Notes 08: Function Notation and Composition

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1 Motivation

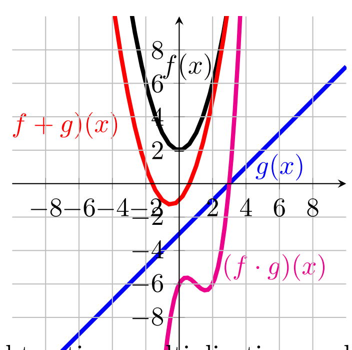
It is not necessary to be a cattle rancher to know how many out of 100 cattle are left in a field if 30 manage to escape, and in the same way, it's not necessary for kaboodles to be real to know how many are left in a box of 100 after throwing 30 from a window; numbers give us a way to talk about relationships without having to actually know anything about what they represent. Numbers are just an abstraction whose behavior lines up with what we see in our physical world, and if we don't even care what the number is, we can "abstract" away with variables because $x + x$ behaves in the same way that $2 + 2$ and $98 + 98$ do, or anything else.

Last week, we talked about function transformations and made a point to mention that $f(x)$ and y have the same meaning. $f(x)$ is read as "f of x" because it means that f is a function acting on x. We usually think of functions as transforming a number into a different number - maybe by adding, dividing, or something else entirely - and maybe even plotting it on a graph. If we have a way to combine numbers, what happens if we want to combine functions?

2 Operations

An operation is a way of saying that we are doing *something* to an object. A binary operation takes two of them and turns them into one thing; for example, addition/subtraction and multiplication/division are classic operations for combining numbers. These operations apply to functions as well, so take $f(x) = x^2+2$ and $g(x) = x-3$. Adding and multiplying them is written as follows

$$\begin{aligned}(f+g)(x) &= f(x)+g(x) \\ &= x^2+2+(x-3) \\ &= x^2+x-1\end{aligned} \qquad \begin{aligned}(f\cdot g)(x) &= f(x)\cdot g(x) \\ &= (x^2+2)(x-3) \\ &= x^3-3x^2+2x-6\end{aligned}$$



This is read as "f plus g of x" because, unsurprisingly, it is a function that acts on x; just by looking, we can see that $(f + g)(x)$ is a shift downwards of $f(x)$ by the value of $g(x)$ at x. So, finding the value of the function is pretty straightforward: plug the value you want in to both f and g, then add them! The same thing applies with subtraction, multiplication, and division. This provides a useful way method to check our work - if we've combined the functions correctly, then we will get the same result as if we combined their outputs directly instead.

The important thing to remark on is that $f(x)$ defines a *rule* to follow. With f as before, writing $f(3)$ means that we substitute 3 in for x wherever we see it, so $f(3) = 3^2 + 1 = 10$. While this isn't too bad, the thing that students find most confusing is something like $f(5x)$ or $f(x+4)$, as the notation is unfamiliar. But this isn't too bad as it just means substituting $5x$ or $(x+4)$ for x:

$$\begin{aligned} f(5x) &= (5x)^2 + 1 \\ &= 25x^2 + 1 \end{aligned} \qquad \begin{aligned} f(x+4) &= (x+4)^2 + 1 \\ &= x^2 + 8x + 17 \end{aligned}$$

One way to avoid this confusion is to rethink the "base" as a function of t, which is the usual way to represent **parameterization**; here, we have $f(t) = t^2 + 1$ and $t = 5x$ and this might make things more clear.

2.1 Composition

Unlike numbers, functions have access to an operation called *composition* and $\circ$ is the symbol we use for it – but bear in mind that this is completely different from the symbol $\cdot$, which is for multiplication. But what *is* composition? In plain English, sugar, flour, eggs, and water are the ingredients that required to bake a cake; we would say that a cake is composed of these things. The idea of composition is similar: one function is plugged into another, with the effect that the output becomes the second function's input. The notation seems a bit backwards: if we write $f \circ g$, what we mean is that the output of g is plugged into f. It might help to think of it as "f after g" and to also think of this as $f(g(x))$. Let's do an example; reusing $f(x)$ from before, let's define a new function h so we have

$$f(x) = x^2 + 2 \quad \text{and} \quad h(x) = \sqrt{2x-7} - 2$$

then compute $f \circ h$ and $h \circ f$:

$$\begin{aligned} (f \circ h)(x) &= f(h(x)) \\ &= \left(\sqrt{2x-7} - 2\right)^2 + 2 \\ &= 2x - 7 - 4\sqrt{2x-7} + 4 + 2 \\ &= 2x - 4\sqrt{2x-7} - 1 \end{aligned} \qquad \begin{aligned} (h \circ f)(x) &= h(f(x)) \\ &= \sqrt{2(x^2+2) - 7} - 2 \\ &= \sqrt{2x^2 - 3} - 2 \end{aligned}$$

This is graphed on Desmos as we can have much more interaction.

3 Piecewise

A piecewise function is simply a function defined in pieces, best demonstrated through example. Consider $f(x)$:

$$f(x) = \begin{cases} x^2, & x < 1 \\ \sqrt{x-1} + 1, & x \geq 1 \end{cases}$$

We have two functions to use if x fits certain criteria, which are given on the right hand side. To find $f(10)$, we use the function with the criteria that 10 satisfies - since $10 \geq 1$, we have $f(10) = \sqrt{10-1} + 1 = 4$. For $f(-4)$, since $-4 < 1$, this time we have $f(-4) = (-4)^2 = 16$. ☕