Notes 02: Fractions & Absolute Values With Inequalities

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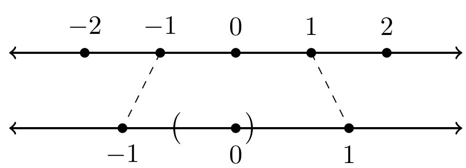
1 Fractions

It should come as no surprise to say that fractions are numbers. This isn't to insult your intelligence: you need to remember that everything we work with is going to follow the same fundamental rules. When it comes to fractions, multiplying and dividing them are the easiest things to do; the only difference between multiplication and division is that there is one extra step that needs to be taken when dividing. Adding and subtracting fractions requires that we multiply them, so we'll tackle that last.

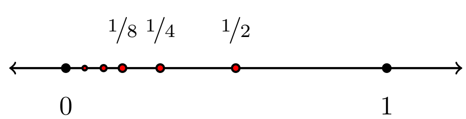
1.1 Motivation

The best way to think about addition, subtract, multiplication, and division is that there are actually only two operations: we have multiplication/division as one and addition/subtraction as the other. In this sense, what we call "division" is just multiplying by the reciprocal while what we call "subtraction" is just adding a negative.

Let's think about whole numbers, also called **integers**. We use symbol $\mathbb{Z}$[1] to mean the set of all integers, but that's not super important. It is easy to see that there's a gap in between each integer, and you can easily construct an interval that contains only one integer (or perhaps none at all):



In regards to fractions, remember that an integer can be written as a fraction; for example, we can write 3 as $^3/_1$. Since fractions are ratios of integers, the proper term is "rational number" and their set is called $\mathbb{Q}$[2]. The distinction between fractions and integers is that we can always find a fraction between any two numbers; in fact, we can find *infinitely many* fractions. We describe this by saying that the rational numbers are dense, where the visual intuition is that there are no gaps:



The real numbers are even more densely packed (in a manner of speaking), but this is a conversation for another day.

[1] The Germans beat us to it. The translation for the word "numbers" is "Zahlen".
[2] $\mathbb{Q}$ is for quotient, so we beat the Germans to it this time.

1.2 Multiplying and Dividing

Just multiply straight across:

$$\frac{9}{5} \cdot \frac{25}{7} = \frac{225}{35}$$

But now we've hit a bit of a roadblock, which is fairly obvious: nobody wants to multiply 9 and 25 when they really don't have to, and this definitely isn't simplified anyway. You could go through the trouble fo simplifying it after you've done all the hard work, but the good news is that you can simplify in the beginning to avoid it when it's annoying. Factor out 25 and cancel out the terms that are the same:

$$\frac{9}{5} \cdot \frac{5 \cdot 5}{7}$$

Since $5/5 = 1$, we're left with

$$\frac{9}{1} \cdot \frac{1 \cdot 5}{7} = \frac{9 \cdot 1 \cdot 5}{1 \cdot 7} = \frac{45}{7}$$

Division is the same process but you flip the fraction on the right. I won't bother to provide an example here, but your intuition should be this: how many halves fit into one whole? Two is the obvious answer. So one divided by one half is two! Remember the phrase "keep, change, flip". Keep the left, change divide to multiply, flip the right. ☕

1.3 Addition and Subtraction

Now imagine that you have a party and order a ton of pizza but not all of it is eaten. You want to consolidate the pizza that remains, and in doing so, you've added fractions. Let's suppose that the pizzas are cut into six slices. Two boxes have one piece left, one box has half remaining, and another has one third left. How many boxes do we need?

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{2} + \frac{1}{3}$$

There are two ways we can do this: brute force and the least common multiple. First, let's do the brute force method. Take each unique denominator (the bottom part) and multiply them together:

$$6 \cdot 2 \cdot 3 = 36$$

This is what we want each denominator to look like. To achieve this, take a fraction multiply it by each of the other fractions' denominators on the top and the bottom; this will ensure that we are multiplying by one and not actually changing anything:

$$\frac{1}{6} \cdot \left(\frac{2}{2} \cdot \frac{3}{3}\right) + \frac{1}{6} \cdot \left(\frac{2}{2} \cdot \frac{3}{3}\right) + \frac{1}{2} \cdot \left(\frac{6}{6} \cdot \frac{3}{3}\right) + \frac{1}{3} \cdot \left(\frac{2}{2} \cdot \frac{6}{6}\right)$$

And our final result[3] is

$$\frac{6}{36} + \frac{6}{36} + \frac{18}{36} + \frac{12}{36} = \frac{42}{36} = \frac{7}{6}$$

The easier way of doing this is to notice that 2 and 3 are both factors of 6. So, all you have to do is multiply the numbers by the appropriate factors:

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{2} \cdot \frac{3}{3} + \frac{1}{3} + \frac{2}{2} = \frac{1}{6} + \frac{1}{6} + \frac{3}{6} + \frac{2}{6} = \frac{7}{6}$$

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[3]Which means we need two boxes, but frankly you should just eat one more slice and only need one box.

2 Variables

Imagine that you have two cows in a pasture. You buy four more; how many cows do you have? Now think about a box of kaboodles: you've got two, but after winning a round of bakoodle, you get four more. How many kaboodles do you have? There are more things we could say, but hopefully you see the point that's being made here. It doesn't matter what we have, what someone else has, or what is actually in existence because we have a way to quantify and describe them.

Numbers are "removed" from reality because they allow us to describe things without needing to know anything about them (or even requiring that they exist at all). The common theme is that they follow the same rules: you have six cows and six kaboodles. When we say 2+4=6, we're saying that *it doesn't even matter* if there's anything in reality that corresponds to these numbers. Similarly, variables follow the same rules as a concrete number: if you add the same variable to itself, you double it; if you multiply it by four, that's like adding it to itself four times. As an example, we have $4 + 4 = 2 \cdot 4$, and $x + x = 2 \cdot x$ to mean the same thing if we say that $x = 4$. With that in mind, fractions containing variables must behave in exactly the same way that you're familiar with. Don't be afraid of them!

3 Inequalities & Absolute Value

Inequalities are solved in the same way that "regular" equations are with the key exception being that multiplying or dividing by a negative number will flip the sign's direction, so I won't give an explicit example. The absolute value is, believe it or not, very similar with the exception that asks the following question: how far away from zero is the stuff in the middle? For example, let's take

$$3 \cdot |2x - 9| + 2 \leq 17$$

The first order of business is to get the absolute value by itself. We're going to subtract two and then divide by three:

$$|2x - 9| \leq 5$$

What numbers, when multiplied by two and then subtracted by nine, are within "five" of zero? What numbers are within "five" of zero in the first place? This is actually straightforward: everything between -5 and 5, including them. We can transform this!

$$-5 \leq 2x - 9 \leq 5$$

Now we solve **both** sides as normal:

$$\begin{aligned} -5 \leq 2x - 9 &\leq 5 \\ 4 \leq 2x &\leq 14 \\ 2 \leq x &\leq 7 \end{aligned}$$

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