Notes 07: Function Transformations

Edward Bensley
Email: ebensle1@charlotte.edu

Office Hours: Thursday 1:00 - 3:00
Fretwell 321

1 Functions

It takes no great effort to notice the similarities between something like x^2 and $x^2 + 4$, but the similarities between x^2 and x^2+4x+3 may not be as clear. In the latter case, completing the square yields $(x+2)^2 - 1$; now it should be apparent that they *do* have something in common. But what? And, furthermore, how can we tell? To answer this question, it is worth first taking a slight detour to talk about function notation; even though we will cover that in the next section, it is easier to do these things with the proper motivation instead of just presenting rules to memorize. Think about these two statements:

1. I had to prove that $\sqrt{2}$ is irrational, and my little brother helped.

2. I had to prove that $\sqrt{2}$ is irrational, but my little brother helped.

Fundamentally, these two statements provide the same exact information. However, it is the second statement that indicates something beyond just providing information - maybe it's that I *needed* help, maybe it's that I'm proud of him for knowing what to do; regardless, we can present the same thing in different ways to highlight what it is that we want to discuss.

As it relates to functions, consider $y = x^2 - 4$ and $f(x) = x^2 - 4$. At $x = 4$, we end up with $y = 12$ and $f(4) = 12$; if we graph the former, we find y-intercepts at -2 and 2 while using the quadratic formula on the latter gives us the same thing. Why do we bother to write $y =$ instead of $f(x) =$ if they mean the same thing? On the one hand, we might want to highlight our ability to graph the equation and so it would be natural to express this in terms of dependent and independent variables. On the other, we might care more about the function itself and its shape isn't important - or maybe we even have a function that is quite difficult to graph, despite being easy to write.

The most important thing to remember is that $f(x) = y$. That is, they are **interchangeable**. A basic function, also called a parent function, is one that passes through the origin, the point $(0, 0)$, meaning that $f(0) = 0$. There are many types, so here is a list that is not exhaustive (but good enough for what we need):

1. Linear: $f(x) = x$

2. Quadratic: $f(x) = x^2$

3. Cubic: $f(x) = x^3$

4. Square root: $f(x) = \sqrt{x}$

5. Absolute value: $f(x) = |x|$

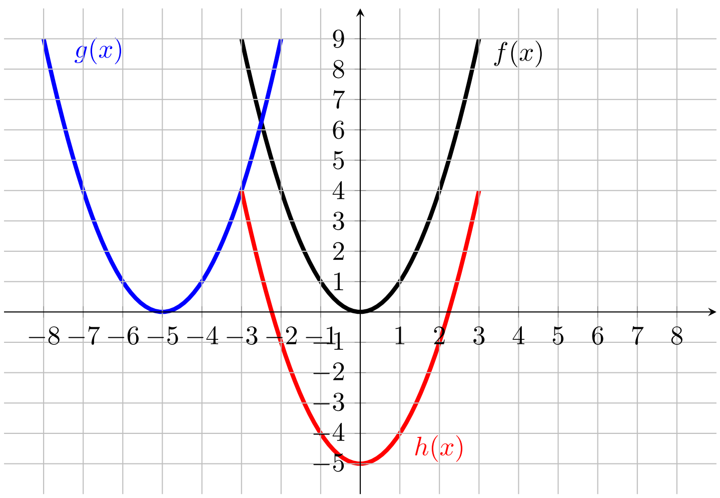
2 Transformations

Think of what it means to transform something. With this in mind, a function can be thought of as taking a number and transforming it to another number; similarly, a function can be transformed into another. The best way to see this is to literally *see* it, so let's actually make a graph of a basic function and its transformations. Take $f(x) = x^2$; first, add 5 *to* the x:

$$f(x) = x^2 \quad \longrightarrow \quad g(x) = (x+5)^2$$

and then subtract 5 from $f(x)$ as a whole:

$$f(x) = x^2 \quad \longrightarrow \quad h(x) = x^2 - 5$$



The function was shifted to the left by five, which runs contrary to what we would expect. For horizontal shifts, the rule is that addition *to* the x shifts left and subtraction shifts right. On the other hand, a vertical shift behaves in the way that we would expect.

Similar to shifts are **compressions** and **stretches**, which behave in the same way: a vertical stretch multiplies $f(x)$ while a compression divides it (or multiplies by a fraction), while a horizontal stretch divides x (or multiplies by a fraction) and a horizontal compression multiplies it instead. Stretching vertically by a factor of 2 means $2 \cdot f(x)$, while compressing horizontally by a factor of 2 means $f(2x)$.

Finally, we have **reflections** over axes. A reflection over the x-axis turns the graph of the function upside down, so it flips the y values, while a reflection over the y-axis flips the x values. Combined with a reflection, it may be difficult to figure out exactly how a function was transformed. Below are all of the ways a function can be transformed in one convenient expression and some rules:

$$g(x) = a \cdot f\big(b(x+c)\big) + d$$

1. $|a| > 0$ is a vertical stretch, and $a < 0$ is a reflection over the x-axis
2. $|b| > 0$ is a horizontal compression, and $b < 0$ is a reflection over the y-axis
3. $c > 0$ is a shift to the left
4. $d > 0$ is a shift up

Caution: It is extremely important that b is factored out. For example, $g(x) = \sqrt{-x+6}$ is shifted to the right!