Budyko Climate Model with Optimal Control

Mathematical Description and Proposed Solution Method

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Abstract

Earth's climate is an incredibly complex system that, at its core, can be represented by equations keeping track of the total energy coming in and the total energy going out. In attempting to minimize and repair the damage humanity has done on the Earth's climate system, we reformulate the Budyko-Sellers Model (3) to an optimal control problem. In doing so, we identify a control variable A, which is the variable representing green house gas emissions. We use numerical solvers and Pontryagin's Maximum Principle to obtain a reasonable optimal control solution to achieving an optimal climate without extreme changes in the current climate. Eventually, we find and compare this optimal control value A for various time domains. Relevant code, plots, and animations for this project are found at, *https://github.com/ebensonm/Climate-Change-Project*

1 Budyko Model Derivation

1.1 Initial Discussion

We have chosen to use the Budyko model first developed by Mikhail I. Budyko in his 1969 article *The Effect of Solar Radiation Variation on the Climate of the Earth*. An important factor that distinguishes the Budyko Energy Balance Model from other similar climate models is the Albedo factor. By definition, albedo is the fraction of light/heat that is reflected by a body or surface. At its base, the Budyko Energy Balance Model seeks to satisfy the simple temperature balance relation

$$\textit{change in temperature} \sim \textit{energy in} - \textit{energy out}.$$

Earth-atmosphere system

The roots of the model are given by the balance equation for a perfectly conducting black body. This equation is

$$R\frac{\partial T}{\partial t} = Q - \sigma T^4.$$

In this model, temperature is in degrees Celcius. We note that R is the heat capacity of the Earth, Q is the solar radiation coming to the outer boundary of Earth's atmosphere, and T is the average mean temperature at a given latitude averaged over all the longitudes. The next step is adding albedo α, and we get,

$$R\frac{\partial T}{\partial t} = Q(1-\alpha) - \sigma T^4.$$

We add a term given by $A + BT$, which is the outgoing heat or radiation at a given latitude and replaces the black body assumption. This is an approximation of both the Stephan-Boltzman's law of black body radiation and the greenhouse gas effect on the atmosphere. Sometimes this term is referred to as the re-radiation or re-emission term. This is one of the balances to the solar radiation Q coming to the outer boundary of Earth's atmosphere.

Transport or Transmission Term

Next, we add the term for the radiation's horizontal dispersal at a certain latitude, this is the other term meant to balance the incoming solar radiation. This is given by

$$C(T - \bar{T}).$$

$\bar{T}$ is the globally averaged mean temperature and C is simply a proportionality constant.

1.2 Final Unadjusted Result

We combine everything and get

$$R\frac{\partial T}{\partial t} = Q(1 - \alpha) - (A + BT) - C(T - \bar{T}).$$

1.3 Further Conditions and Discussion

We make the assumption that both semi-spheres of the earth are symmetric and note that a location can generally be easier to work with if represented as $y = \sin(\text{latitude}) \in [0, 1]$. We rewrite the equation and also add a new term s, which represents the latitudinal distribution of the incoming solar radiation. We can now represent Q as the total input into the atmosphere system, so it no longer is a function of the location. We have the system,

$$R\frac{\partial T(t, y)}{\partial t} = Qs(y)(1 - \alpha) - (A + BT(t, y)) - C(T(t, y) - \bar{T})$$

with

$$\bar{T} = \int_0^1 T(t, y)dy.$$

The variables are yearly averages and when discretely considered, each time step represents a year.

1.4 A little more on the albedo term

One of the commonly used forms of the albedo term is a result of the *Iceline Assumption*. Basically, there is a single iceline located at $t = \eta$ between the equator and the pole. Above η, α is a_1 and below it α is a_2. This equation is meant to represent the different radiation reflection coefficients on ice (a_1) and water (a_2). We have the smooth iceline equation,

$$\alpha(\eta, y) = \frac{\alpha_1 + \alpha_2}{2} + \frac{\alpha_2 - \alpha_1}{2}\left(\tanh\left(M\left(y - \eta\right)\right)\right) \tag{1}$$

1.5 Final form of the equation and parameter values

$$R\frac{\partial T(t, y)}{\partial t} = Qs(y)(1 - \alpha(\eta, y)) - (A + BT(t, y)) - C(T(t, y) - \bar{T}) \tag{2}$$

Getting the parameters of the Budyko equation varies by paper and purpose of the project, meaning it is often a very complicated and varying process. The equation s, we define as

$$s(y) = 1 - 0.482\frac{3y^2 - 1}{2}.$$

This is a Legendre polynomial approximation. This definition of s is used in multiple published papers, including those written by [7] and [8]. To simplify our project, we use the parameters given in [7] as the system variables. These parameter values were derived from approximations using the climate conditions when the paper was written.

$$Q = 343\frac{W}{m^2} : A = 202\frac{W}{m} : B = 1.9\frac{W}{m} : C = 1.6B : \alpha_1 = 0.62 : \alpha_2 = 0.32 : R = 12.6$$

1.6 Budyko-Sellers Model

In addition to the dynamics given by Equation (2), we add a feature that characterizes the movement of the iceline. In order to properly characterize the movement, we have a few assumptions that we need to satisfy.

1. The movement of the iceline corresponds to the temperature at the iceline being greater than or smaller than the critical temperature
2. The equilibrium solutions and analysis should match those given in the original Budyko model
3. The movement of the iceline is much slower than the change in atmospheric temperature, consistent with the fact that glaciers move very slowly

Based on the Budyko-Sellers model and satisfying these assumptions we have,

$$\frac{\partial \eta}{\partial t} = \epsilon(T(\eta) - T_c)$$

and combined,

$$\begin{aligned} R\frac{\partial T(t,y)}{\partial t} &= Qs(y)(1 - \alpha(\eta, y)) - (A + BT(t,y)) - C(T(t,y) - \bar{T}) \\ \frac{\partial \eta}{\partial t} &= \epsilon(T(\eta) - T_c) \end{aligned} \tag{3}$$

T_c is a constant denoting the temperature at which the water will freeze and extend the iceline, and ϵ allows the model to deal with the fact that Earth's climate is very large and thus significant changes in the iceline will take time.

2 Budyko-Sellers Modification for Optimal Control

Using the Budyko-Sellers model describe above, we have two partial differential equations approximating Earth's climate. As mentioned, A is the term that is created to account for excess radiation and green house gas emissions. Instead of a constant A, we allow A to vary with time. For this optimal control problem A will be our control with each $A(t) \in [100, 200]$

2.1 Modification for Optimization

We will also change y to be a constant vector of N dimensions. This means our updated function $\mathbf{T}(t) : \mathbb{R} \to \mathbb{R}^N$. We can justifiably do this because y is just the location at which the temperature is recorded and will be held constant throughout the optimization process.

We can rewrite our differential equations as

$$\mathbf{T}'(t) = \frac{1}{R}\left(Q\mathbf{s}\left(1 - \boldsymbol{\alpha}\left(\eta(t)\right)\right) - (A(t) + B\mathbf{T}(t)) - C\left(\mathbf{T}(t) - \bar{T}\right)\right) \tag{4}$$

$$\eta'(t) = \epsilon(T(t)_{\eta(t)} - T_c) \qquad 0 \leq \eta(t) \leq 1 \tag{5}$$

In this redefinition of the differential equations, the bolded variables represent vector-valued variables or constants. The functions s and α are simply changed to accept a constant vector of locations instead of a value y. Vector multiplication and other vector operations are assumed to be entry-wise unless specified. Also,

$$\bar{T} = \frac{1}{N}\sum_{i=1}^{N} T_i.$$

As a result of the constant $\mathbf{y}$, we can also assume that $T(t)_{\eta(t)}$ in Equation (5), does not actually exist because $\eta(t)$ will never line up exactly with a value in $\mathbf{y}$. We will instead compute $T(t)_{\eta(t)}$, by taking the average of the surrounding temperatures. We write

$$T(t)_{\eta(t)} = \frac{T(t)_k + T(t)_{k-1}}{2} \tag{6}$$

where k is the index of the closest value of $\mathbf{y}$ that is larger than η. In other words

$$k = \operatorname{argmin}\{y_i - \eta : y_i > \eta\} \tag{7}$$

3 Optimal Control Solution Method

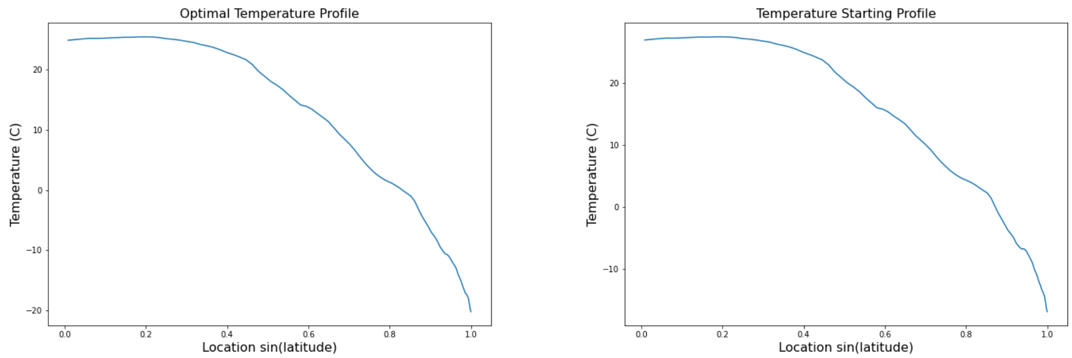
In order to optimize Earth's climate, we need to first decide what the optimal climate is. This is still undetermined, but we will most likely choose an ideal temperature profile from Earth's history, where the human affect on the Earth's climate patterns was not as noticeable. We will call this variable $\mathbf{T}^*$. Ideally, we are minimizing humankind's negative effects on Earth's climate.

We have two things that we will be minimizing; the difference in the final "optimal climate" and the current climate ($\mathbf{T}(t_f) - \mathbf{T}^*$) and the rate at which the temperature profile of the Earth is changing ($\mathbf{T}'$) where t_f is the final time. We define the Lagrangian in our control problem as

$$L(t, \mathbf{T}, A) = \beta\left\|\mathbf{T}'\right\|^2$$

$A(t)$ is the control because it describes the outgoing temperature/radiation of the Earth's climate. In other words, greenhouse gas emissions greatly contribute to $A(t)$ and how it changes with time. We then have the cost functional given by,

$$J[A] = \gamma\left\|\mathbf{T}(t_f) - \mathbf{T}^*\right\|^2 + \int_0^{t_f} \beta\left\|\mathbf{T}'\right\|^2 dt \tag{8}$$



(a) The optimal temperature profile

(b) The starting temperature profile

Figure 1: *Both temperature profile graphs needed for finding the solution of the optimal climate control problem. The difference is subtle, but important. The optimal temperature profile has an overall smaller average and is about* 5 *degrees cooler at the poles.*

where $T^{'}$ is defined above and is a function of the control A and t_f is fixed. The optimal temperature profile T^* and the starting profile T_0 are given in Figure (1). As in standard in optimization problems, we will be finding the control $A(t)$ that minimizes the value of the cost functional

3.1 Solution Method

3.1.1 Hamiltonian Derivation

We first write out the Hamiltonian using the equation

$$H = \mathbf{p} \cdot f - L$$

where $\mathbf{p}$ is the costate variable, f is given by the equations of constraints described in Equations (4) and (5), and L is the cost we are minimizing. So,

$$\begin{aligned} H &= \sum_{i=0}^{N-1} p_i T_i' + p_N \epsilon (T(t)_{\eta(t)} - T_c) - \beta \left\| \mathbf{T}^{'} \right\|^2 \\ &= \sum_{i=0}^{N-1} \left(p_i T_i' - \beta T_i'^2 \right) + p_N \epsilon (T(t)_{\eta(t)} - T_c) \end{aligned}$$

where N is the number of locations at which we are taking the temperature and T and other vector variables are indexed starting at $i = 0$. Finally, using Equation (6) we get the final form of the Hamiltonian as

$$H = \sum_{i=0}^{N-1} \left(p_i T_i' - \beta T_i'^2 \right) + p_N \epsilon \left(\frac{T_k + T_{k-1}}{2} - T_c \right) \tag{9}$$

where k is defined in Equation (7).

3.2 Pontryagin's Maximum Principle

3.2.1 Costate Equations

We use Pontryagin's maximum principle and then compute the costate equations as

$$p_i' = \begin{cases} -\frac{\partial H}{\partial T_i} & \text{if } i < N \\ -\frac{\partial H}{\partial \eta} & \text{if } i = N \end{cases} \tag{10}$$

Recall that $i \in \{0, 1, \ldots, N\}$. Now, we first take the partial derivatives of the different elements of the Hamiltonian (9).

$$\begin{aligned} \frac{\partial}{\partial T_i}\left(\sum_{i=0}^{N-1} p_i T_i'\right) &= \frac{1}{R}\left(p_i\left(-B - C\left(1 - \frac{1}{N}\right)\right) + \sum_{j \neq i} \frac{p_j C}{N}\right) \\ &= -\frac{1}{R}\left(p_i (B + C) - \frac{C}{N}\sum_{j=0}^{N-1} p_j\right) \end{aligned} \tag{11}$$

and

$$\begin{aligned} \frac{\partial}{\partial T_i}\left(\sum_{i=0}^{N-1} \beta T_i'^2\right) &= \frac{2\beta}{R}\left(T_i'\left(-B - C\left(1 - \frac{1}{N}\right)\right) + \sum_{j \neq i} \frac{C T_i'}{N}\right) \\ &= \frac{2\beta}{R}\left(\frac{C}{N}\sum_{j=0}^{N-1} T_j' - T_i' (B + C)\right) \end{aligned} \tag{12}$$

The other partial derivatives of the Hamiltonian are much more straightforward and we do not show the complete computation here. We combine Equations (11), (12) and the other partial derivatives and we get

$$\frac{\partial H}{\partial T_i} = -\frac{1}{R}\left((B + C)\left(p_i - 2\beta T_i'\right) + \frac{C}{N}\sum_{j=0}^{N-1}\left(2\beta T_j' - p_j\right)\right) + p_N \epsilon\left(\frac{\partial}{\partial T_i}\left(\frac{T_k + T_{k-1}}{2}\right)\right)$$

Now we define a new piecewise function

$$T_{\eta\delta} = \frac{\partial}{\partial T_i}\left(\frac{T_k + T_{k-1}}{2}\right) = \begin{cases} \frac{1}{2} & \text{if } k = i \\ \frac{1}{2} & \text{if } k - 1 = i \\ 0 & \text{otherwise} \end{cases}$$

We finally have

$$-\frac{\partial H}{\partial T_i} = \frac{1}{R}\left((B + C)\left(p_i - 2\beta T_i'\right) + \frac{C}{N}\sum_{j=0}^{N-1}\left(2\beta T_j' - p_j\right)\right) - p_N \epsilon T_\eta \delta \tag{13}$$

We compute the second part

$$
\begin{aligned}
-\frac{\partial H}{\partial \eta} &= \sum_{i=0}^{N-1}\left(p_i \frac{Q}{R} s_i \frac{\partial \alpha_i}{\partial \eta} - 2\beta\left(T_i'\left(\frac{Q}{R} s_i \frac{\partial \alpha_i}{\partial \eta}\right)\right)\right) \\
&= \frac{Q}{R}\sum_{i=0}^{N-1}\left(\frac{\partial \alpha_i}{\partial \eta} s_i\left(p_i - 2\beta T_i'\right)\right) \qquad (14)
\end{aligned}
$$

Combining Equations (13) and (14), we get the final form of the costate equations.

Finally, we derive endpoint conditions for the costates p_i given by

$$p_i(t_f) = -\frac{\partial \phi(t_f)}{\partial T_i(t_f)}$$

where

$$\phi(t_f) = \gamma\left\|\mathbf{T}(t_f) - \mathbf{T}^*\right\|^2$$

From here,

$$p_i(t_f) = \begin{cases} -2\gamma(T_i - T_i^*) & \text{if } i < N \\ 0 & \text{if } i = N \end{cases} \qquad (15)$$

As metioned our other boundary condition is given by $\mathbf{T}[0] = T_0$ for some given T_0. The starting value for T_0 was determined from current weather and climate data at *Berkely Earth*.

3.2.2 Optimality Equation Derivation

Now, we compute the optimality equations

$$\frac{\partial H}{\partial A} = \frac{1}{R}\sum_{i=0}^{N-1}\left(2\beta T_i' - p_i\right) = 0.$$

Solving for the $A(t)$ in $\mathbf{T}'$ the control is,

$$A^* = \frac{\sum_{i=0}^{N-1}\left(-\frac{Rp_i}{2\beta} + Qs_i\left(1 - \alpha(\eta(t))_i\right) - BT_i + C\left(\bar{T} - T_i\right)\right)}{N}$$

With control bounds on $A(t) \in [a, b]$ we get

$$A^* = \min\left\{\max\left\{a, \frac{\sum_{i=0}^{N-1}\left(-\frac{Rp_i}{2\beta} + Qs_i\left(1 - \alpha(\eta(t))_i\right) - BT_i + C\left(\bar{T} - T_i\right)\right)}{N}\right\}, b\right\} \qquad (16)$$

The corresponding boundary value problem is created by substituting A^* into Equation (4) for $A(t)$ and propagating those changes to Equation (10).

3.3 Computational Solution Method

In the solving of this problem, we attempted two methods. The first, was simply using scipy's boundary value problem solver to solve the boundary value problem defined in Section (3). Because of the size of this system, testing and attempting to get this method working was tedious and most of the time was spent waiting for the algorithm to run. For this reason, we decided to do a different approach.

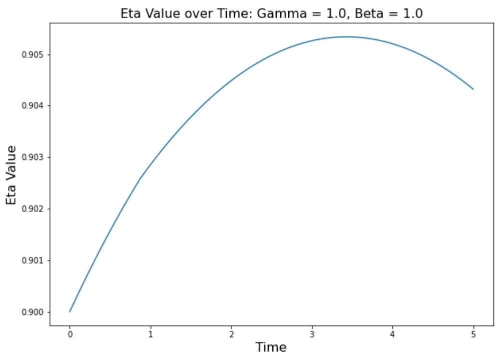
The second method, was an iterative method similar to that done in the HIV lab. We used this iterative method to converge on a solution to the equation and Runge-Kutta methods to solve for the states of the system making sure to enforce the final boundary point of the costates, we used these states to find the costates and then the corresponding optimal control. The final value of the optimal control was then updated. We repeated this process until the optimal control converged in a given tolerance. Refer to the following code block for the implementation of our algorithm. This method was much faster than Scipy's built-in boundary value problem solver. For this reason, we selected the iterative solver in finalzing our results and solutions.

```
for i in range(maxiter):
    optimalControl = optimalControlNew
    state = RK4(stateEquations,initialState)

    #set costate final value
    finalCostate = computeCostateBoundary(state)

    #solve the costate equations with backwards iteration
    costate = RungeKutta(costateEquations,finalCostate)[::-1]

    #compute optimal control
    controlStar = computeOptimalControl(costate, state)

    #update control
    optimalControlNew = 1/2*(controlStar+optimalControl)

    #stopping condition
    if (norm(optimalControlNew-optimalControl) < tol):
        break
```

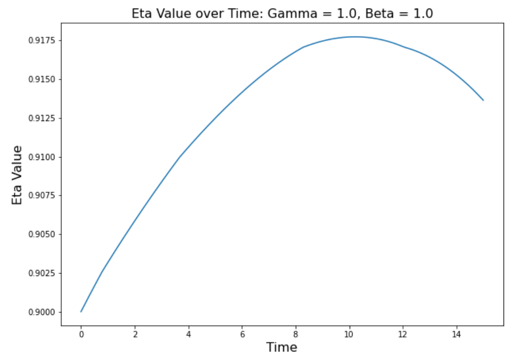
4 Results and Conclusions

4.1 Iterative Method

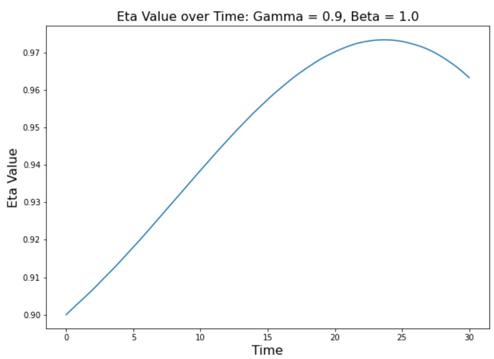
When we first tested the iterative method, especially with longer times, we noticed that the algorithm did a good job when γ was small compared to β. This corresponds to finding an optimal control that favors a non-moving temperature profile over reaching the optimal profile. We determined that a better initial guess to the actual solution could improve the algorithm's success with



(a) η evolution with a final time of $t = 5$



(b) η evolution with a final time of $t = 15$



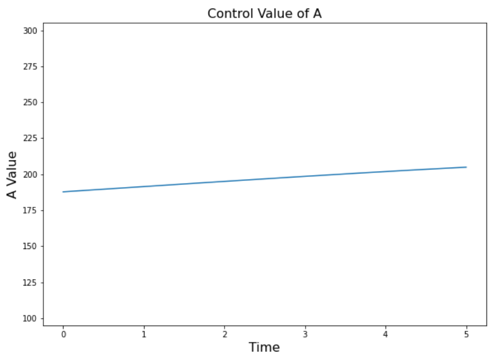
(c) η evolution with a final time of $t = 30$

Figure 2: *This group of figures show the evolution of η (or the iceline) for each of the final times* $t_f \in \{5, 15, 30\}$. The overall structure is similar, but each plot covers a different region. Note the difference of scales on the y-axis. The starting value eta_0 on each of the plots was the same.

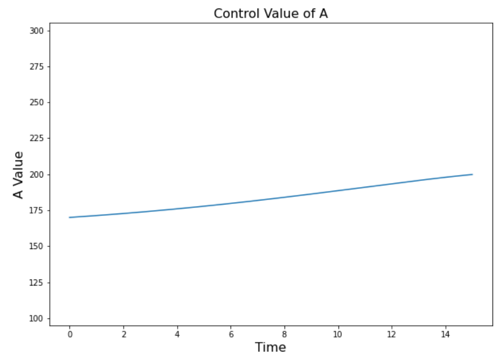
a larger γ value. In order to make better initial guesses, we added a version of continuation. We started with a γ value of 0 and found the optimal control. From here, we increased the value of γ by some step size $\Delta\gamma$ and used the optimal control from the previous step as the initialization point for the next value of γ. This process was repeated until γ reached a certain value or the solver no longer converged on a solution. We computed the optimal control $A(t)$ for the values of $t_f \in \{5, 15, 30\}$. When $t_f = 30$ we were not able to reach $\gamma = 1.0$ before the continuation algorithm stopped converging. Instead, it ended on a value of 0.9. For each t_f we show three different plots.

The first group of plots shows the evolution of η (or the iceline). For these plots, refer to Figure (2) We observe that the general structure is very similar, but each plot is a little different. The main difference is in the actual values of η and not the shape of the graph. The smaller the final time, the smaller the overall change in η. This makes sense, as the movement of η is slow and a larger time window will allow for more η movement.

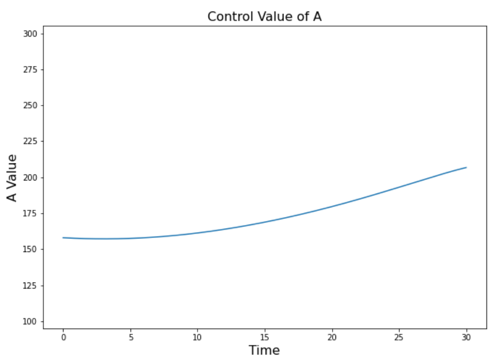
The second group of plots shows the optimal control solution $A(t)$. As mentioned, we are as-



(a) Optimal control A with $t_f = 5$



(b) Optimal control A with $t_f = 15$

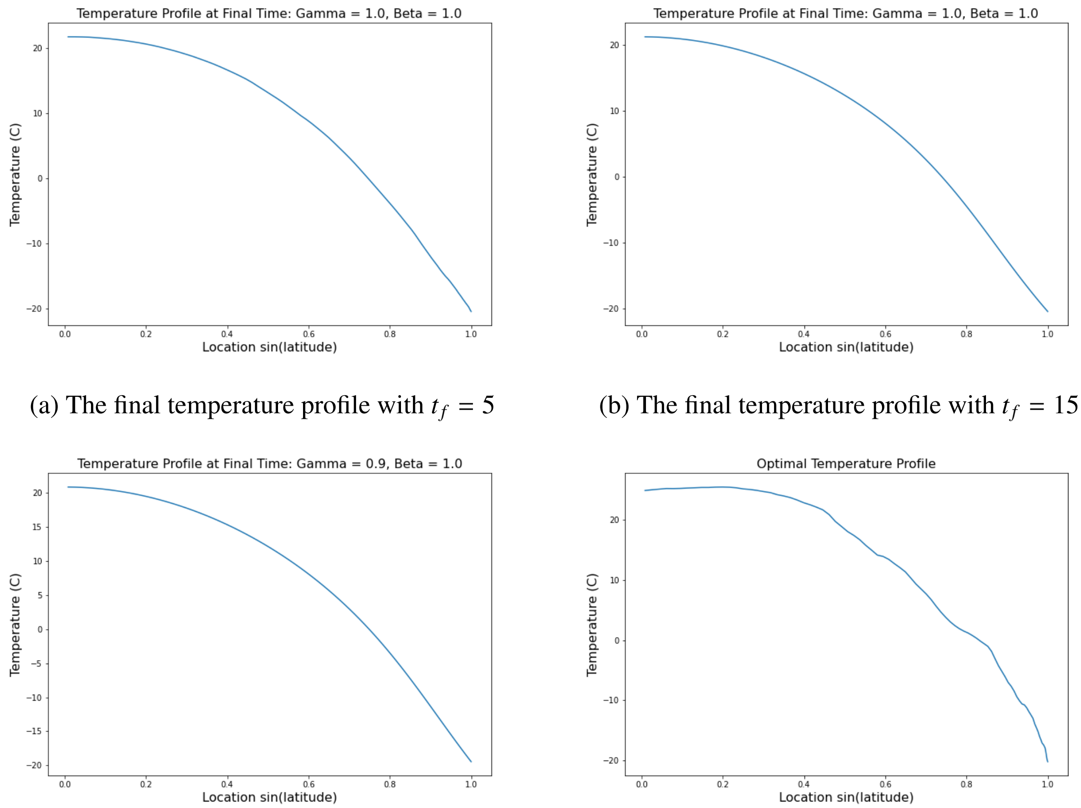


(c) Optimal control A with $t_f = 30$

Figure 3: *The group of plots shows the optimal control for each* $t_f \in \{5, 15, 30\}$. *Like the* η *plots the structure is very similar. In addition, the optimal control values are also around the same region. It appears, that a larger time domain allows for a smaller start in A, meaning that greenhouse gas emissions do not have to start small, but can gradually make there way to an optimal emission.*

suming we can control this variable, because it accounts for greenhouse gas emissions, and humans are often considered a major contributor in excess greenhouse gas emissions. For these plots, refer to Figure (3). Even though the end value for all of the plots is about the same, we observe that a larger final time allows for smaller starting value and then a gradual increase to the final value. This corresponds to more subtle changes in the way that humanity contributes to total green house gas emissions.

The final group of plots displays the final temperature profile at the end of the time domain as well as the desired optimal temperature profile for reference. For these plots, refer to Figure (4). In all cases, we can see that the final temperature profile is able to get relatively close the optimal temperature profile. Though the optimal and starting profiles are not extremely different (*see Figure 1*), we are able to identify a control A within all time limits that decreases the overall temperature profile as desired.



(a) The final temperature profile with $t_f = 5$

(b) The final temperature profile with $t_f = 15$

(c) The final temperature profile with $t_f = 30$

(d) The optimal temperature profile

Figure 4:

For animations of the temperature profiles over the entire time domain, go to our *GitHub* for the project.

4.2 Future Work

In order to further improve the optimal control problem, it could be important to also provide control equations for the iceline η. Much of the climate is affected by the movements of η, which is physically explained by the melting and freezing of the massive glaciers at the poles. Many scientists are concerned of the rate at which these glaciers are melting and many of the proposed solutions, by our optimal control solver, show that the glaciers will melt even more before allowing the Earth's mean average temperature to drop within an optimal range. This could be a problem and to avoid this, we could add a final value cost to η similar to that done on the final temperature profile.

In addition to the η modification to our model, our choice of optimal climate was not based on anything except the temperature profile about 60 years ago. To further improve our accuracy we could find a more ”optimal” optimal climate and use that in our optimal control problem.

Finally, it would be interesting to analyze what the values of A specifically mean with respect to green house gas emissions and what we can do to achieve these A values in real life.

4.3 Conclusion

Though the Budyko Model is not a perfect model for the modeling of Earth’s climate, it is one of the most widely used and considered models when attempting to do mathematical modeling on climate patterns and temperature movements. In our optimal control problem, we were able to find reasonable solutions to our optimal control problem. In other words, according to the Budyko Model and the values of the optimal control $A(t)$, there are things that we can do, either negative or positive, that influence the Earth’s climate.

We do not propose any kind of direct comparison to the value of $A(t)$ and the levels of greenhouse gas emissions, but we are assuming the bounds on A given in $[100, 300]$ are reasonable and in some way able to be applied to how humanity can affect Earth’s climate. We are far from a complete analysis on optimal control applied to the Budyko-Sellers Model (refer to the previous section for more analysis ideas), but our experiments successfully achieved what they were designed for.

Citations

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