor:

$$\begin{aligned}\frac{x^2+4x-5}{x^2+1} \div \frac{x-1}{4x^2+4} &= \frac{(x+5)(x-1)}{x^2+1} \cdot \frac{4(x^2+1)}{x-1} \\ &= 4(x+5) \\ &= 4x+20\end{aligned}$$

The difficulty comes in recognizing that you can apply things that you already know how to do. These individual steps shouldn't be too difficult, but remembering that you know things is where the challenge comes from. In the problem above, all we wanted was to simplify the expression. What if we have an equation?

$$\frac{2x+2}{x^4-x^2} = \frac{6}{x^2} - \frac{5}{x^2-1}$$

Notice that we can factor the bottom of the left-hand side of the equation. We can use this to our advantage and multiply everything by these factors; the best practice, in my opinion, is to get rid of every single denominator. It's very difficult to make mistakes with fractions when you don't even have any!

$$\begin{aligned}\frac{2x+2}{x^2(x^2-1)} &= \frac{6}{x^2} - \frac{5}{x^2-1} \\ x^2 \cdot (x^2-1) \cdot \frac{2x+2}{x^2(x^2-1)} &= x^2 \cdot (x^2-1) \cdot \frac{6}{x^2} - \frac{5}{x^2-1} \cdot x^2 \cdot (x^2-1) \\ 2x+2 &= (x^2-1) \cdot 6 - 5 \cdot x^2 \\ 2x+2 &= x^2-6 \\ 0 &= x^2-2x-8\end{aligned}$$

By now, you know how to find the roots of this polynomial. Let's end the problem here. ☕

The technique of "multiply by every denominator" is the most useful thing to do in these scenarios. It even works in cases like this:

$$\frac{2x+2}{x^4-x^2} = 1 - \frac{5}{x^2-1}$$

Since it's exactly the same process as we did before, this problem doesn't need to be worked out in its entirety. Here's what the next step of the problem will look like, and I leave it to you to solve it if you're sufficiently motivated[1]:

$$2x+2 = x^4 - 6x^2$$

Regardless, the technique never fails. It always results in a problem that you can factor, so you should aim to make that happen each time.

2 Other Techniques

Even if factoring is difficult, another tool at your disposal is the quadratic formula; it's used to find the roots of a quadratic, which you can then use to form the factors. Don't forget that if r is a root of a polynomial, then $(x-r)$ is one of its factors! Here's an example:

$$\frac{21x^2-22x-8}{7x^2+14x} \div \frac{3x^2-10x+8}{7x^2-2x}$$

The very first step we need to do is factor out the easiest stuff; then, we'll do the keep-change-flip to simplify the problem. Doing this gives us

$$\frac{21x^2-22x-8}{7x(x+2)} \cdot \frac{x(7x-2)}{3x^2-10x+8}$$

Notice that we can cancel out the x; we'll do that later. First, we need to find the factors of $3x^2-10x+8$ and $21x^2-22x-8$. It is *not* worth the effort to factor this in the way that we did in the beginning of the course. Let's put the quadratic formula to use; for a quadratic in the form $ax^2+bx+c=0$, the roots are found by $r_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. So,

$$\begin{aligned}
&21x^2-22x-8 \\
&r_{1,2} = \frac{22 \pm \sqrt{(-22)^2-4(21)(-8)}}{2(21)} \\
&r_{1,2} = \frac{22 \pm 34}{42} \\
&\quad r_1 = \frac{4}{3}, \quad r_2 = -\frac{2}{7}
\end{aligned}
\qquad\qquad
\begin{aligned}
&3x^2-10x+8 \\
&r^*_{1,2} = \frac{10 \pm \sqrt{(-10)^2-4(3)(8)}}{2(3)} \\
&r^*_{1,2} = \frac{10 \pm 2}{6} \\
&\quad r^*_1 = 2, \quad r^*_2 = \frac{4}{3}
\end{aligned}$$

We can build the factors from these roots, remembering that they "flip" when we put them to factors; however, these become ugly. We can resolve this by moving the denominator to x:

$$\begin{aligned}
21x^2-22x-8 &= \left(x-\tfrac{4}{3}\right)\left(x+\tfrac{2}{7}\right) = (3x-4)(7x+2) \\
3x^2-10x+8 &= (x-2)\left(x-\tfrac{4}{3}\right) = (x-2)(3x-4)
\end{aligned}$$

Now let's put it back together, cancel all of the terms we can, and simplify:

$$\frac{(3x-4)(7x+2)}{x+2} \cdot \frac{7x-2}{(x-2)(3x-4)} = \frac{(7x+2)(7x-2)}{(x+2)(x-2)} = \frac{49x^2-4}{x^2-4}$$

[1] This is not a pleasant problem to solve! There is a quartic formula you can use, but it's absolutely massive and unwieldy. I have no doubt that there's a way to solve this without it, but I personally wouldn't recommend it. If you're curious, it has two real solutions and two complex solutions.