Notes 08: Function Notation and Composition

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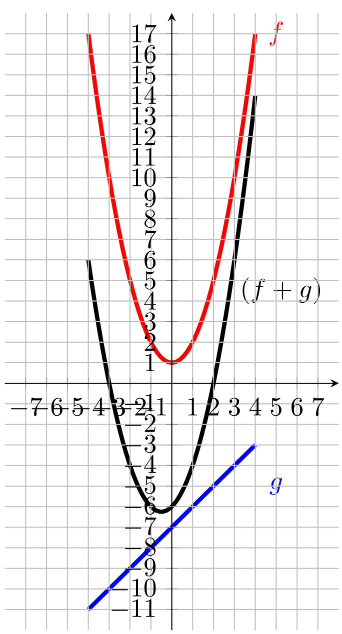
1 Motivation

It is not necessary to be a cattle rancher to know how many out of 100 cattle are left in a field if 30 manage to escape, and in the same way, it's not necessary for kaboodles to be real to know how many are left in a box of 100 after throwing 30 from a window; numbers give us a way to talk about relationships without having to actually know anything about what they represent. Numbers are just an abstraction whose behavior lines up with what we see in our physical world, and if we don't even care what the number is, we can "abstract" away with variables because $x + x$ behaves in the same way that $2 + 2$ and $98 + 98$ do, or anything else.

Last week, we talked about function transformations and made a point to mention that $f(x)$ and y have the same meaning. $f(x)$ is read as "f of x" because it means that f is a function acting on x. We usually think of functions as transforming a number into a different number - maybe by adding, dividing, or something else entirely - and maybe even plotting it on a graph. If we have a way to combine numbers, what happens if we want to combine functions?

2 Operations



From the perspective of transforming numbers, combining functions through addition, subtraction, multiplication, and division is pretty straightforward. For example, let $f(x) = x^2 + 1$ and $g(x) = x - 7$. Adding,

$$\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ &= x^2 + 1 + (x - 7)\\ &= x^2 + x - 6\end{aligned}$$

This is read as "f plus g of x" because, unsurprisingly, it is a function that acts on x; just by looking, we can see that $(f + g)(x)$ is a shift downwards of $f(x)$ by the value of $g(x)$ at x. So, finding the value of the function is pretty straightforward: plug the value you want in to both f and g, then add them! The same thing applies with subtraction, multiplication, and division, so there isn't really a need for further examples here. This provides a useful way method to check our work - if we've combined the functions correctly, then we will get the same result as if we combined their outputs directly instead.

The important thing to remark on is that $f(x)$ defines a *rule* to follow. With f as before, writing $f(3)$ means that we substitute 3 in for x wherever we see it:

$$f(3) = 3^2 + 1 = 10$$

What students find most confusing is something like $f(5x)$ or $f(x+4)$. This is a failure of notation because this just means substituting $5x$ or $(x+4)$ for x in the same way

$$f(5x) = (5x)^2 + 1 = 25x^2 + 1$$
$$f(x+4) = (x+4)^2 + 1 = x^2 + 8x + 17$$

One way to avoid this confusion is to rethink the "base" as a function of t, which is the usual way to represent **parameterization**; here, we have $f(t) = t^2 + 1$ and $t = 5x$ and this might make things more clear.

The motivation to bring up this example in particular is because we have one more operation to discuss: function composition, of which the two above are great examples. Let's demonstrate the notation by giving the above inputs as functions:

$$f(x) = x^2 + 1 \qquad g(x) = 5x \qquad h(x) = x + 4$$

In terms of composition, $f(5x)$ is written as $(f \circ g)(x)$. It is fine to write $f(g(x))$ as well. The proper way to read this would be "f after g of x". So $(f \circ g)(x) = 25x^2 + 1$, and we also have $(f \circ h)(x) = x^2 + 8x + 17$. But function composition has quirks that addition does not! For example,

$$(g \circ f)(x) = 5x^2 + 1$$
$$(h \circ f)(x) = x^2 + 5$$

By simply looking at it, we see that function composition is **not** commutative at all. We can get around this issue in the cases of subtraction and division (just add negatives and multiply fractions), but with composition there is no way to resolve this. Pay close attention and be careful! Properly understanding what the symbol $\circ$ means is vital - it is **not** multiplication! The process of solving is the same though - to find $(f \circ h)(5)$, find $h(5)$ and then plug that value into $f(x)$. We get $h(5) = 9$ and $f(9) = 82$, so $(f \circ h)(5) = 82$.

3 Piecewise

A piecewise function is simply a function defined in pieces, best demonstrated through example. Consider $f(x)$:

$$f(x) = \begin{cases} x^2, & x < 1 \\ \sqrt{x-1} + 1, & x \geq 1 \end{cases}$$

We have two functions to use if x fits certain criteria, which are given on the right hand side. To find $f(10)$, we use the function with the criteria that 10 satisfies - since $10 \geq 1$, we have $f(10) = \sqrt{10-1} + 1 = 4$. For $f(-4)$, since $-4 < 1$, this time we have $f(-4) = (-4)^2 = 16$.