Factoring by Grouping

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We know that the standard form of a quadratic is $ax^2 + bx + c = 0$. To give a name to things, a is the leading coefficient, b is the middle coefficient, and c is the tailing coefficient. Every quadratic has two roots, so let's call them r_1 and r_2 and remember that plugging these in for x gets zero. The roots can be found by factoring the quadratic into $(x - r_1)$ and $(x - r_2)$, which multiply to get the equation we started with. When the leading coefficient is 1, factoring is as simple as finding the factors of the tailing coefficient that add to the middle coefficient.

But we do not always have the luxury of solving a quadratic with such ease - when the leading coefficient is something else, we may have a problem on our hands: one of these roots may be a *fraction*. In this case, I prefer to write

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$$

so that I can keep track of all the parts we need to care about. Now this has roots $r_1 = {}^{-b}/_{a}$ and $r_2 = {}^{-d}/_{c}$; it's simple to be given two factors and make a quadratic, but starting *with* a quadratic and finding the roots is far more challenging because the factors of the lead and tail will mix together *before* they add up. We could make a list of factors and start matching them together, multiplying and adding endlessly, but this will require a **lot** of guess-and-check and is considered a war crime.

Grouping is a method that multiplies the lead and tail together; then, with each of the factors mixed, the next step is doing what we did before: find the factors of this number that add to the middle coefficient. By mixing everything together, we can systematically list the factors and know straight away if we shouldn't even bother checking if they add up properly or not. The method gets its name from the next step: the pair that works will contain one factor of the lead and one factor of the tail, which means that we can group things no matter what order we write them in.

Since we know we're finding factors of the lead and trail, let's put question marks for the middle.

$$acx^2 + (???)x + bd$$

After we mix them and find the factors that add up to (???), we have

$$\begin{aligned} \big(acx^2 + adx\big) + \big(bcx + bd\big) &= \big(acx^2 + bcx\big) + \big(adx + bd\big) \\ ax\big(cx + d\big) + b\big(cx + d\big) &= cx\big(ax + d\big) + d\big(ax + d\big) \end{aligned}$$

On the left, we factor out $cx + d$. On the right, $ax + d$. Either way, we end up with $(ax + b)(cx + d)$, which is what we wanted.