Notes 09: Exponents and Logarithms

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1 Motivation

Our first introduction to math is through recognizing numbers and counting up to them; using one hand, we learn to count in the following way: one, two, three, four, and five - but why not jump straight to five? There are other numbers that we could reach first, so what are we *actually* doing when we go "in order"? By insisting that we go from one step to the next, we are (without realizing it!) saying that there is a number that must come next, a successor, and that there is nothing in between[1] a number and its successor. Repeatedly applying succession is how we define addition and is in fact how most people learn to add in the first place.

Using repeated addition gives us multiplication, but of course, this brings up things that are not so simple; considering $^1/_3$ and $^3/_7$, how do you use succession to add them? How do you add one third to itself for three sevenths of a time? It doesn't make sense, but it's not super important that we discuss how to expand their definitions; rather, it's brought up because exponents are special and we need to discuss what taking an exponent really is.

2 Exponentiation

The usual explanation for how to consider exponents and operations is that it is repeated multiplication, where the exponent (or power) tells us how many times we do it. For example, when we say 4^3, what we really mean is $4 \cdot 4 \cdot 4$, so there are three copies of 4 combined by multiplication. The same things applies to solving $(4^3)^2$; this means that we have two copies of 4^3 multiplied together to give us $(4^3) \cdot (4^3)$, so how many copies in total do we have? This should be easy to figure out and leads us straight into the first two exponent rules.

1. $a^b \cdot a^c = a^{b+c}$
2. $\left(a^b\right)^c = a^{b \cdot c}$

With addition comes subtraction and with multiplication comes division, so what do negative numbers and fractions look like if they're exponents? Let's start with negatives.

$$2^{-2} \cdot 2^4 = 2^{-2+4} = 2^2 = 4$$

We know that we'll have 4 as an answer, so let's examine the parts: $2^4 = 16$, so we can write

$$2^{-2} \cdot 16 = 4$$

[1] This only applies to the set $\mathbb{N}$ of **natural numbers**, which are defined by saying that there is a least thing and using succession to generate the "next" one; in fact, this definition doesn't define numbers as an amount of things, which is actually difficult to describe, but rather as an *order*. From this definition, we get things like infinite numbers like ω (omega). If you finished a race an ωth place, an infinite number of people finished the race...and *then* you did.

Dividing both sides by 16,

$$2^{-2} = \frac{1}{4}$$

This must mean that negative exponents result in fractions! We aren't done yet though - we can break the equation up differently to find more rules.

$$\begin{aligned} 2^{-2} \cdot 2^4 &= \left(2^{-2} \cdot 2^2\right) \cdot 2^2 \\ &= 2^0 \cdot 2^2 \\ &= 2^0 \cdot 4 \end{aligned}$$

It must be that $2^0 \cdot 4 = 4$, and therefore $2^0 = 1$ *must* be true! This is just a consequence of our definitions. Let's add these to our list:

1. $a^b \cdot a^c = a^{b+c}$
2. $\left(a^b\right)^c = a^{b \cdot c}$
3. $a^0 = 1$
4. $a^{-b} = \frac{1}{a^b}$

It's worth explicitly stating that negative exponents change the position of its base in a fraction. For example,

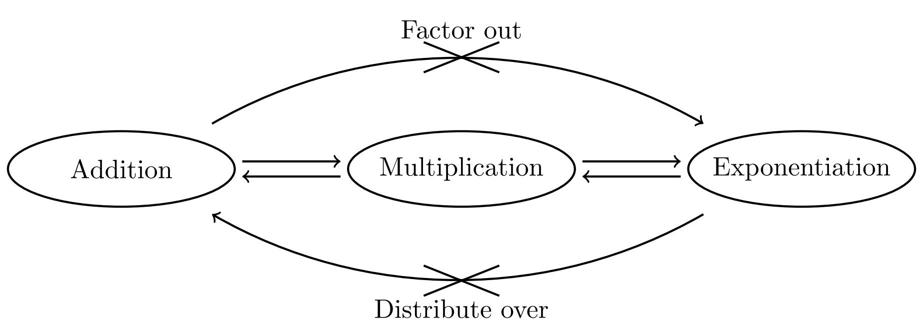
$$\frac{4}{3^{-2}} = \frac{4 \cdot 3^2}{1} = 36$$

Written as we have it, we can freely reorder the multiplication of the exponents. There are a few quirks to keep in mind, so consider 3ˆ3ˆ3 and compare the placement of parentheses:

$$\begin{aligned} &(3\hat{}3)\hat{}3 = 27\hat{}3 = 19,683 \\ &3\hat{}(3\hat{}3) = 3\hat{}27 = 7,625,597,484,987 \end{aligned}$$

Exponentiation behaves very differently from what we've worked with before in the same way that function composition behaves differently. While composition is non-commutative (order matters), exponentiation is non-associative (grouping matters). Think back on how we first think of multiplication and exponentiation - multiplication is introduced as repeated addition and exponentiation is introduced as repeated multiplication. Thanks to these definitions, being able to distribute and factor between multiplication and addition means that we can distribute and factor between exponentiation and multiplication.

In a way, exponentiation is **repeated** repeated addition. We can distribute over the thing below and factor out from the thing above, but we cannot skip steps - these operations are successors too! Below is a diagram to help visualize their relationships:



We've resolved what it means to have a negative exponent; what about fractions? Negative exponents correspond to fractions, so to maintain consistency, anything with an exponent of zero is equal to 1. Let's try to make this happen with fractions, starting with $1/2$; we know that $1/2 \cdot 2 = 1$, so let's use our exponent rules to force that to occur:

$$\left(4^{1/2}\right)^2 = 4^{2 \cdot 1/2} = 4$$

So squaring $4^{1/2}$ means we end up with 4. We know that squaring 2 results in 4, so $4^{1/2} = 2$, and we also know that $2 = \sqrt{4}$ - by chaining these relationships together, we see that $4^{1/2} = \sqrt{4}$. In other words, having fractions as exponents mean we have roots!

We can break up fractions into multiplication, which makes solving problems easier; this is not strictly true in terms of exponents because we have not given a rigorous definition of exponentiation, but it's unlikely that we'll encounter a scenario where it would matter. To round this all out, here's the full list of rules that we've built:

1. $a^b \cdot a^c = a^{b+c}$
2. $\left(a^b\right)^c = a^{b \cdot c}$
3. $a^0 = 1$
4. $a^{-b} = \frac{1}{a^b}$
5. $(a \cdot b)^c = a^c \cdot b^c$
6. $a^{c/b} = \left(\sqrt[b]{a}\right)^c$

2.1 Examples

For this first one, we're going to apply exponents using the rules. The best way to approach these problems is to group together terms with the same base, but it's personal preference whether to move things with negative exponents. The goal is to write this in simplest form.

$$\begin{aligned}
\left(\frac{4x^{-3}y^2}{z^8} \cdot \frac{x^2}{8z^{-2}}\right)^{-4} &= \left(\frac{4}{8} \cdot \frac{x^{-3}x^2}{1} \cdot \frac{y^2}{1} \cdot \frac{1}{z^8 z^{-2}}\right)^{-4} \\
&= \left(\frac{1}{2} \cdot \frac{x^{-1}}{1} \cdot \frac{y^2}{1} \cdot \frac{1}{z^6}\right)^{-4} \\
&= \left(\frac{y^2}{2xz^6}\right)^{-4} \\
&= \left(\frac{2xz^6}{y^2}\right)^4 \\
&= \frac{2^4 x^4 z^{24}}{y^8}
\end{aligned}$$

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Next, we're going to simplify a number with a fraction for an exponent. We'll rearrange the multiplication in the exponent (the fraction!) to simplify things that we can actually work with, and we won't even need a calculator to do it.

$$81^{\frac{3}{4}} = 81^{3 \cdot 1/4}$$

From here, we could chose to solve 81^3 and then take its fourth root. This would be awful to do by hand, so instead we should see what we can do with the fourth root. Since $1/4 = 1/2 \cdot 1/2$, we just take the square root of 81 twice:

$$\left(\left(81^{1/2}\right)^{1/2}\right)^3 = \left(9^{1/2}\right)^3 = 3^3 = 27$$

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3 Logarithms

Logarithms and exponents are interchangeable. In fact, logarithms were invented to turn large multiplication problems into addition problems by converting both into the proper base, adding the result, then converting back. The relationship is simple:

$$\log_a b = x \quad \longleftrightarrow \quad a^x = b$$

An easy way to remember the conversion is by reading left to right: a is the base, b is the answer, and x is the exponent we need to get there. Here are the rules they follow and the ways to combine them:

1. $\log(a) + \log(b) = \log(a \cdot b)$
2. $b \cdot \log(a) = \log(a^b)$
3. $\log_a(a) = 1$
4. $a^{\log_a(b)} = b$
5. $\ln(x) = \log_e(x)$

This list isn't entirely exhaustive, but the others are consequences of the definition and the previous rules - for example, subtraction, the log of zero or one, and so on. If the way to solve a problem is unclear or seems difficult, then write it in both forms.

3.1 Example

To solve the problem below, first we need to convert it into exponential form. Then we need to factor and move around exponents until the left and right side of the equation both have the same base; when this is the case, we can just set the exponents equal to each other.

$$\log_{\frac{1}{8}} 128 = x \quad \longrightarrow \quad \left(\frac{1}{8}\right)^x = 128$$

The most important and least obvious step is figuring out how to find an equivalent expression. It's best to start with powers of 2 or 3 and list them until we find the numbers needed; all powers of 4 can be written as powers of 2, and it is exceedingly unlikely that there will be a power of 5. Here, we have $8 = 2^3$ and $128 = 2^7$, so let's do some more conversion:

$$\begin{aligned}
\left(\frac{1}{2^3}\right)^x &= 2^7 \\
\left(\frac{1}{2}\right)^{3x} &= 2^7 \\
\left(2^{-1}\right)^{3x} &= 2^7 \\
2^{-3x} &= 2^7
\end{aligned}$$

Since the bases are equal, the exponents must be equal too. So we can write $-3x = 7$; therefore, $x = -\frac{7}{3}$. ☕