Notes 05: Polynomial Division and Asymptotes

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1 Rational Functions

Rational functions are best thought of as fractions made of polynomials; to use symbols, for any polynomials $p(x)$ and $q(x)$, we say $r(x) = {}^{p(x)}/_{q(x)}$ is a rational function. For example,

$$\frac{p(x)}{q(x)} = \frac{x^5 + 6x^3 + 2x + 9}{x^2 + 1}$$

In some sense, we can think of these as the "true" forms of real numbers. In fact, the real numbers $\mathbb{R}$ are what's called a field (long story short - everything except zero can divide to 1) and so are the rational functions. There are differences[1] of course, which we will explore later, but for now we want to emphasize their similarities: in the same way that we can reduce fractions by division, we can reduce these as well. For example, we can easily factor and cancel things:

$$\begin{aligned}\frac{2x^2 + 14x + 24}{x^2 - x - 20} = \frac{2(x^2 + 7 + 12)}{x^2 - x - 20} &= \frac{2(x+3)(x+4)}{(x+4)(x-5)} \\ &= \frac{2(x+3)}{x-5}\end{aligned}$$

However, it is not always obvious how to factor these things, and furthermore we may not even be able to factor them. The tool we will use is called *synthetic division* and is similar to the division you should be familiar with; here, we are concerned with remainders as well.

1.1 Division

Consider the following problem:

$$\frac{x^4 - 3x^3 - 8x^2 - 13x + 21}{x - 5}$$

On the next page there is a table demonstrating exactly what we're about to do, so don't worry if the text here is difficult to follow. Notice that the root of the divisor is 5, which is what we are going to use; if it was $x + 5$, we would use -5 instead. Notice that the coefficient for x^4 is 1. Our first step will be to bring this down to the new x^3 spot. Next, we multiply this by 5 and add it to the

[1]Namely, that the rational functions are not *Archimedean*, which is to say that there is nothing larger than any natural number; proving this requires that we define how to tell if a rational function is larger than another. Using this definition, we find that for all $r \in \mathbb{R}$, we have $r < x$ while ${}^{1}/_{x} < r$. In my opinion, this is very interesting - but it is not the point of this class. We can construct any field of fractions using special types of rings called integral domains; these are abstract structures that follow certain rules. The study of these structures is also called algebra, which is dear to my heart.

coefficient of the old x^3; this result will be the coefficient for our new x^2, which is 2. Once again, we will take the new x^2 coefficient, multiply it by 5, and add it to the coefficient of the old x^2, which gives us 2 again, and we will repeat this process until we're done.

5	x^4	x^3	x^2	x	x^0
	1	-3	-8	-13	21
+		5(1)	5(2)	5(2)	5(−3)
	1	2	2	-3	4
	x^3	x^2	x	x^0	$\frac{}{x-5}$

$$\longrightarrow \qquad x^3 + 2x^2 + 2x - 3 + \frac{4}{x-5}$$

The table is set up this way to help you understand what's happening. We are, ultimately, dividing by x and some junk, which means that we are reducing the degree of the top polynomial by one in each term. Everything to the zero power is 1 (excluding 0^0 because it's meaningless and dumb), so denoting the constant term as x to the zero power helps us remember that it's our remainder.

2 Properties of Rational Functions

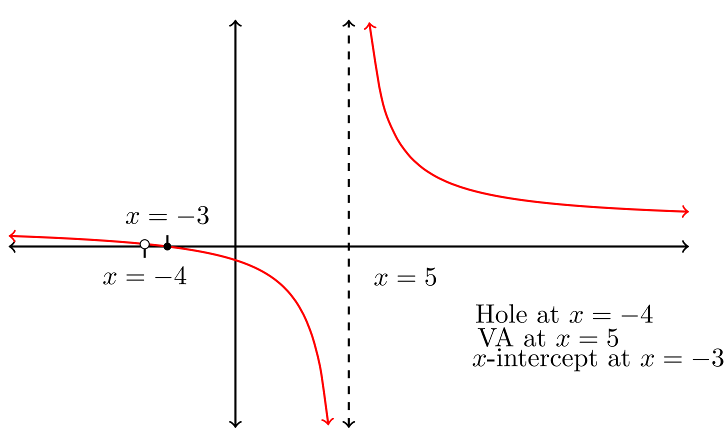
The discussions on domains apply here, so we will put this knowledge to use in order to answer the most important question that we are going to ask: where do we divide by zero, and if we do, does it cancel out? Answering these questions does not require any new information because all we are going to do is factor the top and bottom. However, we also want to know the behavior of the function "at infinity"; determining this *will* require new information.

2.1 Intercepts, Holes, and Vertical Asymptotes

To make a long story short, vertical asymptotes and holes exist wherever $r(x)$ ends up dividing by zero. This amounts to factoring $q(x)$; let's take our second example since we've already factored it:

$$r(x) = \frac{p(x)}{q(x)} = \frac{2x^2 + 14x + 24}{x^2 - x - 20} \qquad \longrightarrow \qquad p(x) = 2(x+3)(x+4), \quad q(x) = (x+4)(x-5)$$

We have two places where there may be a hole or vertical asymptotes, namely $x = -4$ and $x = 5$. The rule to determine which is very simple: if it cancels, then it is a hole, and if it does not, then it is a vertical asymptote. Since $x + 4$ cancels, that means there is a hole at $x = -4$ and a vertical asymptote at $x = 5$ since $x - 5$ does not. Visually, there is quite literally a hole at $x = -4$, meaning that there is no corresponding y value, and that y grows larger as we get closer to $x = 5$ - negative infinity on the left, positive on the right. For the x-intercepts, we look at the numerator *after* we've factored.



2.2 Behavior at Infinity

Finally, we want to talk about the behavior at infinity; that is, we are imagining what would happen if we could actually plug in “infinity” for x. In the graph above, we see that it flattens out as x gets bigger (negative or positive); this implies that we are getting closer to a horizontal line at some y but we won’t actually reach it - this is called a *horizontal asymptote*. This much easier to describe and explain with calculus and the concept of limits, but we don’t have that language yet - meaning that this is one of the few times where you will have to simply memorize the rule.

The thing that we care about is the degree of a polynomial, which is the actual value of the largest exponent. The coefficient **does not matter**, nor does the order that the polynomial is written in. Here are a few examples

$$\begin{aligned}
f(x) &= x^6 + 98x^3 + x^4 - 1776 && \longrightarrow && \deg(f) = 6 \\
g(x) &= x^8 + 500x^3 + 45x^{17} + x^7 + 5 && \longrightarrow && \deg(g) = 17 \\
h(x) &= 46{,}898{,}070{,}491{,}627x && \longrightarrow && \deg(h) = 1 \\
p(x) &= 300 && \longrightarrow && \deg(p) = 0
\end{aligned}$$

To determine the horizontal asymptotes, we need to compare the degrees of the numerator and denominator. This means that there are three possibilities: for $r(x) = {}^{p(x)}/_{q(x)}$, the horizontal asymptote

1. does not exist if $\deg(p) > \deg(q)$,
2. is $y = 0$ if $\deg(p) < \deg(q)$, or
3. is $y = {}^{p_n}/_{q_n}$ if $\deg(p) = \deg(q)$

By p_n and q_n we mean the coefficients that go with the highest degree n. For example, in $g(x)$ above, $g_n = g_{17} = 45$ since $\deg(g) = 17$. It’s good to have notation, but it shouldn’t be confusing - the right way to think about it is the way that makes the most sense to you, so don’t get caught up in writing down things “correctly” when you’re working through a problem. For the example we have above, we see pretty clearly that $\deg(p) = \deg(q) = 2$. So, we find $p_2 = 2$ and $q_2 = 1$, which means that we have a horizontal asymptote at $y = 2$. Visually, this looks like