Notes 03: Factoring, Quadratics, & Linear Systems

Edward Bensley
Email: ebensle1@charlotte.edu

Office Hours: Thursday 1:00 - 3:00
Fretwell 321

1 Greatest Common Factor

We learn to think of multiplication as repeated addition, and this is a perfectly fine way to think about things - this hints at the relationship between addition and multiplication, which we call the **distributive** property. Essentially, the distributive property tells us that if adding two things together and then multiplying the result by some number the same as multiplying each individually, then adding them. As an example,

$$\begin{aligned} 2(4+3) &= 2 \cdot 7 \\ &= 14 \\ 2 \cdot 4 + 2 \cdot 3 &= 8 + 6 \\ &= 14 \end{aligned}$$

We should put this relationship into symbols to see the different ways that we can apply it. Remember, the equality sign works *both* ways - if we say that two things are equal, then we can switch freely between them.

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

What we can glean from this is that we can "pull out" a number if it is a multiplicative factor of both numbers that we add (or subtract) together, and we call this process *factoring*. It's easy to think of distributing and factoring as two things that you have to memorize and do. But, as mentioned in the last workshop, what we actually have is another dual relationship: these are two sides of the same coin where factoring is just un-distributing in the same way that subtracting is un-adding. Let's make an example and go through the steps of solving it. First, find the multiplicative factors and identify which ones they share:

$$26x^4 + 39x^3$$
$$(13x^3) \cdot 2x + (13x^3) \cdot 3$$

Then bring them out to the front, multiply them, and put parenthesis around what's left:

$$13x^3(2x + 3)$$

The key thing to remember is that powers of a variable just means that you're multiplying it that many times. Since we brought out three of the four powers of x in the first term, there is just one left when we're done. In our case, $13x^3$ is the **greatest common factor**.

2 Quadratics

An equation (or expression) is called quadratic if there is a variable raised to the second power, i.e. x^2. A quadratic can be binomial, like $x^2 + 1$, or trinomial like $x^2 + 4x - 5$; in any case, solving a quadratic amounts to finding the values of x that make the equation true. Most quadratic equations will have multiple solutions! For example, $x^2 = 4$ is true when $x = 2$ and when $x = -2$ (we could write this as $x = \pm 2$ but it doesn't matter) since a negative times a negative is positive - if you think of multiplying a negative as turning around, then turning around twice means you haven't changed which way you're facing. All of the techniques used to solve quadratics **require** that it be set equal to zero when solving it. The general form of a quadratic is given below.

$$ax^2 + bx + c = 0$$

Given that a quadratic usually has two solutions and that it is set equal to zero, remember what you know about factoring: you end up with two terms multiplying each other. Let's go back to our previous example and set it equal to zero:

$$13x^3(2x + 3) = 0$$

If one of these terms is equal to zero, either $13x^3$ or $2x + 3$, then it doesn't matter what the other term equals because zero multiplied by anything is still zero. This means that we will have two equations to solve:

$$\begin{aligned} 13x^3 &= 0 & \qquad\qquad 2x + 3 &= 0 \\ x &= 0 & x &= -\tfrac{3}{2} \end{aligned}$$

So we have $x = 0$ and $x = -3/2$ as our solutions.

2.1 Simple Case

Factorizing a quadratic means turning a trinomial into the product of two binomials. We will go through the technique and reasoning in the following example:

$$x^2 - 2x - 15 = 0$$

We know that this breaks down into two factors, but we don't know what they are. Let's just use a and b as the unknowns, then multiply the factors together:

$$\begin{aligned} x^2 - 2x - 15 &= (x + a)(x + b) \\ &= x^2 + ax + bx + ab \end{aligned}$$

We can't combine terms unless their variables are the same and have the same degree (which is another way of saying "power"), so let's arrange our equations vertically:

$$\begin{array}{ccc} x^2 & +(a+b)x & +ab \\ x^2 & -2x & -15 \end{array}$$

What we have here is $a + b = -2$ and $ab = -15$. In other words, the factors of 15 (one of which is negative) will add together to -2, so the next step is to find the factors of 15 and work with them to figure out which pair you need and which of them is negative. We have the pairs 1 and 15 and 3 and 5 as factors of 15; it seems obvious that we need -5 and 3. Now this is completely factored, and there's nothing else to do unless we need to solve for x:

$$x^2 - 2x - 15 = (x + 3)(x - 5)$$

☕

2.2 Difficult Case

It's fairly easy to construct a cruel polynomial that is difficult to factor. To see this, we have

$$(ax + b)(cx + d) = (ac)x^2 + (ad + bc)x + bd$$

It is our task to find the left side of the equation from the right side, which is easy when ac equals one or when ac equals a prime number (whose only factors are itself and one such as 3, 5, 7, and so on). For this example,

$$6x^2 + 11x - 10$$

we have a bit of work ahead of us, but there is an amount of deduction that can be used to eliminate possible combinations of pairs. Keeping in mind that we want to find $(ax+b)(cx+d)$, we write $(1, 6)$ to mean that we are guessing that 1 and 6 are the factors of 6 used to build the expression above. So, our guess of the factorization would look like $(x + b)(6x + d)$. This may or may not be correct, but what's important is that we have a way to express things. For now, let's align our quadratic with its general form and make a table of factors:

x^2	x	1
6	11	-10
(ac)	$(ad + bc)$	bd

6	-10
$(1, 6)$	$(1, -10)$
$(2, 3)$	$(-1, 10)$
	$(-2, 5)$
	$(2, -5)$

There is no hard and fast rule that $(1, 6)$ means that $a = 1$ and $c = 6$; we could just as easily say that $a = 6$ and $c = 1$ if we wanted to. At any rate, let's finally get started with working this out. According to the table, we have

$$\begin{aligned} 1 \cdot d + b \cdot 6 &= 11 \\ d + 6b &= 11 \end{aligned}$$

Now we need to figure out if any pairs for -10 work, which may involve testing both numbers in each pair for b and d. Luckily, we don't have to check much: using some deduction, b cannot be any of the numbers on the right of the pairs because it would be too big, and what's more is that it can't be negative either. That means we only need to see if $b = 1$ or if $b = 2$ work. Let's check in that order:

$$-10 + 6 = 4 \qquad -5 + 6(2) = 7$$

Apparently not. So $(1, 6)$ isn't the pair that we're looking for, which means we need to figure out $(2, 3)$:

$$2d + 3b = 11$$

For the same reasons as before, we can actually eliminate the (1,10) pairs completely since they're way too big regardless and in fact -5 can't work at all for b or d. This means we only need to see if $b = 5$ or if $d = 5$:

$$2(-2) + 2(5) = -4 + 15 = 11$$

So there we have it: $d = -2$ and $b = 5$. With $a = 2$ and $c = 3$, we are finally done:

$$6x^2 + 11x - 10 = (2x + 5)(3x - 2)$$

2.3 Common Forms

Here we will show the common forms and their solutions and prove that they actually work. This provides a useful shortcut to factoring when you recognize the forms.

2.3.1 Square of Sums

Some trinomials come from taking a binomial and squaring it:

$$\begin{aligned}(ax+b)^2 &= (ax+b)(ax+b)\\ &= (ax)^2 + (ax)b + b(ax) + b^2\\ &= a^2x^2 + 2(ab)x + b^2\end{aligned}$$

If given a polynomial the first and last terms are square numbers (remember that 1 is a square!), check the middle: if it is equal to the product of the un-squared parts and two, the factor is the form on the left. For example,

$$4x^2 + 12x + 9$$

Since $4 = 2^2$ and $9 = 3^2$, multiply the bases by each other and by two. We get $(2\cdot3)\cdot2 = 12$, which is what we have in the middle term. So our "square of sums" rule tells us that $4x^2+12x+9 = (2x+3)^2$.

2.3.2 Difference of Squares

All quadratic binomials with a subtracted constant can be factored like this:

$$a^2x^2 - b^2 = (ax+b)(ax-b)$$

If the last terms is negative, then the quadratic factors into the left hand side of the equation. For example,

$$81x^2 - 64 = (9x+8)(9x-8)$$

We could also have something ugly, but it works either way:

$$5x^2 - 7 = (\sqrt{5}\cdot x - \sqrt{7})(\sqrt{5}\cdot x + \sqrt{7})$$

3 Linear Systems

From a visual standpoint, linear systems are solved by determining where two lines intersect (if it all). Two lines that intersect have a single solution and the answer is a coordinate, also called an ordered pair. On the other hand, a system of equations could be inconsistent or dependent. Again from a visual standpoint, this would be two parallel lines (no solution, so **inconsistent**) and two lines that are actually the same (infinitely many solutions, so **dependent**). If a system is inconsistent, then the variables will cancel but you will be left with something like $5 = 3$. A dependent system will have the variables canceling, as well as each term, so you will be left with $0 = 0$ - a shortcut determining dependence is noticing that the two equations are the same except that one of them is multiplied by some number. There are two methods to solve these, but neither are particularly difficult. We will solve the following example in both ways:

$$2x + 5y = 10 \tag{1}$$

$$3x - 9y = 48 \tag{2}$$

3.1 Substitution

Substitution is nothing more than solving one equation for one variable and then substituting it into the other. It important to understand that we **must** plug it into the **other** equation; plugging this into the equation we started with is essentially saying that the equation is equal to itself. Since everything is divisible by three in (2), we should use it to solve for x:

$$x = 3y + 16$$

which we plug into (1) and solve for way y:

$$\begin{aligned} 2(3y + 16) + 5y &= 10 \\ 6y + 32 + 5y &= 10 \\ 11y &= -22 \\ y &= -2 \end{aligned}$$

Now plug $y = -2$ into wherever it is most convenient. Since we already solved for x, let's use it: $x = 3(-2) + 16 = 10$. So our final answer here is (10, -2).

3.2 Elimination

Elimination is done by adding the equations together in order to cancel one of the variables. Since x is again easier, we should multiply (1) by 3 and (2) by -2; to abuse some notation, we are solving the "equation" $3[1] - 2[2]$. The best way to do this in practice is vertically:

$$\begin{array}{r} 3 \cdot (2x + 5y = 10) \\ -2 \cdot (3x - 9y = 48) \end{array} \quad \Longrightarrow \quad \begin{array}{rrr} 6x & +15y = & 30 \\ -6x & +18y = & -96 \\ \hline 0 & +33y = & -66 \end{array}$$

It's obvious that $y = -2$. Since we combined the equations, we can plug it in to either of them to find x:

$$\begin{aligned} 2x + 5(-2) &= 10 \\ 2x - 10 &= 10 \\ 2x &= 20 \\ x &= 10 \end{aligned}$$