Budyko Model Derivation

Initial Discussion

We have chosen to take a close look at the Budkyo model first developed by Mikhail I. Budyko in his 1969 article *The Effect of Solar Radiation Variation on the Climate of the Earth*. An important factor that distinguishes the Budyko Energy Balance Model from other similar climate models is the Albedo factor. By definition, albedo is the fraction of light/heat that is reflected by a body or surface. At its base, the Budyko Energy Balance Model seeks to satisfy the simple temperature balance relation

$$\textit{change in temperature} \sim \textit{energy in} - \textit{energy out}.$$

Earth-atmosphere system

The roots of the mode can be taken down as far is the balance equation for a perfectly conducting black body, this equation is given by

$$R\frac{\partial T}{\partial t} = Q - \sigma T^4.$$

We note that R is the heat capacity of the Earth, Q is the solar radiation coming to the outer boundary of Earth's atmosphere, and T is the average mean temperature at a given latitude also averaged over all the longitudes. The next step is adding albedo a, and we get,

$$R\frac{\partial T}{\partial t} = Q(1 - \alpha) - \sigma T^4.$$

We now move on from the elementary assumption that the earth acts like perfectly conducting black body. Instead we add a term given by $A + BT$, which is the outgoing heat or radiation at a given latitude. This is an approximation of both the Stephan-Boltzman's law of black body radiation and the greenhouse gas effect on the atmosphere. Sometimes this term is referred to as the re-radiation or re-emission term. This is one of the balances to the solar radiation coming to the outer boundary of Earth's atmosphere.

Transport or Transmission Term

We next add the term for how the radiation disperses horizontally at a certain latitude, this is the other term for the balancing of the incoming solar radiation. This can be given by

$$C(T - \bar{T}).$$

$\bar{T}$ is the globally averaged mean temperature and C is simply a proportionality constant.

Final Unadjusted Result

We combine everything and get

$$R\frac{\partial T}{\partial t} = Q(1 - \alpha) - (A + BT) - C(T - \bar{T}).$$

Further Conditions and Discussion

We make the assumption that both semi-spheres of the earth are symmetric and note that a location can generally be easier to work with if represented as $y = \sin(\text{latitude}) \in [0, 1]$. We rewrite the equation, and also add a new term s, which represents that latitudinal distribution of the incoming sun radiation. We can now represent Q as the total input into the atmosphere system, instead at the given point, so it no longer is a function of the location. We get, with variables added

$$R\frac{\partial T(t,y)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(t,y)) - C(T(t,y) - \bar{T})$$

with

$$\hat{T} = \int_0^1 T(t,y)dy.$$

The variables are yearly averages, and when discretely considered, each time step represents a year.

A little more on the albedo term

One of the commonly used forms of the albedo term is given from the *Iceline Assumption*. Basically, there is a single iceline located at $t = \eta$ between the equator and the pole. Above η α is a_1 and below it α is a_2. This equation is meant to represent the different radiation reflections on ice (a_1) versus water (a_2). So the assumption is being made that the surface is either water or ice. We have,

$$\alpha(\eta, y) = \begin{cases} a_1, & \text{if } y < \eta \\ a_2, & \text{if } y > \eta \end{cases}$$

Final form of the Equation

$$R\frac{\partial T(t,y)}{\partial t} = Qs(y)(1-\alpha(\eta,y)) - (A + BT(t,y)) - C(T(t,y) - \bar{T}) \tag{1}$$

Setting Parameters

Getting the parameters of the Budyko equation varies by paper and purpose of the project, meaning it is often a very complicated and varying process. The equation, we first define

$$s(y) = 1 - 0.482\frac{3y^2 - 1}{2}$$

Which is the Legendre polynomial approximation. This is the method is in multiple papers including those written by Tchung[25] and North[17] To simplify our project we use the parameters given in Tchung[25] as the system variables. These parameter values were derived from approximations using present climate conditions.

$$Q = 343\frac{W}{m^2} : A = 202\frac{W}{m} : B = 1.9\frac{W}{m} : C = 1.6B : a_1 = 0.62 : a_2 = 0.32 : R = 12.6$$

Equilibrium Solutions

We first examine the different equillibrium solutions based on the current parameters. A system is at equilibrium when the mean temperature does not change over time,

$$Qs(y)(1-\alpha(\eta,y)) - (A+BT(y)) - C(T(y)-\bar{T}) = 0$$

We solve for the equilibrium global mean temperature,

$$\int_0^1 (Qs(y)(1-\alpha(\eta,y)) - (A+BT(y)) - C(T(y)-\bar{T}))dy = 0$$

$$= Q\int_0^1 s(y)dy - Q\int_0^1 s(y)\alpha(\eta,y))dy - \int_0^1 Ady - B\int_0^1 T(y)dy - C\int_0^1 (T(y))dy - C\int_0^1 \bar{T}dy$$

$$= Q(1-\bar{\alpha}(\eta)) - A - B\bar{T} = 0 \rightarrow \bar{T} = \frac{1}{B}\left(Q(1-\bar{\alpha}(\eta))) - A\right).$$

Where $\bar{\alpha} = \int_0^1 s(y)\alpha(\eta,y)dy$. We let this equilibrium global mean temperature be given by the variable $\bar{T}^*$, and now we have the Temperature equilibrium,

$$T^*(y) = \frac{1}{B+C}\left(Qs(y)(1-\alpha(\eta,y)) - A + C\bar{T}^*\right).$$

Before further analyze on the equilibrium solution we need to determine, where the η is. The standard assumption is that the ice line forms if the yearly mean temperature is below $T_c = -10°C$. For each set of parameters, we define a new equation

$$h(\eta) = T^*(y) - T_c = \frac{1}{B+C}(Qs(\eta)(1-a_0) - A + C\bar{T}^*_n) - T_c.$$

Possible η values for equilibrium are given by solutions to $h(\eta) = 0$. We get the figure 1, when we plot $h(\eta)$. It turns out with the given parameters and T_c, there are two stable solutions for η. These solutions are

$$\eta \approx 0.2, 0.95.$$

From here, we have four different equilibrium situations. The first is when the earth is entirely covered in ice. This takes the η dependency out of α, and $\alpha = a_1$. The second is when the earth has not ice line. This means that $\alpha = a_2$. The final conditions are with an existing ice line. One, is with $\eta = 0.95$ and it is often called the large ice cap condition and the other with $\eta = 0.2$ is often called the small ice cap condition (for obvious reasons). This uses the $\alpha(\eta, y)$ used above. We get the figure 2 when we plot with the above defined parameters.

Budyko-Sellers

In addition to the dynamics given by equation 1, we add an additional feature that attempts to characterize the movement of the iceline. In order to properly characterize the slowness of the movement, we have a few assumptions that we make.

1. The movement of the iceline corresponds to the temperature at the iceline being greater than or smaller than the critical temperature

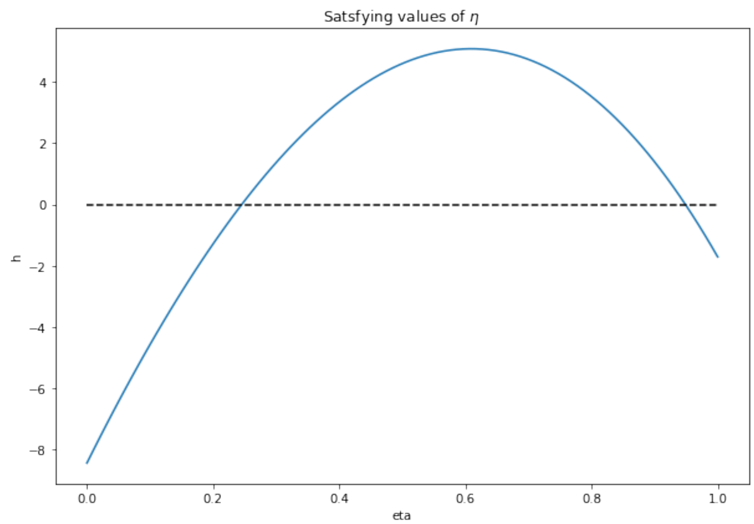

Figure 1: Values of η that contribute to equilibrium

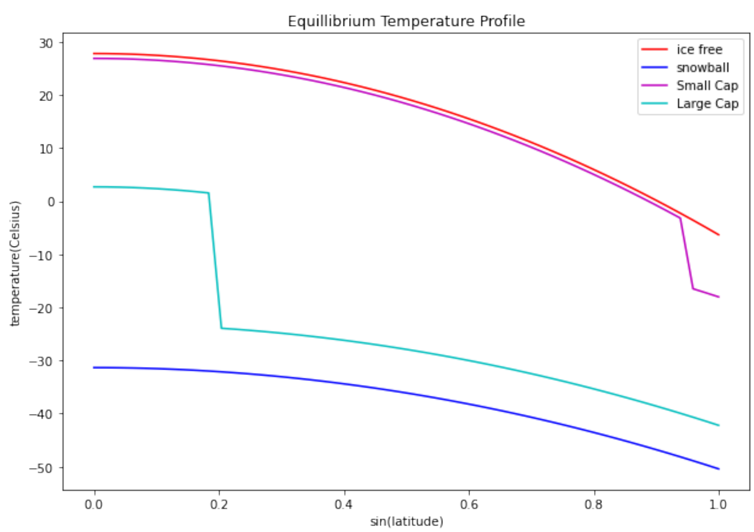

Figure 2: The Temperature Profile of the Equilibrium

2. The equilibrium solutions and analysis should match that given in the original Budyko model
3. The movement of the iceline is much slower than the change in atmospheric temperature, consistent with the fact that glaciers move very slowly

Based on the Budyko-Sellers model we get,

$$\frac{\partial \eta}{\partial t} = \epsilon(T(\eta) - T_c)$$

and combined,

$$\begin{aligned} R\frac{\partial T(t,y)}{\partial t} &= Qs(y)(1-\alpha(\eta,y)) - (A + BT(t,y)) - C(T(t,y) - \bar{T}) \\ \frac{\partial \eta}{\partial t} &= \epsilon(T(\eta) - T_c) \end{aligned} \tag{2}$$

Various animations are given with starting temperature function

$$T_0 = 14 - 54y^2$$

where $y = \sin(latitude)$. Go to *this link* to view the animations. Animations are given for the following hyperparameters:

η_0	ϵ
0.5	0.01
0.5	0.0
0.1	0.01
0.1	0.0
1	0.01
1	0.0

If you look at the animations, you notice that the temperature profile finds an equilibrium and stops changing. But depending on the parameters it finds a different equilibrium.

Proposed Alteration

In reality, none of the constants A, B, Q, C are constant in time. Most of the time, the constants are determined by observing and recording current data on the climate patterns of the Earth and then held constant for the experiments. The term in the original Budyko Model (1) given by $A + BT$ is the part that is likely most affected by the change in carbon emissions. We note that the greenhouse effect is modeled by the fact that $A + BT$, which is the outgoing radiation, decreases as greenhouse gases increase. This relatively quick change in carbon emissions is greatly associated with climate change and global warming. If we add an extra equation to the Budyko-Sellers (2) to describe the change in the carbon emissions

Adding a changing A has been done in other models, but we propose that we also couple this with the changing η to more accurately represent how changing carbon emissions and humans effects on them could change the equilibrium of the model. We have already determined, through our analysis of the Budyko-Model, that the Earth's climate is fairly sensitive to changes in any of the parameters. Using the Budyko-Widiasih model assumptions and reasoning based on the fact that when Earth is mostly covered by ice, the CO_2 builds up in the atmosphere, and when the earth is mostly free of ice the CO_2 is drawn into the atmosphere. We start with

$$\frac{\partial A}{\partial t} = \hat{\epsilon}(\eta - \eta_c)$$

which comes from the above described relation dealing with the iceline. Adding an additional term, will allow us to better characterize additional human interactions affecting carbon emissions. We get

$$\frac{\partial A}{\partial t} = \hat{\epsilon}(\eta - \eta_c) - \alpha$$

where α is the parameter representing the growth (or decrease) of the rate of carbon emissions. There are many options for defining the function α, but in the testing of this model, we will begin by assuming that it is a small positive increase and look at the simulations.

Future Work

First, we will begin by looking at various values of α and their affect on the current Budyko Sellers model. For the future it could prove useful to have a changing α depending on a carbon tax, an influx of electrical cars, more vegetarians, etc.